The local geometry of idempotent Schur multipliers

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Abstract

A Schur multiplier is a linear map on matrices which acts on its entries by multiplication with some function, called the symbol. We consider idempotent Schur multipliers, whose symbols are indicator functions of smooth Euclidean domains. Given $1 < p \neq 2 < \infty$, we provide a local characterization (under some mild transversality condition) for the boundedness on Schatten p-classes of Schur idempotents in terms of a lax notion of boundary flatness. We prove in particular that all Schur idempotents are modeled on a single fundamental example: the triangular projection. As an application, we fully characterize the local L_p-boundedness of smooth Fourier idempotents on connected Lie groups. They are all modeled on one of three fundamental examples: the classical Hilbert transform, and two new examples of Hilbert transforms that we call affine and projective. Our results in this paper are vast noncommutative generalizations of Fefferman's celebrated ball multiplier theorem. They confirm the intuition that Schur multipliers share profound similarities with Euclidean Fourier multipliers —even in the lack of a Fourier transform connection— and complete, for Lie groups, a longstanding search of Fourier L_p-idempotents.

Introduction

Schur multipliers are linear maps on matrix algebras with a great impact on geometric group theory, operator algebras, and functional analysis. Their definition is rather simple on discrete spaces $S_M(A) = (M(j,k)A_{jk})_{jk}$. It easily extends to nonatomic σ-finite measure spaces (Ω, μ), by restricting to operators A in $L_2(\Omega, \mu)$ admitting a kernel representation over $\Omega \times \Omega$. Their role in geometric group theory and operator algebras was first analyzed by Haagerup. His pioneering work on free groups [19] and the research thereafter on semisimple lattices [4, 13] encoded deep geometric properties via approximation properties with Schur multipliers. Other interesting links can be found in [1, 35, 37, 39, 40, 42].

In 2011, stronger rigidity properties of high rank lattices were discovered by studying L_p-approximations [25, 26]. In first place, there are no L_p-approximations by means of Fourier or Schur multipliers over $\mathrm{SL}_n(\mathbf{R})$ for $p > 2 + \alpha_n$, with $\alpha_n \to 0$ as $n \to \infty$. Secondly, it turns out that this unprecedented pathology leads to a strong form of nonamenability which is potentially useful to distinguish the group von Neumann algebras of $\mathrm{PSL}_n(\mathbf{Z})$ for different values of $n \geq 3$, the most iconic form of Connes' rigidity conjecture. This has strongly motivated our recent work [10, 33] with several forms of the Hörmander-Mikhlin theorem. Nevertheless, there is still much to learn about less regular multipliers. A key point in [26] was a careful analysis of Schur multipliers over the n-sphere for symbols of the form $M_\varphi(x, y) = \varphi(\langle x, y \rangle)$. More precisely, the boundedness of S_{M_φ} on the Schatten class S_p for $p > 2 + \frac{2}{n-1}$ imposes Hölder regularity conditions on φ. This article grew from the analysis of the *spherical Hilbert transform*

$$H_{\mathbf{S}} : A \mapsto \left(\mathrm{sgn}\langle x, y \rangle A_{xy} \right)_{xy}.$$

Is it S_p-bounded for some $\frac{2n}{n+1} < p \neq 2 < \frac{2n}{n-1}$? It is worth noting the analogy with the ball multiplier problem, which was only known to be unbounded for p outside this range before Fefferman's celebrated contribution [15]. Our main result completely solves this problem: $H_{\mathbf{S}}$ is S_p-unbounded unless $n = 1$ or $p = 2$. In fact we characterize S_p-boundedness for a much larger class of idempotents.

Let M, N be two differentiable manifolds with the Lebesgue measure coming from any Riemmanian structure on them. Consider a $\mathcal{C}^1$-domain $\Sigma \subset M \times N$ so that its boundary $\partial\Sigma$ is a smooth hypersurface, which is locally represented by level sets of some real-valued $\mathcal{C}^1$-functions with nonvanishing gradients. We say that Σ is transverse at a point (x, y) in its boundary when the tangent space of $\partial\Sigma$ at (x, y) maps surjectively on each factor $T_x M$ and $T_y N$. In that case, both sections $\partial\Sigma_x = \{y' \in N \mid (x, y') \in \partial\Sigma\}$ and $\partial\Sigma^y = \{x' \in M \mid (x', y) \in \partial\Sigma\}$ become codimension 1 manifolds on a neighbourhood of y and x respectively.

Theorem A. *Let $p \in (1, \infty) \setminus \{2\}$ and consider a $\mathcal{C}^1$-domain $\Sigma \subset M \times N$. Then the following statements are equivalent for any transverse point $(x_0, y_0) \in \partial\Sigma$:*

(1) S_p-*boundedness. The idempotent Schur multiplier S_Σ whose symbol equals 1 on Σ and 0 elsewhere is bounded on $S_p(L_2(U), L_2(V))$ for some pair of neighbourhoods U, V of x_0, y_0 in M, N.*

(2) *Zero-curvature condition. There are neighbourhoods U, V of x_0, y_0 in M, N such that the tangent spaces $T_y(\partial\Sigma_{x_1})$ and $T_y(\partial\Sigma_{x_2})$ coincide for any pair of points $(x_1, y), (x_2, y) \in \partial\Sigma \cap (U \times V)$.*

(3) *Triangular truncation representation. There are neighbourhoods U, V of the points x_0, y_0 in M, N and $\mathcal{C}^1$-functions $f_1 : U \to \mathbf{R}$ and $f_2 : V \to \mathbf{R}$, such that the domain $\Sigma \cap (U \times V) = \{(x, y) \in U \times V : f_1(x) > f_2(y)\}$.*

Theorem A characterizes the local behavior of S_p-bounded idempotent Schur multipliers at transverse points. This includes spherical Hilbert transforms and vastly amplifies Fefferman's ball multiplier theorem [15]. When $M = N = \mathbf{R}^n$ and $\mathbf{n}(x, y) = (\mathbf{n}_1(x, y), \mathbf{n}_2(x, y))$ is a normal vector to $\partial\Sigma$ at (x, y), transversality means that both n-dimensional components $\mathbf{n}_1, \mathbf{n}_2$ are nonzero. The zero-curvature condition means that $\mathbf{n}_2(x_1, y)$ and $\mathbf{n}_2(x_2, y)$ are parallel—equivalent forms in terms of $\mathbf{n}_1(x, y_1)$ and $\mathbf{n}_1(x, y_2)$ instead, or simpler formulations for $\mathcal{C}^2$-domains will be also discussed after the proof of Theorem A. Fefferman's Fourier analytic framework corresponds to sets of the form

$$\Sigma = \Big\{(x, y) : x - y \in \Omega\Big\}$$

for an Euclidean domain Ω, which turn out to be fully transverse and partly explains why transversality did not appear so far. It is also quite remarkable that Theorem A holds beyond Euclidean spaces, since Schur multipliers on general manifolds lack to admit a Fourier transform connection. It has come as a surprise that even after substantial and necessary differences with Fefferman's formulation, a similar phenomenon applies in such a general context. In a different direction, a global nonlocal characterization of S_p-bounded idempotent Schur multipliers also follows for relatively compact fully transverse domains Σ. Let us note that this class of domains has no representatives in the Fourier analytic setting, since Toeplitz-like domains $\Sigma = \{(x, y) : x - y \in \Omega\}$ are not relatively compact.
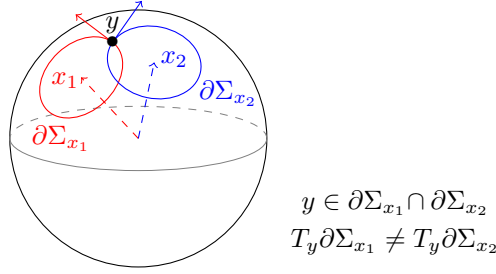
$$y \in \partial\Sigma_{x_1} \cap \partial\Sigma_{x_2}$$
$$T_y\partial\Sigma_{x_1} \neq T_y\partial\Sigma_{x_2}$$

FIGURE 1. Failure of (2) for spherical Hilbert transforms $H_{\mathbf{S},\delta}$
Here $H_{\mathbf{S},\delta} = S_\Sigma$ with $\Sigma = \{(x,y) \in \mathbf{S}^n \times \mathbf{S}^n : \langle x, y \rangle > \delta\}$ for $n = 2$.

Theorem A has profound consequences for Fourier multipliers on Lie group von Neumann algebras. Smooth Fourier multipliers on group algebras were intensively investigated over the last decade [5, 11, 17, 22, 23, 30, 33]. The nonsmooth theory concerns a longstanding search to classify idempotent Fourier L_p-multipliers, but the geometric behavior of their symbols is very sensitive to the underlying group. Bożejko and Fendler [3] studied an analog of Fefferman's ball multiplier theorem in the free group for $|1/p - 1/2| > 1/6$, though the general case $p \neq 2$ stands open since then. More recently, Mei and Ricard found a large class of free Hilbert transforms in their remarkable work [29]. The search for Hilbert transforms on general groups also includes crossed products and groups acting on tree-like structures [18, 34].

In this paper, we shall give a complete characterization of the local boundary behavior for completely bounded idempotent Fourier multipliers on arbitrary Lie groups. Our result is more easily stated for simply connected groups. We refer to Section 2 for the statement of the result on general Lie groups (Theorem 2.3) and for the precise definitions of local Fourier multipliers.

Theorem B. *Let $p \in (1,\infty) \setminus \{2\}$. Let G be a simply connected Lie group, $\Omega \subset G$ a C^1-domain and $g_0 \in \partial\Omega$ a point in the boundary of Ω. The following are equivalent:*

   (1) *χ_Ω defines locally at g_0 a completely bounded Fourier L_p-multiplier.*

   (2) *There is a smooth action $G \to \mathrm{Diff}(\mathbf{R})$ by diffeomorphisms on the real line, such that Ω coincides on a neighbourhood of g_0 with $\{g \in G \mid g \cdot 0 > g_0 \cdot 0\}$.*

Lie himself classified Lie groups admitting (local) actions by diffeomorphisms on the real line [28]. This classification into three types (translation, affine, and projective) gives rise to the following three fundamental examples of a group G with a smooth domain Ω:

   i) The real line $G_1 = \mathbf{R}$ with $\Omega_1 = (0, \infty)$.

   ii) The affine group $G_2 = \mathrm{Aff}_+(\mathbf{R})^\star$ and $\Omega_2 = \{ax + b : b > 0\}$.

   iii) The universal covering group $G_3 = \widetilde{\mathrm{PSL}}_2(\mathbf{R})^\dagger$ with $\Omega_3 = \{g : \alpha_g(0) > 0\}$.

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*Affine increasing bijections $x \mapsto ax + b$ for $a \in \mathbf{R}^*_+$ and $b \in \mathbf{R}$, isomorphic to $\mathbf{R} \rtimes \mathbf{R}^*_+$.

