SINGULAR INTEGRALS
IN QUANTUM EUCLIDEAN SPACES

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ABSTRACT. In this paper, we establish the core of singular integral theory and pseudodifferential calculus over the archetypal algebras of noncommutative geometry: quantum forms of Euclidean spaces and tori. Our results go beyond Connes' pseudodifferential calculus for rotation algebras, thanks to a new form of Calderón-Zygmund theory over these spaces which crucially incorporates nonconvolution kernels. We deduce L_p-boundedness and Sobolev p-estimates for regular, exotic and forbidden symbols in the expected ranks. In the L_2 level both Calderón-Vaillancourt and Bourdaud theorems for exotic and forbidden symbols are also generalized to the quantum setting. As a basic application of our methods, we prove L_p-regularity of solutions for elliptic PDEs.

CONTENTS

Introduction

Harmonic analysis and PDEs over Riemannian manifolds are paramount for the solution of many important problems in differential geometry, fluid mechanics or theoretical physics. In this paper, we establish the core of singular integral theory and pseudodifferential calculus over the archetypal algebras of noncommutative geometry. This includes the Heisenberg-Weyl algebra, quantum tori and other noncommutative deformations of Euclidean spaces of great interest in quantum field theory and quantum probability. Our approach crucially relies on a quantum form of the fruitful interplay

$$\text{KERNELS}$$
$$\swarrow \qquad \searrow$$
$$\text{SYMBOLS} \longrightarrow \text{OPERATORS}$$
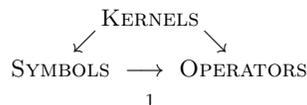
at the interface of analysis and geometry. Strong reasons to develop such a program over matrix algebras and other noncommutative manifolds are also in connection to string theory, where several PDEs arise naturally over quantum spaces. We obtain optimal smoothness conditions for L_p-boundedness of singular integrals and corresponding Sobolev p-estimates for pseudodifferential operators. This is crucial for applications to PDEs, which we shall briefly discuss. In the line of the harmonic analysis school, a key point has been a profound analysis of the associated kernels which is specially challenging for noncommutative algebras.

Let Θ be an anti-symmetric real $n \times n$ matrix. Roughly speaking, the quantum Euclidean space $\mathcal{R}_\Theta$ is the von Neumann algebra generated by certain family of unitaries $\{u_j(s) : 1 \le j \le n,\, s \in \mathbb{R}\}$ satisfying

$$u_j(s)u_j(t) = u_j(s+t),$$

$$u_j(s)u_k(t) = e^{2\pi i \Theta_{jk} st} u_k(t) u_j(s).$$

Set $\lambda_\Theta(\xi) = u_1(\xi_1) u_2(\xi_2) \cdots u_n(\xi_n)$ for $\xi \in \mathbb{R}^n$ and

$$\lambda_\Theta : \mathcal{C}_c(\mathbb{R}^n) \ni f \longmapsto \int_{\mathbb{R}^n} f(\xi) \lambda_\Theta(\xi) \, d\xi \in \mathcal{R}_\Theta.$$

Consider the trace determined by $\tau_\Theta(\lambda_\Theta(f)) = f(0)$ and the corresponding L_p spaces $L_p(\mathcal{R}_\Theta, \tau_\Theta)$ [62]. Of course, $\Theta = 0$ yields the Euclidean L_p-space in $\mathbb{R}^n$ with the Lebesgue measure and $(\mathcal{R}_\Theta, \tau_\Theta)$ should be understood as a noncommutative deformation of it. Section 1 includes a careful presentation of $(\mathcal{R}_\Theta, \tau_\Theta)$ for those potential readers not familiar with them. Our approach also contains a key Poincaré type inequality and a few more crucial results, maybe some known to experts. The lack of appropriate literature justifies a self-contained presentation.

The algebraic structure of these operator (type I) algebras is quite simple, but the connection to Euclidean spaces make them indispensable in a great variety of scenarios. If $\hbar$ stands for Planck's constant, the choice

$$\Theta = 2\pi\hbar \ id_{M_m} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

yields the Heisenberg-Weyl algebra in quantum mechanics. Another description arises from the unbounded generators $x_{\Theta,j}$ of u_j —a quantum analogue of the Euclidean variables— which satisfy $2\pi i [x_{\Theta,j}, x_{\Theta,k}] = \Theta_{jk}$ and provide additional insight in our kernel manipulations below. Considering a Fock space representation $\mathcal{R}_\Theta$ becomes the CCR algebra associated to the symplectic form Θ, thoroughly studied in quantum probability and quantum field theory. In this setting, it is simple to find a (nontracial) gaussian state with respect to which these $x_{\Theta,j}$ admit a gaussian distribution. In the physics literature, higher dimensional deformations are usually referred to as Moyal deformations of $\mathbb{R}^n$. An important instance in string theory is given by the noncommutative deformation of $\mathbb{R}^4$ associated to an invertible symbol Θ, which leads to instantons on a noncommutative space in the influential papers [21, 55, 67]. In view of so many names for the same object, we have decided to rebaptize these algebras as *quantum Euclidean spaces*, in consonance with quantum tori $\mathcal{A}_\Theta$ —also known in the literature as noncommutative tori or rotation algebras— which appear in turn as the subalgebra generated by $\lambda_\Theta(\xi)$ with ξ running along $\mathbb{Z}^n$ or any other lattice of $\mathbb{R}^n$. Our main results in this paper about pseudodifferential operators hold for $\mathcal{A}_\Theta$ and $\mathcal{R}_\Theta$.

Calderón-Zygmund extrapolation

In harmonic analysis, integral kernel representations play a central role to study the most relevant operators. In this particular form, pseudodifferential operators become well-behaved singular integrals, which admit a fruitful L_p-theory [66]. A singular integral operator in a Riemannian manifold (X, d, μ) admits the kernel representation

$$T_k f(x) = \int_X k(x,y) f(y) \, d\mu(y) \quad \text{for} \quad x \notin \text{supp} f.$$

Namely, T_k is only assumed a priori to send test functions into distributions, so that it admits a distributional kernel in $X \times X$ which coincides in turn with a locally integrable function k away from the diagonal $x = y$, where the kernel presents certain singularity. This already justifies the assumption $x \notin \text{supp} f$ in the kernel representation. The paradigm of singular integral theory is the Hilbert transform in $\mathbb{R}$, paramount to study the convergence of Fourier series and integrals. The challenge in higher dimensions required new real variable methods which culminated in the celebrated theorem of Calderón and Zygmund [10], who established sufficient conditions on a singular integral operator in $\mathbb{R}^n$ for its L_p-boundedness:

   i) Cancellation

$$\left\| T_k : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \right\| \leq A_1.$$

   ii) Kernel smoothness

$$\left| \nabla_x \, k(x,y) \right| + \left| \nabla_y \, k(x,y) \right| \leq \frac{A_2}{|x-y|^{n+1}}.$$

The same holds in Riemannian manifolds with nonnegative Ricci curvature [3].

Noncommutative L_p methods in harmonic analysis have gained a considerable momentum in recent years. The fast development of Fourier L_p multiplier theory on group von Neumann algebras [11, 32, 42, 43, 47, 48, 57] has been possible in part due to a deeper comprehension of the involved kernels. In spite of this, the validity of Calderón-Zygmund extrapolation principle over noncommutative manifolds is still widely open. Noncommutative martingale methods were used in [56] to establish endpoint estimates for singular integrals over tensor product von Neumann algebras with an Euclidean factor, which have been the key for the recent solution in [12] of the Nazarov-Peller conjecture. Other results in this direction include a CZ theory for group algebras over orthogonal crossed products $\mathbb{R}^n \rtimes G$, operator-valued kernels acting by left/right or Schur multiplication, other BMO spaces in a new approach towards nondoubling CZ theory, Littlewood-Paley estimates, Hörmander-Mihlin multipliers or directional Hilbert transforms [15, 16, 37, 42, 54].

The Calderón-Zygmund theory presented below is the first form over a 'fully noncommutative' von Neumann algebra. In other words, the singular integral acts on the whole algebra $\mathcal{M}$, not just over copies of $\mathbb{R}^n$ as tensor or crossed product factors in $\mathcal{M}$. A major challenge for such a von Neumann algebra $(\mathcal{M}, \tau)$ is to understand what it means to be a singular kernel. One has to identify the diagonal where the kernel singularity should be located, the quantum metric which measures the distance to it and its relation to the trace. A crucial point, undistinguishable in abelian algebras or the work cited so far, is to define kernels over $\mathcal{M} \bar{\otimes} \mathcal{M}_{\text{op}}$ with

the op-structure (reversed product law) in the second copy, see also [44]. In the case of $\mathcal{R}_\Theta$, this is justified from the important map

$$\pi_\Theta : L_\infty(\mathbb{R}^n) \to \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}},$$

$$\exp(2\pi i \langle \xi, \cdot \rangle) \longmapsto \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*,$$

which extends to a normal $*$-homomorphism, for which the op-structure is strictly necessary. Note that $\pi_\Theta(f)(x,y) = f(x-y)$ for $\Theta = 0$. In particular, if $|\cdot|$ stands for the Euclidean distance to 0, the operator

$$\mathrm{d}_\Theta = \pi_\Theta(|\cdot|)$$

is affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ and implements the *distance to the diagonal* as an unbounded operator. Similarly, the diagonal bands $\mathrm{b}_\Theta(\mathrm{R}) = \pi_\Theta(\chi_{|\cdot| \leq \mathrm{R}})$ or smoothings of them will be indispensable to produce kernel truncations. An integral representation in $\mathcal{R}_\Theta$ is formally given by

$$T_k(\lambda_\Theta(f)) = (id \otimes \tau_\Theta)\big(k(\mathbf{1} \otimes \lambda_\Theta(f))\big)$$

for some kernel k affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$. We shall work with more general singular kernels which lead to $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$, a map which sends the quantum Schwartz class $\mathcal{S}_\Theta = \lambda_\Theta(\mathcal{S}(\mathbb{R}^n))$ in $\mathcal{R}_\Theta$ into its tempered distribution class $\mathcal{S}_\Theta'$. We shall also use the 'free gradient'

$$\nabla_\Theta = \sum_{j=1}^n s_j \otimes \partial_\Theta^j$$

associated to the partial derivatives $\partial_\Theta^j(\lambda_\Theta(\xi)) = 2\pi i \xi_j \lambda_\Theta(\xi)$ and a free family $s_1, s_2, \ldots, s_n$ of semicircular random variables living in the free group algebra $\mathcal{L}(\mathbb{F}_n)$.

Theorem A. *Let $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ and assume:*

  i) Cancellation

$$\big\| T_k : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \big\| \leq \mathrm{A}_1.$$

  ii) Kernel smoothness

$$\left| \mathrm{d}_\Theta^\alpha \bullet (\nabla_\Theta \otimes id)(k) \bullet \mathrm{d}_\Theta^\beta \right| + \left| \mathrm{d}_\Theta^\alpha \bullet (id \otimes \nabla_\Theta)(k) \bullet \mathrm{d}_\Theta^\beta \right| \leq \mathrm{A}_2,$$

for $(\alpha, \beta) = (n+1, 0)$, $(\alpha, \beta) = (0, n+1)$ and $(\alpha, \beta) = (\frac{n+1}{2}, \frac{n+1}{2})$.

Then, $T_k : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$ is completely bounded for every $1 < p < \infty$.

A more general statement is proved in Theorem 2.18. Our argument establishes $L_\infty \to \mathrm{BMO}$ endpoint estimates for a suitable noncommutative BMO. Interpolation with L_p spaces is deduced in [41] from the theory of noncommutative martingales with continuous index set and a theory of Markov dilations. The convolution kernel case —in other words, quantum Fourier multipliers— is much easier to prove by transference methods stablished in [13, 63]. In the terminology of pseudodifferential operators, Fourier multipliers correspond to differential operators with constant coefficients. Of course, we aim to include nonconstant coefficients which leads to the analysis of the harder nonconvolution quantum kernels. Our statement above is very satisfactory and crucial for applications to pseudodifferential operator theory below. We shall also use other methods to justify that every CZ operator differs from its principal value by a left/right pointwise multiplier. This is fundamental in classical CZ theory and therefore of independent interest.

The L_p pseudodifferential calculus

The theory of pseudodifferential operators goes back to the mid 1960s with the work of Kohn, Nirenberg and Hörmander. The basic idea is to exploit properties of the Fourier transform to produce a suitable representation Ψ_L of partial differential operators $L = \sum_{|\alpha| \le m} a_\alpha(x)\partial_x^\alpha$ which can be inverted up to a controllable error term. This representation looks like

$$\Psi_a f(x) = \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\, d\xi$$

for a smooth symbol $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfying

$$(S_{\rho,\delta}^m) \qquad \big|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\big| \le C_{\alpha\beta}\big(1+|\xi|\big)^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for all} \quad \alpha, \beta \in \mathbb{Z}_+^n,$$

some $m \in \mathbb{R}$ and some $0 \le \delta \le \rho \le 1$. The realization of Ψ_a as singular integral is given by partial Fourier inversion $k(x,y) = (id \otimes \mathcal{F}^{-1})(a)(x, x-y)$, which opens a door to CZ theory for Sobolev p-estimates of parametrices and error terms.

In the noncommutative setting, this line took off in 1980 with Connes' work on pseudodifferential calculus for C*-dynamical systems [19], originally conceived to extend the Atiyah-Singer index theorem for Lie group actions on C*-algebras, see also [1, 2, 50] for related results. Other applications in the context of quantum tori include a well-established elliptic operator theory [17], the Gauss-Bonnet theorem for 2D quantum tori [23, 28] and recent results on the local differential geometry of non-flat noncommutative tori [6, 22]. Unfortunately, the work of Connes and his collaborators does not include L_p estimates for parametrices and error terms, which are paramount in harmonic analysis and partial differential equations. On the other hand, the only approach [13, 63, 75] to harmonic L_p-analysis in quantum tori does not include pseudodifferential calculus, which requires Calderón-Zygmund estimates in $\mathcal{R}_\Theta$. In comparison with Connes' work —which focuses on the smallest Hörmander class $S_{\rho,\delta}^m$ with $(\rho,\delta) = (1,0)$— our main contributions in this direction include all classes of symbols and L_p-estimates:

i) An L_2-theory for exotic and forbidden symbols $0 < \delta = \rho \le 1$.
ii) An L_p-theory for arbitrary Hörmander classes and $1 < p < \infty$.

We refer to [66, 69, 70] for the applications of these results in the Euclidean context.

Pseudodifferential operators over quantum Euclidean spaces are easier to define than Calderón-Zygmund operators. The symbol $a(x,\xi)$ is now understood as an smooth function $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ since ξ is still (dual) Euclidean, while x becomes its Θ-deformed analog $x_\Theta = (x_{\Theta,1}, x_{\Theta,2}, \ldots, x_{\Theta,n})$ as introduced above. We shall deal in this paper with two quantum forms of the Hörmander classes:

- We say that $a \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ when

$$\big|\partial_\Theta^\beta \partial_\xi^\alpha a(\xi)\big| \le C_{\alpha\beta}\big(1+|\xi|\big)^{m-\rho|\alpha|+\delta|\beta|}.$$

This is probably the most natural definition that comes to mind.

- We say that $a \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ when

$$\big|\partial_\Theta^\beta \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} a(\xi)\big| \le C_{\alpha_1\alpha_2\beta}\big(1+|\xi|\big)^{m-\rho|\alpha_1+\alpha_2|+\delta|\beta|}.$$

Here $\partial_{\Theta,\xi}$ is a Θ-deformation of ∂_ξ by ∂_Θ's. More precisely, we have

$$
\begin{aligned}
\partial_{\Theta,\xi}^j a(\xi) &= \partial_\xi^j a(\xi) + 2\pi i \big[x_{\Theta,j}, a(\xi) \big] \\
&= \partial_\xi^j a(\xi) + \frac{1}{2\pi i} \sum_{k=1}^n \Theta_{jk} \, \partial_\Theta^k a(\xi) \\
&= \lambda_\Theta(\xi)^* \partial_\xi^j \big\{ \lambda_\Theta(\xi) a(\xi) \lambda_\Theta(\xi)^* \big\} \lambda_\Theta(\xi).
\end{aligned}
$$

We clearly have $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta) \subset S_{\rho,\delta}^m(\mathcal{R}_\Theta)$. It is very important to recall that both classes collapse into Hörmander classical set of symbols $S_{\rho,\delta}^m$ when $\Theta = 0$, so that both definitions above are a priori valid to generalize the Euclidean theory. It turns out that the L_2-theory holds for $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$, while the more involved class $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ makes the L_p-theory valid. The reason has to do with the link to CZ theory and the two-sided nature of our Calderón-Zygmund conditions. Indeed, in all our past experiences with noncommutative Calderón-Zygmund theory certain amount of modularity is required. In this case, the bilateral form of our kernel conditions in Theorem A ultimately imposes the mixed quantum-classical derivatives $\partial_{\Theta,\xi}$.

The pseudodifferential operator associated to $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ has the form

$$
\begin{aligned}
\Psi_a(\lambda_\Theta(f)) &= \int_{\mathbb{R}^n} a(\xi) f(\xi) \lambda_\Theta(\xi) \, d\xi \\
&= (id \otimes \tau_\Theta) \Big[\Big(\underbrace{\int_{\mathbb{R}^n} (a(\xi) \otimes \mathbf{1})(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*) d\xi}_{\text{The kernel } k} \Big) \big(\mathbf{1} \otimes \lambda_\Theta(f) \big) \Big].
\end{aligned}
$$

The algebra of pseudodifferential operators is formally generated by the derivatives ∂_Θ^j and the left multiplication maps $\lambda_\Theta(f) \mapsto x_{\Theta,j} \lambda_\Theta(f)$. The kernels affiliated to $\pi_\Theta(L_\infty(\mathbb{R}^n)) \subset \mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}$ implement Fourier multipliers $\lambda_\Theta(\xi) \mapsto m(\xi) \lambda_\Theta(\xi)$ in this setting, which correspond to the closure of pseudodifferential operators $\sum_\alpha a_\alpha \partial^\alpha$ with constant coefficients a_α. A very subtle transference method —which avoids properly supported symbols— is required to obtain adjoint and product formulae in Section 3.1. Our main L_p results are collected in the following statement.

Theorem B. *Let $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ and $1 < p < \infty$:*

  i) *If $a \in S_{\rho,\rho}^0(\mathcal{R}_\Theta)$ with $0 \le \rho < 1$, $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$.*

  ii) *If $a \in S_{1,1}^0(\mathcal{R}_\Theta) \cap S_{1,1}^0(\mathcal{R}_\Theta)^*$, then $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$.*

  iii) *If $a \in \Sigma_{1,1}^0(\mathcal{R}_\Theta) \cap \Sigma_{1,1}^0(\mathcal{R}_\Theta)^*$, then $\Psi_a : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$.*

Using $S_{\rho,\delta}^m(\mathcal{R}_\Theta) \subset S_{\delta,\delta}^m(\mathcal{R}_\Theta) \cap S_{\rho,\rho}^m(\mathcal{R}_\Theta)$ for $0 \le \delta \le \rho \le 1$ —same inclusions for Σ-classes— we get L_p-estimates for regular, exotic and forbidden symbols in the expected ranks and Theorem B opens the core of the pseudodifferential L_p-calculus [66, 69] to the context of quantum Euclidean spaces:

  • Theorem B i). Calderón-Vaillancourt theorem [9] on L_2-boundedness for exotic symbols quickly obtained a spectacular application [4] for $\rho = 1/2$. Our proof of its quantum form for $\rho = 0$ requires a careful approach due to the presence of a Θ-phase. The case $\rho > 0$ also imposes an unexpected dilation argument among different deformed algebras $\mathcal{R}_\Theta$.

- Theorem B ii). Bourdaud's theorem [7] yields a form of the $T(1)$-theorem for pseudodifferential operators when $\rho = \delta = 1$: Ψ_a is L_2-bounded iff the symbol $a_\dagger^*$ of Ψ_a^* remains in the same Hörmander class. Our proof follows the classical one by showing that Ψ_a is bounded in the Sobolev space $\mathrm{W}_{2,s}(\mathcal{R}_\Theta)$ under a minimal amount of regularity $s > 0$.
- Theorem B iii). Our L_p-results follow by showing that any such symbol is a Calderón-Zygmund operator which fulfills all the hypotheses of Theorem A, the L_2-boundedness being assured by Theorem B ii). It is our CZ kernel condition what imposes the mixed quantum-classical derivatives $\partial_{\Theta,\xi}$ and the corresponding "forbidden" Hörmander symbol classes $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$.
- Related estimates. Our L_p-inequalities give rise to Sobolev p-estimates for symbols of arbitrary order m, we shall recollect these estimates in the body of the paper. On the other hand, the L_p-theory for symbols with $\rho < 1$ requires a negative degree to compensate lack of regularity. Fefferman proved in [29] the L_p-bounds for the critical index $m = -(1-\rho)\frac{n}{2}$. We shall obtain nonoptimal L_p-estimates of this kind in $\mathcal{R}_\Theta$. Interpolation yields even finer results for intermediate values of m.

The analogue of Theorem B for quantum tori $\mathcal{A}_\Theta$ is proved by transference in Appendix A. The Hörmander classes $S_{\rho,\delta}^m(\mathcal{A}_\Theta)$ and $\Sigma_{\rho,\delta}^m(\mathcal{A}_\Theta)$ involve discrete derivations over $\mathbb{Z}^n$ in the dual variable. In the line of Connes definition, we could also proceed by restriction to $\mathbb{Z}^n$ of symbols $\mathbb{R}^n \to \mathcal{A}_\Theta \subset \mathcal{R}_\Theta$ in the corresponding Hörmander classes. As in $\mathbb{T}^n$ both definitions turn out to be equivalent and this will be the source of our transference approach. The discrete form of difference operators has the advantage of being easier to be calculated with computers.

An illustration for elliptic PDEs

Pseudodifferential operators are a very powerful tool for linear and nonlinear partial differential equations [69, 70]. The existence, uniqueness and qualitative behavior of solutions for many PDEs are frequently understood by application of these methods. After the announced results so far, the potential applications for PDEs over quantum Euclidean spaces and tori are vast and beyond the scope of this paper. As a small but basic illustration, we prove in Theorem 4.5 the L_p regularity for solutions of elliptic PDEs over quantum Euclidean spaces. We do not include this statement in the Introduction to avoid more terminology at this point. A profound analysis of partial differential equations over quantum spaces —$\mathcal{A}_\Theta$, $\mathcal{R}_\Theta$ or even more general noncommutative manifolds— constitutes a long term program with conceivable implications for the geometry of such objects.