†The action $\alpha : \widetilde{\mathrm{PSL}}_2(\mathbf{R}) \curvearrowright \mathbf{R}$ is obtained by lifting the standard action of $\mathrm{PSL}_2(\mathbf{R})$ on the projective line to the universal covers. If $p : \mathbf{R} \to P^1(\mathbf{R})$ denotes the universal cover, then the universal cover of $\mathrm{SL}_2(\mathbf{R})$ is identified with the group of homeomorphisms $g : \mathbf{R} \to \mathbf{R}$ for which there is $A \in \mathrm{PSL}_2(\mathbf{R})$ such that $p \circ g = A \cdot p$.

Lie's classification implies the following interesting consequence.

Corollary B1. *Conditions* (1) *and* (2) *in Theorem* B *are equivalent to*:

> (3) *There is* $j \in \{1, 2, 3\}$ *and a smooth surjective homomorphism* $f \colon \mathrm{G} \to \mathrm{G}_j$
> *such that the domain* Ω *coincides on a neighbourhood of* g_0 *with* $g_0 f^{-1}(\Omega_j)$.

Therefore, there are three fundamental models for Hilbert transforms on Lie groups.

The above examples define (not only local, but) global completely bounded L_p Fourier multipliers for $1 < p < \infty$. Example i) is the classical Hilbert transform and Example iii) follows from recent Cotlar identities for unimodular groups [18]. The nonunimodular Example ii) will be properly justified in Example 2.8.

Theorem B and Corollary B1 give very satisfactory descriptions of completely bounded Fourier idempotents in arbitrary Lie groups. It is certainly surprising that these multipliers are modeled out of exactly three fundamental examples, the classical Hilbert transform and its affine and projective variants. It also shows that every $\mathcal{C}^1$-idempotent is automatically $\mathcal{C}^\infty$. This rigidity property collides head-on with the much more flexible scenario of Theorem A. In the following result, we further describe Fourier L_p-idempotents for large classes of Lie groups.

Corollary B2. *Let* $p \in (1, \infty) \setminus \{2\}$ *and let* G *be a Lie group*:

> i) *If* G *is simply connected and nilpotent, every cb-L_p-bounded smooth Fourier idempotent is locally of the form* $H \circ \varphi$, *for the Hilbert transform* H *and some continuous homomorphism* $\varphi \colon \mathrm{G} \to \mathbf{R}$.
>
> ii) *If* G *is a simple Lie group which is not locally isomorphic to* $\mathrm{SL}_2(\mathbf{R})$, *then* G *does not carry any smooth Fourier idempotent which is locally completely L_p-bounded on its group von Neumann algebra.*
>
> iii) *If* G *is locally isomorphic to* $\mathrm{SL}_2(\mathbf{R})$, *then* G *carries a unique local Fourier idempotent which is completely L_p-bounded on its group algebra (up to left/right translations) given by* $g \mapsto \frac{1}{2}\big(1 + \operatorname{sgn} \operatorname{Tr}(g e_{12})\big)$.

As an illustration for stratified Lie groups, the homomorphism φ corresponds on the Lie algebra level with the projection onto any 1-dimensional subspace of the first stratum. The second statement above spotlights the singular nature of harmonic analysis over simple Lie groups. It also yields an alternative way to answer our motivating question: the spherical Hilbert transform $H_{\mathbf{S}}$ is not L_p-bounded for any $p \neq 2$. Finally, as we shall justify, the third statement gives a straightforward solution (in the negative) to Problem A in [18]. We refer to [36, 38] for the operator space background necessary for this paper.

The plan of the paper is as follows. Section 1 is devoted to idempotent Schur multipliers. It contains the proof of Theorem A and several discussions, including our analysis of the spherical Hilbert transform. Section 2 is devoted to Fourier multipliers. It contains the proof of Theorem B and its corollaries. The proof relies on a result of independent interest on the local transference between Fourier and Schur multipliers for arbitrary locally compact groups, Theorem 2.1.

1. **Idempotent Schur multipliers**

In this section we give a complete proof of Theorem A. We begin by recalling some particularly flexible changes of variables for Schur symbols, which preserve the S_p-norm of the corresponding Schur multipliers on nonatomic spaces. Then, we prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ in Theorem A separately. We shall finish with some comments and applications to spherical Hilbert transforms.

1.1. **Schur multipliers.** Let (X, μ) and (Y, ν) be σ-finite measure spaces. Given $1 \leq p < \infty$, let $S_p(L_2(X), L_2(Y))$ be the space all of bounded linear operators $T : L_2(X) \to L_2(Y)$ with $\operatorname{Tr} |T|^p < \infty$, which is a Banach space for the norm below

$$\|T\|_{S_p} = \left(\operatorname{Tr} |T|^p \right)^{\frac{1}{p}}.$$

When $p = 2$, the Schatten class $S_2(L_2(X), L_2(Y))$ is the space of Hilbert-Schmidt operators $L_2(X) \to L_2(Y)$. It coincides with $L_2(X \times Y)$, regarding any L_2-function $(x, y) \mapsto K(x, y)$ as the kernel of the corresponding Hilbert-Schmidt operator

$$T_K f(y) = \int_X K(x, y) f(x) d\mu(x).$$

Given $m \in L_\infty(X \times Y)$, the Schur S_p-multiplier with symbol m is defined (when it exists) as the unique bounded linear map S_m on $S_p(L_2(X), L_2(Y))$ which assigns $T_K = (K(x,y))_{x \in X, y \in Y} \in S_2 \cap S_p$ to $(m(x,y)K(x,y))_{x \in X, y \in Y} = S_m(T_K)$. We shall write $\|m\|_{MS_p}$ for its norm, with the convention $\|m\|_{MS_p} = \infty$ if S_m does not exist.

The following general fact will be crucial in our proof of Theorem A. It evidences a much greater flexibility of Schur multipliers compared to Fourier multipliers. The proof follows from [26], we include the argument below.

Lemma 1.1. *Let $(X, \mu), (X', \mu'), (Y, \nu), (Y', \nu')$ be atomless σ-finite measure spaces and $f \colon X \to X'$ and $g \colon Y \to Y'$ be measurable maps. Assume the pushforward measures $f_* \mu$ and $g_* \nu$ are absolutely continuous with respect to the measures μ' and ν' respectively. Then, for every $m \in L_\infty(X' \times Y')$*

$$\|m \circ (f \times g)\|_{MS_p(L_2(X,\mu), L_2(Y,\nu))} = \|m\|_{MS_p(L_2(X', f_* \mu), L_2(Y', g_* \nu))}$$
$$\leq \|m\|_{MS_p(L_2(X', \mu'), L_2(Y', \nu'))}.$$

Proof. The last inequality follows directly from [26, Lemma 1.9] and the absolute continuity assumption. To prove the first identity and lighten the notation, let us assume for simplicity that $(X, \mu) = (Y, \nu)$, $(X', \mu') = (Y', \nu')$ and $f = g$. Let $\mathcal{B}, \mathcal{B}'$ be the underlying σ-algebras, and consider $\mathcal{A} := f^{-1}(\mathcal{B}')$. Then f allows to identify $L_2(X', \mathcal{B}', f_* \mu)$ with $L_2(X, \mathcal{A}, \mu)$. In particular we have

$$\|m \circ (f \times f)\|_{MS_p(L_2(X, \mathcal{A}, \mu))} = \|m\|_{MS_p(L_2(X', \mathcal{B}', f_* \mu))},$$

and similarly for the cb-norm. On the other hand, [26, Lemma 1.13] implies that the cb norms of $m \circ (f \times f)$ on $S_p(L_2(\mathcal{A}, \mu))$ and $S_p(L_2(\mathcal{B}, \mu))$ coincide, so we deduce

$$\|m \circ (f \times f)\|_{cbMS_p(L_2(X, \mathcal{B}, \mu))} = \|m\|_{cbMS_p(L_2(X', \mathcal{B}', f_* \mu))}.$$

Then [26, Theorem 1.18] allows us to conclude. Indeed, our assumptions that μ and μ' have no atoms imply that both cb-norms are equal to their norms. $\qquad\square$

1.2. **Proof of Theorem A: Boundedness implies zero-curvature.** In this paragraph we prove $(1) \Rightarrow (2)$ from the statement of Theorem A. The key new idea we introduce is an amplification of Meyer's lemma [15, Lemma 1], which directly connects Schur idempotents with Meyer's classical condition, to which Fefferman's construction may be applied. Taking charts, we can and will assume that M and N are open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. We shall further assume for simplicity that $m = n$, our argument applies as well when $m \neq n$. Given $z \in \partial\Sigma$, let $\mathbf{n}(z) = (\mathbf{n}_1(z), \mathbf{n}_2(z)) \in \mathbf{R}^n \oplus \mathbf{R}^n$ be a normal to $\partial\Sigma$ at z pointing towards Σ.

Lemma 1.2. *Consider a $\mathcal{C}^1$-domain $\Sigma \subset \mathbf{R}^n \times \mathbf{R}^n$ and open sets $U, V \subset \mathbf{R}^n$ such that $\partial\Sigma$ intersects $U \times V$. Let S_Σ be the idempotent Schur multiplier whose symbol is the characteristic function of Σ and assume that it is bounded on $S_p(L_2(U), L_2(V))$ with norm C. Let $x_1, x_2, \ldots, x_N \in U$ and $y \in V$ such that $z_j = (x_j, y)$ is a transverse point in the boundary $\partial\Sigma$ for every $j = 1, 2, \ldots, N$. Define $u_j = \mathbf{n}_2(z_j)$ and consider functions $f_1, f_2, \ldots, f_N \in L_p(\mathbf{R}^n)$. Then, we have*

$$\left\| \left(\sum_{j=1}^N |H_{u_j}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbf{R}^n)} \leq C \left\| \left(\sum_{j=1}^N |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbf{R}^n)}.$$

Here we write H_u for the u-directional Hilbert transform $\widehat{H_u f}(\xi) = \chi_{\langle \xi, u \rangle > 0} \widehat{f}(\xi)$.

The implication $(1) \Rightarrow (2)$ in Theorem A follows from Lemma 1.2. Indeed, by the transversality assumption, the map $z \mapsto \mathbf{n}_2(z)/\|\mathbf{n}_2(z)\|$ is continuous on a neighbourhood of the transverse point (x_0, y_0) in Theorem A. Moreover, for y close to y_0 we have that $\partial\Sigma^y$ is locally a manifold, so is connected. Therefore, if (2) was not true, there would exist y close to y_0 such that the subset of the sphere $X = \{\mathbf{n}_2(x', y)/\|\mathbf{n}_2(x', y)\| : x' \in \partial\Sigma^y \cap U\}$ contains a connected subset not reduced to a point. According to (1) and Lemma 1.2, this would imply that the square function inequality there holds uniformly in $L_p(\mathbf{R}^n)$ for any finite set in a continuum of directions in the $(n-1)$-sphere. However, Fefferman's main result in his proof of the ball multiplier theorem [15] claims that such a uniform inequality cannot hold. In fact, Fefferman stated it for $n = 2$ but the result in arbitrary dimension follows from the 2-dimensional case by K. de Leeuw's restriction theorem [27]. In particular, the zero-curvature condition (2) must hold.

Proof of Lemma 1.2. The proof relies on the following two claims:

(A) Let (x, y) be a transverse point in the boundary of $\partial\Sigma$ and let $T \in \mathrm{GL}_n(\mathbf{R})$ be such that $T^* \mathbf{n}_1(x, y) = -\mathbf{n}_2(x, y)$. Then, the following identity holds for almost every $\xi, \eta \in \mathbf{R}^n$

$$\lim_{\varepsilon \to 0^+} \chi_\Sigma \big(x + \varepsilon T\xi, y + \varepsilon\eta \big) = \frac{1}{2} \big(1 + \mathrm{sgn}\langle \mathbf{n}_2(x, y), \eta - \xi \rangle \big).$$

(B) Let u_j be as in the statement. Then the Schur multiplier

$$\big((\xi, j), \eta \big) \in \big(\mathbf{R}^n \times \{1, \ldots, N\} \big) \times \mathbf{R}^n \mapsto \frac{1}{2} \big(1 + \mathrm{sgn}\langle \eta - \xi, u_j \rangle \big)$$

is bounded on $S_p(L_2(\mathbf{R}^n), L_2(\mathbf{R}^n \times \{1, 2, \ldots, N\}))$ with norm $\leq \|\chi_\Sigma\|_{MS_p}$.

Assuming the validity of the above claims, we may now conclude the proof using standard transference ideas that go back at least to the work of Bożejko and Fendler

[2]. We know from [7, Theorem 5.2] that there is an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ and a completely isometric map

$$j_p\colon L_p(\mathbf{R}^n) \to \prod_{\mathcal{U}} S_p(L_2(\mathbf{R}^n))$$

that intertwines Fourier and Schur multipliers. This gives $j_p(H_u(f)) = S_{m_u}(j_p(f))$ for every $u \in \mathbf{R}^n$ and $f \in L_p(\mathbf{R}^n)$. Here $m_u(\xi,\eta) = \frac{1}{2}(1 + \operatorname{sgn}\langle \eta - \xi, u\rangle)$. As a consequence we have for every $f_1, f_2, \ldots, f_N \in L_p(\mathbf{R}^n)$

$$\left\|\Big(\sum_{j=1}^{N} |H_{u_j}(f_j)|^2\Big)^{\frac{1}{2}}\right\|_{L_p(\mathbf{R}^n)} = \left\|\sum_{j=1}^{N} H_{u_j}(f_j) \otimes e_{j,1}\right\|_{L_p(\mathbf{R}^n;S_p)}$$

$$= \left\|\sum_{j=1}^{N} j_p(H_{u_j}(f_j)) \otimes e_{j,1}\right\|_{S_p}$$

$$= \left\|\sum_{j=1}^{N} S_{m_{u_j}}(j_p(f_j)) \otimes e_{j,1}\right\|_{S_p}.$$

According to claim (B) we deduce

$$\left\|\Big(\sum_{j=1}^{N} |H_{u_j}(f_j)|^2\Big)^{\frac{1}{2}}\right\|_{L_p(\mathbf{R}^n)} \leq \|\chi_\Sigma\|_{MS_p} \left\|\sum_{j=1}^{N} j_p(f_j) \otimes e_{j,1}\right\|_{S_p}$$

$$= \|\chi_\Sigma\|_{MS_p} \left\|\Big(\sum_{j=1}^{N} |f_j|^2\Big)^{\frac{1}{2}}\right\|_{L_p(\mathbf{R}^n)}.$$

Thus, the assertion is a consequence of claim (B), for which we need to justify claim (A) first. To do so, we can assume that $\Sigma = f^{-1}(0, \infty)$ for a $\mathcal{C}^1$-submersion $f\colon U \times V \to \mathbf{R}$. Then $\nabla f(x,y) = (\nabla_x f(x,y), \nabla_y f(x,y))$ is a normal vector to the boundary $\partial \Sigma = f^{-1}(0)$ at every $(x,y) \in \partial \Sigma$. Thus, replacing f by a multiple, we can assume that its gradient is $(\mathbf{n}_1(x,y), \mathbf{n}_2(x,y))$. Then, the Taylor expansion of f gives

$$f\big(x + \varepsilon T\xi, y + \varepsilon\eta\big) = \varepsilon\langle \mathbf{n}_1(x,y), T\xi\rangle + \varepsilon\langle \mathbf{n}_2(x,y), \eta\rangle + o(\varepsilon)$$

$$= \varepsilon\langle \mathbf{n}_2(x,y), \eta - \xi\rangle + o(\varepsilon).$$

Therefore, if $\eta - \xi$ is not orthogonal to $\mathbf{n}_2(x,y)$ (a condition that holds for almost every ξ and η), we have $\chi_\Sigma(x + \varepsilon T\xi, y + \varepsilon\eta) = \frac{1}{2}(1 + \operatorname{sgn}\langle \mathbf{n}_2(x,y), \eta - \xi\rangle)$ for every $\varepsilon > 0$ small enough. This proves claim (A).

To prove claim (B) we apply (A). More precisely, let $T_j \in \mathrm{GL}_n(\mathbf{R})$ be such that $T_j^* \mathbf{n}_1(x_j, y) = -\mathbf{n}_2(x_j, y) = -u_j$ for every $j = 1, 2, \ldots, N$. The existence of these maps is clear, because by the transversality assumption both $\mathbf{n}_1(x_j, y)$ and $\mathbf{n}_2(x_j, y)$ are nonzero vectors in $\mathbf{R}^n$ and $\mathrm{GL}_n(\mathbf{R})$ acts transitively on them. By Lemma 1.1 the Schur multiplier with symbol

$$m_\varepsilon\big((\xi, j), \eta\big) = \chi_\Sigma\big(x_j + \varepsilon T_j\xi, y + \varepsilon\eta\big)$$

is bounded with norm $\leq \|\chi_\Sigma\|_{MS_p}$ for every $\varepsilon > 0$. Taking $\varepsilon \to 0^+$, we obtain that the almost everywhere limit of m_ε is S_p-bounded with norm $\leq \|\chi_\Sigma\|_{MS_p}$. However this limit is $(1 + \operatorname{sgn}\langle \eta - \xi, u_j\rangle)/2$ from claim (A). This proves claim (B). $\qquad\square$

Remark 1.3. Taking
$$\Sigma = \big\{(x,y) : x - y \in \Omega\big\}$$
for some smooth domain Ω, Lemma 1.2 reduces to the classical Meyer's lemma.

1.3. **Proof of Theorem A: Zero-curvature implies triangular truncations.**
The implication $(2)\Rightarrow(3)$ is a general geometric statement concerning transverse hypersurfaces in manifolds of product type. Note that both conditions are invariant by diffeomorphisms of product type, that is of the form $(x,y) \mapsto (\phi(x), \psi(y))$. It will be useful to have a description of a local normal form, analogous to the fact that hypersurfaces are locally the diffeomorphic image of a hyperplane.

Lemma 1.4. *Let $\Sigma \subset \mathbf{R}^n \times \mathbf{R}^n$ be a $\mathcal{C}^1$-domain and (x_0, y_0) be a transverse point of its boundary. Then, there are diffeomorphisms ϕ and ψ from neighbourhoods U and V of x_0 and y_0 into $\mathbf{R}^n$ satisfying $\phi(x_0) = \psi(y_0) = 0$ and*
$$\Sigma \cap (U \times V) = (\phi \times \psi)^{-1}\Big\{\big((s,\tilde{x}),y\big) : s > g(\tilde{x},y)\big\}\Big\}$$
for some $\mathcal{C}^1$-function $g \colon \mathbf{R}^{n-1} \times \mathbf{R}^n \to \mathbf{R}$ satisfying $g(0,y) = \langle y, e_1\rangle$ for every y.

Proof. Fix $z_0 = (x_0, y_0)$, by the transversality assumption that $T_{z_0}\partial\Sigma$ surjects onto the second coordinate, $T_{z_0}\partial\Sigma \cap (\mathbf{R}^n \oplus 0) \neq \mathbf{R}^n \oplus 0$. Equivalently, we have $\mathbf{n}_1(z_0) \neq 0$. In particular, we may construct a local diffeomorphism ϕ on the first variable, satisfying that $\phi(x_0) = 0$ and $(e_1, 0) \notin T_{z_0}\partial\Sigma$. Then, according to the implicit function theorem, there is a $\mathcal{C}^1$-function $h \colon \mathbf{R}^{n-1} \times \mathbf{R}^n \to \mathbf{R}$ such that
$$\phi \times \mathrm{id}(\Sigma) = \Big\{\big((s,\tilde{x}),y\big) \mid s > h(\tilde{x},y)\Big\}$$
on a neighbourhood of $(0, y_0)$. Moreover, the function $h(0, \cdot)$ vanishes at y_0 and by the second half of the transversality assumption, it has nonzero differential at y_0. By the implicit function theorem again, there is a diffeomorphism ψ from a neighbourhood of y_0 into $\mathbf{R}^n$ vanishing at y_0 and such that $h(0,y) = \langle\psi(y), e_1\rangle$ for every y close enough to y_0. Taking $g(\tilde{x},y) = h(\tilde{x}, \psi^{-1}(y))$, we get
$$\phi \times \psi(\Sigma) = \Big\{\big((s,\tilde{x}),y\big) : s > g(\tilde{x},y)\Big\}$$
in a neighbourhood $\phi(U) \times \psi(V)$ of $(0,0)$ as expected. Also $g(0,y) = \langle y, e_1\rangle$. $\square$

With this in hand, we can prove the implication $(2)\Rightarrow(3)$. Both conditions are unchanged if we replace (z_0, Σ) by $(\phi \times \psi(z_0), \phi \times \psi(\Sigma))$ for any pair (ϕ, ψ) of local diffeomorphisms. By Lemma 1.4, we may assume without loss of generality that $\Sigma = \{(x,y) : s > g(\tilde{x},y)\}$ locally around $z_0 = (0,0)$ for $g \colon \mathbf{R}^{n-1} \times \mathbf{R}^n \to \mathbf{R}$ a $\mathcal{C}^1$-function satisfying $g(0,y) = \langle y, e_1\rangle$. Then, for every point $(x,y) \in \partial\Sigma$ where $x = (s, \tilde{x})$, we have $T_y\partial\Sigma_x = \ker(d_y g(\tilde{x},y))$. Consider $\tilde{U} \subset U$ and $\tilde{V} \subset V$ convex neighbourhoods of 0 such that $(g(\tilde{x},y),\tilde{x}) \in U$ for every $(x,y) \in \tilde{U} \times \tilde{V}$. Then, (2) applied to $x_1 = (g(\tilde{x},y),\tilde{x})$ and $x_2 = (g(0,y),0)$ yield
$$\ker d_y g(\tilde{x},y) = \ker d_y g(0,y) = \mathrm{span}\{e_2, e_3, \ldots, e_n\} \quad \text{for every } (x,y) \in \tilde{U} \times \tilde{V}.$$

In particular, $\partial_{y_j} g(\tilde{x},y) = 0$ for every $j \geq 2$, so $g(\tilde{x},y) = w(\tilde{x}, \langle y, e_1\rangle)$ (since $\tilde{V}$ is convex) for a $\mathcal{C}^1$-function $w \colon \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}$ satisfying $w(0,t) = t$ for all t. By the implicit function theorem, we obtain that locally $\{(s,\tilde{x},t) : s > w(\tilde{x},t)\}$ is of the form $\{(s,\tilde{x},t) : t > u(s,\tilde{x})\}$ for a $\mathcal{C}^1$-function $u \colon \mathbf{R}^n \to \mathbf{R}$. This proves (3).