1. **Quantum Euclidean spaces**

Given an integer $n \geq 1$, fix an anti-symmetric $\mathbb{R}$-valued $n \times n$ matrix Θ. Let us write $A_n(\mathbb{R})$ to denote this class of matrices. We define A_Θ as the universal C*-algebra generated by a family $u_1(s), u_2(s), \ldots, u_n(s)$ of one-parameter unitary groups in $s \in \mathbb{R}^n$ which are strongly continuous and satisfy the Θ-commutation relations below

$$u_j(s)u_k(t) = e^{2\pi i \Theta_{jk} st} u_k(t) u_j(s).$$

If $\Theta = 0$ and by Stone's theorem we may take $u_j(s) = \exp(2\pi i s \langle e_j, \cdot \rangle)$ and A_Θ is the space of bounded continuous functions $\mathbb{R}^n \to \mathbb{C}$. Moreover, since $\mathbb{R}^n$ is amenable A_Θ may be described as an spatial crossed product $\mathbb{C} \rtimes_\Theta \mathbb{R}^n \subset \mathcal{B}(L_2(\mathbb{R}^n))$ twisted by the 2-cocycle determined by Θ, as introduced by Zeller-Meier [76]. In general given $\xi \in \mathbb{R}^n$ we shall extensively use the unitaries $\lambda_\Theta(\xi) = u_1(\xi_1)u_2(\xi_2) \cdots u_n(\xi_n)$ and we define E_Θ as the closure in A_Θ of $\lambda_\Theta(L_1(\mathbb{R}^n))$ with

$$\lambda_\Theta(f) = \int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi)\,d\xi.$$

If $\Theta = 0$, we find $E_\Theta = \mathcal{C}_0(\mathbb{R}^n)$. Given any Θ, we easily see that

- $\lambda_\Theta(\xi)^* = e^{2\pi i \sum_{j>k} \Theta_{jk}\xi_j\xi_k} \lambda_\Theta(-\xi)$,
- $\lambda_\Theta(\xi)\lambda_\Theta(\eta) = e^{2\pi i \langle \xi, \Theta\eta \rangle} \lambda_\Theta(\eta)\lambda_\Theta(\xi)$,
- $\lambda_\Theta(\xi)\lambda_\Theta(\eta) = e^{2\pi i \sum_{j>k} \Theta_{jk}\xi_j\eta_k} \lambda_\Theta(\xi + \eta)$,
- $\lambda_\Theta(f_1)\lambda_\Theta(f_2) = \lambda_\Theta(f_1 *_\Theta f_2)$ with Θ-convolution given by

$$f_1 *_\Theta f_2(\xi) = \int_{\mathbb{R}^n} f_1(\xi - \eta)f_2(\eta)e^{2\pi i \sum_{j>k} \Theta_{jk}(\xi_j - \eta_j)\eta_k}\,d\eta.$$

Note that $\sum_{j>k} \Theta_{jk}\xi_j\eta_k = \langle \xi, \Theta_\downarrow \eta \rangle$ for the lower triangular truncation $\Theta_\downarrow$ of Θ.

1.1. Crossed product form. Let

$$\tau_\Theta(\lambda_\Theta(f)) = \tau_\Theta\Big(\int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi)\,d\xi \Big) = f(0)$$

for $f : \mathbb{R}^n \to \mathbb{C}$ smooth and integrable. As we shall see, τ_Θ extends to a normal faithful semifinite trace on E_Θ. Let $\mathcal{R}_\Theta = A_\Theta'' = E_\Theta''$ be the von Neumann algebra generated by E_Θ in the GNS representation of τ_Θ. We obtain $\mathcal{R}_\Theta = L_\infty(\mathbb{R}^n)$ for $\Theta = 0$. In general, we call the Θ-*deformation* $\mathcal{R}_\Theta$ a quantum Euclidean space.

1.1.1. *Crossed products and trace.* A C*-dynamical system is a triple formed by a C*-algebra A, a locally compact group G and a continuous action $\beta : G \to \mathrm{Aut}(A)$ by $*$-automorphisms. The reduced crossed product $A \rtimes_{\beta,\mathrm{red}} G$ is the norm closure in $A \bar\otimes \mathcal{B}(L_2(G))$ of the $*$-algebra generated by the representations $\rho : A \to L_\infty(G; A)$ and $\lambda : G \to \mathcal{U}(L_2(G))$, given by

$$\rho(a)(g) = \beta_{g^{-1}}(a) \quad \text{and} \quad [\lambda(g)f](h) = f(g^{-1}h).$$

The full crossed product $A \rtimes_{\beta,\mathrm{full}} G$ is the C*-algebra generated by all covariant representations $\gamma : A \to \mathcal{B}(\mathcal{H})$ and $u : G \to \mathcal{U}(\mathcal{H})$ over some Hilbert space $\mathcal{H}$: $u(g)\gamma(a)u(g)^* = \gamma(\beta_g(a))$. Given $f : G \to A$ continuous and integrable

$$\Big\| \int_G f_g \rtimes g\,d\mu(g) \Big\|_{A \rtimes_{\beta,\mathrm{full}} G} = \sup_{\substack{\gamma,\,u \\ \mathrm{covariant}}} \Big\| \int_G \gamma(f_g)u(g)\,d\mu(g) \Big\|_{\mathcal{B}(\mathcal{H})}.$$

It is a very well-known result [8] that $A \rtimes_{\beta,\mathrm{full}} G = A \rtimes_{\beta,\mathrm{red}} G$ when G is amenable.

Given a pair $(\mathcal{M}, \tau)$ formed by a von Neumann algebra $\mathcal{M}$ equipped with a normal faithful semifinite trace τ —noncommutative measure space— and a locally compact unimodular group G acting on $(\mathcal{M}, \tau)$ by trace preserving automorphisms $\beta : G \to \mathrm{Aut}(\mathcal{M}, \tau)$, the crossed product von Neumann algebra $\mathcal{M} \rtimes_\beta G$ is the von Neumann subalgebra of $\mathcal{M} \bar\otimes \mathcal{B}(L_2(G))$ generated by $\rho(\mathcal{M})$ and $\lambda(G)$, defined as above. In other words, $\mathcal{M} \rtimes_\beta G$ is the weak-$*$ closure of $\mathcal{M} \rtimes_{\beta,\mathrm{red}} G$.

Given $f : \mathrm{G} \to \mathcal{M}$ continuous and integrable, set

$$\tau_{\rtimes} \Big(\int_{\mathrm{G}} f_g \rtimes \lambda(g) \, d\mu(g) \Big) = \tau(f_e)$$

where μ and e stand for the Haar measure and the identity in the unimodular group G. This determines a normal faithful semifinite trace which extends to the crossed product von Neumann algebra $\mathcal{M} \rtimes_\beta \mathrm{G}$, see Takesaki [68]. In the following result, we provide an iterated crossed product characterization of quantum Euclidean spaces and construct a normal faithful semifinite trace on them.

Proposition 1.1. *The following results hold*:

i) *If $n = 2$ and $\Theta \neq 0$, we have*

$$\mathrm{E}_\Theta \simeq \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}.$$

   In this case, the crossed product action is given by $\mathbb{R}$-translations.

ii) *If $n \geq 2$, let us define*

$$\tau_\Theta(\lambda_\Theta(f)) = \tau_\Theta \Big(\int_{\mathbb{R}^n} f(\xi) \lambda_\Theta(\xi) \, d\xi \Big) = f(0)$$

   for $f : \mathbb{R}^n \to \mathbb{C}$ smooth and integrable. Then, τ_Θ extends to a normal faithful semifinite trace on E_Θ. Moreover, let Ξ denote the $(n-1) \times (n-1)$ upper left corner of $\Theta \in A_n(\mathbb{R})$. Then there exists a continuous group action $\beta_{n-1} : \mathbb{R} \to \mathrm{Aut}(\mathrm{E}_\Xi)$ satisfying

$$\mathrm{E}_\Theta \simeq \mathrm{E}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R}.$$

iii) *Let $\mathcal{R}_\Theta = \mathrm{E}_\Theta''$ be the von Neumann algebra generated by E_Θ in the GNS representation determined by τ_Θ. We have $\mathcal{R}_\Theta \simeq L_\infty(\mathbb{R}) \rtimes \mathbb{R} \simeq \mathcal{B}(L_2(\mathbb{R}))$ when $n = 2$ and $\Theta \neq 0$, with $\rtimes$-action given by $\mathbb{R}$-translations. Moreover τ_Θ extends to a n.f.s. trace on $\mathcal{R}_\Theta$, and the action β_{n-1} is trace preserving on $(\mathcal{R}_\Xi, \tau_\Xi)$. Induction on n and iteration give*

$$\mathcal{R}_\Theta \simeq \mathcal{R}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R},$$
$$\mathcal{R}_\Theta \simeq \Big(\big(L_\infty(\mathbb{R}) \rtimes_{\beta_1} \mathbb{R} \big) \cdots \rtimes_{\beta_{n-1}} \mathbb{R} \Big).$$

Proof. Given $\Theta \in A_n(\mathbb{R})$ and a Hilbert space $\mathcal{H}_\pi$, every set of one-parameter unitary groups $\{u_{\pi j}(s) : 1 \leq j \leq n, s \in \mathbb{R}\}$ in $\mathcal{B}(\mathcal{H}_\pi)$ satisfying the Θ-relations yields a $*$-representation $\pi : \mathrm{E}_\Theta \to \mathcal{B}(\mathcal{H}_\pi)$. In order to work with a concrete set of unitary groups $u_j(s)$, we consider the universal representation in the direct sum $\bigoplus_\pi \mathcal{H}_\pi$ over the set of all cyclic representations π as above. We shall use in what follows —with no further reference— that $\mathcal{C}_0(\mathbb{R})$ is the closure of $\mathcal{F}(L_1(\mathbb{R}))$, which can also be understood replacing the characters $\exp_s = \exp(2\pi\langle s, \cdot\rangle)$ in the Fourier transform $\mathcal{F}$ by $u_j(s)$ for any fixed $1 \leq j \leq n$, since $\{u_j(s) : s \in \mathbb{R}\}$ forms a non-trivial one-parameter group of unitaries.

i) If $n = 2$ and $\Theta \neq 0$, there must exist $\delta \neq 0$ with $\Theta = \delta(e_{12} - e_{21})$. We may rescale $u_1(s), u_2(t)$ and assume without loss of generality that $\delta = 1$. Now, consider the map

$$\mathrm{E}_\Theta \ni z = \int_{\mathbb{R}^2} z(s,t) u_1(s) u_2(t) \, ds dt \mapsto \int_{\mathbb{R}} f_t \rtimes t \, dt = f \in \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}$$

with $f_t \in \mathcal{C}_0(\mathbb{R})$ given by

$$f_t = \int_{\mathbb{R}} z(s,t)e^{2\pi i s \cdot}\, ds \simeq \int_{\mathbb{R}} z(s,t)u_1(s)\, ds.$$

If $\mathcal{H} = \oplus_\pi \mathcal{H}_\pi$, define $u : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ and $\gamma : \mathcal{C}_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ by

$$u(t) = u_2(t) \quad \text{and} \quad \gamma\Big(\int_{\mathbb{R}} \widehat{a}(s)e^{2\pi i s \cdot}\, ds \Big) = \int_{\mathbb{R}} \widehat{a}(s)u_1(s)\, ds.$$

The pair (γ, u) forms a covariant representation since we have

$$
\begin{aligned}
u(t)\gamma(a)u(t)^* &= \int_{\mathbb{R}} \widehat{a}(s)u_2(t)u_1(s)u_2(-t)\, ds \\
&= \int_{\mathbb{R}} \widehat{a}(s)e^{-2\pi i s t}u_1(s)\, ds \ = \ \gamma\Big(\int_{\mathbb{R}} \widehat{a}(s)e^{2\pi i s(\cdot - t)}\, ds \Big) \ = \ \gamma(\beta_t(a))
\end{aligned}
$$

where $\beta_t(a) = \lambda(t)[a]$ is the left regular representation at t acting on a. This gives

$$\|f\|_{\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}} \geq \Big\| \int_{\mathbb{R}} \gamma(f_t)u(t)\, dt \Big\|_{\mathcal{B}(\mathcal{H})} = \Big\| \int_{\mathbb{R}^2} z(s,t)u_1(s)u_2(t)\, dsdt \Big\|_{\mathcal{B}(\mathcal{H})} = \|z\|_{\mathrm{E}_\Theta}.$$

The reverse inequality is proved similarly. Indeed, let us consider the following map

$$\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R} \ni f = \int_{\mathbb{R}} f_t \rtimes t\, dt \mapsto \int_{\mathbb{R}^2} z(s,t)u_1(s)u_2(t)\, dsdt = z \in \mathrm{E}_\Theta$$

with $z(s,t) = \widehat{f}_t(s)$. Fix a Hilbert space $\mathcal{K}_{\gamma u}$ and a covariant representation (γ, u) of the pair $(\mathcal{C}_0(\mathbb{R}), \mathbb{R})$ in $\mathcal{B}(\mathcal{K}_{\gamma u})$. Define $w_1(s) = \gamma(e^{2\pi i s \cdot})$ and $w_2(t) = u(t)$. This shows that covariant representations of $(\mathcal{C}_0(\mathbb{R}), \mathbb{R})$ with action given by translations are in one-to-one correspondence with $*$-representations of E_Θ for the deformation $\Theta = e_{12} - e_{21}$. Indeed, $w_1(s)$ and $w_2(t)$ are one-parameter groups of unitaries since γ is a $*$-representation and u a unitary representation. Moreover, the commutation relations hold as a consequence of the covariant property

$$w_1(s)w_2(t) = \gamma(\exp_s)u(t) = u(t)\gamma(\beta_{-t}(\exp_s)) = e^{2\pi i s t}w_2(t)w_1(s)$$

for $\exp(s) = \exp(2\pi i s \cdot)$. In particular

$$\|z\|_{\mathrm{E}_\Theta} \geq \Big\| \int_{\mathbb{R}^2} z(s,t)w_1(s)w_2(t)\, dsdt \Big\|_{\mathcal{B}(\mathcal{K}_{\gamma u})} = \Big\| \int_{\mathbb{R}} \gamma(f_t)u(t)\, dt \Big\|_{\mathcal{B}(\mathcal{K}_{\gamma u})}.$$

Taking the supremum over (γ, u) covariant, we see that $\|z\|_{\mathrm{E}_\Theta} \geq \|f\|_{\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}}$.

ii) When $n \geq 2$ we proceed by induction. To prove ii) for $n = 2$, it suffices from i) to justify that τ_Θ extends to a n.s.f. trace on E_Θ. Note that $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}$ is generated by $\exp_\eta \rtimes \lambda(\zeta)$ for $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}$ where $\exp_\eta(x) = \exp(2\pi i x \eta)$ and $\lambda(\zeta)f(x) = f(x - \zeta)$. According to i), this gives

$$\lambda_\Theta(f) = \int_{\mathbb{R}^2} f(\xi)\lambda_\Theta(\xi)\, d\xi = \int_{\mathbb{R}} \Big[\underbrace{\int_{\mathbb{R}} f(\eta, \zeta) \exp_\eta\, d\eta}_{\varphi_\zeta} \Big] \rtimes \lambda(\zeta)\, d\zeta.$$

This means that the crossed product trace

$$\tau_\rtimes \Big(\int_{\mathbb{R}} \varphi_\zeta \rtimes \lambda(\zeta)\, d\zeta \Big) = \int_{\mathbb{R}} \varphi_0(x)\, dx = \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} f(\eta, 0) \exp_\eta(x)\, d\eta \Big] dx = f(0)$$

coincides with τ_Θ in E_Θ. Since $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}$ embeds faithfully in $L_\infty(\mathbb{R}) \rtimes \mathbb{R}$, it turns out that τ_Θ is a n.f.s. trace and completes the argument for the case $n = 2$. Once

this is settled, consider $\Theta \in A_n(\mathbb{R})$ whose upper left $(n-1) \times (n-1)$ corner is denoted by Ξ. Assume ii) holds for any dimension smaller than n, and set

$$\beta_{n-1}(s)\Big(\int_{\mathbb{R}^{n-1}} \varphi(z)\lambda_\Xi(z)\,dz\Big) = \int_{\mathbb{R}^{n-1}} \varphi(z)e^{-2\pi i \sum_{j<n} \Theta_{jn}z_j s}\lambda_\Xi(z)\,dz.$$

Then, β_{n-1} trivially yields a τ_Ξ-preserving action on $(\mathrm{E}_\Xi, \tau_\Xi)$. Moreover, the map $\lambda_\Theta(\xi) \mapsto \lambda_\Xi(\xi_1, \xi_2, \ldots, \xi_{n-1}) \rtimes \lambda(\xi_n)$ also gives rise to $\mathrm{E}_\Theta \simeq \mathrm{E}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R}$ and $\tau_\Theta = \tau_\rtimes|_{\mathrm{E}_\Theta}$ by arguing as above for $n = 2$, details are left to the reader.

iii) Now, for $n = 2$ and $\Theta \neq 0$ we get

$$\mathcal{R}_\Theta = \mathrm{E}_\Theta'' = \big(\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{R}\big)'' = \mathcal{C}_0(\mathbb{R})'' \rtimes \mathbb{R} = L_\infty(\mathbb{R}) \rtimes \mathbb{R},$$

$\tau_\Theta = \tau_\rtimes$ on $\mathcal{R}_\Theta$ and E_Θ sits faithfully in $\mathcal{R}_\Theta$. Moreover, $L_\infty(\mathbb{R}) \rtimes \mathbb{R} \subset \mathcal{B}(L_2(\mathbb{R}))$ acts on $L_2(\mathbb{R})$ by modulation and translation, which implies $\mathcal{R}_\Theta \simeq \mathcal{B}(L_2(\mathbb{R}))$ since only constant multiples of the identity map commute with all modulations and translations. When $n > 2$ we proceed by induction one more time to conclude that β_{n-1} is τ_Ξ-preserving, $\mathcal{R}_\Theta \simeq \mathcal{R}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R}$, $\tau_\Theta = \tau_\rtimes$ and $\mathrm{E}_\Theta \subset \mathcal{R}_\Theta$ faithfully. The last assertion follows trivially by iteration. This completes the proof. $\qquad\square$

Remark 1.2. The map

$$\lambda_\Theta : L_2(\mathbb{R}^n) \to L_2(\mathcal{R}_\Theta, \tau_\Theta)$$

is an isometric isomorphism, extending Plancherel theorem for $\Theta = 0$. Indeed, once we know τ_Θ is a trace, it follows from the density of the quantum Schwartz class $\mathcal{S}_\Theta = \lambda_\Theta(\mathcal{S}(\mathbb{R}^n))$ in $L_2(\mathcal{R}_\Theta)$ and the identity $\lambda_\Theta(f_1)\lambda_\Theta(f_2) = \lambda_\Theta(f_1 *_\Theta f_2)$.

Remark 1.3. When $n = 1$, $\mathcal{R}_\Theta = L_\infty(\mathbb{R})$ generated by $u(s) = \exp(2\pi i s \cdot)$. In the 2D case, we find one more time $\mathcal{R}_\Theta = L_\infty(\mathbb{R}^2)$ for $\Theta = 0$. Otherwise, there exists $\delta \neq 0$ such that $\Theta = \delta(e_{12} - e_{21})$. Rescaling $\delta = 1$ and arguing as in Proposition 1.1 iii) gives

$$\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\Rightarrow\; \mathcal{R}_\Theta \simeq \mathcal{B}(L_2(\mathbb{R})) \simeq L_\infty(\mathbb{R}) \rtimes \mathbb{R}$$

generated by modulations $\exp_\eta \rtimes \mathbf{1}$ and translations $\mathbf{1} \rtimes \lambda(\zeta)$ for $\eta, \zeta \in \mathbb{R}$. These are the standard time/frequency unitaries in Fourier analysis. If we set Ξ to be the $n \times n$ matrix with all its entries equal to 1, then the analogous space in dimension $2n$ is given by $\Theta = \Xi \otimes (e_{12} - e_{21})$ with $\mathcal{R}_\Theta \simeq \mathcal{B}(L_2(\mathbb{R}^n)) \simeq L_\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n$. The 3D case admits other models. By Proposition 1.1 iii)

- If $\Theta = 0$, then $\mathcal{R}_\Theta = L_\infty(\mathbb{R}^3) \simeq L_\infty(\mathbb{R})\bar\otimes L_\infty(\mathbb{R})\bar\otimes L_\infty(\mathbb{R})$,

- If $\Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix} \;\Rightarrow\; \mathcal{R}_\Theta \simeq L_\infty(\mathbb{R})\bar\otimes\big(L_\infty(\mathbb{R}) \rtimes \mathbb{R}\big),$

- If $\Theta = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ -\beta & -\alpha & 0 \end{pmatrix} \;\Rightarrow\; \mathcal{R}_\Theta \simeq \big(L_\infty(\mathbb{R})\bar\otimes L_\infty(\mathbb{R})\big) \rtimes \mathbb{R},$

- If $\Theta = \begin{pmatrix} 0 & \gamma & \beta \\ -\gamma & 0 & \alpha \\ -\beta & -\alpha & 0 \end{pmatrix} \;\Rightarrow\; \mathcal{R}_\Theta \simeq \big(L_\infty(\mathbb{R}) \rtimes \mathbb{R}\big) \rtimes \mathbb{R},$

for $\alpha, \beta, \gamma \neq 0$. Higher dimensions are treated similarly. When $\alpha \neq 0 = \beta = \gamma$, the $\rtimes$-action is $t \cdot f(s) = f(s - \alpha t)$. In the second case $\alpha, \beta \neq 0 = \gamma$, the $\rtimes$-action in $L_\infty(\mathbb{R}^2)$ is $t \cdot f(x, y) = f(x - \beta t, y - \alpha t)$. In the third case $\alpha, \beta, \gamma \neq 0$, both actions yield $t \cdot_{\text{ext}} \big((s \cdot_{\text{int}} f)(r)\big) = \big((s - \alpha t) \cdot_{\text{int}} f\big)(r - \beta t) = f\big(r - \gamma s + (\alpha\gamma - \beta)t\big)$. In the particular case $\alpha = \beta = \gamma = 1$, we have full symmetry under the action of the permutation group. In this case, the time/frequency dichotomy described above for $n = 2$ is replaced by three indistinguishable sets of unitaries.

1.1.2. *The corepresentation map.* Let us now recall a very useful consequence of the crossed product characterization of $\mathcal{R}_\Theta$ given above, the normality (weak-$*$ continuity) of the corepresentation map $\sigma_\Theta : \lambda_\Theta(\xi) \mapsto \exp_\xi \otimes \lambda_\Theta(\xi)$, where $\exp_\xi$ stands for the character $x \mapsto \exp(2\pi i \langle x, \xi \rangle)$ in $L_\infty(\mathbb{R}^n)$. This will be the source of several metric and differentiability considerations over quantum Euclidean spaces.

Corollary 1.4. $\sigma_\Theta : \mathcal{R}_\Theta \to L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *is a normal injective $*$-homomorphism.*

Proof. The assertion is a simple exercise in the C*-algebra level, so that we shall only justify normality. We proceed by induction on n, the case $n = 1$ is nothing but comultiplication in $L_\infty(\mathbb{R}^n)$. In higher dimensions, σ_Θ factorizes as follows

$$
\begin{array}{ccc}
\mathcal{R}_\Theta & \xrightarrow{\;\simeq\;} & \mathcal{R}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R} \\
\downarrow{\scriptstyle \sigma_\Theta} & & \downarrow{\scriptstyle \widetilde{\sigma}_\Xi = \sigma_\Xi \rtimes id_\mathbb{R}} \\
L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta & & L_\infty(\mathbb{R}^{n-1}) \bar{\otimes} \mathcal{R}_\Xi \rtimes_{\widehat{\beta}_{n-1}} \mathbb{R} \\
& \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \Omega} \\
& & L_\infty(\mathbb{R}) \bar{\otimes} \big((L_\infty(\mathbb{R}^{n-1}) \bar{\otimes} \mathcal{R}_\Xi) \rtimes_{\widehat{\beta}_{n-1}} \mathbb{R}\big)
\end{array}
$$

where $\widehat{\beta}_{n-1} = id_{\mathbb{R}^{n-1}} \otimes \beta_{n-1}$ and the map Ω is given by

$$
\mathcal{M} \rtimes_\beta \mathbb{R} \ni \int_\mathbb{R} f_s \rtimes \lambda(s)\, ds \xrightarrow{\;\Omega\;} \int_\mathbb{R} \exp_s \otimes \big(f_s \rtimes \lambda(s)\big)\, ds \in L_\infty(\mathbb{R}) \bar{\otimes} (\mathcal{M} \rtimes_\beta \mathbb{R}).
$$

By such factorization, it suffices to justify the normality of $\widetilde{\sigma}_\Xi$ and Ω:

- The map σ_Ξ is equivariant

$$
\sigma_\Xi(\beta_{n-1}(s)(a)) = \widehat{\beta}_{n-1}(s)(\sigma_\Xi(a)).
$$

  Let $j = \rho \rtimes \lambda : \mathcal{R}_\Xi \rtimes \mathbb{R} \to \mathcal{R}_\Xi \bar{\otimes} \mathcal{B}(L_2(\mathbb{R}))$ be the natural injection. By the above equivariance, j intertwines $\sigma_\Xi \rtimes id$ and $\sigma_\Xi \otimes id$

$$
\widetilde{\sigma}_\Xi = \sigma_\Xi \rtimes id = (id_{L_\infty(\mathbb{R}^{n-1})} \otimes j)^{-1} \circ (\sigma_\Xi \otimes id_{\mathcal{B}(L_2(\mathbb{R}))}) \circ j.
$$

  Since $\sigma_\Xi \otimes id$ is normal by induction hypothesis, the same holds for $\widetilde{\sigma}_\Xi$.

- The fundamental unitary on $\mathbb{R}^2$

$$
\mathrm{W}f(x, y) = f(x + y, y)
$$

  satisfies $\mathrm{W}^*(\mathbf{1} \otimes \lambda(s))\mathrm{W} = \lambda(s) \otimes \lambda(s)$. Using the isometric isomorphism $\Lambda : \mathcal{L}(\mathbb{R}) \ni \lambda(s) \mapsto \exp_s \in L_\infty(\mathbb{R})$, we get

$$
\Omega(f) \;=\; \int_\mathbb{R} \exp_s \otimes \big(f_s \rtimes \lambda(s)\big)\, ds
$$

$$= (\Lambda \otimes id_{\mathcal{M} \rtimes \mathbb{R}})(\mathbf{1} \rtimes \mathrm{W}^*)\Big(\int_{\mathbb{R}} \mathbf{1} \otimes \big(f_s \rtimes \lambda(s)\big) \, ds \Big)(\mathbf{1} \rtimes \mathrm{W}).$$

Thus $\Omega(f) = (\Lambda \otimes id_{\mathcal{M} \rtimes \mathbb{R}})(\mathbf{1} \rtimes \mathrm{W}^*)(\mathbf{1} \otimes f)(\mathbf{1} \rtimes \mathrm{W})$ and Ω is normal. $\qquad \square$

1.2. Metrics and derivations. In this paragraph, we exploit the corepresentation σ_Θ to introduce some other auxiliary operators which will help us to equip $\mathcal{R}_\Theta$ with an induced metric, a natural BMO space and a differential structure.

1.2.1. A metric in $\mathcal{R}_\Theta$ and BMO. Given a von Neumann algebra $\mathcal{M}$, its opposite algebra $\mathcal{M}_{\mathrm{op}}$ is obtained by preserving linear and adjoint structures but reversing the product $a_1 \cdot a_2 = a_2 a_1$. Several reasons justify why noncommutative singular integral operators require to understand the singular kernels as operators affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$, see [44] and Section 2.1 below. We shall use from now on $\cdot$ for the $\mathcal{M}_{\mathrm{op}}$-product, as well as $\bullet$ for the product in $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$, so that

$$(a_1 \otimes a_2) \bullet (a_1' \otimes a_2') = (a_1 a_1') \otimes (a_2 \cdot a_2') = (a_1 a_1') \otimes (a_2' a_2).$$

Let us consider the linear map π_Θ, determined by

$$\exp_\xi \xmapsto{\pi_\Theta} \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*$$

where, as usual, we write $\exp_\xi$ for the Fourier characters $\exp(2\pi i \langle \xi, \cdot \rangle)$ in $\mathbb{R}^n$. As an illustration, recall that for $\Theta = 0$ we may expect to get the following identity for any (say) Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$

$$\pi_0(f)(x,y) = \pi_0\Big(\int_{\mathbb{R}^n} \widehat{f}(\xi) \exp_\xi d\xi \Big)(x,y) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \exp_\xi(x-y) \, d\xi = f(x-y).$$

Of course, this requires to justify the continuity properties of the map π_Θ which we shall do in Lemma 1.5 below. The quantum analogue of this map is particularly useful to identify the *diagonal* in $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$, where the kernel singularities of our operators are expected to live. Of particular relevance is the induced metric which we define by

$$\mathrm{d}_\Theta = \pi_\Theta(|\cdot|)$$

for the Euclidean norm $|\cdot|$ or the bands around the diagonal $\mathrm{b}_\Theta(\mathrm{R}) = \pi_\Theta(\chi_{|\cdot| \leq \mathrm{R}})$.

It is worth recalling that both σ_Θ and π_Θ take $L_2(\mathbb{R}^n)$ into $L_\infty(\mathbb{R}^n; L_2(\mathbb{R}^n))$ when $\Theta = 0$. The quantum analogue for $\Theta \neq 0$ requires noncommutative forms of mixed-norm $L_\infty(L_2)$-spaces, whose construction we briefly recall. Given a Hilbert space $\mathcal{H}$ and $x = \sum_j m_j \otimes h_j \in \mathcal{M} \otimes_{\mathrm{alg}} \mathcal{H}$, we define

$$\|x\|_{\mathcal{M} \bar{\otimes} \mathcal{H}^r} = \big\| \langle x, x \rangle_r \big\|_{\mathcal{M}}^{\frac{1}{2}} = \Big\| \sum_{j,k} m_j m_k^* \langle h_j, h_k \rangle_{\mathcal{H}} \Big\|_{\mathcal{M}}^{\frac{1}{2}}$$

$$\|x\|_{\mathcal{M} \bar{\otimes} \mathcal{H}^c} = \big\| \langle x, x \rangle_c \big\|_{\mathcal{M}}^{\frac{1}{2}} = \Big\| \sum_{j,k} m_j^* m_k \langle h_j, h_k \rangle_{\mathcal{H}} \Big\|_{\mathcal{M}}^{\frac{1}{2}}.$$

Given $\dagger \in \{r, c\}$, the space $\mathcal{M} \bar{\otimes} \mathcal{H}^\dagger$ —also denoted by $\mathcal{H}^\dagger \bar{\otimes} \mathcal{M}$ or $L_\infty(\mathcal{M}; \mathcal{H}^\dagger)$— is defined as the closure of $\mathcal{M} \otimes_{\mathrm{alg}} \mathcal{H}$ with respect to the weak topology generated by the functionals

$$p_\omega(x) = \omega\big(\langle x, x \rangle_\dagger^{\frac{1}{2}} \big) \quad \text{for every } \omega \in \mathcal{M}_*.$$

Alternatively, $\mathcal{M} \bar{\otimes} \mathcal{H}^\dagger$ is the weak-$*$ closed tensor product of the dual operator spaces $\mathcal{M}$ and $\mathcal{H}^\dagger$, the latter space representing the row or column operator space structure on $\mathcal{H}$. Indeed, if X and Y are dual operator spaces, there are completely

isometric and weak-$*$ continuous injections $\pi_X : X \to \mathcal{B}(\mathcal{H}_X)$, $\pi_Y : Y \to \mathcal{B}(\mathcal{H}_Y)$ and we define $X \bar{\otimes} Y$ as

$$\overline{\pi_X[X] \otimes_{\mathrm{alg}} \pi_Y[Y]^{\mathrm{w}^*}} \subset \mathcal{B}(\mathcal{H}_X \otimes_2 \mathcal{H}_X).$$

It is well-known that such construction is representation-independent and when one of the tensor components is a von Neumann algebra, the predual is given by the projective tensor product $X_* \widehat{\otimes} Y_*$, see [26, 60] for further details. Noncommutative mixed-norm spaces have also been studied in [45, 59].

Lemma 1.5. π_Θ *extends to a normal $*$-homomorphism*

$$\pi_\Theta : L_\infty(\mathbb{R}^n) \to \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}} \quad \textit{satisfying} \quad (\sigma_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}}) \circ \pi_\Theta = (id_{\mathbb{R}^n} \otimes \pi_\Theta) \circ \Delta_{\mathbb{R}^n}$$

where $\Delta_{\mathbb{R}^n}(\exp_\xi) = \exp_\xi \otimes \exp_\xi$ *is the comultiplication map in* $\mathbb{R}^n$. *This shows in particular that* $\mathrm{d}_\Theta = \pi_\Theta(|\cdot|)$ *is a well-defined operator affiliated to* $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ *as an increasing limit of the bounded operators* $\mathrm{d}_\Theta(\mathrm{R}) = \pi_\Theta(\chi_{|\cdot| \le \mathrm{R}} |\cdot|)$. *Moreover, the map* π_Θ *also extends to a complete isometry* $\pi_\Theta : L_2^c(\mathbb{R}^n) \to L_2^c(\mathcal{R}_\Theta) \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$.

Proof. That $\pi_\Theta : \exp_\xi \mapsto \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*$ extends to a $*$-homomorphism is a simple consequence of the product in $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$, details are left to the reader. Let us then prove that π_Θ is weak-$*$ continuous. It is tedious but straightforward to check that

$$\Lambda_\Theta : h \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} h(\xi_1, \xi_2) \lambda_\Theta(\xi_1) \lambda_\Theta(\xi_2) \otimes \lambda_\Theta(\xi_1)^* \, d\xi_1 d\xi_2$$

yields an isometry $L_2(\mathbb{R}^n) \otimes_2 L_2(\mathbb{R}^n) \to L_2(\mathcal{R}_\Theta) \otimes_2 L_2(\mathcal{R}_\Theta^{\mathrm{op}})$. Indeed, by density it suffices to expand $\tau_\Theta \otimes \tau_\Theta(\Lambda_\Theta(h)^* \Lambda_\Theta(h))$ for h smooth, then calculate the trace applying twice the simple identity $\tau_\Theta(\lambda_\Theta(f) \lambda_\Theta(\xi)^*) = f(\xi)$ for a smooth integrable function f in $\mathbb{R}^n$. Moreover, given any $z = \Lambda_\Theta(h) \in L_2(\mathcal{R}_\Theta) \otimes_2 L_2(\mathcal{R}_\Theta^{\mathrm{op}})$ it turns out that

$$
\begin{aligned}
\pi_\Theta(\exp_\xi)(z) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} h(\xi_1, \xi_2) \, \lambda_\Theta(\xi) \lambda_\Theta(\xi_1) \lambda_\Theta(\xi_2) \otimes \lambda_\Theta(\xi)^* \cdot \lambda_\Theta(\xi_1)^* \, d\xi_1 d\xi_2 \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} h(\xi_1, \xi_2) \, \lambda_\Theta(\xi_1 + \xi) \lambda_\Theta(\xi_2) \otimes \lambda_\Theta(\xi_1 + \xi)^* \, d\xi_1 d\xi_2 \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} h(\xi_1 - \xi, \xi_2) \, \lambda_\Theta(\xi_1) \lambda_\Theta(\xi_2) \otimes \lambda_\Theta(\xi_1)^* \, d\xi_1 d\xi_2 \\
&= \big(\Lambda_\Theta \circ (\lambda_{\mathbb{R}^n}(\xi) \otimes id_{\mathbb{R}^n}) \circ \Lambda_\Theta^{-1}\big)(z),
\end{aligned}
$$

where $\lambda_{\mathbb{R}^n}$ denotes the left regular representation on $\mathbb{R}^n$. This shows that π_Θ is weak-$*$ continuous and satisfies the identity $\pi_\Theta(f) = \Lambda_\Theta \circ (f \otimes id_{\mathbb{R}^n}) \circ \Lambda_\Theta^{-1}$ for all $f \in L_\infty(\mathbb{R}^n)$, after identifying $\exp_\xi$ with $\lambda_{\mathbb{R}^n}(\xi)$. Once we have justified the weak-$*$ continuity, the relation $(\sigma_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}}) \circ \pi_\Theta = (id_{\mathbb{R}^n} \otimes \pi_\Theta) \circ \Delta_{\mathbb{R}^n}$ follows since it trivially holds when acting on $\exp_\xi$ for any $\xi \in \mathbb{R}^n$. Also, it implies that d_Θ is affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ and arises as an increasing limit of bounded operators $\mathrm{d}_\Theta(\mathrm{R})$ for $\mathrm{R} > 0$. It remains to show that $\pi_\Theta : L_2^c(\mathbb{R}^n) \to L_2^c(\mathcal{R}_\Theta) \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$. Recall that the norm in $L_2^c(\mathcal{M}) \bar{\otimes} \mathcal{M}_{\mathrm{op}}$ is given by

$$a \mapsto \left\| (\tau \otimes id_{\mathcal{M}_{\mathrm{op}}})(a^* a) \right\|_{\mathcal{M}_{\mathrm{op}}}^{\frac{1}{2}}.$$

When $\mathcal{M} = \mathcal{R}_\Theta$ and $f \in L_2(\mathbb{R}^n)$ is smooth we find

$$(\tau_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}})(\pi_\Theta(f)) = (\tau_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}})\Big(\int_{\mathbb{R}^n} \widehat{f}(\xi) \, \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \, d\xi \Big) = \widehat{f}(0) \mathbf{1}_{\mathcal{R}_\Theta^{\mathrm{op}}}.$$

Therefore, taking $f = \sum_{jk} f_{jk} \otimes e_{jk} \in M_n(L_2^c(\mathbb{R}^n))$ smooth, we obtain

$$
\begin{aligned}
\left\| \left(\pi_\Theta(f_{jk}) \right)_{jk} \right\|_{M_n(L_2^c(\mathcal{R}_\Theta)\bar{\otimes}\mathcal{R}_\Theta)} &= \left\| \left(\left(\tau_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}} \right) \left(\pi_\Theta(f^*f) \right) \right)_{jk} \right\|_{M_n \otimes_{\min} \mathcal{R}_\Theta^{\mathrm{op}}}^{\frac{1}{2}} \\
&= \left\| \left(\int_{\mathbb{R}^n} (f^*f)_{jk}(\xi)\,d\xi \right) \right\|_{M_n}^{\frac{1}{2}} = \|f\|_{M_n(L_2^c(\mathbb{R}^n))}.
\end{aligned}
$$

By density, we see that $\pi_\Theta : L_2^c(\mathbb{R}^n) \to L_2^c(\mathcal{R}_\Theta)\bar{\otimes}\mathcal{R}_\Theta^{\mathrm{op}}$ is a complete isometry. $\quad\square$

Let

$$
S_t f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-t|\xi|^2} e^{2\pi i \langle x, \xi \rangle}\,d\xi
$$

denote the heat semigroup acting on $f : \mathbb{R}^n \to \mathbb{C}$. Consider the induced semigroup $S_\Theta = (S_{\Theta,t})_{t\geq 0}$ on $\mathcal{R}_\Theta$ determined by $\sigma_\Theta \circ S_{\Theta,t} = (S_t \otimes id_{\mathcal{R}_\Theta}) \circ \sigma_\Theta$. This yields a Markov semigroup which formally acts as

$$
S_{\Theta,t}(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} f(\xi) e^{-t|\xi|^2} \lambda_\Theta(\xi)\,d\xi.
$$

Consider the corresponding column BMO norm

$$
\begin{aligned}
\|a\|_{\mathrm{BMO}_c(\mathcal{R}_\Theta)} &= \sup_{t>0} \left\| \left(S_{\Theta,t}(a^*a) - S_{\Theta,t}(a)^* S_{\Theta,t}(a) \right)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} \\
&\sim \sup_{Q \in \mathcal{Q}} \left\| \left(\fint_Q |\sigma_\Theta(a) - \sigma_\Theta(a)_Q|^2\,d\mu \right)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} = \|\sigma_\Theta(a)\|_{\mathrm{BMO}_c(\mathcal{Q}_\Theta)}
\end{aligned}
$$

where $\mathcal{Q}$ denotes the set of all Euclidean balls in $\mathbb{R}^n$, μ stands for the Lebesgue measure and $\sigma_\Theta(a)_Q$ is the average of $\sigma_\Theta(a)$ over the ball Q. The norm equivalence above —which holds up to constants depending on the dimension n— is a simple consequence of the intertwining identity $\sigma_\Theta \circ S_{\Theta,t} = (S_t \otimes id_{\mathcal{R}_\Theta}) \circ \sigma_\Theta$ and the equivalence between the BMO norms respectively associated to the heat semigroup and the Euclidean metric in $\mathbb{R}^n$, see [42, Section 1.2] for further details. The space $\mathrm{BMO}_c(\mathcal{Q}_\Theta)$ is an illustration of the operator-valued spaces $\mathrm{BMO}_c(\mathbb{R}^n; \mathcal{B}(\mathcal{H}))$ which were extensively studied by Tao Mei in his PhD Thesis [51].

We may use these latter spaces to properly define the column space $\mathrm{BMO}_c(\mathcal{R}_\Theta)$. Indeed, we know from Corollary 1.4 that $\sigma_\Theta(\mathcal{R}_\Theta)$ is a subalgebra of $L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$, which in turn is included in $\mathrm{BMO}_c(\mathcal{Q}_\Theta)$. Since we know from [51] that $\mathrm{BMO}_c(\mathcal{Q}_\Theta)$ admits a predual $\mathrm{H}_1^c(\mathcal{Q}_\Theta)$, we may define

$$
\mathrm{BMO}_c(\mathcal{R}_\Theta) = \overline{\sigma_\Theta(\mathcal{R}_\Theta)^{\mathrm{w}^*}}
$$

where the weak-$*$ closure is taken with respect to the pair $(\mathrm{H}_1^c(\mathcal{Q}_\Theta), \mathrm{BMO}_c(\mathcal{Q}_\Theta))$. This kind of BMO spaces over Markov semigroups have been deeply investigated in [41] for finite von Neumann algebras. The semifinite case is more subtle and we shall give in Appendix B a self-contained argument for $\mathcal{R}_\Theta$.

1.2.2. *A Poincaré type inequality.* Let

$$
\mathcal{S}_\Theta = \left\{ \lambda_\Theta(f) \, : \, f \in \mathcal{S}(\mathbb{R}^n) = \mathbb{R}^n\text{-Schwartz class} \right\}.
$$

Define ∂_Θ^j as the linear extension of the map

$$
\partial_\Theta^j(\lambda_\Theta(\xi)) = 2\pi i \xi_j \lambda_\Theta(\xi)
$$

over the quantum Schwartz class $\mathcal{S}_\Theta$ for $1 \leq j \leq n$. Recall that $\mathcal{S}_\Theta$ is an $*$-algebra since $\lambda_\Theta(f_1)\lambda_\Theta(f_2) = \lambda_\Theta(f_1 *_\Theta f_2)$ and $\lambda_\Theta(f)^* = \lambda_\Theta(f_\Theta^*)$ are stable in $\mathcal{S}_\Theta$. In what follows, we shall be working with this and other natural differential operators in $\mathcal{R}_\Theta$. The following one is a free analogue of the gradient operator associated to the partial derivatives considered above. Let $\mathcal{L}(\mathbb{F}_n)$ denote the group von Neumann algebra associated to the free group over n generators $\mathbb{F}_n$. It is well-known from (say) [72] that $\mathcal{L}(\mathbb{F}_n)$ is generated by n semicircular random variables $s_1, s_2, \ldots, s_n$. Let us consider the map $s : \mathbb{R}^n \to \mathcal{L}(\mathbb{F}_n)$ given by

$$s(\xi) = \sum_{j=1}^{n} \langle \xi, e_j \rangle s(e_j) = \sum_{j=1}^{n} \xi_j s_j.$$

Then we introduce the Θ-deformed free gradient

$$\nabla_\Theta = \sum_{k=1}^{n} s_k \otimes \partial_\Theta^k : \mathcal{S}_\Theta \to \mathcal{L}(\mathbb{F}_n) \bar{\otimes} \mathcal{R}_\Theta.$$

If ∇ denotes the free gradient for $\Theta = 0$, is easily checked that

$$(id_{\mathcal{L}(\mathbb{F}_n)} \otimes \sigma_\Theta) \circ \nabla_\Theta = \sum_{k=1}^{n} s_k \otimes (\sigma_\Theta \circ \partial_\Theta^k) = \sum_{k=1}^{n} s_k \otimes (\partial_k \circ \sigma_\Theta) = (\nabla \otimes id_{\mathcal{R}_\Theta}) \circ \sigma_\Theta.$$

Moreover, let us recall that $\nabla_\Theta(\lambda_\Theta(\xi)) = \sum_k s_k \otimes 2\pi i \xi_k \lambda_\Theta(\xi) = 2\pi i s(\xi) \otimes \lambda_\Theta(\xi)$.