1.4. **Proof of Theorem A: Transference on triangular truncations.** The implication (3) $\Rightarrow$ (1) in Theorem A is immediate from the boundedness of the triangular projection on Schatten p-classes for $1 < p < \infty$ and the transference Lemma 1.1 above. This completes the proof of Theorem A. $\qquad\square$

1.5. **Relatively compact domains.** Using a partition of unity argument, it is not difficult to prove that Theorem A holds globally for relatively compact fully transverse domains Σ. More precisely, let $p \in (1, \infty) \setminus \{2\}$ and consider a relatively compact domain Σ in $M \times N$ which is transverse at every point of $\partial\Sigma$. Then S_Σ is an S_p-bounded multiplier if and only if any of the equivalent conditions (2) and (3) in the statement of Theorem A holds at every point of the boundary.

Remark 1.5. The fact that Σ is relatively compact is crucial in the preceding argument. For instance, (2) holds trivially at every boundary point for every fully transverse $\mathcal{C}^1$-domain of $\mathbf{R} \times \mathbf{R}$. But there are examples of such domains —which are Toeplitz, arising from Fourier symbols— that do not define an S_p multiplier for any $p \neq 2$. An explicit construction is given in [6, Appendix A].

At this point, it is interesting to observe the difference here between Fourier and Schur idempotents. We know from Fefferman's theorem [15] that there are no Fourier L_p-idempotents associated to smooth compact domains. However, there are plenty such Schur idempotents: necessarily nonToeplitz, since Toeplitz symbols give rise to Fourier idempotents. A funny instance is precisely given by other forms of ball multipliers $\Sigma_R = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \colon |x|^2 + |y|^2 < R^2\}$, which are clearly S_p-bounded and have been recently used by Chuah-Liu-Mei in their recent paper [9, Example 4.4]. Theorem A proves in addition that the spheres $\partial\Sigma_R$ satisfy the zero-curvature condition (2). More intriguing examples are the spherical Hilbert transforms defined in the Introduction as

$$H_{\mathbf{S}} : A \mapsto \Big(\mathrm{sgn}\langle x, y\rangle A_{xy}\Big)_{x,y \in \mathbf{S}^n}.$$

More generally, we also define $H_{\mathbf{S},\delta}(A) = (\chi_{\langle x,y\rangle > \delta} A_{xy})$ for any $\delta \in (-1, 1)$. The case $\delta = 0$ corresponds to $\frac{1}{2}(1 + \mathrm{sgn}\langle x, y\rangle)$ which is formally equivalent to $H_{\mathbf{S}}$ above.

Corollary 1.6. *Let us fix $1 < p \neq 2 < \infty$. Then, the n-dimensional spherical Hilbert transforms $H_{\mathbf{S},\delta}$ are all S_p-bounded for $n = 1$ and S_p-unbounded for $n \geq 2$.*

Proof. Spherical Hilbert transforms arise from relatively compact domains whose boundary is fully transverse. In particular, we may apply Theorem A. Next, in dimension 1 the assertion follows since the zero-curvature condition (2) is trivially satisfied. Alternatively, the symbol can be expressed as a triangular truncation in terms of the polar coordinates of x and y. When $n \geq 2$, it is easily checked that the tangent spaces at $\partial\Sigma_{x_1}$ and $\partial\Sigma_{x_2}$ differ at their intersection points. This was illustrated for $n = 2$ in Figure 1. Theorem A implies the assertion. $\qquad\square$

Remark 1.7. Alternatively, Corollary 1.6 also follows as a special case of Corollary B2. Indeed, Lemma 1.1 implies that $H_{\mathbf{S},\delta}$ has the same norm as the Schur multiplier on $\mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$ with symbol $(g, h) \mapsto \mathrm{sgn}((g^{-1}h)_{1,1})$, which by [7] coincides with the cb-norm of the Fourier multipler with symbol $g \mapsto \mathrm{sgn}(g_{1,1})$. But for $n \geq 2$ $\mathrm{SO}(n + 1)$ is a simple Lie group not locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$, so it does not carry *any* idempotent multiplier.

Remark 1.8. We may also consider the symbols $\Sigma_\delta = \{(x,y) \in \mathbf{R}^n : \langle x,y \rangle > \delta\}$ in the full Euclidean space for $n \geq 2$. In this case, Theorem A gives S_p-unboundedness for $(n,\delta) \neq (2,0)$. By [26, Theorem 1.18] and since $S_{\Sigma_0} = H_{\mathbf{S},0} \otimes \mathrm{id}_{\mathbf{R}_+}$, it turns out that S_p-boundedness for $(n,\delta) = (2,0)$ follows from Corollary 1.6.

1.6. **Curvature on smoother domains.** Our curvature condition (2) admits an alternative formulation under additional regularity. Let Σ be a $\mathcal{C}^2$-domain. Then $\Sigma \cap (U \times V) = \big\{(x,y) : F(x,y) > 0\big\}$ for some $\mathcal{C}^2$-function $F : M \times N \to \mathbf{R}$ and small enough neighbourhoods U, V. Our curvature condition holds if and only if we have

$$\Big\langle d_x d_y F(x,y), u \otimes v \Big\rangle := u^{\mathrm{t}} \cdot \Big(\partial_{x_j} \partial_{y_k} F(x,y) \Big)_{j,k} \cdot v = 0$$

for $(u,v) \in \ker d_x F(x,y) \times \ker d_y F(x,y)$ at every $(x,y) \in \partial\Sigma \cap (U \times V)$. The argument is quite simple. By fixing boundary points (x,y) and vectors (u,v) as specified above, let $\gamma : [0,1] \to \partial\Sigma^y \cap U$ be a curve with $\gamma(0) = x$ and $\gamma'(0) = u$, and set $h(s) = d_y F(\gamma(s), y)$. The curvature condition (2) means that $h(s) = \alpha(s)h(0)$ for some nonvanishing function $\alpha : [0,1] \to \mathbf{R}$. In particular, we get

$$\Big\langle d_x d_y F(x,y), u \otimes v \Big\rangle = \langle h'(0), v \rangle = \alpha'(0) \langle h(0), v \rangle = 0.$$

Reciprocally, assume that the $\mathcal{C}^2$-curvature condition above holds. Consider a curve $\gamma \colon [0,1] \to \partial\Sigma^y \cap U$ and define h as above. Since we have $\gamma'(s) \in \ker d_x F(\gamma(s), y)$ and $h'(s) = \gamma'(s)^{\mathrm{t}} \cdot d_x d_y F(\gamma(s), y)$ by construction, it turns out that $\langle h'(s), v \rangle$ equals $\langle d_x d_y F(\gamma(s), y), \gamma'(s) \otimes v \rangle$ for any $v \in \ker d_y F(\gamma(s), y)$. Applying the $\mathcal{C}^2$-curvature condition, this implies that $h'(s)$ is parallel to $h(s)$ for every s, which leads to the ODE

$$\left. \begin{array}{rcl} h'(s) &=& \lambda(s)h(s) \\ h(0) &=& d_y F(x,y) \end{array} \right\} \Rightarrow h(s) = \exp\Big(\int_0^s \lambda(t)dt \Big) h(0) = \alpha(s)h(0)$$

for a nonvanishing $\alpha : [0,1] \to \mathbf{R}$. This implies condition (2) in Theorem A.

Remark 1.9. In this form, (2) is invariant under exchanging x and y, which is clear a posteriori without the $\mathcal{C}^2$ assumption, since both (1) and (3) are. On the other hand, condition (2) in Theorem A seems new, while its $\mathcal{C}^2$-form above is quite similar to the *rotational curvature* $\det[d_x d_y F(x,y)]$ defined by Stein in [43, XI.3.1].

1.7. **On the transversality condition.** The transversality assumption has been essential in our proofs of $(1) \Rightarrow (2) \Rightarrow (3)$ in Theorem A, but it is not clear that it is really needed for the statement. Indeed, conditions (1) and (3) make sense without it, and (2) is already meaningful if one only assumes that $\mathbf{n}_2(x_0, y_0) \neq 0$, and we do have an example where the equivalence fails. It is likely that such examples can be found, but probably not for domains with analytic boundary. We leave these questions as open problems. In the degenerate case where $\mathbf{n}_1$ is identically 0, or equivalently when Σ is locally of the form $\Sigma = \{(x,y) : y \in \Omega\}$, all conditions in Theorem A hold. The S_p-boundedness is in that case even true for $1 \leq p \leq \infty$ because the Schur multiplier whose symbol is the indicator function of Σ is just the right-multiplication by the orthogonal projection on $L_2(\Omega)$.

2. **Idempotent Fourier multipliers on Lie groups**

Let G be a Lie group, that we equip with a left Haar measure. As to every locally compact group, we can associate to it:

- Its von Neumann algebra $\mathcal{L}$G.
- The noncommutative L_p spaces $L_p(\mathcal{L}G)$ for $1 \leq p < \infty$.
- The Fourier L_p-multipliers T_m with symbol $m \colon G \to \mathbf{C}$.

The group von Neumann algebra $\mathcal{L}$G is the weak-$*$ closure in $B(L_2(G))$ of the algebra of convolution operators $\lambda(f) : \xi \in L_2(G) \mapsto f * \xi$ for $f \in \mathcal{C}_c(G)$. When G is unimodular, its L_p-theory is quite elementary: $\mathcal{L}$G carries a natural semifinite trace τ given by $\tau(\lambda(f)^*\lambda(f)) = \int |f(g)|^2 dg$ for every $f \in L_2(G)$ with $\lambda(f) \in \mathcal{L}$G and $L_p(\mathcal{L}G)$ is then defined as the completion of $\{x \in \mathcal{L}G : \|x\|_p < \infty\}$ for the norm $\|x\|_p = \tau(|x|^p)^{1/p}$. It turns out that $L_p(\mathcal{L}G)$ contains $\{\lambda(f) : f \in \mathcal{C}_c(G)*\mathcal{C}_c(G)\}$ as a dense subspace. A bounded measurable $m \colon G \to \mathbf{C}$ defines a Fourier L_p-multiplier if $\lambda(f) \mapsto \lambda(mf)$ extends to a bounded map T_m on $L_p(\mathcal{L}G)$. These definitions are more involved for nonunimodular groups and will be recalled in Section 2.3 below.

When $p = 1, \infty$, a bounded measurable function $m \colon G \to \mathbf{C}$ defines a completely bounded Fourier L_p-multiplier if and only if the Schur multiplier associated to the symbol $(g, h) \mapsto m(gh^{-1})$ —called the Herz-Schur multiplier with symbol m and denoted S_m— is completely S_p-bounded, with same norms [3]. For amenable groups, the same holds for $1 < p < \infty$ [7, 32], and it is an intriguing open problem whether this holds beyond amenable groups. We shall use that this always holds locally. This phenomenon was discovered recently [33, Theorem 1.4] when p is an even integer and G unimodular, and the following generalizes this to the general case, see [8] for other local results of similar nature. In what follows, the Fourier support of an element $x \in L_p(\mathcal{L}G)$ will refer to the smallest closed subset Λ such that $T_m(x) = x$ for every Fourier L_p-multiplier with symbol m which is equal to 1 on Λ. When G is unimodular and $x = \lambda(f)$ for $f \in \mathcal{C}_c(G) * \mathcal{C}_c(G)$, it is easy to see that this coincides with the support of the function f.

Theorem 2.1. *Let* G *be a locally compact group and consider a bounded measurable function* $m \colon G \to \mathbf{C}$. *Then, the following are equivalent for* $p \in (1, \infty)$ *and* $g_0 \in G$:

(1) *There is a neighbourhood U of g_0 such that the restriction $T_{m,U}$ of T_m to the space of elements of $L_p(\mathcal{L}G)$ Fourier supported by U is completely bounded.*

(2) *There exists a function $\varphi \colon G \to \mathbf{C}$ which equals 1 on a neighbourhood of g_0 such that φm defines a completely bounded Fourier multiplier on $L_p(\mathcal{L}G)$.*

(3) *There are open sets $V, W \subset G$ with $g_0 \in VW^{-1}$ such that the function $(g, h) \in V \times W \mapsto m(gh^{-1})$ defines a completely bounded Schur multiplier on the Schatten class $S_p(L_2(V), L_2(W))$.*

When these conditions hold, we say that m defines locally at g_0 a completely bounded Fourier L_p-multiplier. The proof is given in Section 2.2. We can record the following consequence, which is immediate by looking at condition (3).

Corollary 2.2. *Let* G *be a connected Lie group and denote by* $\widetilde{G}$ *its universal cover. Let $\tilde{g}_0 \in \widetilde{G}$ be any lift of $g_0 \in G$. Then $m \colon G \to \mathbf{C}$ defines locally at g_0 a completely bounded Fourier L_p-multiplier over $\mathcal{L}G$ if and only its lift $\widetilde{m}$ defines locally at $\tilde{g}_0$ a completely bounded Fourier L_p-multiplier as well.*

2.1. **Idempotent multipliers.** Now we are ready to prove Theorem B and also Corollaries B1 and B2 from the Introduction. In fact, we shall prove a slightly expanded version of Theorem B which includes non simply connected groups and Corollary B1 at once. The groups G_1, G_2, G_3 in the statement below refer to the real line $\mathbf{R}$, $\mathrm{Aff}_+(\mathbf{R})$ and $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$ as in the Introduction.

Theorem 2.3. *Let $p \in (1, \infty) \setminus \{2\}$. Let G be a connected Lie group, $\Omega \subset G$ a $\mathcal{C}^1$-domain and $g_0 \in \partial\Omega$ a point in the boundary of Ω. Consider the following conditions:*

(1) *χ_Ω defines locally at g_0 a completely bounded Fourier L_p-multiplier.*

(2) *There is a smooth action $G \to \mathrm{Diff}(\mathbf{R})$ by diffeomorphisms on the real line, such that Ω coincides on a neighbourhood of g_0 with $\{g \in G \mid g \cdot 0 > g_0 \cdot 0\}$.*

(3) *There is $j \in \{1, 2, 3\}$ and a smooth surjective homomorphism $f \colon G \to G_j$ such that the domain Ω coincides on a neighbourhood of g_0 with $g_0 f^{-1}(\Omega_j)$.*

(4) *$\partial\Omega = g_0 \exp(\mathfrak{h})$ locally near g_0 for some codimension 1 Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.*

Then $(1) \Leftrightarrow (4) \Leftarrow (2) \Leftrightarrow (3)$. If G is simply connected, then we also have $(4) \Rightarrow (2)$.

Proof. The main difficulty is to prove the equivalence $(1) \Leftrightarrow (4)$, which we leave to the end of the proof. The implication $(2) \Rightarrow (4)$ is clear, with $H = \exp(\mathfrak{h})$ the stabilizer of 0. Under the assumption that G is simply connected, the converse $(4) \Rightarrow (2)$ holds by a Theorem of Mostow [31], which implies that H is a closed subgroup. Therefore, G/H is a 1-dimensional manifold that is simply connected, so is diffeomorphic to $\mathbf{R}$. The implication $(3) \Rightarrow (2)$ is also clear because G_j is given as a group of diffeomorphisms of $\mathbf{R}$ with $\Omega_j = \{g \in G_j \mid g \cdot 0 > 0\}$. The converse $(2) \Rightarrow (3)$ follows from Lie's classification of (local) actions by diffeomorphism on the real line [28], see also [45] for modern presentations and [16] for the global aspect. More precisely, the fact that g_0 belongs to the boundary of Ω implies that 0 is not fixed by G —here we use that G is connected— and the G-orbit of 0 is an open interval, so by identifying it with $\mathbf{R}$ we can assume that the G-action is transitive. In that case, the image of G in $\mathrm{Diff}(\mathbf{R})$ is one of the three groups in condition (3) or Corollary B1, see [16, Section 4.1] for the details.

Next, let us focus on the equivalence $(1) \Leftrightarrow (4)$ for general Lie groups. If we translate Ω by g_0^{-1}, we may assume that $g_0 = e$ and the tangent space of G at g_0 identifies with its Lie algebra $\mathfrak{g}$. Also, the tangent space of $\partial\Omega$ at g_0 identifies with a codimension 1 subspace $\mathfrak{h}$ of $\mathfrak{g}$. Define the $\mathcal{C}^1$-manifold

$$\widetilde{\Omega} = \Big\{(g, h) \in G \times G \mid gh \in \Omega\Big\}.$$

Its sections $\widetilde{\Omega}_g$ and $\widetilde{\Omega}^h$ are left and right translates of the $\mathcal{C}^1$-domain Ω

$$(2.1) \qquad \widetilde{\Omega}_g := \big\{h : (g, h) \in \widetilde{\Omega}\big\} = g^{-1}\Omega \quad \text{and} \quad \widetilde{\Omega}^h := \big\{g : (g, h) \in \widetilde{\Omega}\big\} = \Omega h^{-1}.$$

In particular, $\widetilde{\Omega}$ is transverse at every point of its boundary. By Lemma 1.1 and Theorem 2.1 we know that (1) is equivalent to the existence of a neighbourhood of the identity $U \subset G$ such that $\chi_{\widetilde{\Omega}}$ defines a Schur multiplier on $S_p(L_2(U))$. By Theorem A this is equivalent to the existence of a neighbourhood of the identity $V \subset G$ such that both conditions below hold:

$$(2.2) \qquad T_h \partial\widetilde{\Omega}_{g_1} = T_h \partial\widetilde{\Omega}_{g_2} \text{ for every } g_1, g_2, h \in V \text{ such that } g_1 h, g_2 h \in \partial\Omega.$$

$$(2.3) \qquad T_g \partial \widetilde{\Omega}^{h_1} = T_g \partial \widetilde{\Omega}^{h_2} \text{ for every } g, h_1, h_2 \in V \text{ such that } gh_1, gh_2 \in \partial \Omega.$$

By the above idenfications (2.1), if we denote by $L_x, R_x \colon \mathrm{G} \to \mathrm{G}$ the left and right multiplication by x, these conditions are equivalent to the existence of a neighbourhood of the identity W such that

$$(2.4) \qquad d_{x_1} L_{x_2 x_1^{-1}} (T_{x_1} \partial \Omega) = T_{x_2} \partial \Omega \text{ for every } x_1, x_2 \in \partial \Omega \cap W.$$

$$(2.5) \qquad d_{x_1} R_{x_1^{-1} x_2} (T_{x_1} \partial \Omega) = T_{x_2} \partial \Omega \text{ for every } x_1, x_2 \in \partial \Omega \cap W.$$

Indeed, taking $x_j = g_j h$ for $j = 1, 2$ we have

$$T_h (g_j^{-1} \partial \Omega) = d_{x_j} L_{g_j^{-1}} (T_{x_j} \partial \Omega).$$

Composing by $(d_{x_2} L_{g_2^{-1}})^{-1} = d_h L_{g_2}$, and using $(d_h L_{g_2}) \circ (d_{x_1} L_{g_1^{-1}}) = d_{x_1} L_{x_2 x_1^{-1}}$ by the chain rule, we see that (2.2) is equivalent to (2.4). The equivalence for right multiplication maps is entirely similar. Next, recalling that $T_e \partial \Omega = \mathfrak{h}$ the above conditions can be written in the equivalent forms

$$(2.6) \qquad d_e L_x (\mathfrak{h}) = T_x \partial \Omega \text{ for every } x \in \partial \Omega \cap W.$$

$$(2.7) \qquad d_e R_x (\mathfrak{h}) = T_x \partial \Omega \text{ for every } x \in \partial \Omega \cap W.$$

If we remember that $\mathrm{Ad}_x = d_e (R_{x^{-1}} L_x)$, we obtain that this system is equivalent to $d_e L_x (\mathfrak{h}) = T_x \partial \Omega$ and $\mathrm{Ad}_x \mathfrak{h} = \mathfrak{h}$ for every $x \in \partial \Omega \cap W$. Therefore, we have proved that (1) at $g_0 = e$ is equivalent to the existence of a neighbourhood of the identity W such that $T_x \partial \Omega = d_e L_x (\mathfrak{h})$ and $\mathrm{Ad}_x \mathfrak{h} = \mathfrak{h}$ for every $x \in \partial \Omega \cap W$. These conditions clearly hold if $\mathfrak{h}$ is a Lie algebra and $\partial \Omega$ locally coincides with the exponential of a neighbourhood of 0 in $\mathfrak{h}$. Conversely, assume $T_x \partial \Omega = d_e L_x (\mathfrak{h})$ and $\mathrm{Ad}_x (\mathfrak{h}) = \mathfrak{h}$ for every $x \in \partial \Omega \cap W$. Making x go to the identity element e in the second condition, we deduce that $\mathrm{ad}_X (\mathfrak{h}) \subset \mathfrak{h}$ for every $X \in \mathfrak{h}$. That is, $\mathfrak{h}$ is a Lie subalgebra. By the local uniqueness of a manifold in G containing e and whose tangent space at x is $d_e L_x (\mathfrak{h})$ (Frobenius' theorem), we deduce that $\partial \Omega$ is locally the exponential of a neighbourhood of 0 in $\mathfrak{h}$. This completes the proof. $\qquad \square$

Remark 2.4. By a partition of the unity argument, the following global form of Theorem 2.3 holds: if $p \in (1, \infty) \setminus \{2\}$, G is a connected Lie group and $\Omega \subset \mathrm{G}$ a relatively compact $\mathcal{C}^1$-domain, then χ_Ω defines a Fourier cb-L_p-multiplier if and only if the condition (4) holds for every point g_0 in the boundary of Ω.