Proposition 1.6. *Let* B_R *and* q_R *stand for any ball of radius* R *in* $\mathbb{R}^n$ *and the characteristic function of it. Given a noncommutative measure space* $(\mathcal{M}, \tau)$ *and* $\varphi : B_R \to \mathcal{M}$ *smooth with* B_R*-average denoted by* φ_{B_R}*, the following inequality holds for the free gradient* ∇ *in* $\mathbb{R}^n$

$$\left\| \fint_{B_R} |\varphi - \varphi_{B_R}|^2 d\mu \right\|_{\mathcal{M}}^{\frac{1}{2}} \leq 2\sqrt{2}R \left\| (\mathbf{1} \otimes q_R \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{M}})(\varphi) \right\|_{\mathcal{L}(\mathbb{F}_n) \bar{\otimes} L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{M}}.$$

Proof.   Consider the derivation map $\delta(f) = f \otimes \mathbf{1} - \mathbf{1} \otimes f$. We shall use the following straightforward algebraic identity, which is valid for any normal state ϕ on any von Neumann algebra

$$\phi\big((f - \phi(f))^*(f - \phi(f))\big) = \frac{1}{2}\phi \otimes \phi\big(\delta(f)^*\delta(f)\big).$$

Applying it for $\varphi = \sum_j f_j \otimes y_j$ we obtain

$$(\phi \otimes id_{\mathcal{M}})\Big(|\varphi - (\phi \otimes id_{\mathcal{M}})(\varphi)|^2\Big) = \frac{1}{2}(\phi \otimes \phi \otimes id_{\mathcal{M}})\Big(|(\delta \otimes id_{\mathcal{M}})(\varphi)|^2\Big).$$

If c_R denotes the center of B_R, we observe that

$$(\delta \otimes id_{\mathcal{M}})(\varphi) = (\varphi - \varphi(c_R)) \otimes \mathbf{1}_{\mathbb{R}^n} - \mathbf{1}_{\mathbb{R}^n} \otimes (\varphi - \varphi(c_R)).$$

Then, letting ϕ be the average over B_R we deduce the following inequality

$$\begin{aligned}
\left\| \fint_{B_R} |\varphi - \varphi_{B_R}|^2 d\mu \right\|_{\mathcal{M}}^{\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left\| (\phi \otimes \phi \otimes id_{\mathcal{M}})\Big(|(\delta \otimes id_{\mathcal{M}})(\varphi)|^2\Big) \right\|_{\mathcal{M}}^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}}\left\| (\phi \otimes id_{\mathbb{R}^n} \otimes id_{\mathcal{M}})\Big(|(\varphi - \varphi(c_R)) \otimes \mathbf{1}_{\mathbb{R}^n}|^2\Big) \right\|_{\mathcal{M}}^{\frac{1}{2}} \\
&\quad + \frac{1}{\sqrt{2}}\left\| (id_{\mathbb{R}^n} \otimes \phi \otimes id_{\mathcal{M}})\Big(|\mathbf{1}_{\mathbb{R}^n} \otimes (\varphi - \varphi(c_R))|^2\Big) \right\|_{\mathcal{M}}^{\frac{1}{2}}
\end{aligned}$$

$$= \sqrt{2}\left\|(\phi \otimes id_{\mathcal{M}})\left(\left|\varphi - \varphi(c_{\mathrm{R}})\right|^2\right)\right\|_{\mathcal{M}}^{\frac{1}{2}}.$$

In order to estimate the latter term, we use integration by parts to obtain

$$\varphi(x) - \varphi(c_{\mathrm{R}}) = \int_0^1 \sum_{k=1}^n \partial_k \varphi\big(t(x - c_{\mathrm{R}}) + c_{\mathrm{R}}\big)\langle x - c_{\mathrm{R}}, e_k\rangle \, dt$$

$$= \int_0^1 \underbrace{(\tau_{\mathcal{L}(\mathbb{F}_n)} \otimes id_{\mathbb{R}^n})}_{\mathsf{E}_{\mathbb{R}^n}}\big(\underbrace{q_{\mathrm{R}}(x)\nabla\varphi(t(x - c_{\mathrm{R}}) + c_{\mathrm{R}})}_{\mathrm{A}(t)}\,\underbrace{q_{\mathrm{R}}(x)s(x - c_{\mathrm{R}})}_{\mathrm{B}}\big)\,dt$$

for $x \in \mathrm{B}_{\mathrm{R}}$. By the operator-convexity of $|\cdot|^2$, we find the inequality below

$$\left\|\fint_{\mathrm{B}_{\mathrm{R}}} \left|\varphi - \varphi_{\mathrm{B}_{\mathrm{R}}}\right|^2 d\mu\right\|_{\mathcal{M}}^{\frac{1}{2}} \leq \sqrt{2}\Big(\int_0^1 \left\|(\phi \otimes id_{\mathcal{M}})\big(|\mathsf{E}_{\mathbb{R}^n}(\mathrm{A}(t)\mathrm{B})|^2\big)\right\|_{\mathcal{M}} dt\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{2}\Big(\int_0^1 \|\mathrm{A}(t)\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}}^2 dt\Big)^{\frac{1}{2}}\|\mathrm{B}\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n)}.$$

Now we observe that $\|\mathrm{A}(t)\| \leq \|\mathrm{A}(1)\|$ for all $0 \leq t \leq 1$, so we conclude that

$$\left\|\fint_{\mathrm{B}_{\mathrm{R}}} \left|\varphi - \varphi_{\mathrm{B}_{\mathrm{R}}}\right|^2 d\mu\right\|_{\mathcal{M}}^{\frac{1}{2}} \leq \sqrt{2}\|\mathrm{B}\|_{\mathcal{L}(\mathbb{F}_n)}\left\|(\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{M}})(\varphi)\right\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}}.$$

Finally, Voiculescu's inequality [72] claims

$$\|s(h)\|_{\mathcal{L}(\mathbb{F}_n)} = 2\|h\|_{\mathbb{R}^n},$$

so that $\|\mathrm{B}\|_{\mathcal{L}(\mathbb{F}_n)} \leq 2 \sup_{x \in \mathrm{B}_{\mathrm{R}}} \|x - c_{\mathrm{R}}\| = 2\mathrm{R}$ and the proof is complete. $\qquad\square$

Remark 1.7. Recall that

$$\|a\|_{\mathrm{BMO}_c(\mathcal{R}_\Theta)} \sim \|\sigma_\Theta(a)\|_{\mathrm{BMO}_c(\mathcal{Q}_\Theta)}$$

$$= \sup_{\mathrm{R}>0}\left\|\Big(\fint_{\mathrm{B}_{\mathrm{R}}} \left|\sigma_\Theta(a) - \sigma_\Theta(a)_{\mathrm{B}_{\mathrm{R}}}\right|^2 d\mu\Big)^{\frac{1}{2}}\right\|_{\mathcal{R}_\Theta}.$$

According to Proposition 1.6 for $\mathcal{M} = \mathcal{R}_\Theta$, we deduce

$$\|a\|_{\mathrm{BMO}_c(\mathcal{R}_\Theta)} \lesssim \sup_{\mathrm{R}>0}\mathrm{R}\left\|(\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{R}_\Theta}) \circ \sigma_\Theta(a)\right\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta}$$

$$= \sup_{\mathrm{R}>0}\mathrm{R}\left\|(\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1})(id_{\mathcal{L}(\mathbb{F}_n)} \otimes \sigma_\Theta) \circ \nabla_\Theta(a)\right\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta}.$$

Remark 1.8. Given $2 \leq p \leq \infty$, we have

$$\left\|\nabla_\Theta(a)\right\|_{L_p(\mathcal{L}(\mathbb{F}_n)\bar{\otimes}\mathcal{R}_\Theta)} \sim \left\|\Big(\sum_{k=1}^n (\partial_\Theta^k a)(\partial_\Theta^k a)^*\Big)^{\frac{1}{2}}\right\|_{L_p(\mathcal{R}_\Theta)}$$

$$+ \left\|\Big(\sum_{k=1}^n (\partial_\Theta^k a)^*(\partial_\Theta^k a)\Big)^{\frac{1}{2}}\right\|_{L_p(\mathcal{R}_\Theta)}$$

from the operator-valued form of Voiculescu's inequality [46, 71]. Let us recall in passing that this norm equivalence holds in the category of operator spaces and moreover, the constants do not depend on the dimension n. This justifies our choice of free generators in the definition of ∇_Θ. An alternative choice would have been

to work with Rademacher variables or matrix units, but the former does not lead to the same norm equivalences for $p = \infty$. If $\Theta = 0$ we get

$$\big\|\nabla(f)\big\|_{L_p(\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_\infty(\mathbb{R}^n))} \sim \Big\|\Big(\sum_{k=1}^n |\partial_x^k f|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbb{R}^n)}.$$

1.3. **Quantum Euclidean variables.** We consider other characterizations of $\mathcal{R}_\Theta$ in terms of the infinitesimal generators of $u_j(s)$. These unbounded operators play the role in $\mathcal{R}_\Theta$ of the Euclidean variables x_j in $\mathbb{R}^n$. We will use them to study the quantum analogue of the Schwartz class, to give an intrinsic characterization of the quantum distance d_Θ and to deduce the algebraic structure of $\mathcal{R}_\Theta$.

1.3.1. *Another approach towards $\mathcal{R}_\Theta$.* Define

$$x_{\Theta,j} = \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}(u_j(s)) \quad \text{for} \quad 1 \le j \le n,$$

with $u_j(s)$ the generating unitaries of the quantum Euclidean space $\mathcal{R}_\Theta$. These are the (self-adjoint) infinitesimal generators of the one-parameter groups of unitaries $(u_j(s))_{s\in\mathbb{R}}$ given by Stone's theorem and may be regarded as quantum forms of the Euclidean variables. Namely, when $\Theta = 0$ the one-parameter unitary group $u_j(s)$ is composed of multiplication operators by the Fourier characters $x \mapsto \exp(2\pi i s x_j)$ and

$$2\pi i x_j = \partial_s(e^{2\pi i s x_j})_{|s=0}.$$

The operators $x_{\Theta,j}$ enjoy some fundamental properties of the Euclidean variables.

Proposition 1.9. *The following results hold*:

i) *The generators $x_{\Theta,j}$ satisfy for $1 \le j, k \le n$*

$$[x_{\Theta,j}, x_{\Theta,k}] = \frac{1}{2\pi i}\,\Theta_{jk} \;\Leftrightarrow\; u_j(s)u_k(t) = e^{2\pi i \Theta_{jk}st}u_k(t)u_j(s).$$

ii) *Recall the definition of the quantum Schwartz class*

$$\mathcal{S}_\Theta = \Big\{\lambda_\Theta(f) : f \in \mathcal{S}(\mathbb{R}^n)\Big\}.$$

The infinitesimal generators $x_{\Theta,j}$ are densely defined unbounded operators affiliated to $\mathcal{R}_\Theta$. Moreover, in the GNS representation on $L_2(\mathcal{R}_\Theta)$ we find $\mathcal{S}_\Theta \subset \mathrm{dom}(x_{\Theta,j})$ and $x_{\Theta,j}\mathcal{S}_\Theta, \mathcal{S}_\Theta x_{\Theta,j} \subset \mathcal{S}_\Theta$. More precisely

$$x_{\Theta,j}\lambda_\Theta(f) = \lambda_\Theta(D_{\Theta,j}^\ell f) \quad \text{where} \quad D_{\Theta,j}^\ell = \sum_{k=1}^{j-1}\Theta_{jk}M_{\xi_k} - \frac{1}{2\pi i}\partial_\xi^j,$$

$$\lambda_\Theta(f)x_{\Theta,j} = \lambda_\Theta(D_{\Theta,j}^r f) \quad \text{where} \quad D_{\Theta,j}^r = \sum_{i=j+1}^n\Theta_{ij}M_{\xi_i} - \frac{1}{2\pi i}\partial_\xi^j,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and $M_{\xi_k}f(\xi) = \xi_k f(\xi)$. In addition we get $[D_{\Theta,j}^\ell, D_{\Theta,k}^r] = 0$.

iii) *Let $(x_{\Theta,j})_j$ and $(y_{\Theta,j})_j$ be the infinitesimal generators associated to $\mathcal{R}_\Theta \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{R}_\Theta^{\mathrm{op}}$ respectively. Then, we may relate the quantum distance d_Θ with these quantum variables as follows*

$$d_\Theta = \Big(\sum_{j=1}^n \big(x_{\Theta,j} - y_{\Theta,j}\big)^2\Big)^{\frac{1}{2}}.$$

Proof. All the assertions are quite standard. Assume $[x_{\Theta,j}, x_{\Theta,k}] = \frac{\Theta_{jk}}{2\pi i}$ and define the maps $\varphi_j(s)[X] = \exp(2\pi i s x_{\Theta,j}) X \exp(-2\pi i s x_{\Theta,j}) = u_j(s) X u_j(-s)$. Recalling that $\varphi_j(0)$ is the identity map and noticing that

$$\frac{d}{ds}\big(\varphi_j(s)\big) = 2\pi i\Big(x_{\Theta,j}\varphi_j(s) - \varphi_j(s)x_{\Theta,j}\Big),$$

we deduce $\varphi_j(s)[X] = \exp\big(2\pi i s \delta_j(X)\big)$ for the derivation $\delta_j(X) = [x_{\Theta,j}, X]$, so that

$$u_j(s)u_k(t)u_j(-s) = \sum_{n\geq 0} \frac{(2\pi i s)^n}{n!}\delta_j^n(u_k(t)).$$

Consider the map given by $\Theta_{jk}z_{jk} = 2\pi i x_{\Theta,k}$, so that $\delta_j(z_{jk}) = [x_{\Theta,j}, z_{jk}] = 1$ and $\delta_j(h(z_{jk})) = h'(z_{jk})$. Since $u_k(t) = \exp(2\pi i t x_{\Theta,k}) = \exp(t\Theta_{jk}z_{jk}) = h(z_{jk})$ we deduce the identity

$$\delta_j^n(u_k(t)) = h^{(n)}(z_{jk}) = (t\Theta_{jk})^n u_k(t)$$

which yields $u_j(s)u_k(t) = e^{2\pi i \Theta_{jk}st}u_k(t)u_j(s)$ as desired. Reciprocally, assuming this commutation relation we express the commutator of $x_{\Theta,j}$ and $x_{\Theta,k}$ as follows

$$
\begin{aligned}
[x_{\Theta,j}, x_{\Theta,k}] &= \frac{1}{-4\pi^2}\frac{d^2}{dsdt}\Big|_{s=t=0}\big(u_j(s)u_k(t) - u_k(t)u_j(s)\big)\\
&= \frac{1}{-4\pi^2}\frac{d^2}{dsdt}\Big|_{s=t=0}\big(e^{2\pi i\Theta_{jk}st} - 1\big)u_k(t)u_j(s) = \frac{2\pi i}{-4\pi^2}\Theta_{jk} = \frac{\Theta_{jk}}{2\pi i}.
\end{aligned}
$$

Regarding the second assertion ii), the first identity can be justified as follows

$$
\begin{aligned}
x_{\Theta,j}\lambda_\Theta(f) &= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}(u_j(s))\lambda_\Theta(f)\\
&= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n} f(\xi)u_j(s)\lambda_\Theta(\xi)d\xi\\
&= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n} f(\xi)e^{2\pi i\sum_{k<j}\Theta_{jk}s\xi_k}\lambda_\Theta(\xi + se_j)d\xi\\
&= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n} f(\xi - se_j)e^{2\pi i\sum_{k<j}\Theta_{jk}s\xi_k}\lambda_\Theta(\xi)d\xi\\
&= \frac{1}{2\pi i}\Big[\int_{\mathbb{R}^n}\Big(2\pi i\sum_{k<j}\Theta_{jk}\xi_k\Big)f(\xi)\lambda_\Theta(\xi)d\xi - \int_{\mathbb{R}^n}\partial_\xi^j f(\xi)\lambda_\Theta(\xi)d\xi\Big].
\end{aligned}
$$

The second identity is proved similarly. This shows that $\mathcal{S}_\Theta$ is a common core of the $x_{\Theta,j}$ for $1 \leq j \leq n$. Thus, it just remains to show that $[D_{\Theta,j}^\ell, D_{\Theta,k}^r] = 0$ to complete the proof of ii). This is clear for $j \leq k$, as for $j > k$

$$[D_{\Theta,j}^\ell, D_{\Theta,k}^r] = -\frac{\Theta_{jk}}{2\pi i}\Big([\partial_\xi^j, M_{\xi_j}] + [M_{\xi_k}, \partial_\xi^k]\Big) = 0.$$

Finally, since $d_\Theta = \pi_\Theta(|\cdot|)$ and π_Θ is a $*$-homomorphism, assertion iii) reduces to show that $\pi_\Theta(x_j) = x_{\Theta,j} - y_{\Theta,j}$ for $1 \leq j \leq n$. This can be proved again with a differentiation argument as follows

$$
\begin{aligned}
\pi_\Theta(x_j) &= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\pi_\Theta\big(e^{2\pi i s\langle\cdot, e_j\rangle}\big)\\
&= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\big(\lambda_\Theta(se_j) \otimes \lambda_\Theta(se_j)^*\big)\\
&= \Big(\frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}u_j(s)\Big) \otimes \mathbf{1} + \mathbf{1} \otimes \Big(\frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}u_j(-s)\Big) = x_{\Theta,j} - y_{\Theta,j},
\end{aligned}
$$

according to our definition of $x_{\Theta,j}$ and $y_{\Theta,j}$ in $\mathcal{R}_\Theta \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{R}_\Theta^{\mathrm{op}}$ respectively. $\quad\square$

Remark 1.10. A few comments are in order:

- $\mathcal{R}_\Theta$ is generated by the spectral projections of the quantum variables $x_{\Theta,j}$.
- The Euclidean Schwartz $\mathcal{S}(\mathbb{R}^n)$ class is the space of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ which satisfy that f and its derivatives decay at infinity faster than polynomials. In $\mathcal{R}_\Theta$, we find

$$
\Big(\overrightarrow{\prod_{1 \le r \le m_\ell}} x_{\Theta,j_r} \Big) \partial_\Theta^\beta \big(\lambda_\Theta(f) \big) \Big(\overrightarrow{\prod_{1 \le s \le m_r}} x_{\Theta,k_s} \Big)
$$
$$
= \ \lambda_\Theta \Big[\Big(\overrightarrow{\prod_{1 \le r \le m_\ell}} D_{\Theta,j_r}^\ell \Big) \Big(\overrightarrow{\prod_{1 \le s \le m_r}} D_{\Theta,k_s}^r \Big) M_{(2\pi i \xi)^\beta} f \Big] \ \in \ \mathcal{S}_\Theta,
$$

  which admits other representations since D_{Θ,j_r}^ℓ and D_{Θ,k_s}^r commute. It shows that the quantum Schwartz class is also closed under differentiation and left/right multiplication by quantum polynomials.

- Proposition 1.9 iii) establishes a canonical Pythagorean formula for the quantum Euclidean distance d_Θ in terms of quantum variables. This shows that the metric d_Θ that we shall be using along the rest of the paper is not *induced* but somehow *intrinsic* to $\mathcal{R}_\Theta$. This gives some evidence that our main results in this paper are formulated in their most natural way.

- We also note in passing that the quantum variables $z_{\Theta,j} = x_{\Theta,j} - y_{\Theta,j}$ from Proposition 1.9 iii) are pairwise commuting for different values of $1 \le j \le n$ since

$$
\begin{aligned}
[z_{\Theta,j}, z_{\Theta,k}] \ &= \ z_{\Theta,j} \bullet z_{\Theta,k} - z_{\Theta,k} \bullet z_{\Theta,j} \\
&= \ [x_{\Theta,j}, x_{\Theta,k}]_{\mathcal{R}_\Theta} \otimes \mathbf{1} + \mathbf{1} \otimes [y_{\Theta,j}, y_{\Theta,k}]_{\mathcal{R}_\Theta^{\mathrm{op}}} \ = \ 0.
\end{aligned}
$$

1.3.2. *On the quantum Schwartz class.* Using quantum variables, we are ready to prove some fundamental properties of the quantum Schwartz class. The analogues in the commutative case $\Theta = 0$ are rather easy to prove. Let us consider the map $j_\Theta : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}_\Theta$ given by

$$
j_\Theta \Big(\int_{\mathbb{R}^n} f(\xi) e^{2\pi i \langle \cdot, \xi \rangle} \, d\xi \Big) = \int_{\mathbb{R}^n} f(\xi) \lambda_\Theta(\xi) \, d\xi,
$$

so that $j_\Theta(f) = \lambda_\Theta(\widehat{f})$. By Remark 1.2 and Plancherel theorem, j_Θ extends to an isometric isomorphism $L_2(\mathbb{R}^n) \to L_2(\mathcal{R}_\Theta)$. We shall also need the space $\mathcal{S}_\Theta'$ of continuous linear functionals on $\mathcal{S}_\Theta$, tempered quantum distributions. Finally recalling that the quantum variables $x_{\Theta,j}$ are affiliated to $\mathcal{R}_\Theta$, we set for $1 \le j \le n$

$$
\mathcal{R}_{\Theta,j} = \Big\langle \text{spectral projections of } x_{\Theta,j} \Big\rangle'' \subset \mathcal{R}_\Theta.
$$

We write $\mathcal{R}_j$ for $\mathcal{R}_{\Theta,j}$ with $\Theta = 0$. We begin with an elementary auxiliary result.

Lemma 1.11. *We have*:

  i) $j_\Theta(x_j^k) = x_{\Theta,j}^k$ *in the sense of distributions.*
  ii) $j_\Theta : \mathcal{R}_j \to \mathcal{R}_{\Theta,j}$ *is a normal $*$-homomorphism.*

Proof. Every element in $\mathcal{S}_\Theta$ may be represented in the form $j_\Theta(f)$ for some f in the Schwartz class of $\mathbb{R}^n$. On the other hand, since $j_\Theta : L_2(\mathbb{R}^n) \to L_2(\mathcal{R}_\Theta)$ is an isometric isomorphism, we define $j_\Theta(x_j^k) \in \mathcal{S}'_\Theta$ by

$$\langle j_\Theta(x_j^k), j_\Theta(f) \rangle = \int_{\mathbb{R}^n} x_j^k f(x)\, dx.$$

Thus, it suffices to see that this quantity coincides with

$$\tau_\Theta\big(x_{\Theta,j}^k j_\Theta(f)\big) = \tau_\Theta\big(x_{\Theta,j}^k \lambda_\Theta(\widehat{f})\big) = \tau_\Theta\Big(\lambda_\Theta\big((D_{\Theta,j}^\ell)^k \widehat{f}\,\big)\Big) = (D_{\Theta,j}^\ell)^k \widehat{f}(0)$$

for $D_{\Theta,j}^\ell = \sum_{s<j} \Theta_{js} M_{\xi_s} - \frac{1}{2\pi i}\partial_j^\xi$, by Proposition 1.9. A simple computation shows that this is indeed the case. Next, assertion ii) follows from the fact that $\mathcal{R}_{\Theta,j} \simeq L_\infty(\mathbb{R})$ for $1 \le j \le n$ no matter which is the deformation Θ. Indeed, in order to use the same terminology as in the proof of Proposition 1.1, we shall assume for convenience that $j = n$. Then, we may identify $\mathcal{R}_{\Theta,n}$ with the subalgebra $\mathbf{1} \rtimes_{\beta_{n-1}} \mathbb{R}$ of the von Neumann algebra $\mathcal{R}_\Xi \rtimes_{\beta_{n-1}} \mathbb{R}$, which in turn is isomorphic to $\mathcal{R}_\Theta$. Then, it is a well-known fact that we have $\mathbf{1} \rtimes_{\beta_{n-1}} \mathbb{R} \simeq L_\infty(\mathbb{R})$. $\qquad\square$

Proposition 1.12. *If $\Theta \in A_n(\mathbb{R})$ and $\gamma > \frac{1}{2}$, we find*

$$\prod_{1 \le j \le n}^{\rightarrow} \big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)^{-1} \in L_2(\mathcal{R}_\Theta).$$

In particular, the quantum Schwartz class $\mathcal{S}_\Theta \subset L_p(\mathcal{R}_\Theta)$ for all $p > 0$.

Proof. According to Lemma 1.11

$$\big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)^{-1} = j_\Theta\Big(\frac{1}{1 + |x_j|^\gamma}\Big).$$

Let us proceed by induction on n, the case $n = 1$ being trivial. According to Proposition 1.1, τ_Θ coincides with the crossed product trace $\tau_\rtimes$ in $\mathcal{R}_\Xi \rtimes \mathbb{R}$ which in turn factorizes for operators with separated variables. This means that

$$\tau_\Theta\Big(\big| \prod_{1 \le j \le n}^{\rightarrow} \big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)^{-1}\big|^2\Big)$$

$$= \tau_\Xi\Big(\big| \prod_{1 \le j \le n-1}^{\rightarrow} \big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)^{-1}\big|^2\Big) \int_{\mathbb{R}} \frac{dx}{(1 + |x|^\gamma)^2}$$

and we conclude by induction. To prove the last assertion, since $\mathcal{S}_\Theta \subset \mathcal{R}_\Theta$ it clearly suffices to show that $\mathcal{S}_\Theta \subset L_p(\mathcal{R}_\Theta)$ for p small. Assume $p = 1/m$ for $m \in \mathbb{Z}_+$ and let

$$Q = \Big| \prod_{1 \le j \le n}^{\rightarrow} \big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)\Big|^{2m}.$$

According to Hölder's inequality, we find for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\big\|\lambda_\Theta(f)\big\|_p \le \big\|Q^{-1}\big\|_p \big\|Q\lambda_\Theta(f)\big\|_\infty$$

$$= \tau_\Theta\Big[\big| \prod_{1 \le j \le n}^{\rightarrow} \big(\mathbf{1} + |x_{\Theta,j}|^\gamma\big)^{-1}\big|^2\Big]^{\frac{1}{p}} \big\|\lambda_\Theta(Q[D_{\Theta,j}^\ell]f)\big\|_\infty$$

where $Q[D_{\Theta,j}^\ell]$ is the differential operator associated to Q according to the second point of Remark 1.10. Since $Q[D_{\Theta,j}^\ell]f \in \mathcal{S}(\mathbb{R}^n)$, the finiteness of the quantity in the right hand side is guaranteed by the first assertion in the statement. $\qquad\square$

Proposition 1.13. *We have*:

  i) $\mathcal{S}_\Theta$ *is weak-$*$ dense in* $\mathcal{R}_\Theta$.
  ii) $\mathcal{S}_\Theta$ *is dense in* $L_p(\mathcal{R}_\Theta)$ *for all* $p > 0$.