Proof of Corollary B2. Assertion i) follows since the quotient of a nilpotent Lie algebra remains nilpotent, so the nonnilpotent examples in Theorem 2.3 (3) cannot happen when G is nilpotent. Assertions ii) and iii) follow immediately from Lie's classification [28]: up to isomorphism, there is a unique pair $(\mathfrak{h}, \mathfrak{g})$ where $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ is a codimension 1 subalgebra. It is given by $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{h}$ the subalgebra of upper-triangular matrices. This completes the proof. $\qquad \square$

2.2. **Local Fourier-Schur transference.** The rest of this paper will be devoted to justify Theorem 2.1. A form of the following lemma was proved in [33, Lemma 1.3] for p an even integer and G unimodular, which was enough for the applications there. Here we need a form valid for every p and every locally compact group.

Lemma 2.5. *Let $V, W \subset$ G be open sets and $U \subset VW^{-1}$ be a relatively compact open subset. There are maps $J_p \colon L_p(\mathcal{L}G) \to S_p(L_2(V), L_2(W))$ for $1 \le p \le \infty$ intertwining Fourier and Herz-Schur multipliers and such that*

$$C^{-1}\|x\|_p \le \|J_p(x)\|_p \le C\|x\|_p$$

for some constant C, every $n \ge 1$ and $x \in M_n \otimes L_p(\mathcal{L}G)$ Fourier supported in U.

This is the key point in the proof of Theorem 2.1. For the proof, we will need the following form of the Tauberian theorem, due to Eymard [14]. Let $A(G)$ be the Fourier algebra of G. That is

$$A(G) = \Big\{ g \mapsto \int \phi(gh)\psi(h)\, dh : \phi, \psi \in L_2(G) \Big\}.$$

Lemma 2.6. *Let $f \in A(G)$ be a function which does not vanish on some compact set $Q \subset$ G. Then, there exists another function $\rho \in A(G)$ such that $\rho f = 1$ over Q.*

Proof. The set $A_{|Q}(G)$ of restrictions to Q of elements of $A(G)$ is a commutative Banach algebra —as a quotient of the Banach algebra $A(G)$— that is unital because Q is compact. Moreover, by [14, Théorème 3.34] its spectrum coincides with Q through the evaluation maps $f \mapsto f(q)$. So by Gelfand's theorem, the condition that f does not vanish on Q exactly means that f's image in $A_{|Q}(G)$ is invertible. $\square$

For the reader's convenience, we first prove Lemma 2.5 and Theorem 2.1 for unimodular groups, and explain in the next paragraph how to modify the definition of Fourier multiplier and the argument for nonunimodular groups.

Proof of Lemma 2.5 for G *unimodular.* Let $\phi \in \mathcal{C}_c(V)_+$ and $\psi \in \mathcal{C}_c(W)_+$ such that $\int \phi(gh)\psi(h)dh \ge 1$ for every $g \in U$. They exist by the assumption that the open set U is relatively compact in VW^{-1}. Consider the map

$$J_p \colon L_p(\mathcal{L}G) \ni x \mapsto \phi^{\frac{1}{p}} x \psi^{\frac{1}{p}} \in S_p(L_2(V), L_2(W)),$$

where we identify ϕ, ψ with the operators of multiplication by ϕ, ψ. The convention is that $0^{\frac{1}{\infty}} = 0$. We claim that the maps J_p are completely bounded with cb-norm

$$\|J_p\|_{\mathrm{cb}(L_p, S_p)} \le \|\phi\|_{L_2(G)}^{\frac{1}{p}} \|\psi\|_{L_2(G)}^{\frac{1}{p}}$$

whenever $1 \le p \le \infty$. By interpolation, it is enough to justify the extreme cases $p = 1$ and $p = \infty$. The case $p = \infty$ is clear. For the case $p = 1$, we factorize $x = x_1 x_2$ so that $J_1(x) = \phi x_1 \cdot x_2 \psi$. Take them so that $\|x\|_1 = \|x_1\|_2 \|x_2\|_2$, and it suffices to show that both factors are bounded in $S_2(L_2(G)) \simeq L_2(G \times G)$ by $\|\phi\|_{L_2(G)} \|x_1\|_{L_2(\mathcal{L}G)}$ and $\|\psi\|_{L_2(G)} \|x_2\|_{L_2(\mathcal{L}G)}$ respectively. Using that

$$J_\infty : x \mapsto \Big(\widehat{x}(gh^{-1}) \Big),$$

the expected bounds follow from Plancherel theorem $L_2(\mathcal{L}G) \simeq L_2(G)$. Moreover when $x = \lambda(f)$, the operator $J_p(x)$ has kernel $(\phi(g)^{1/p} f(gh^{-1}) \psi(h)^{1/p})$. Thus it is clear that the map J_p intertwines the Fourier multiplier with symbol $g \mapsto m(g)$ and the Schur multiplier with symbol $(g, h) \mapsto m(gh^{-1})$.

The inequality $C^{-1}\|x\|_p \le \|J_p(x)\|_p$ is a bit more involved. Let $f \in M_n \otimes \mathcal{C}_c(U)$ with $\lambda(f) \in L_p(M_n \otimes \mathcal{L}G)$ and assume that $x = \lambda(f)$ by density. Let q be the

conjugate exponent of p and $\gamma \in M_n \otimes \mathcal{C}_c(G)$ with $\lambda(\gamma) \in L_q(M_n \otimes \mathcal{L}G)$. By Lemma 2.6, there is $\rho \in A(G)$ such that

$$\rho(g) \int_G \phi(gh)\psi(h)\,dh = 1 \quad \text{for all} \quad g \in U.$$

The reason why we require that ρ belongs to $A(G)$ is because we want to ensure that ρ defines a completely bounded Fourier L_p-multiplier. Then we have

$$\operatorname{Tr} \otimes \operatorname{Tr}_n \left(J_p(\lambda(f)) J_q(\lambda(\rho\gamma))^* \right) = \operatorname{Tr}_n \int_{G \times G} \phi(g) f \gamma^*(gh^{-1}) \rho(gh^{-1}) \psi(h)\,dsdt$$

$$= \int_G \operatorname{Tr}_n \left(f(g)\gamma(g)^* \right) \Big[\rho(g) \int_G \phi(gh)\psi(h)\,dh \Big]\,dg$$

$$= \int_G \operatorname{Tr}_n \left(f(g)\gamma(g)^* \right)\,dg = \tau \otimes \operatorname{Tr}_n \left(\lambda(f)\lambda(\gamma)^* \right).$$

In the last line, we used Lemma 2.6 and $\operatorname{supp} f \subset U$. By Hölder's inequality, we get

$$\left| \tau \otimes \operatorname{Tr}_n(\lambda(f)\lambda(\gamma)^*) \right| \le \|J_p(\lambda(f))\|_p \|J_q(\lambda(\rho\gamma))\|_q$$

$$\le \|J_q\|_{\mathrm{cb}} \|J_p(\lambda(f))\|_p \|\lambda(\rho\gamma)\|_q$$

$$\le \|J_q\|_{\mathrm{cb}} \|T_\rho\|_{\mathrm{cb}} \|\lambda(\gamma)\|_q \|J_p(\lambda(f))\|_p.$$

Taking the sup over γ, we get $C^{-1}\|x\|_p \le \|J_p(x)\|_p$ for $C = \|J_q\|_{\mathrm{cb}} \|T_\rho\|_{\mathrm{cb}} < \infty$. $\square$

With the same argument as in [33], we deduce:

Proof of Theorem 2.1. The implication (1)$\Rightarrow$(2) is easy. Indeed, if (1) holds and $\varphi \in A(G)$ is supported in U and equal to 1 on a neighbourhood of g_0 (its existence is guaranteed by Lemma 2.6), then T_φ is completely bounded on $L_p(\mathcal{L}G)$ for every $1 \le p \le \infty$, and takes values in the space of elements Fourier supported in U. In particular, $T_{m\varphi} = T_{m,U} \circ T_\varphi$ is also completely bounded. The implication (2)$\Rightarrow$(3) follows from [7, Theorem 4.2] which implies that $(g,h) \in G \times G \mapsto m(gh^{-1})\varphi(gh^{-1})$ defines a completely bounded Fourier L_p-multiplier. Thus, we get (3) if V, W are chosen so that $\varphi = 1$ on VW^{-1}. Finally, (3)$\Rightarrow$(1) follows from Lemma 2.5. $\square$

2.3. **Nonunimodular groups.** Let G be an arbitrary locally compact group with modular function $\Delta \colon G \to \mathbf{R}_+$. Our choice of Δ is characterized by the following identity for all $f \in \mathcal{C}_c(G)$

$$\int_G f(hg)\,dh = \Delta(g)^{-1} \int_G f(h)\,dh.$$

When G is not unimodular —that is, Δ is not the constant 1 function— the natural weight $\lambda(f)^*\lambda(f) \mapsto \int |f|^2$ on $\mathcal{L}G$ is not tracial. Even when $\mathcal{L}G$ is semifinite, it is better to work with the general definition of L_p spaces associated to a von Neumann algebra. Several concrete descriptions are possible: Haagerup's original one [20], Kosaki's complex interpolation [24], Connes-Hilsum's [12, 21]... see [41]. Here we will use the Connes-Hilsum spatial description because we want to rely on some results from [7, 44], to which we refer for precise definitions. In that case, $L_p(\mathcal{L}G)$ is realized as a space of unbounded operators on $L_2(G)$.

In [7], Caspers and the second-named author defined Fourier L_p-multipliers for symbols that ensure that the Fourier multiplier is completely bounded for every $1 \le p \le \infty$. Here we extend the definition, allowing to talk about Fourier multipliers

for a single p, and possibly only bounded. The shortest way to do so properly in this context is by using Terp's Hausdorff-Young inequality [44].