In particular, the same density results hold for $\lambda_\Theta(L_1(\mathbb{R}^n)) \subset \mathcal{R}_\Theta$.

Proof. Since finite sums of the elementary frequencies $\lambda_\Theta(\xi)$ are weak-$*$ dense in $\mathcal{R}_\Theta$ by construction, it suffices to approximate $\lambda_\Theta(\xi)$ by elements of $\mathcal{S}_\Theta$ in the weak-$*$ topology. In other words, we need to find a family of functions $\phi_{\xi,\varepsilon} \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\lim_{\varepsilon\to 0} \tau_\Theta\big((\lambda_\Theta(\phi_{\xi,\varepsilon}) - \lambda_\Theta(\xi))a\big) = 0 \quad \text{for all} \quad a \in L_1(\mathcal{R}_\Theta).$$

If $\mathrm{B}_\varepsilon(\xi)$ denotes the Euclidean ball around ξ of radius ε, let $\phi_{\xi,\varepsilon}$ be a smoothing of the function $|\mathrm{B}_\varepsilon(\xi)|^{-1}\chi_{\mathrm{B}_\varepsilon(\xi)}$, so that Lebesgue differentiation theorem holds for the family $\{\phi_{\xi,\varepsilon} : \varepsilon > 0\}$. Now, since $\mathcal{S}_\Theta$ is dense in $L_2(\mathcal{R}_\Theta)$, the same holds for $\mathcal{S}_\Theta\mathcal{S}_\Theta \subset L_2(\mathcal{R}_\Theta)L_2(\mathcal{R}_\Theta) = L_1(\mathcal{R}_\Theta)$ and we may approximate a by a sequence $\lambda_\Theta(f_j) \in \mathcal{S}_\Theta$. Recall that

$$\big\|\lambda_\Theta(\phi_{\xi,\varepsilon}) - \lambda_\Theta(\xi)\big\|_{\mathcal{R}_\Theta} \leq 1 + \int_{\mathbb{R}^n} \phi_{\xi,\varepsilon}(\zeta)d\zeta = 2.$$

Given $\delta > 0$, there exists $j_\delta \geq 1$ so that $\|a - \lambda_\Theta(f_{j_\delta})\|_1 < \delta/2$. Thus

$$\left| \lim_{\varepsilon\to 0} \tau_\Theta\big((\lambda_\Theta(\phi_{\xi,\varepsilon}) - \lambda_\Theta(\xi))a\big)\right| \leq \delta + \left| \lim_{\varepsilon\to 0} \tau_\Theta\big((\lambda_\Theta(\phi_{\xi,\varepsilon}) - \lambda_\Theta(\xi))\lambda_\Theta(f_{j_\delta})\big)\right|.$$

On the other hand, since $\lambda_\Theta(\xi)^* = e^{2\pi i\langle\xi,\Theta_\downarrow\xi\rangle}\lambda_\Theta(-\xi)$ we find

$$\tau_\Theta\big((\lambda_\Theta(\phi_{\xi,\varepsilon}) - \lambda_\Theta(\xi))\lambda_\Theta(f_{j_\delta})\big)$$

$$= \phi_{\xi,\varepsilon} *_\Theta f_{j_\delta}(0) - e^{-2\pi i\langle\xi,\Theta_\downarrow\xi\rangle}f_{j_\delta}(-\xi)$$

$$= \int_{\mathbb{R}^n} \phi_{\xi,\varepsilon}(\zeta)e^{-2\pi i\langle\zeta,\Theta_\downarrow\zeta\rangle}f_{j_\delta}(-\zeta)d\zeta - e^{-2\pi i\langle\xi,\Theta_\downarrow\xi\rangle}f_{j_\delta}(-\xi),$$

where $\Theta_\downarrow$ is the lower triangular part of Θ. The expression above converges to 0 as $\varepsilon \to 0$. Letting $\delta \to 0$ we conclude that $\mathcal{S}_\Theta$ is weak-$*$ dense in $\mathcal{R}_\Theta$. Let us now prove that $\mathcal{S}_\Theta$ is norm dense in $L_p(\mathcal{R}_\Theta)$ for all $p > 0$. Since $\mathcal{S}_\Theta\mathcal{S}_\Theta \subset \mathcal{S}_\Theta$, it suffices from Hölder inequality to prove norm density in the case $p > 2$. Given $a \in L_p(\mathcal{R}_\Theta)$ for some $p > 2$, we may approximate it in the L_p-norm by another element in $\mathcal{R}_\Theta$ which is left/right supported by a finite projection. In other words, we may assume that a itself belongs to $\mathcal{R}_\Theta$ and $a = qaq$ for some projection q satisfying $\tau_\Theta(q) < \infty$. Pick two sequences $f_j, g_k \in \mathcal{S}(\mathbb{R}^n)$ satisfying that

$$\mathrm{w}^*\text{-}\lim_{j\to\infty} \lambda_\Theta(f_j) = a \quad \text{and} \quad \mathrm{w}^*\text{-}\lim_{k\to\infty} \lambda_\Theta(g_k) = q.$$

By Kaplanski density theorem, we may also assume that

$$\sup_{j,k\geq 1} \Big(\|\lambda_\Theta(f_j)\|_{\mathcal{R}_\Theta} + \|\lambda_\Theta(g_k)\|_{\mathcal{R}_\Theta}\Big) \leq 1 + \|a\|_{\mathcal{R}_\Theta} < \infty$$

and both convergences hold strongly. Therefore, since $a \in L_2(\mathcal{R}_\Theta)$, given $\delta > 0$ there must exists k_δ satisfying $\|a(q - \lambda_\Theta(g_{k_\delta}))\|_2 < \frac{\delta}{2}$. Moreover, once the index

k_δ is fixed and since $\lambda_\Theta(g_{k_\delta}) \in L_2(\mathcal{R}_\Theta)$ there must exists an index j_δ satisfying the inequality $\|(a - \lambda_\Theta(f_{j_\delta}))\lambda_\Theta(g_{k_\delta})\|_2 < \frac{\delta}{2}$. Combining these estimates

$$\left\|a - \lambda_\Theta(f_{j_\delta})\lambda_\Theta(g_{k_\delta})\right\|_2 \leq \left\|a(q - \lambda_\Theta(g_{k_\delta}))\right\|_2 + \left\|(a - \lambda_\Theta(f_{j_\delta}))\lambda_\Theta(g_{k_\delta})\right\|_2 < \delta.$$

On the other hand, by the three lines lemma

$$
\begin{aligned}
\left\|a - \lambda_\Theta(f_{j_\delta})\lambda_\Theta(g_{k_\delta})\right\|_p &\leq \left\|a - \lambda_\Theta(f_{j_\delta})\lambda_\Theta(g_{k_\delta})\right\|_\infty^{1-\frac{2}{p}} \left\|a - \lambda_\Theta(f_{j_\delta})\lambda_\Theta(g_{k_\delta})\right\|_2^{\frac{2}{p}} \\
&\leq \delta^{\frac{2}{p}} \left\|a - \lambda_\Theta(f_{j_\delta})\lambda_\Theta(g_{k_\delta})\right\|_\infty^{1-\frac{2}{p}} \leq \left(2\|a\|_{\mathcal{R}_\Theta}\right)^{1-\frac{2}{p}} \delta^{\frac{2}{p}}.
\end{aligned}
$$

Taking $\delta \to 0$ we see that $\mathcal{S}_\Theta$ is norm dense in $L_p(\mathcal{R}_\Theta)$ for $p > 2$. □

1.3.3. *Structure of* $\mathcal{R}_\Theta$. We start by showing the very simple algebraic structure of quantum Euclidean spaces $\mathcal{R}_\Theta$. Indeed, given $\Theta \in A_n(\mathbb{R})$ and according to the spectral theorem, there exist $d_1, d_2 \in \mathbb{Z}_+$ with $d_1 + 2d_2 = n$ and $\kappa_1, \kappa_2, \ldots \kappa_{d_2} \in \mathbb{R} \setminus \{0\}$ satisfying the following relation for some orthogonal matrix $B \in \mathrm{SO}(n)$ and for Φ the $d_1 \times d_1$ 0-matrix

$$\Theta = B \left[\Phi \oplus \underbrace{\bigoplus_{j=1}^{d_2} \kappa_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\Delta} \right] B^* = B \, \Delta \, B^*.$$

Arguing as in Proposition 1.1 iii) for $n = 2$, we find that

$$
\begin{aligned}
\mathcal{R}_\Delta &\simeq L_\infty(\mathbb{R}^{d_1}) \bar\otimes \left(\bigotimes_{j=1}^{d_2} \mathcal{B}(L_2(\mathbb{R})) \right) \simeq L_\infty\left(\mathbb{R}^{d_1}; \mathcal{B}(L_2(\mathbb{R}^{d_2}))\right) \\
&\simeq L_\infty(\mathbb{R}^{d_1}) \bar\otimes \left(\bigotimes_{j=1}^{d_2} L_\infty(\mathbb{R}) \rtimes \mathbb{R} \right) \simeq L_\infty\left(\mathbb{R}^{d_1}; L_\infty(\mathbb{R}^{d_2}) \rtimes \mathbb{R}^{d_2}\right)
\end{aligned}
$$

is a type I von Neumann algebra. Since the commutation relations are determined by $\lambda_\Theta(\xi)\lambda_\Theta(\eta) = \exp(2\pi i\langle \xi, \Theta\eta \rangle)\lambda_\Theta(\eta)\lambda_\Theta(\xi)$ it is tempting to set $\lambda_\Delta(\xi) = \lambda_\Theta(B\xi)$ to conclude that $\mathcal{R}_\Theta \simeq \mathcal{R}_\Delta$ is also a type I von Neumann algebra. This choice of unitaries do not arise however from a family of one-parameter groups of unitaries as expected. The right change of variables is $x_\Theta \mapsto Bx_\Theta$, where x_Θ stands for $(x_{\Theta,1}, x_{\Theta,2}, ..., x_{\Theta,n})$, at the level of infinitesimal generators. If we want to take exponentials to generate one-parameter groups of unitaries $s \mapsto \exp(2\pi i s(Bx_\Theta)_j)$ new extra terms appear due to nonvanishing commutators.

Proposition 1.14. *The unitaries*

$$\lambda_\Delta(\xi) = \exp\left(\pi i \sum_{j<k} \left(\xi_j \xi_k \Delta_{jk} - (B\xi)_j (B\xi)_k \Theta_{jk} \right) \right) \lambda_\Theta(B\xi)$$

generate $\mathcal{R}_\Delta$. *In particular,* $\mathcal{R}_\Theta \simeq \mathcal{R}_\Delta$ *so that quantum Euclidean spaces* $\mathcal{R}_\Theta$ *are always type* I *von Neumann algebras which are invariant under conjugation by* $\mathrm{SO}(n)$. *Moreover, the traces coincide* $\tau_\Theta = \tau_\Delta$ *and the one-parameter unitary groups* $w_j(s) = \exp(2\pi i s x_{\Delta,j}) = \lambda_\Delta(se_j)$ *have the form*

$$w_j(s) = \exp\left(-\pi i s^2 \sum_{\alpha<\beta} B_{j\alpha}^* \Theta_{\alpha\beta} B_{\beta j} \right) \lambda_\Theta(sBe_j).$$

Proof. Consider the self-adjoint operators

$$x_{\Delta,j} = \sum_{k=1}^{n} B_{jk}^{*}\, x_{\Theta,k} = \sum_{k=1}^{n} B_{kj} x_{\Theta,k}.$$

It follows from Proposition 1.9 that the quantum Schwartz class $\mathcal{S}_{\Theta}$ is a common core for the family $x_{\Delta,j}$ with $1 \leq j \leq n$. In particular, these operators are densely defined in the Hilbert space $L_2(\mathcal{R}_{\Theta}) \simeq L_2(\mathbb{R}^n) \simeq L_2(\mathcal{R}_{\Delta})$. On the other hand, the commutators are

$$[x_{\Delta,j}, x_{\Delta,k}] = \sum_{1 \leq \alpha,\beta \leq n} B_{j\alpha}^{*}[x_{\Theta,\alpha}, x_{\Theta,\beta}]B_{\beta k} = \frac{1}{2\pi i} \sum_{1 \leq \alpha,\beta \leq n} B_{j\alpha}^{*}\Theta_{\alpha\beta}B_{\beta k} = \frac{1}{2\pi i}\Delta_{jk}.$$

Therefore, Proposition 1.9 implies that $\mathcal{R}_{\Delta}$ is the weak-$*$ closure of the C*-algebra generated by the one-parameter unitary groups $w_j(s) = \exp(2\pi i s x_{\Delta,j})$ for $j \leq n$ or equivalently by the products

$$w_1(\xi_1)w_2(\xi_2)\cdots w_n(\xi_n) = \prod_{1 \leq j \leq n}^{\rightarrow} \exp(2\pi i \xi_j x_{\Delta,j}).$$

Consequently, if we can justify the equality

$$\prod_{1 \leq j \leq n}^{\rightarrow} \exp(2\pi i \xi_j x_{\Delta,j}) = \exp\Big(\pi i \sum_{j<k} \big(\xi_j \xi_k \Delta_{jk} - (B\xi)_j (B\xi)_k \Theta_{jk}\big)\Big)\lambda_{\Theta}(B\xi)$$

it will follow automatically that $\mathcal{R}_{\Theta} \simeq \mathcal{R}_{\Delta}$ as expected. The identity $\tau_{\Theta} = \tau_{\Delta}$ and the expression given for $w_j(s)$ also follow easily from the above equality. This is proved from the Baker-Campbell-Hausdorff formula. Namely, since we know that $[x_{\Delta,j}, x_{\Delta,k}] = \frac{1}{2\pi i}\Delta_{jk}$ we may use the simple identity below for operators X, Y with vanishing iterated brackets

$$\log\big(\exp X \exp Y\big) = X + Y + \frac{1}{2}[X, Y].$$

Taking $X_j = 2\pi i \xi_j x_{\Delta,j}$ we have $[X_j, X_k] = 2\pi i \xi_j \xi_k \Delta_{jk}$, so that

$$
\begin{aligned}
\prod_{1 \leq j \leq n}^{\rightarrow} \exp(2\pi i \xi_j x_{\Delta,j}) &= \prod_{1 \leq j \leq n}^{\rightarrow} \exp X_j \\
&= \exp\Big(\frac{1}{2}\sum_{j<k}[X_j, X_k]\Big)\exp\Big(\sum_{j=1}^{n} X_j\Big) \\
&= \exp\Big(\pi i \sum_{j<k}\xi_j \xi_k \Delta_{jk}\Big)\exp\Big(2\pi i \sum_{j=1}^{n}\xi_j x_{\Delta,j}\Big) \\
&= \exp\Big(\pi i \sum_{j<k}\xi_j \xi_k \Delta_{jk}\Big)\exp\Big(2\pi i \sum_{k=1}^{n}(B\xi)_k x_{\Theta,k}\Big).
\end{aligned}
$$

Using the same formula for the family $Z_j = 2\pi i (B\xi)_j x_{\Theta,j}$ we may conclude. $\qquad\square$

1.3.4. *A Θ-deformation of ∂_ξ.* We finish this section with another local operator acting on a given symbol $a : \mathbb{R}^n \to \mathcal{R}_{\Theta}$. It plays a crucial role in the Hörmander classes $\Sigma_{\rho,\delta}^{m}(\mathcal{R}_{\Theta})$ from the Introduction. The *mixed classical-quantized derivative* is given by

$$\partial_{\Theta,\xi}^{j} a(\xi) = \lambda_{\Theta}(\xi)^{*}\partial_{\xi}^{j}\big\{\lambda_{\Theta}(\xi)a(\xi)\lambda_{\Theta}(\xi)^{*}\big\}\lambda_{\Theta}(\xi).$$

Lemma 1.15. *We have*

$$\partial^j_{\Theta,\xi} a(\xi) = \partial^j_\xi a(\xi) + 2\pi i \big[x_{\Theta,j}, a(\xi) \big].$$

Proof. Note that

$$\frac{d}{ds}\Big|_{s=0} \lambda_\Theta(\xi + se_j) \; = \; \lambda_\Theta(\xi)\Big(\frac{d}{ds}\Big|_{s=0} e^{-2\pi i s \langle \xi, \Theta_\downarrow e_j \rangle} \lambda_\Theta(se_j) \Big) ,$$

$$\frac{d}{ds}\Big|_{s=0} \lambda_\Theta(\xi + se_j)^* \; = \; \Big(\frac{d}{ds}\Big|_{s=0} e^{2\pi i s \langle \xi, \Theta_\downarrow e_j \rangle} \lambda_\Theta(-se_j) \Big) \lambda_\Theta(\xi)^*.$$

A simple calculation then yields that

$$\partial^j_{\Theta,\xi} a(\xi) \; = \; \lambda_\Theta(\xi)^* \Big(\frac{d}{ds}\Big|_{s=0} \lambda_\Theta(\xi + se_j) a(\xi + se_j) \lambda_\Theta(\xi + se_j)^* \Big) \lambda_\Theta(\xi)$$

$$= \; \big(2\pi i x_{\Theta,j} - 2\pi i \langle \xi, \Theta_\downarrow e_j \rangle \big) a(\xi) + \partial^j_\xi a(\xi) + a(\xi) \big(2\pi i \langle \xi, \Theta_\downarrow e_j \rangle - 2\pi i x_{\Theta,j} \big).$$

Eliminating vanishing terms and rearranging gives the desired identity.   □

The commutator vanishes in the Euclidean setting $\Theta = 0$. Therefore, we should understand $\partial_{\Theta,\xi}$ as a Θ-deformation of the classical derivative ∂_ξ which —as it is indicated by the result below— is also very much related to the quantum derivatives ∂_Θ. Thus, we get a Θ-deformation of ∂_ξ by ∂_Θ.

Lemma 1.16. *Given* $\varphi \in \mathcal{S}_\Theta \subset \mathcal{R}_\Theta$ *we have*

$$\big[x_{\Theta,j}, \varphi \big] = \frac{1}{2\pi i} \sum_{k=1}^n \Theta_{jk} \partial^k_\Theta \varphi.$$

In particular, we obtain the following estimate

$$\big\| \big[x_{\Theta,j}, \varphi \big] \big\|_{\mathcal{R}_\Theta} \leq \frac{1}{2\pi} \Big(\sum_{k=1}^n |\Theta_{jk}|^2 \Big)^{\frac{1}{2}} \Big\| \Big(\sum_{k=1}^n |\partial^k_\Theta \varphi|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{R}_\Theta}.$$

Proof. Observe that

$$\big[x_{\Theta,j}, \varphi \big] = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) \big[x_{\Theta,j}, \lambda_\Theta(\xi) \big] d\xi = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) \frac{1}{2\pi i} \frac{d}{ds}\Big|_{s=0} \big[\lambda_\Theta(se_j), \lambda_\Theta(\xi) \big] d\xi.$$

Using $[\lambda_\Theta(se_j), \lambda_\Theta(\xi)] = \lambda_\Theta(se_j)\lambda_\Theta(\xi)(1 - e^{2\pi i s \langle \xi, \Theta e_j \rangle})$ and applying the Leibniz rule, we easily deduce the first assertion. The second one is straightforward.   □

The last estimate above provides a uniform and linear bound that explicitly gives the convergence $\partial^j_{\Theta,\xi} \to \partial^j_\xi$ in the point-operator norm when $\Theta \to 0^+$. We also recall that the assumption $\varphi \in \mathcal{S}_\Theta$ is just needed a priori and can be extended to any φ in the weak-$*$ closed domain of ∇_Θ.

2. Calderón-Zygmund L_p theory

We now introduce kernel representations of linear operators acting on a von Neumann algebra and develop a very satisfactory Calderón-Zygmund theory over quantum Euclidean spaces. We shall prove L_p-boundedness from L_2-boundedness and Calderón-Zygmund conditions for the kernel. L_2-boundedness will be analyzed below in this paper for pseudodifferential operators. Our kernel conditions are given in terms of the intrinsic metric and gradient introduced above and resemble very

neatly the classical conditions. This is the first form of Calderón-Zygmund theory over a fully noncommutative von Neumann algebra. We refer to [42, 56] for related results over tensor product and crossed product algebras containing an abelian factor. An algebraic/probabilistic approach —lacking the geometric aspects of the present one— will be presented in [44] for more general von Neumann algebras.

2.1. **Kernels and symbols.** Given a measure space (Ω, μ) and a linear map T acting on certain function space X over Ω, a kernel representation has the following form for functions f living in some dense domain in X

$$T_k f(x) = \int_\Omega k(x, y) f(y) \, d\mu(y),$$

where the kernel $k : \Omega \times \Omega \to \mathbb{C}$ is only assumed a priori to be defined almost everywhere and measurable. Now, given a noncommutative measure space $(\mathcal{M}, \tau)$ composed by a semifinite von Neumann algebra $\mathcal{M}$ and normal faithful semifinite trace τ, the kernel representation takes the analogous form

$$T_k \varphi = \big(id \otimes \tau\big)\big(k(\mathbf{1} \otimes \varphi)\big) = \big(id \otimes \tau\big)\big((\mathbf{1} \otimes \varphi)k\big)$$

with the only difference that k is now an operator affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$, instead of $\mathcal{M} \bar{\otimes} \mathcal{M}$ as one could have expected. This novelty —undistinguishable in the abelian case, where $\mathcal{M} = \mathcal{M}_{\mathrm{op}}$— is crucial to develop a consistent theory. Let us begin by showing the fundamental properties of these kernel representations. This will simplify the task of justifying our choice of $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$. Recall the products $\cdot$ and $\bullet$ in $\mathcal{M}_{\mathrm{op}}$ and $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$ respectively from Section 1.2.1 above.

Remark 2.1. Rigorously speaking, the map T_k so defined should send operators φ affiliated to $\mathcal{M}_{\mathrm{op}}$ to another operator $T_k \varphi$ affiliated to $\mathcal{M}$. Of course, this is not an obstruction since the set of affiliated operators coincides for $\mathcal{M}$ and $\mathcal{M}_{\mathrm{op}}$ and $\tau_{\mathcal{M}} = \tau_{\mathcal{M}_{\mathrm{op}}}$. We will regard φ as affiliated to $\mathcal{M}$, so that T_k becomes a linear map over the noncommutative measure space $(\mathcal{M}, \tau)$.