Informally, a typical element of $L_p(\mathcal{L}G)$ is of the form $\lambda(f)\Delta^{\frac{1}{p}}$ —where we are identifying the function Δ with the densely defined operator of multiplication by Δ on $L_2(G)$— for some *suitable* function f. Keeping at the informal level, the Fourier multiplier with symbol $m \colon G \to \mathbf{C}$ should be, whenever it exists, the operator acting as follows

$$\lambda(f)\Delta^{\frac{1}{p}} \mapsto \lambda(mf)\Delta^{\frac{1}{p}}.$$

Making this definition precise requires some lengthy and unpleasant discussions about domains/cores of unbounded operators, but fortunately we can rely on the results from [44], where these discussions have been performed. We shall need to distinguish the cases $p \geq 2$ and $p \leq 2$. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p. When $p \geq 2$, the Fourier transform

$$\mathcal{F}_q \colon L_q(G) \to L_p(\mathcal{L}G)$$

is an injective norm 1 linear map with dense image, where $\mathcal{F}_q(f)$ is defined as a suitable extension of $\lambda(f)\Delta^{1/p}$, see [44, Theorem 4.5]. When $p \leq 2$, the adjoint of $\mathcal{F}_p$ gives a norm 1 injective map with dense image $\overline{\mathcal{F}}_p \colon L_p(\mathcal{L}G) \to L_q(G)$. If I_q denotes the isometry of $L_q(G)$ defined by $I_q(f)(g) = f(g^{-1})\Delta(g)^{-1/q}$, we know from [44, Proposition 1.15] that every element x of $L_p(\mathcal{L}G)$ is a suitable extension of $\lambda(f)\Delta^{1/p}$ for $f = I_q \circ \overline{\mathcal{F}}_p(x) \in L_q(G)$. In the particular case $p = 2$, these two statements together yield Plancherel's formula : $\mathcal{F}_2$ is a unitary. If $p = 1$, the image of $\overline{\mathcal{F}}_1$ is the Fourier algebra $A(G)$, and following standard notation we write

$$(2.8) \qquad\qquad \mathrm{tr}(x) = \varphi(e) \quad \text{if} \quad \overline{\mathcal{F}}_1(x) = \varphi.$$

Definition 2.7. Let $1 < p < \infty$ and $m \in L_\infty(G)$. We say that m defines a bounded Fourier L_p-multiplier when the condition below holds according to the value of p:

- Case $p \geq 2$. The map

$$\mathcal{F}_q(f) \mapsto \mathcal{F}_q(mf)$$

  (densely defined on $\mathcal{F}_q(L_q(G))$) extends to a bounded map T_m on $L_p(\mathcal{L}G)$.

- Case $p \leq 2$. The multiplication by m preserves the image of $I_q \circ \overline{\mathcal{F}}_p$ when the map $T_m : x \mapsto (I_q \circ \overline{\mathcal{F}}_p)^{-1}(m(I_q \circ \overline{\mathcal{F}}_p(x)))$ is a bounded map on $L_p(\mathcal{L}G)$.

We say that m defines a completely bounded Fourier L_p-multiplier when m defines a bounded Fourier L_p-multiplier and the Fourier multiplier T_m is completely bounded.

It follows from the above definition that m defines a (completely) bounded L_p multiplier if and only it defines a (completely) bounded L_q multiplier, and in that case

$$\mathrm{tr}(T_m(x)y^*) = \mathrm{tr}(x(T_{\overline{m}}(y))^*) \quad \text{for all} \quad x \in L_p(\mathcal{L}G), y \in L_q(\mathcal{L}G).$$

Once we have polished the definition of Fourier L_p-multipliers in nonunimodular group von Neumann algebras, we can extend a Cotlar identity from [18] to arbitrary locally compact groups.

Example 2.8. *Let* $G \to \mathrm{Homeo}_+(\mathbf{R})$ *be a continuous action of a connected Lie group. Then, the indicator function m of $\{g \in G \mid g \cdot 0 > 0\}$ defines a completely bounded L_p Fourier multiplier on G with completely bounded norm $\leq 2\max\{p, \frac{p}{p-1}\}$.*

Proof. Let m be the indicator function of $\{g \in \mathrm{G} \mid g \cdot 0 > 0\}$. It suffices to prove the following implication for every $2 \le p < \infty$: if m defines a completely bounded Fourier L_p-multiplier with norm $\le C_p$, then it defines a completely bounded Fourier L_{2p}-multiplier with norm $\le 2C_p$. Indeed, using that $C_2 = \|m\|_\infty = 1$, we deduce $C_{2^N} \le 2^N$ for every integer N, so by interpolation $C_p \le 2p$ for all $p \ge 2$. By duality the conclusion also holds for $p \le 2$.

Let r be the dual exponent of $2p$. Let $f \in \mathcal{C}_c(\mathrm{G})$ and consider $X = \mathcal{F}_r(f)$ and $Y = \mathcal{F}_r(mf)$; these are well-defined elements of $L_{2p}(\mathcal{L}G)$ by [44]. Then, we claim that the equality below holds

$$(2.9) \qquad\qquad Y^*Y = T_m(Y^*X) + T_m(Y^*X)^*.$$

Indeed, this inequality is equivalent to the almost everywhere equality

$$(mf)^* * (mf) = m\big((mf)^* * f\big) + \big(m((mf)^* * f)\big)^*,$$

which follows for the fact that $m(g^{-1})m(g^{-1}h) = m(h)m(g^{-1}) + m(h^{-1})m(g^{-1}h)$ for almost every $g, h \in \mathrm{G}$. If the whole group G fixes 0, this is obvious because m is identically 0. Otherwise, the stabilizer of 0 is a closed subgroup, so it has measure 0 and it is enough to justify the equality for $h \cdot 0 \ne 0$. Set $(\alpha, \beta) = (g \cdot 0, h \cdot 0)$ and observe that $m(g^{-1})m(g^{-1}h) = 1$ if and only if $\alpha < \min\{0, \beta\}$. Similarly, we have $m(h)m(g^{-1}) = 1$ iff $\alpha < 0 < \beta$ and $m(h^{-1})m(g^{-1}h) = 1$ iff $\alpha < \beta < 0$. Therefore the expected identity reduces to the trivial one $\chi_{\alpha<0\wedge\beta} = \chi_{\alpha<0<\beta} + \chi_{\alpha<\beta<0}$. This justifies (2.9), both sides of which are in $L_p(\mathcal{L}\mathrm{G})$. Thus, taking the norm and applying the triangle inequality, the hypothesis and Hölder's inequality leads to

$$\|Y\|_{2p}^2 \le 2C_p \|X\|_{2p} \|Y\|_{2p}.$$

We deduce $\|Y\|_{2p} \le 2C_p \|X\|_{2p}$. Since $\mathcal{C}_c(\mathrm{G})$ is dense in $L_r(\mathrm{G})$, we obtain that m defines a Fourier L_{2p}-multiplier with norm $\le 2C_p$. A similar argument gives the same bound for the completely bounded norm, which concludes the proof. $\qquad\square$

Remark 2.9. The Cotlar-type identity from [18] is refined in some cases by (2.9).

The following summarizes the properties that we need.

Lemma 2.10. *Let $1 \le p \le \infty$ and consider functions $\phi, \psi \in L_{2p}(\mathrm{G})$, which we identify with (possibly unbounded) multiplication operators on $L_2(\mathrm{G})$. Then*

- *Given $x \in L_{2p}(\mathcal{L}G)$, $x\phi$ is densely defined and closable. In fact, its closure $[x\phi]$ belongs to $S_{2p}(L_2(\mathrm{G}))$ and has S_{2p}-norm $\le \|\phi\|_{L_{2p}(\mathrm{G})} \|x\|_{L_{2p}(\mathcal{L}G)}$.*
- *There exists a bounded linear map $L_p(\mathcal{L}G) \to S_p(L_2(\mathrm{G}))^\dagger$ sending y^*x to $[y\psi^*]^*[x\phi]$ for every $x, y \in L_{2p}(\mathcal{L}G)$. It has norm $\le \|\phi\|_{L_{2p}(\mathrm{G})} \|\psi\|_{L_{2p}(\mathrm{G})}$.*
- *If q denotes the conjugate exponent of p, consider $\phi', \psi' \in L_{2q}(\mathrm{G})$ and $y \in L_q(\mathcal{L}G)$. Then, we have*

$$(2.10) \qquad\qquad \mathrm{Tr}\big(\phi x \psi (\phi' y \psi')^*\big) = \mathrm{tr}(T_m(x)y^*),$$

where $m \in A(\mathrm{G})$ is the function $m(g) = \int_{\mathrm{G}} (\phi\overline{\phi'})(h)(\psi\overline{\psi'})(g^{-1}h)\,dh$.

———————

†That, with a slight abuse of notation, we denote $z \mapsto \psi z \phi$.

Proof. When $\phi = \psi$ are indicator functions, the first two points were proved in [7, Proposition 3.3, Theorem 5.2]. The same argument applies in our case. Let us justify identity (2.10). First observe that $\phi\overline{\phi'}$ and $\psi\overline{\phi'}$ belong to $L_2(G)$ by Hölder's inequality, so that m indeed belongs to $A(G)$. In particular, m defines a completely bounded L_1 and L_∞ Fourier multiplier. Thus, it also defines a Fourier L_p-multiplier [7, Definition-Proposition 3.5]. Therefore, by interpolation [7, Section 6] it suffices to prove (2.10) for $p = 1$ and $p = \infty$. These two cases are formally equivalent and we just consider $p = \infty$. In that case, $y \in L_1(\mathcal{L}G)$ corresponds to an element $f \in A(G)$ and $\phi'y\psi'$ is the trace class operator with kernel

$$\Big(\phi'(g)f(hg^{-1})\psi'(h)\Big)_{g,h\in G},$$

see [7, Lemma 3.4]. By a weak-$*$ density argument, it is enough to prove (2.10) for $x = \lambda(g_0)$ for some $g_0 \in G$. In that case, $T_m(x) = m(g_0)\lambda(g_0)$ and we obtain $\mathrm{tr}(T_m(x)y^*) = m(g_0)\overline{f(g_0^{-1})}$. We can compute

$$
\begin{aligned}
\mathrm{Tr}\left(\phi x\psi(\phi'y\psi')^*\right) &= \mathrm{Tr}\left(\lambda(g_0)\psi\overline{\psi'}y^*\phi\overline{\phi'}\right)\\
&= \mathrm{Tr}\left[\Big((\psi\overline{\psi'})(g_0^{-1}g)\overline{f(g_0^{-1}gh^{-1})}(\phi\overline{\phi'})(h)\Big)_{g,h\in G}\right]\\
&= \int_G (\psi\overline{\psi'})(g_0^{-1}g)\overline{f(g_0^{-1})}(\phi\overline{\phi'})(g)\,dg = m(g_0)\overline{f(g_0^{-1})}.
\end{aligned}
$$

This justifies the identity (2.10) and completes the proof of the lemma. $\square$

Lemma 2.10 allows us to adapt the proof of Lemma 2.5 from the unimodular case.

Proof of Lemma 2.5, general case. We take $\phi, \psi \in \mathcal{C}_c(G)$ as in the proof in the unimodular case, and set $m(g) = \int \phi(h)\psi(g^{-1}h)dh$. Let $\rho \in A(G)$ such that $\rho m = 1$ on U, as provided by the Tauberian theorem in Lemma 2.6. By Lemma 2.10, we can define completely bounded maps

$$J_p \colon L_p(\mathcal{L}G) \ni x \mapsto \phi^{\frac{1}{p}}x\psi^{\frac{1}{p}} \in S_p(L_2(V), L_2(W)),$$

which intertwine Fourier and Schur multipliers. Now, if $x \in L_p(M_n \otimes \mathcal{L}G)$ is Fourier supported in U and $y \in L_q(M_n \otimes \mathcal{L}G)$ —for q being the dual exponent of p— we get

$$
\begin{aligned}
\mathrm{tr}(xy^*) = \mathrm{tr}\left(T_{m\rho}(x)y^*\right) = \mathrm{tr}\left(T_m(x)(T_\rho(y))^*\right) = \mathrm{Tr}\left(J_p(x)J_q(T_\rho(y))^*\right)\\
\leq \|J_p(x)\|_{S_p}\|J_q(T_\rho(y))\|_{S_q} \leq \|J_q\|_{\mathrm{cb}}\|\rho\|_{A(G)}\|y\|_{L_q(\mathcal{L}G)}\|J_p(x)\|_{S_p}.
\end{aligned}
$$

The first line is by definition of x being Fourier supported in U, the duality for Fourier multipliers, and (2.10). The last line is Hölder's inequality. Taking suprema over y in the unit ball of $L_q(\mathcal{L}G)$ gives $\|x\|_{L_p(\mathcal{L}G)} \leq \|J_q\|_{\mathrm{cb}}\|\rho\|_{A(G)}\|J_p(x)\|_{S_p}$. $\square$

2.4. **The group** $\mathrm{SL}_2(\mathbf{R})$. Consider the symbol

$$m_0\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \frac{1}{2}\Big(1 + \mathrm{sgn}(ac + bd)\Big).$$

This was identified in [18] as the canonical Hilbert transform in $\mathrm{SL}_2(\mathbf{Z})$. Its complete L_p-boundedness follows for $1 < p < \infty$ from a Cotlar-type identity. The same problem in $\mathrm{SL}_2(\mathbf{R})$ was left open in [18, Problem A]. Now this is solved by condition

(4) in Theorem 2.3, which disproves cb-L_p-boundedness for any $p \neq 2$. On the other hand, according to Corollary B2, the map

$$m(g) = m\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] := \frac{1}{2}\big(1 + \operatorname{sgn}(c)\big) = m_0(gg^{\mathrm{t}})$$

does define, locally at every point of its boundary, a completely bounded Fourier L_p-multiplier for every $1 < p < \infty$. But, is it globally L_p-bounded? Is it completely L_p-bounded as well? We leave these problems open for future attempts.

2.5. **Stratified Lie groups.** A Lie algebra $\mathfrak{g}$ is called graded when there exists a finite family of subspaces $W_1, W_2, \ldots, W_N$ of the Lie algebra satisfying conditions below

$$\mathfrak{g} = \bigoplus_{j=1}^{N} W_j \qquad \text{and} \qquad [W_j, W_k] \subset W_{j+k}.$$

A simply connected Lie group G is called stratified when its Lie algebra $\mathfrak{g}$ is graded and the first stratum W_1 generates $\mathfrak{g}$ as an algebra. Stratified Lie groups are nilpotent and include, among many other examples, Heisenberg groups. According to Corollary B2 Hilbert transforms are of the form $H \circ \varphi$, for the classical Hilbert transform H and some continuous homomorphism $\varphi : G \to \mathbf{R}$. A quick look at Theorem 2.3 shows that φ corresponds on the Lie algebra with the projection onto any 1-dimensional subspace of the first stratum, since codimension 1 Lie subalgebras are exactly those codimension 1 subspaces leaving out a vector in the first stratum.

References

1. G. Bennett G, Schur multipliers. Duke Math J. **44** (1977), 603-639.

2. M. Bożejko and G. Fendler, Herz–Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. Boll. Un. Mat. Ital. A **3** (1984), 297-302.

3. M. Bożejko and G. Fendler, A note on certain partial sum operators. Banach Center Publications **73** (2006) 117-225.

4. J. de Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math. **107** (1985), 455-500.

5. M. Caspers, A Sobolev estimate for radial L_p-multipliers on a class of semi-simple Lie groups. Trans. Amer. Math. Soc. To appear.

6. M. Caspers, J. Parcet, M. Perrin and É. Ricard, Noncommutative de Leuuw theorems. Forum Math. Σ **3** (2015), e21.

7. M. Caspers and M. de la Salle, Schur and Fourier multipliers of an amenable group acting on non-commutative L_p-spaces. Trans. Amer. Math. Soc. **367** (2015), 6997-7013.

8. M. Caspers, B. Janssens, A. Krishnaswamy-Usha and L Miaskiwskyi, Local and multilinear noncommutative de Leeuw theorems. Math. Ann. (2023).

9. C.Y. Chuah, Z-C. Liu and T. Mei, A Marcinkiewicz multiplier theory for Schur multipliers. ArXiv: 2209.13108.

10. J.M. Conde-Alonso, A.M. González-Pérez, J. Parcet and E. Tablate, Schur multipliers in Schatten-von Neumann classes. Ann. of Math. **198** (2023), 1229-1260.

11. J.M. Conde-Alonso, A.M. González-Pérez, J. Parcet and E. Tablate, A Hörmander-Mikhlin theorem for higher rank simple Lie groups. ArXiv: 2201.08740.

12. A. Connes, On the spatial theory of von Neumann algebras, J. Funct. Anal. 35 (1980), no. 2, 153–164.

13. M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math. **96** (1989), 507-549.

14. P. Eymard, L'algèbre de Fourier d'un groupe localement compact. Bull. Soc. Math. France 92 (1964), 181–236.

15. C. Fefferman, The multiplier problem for the ball. Ann. of Math. **94** (1971), 330-336.

16. E. Ghys, Groups acting on the circle. Enseign. Math. (2) 47 (2001), no. 3-4, 329-407.

17. A.M. González-Pérez, M. Junge and J. Parcet. Smooth Fourier multipliers in group algebras via Sobolev dimension. Ann. Sci. École Norm. Sup. **50** (2017), 879-925.

18. A.M. González-Pérez, J. Parcet and R. Xia. Noncommutative Cotlar identities for groups acting on tree-like structures. ArXiv: 2209.05298.

19. U. Haagerup, An example of a non nuclear C^*-algebra, which has the metric approximation property. Invent. Math. **50** (1979), 279-293.

20. U. Haagerup, L^p-spaces associated with an arbitrary von Neumann algebra. Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977), pp. 175–184.

21. M. Hilsum, Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, J. Funct. Anal. 40 (1981), no. 2, 151–169.

22. M. Junge, T. Mei and J. Parcet, Smooth Fourier multipliers on group von Neumann algebras. Geom. Funct. Anal. **24** (2014), 1913-1980.

23. M. Junge, T. Mei and J. Parcet, Noncommutative Riesz transforms – Dimension free bounds and Fourier multipliers. J. Eur. Math. Soc. **20** (2018), 529-595.

24. H. Kosaki, Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p-spaces, J. Funct. Anal. 56 (1984), no. 1, 29–78

25. T. de Laat and M. de la Salle, Approximation properties for noncommutative L_p of high rank lattices and nonembeddability of expanders. J. Reine Angew. Math. **737** (2018), 46-69.

26. V. Lafforgue and M. de la Salle, Noncommutative L_p-spaces without the completely bounded approximation property. Duke. Math. J. **160** (2011), 71-116.

27. K. de Leeuw, On L_p multipliers. Ann. of Math. **81** (1965), 364-379.

28. S. Lie, Theorie der Transformationsgruppen. Dritter (und letzten) Abschnitt. Unter Mitwirkung von Fr. Engel bearbeitet., Leipzig. B. G. Teubner. XXVII + 830 S. 8° (1893).

29. T. Mei and É. Ricard, Free Hilbert transforms. Duke Math. J. **166** (2017), 2153-2182.

30. T. Mei, É. Ricard and Q. Xu, A Hörmander-Mikhlin multiplier theory for free groups and amalgamated free products of von Neumann algebras. Adv. Math. **403** (2022), 108394.

31. G. Mostow, The extensibility of local Lie groups of transformations and groups on surfaces. Ann. of Math. (2) 52 (1950), 606–636.

32. S. Neuwirth and É. Ricard, Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group. Canad. J. Math. **63** (2011), 1161-1187.

33. J. Parcet, É. Ricard and M. de la Salle, Fourier multipliers in $SL_n(\mathbf{R})$. Duke Math. J. **171** (2022), 1235-1297.

34. J. Parcet and K. Rogers, Twisted Hilbert transforms vs Kakeya sets of directions. J. Reine Angew. Math. **710** (2016), 137-172.

35. G. Pisier, Multipliers and lacunary sets in non-amenable groups. Amer. J. Math. **117** (1995), 337-376.

36. G. Pisier, Non-commutative vector valued L_p-spaces and completely p-summing maps. Asrérisque **247**. Soc. Math. France, 1998.

37. G. Pisier, Similarity Problems and Completely Bounded Maps. Lecture Notes in Mathematics **1618**. Springer-Verlag, 2001.

38. G. Pisier, Introduction to Operator Space Theory. Cambridge University Press, 2003.

39. G. Pisier, Grothendieck's Theorem, past and present. Bull. Amer. Math. Soc. **49** (2012), 237-323.

40. G. Pisier and D. Shlyakhtenko, Grothendieck's theorem for operator spaces. Invent. Math. **150** (2002), 185-217.

41. G. Pisier and Q. Xu, Non-commutative Lp-spaces. Handbook of the Geometry of Banach Spaces II (Eds. W.B. Johnson and J. Lindenstrauss) North-Holland (2003), 1459-1517.

42. D. Potapov and F. Sukochev, Operator-Lipschitz functions in Schatten-von Neumann classes. Acta Math. **207** (2011), 375-389.

43. E.M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Math. Ser. **43**. Princeton Univ. Press., NJ, 1993.

44. M. Terp, L_p Fourier transformation on non-unimodular locally compact groups. Adv. Oper. Theory 2 (2017), no.4, 547–583.

45. J. Tits, Sur une classe de groupes de Lie résolubles, Bull. Soc. Math. Belg. 11 (2) (1959), 100–115.

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