Lemma 2.2. *The following properties hold*:

    i) *Adjoints and composition*

$$T_k^* = T_{\mathrm{flip}(k)^*} \quad with \quad \mathrm{flip}(a \otimes b) = b \otimes a,$$

$$T_{k_1} \circ T_{k_2} = T_k \quad with \quad k = \big(id \otimes \tau \otimes id\big)\Big(\big(k_1 \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes k_2\big)\Big).$$

    ii) *Schur lemma and factorization*

$$T_{k_{\mathrm{A}} \bullet k_{\mathrm{B}}} \varphi = \big(id \otimes \tau\big)\big(k_{\mathrm{A}}\big(\mathbf{1} \otimes \varphi\big)k_{\mathrm{B}}\big),$$

$$\big\| T_{k_{\mathrm{A}} \bullet k_{\mathrm{B}}} : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \big\| \leq \big\|(id \otimes \tau)(k_{\mathrm{A}} k_{\mathrm{A}}^*)\big\|_{\mathcal{M}}^{\frac{1}{2}} \big\|(\tau \otimes id)(k_{\mathrm{B}} k_{\mathrm{B}}^*)\big\|_{\mathcal{M}}^{\frac{1}{2}},$$

$$\big\| T_{k_{\mathrm{A}} \bullet k_{\mathrm{B}}} : L_\infty(\mathcal{M}) \to L_\infty(\mathcal{M}) \big\| \leq \big\|(id \otimes \tau)(k_{\mathrm{A}} k_{\mathrm{A}}^*)\big\|_{\mathcal{M}}^{\frac{1}{2}} \big\|(id \otimes \tau)(k_{\mathrm{B}}^* k_{\mathrm{B}})\big\|_{\mathcal{M}}^{\frac{1}{2}}.$$

    iii) T_k *is completely positive if and only if k is positive as affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$.*

Proof. We have

$$\begin{aligned}
\big\langle T_k \varphi_1, \varphi_2 \big\rangle &= \tau\Big(\big(id \otimes \tau\big)\big(k(\mathbf{1} \otimes \varphi_1)\big)^* \varphi_2\Big) \\
&= \tau\Big((\tau \otimes id)\big((\mathbf{1} \otimes \varphi_1^*)k^*(\varphi_2 \otimes \mathbf{1})\big)\Big)
\end{aligned}$$

$$= \tau\Big(\varphi_1^*(id \otimes \tau)\big(\mathrm{flip}(k)^*(\mathbf{1} \otimes \varphi_2)\big)\Big) = \langle \varphi_1, T_k^* \varphi_2 \rangle,$$

which proves the kernel formula for the adjoint. Regarding the composition

$$
\begin{aligned}
T_{k_1}(T_{k_2}\varphi) &= (id \otimes \tau)\Big(k_1\big(\mathbf{1} \otimes (id \otimes \tau)(k_2(\mathbf{1} \otimes \varphi))\big)\Big) \\
&= (id \otimes \tau \otimes \tau)\Big(\big(k_1 \otimes \mathbf{1})(\mathbf{1} \otimes k_2)(\mathbf{1} \otimes \mathbf{1} \otimes \varphi)\big)\Big) \\
&= (id \otimes \tau)\Big(\big[(id \otimes \tau \otimes id)(k_1 \otimes \mathbf{1})(\mathbf{1} \otimes k_2)\big](\mathbf{1} \otimes \varphi)\Big).
\end{aligned}
$$

In both cases —adjoints and compositions— we have regarded one more time the involved operators as affiliated to $\mathcal{M}$ or $\mathcal{M}_{\mathrm{op}}$ according to the context, as we explain in Remark 2.1. Next, the factorization identity in ii) uses in a fundamental way the product $\bullet$ in $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$

$$T_{k_A \bullet k_B}\varphi = (id \otimes \tau)\big(k_A \bullet k_B(\mathbf{1} \otimes \varphi)\big) = (id \otimes \tau)\big(k_A(\mathbf{1} \otimes \varphi)k_B\big).$$

Namely, in the last identity above the first coordinate remains unchanged since $\mathbf{1} \otimes \varphi$ does not affect the product in $\mathcal{M} \otimes \mathbf{1}$, whereas the second coordinate in $\mathbf{1} \otimes \mathcal{M}_{\mathrm{op}}$ is explained using its product $\cdot$ as follows

$$\tau(\alpha \cdot \beta \cdot \varphi) = \tau(\alpha \cdot \varphi\beta) = \tau(\alpha\varphi\beta).$$

Let us now prove the announced inequalities. By the Cauchy-Schwarz inequality for the operator-valued inner product $(x, y) = (id \otimes \tau)(x^*y)$ over the von Neumann algebra $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$ we note that

$$\big|(id \otimes \tau)(x^*y)\big|^2 \leq \big\|(id \otimes \tau)(x^*x)\big\|_{\mathcal{M}}(id \otimes \tau)(y^*y),$$

see for instance [49, Proposition 1.1]. In particular

$$
\begin{aligned}
\big|T_{k_A \bullet k_B}\varphi\big|^2 &= \Big|(id \otimes \tau)\big(k_A(\mathbf{1} \otimes \varphi)k_B\big)\Big|^2 \\
&\leq \big\|(id \otimes \tau)(k_A k_A^*)\big\|_{\mathcal{M}}(id \otimes \tau)\Big(k_B^*(\mathbf{1} \otimes \varphi^*\varphi)k_B\Big).
\end{aligned}
$$

The L_∞-estimate announced for $T_{k_A \bullet k_B}\varphi$ follows immediately from the inequality above. In order to prove the L_2-estimate, pick a unit vector $\varphi \in L_2(\mathcal{M})$. We just need to take the trace and apply Fubini

$$
\begin{aligned}
\tau\big(|T_{k_A \bullet k_B}\varphi|^2\big) &\leq \big\|(id \otimes \tau)(k_A k_A^*)\big\|_{\mathcal{M}}(\tau \otimes \tau)\Big(k_B^* \bullet k_B(\mathbf{1} \otimes \varphi^*\varphi)\Big) \\
&= \big\|(id \otimes \tau)(k_A k_A^*)\big\|_{\mathcal{M}}\tau\Big((\tau \otimes id)(k_B^* \bullet k_B)\varphi^*\varphi\Big) \\
&= \big\|(id \otimes \tau)(k_A k_A^*)\big\|_{\mathcal{M}}\tau\Big((\tau \otimes id)(k_B k_B^*)\varphi^*\varphi\Big) \\
&\leq \big\|(id \otimes \tau)(k_A k_A^*)\big\|_{\mathcal{M}}\big\|(\tau \otimes id)(k_B k_B^*)\big\|_{\mathcal{M}}.
\end{aligned}
$$

It remains to prove the last assertion iii). As an operator affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$, the kernel k is positive iff there exists κ also affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$ so that $k = \kappa^* \bullet \kappa$ and the factorization identity above gives in that case

$$T_k f = (id \otimes \tau)\big(\kappa^*(\mathbf{1} \otimes \varphi)\kappa\big)$$

which is clearly a completely positive map. Reciprocally, let T_k be completely positive. Assume for simplicity that T_k is well-defined over projections in $\mathcal{M}$.

Then, given any pair of projections $p, q \in \mathcal{M}$ we know from positivity of T_k that $\tau(T_k(q)p) \geq 0$. However

$$
\begin{aligned}
\tau(T_k(q)p) &= \tau\Big((id \otimes \tau)\big(k(\mathbf{1} \otimes q)\big)p\Big) \\
&= \tau\Big((id \otimes \tau)\big(k \bullet (\mathbf{1} \otimes q)\big)p\Big) \\
&= \tau\Big((id \otimes \tau)\big(k \bullet (p \otimes q)\big)\Big) = \tau \otimes \tau\big(k \bullet (p \otimes q)\big).
\end{aligned}
$$

The positivity of the last term for arbitrary projections implies the assertion. $\square$

Remark 2.3. Lemma 2.2 i) also holds for kernels affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}$, contrary to points ii) and iii). On the other hand, recall that the norm of a completely positive map is determined by its value at $\mathbf{1}$. The L_∞-estimate in Lemma 2.2 ii) rephrases it in terms of kernels when $k \geq 0$ and $k_{\mathrm{A}} = k_{\mathrm{B}} = \sqrt{k}$, so this estimate provides a generalization for nonpositive maps. Also, the L_2-estimate generalizes a classical result for integral kernels known as Schur lemma [66, Lemma in page 284]. Finally, the use of kernels k affiliated to $\mathcal{M} \bar{\otimes} \mathcal{M}_{\mathrm{op}}$ —essential for the properties in Lemma 2.2 ii) and iii) above— is consistent with the duality

$$
L_1(\mathcal{M})^* = \mathcal{M}_{\mathrm{op}}
$$

via the pairing $\langle x, y \rangle = \tau(xy)$, we refer to Pisier's book [60] for further details.

Remark 2.4. Ignoring for the moment more general kernels which will arise as tempered distributions, let us assume that k is affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ and admits an expression

$$
k = \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{k}(\xi, \eta) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)^* \, d\mu(\xi, \eta)
$$

for some measure μ on $\mathbb{R}^n \times \mathbb{R}^n$. Noticing that $\tau_\Theta(\lambda_\Theta(f)\lambda_\Theta(\xi)^*) = f(\xi)$ for f smooth we may interpret the kernel k as a bilinear form where —regardless $\lambda_\Theta(\xi), \lambda_\Theta(\eta)$ are not in $L_2(\mathcal{R}_\Theta)$— we put $\lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)^* \sim |\lambda_\Theta(\xi)\rangle\langle\lambda_\Theta(\eta)|$ following the bra-ket notation. This is easily checked for Dirac measures $\mu = \delta_{\xi_0, \eta_0}$

$$
\big\langle T_k(\lambda_\Theta(f_1)), \lambda_\Theta(f_2) \big\rangle = \tau_\Theta\big(T_k(\lambda_\Theta(f_1))^* \lambda_\Theta(f_2)\big) = \widehat{k}(\xi_0, \eta_0)\overline{f_1(\eta_0)}f_2(\xi_0).
$$

We will approximate general measures as limits of finite sums of Dirac measures.

This paper is devoted to investigate singular integral operators over quantum Euclidean spaces, both in terms of Calderón-Zygmund conditions for the kernel and Hörmander smoothness for the associated symbol. Let us therefore briefly describe the kernels and symbols we will be working with. There exists a very well-known relation between kernels and symbols of classical pseudodifferential operators, the reader can look for instance in [66, 69] or almost any textbook on pseudodifferential operators. Indeed, given

$$
\Psi_a f(x) = \int_{\mathbb{R}^n} a(x, \xi)\widehat{f}(\xi)e^{2\pi i \langle x, \xi \rangle} \, d\xi,
$$

it turns out that $\Psi_a f = T_k f$ for

$$
k = (id \otimes \mathcal{F}^{-1})(a)(x, x - y) = \int_{\mathbb{R}^n} a(x, \xi)e^{2\pi i \langle x - y, \xi \rangle} \, d\xi.
$$

Given $n \geq 2$, let us know consider a $n \times n$ deformation Θ. As explained in the Introduction, noncommutative symbols over quantum Euclidean spaces are smooth functions $a : \mathbb{R}^n \to \mathcal{S}_\Theta$ and pseudodifferential operators look like

$$
\begin{aligned}
\Psi_a(\lambda_\Theta(f)) &= \int_{\mathbb{R}^n} a(\xi) f(\xi) \lambda_\Theta(\xi) \, d\xi \\
&= \int_{\mathbb{R}^n} a(\xi) \tau_\Theta \big(\lambda_\Theta(f) \lambda_\Theta(\xi)^* \big) \lambda_\Theta(\xi) \, d\xi \\
&= \big(id \otimes \tau_\Theta \big) \Big[\Big(\underbrace{\int_{\mathbb{R}^n} (a(\xi) \otimes \mathbf{1})(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*) d\xi}_{\text{The kernel } k} \Big) \big(\mathbf{1} \otimes \lambda_\Theta(f) \big) \Big].
\end{aligned}
$$

Thus, we find formally that $\Psi_a(\lambda_\Theta(f)) = T_k(\lambda_\Theta(f))$ for

$$
k = \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{k}(\xi, \eta) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)^* \, d\mu(\xi, \eta) = \int_{\mathbb{R}^n} a(\xi) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* d\xi.
$$

Reciprocally, we also have the following expression for a

$$
a(\xi) = \int_{\mathbb{R}^n} \widehat{k}(\xi, \eta) \big(\mathbf{1} \otimes \lambda_\Theta(\eta)^* \lambda_\Theta(\xi) \big) d\eta
$$

when μ is the Lebesgue measure in $\mathbb{R}^n \times \mathbb{R}^n$. The algebra of pseudodifferential operators is formally generated by the derivatives ∂_Θ^j and the left multiplication maps $\lambda_\Theta(f) \mapsto x_{\Theta,j} \lambda_\Theta(f)$.

Remark 2.5. Given $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ define

$$
\Psi_a^{\mathrm{right}}(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} f(\xi) \lambda_\Theta(\xi) a(\xi) \, d\xi.
$$

Both theories for right and left pseudodifferential operators are analogous. In fact, if we denote by S the bounded operator in $L_p(\mathcal{R}_\Theta)$ given by extension of $S(x) = x^*$ we have that

$$
S \Psi_a^{\mathrm{left}} = \Psi_b^{\mathrm{right}} S \quad \text{where} \quad b(\xi) = a(-\xi)^*.
$$

The proof of such intertwining identity amounts to a straightforward calculation.

2.2. **CZ extrapolation: Model case.** We are ready to prove L_p-boundedness of operators associated to elementary kernels satisfying cancellation and smoothness conditions of Calderón-Zygmund type. Our kernels will belong along this paragraph to $\mathcal{S}_\Theta \otimes_{\mathrm{alg}} \mathcal{S}_\Theta^{\mathrm{op}}$, so that

$$
k = \sum_j \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \kappa_{1j}(\xi) \kappa_{2j}(\eta) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)^* d\xi d\eta,
$$

where the sum above is finite and $\kappa_{ij} \in \mathcal{S}(\mathbb{R}^n)$. We will temporarily refer to these kernels as *algebraic kernels*. Of course, in this case T_k is L_p-bounded for $1 \leq p \leq \infty$ with constants a priori depending on the family κ_{ij}. Our goal is to provide $L_\infty \to \mathrm{BMO}$ estimates with constants which only depend on structural properties of the whose k since this will allow us to include general singular kernels below. The following result is the basic core of this paper. We shall use the quantum metric d_Θ defined in Section 1.2.1, the notation $\nabla_\Theta k$ to denote the operator

$$
\big(\nabla_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}} \big)(k) \in \mathcal{L}(\mathbb{F}_n) \bar\otimes \mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}
$$

for $k \in \mathcal{S}_\Theta \otimes_{\mathrm{alg}} \mathcal{S}_\Theta^{\mathrm{op}}$ and the dimensional constant $\mathrm{K}_n = \frac{1}{2}(n+1)$.

Theorem 2.6. *Let* $k \in \mathcal{S}_\Theta \otimes_{\mathrm{alg}} \mathcal{S}_\Theta^{\mathrm{op}}$ *and assume*:

i) Cancellation
$$\big\| T_k : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \big\| \leq \mathrm{A}_1.$$

ii) Kernel smoothness. *There exists*
$$\alpha < \mathrm{K}_n - \frac{1}{2} < \beta < \mathrm{K}_n + \frac{1}{2}$$

satisfying the gradient conditions below for $\rho = \alpha, \beta$
$$\Big| \mathrm{d}_\Theta^\rho \bullet (\nabla_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}})(k) \bullet \mathrm{d}_\Theta^{n+1-\rho} \Big| \leq \mathrm{A}_2.$$

Then, we find the following $L_\infty \to \mathrm{BMO}_c$ *estimate*
$$\big\| T_k : L_\infty(\mathcal{R}_\Theta) \to \mathrm{BMO}_c(\mathcal{R}_\Theta) \big\|_{\mathrm{cb}} \leq C_n(\alpha, \beta)\big(\mathrm{A}_1 + \mathrm{A}_2 \big).$$

Proof. Since k is an algebraic kernel, T_k is bounded on $\mathcal{R}_\Theta$. Moreover, T_k^* is also bounded on $L_1(\mathcal{R}_\Theta^{\mathrm{op}})$ and T_k is a normal operator. On the other hand, the weak-$*$ topology in $\mathcal{R}_\Theta$ is stronger than the inherited one from the weak-$*$ topology in $\mathrm{BMO}_c(\mathcal{R}_\Theta)$. Therefore, by Kaplansky density theorem, it suffices to estimate the norm of T_k on the weak-$*$ dense subalgebra $\mathcal{S}_\Theta$. Given $\varphi = \lambda_\Theta(f) \in \mathcal{S}_\Theta$

$$\|T_k\varphi\|_{\mathrm{BMO}_c(\mathcal{R}_\Theta)} \sim \sup_{\mathrm{R}>0} \sup_{\mathrm{B_R} \in \mathcal{Q}_\mathrm{R}} \bigg\| \Big(\fint_{\mathrm{B_R}} \big| \sigma_\Theta(T_k\varphi) - \sigma_\Theta(T_k\varphi)_{\mathrm{B_R}} \big|^2 \, d\mu \Big)^{\frac{1}{2}} \bigg\|_{\mathcal{R}_\Theta}$$

where the second supremum runs over the set $\mathcal{Q}_\mathrm{R}$ of Euclidean balls $\mathrm{B_R}$ of radius R and center $c_{\mathrm{B_R}}$. Recall that $\sigma_\Theta(T_k\varphi) = T_{k_\sigma}\varphi$ for $k_\sigma = (\sigma_\Theta \otimes id)(k)$. Now let $\psi : \mathbb{R}^n \to [0,1]$ be a Schwartz function which is identically 1 over the unit ball $\mathrm{B}_1(0)$ and identically 0 outside its concentric 2-dilation $\mathrm{B}_2(0)$. Define

$$\psi_\mathrm{R}(x) = \psi\Big(\frac{x + c_{\mathrm{B_R}}}{2\mathrm{R}} \Big)$$

to decompose the kernel $k_\sigma = (\sigma_\Theta \otimes id)(k)$ as follows

$$k_\sigma = \underbrace{k_\sigma \bullet \pi_\Theta(\psi_\mathrm{R})}_{k_{\sigma 1}(\mathrm{R})} + \underbrace{k_\sigma \bullet \big(\mathbf{1} - \pi_\Theta(\psi_\mathrm{R}) \big)}_{k_{\sigma 2}(\mathrm{R})}.$$

Note here that $\pi_\Theta(\psi_\mathrm{R}) \simeq \mathbf{1} \otimes \pi_\Theta(\psi_\mathrm{R})$ is an element of $\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}$ and only k_σ has a component in $L_\infty(\mathbb{R}^n)$. We claim that the following inequality for $k_{\sigma 1}(\mathrm{R})$ holds up to constants independent of the ball $\mathrm{B_R} \in \mathcal{Q}_\mathrm{R}$ and the radius R

$$(2.1) \qquad \bigg\| \Big(\fint_{\mathrm{B_R}} \big| T_{k_{\sigma 1}(\mathrm{R})}\varphi - (T_{k_{\sigma 1}(\mathrm{R})}\varphi)_{\mathrm{B_R}} \big|^2 \, d\mu \Big)^{\frac{1}{2}} \bigg\|_{\mathcal{R}_\Theta} \leq C_n \mathrm{A}_1 \|\varphi\|_{\mathcal{R}_\Theta}.$$

Before proving this first claim, let us continue with the argument. Of course, it would suffice to give a similar estimate for $k_{\sigma 2}(\mathrm{R})$. To do so and according to the Poincaré type inequality in Proposition 1.6 and its relation to $\mathrm{BMO}_c(\mathcal{R}_\Theta)$ outlined in Remark 1.7, it suffices to estimate

$$\mathrm{R} \Big\| \underbrace{(\mathbf{1} \otimes q_\mathrm{R} \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{R}_\Theta})(T_{k_{\sigma 2}(\mathrm{R})}\varphi)}_{T_\mathbf{K}(\varphi)} \Big\|_{\mathcal{L}(\mathbb{F}_n) \bar\otimes L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta}$$

for $q_\mathrm{R} = 1_{\mathrm{B_R}}$ with constants independent of R. Since

$$(\nabla \otimes id_{\mathcal{R}_\Theta}) \circ \sigma_\Theta = (id_{\mathcal{L}(\mathbb{F}_n)} \otimes \sigma_\Theta) \circ \nabla_\Theta$$

we may rewrite $T_{\mathbf{K}}(\varphi)$ as follows

$$
\begin{aligned}
T_{\mathbf{K}}(\varphi) &= (\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1})(\nabla \otimes id)\Big((id^{\otimes 2} \otimes \tau_{\Theta})\big(k_{\sigma 2}(\mathrm{R})(\mathbf{1}^{\otimes 2} \otimes \varphi)\big)\Big) \\
&= (\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1})\big(id^{\otimes 3} \otimes \tau_{\Theta}\big)\Big((\nabla \otimes id^{\otimes 2})(\sigma_{\Theta} \otimes id)[k] \bullet (\mathbf{1} - \Psi_{\mathrm{R}})(\mathbf{1}^{\otimes 3} \otimes \varphi)\Big) \\
&= \big(id^{\otimes 3} \otimes \tau_{\Theta}\big)\Big(\underbrace{(\mathbf{1} \otimes q_{\mathrm{R}} \otimes \mathbf{1}^{\otimes 2})(id \otimes \sigma_{\Theta} \otimes id)[\nabla_{\Theta} k] \bullet (\mathbf{1} - \Psi_{\mathrm{R}})}_{\mathbf{K}}(\mathbf{1}^{\otimes 3} \otimes \varphi)\Big)
\end{aligned}
$$

with $\mathbf{K} \in \mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_{\infty}(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}\bar{\otimes}\mathcal{R}_{\Theta}^{\mathrm{op}}$ and $\Psi_{\mathrm{R}} = \pi_{\Theta}(\psi_{\mathrm{R}})$. For simplicity, we shall use a more compact terminology and write $\mathbf{K} = q_{\mathrm{R}}\nabla_{\Theta}^{\sigma}k \bullet (\mathbf{1} - \Psi_{\mathrm{R}})$. We may now decompose $\mathbf{K}$ as follows

$$
\mathbf{K} = \underbrace{q_{\mathrm{R}}\Psi_{\mathrm{R}} \bullet \nabla_{\Theta}^{\sigma}k \bullet (\mathbf{1} - \Psi_{\mathrm{R}})}_{\mathbf{K}_1} + \underbrace{q_{\mathrm{R}}(\mathbf{1} - \Psi_{\mathrm{R}}) \bullet \nabla_{\Theta}^{\sigma}k \bullet (\mathbf{1} - \Psi_{\mathrm{R}})}_{\mathbf{K}_2}.
$$

We claim that the following inequality holds for $j = 1, 2$

$$(2.2) \qquad \big\|T_{\mathbf{K}_j}(\varphi)\big\|_{\mathcal{L}(\mathbb{F}_n)\bar{\otimes}L_{\infty}(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}} \le C_n(\alpha, \beta)\frac{\mathrm{A}_2}{\mathrm{R}}\|\varphi\|_{\mathcal{R}_{\Theta}}.$$

Our discussion so far has reduced the proof to justifying (2.1) and (2.2):

Proof of (2.1). It is clear that

$$
\Big\|\Big(\fint_{\mathrm{B_R}} \big|T_{k_{\sigma 1}(\mathrm{R})}\varphi - (T_{k_{\sigma 1}(\mathrm{R})}\varphi)_{\mathrm{B_R}}\big|^2 d\mu\Big)^{\frac{1}{2}}\Big\|_{\mathcal{R}_{\Theta}} \le C_n\mathrm{R}^{-\frac{n}{2}}\Big\|\Big(\int_{\mathbb{R}^n} \big|T_{k_{\sigma 1}(\mathrm{R})}\varphi\big|^2 d\mu\Big)^{\frac{1}{2}}\Big\|_{\mathcal{R}_{\Theta}}.
$$

On the other hand, given $T : \mathcal{S}_{\Theta} \to L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}$ we need to introduce its module extension $T' : \mathcal{S}_{\Theta}\bar{\otimes}\mathcal{R}_{\Theta} \ni \lambda_{\Theta}(f) \otimes \varphi \mapsto T(\lambda_{\Theta}(f))(\mathbf{1} \otimes \varphi) \in L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}$. Recall the following (elementary) algebraic identity

$$
(id \otimes \tau_{\Theta})\big(k_{\sigma} \bullet (a \otimes b) \bullet (\mathbf{1} \otimes \varphi)\big) = T_{k_{\sigma}}(\varphi b)(\mathbf{1} \otimes a) = T'_{k_{\sigma}}\big((\varphi \otimes \mathbf{1})\mathrm{flip}(a \otimes b)\big).
$$

Then, noticing that k is assumed to be an algebraic kernel in $\mathcal{S}_{\Theta} \otimes_{\mathrm{alg}} \mathcal{S}_{\Theta}^{\mathrm{op}}$, it is not difficult to check that the above formula extends from elementary tensors $a \otimes b$ to arbitrary elements in $\mathcal{R}_{\Theta}\bar{\otimes}\mathcal{R}_{\Theta}^{\mathrm{op}}$. This yields

$$
T_{k_{\sigma 1}(\mathrm{R})}\varphi = T_{k_{\sigma} \bullet \Psi_{\mathrm{R}}}\varphi = T'_{k_{\sigma}}\big((\varphi \otimes \mathbf{1})\mathrm{flip}(\pi_{\Theta}(\psi_{\mathrm{R}}))\big).
$$

In particular, we easily obtain the following estimate

$$
\begin{aligned}
&\Big\|\Big(\int_{\mathbb{R}^n} \big|T_{k_{\sigma 1}(\mathrm{R})}\varphi\big|^2 d\mu\Big)^{\frac{1}{2}}\Big\|_{\mathcal{R}_{\Theta}} \\
&\le \Big\|T'_{k_{\sigma}} : L_2^c(\mathcal{R}_{\Theta})\bar{\otimes}\mathcal{R}_{\Theta} \to L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}\Big\|_{\mathrm{cb}} \big\|\mathrm{flip}(\pi_{\Theta}(\psi_{\mathrm{R}}))\big\|_{L_2^c(\mathcal{R}_{\Theta})\bar{\otimes}\mathcal{R}_{\Theta}} \|\varphi\|_{\mathcal{R}_{\Theta}}.
\end{aligned}
$$

According to Lemma 1.5

$$
\big\|\mathrm{flip}(\pi_{\Theta}(\psi_{\mathrm{R}}))\big\|_{L_2^c(\mathcal{R}_{\Theta})\bar{\otimes}\mathcal{R}_{\Theta}} \le \big\|\psi_{\mathrm{R}}\big\|_{L_2(\mathbb{R}^n)} \le C_n\mathrm{R}^{\frac{n}{2}}
$$

since the argument given there for π_{Θ} also holds for $\mathrm{flip} \circ \pi_{\Theta}$. Therefore, it remains to estimate the cb-norm of $T'_{k_{\sigma}}$. We claim that it is bounded by A_1, the L_2-norm of T_k. To justify it, we introduce the map

$$
W : L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta} \ni \int_{\mathbb{R}^n} \exp_{\xi} \otimes a(\xi)\, d\xi \mapsto \int_{\mathbb{R}^n} \exp_{\xi} \otimes \lambda_{\Theta}(\xi)a(\xi)\, d\xi \in L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_{\Theta}.
$$

It is straightforward to show that W extends to an isometry. On the other hand, letting $j_\Theta : \exp_\xi \mapsto \lambda_\Theta(\xi)$ be the L_2-isometry introduced in Section 1.3.2, we observe that

$$
\begin{aligned}
W(j_\Theta^* \otimes id)(\lambda_\Theta(f) \otimes a) &= W\Big(\int_{\mathbb{R}^n} f(\xi) \exp_\xi \otimes a \, d\xi \Big) \\
&= \int_{\mathbb{R}^n} f(\xi) \exp_\xi \otimes \lambda_\Theta(\xi) a \, d\xi \ = \ \sigma_\Theta(\lambda_\Theta(f))(\mathbf{1} \otimes a).
\end{aligned}
$$

Identifying $T_k \varphi$ with $\lambda_\Theta(f)$ for some smooth $f : \mathbb{R}^n \to \mathbb{C}$ we obtain the identity $W(j_\Theta^* T_k \otimes id)(\varphi \otimes a) = T'_{k_\sigma}(\varphi \otimes a)$. This a fortiori implies that the cb-norm of the map $T'_{k_\sigma} : L_2^c(\mathcal{R}_\Theta) \bar\otimes \mathcal{R}_\Theta \to L_2^c(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta$ is dominated by the $L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ norm of T_k, as desired. This completes the proof of claim (2.1).

Proof of (2.2)**.** Let

$$
\mathrm{D}_\Theta = (\sigma_\Theta \otimes id)(\mathrm{d}_\Theta)
$$

and decompose the kernels $\mathbf{K}_j$ for $j = 1, 2$ as follows

$$
\mathbf{K}_1 = \Big(q_\mathrm{R} \, \Psi_\mathrm{R} \bullet \mathrm{D}_\Theta^{-\alpha} \Big) \bullet \Big(\mathrm{D}_\Theta^\alpha \bullet \nabla_\Theta^\sigma k \bullet \mathrm{D}_\Theta^{\alpha'} \Big) \bullet \Big(\mathrm{D}_\Theta^{-\alpha'} \bullet (\mathbf{1} - \Psi_\mathrm{R}) q_\mathrm{R} \Big),
$$

$$
\mathbf{K}_2 = \Big(q_\mathrm{R} \, (\mathbf{1} - \Psi_\mathrm{R}) \bullet \mathrm{D}_\Theta^{-\beta} \Big) \bullet \Big(\mathrm{D}_\Theta^\beta \bullet \nabla_\Theta^\sigma k \bullet \mathrm{D}_\Theta^{\beta'} \Big) \bullet \Big(\mathrm{D}_\Theta^{-\beta'} \bullet (\mathbf{1} - \Psi_\mathrm{R}) q_\mathrm{R} \Big),
$$

with $\alpha' = n + 1 - \alpha$ and $\beta' = n + 1 - \beta$. Using the terminology $\mathbf{K}_j = \mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j$ for the brackets above and according to the operator-valued (trivial) extension of Lemma 2.2, we find for $\mathcal{B}_\Theta = \mathcal{L}(\mathbb{F}_n) \bar\otimes L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta$ that

$$
\big\| T_{\mathbf{K}_j} : \mathcal{R}_\Theta \to \mathcal{B}_\Theta \big\| \leq \big\| (id \otimes \tau_\Theta)(\mathbf{A}_j \mathbf{A}_j^*) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} \big\| \mathbf{B}_j \big\|_{\mathcal{B}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \big\| (id \otimes \tau_\Theta)(\mathbf{C}_j^* \mathbf{C}_j) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}}.
$$

Since $\mathbf{B}_j = (\sigma_\Theta \otimes id)(\mathrm{d}_\Theta^\rho \bullet \nabla_\Theta k \bullet \mathrm{d}_\Theta^{n+1-\rho})$ for $\rho = \alpha, \beta$ and σ_Θ is a $*$-homomorphism we deduce from the hypotheses that $\|\mathbf{B}_1\| + \|\mathbf{B}_2\| \leq \mathrm{A}_2$. Therefore, recalling that we have $\alpha + \alpha' = \beta + \beta' = n + 1$, it suffices to prove the following inequalities for the terms associated to $\mathbf{A}_j$ and $\mathbf{C}_j$

$$
\begin{aligned}
\big\| (id \otimes \tau_\Theta)(\mathbf{A}_1 \mathbf{A}_1^*) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} &\leq C_n(\alpha) \mathrm{R}^{\frac{n}{2} - \alpha}, \\
\big\| (id \otimes \tau_\Theta)(\mathbf{C}_1^* \mathbf{C}_1) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} &\leq C_n(\alpha') \mathrm{R}^{\frac{n}{2} - \alpha'}, \\
\big\| (id \otimes \tau_\Theta)(\mathbf{A}_2 \mathbf{A}_2^*) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} &\leq C_n(\beta) \mathrm{R}^{\frac{n}{2} - \beta}, \\
\big\| (id \otimes \tau_\Theta)(\mathbf{C}_2^* \mathbf{C}_2) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} &\leq C_n(\beta') \mathrm{R}^{\frac{n}{2} - \beta'},
\end{aligned}
$$

for any $\alpha < \mathrm{K}_n - \frac{1}{2} < \beta < \mathrm{K}_n + \frac{1}{2}$. We will justify the above estimates for $\mathbf{A}_1$ and $\mathbf{C}_1$, the proof of the others is very similar. Let $\Delta_{\mathbb{R}^n} f(x, y) = f(x + y)$ be the comultiplication map on $\mathbb{R}^n$. According to Lemma 1.5

$$
(\sigma_\Theta \otimes id) \circ \pi_\Theta = (id \otimes \pi_\Theta) \circ \Delta_{\mathbb{R}^n}.
$$

In particular, $\mathbf{A}_1^* = (id \otimes \pi_\Theta)\big(\Delta_{\mathbb{R}^n}(| \cdot |^{-\alpha}) q_\mathrm{R} (\mathbf{1} \otimes \psi_\mathrm{R}) \big)$ and we find

$$
\begin{aligned}
\big\| (id \otimes \tau_\Theta)(\mathbf{A}_1 \mathbf{A}_1^*) \big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} &= \big\| \mathbf{A}_1^* \big\|_{L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta \bar\otimes L_2^c(\mathcal{R}_\Theta^{\mathrm{op}})} \\
&\leq \big\| id \otimes \pi_\Theta \big\| \, \big\| \Delta_{\mathbb{R}^n}(| \cdot |^{-\alpha}) q_\mathrm{R} (\mathbf{1} \otimes \psi_\mathrm{R}) \big\|_{L_\infty(\mathbb{R}^n) \bar\otimes L_2^c(\mathbb{R}^n)},
\end{aligned}
$$

where $id \otimes \pi_\Theta : L_\infty(\mathbb{R}^n) \bar\otimes L_2^c(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta \bar\otimes L_2^c(\mathcal{R}_\Theta^{\mathrm{op}})$. Its norm is dominated by the cb-norm of $\pi_\Theta : L_2^c(\mathbb{R}^n) \to \mathcal{R}_\Theta \bar\otimes L_2^c(\mathcal{R}_\Theta^{\mathrm{op}})$. We already proved in Lemma 1.5

the cb-contractivity of $\pi_\Theta : L_2^c(\mathbb{R}^n) \to L_2^c(\mathcal{R}_\Theta) \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ and the exact same argument can be trivially adapted to show that $\|id \otimes \pi_\Theta\| \leq 1$ in the right hand side of the above inequality. It then remains to estimate

$$\sup_{x \in \mathbb{R}^n} q_{\mathrm{R}}(x) \Big(\int_{\mathbb{R}^n} |x + y|^{-2\alpha} \psi_{\mathrm{R}}^2(y) \, dy \Big)^{\frac{1}{2}} \leq \Big(\int_{\mathrm{B}_{5\mathrm{R}}(0)} |y|^{-2\alpha} dy \Big)^{\frac{1}{2}} \lesssim \mathrm{R}^{\frac{n}{2} - \alpha}$$

for $\alpha < \frac{n}{2}$. In the case of $\mathbf{C}_1 = (id \otimes \pi_\Theta) \big(\Delta_{\mathbb{R}^n}(|\cdot|^{-\alpha'}) q_{\mathrm{R}} \otimes (1 - \psi_{\mathrm{R}}) \big)$ we get

$$\sup_{x \in \mathbb{R}^n} q_{\mathrm{R}}(x) \Big(\int_{\mathbb{R}^n} |x + y|^{-2\alpha'} (1 - \psi_{\mathrm{R}}(y))^2 \, dy \Big)^{\frac{1}{2}} \leq \Big(\int_{\mathrm{B}_{\mathrm{R}}^{\mathrm{c}}(0)} |y|^{-2\alpha'} dy \Big)^{\frac{1}{2}} \lesssim \mathrm{R}^{\frac{n}{2} - \alpha'}$$

for $\alpha' > \frac{n}{2}$. The same argument applies for $\mathbf{A}_2$ and $\mathbf{C}_2$. This proves claim (2.2).

Conclusion. The argument above proves that $T_k : L_\infty(\mathcal{R}_\Theta) \to \mathrm{BMO}_c(\mathcal{R}_\Theta)$ defines a bounded operator with norm dominated by $C_n(\alpha, \beta)(\mathrm{A}_1 + \mathrm{A}_2)$. The exact same argument can be used after matrix amplification to prove that the cb-norm of T_k satisfies the same upper bound. This completes the proof of the theorem. $\square$

Remark 2.7. Our kernel conditions in Theorem 2.6 are natural extensions of the classical ones [34, 66]. The price for noncommutativity is a concrete and balanced left/right location of the exponents α, β, which of course is meaningless in the commutative setting. These surprisingly transparent Calderón-Zygmund kernel conditions are possible due to the very precise geometric information on $\mathcal{R}_\Theta$ that we collected in Section 1. Our results below for pseudodifferential operators crucially rely on these conditions. For more general von Neumann algebras, the resulting conditions are necessarily less transparent [44].

2.3. **CZ extrapolation: General case.** If $\mathcal{S}_\Theta'$ denotes the space of continuous linear functionals on $\mathcal{S}_\Theta$ —tempered distributions— the aim of this section is to generalize our $L_\infty \to \mathrm{BMO}_c$ estimate in Theorem 2.6 for continuous linear maps $T \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ to incorporate actual Calderón-Zygmund kernels. This imposes a careful analysis of tempered Θ-distributions and how this affects our former kernel manipulations. By symmetrization, interpolation and duality, we shall conclude by proving the L_p-boundedness of quantum Calderón-Zygmund operators.

2.3.1. *Tempered distributions.* The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ comes equipped with the locally convex topology determined by the seminorms $p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|$ for $f : \mathbb{R}^n \to \mathbb{C}$ and all $\alpha, \beta \in \mathbb{N}^n$. The quantum analogue for $\varphi = \lambda_\Theta(f)$ in $\mathcal{S}_\Theta$ was described in Remark 1.10. In particular, a sequence $\varphi_j = \lambda_\Theta(f_j)$ converges to 0 in $\mathcal{S}_\Theta$ as $j \to \infty$ when

$$\lim_{j \to \infty} \big\| P(x_\Theta) \partial_\Theta^\beta \big(\lambda_\Theta(f_j) \big) Q(x_\Theta) \big\|_{\mathcal{R}_\Theta} = 0$$

for all $\beta \in \mathbb{N}^n$ and all quantum monomials

$$P(x_\Theta) = \prod_{1 \leq r \leq m_\ell}^{\rightarrow} x_{\Theta, j_r} \quad \text{and} \quad Q(x_\Theta) = \prod_{1 \leq s \leq m_r}^{\rightarrow} x_{\Theta, k_s}.$$

By Remark 1.10, this holds iff

$$\lim_{j \to \infty} \Big\| \lambda_\Theta \Big[\Big(\prod_{1 \leq r \leq m_\ell}^{\rightarrow} D_{\Theta, j_r}^\ell \Big) \Big(\prod_{1 \leq s \leq m_r}^{\rightarrow} D_{\Theta, k_s}^r \Big) M_{(2\pi i \xi)^\beta} f_j \Big] \Big\|_{\mathcal{R}_\Theta} = 0.$$

Lemma 2.8. *If $\Theta_1, \Theta_2 \in A_n(\mathbb{R})$ we find that*

$$\lim_{j \to \infty} \lambda_{\Theta_1}(f_j) = 0 \ in \ \mathcal{S}_{\Theta_1} \quad \Leftrightarrow \quad \lim_{j \to \infty} \lambda_{\Theta_2}(f_j) = 0 \ in \ \mathcal{S}_{\Theta_2}.$$

Proof. Let us set

$$p_{P,Q,\beta}^{\Theta}(\lambda_{\Theta}(f)) = \left\| P(x_{\Theta}) \partial_{\Theta}^{\beta}(\lambda_{\Theta}(f)) Q(x_{\Theta}) \right\|_{\mathcal{R}_{\Theta}}$$

and assume that $\lim_{j \to \infty} p_{P_1,Q_1,\beta_1}^{\Theta_1}(\lambda_{\Theta_1}(f_j)) = 0$ for all $\beta_1 \in \mathbb{N}^n$ and all quantum Θ_1-monomials P_1, Q_1. Given $\beta_2 \in \mathbb{N}^n$ and quantum Θ_2-monomials P_2, Q_2 it then suffices to show that

$$\lim_{j \to \infty} p_{P_2,Q_2,\beta_2}^{\Theta_2}(\lambda_{\Theta_2}(f_j)) = 0.$$

According to Remark 1.10, we may find two commuting operators $P_2[D_{\Theta_2,j}^{\ell}]$ and $Q_2[D_{\Theta_2,j}^{r}]$ —sums of modulations and derivations— satisfying the following identity

$$p_{P_2,Q_2,\beta_2}^{\Theta_2}(\lambda_{\Theta_2}(f_j)) = \left\| \lambda_{\Theta_2}\left(P_2[D_{\Theta_2,j}^{\ell}]Q_2[D_{\Theta_2,j}^{r}]M_{(2\pi i\xi)^{\beta_2}}f_j\right) \right\|_{\mathcal{R}_{\Theta_2}}.$$

Applying Riemann-Lebesgue and Cauchy-Schwartz with $B(\xi) = 1 + |\xi|^n$ and $B^{-1}(\xi)$

$$
\begin{aligned}
p_{P_2,Q_2,\beta_2}^{\Theta_2}(\lambda_{\Theta_2}(f_j)) &\leq \left\| P_2[D_{\Theta_2,j}^{\ell}]Q_2[D_{\Theta_2,j}^{r}]M_{(2\pi i\xi)^{\beta_2}}f_j \right\|_{L_1(\mathbb{R}^n)} \\
&\lesssim \left\| M_{B(\xi)}P_2[D_{\Theta_2,j}^{\ell}]Q_2[D_{\Theta_2,j}^{r}]M_{(2\pi i\xi)^{\beta_2}}f_j \right\|_{L_2(\mathbb{R}^n)} \\
&= \left\| \lambda_{\Theta_1}\left(M_{B(\xi)}P_2[D_{\Theta_2,j}^{\ell}]Q_2[D_{\Theta_2,j}^{r}]M_{(2\pi i\xi)^{\beta_2}}f_j\right) \right\|_{L_2(\mathcal{R}_{\Theta_1})}
\end{aligned}
$$

By Proposition 1.12, there exists a quantum Θ_1-polynomial $R_1(x_{\Theta_1})$, whose inverse lives in $L_2(\mathcal{R}_{\Theta_1})$. Therefore, multiplying and dividing by it on the left hand side yields

$$
\begin{aligned}
p_{P_2,Q_2,\beta_2}^{\Theta_2}(\lambda_{\Theta_2}(f_j)) &\lesssim \left\| R_1(x_{\Theta_1})\lambda_{\Theta_1}\left(M_{B(\xi)}\mathbf{D}_{\Theta_2}(P_2)\mathbf{D}_{\Theta_2}(Q_2)M_{(2\pi i\xi)^{\beta_2}}f_j\right) \right\|_{\mathcal{R}_{\Theta_1}} \\
&= \left\| \lambda_{\Theta_1}\Big(\underbrace{R_1[D_{\Theta_1,j}^{\ell}]M_{B(\xi)}P_2[D_{\Theta_2,j}^{\ell}]Q_2[D_{\Theta_2,j}^{r}]M_{(2\pi i\xi)^{\beta_2}}f_j}_{\mathbf{A}(f_j)} \Big) \right\|_{\mathcal{R}_{\Theta_1}}.
\end{aligned}
$$

Using standard commutation relations, $\mathbf{A}(f_j)$ may be written as

$$\mathbf{A}(f_j) = \sum_k P_{1k}[D_{\Theta_1,j}^{\ell}]Q_{1k}[D_{\Theta_1,j}^{r}]M_{(2\pi i\xi)^{\beta_{1k}}}f_j$$

for finitely many $\beta_{1k} \in \mathbb{N}^n$ and quantum Θ_1-monomials P_{1k} and Q_{1k}. $\qquad \square$

A linear functional $L : \mathcal{S}_{\Theta} \to \mathbb{C}$ is in $\mathcal{S}'_{\Theta}$ when it satisfies that $\lim_{j \to \infty} \langle L, \varphi_j \rangle = 0$ for any sequence $\varphi_j \in \mathcal{S}_{\Theta}$ converging to 0. Using the unitaries $j_{\Theta} : \exp_{\xi} \mapsto \lambda_{\Theta}(\xi)$ we construct $j_{\Theta_1\Theta_2} = j_{\Theta_2} \circ j_{\Theta_1}^* : \lambda_{\Theta_1}(\xi) \mapsto \lambda_{\Theta_2}(\xi)$. According to Lemma 2.8 and given $L \in \mathcal{S}'_{\Theta_1}$, this means that $\langle j_{\Theta_1\Theta_2}L, \lambda_{\Theta_2}(f) \rangle := \langle L, \lambda_{\Theta_1}(f) \rangle$ defines a tempered distribution in $\mathcal{S}'_{\Theta_2}$. Since this process is invertible, it turns out that the theory of tempered distributions in $\mathcal{R}_{\Theta}$ is formally equivalent to the classical theory.

Let us now consider continuous linear operators $T \in \mathcal{L}(\mathcal{S}_{\Theta}, \mathcal{S}'_{\Theta})$. Of course, since the topology in $\mathcal{S}'_{\Theta}$ is that of pointwise convergence, a linear map $T : \mathcal{S}_{\Theta} \to \mathcal{S}'_{\Theta}$ is continuous whenever $\lim_j \langle T(\lambda_{\Theta}(f_j)), \lambda_{\Theta}(g) \rangle = 0$ for any family $\lambda_{\Theta}(f_j)$ which converges to 0 in $\mathcal{S}_{\Theta}$ and any $\lambda_{\Theta}(g) \in \mathcal{S}_{\Theta}$. To identify the kernel of $T \in \mathcal{L}(\mathcal{S}_{\Theta}, \mathcal{S}'_{\Theta})$ consider $j_{\Theta}^* T j_{\Theta} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)')$ where $j_{\Theta} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}_{\Theta}$ and $j_{\Theta}^* : \mathcal{S}'_{\Theta} \to \mathcal{S}(\mathbb{R}^n)'$

by our discussion above. Then, by a well-known result of Schwartz, there exists a unique kernel $m \in \mathcal{S}'(\mathbb{R}^{2n}) = (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}(\mathbb{R}^n))'$ satisfying

$$\langle j_\Theta^* T j_\Theta f, g \rangle = \langle m, g \otimes f \rangle \quad \text{for all} \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

Therefore, given $T \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ we find its associated kernel

$$k = j_\Theta \otimes j_\Theta(m) \in \mathcal{S}_{\Theta \oplus \Theta}' \simeq (\mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta)'$$

such that

$$\begin{aligned} \langle T(\lambda_\Theta(f)), \lambda_\Theta(g) \rangle &= \langle j_\Theta^* T j_\Theta(\lambda_0(f)), \lambda_0(g) \rangle \\ &= \langle m, \lambda_0(g) \otimes \lambda_0(f) \rangle = \langle k, \lambda_\Theta(g) \otimes \lambda_\Theta(f) \rangle. \end{aligned}$$

Now, according to the density of the quantum Schwartz class $\mathcal{S}_{\Theta \oplus \Theta}$ in $\mathcal{S}_{\Theta \oplus \Theta}'$ —since the same result also holds in the commutative case— we easily conclude the density of the algebraic tensor product $\mathcal{S}_\Theta \otimes_{\text{alg}} \mathcal{S}_\Theta$ in $\mathcal{S}_{\Theta \oplus \Theta}'$. This proves that the family of algebraic kernels we considered for the proof of Theorem 2.6 are dense in the space $\mathcal{S}_{\Theta \oplus \Theta}'$ of arbitrary kernels for maps in $\mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$. Moreover, by the weak-$*$ density of trigonometric polynomials in $\mathcal{R}_{\Theta \oplus \Theta}$, we may also approximate $\mathcal{S}_{\Theta \oplus \Theta}'$ by finite sums of the form

$$k = \sum_j w(\xi_j, \eta_j) \lambda_\Theta(\xi_j)^* \otimes \lambda_\Theta(\eta_j)^*.$$

According to our expression above —Section 2.1— for kernels affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\text{op}}$

$$k = \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{k}(\xi, \eta) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)^* \, d\mu(\xi, \eta),$$

this result amounts to say (not surprising) that kernels associated to finite sums of Dirac deltas are dense. Note that identity $\langle T(\lambda_\Theta(f)), \lambda_\Theta(g) \rangle = \langle k, \lambda_\Theta(g) \otimes \lambda_\Theta(f) \rangle$ takes the following form for finite sums of Dirac deltas

$$\langle T(\lambda_\Theta(f)), \lambda_\Theta(g) \rangle = \sum_j w(\xi_j, \eta_j) f(\eta_j) g(\xi_j) = \langle k, \lambda_\Theta(g) \otimes \lambda_\Theta(f) \rangle.$$

2.3.2. *Kernel manipulations and derivations.* In the model case treated previously we decomposed the kernel $k_\sigma = (\sigma_\Theta \otimes id)(k)$ as $k_\sigma \bullet \pi_\Theta(\psi_{\text{R}}) + k_\sigma \bullet (\mathbf{1} - \pi_\Theta(\psi_{\text{R}}))$ and, after applying our Poincaré type inequality to the second term, we further decomposed the resulting kernel $\mathbf{K}$ as $\pi_\Theta(\psi_{\text{R}}) \bullet \mathbf{K} + (\mathbf{1} - \pi_\Theta(\psi_{\text{R}})) \bullet \mathbf{K}$. This leads us to understand the same operation for general kernels in $\mathcal{S}_{\Theta \oplus \Theta}'$. To that end, we introduce the following operations for $L \in \mathcal{S}_\Theta'$

$$\langle \lambda_\Theta(\xi) L, \varphi \rangle = \langle L, \varphi \lambda_\Theta(\xi) \rangle \quad \text{and} \quad \langle L \lambda_\Theta(\xi), \varphi \rangle = \langle L, \lambda_\Theta(\xi) \varphi \rangle.$$

Lemma 2.9. *Given $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$, the maps*

$$\begin{aligned} M_\psi^\ell(T)(\lambda_\Theta(f)) &= \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \lambda_\Theta(\xi) T(\lambda_\Theta(\xi)^* \lambda_\Theta(f)) \, d\xi, \\ M_\psi^r(T)(\lambda_\Theta(f)) &= \int_{\mathbb{R}^n} \widehat{\psi}(\xi) T(\lambda_\Theta(f) \lambda_\Theta(\xi)^*) \lambda_\Theta(\xi) \, d\xi, \end{aligned}$$

belong to $\mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ and their kernels extend $\pi_\Theta(\psi) \bullet k$ and $k \bullet \pi_\Theta(\psi)$ respectively.

Proof. We shall prove the assertion only for $M_\psi^\ell(T)$, since both operators can be handled similarly. In order to prove continuity, assume that $\lambda_\Theta(f_j) \to 0$ in $\mathcal{S}_\Theta$ as $j \to \infty$. Then we need to show that

$$\lim_{j \to \infty} \langle M_\psi^\ell(T)(\lambda_\Theta(f_j)), \lambda_\Theta(g) \rangle = \lim_{j \to \infty} \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \langle T(\lambda_\Theta(\xi)^* \lambda_\Theta(f_j)), \lambda_\Theta(g) \lambda_\Theta(\xi) \rangle d\xi$$

$$= \lim_{j \to \infty} \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \langle k, \lambda_\Theta(g) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \lambda_\Theta(f_j) \rangle d\xi$$

vanishes for all $g \in \mathcal{S}(\mathbb{R}^n)$. We first note that

$$\lim_{j \to \infty} \langle T(\lambda_\Theta(\xi)^* \lambda_\Theta(f_j)), \lambda_\Theta(g) \lambda_\Theta(\xi) \rangle = 0$$

for all $\xi \in \mathbb{R}^n$, since $\lambda_\Theta(\xi)^* \lambda_\Theta(f_j) \to 0$ in $\mathcal{S}_\Theta$ as $j \to \infty$. Indeed, we have $\lambda_\Theta(\xi)^* \lambda_\Theta(f_j) = \lambda_\Theta(f_{j\xi})$ for $f_{j\xi}(\eta) = f_j(\eta + \xi) e^{-2\pi i \langle \xi, \Theta_\downarrow \eta \rangle}$ and we then use Lemma 2.8 with $(\Theta_1, \Theta_2) = (\Theta, 0)$. Once this is known, we use the dominated convergence theorem, for which we need an integrable upper bound of

$$\Phi(\xi) = \sup_{j \geq 1} \left| \widehat{\psi}(\xi) \langle k, \lambda_\Theta(g) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \lambda_\Theta(f_j) \rangle \right|.$$

Since $\lambda_\Theta(g) \lambda_\Theta(\xi) = \lambda_\Theta(g_\xi)$ for $g_\xi(\eta) = g(\eta - \xi) e^{2\pi i \langle \eta - \xi, \Theta_\downarrow \xi \rangle}$, we have

$$\langle k, \lambda_\Theta(g) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \lambda_\Theta(f_j) \rangle = \langle j_\Theta^* \otimes j_\Theta^*(k), \widehat{g}_\xi \otimes \widehat{f}_{j\xi} \rangle$$

with $j_\Theta^* \otimes j_\Theta^*(k) \in \mathcal{S}(\mathbb{R}^{2n})'$. According to [33, Proposition 2.3.4], there exists an absolute constant C and positive integers k, m such that the following inequality holds

$$\begin{aligned}
\Phi(\xi) &\leq& C |\widehat{\psi}(\xi)| \sup_{j \geq 1} \sum_{\substack{|\alpha| \leq k \\ |\beta| \leq m}} p_{\alpha,\beta} \left(\widehat{g}_\xi \otimes \widehat{f}_{j\xi} \right) \\
&\leq& C |\widehat{\psi}(\xi) P(\xi)| \sup_{j \geq 1} \sum_{\substack{|\alpha| \leq k \\ |\beta| \leq m}} p_{\alpha,\beta} \left(\widehat{g} \otimes \widehat{f}_j \right) \lesssim |\widehat{\psi}(\xi) P(\xi)| \in \mathcal{S}(\mathbb{R}^n)
\end{aligned}$$

for certain polynomial P. On the other hand, for integrable kernels

$$\begin{aligned}
M_\psi^\ell(T)(\lambda_\Theta(f)) &=& \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \lambda_\Theta(\xi) (id \otimes \tau_\Theta) \left((\mathbf{1} \otimes \lambda_\Theta(\xi)^* \lambda_\Theta(f)) \bullet k \right) d\xi \\
&=& \int_{\mathbb{R}^n} \widehat{\psi}(\xi) (id \otimes \tau_\Theta) \left((\mathbf{1} \otimes \lambda_\Theta(f)) \bullet (\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*) \bullet k \right) d\xi \\
&=& (id \otimes \tau_\Theta) \left((\mathbf{1} \otimes \lambda_\Theta(f)) \bullet \pi_\Theta(\psi) \bullet k \right) = T_{\pi_\Theta(\psi) \bullet k}(\lambda_\Theta(f)).
\end{aligned}$$

Interchanging trace and integral is justified for finite tensors by evaluation against a test function, and a fortiori by density of these kernels in $\mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta') \simeq \mathcal{S}_{\Theta \oplus \Theta}'$. $\square$

Remark 2.10. Given $\dagger \in \{\ell, r\}$, it is also clear that

$$\left\| M_\psi^\dagger(T) : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \right\| \leq \left(\int_{\mathbb{R}^n} |\widehat{\psi}(\xi)| \, d\xi \right) \left\| T : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \right\|.$$

Remark 2.11. It will also be relevant to observe that

$$\begin{aligned}
\left\langle M_\psi^\ell(T)(\lambda_\Theta(f)), \lambda_\Theta(g) \right\rangle &=& \left\langle k, (\lambda_\Theta(g) \otimes \lambda_\Theta(f)) \bullet \pi_\Theta(\psi) \right\rangle, \\
\left\langle M_\psi^r(T)(\lambda_\Theta(f)), \lambda_\Theta(g) \right\rangle &=& \left\langle k, \pi_\Theta(\psi) \bullet (\lambda_\Theta(g) \otimes \lambda_\Theta(f)) \right\rangle,
\end{aligned}$$

for the kernel $k \in \mathcal{S}_{\Theta \oplus \Theta}'$ associated to T and any $\psi \in \mathcal{S}(\mathbb{R}^n)$. Indeed, we have

$$\begin{aligned}
\left\langle M_\psi^\ell(T)(\lambda_\Theta(f)), \lambda_\Theta(g) \right\rangle &=& \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \left\langle T(\lambda_\Theta(\xi)^* \lambda_\Theta(f)), \lambda_\Theta(g) \lambda_\Theta(\xi) \right\rangle d\xi \\
&=& \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \left\langle k, \lambda_\Theta(g) \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \lambda_\Theta(f) \right\rangle d\xi
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \Big\langle k, \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)\big) \bullet \big(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*\big) \Big\rangle \, d\xi$$

which gives the desired identity. Moreover, these identities hold for any function ψ for which both $(\lambda_\Theta(g) \otimes \lambda_\Theta(f)) \bullet \pi_\Theta(\psi)$ and $\pi_\Theta(\psi) \bullet (\lambda_\Theta(g) \otimes \lambda_\Theta(f))$ stay in $\mathcal{S}_{\Theta \oplus \Theta}$.

Again as we did in the model case above, we shall need to operate with module extensions. Given a linear map $T : \mathcal{S}_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$ we will use its module extension $T' : \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$ given by

$$T' : \lambda_\Theta(f) \otimes \lambda_\Theta(g) \mapsto T(\lambda_\Theta(f))(\mathbf{1} \otimes \lambda_\Theta(g)),$$

where $\big\langle T(\lambda_\Theta(f))(\mathbf{1} \otimes \lambda_\Theta(g)), (a \otimes b) \big\rangle = \big\langle T(\lambda_\Theta(f)), (\mathbf{1} \otimes \lambda_\Theta(g))(a \otimes b) \big\rangle$.

Lemma 2.12. *There exists a continuous map $\sigma_\Theta : \mathcal{S}'_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$ which extends the corepresentation $\sigma_\Theta : \mathcal{R}_\Theta \ni \lambda_\Theta(\xi) \mapsto \exp_\xi \otimes \lambda_\Theta(\xi) \in L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$. In particular, given $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}'_\Theta)$ with kernel $k \in \mathcal{S}'_{\Theta \oplus \Theta}$ the composition $T_{k_\sigma} = \sigma_\Theta T_k$ belongs to $\mathcal{L}(\mathcal{S}_\Theta, (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)')$ with kernel $k_\sigma = (\sigma_\Theta \otimes id)k$. Moreover, T'_{k_σ} extends to a continuous right modular map $\mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$ with*

$$T'_{k_\sigma}\big((\lambda_\Theta(f) \otimes \mathbf{1})\mathrm{flip}(\pi_\Theta(\psi))\big) = M^r_\psi(T_{k_\sigma})(\lambda_\Theta(f)).$$

Proof. The map $q(\widehat{f}, \widehat{g}) = \widehat{fg}$ arises as the conjugation of the multiplication map $(f, g) \mapsto fg$ by the Fourier transform. It thus follows from the Leibniz rule that it defines a continuous map $\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Letting $q_\Theta = j_\Theta \circ q \circ (id \otimes j^*_\Theta)$ we define for $L \in \mathcal{S}'_\Theta$

$$\big\langle \sigma_\Theta L, \lambda_0(f) \otimes \lambda_\Theta(g) \big\rangle = \big\langle L, q_\Theta(\lambda_0(f) \otimes \lambda_\Theta(g)) \big\rangle.$$

It is clear that $\sigma_\Theta : \mathcal{S}'_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$ is continuous and we find

$$\begin{aligned}
\big\langle \sigma_\Theta(\lambda_\Theta(\xi)^*), \lambda_0(f) \otimes \lambda_\Theta(g) \big\rangle &= \tau_\Theta\big(\lambda_\Theta(\xi)^* \lambda_\Theta(fg)\big) = f(\xi)g(\xi) \\
&= \big\langle \exp_{-\xi} \otimes \lambda_\Theta(\xi)^*, \lambda_0(f) \otimes \lambda_\Theta(g) \big\rangle
\end{aligned}$$

and thus σ_Θ so defined extends the corepresentation σ_Θ introduced in Section 1.1.2. This immediately implies that $T_{k_\sigma} = \sigma_\Theta T_k$ belongs to $\mathcal{L}(\mathcal{S}_\Theta, (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)')$ and its kernel $k_\sigma = (\sigma_\Theta \otimes id)(k)$. Let us now justify the continuity of the module extension T'_{k_σ}. Indeed, the module extension of $\sigma_\Theta j_\Theta$, defines a continuous linear map

$$W : \mathcal{S}(\mathbb{R}^n)' \otimes \mathcal{S}_\Theta \to (\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}_\Theta)'$$

satisfying $W(\exp_\xi \otimes \lambda_\Theta(\eta)) = \exp_\xi \otimes \lambda_\Theta(\xi)\lambda_\Theta(\eta)$. Its continuity follows easily from the continuity of σ_Θ. Next, observe that $T'_{k_\sigma} = W \circ (j^*_\Theta T_k \otimes id)$ since it trivially holds for the dense class of finite sums $k = \sum_j w(\xi_j, \eta_j) \lambda_\Theta(\xi_j)^* \otimes \lambda_\Theta(\eta_j)^*$. This automatically implies the continuity of the module extension T'_{k_σ}. It remains to justify the given identity for T'_{k_σ}

$$\begin{aligned}
T'_{k_\sigma}\big((\lambda_\Theta(f) \otimes \mathbf{1})\mathrm{flip}(\pi_\Theta(\psi))\big) &= \int_{\mathbb{R}^n} \widehat{\psi}(\xi) T'_{k_\sigma}\big(\lambda_\Theta(f)\lambda_\Theta(\xi)^* \otimes \lambda_\Theta(\xi)\big) \, d\xi \\
&= \int_{\mathbb{R}^n} \widehat{\psi}(\xi) T_{k_\sigma}\big(\lambda_\Theta(f)\lambda_\Theta(\xi)^*\big)(\mathbf{1} \otimes \lambda_\Theta(\xi)) \, d\xi,
\end{aligned}$$

which is the definition of $M^r_\psi(T_{k_\sigma})(\lambda_\Theta(f))$. This completes the proof. $\qquad \square$

Our next goal is to generalize the Poincaré type inequality in Proposition 1.6 to the context of tempered distributions. Of course, the free Θ-gradient can be understood as a map $\nabla_\Theta : \mathcal{S}'_\Theta \to \mathcal{L}(\mathbb{F}_n) \otimes \mathcal{S}'_\Theta$ in the canonical way

$$\nabla_\Theta L = \sum_{k=1}^n s_k \otimes \partial_\Theta^k L \quad \text{for} \quad L \in \mathcal{S}'_\Theta,$$

where $\langle \partial_\Theta^k L, \lambda_\Theta(f) \rangle = -\langle L, \partial_\Theta^k \lambda_\Theta(f) \rangle = -2\pi i \langle L, \lambda_\Theta(f_{[k]}) \rangle$ and $f_{[k]}(\xi) = \xi_k f(\xi)$. Now, given a $\mathbb{R}^n$-ball B_R of radius R with characteristic function q_R, Proposition 1.6 gives an upper bound for

$$\big\| \varphi - \varphi_{B_R} \big\|_{\mathcal{R}_\Theta \bar\otimes L_2^c(\phi)} = \Big\| \Big(\fint_{B_R} \big(\varphi - \varphi_{B_R} \big)^* \big(\varphi - \varphi_{B_R} \big) d\mu \Big)^{\frac{1}{2}} \Big\|_{\mathcal{R}_\Theta}$$

in terms of the operator norm of the gradient of φ localized at B_R. Let us recall that the predual of $\mathcal{R}_\Theta \bar\otimes L_2^c(\phi)$ with respect to the linear bracket is given by the space $\mathcal{A}_\Theta(B_R) = L_1(\mathcal{R}_\Theta^{\mathrm{op}}) \widehat\otimes L_2^r(\phi)$, whose norm is

$$\big\| \psi - \psi_{B_R} \big\|_{L_1(\mathcal{R}_\Theta^{\mathrm{op}}) \widehat\otimes L_2^r(\phi)} = \Big\| \Big(\fint_{B_R} \big(\psi - \psi_{B_R} \big) \big(\psi - \psi_{B_R} \big)^* d\mu \Big)^{\frac{1}{2}} \Big\|_{L_1(\mathcal{R}_\Theta)}.$$

It is a simple exercise to show that $\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta$ is norm dense in $L_1(\mathcal{R}_\Theta^{\mathrm{op}}) \widehat\otimes L_2^r(\phi)$. In particular, the following result gives an extension of our Poincaré type inequality.

Proposition 2.13. *Given* $L \in (\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta)'$, *assume*

$$\big(\partial_k \otimes id_{\mathcal{R}_\Theta} \big)(L) \in L_\infty(B_R) \bar\otimes \mathcal{R}_\Theta$$

for $1 \le k \le n$. *Then, the following Poincaré type inequality holds*

$$\sup_{\substack{\psi \in \mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta \\ \|\psi - \psi_{B_R}\|_{\mathcal{A}_\Theta(B_R)} \le 1}} \big| \big\langle q_R L, \psi - \psi_{B_R} \big\rangle \big|$$

$$\le \quad 2\sqrt{2} R \Big\| (\mathbf{1} \otimes q_R \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{R}_\Theta})(L) \Big\|_{\mathcal{L}(\mathbb{F}_n) \bar\otimes L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta}.$$

Proof. Assume for clarity that B_R is centered at the origin, see Proposition 1.6 for the standard modifications in the general case. Since $\partial_k L \in L_\infty(\mathbb{R}^n) \bar\otimes \mathcal{R}_\Theta$, we may define

$$\widetilde{L}(x) \quad = \quad \int_0^1 \sum_{k=1}^n \partial_k L(tx) x_k \, dt$$

$$= \quad \int_0^1 (\underbrace{\tau_{\mathcal{L}(\mathbb{F}_n)} \otimes id_{\mathbb{R}^n}}_{\mathsf{E}_{\mathbb{R}^n}}) \big(\underbrace{q_R(x) \nabla L(tx)}_{\mathsf{A}(t)} \underbrace{q_R(x) s(x)}_{\mathsf{B}} \big) \, dt$$

for $x \in B_R$. Now let $\varphi_j \in \mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta$ be an approximating sequence for L and define the functions $\widetilde{\varphi}_j(x) = \varphi_j(x) - \varphi_j(0)$ accordingly. In particular, the following identity holds for every test function $\psi \in \mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta$

$$\int_{B_R} \tau_\Theta \big(\widetilde{\varphi}_j(x)(\psi(x) - \psi_{B_R}) \big) \, dx = \int_{B_R} \tau_\Theta \big(\varphi_j(x)(\psi(x) - \psi_{B_R}) \big) \, dx.$$

By approximation we get

$$\big| \big\langle q_R L, \psi - \psi_{B_R} \big\rangle \big| \quad = \quad \big| \big\langle q_R \widetilde{L}, \psi - \psi_{B_R} \big\rangle \big|$$

$$= \left| \int_0^1 \left\langle \mathsf{E}_{\mathbb{R}^n}(\mathrm{A}(t)\mathrm{B}), \psi - \psi_{\mathrm{B_R}} \right\rangle dt \right|$$

$$\leq \left(\int_0^1 \left\| \mathsf{E}_{\mathbb{R}^n}(\mathrm{A}(t)\mathrm{B}) \right\|_{\mathcal{R}_\Theta \bar\otimes L_2^c(\phi)} dt \right) \left\| \psi - \psi_{\mathrm{B_R}} \right\|_{\mathcal{A}_\Theta(\mathrm{B_R})}.$$

Now we may complete the argument as we did in the proof of Proposition 1.6. $\quad\square$

2.3.3. A Calderón-Zygmund extrapolation theorem. We are finally ready to prove an estimate for general Calderón-Zygmund operators T_k. According to the classical theory, we impose cancellation and smoothness conditions on the kernel. To be more precise, let $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}'_\Theta)$ admit a kernel $k \in \mathcal{S}'_{\Theta \oplus \Theta}$ with gradient $(\nabla_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}})(k)$ affiliated to $\mathcal{L}(\mathbb{F}_n) \bar\otimes \mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}$. Then, we shall call T_k a *column Calderón-Zygmund operator* with parameters $(\mathrm{A}_j, \alpha_j, \beta_j)$ when:

  i) *Cancellation*

$$\left\| T_k : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \right\| \leq \mathrm{A}_1.$$

  ii) *Kernel smoothness.* If $\mathrm{K}_n = \frac{1}{2}(n+1)$, there exists

$$\alpha < \mathrm{K}_n - \frac{1}{2} < \beta < \mathrm{K}_n + \frac{1}{2}$$

   satisfying the gradient conditions below for $\rho = \alpha, \beta$

$$\left| \mathrm{d}_\Theta^\rho \bullet (\nabla_\Theta \otimes id_{\mathcal{R}_\Theta^{\mathrm{op}}})(k) \bullet \mathrm{d}_\Theta^{n+1-\rho} \right| \leq \mathrm{A}_2.$$

Remark 2.14. We implicitly use that

$$\left\langle \mathrm{d}_\Theta^\gamma \bullet \nabla_\Theta k \bullet \mathrm{d}_\Theta^\eta, z \right\rangle = \left\langle \nabla_\Theta k, \mathrm{d}_\Theta^\eta \bullet z \bullet \mathrm{d}_\Theta^\gamma \right\rangle,$$

$$\left\| \mathrm{d}_\Theta^\gamma \bullet \nabla_\Theta k \bullet \mathrm{d}_\Theta^\eta \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} = \sup_{\substack{z \in \mathcal{L}(\mathbb{F}_n) \bar\otimes \mathcal{S}_{\Theta \oplus \Theta} \\ \|z\|_{L_1(\mathcal{L}(\mathbb{F}_n) \bar\otimes \mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}})} \leq 1}} \left\langle \nabla_\Theta k, \mathrm{d}_\Theta^\eta \bullet z \bullet \mathrm{d}_\Theta^\gamma \right\rangle.$$

As explained in Remark 2.11, this is justified for any tempered distribution when $\gamma, \eta \in 2\mathbb{Z}$, but not for $\gamma, \eta < 2$ since $\mathrm{d}_\Theta^\eta \bullet z \bullet \mathrm{d}_\Theta^\gamma$ does not stay in the test space $\mathcal{S}_{\Theta \oplus \Theta}$. The necessity of using these values —only for $n = 2$ in the simpler statement of Theorem A— forces us to impose that $\nabla_\Theta k$ is, in addition, affiliated to the algebra. Although our assumption is admissible in view of the classical theory we could have alternatively used an approximation argument $\mathrm{d}_\Theta^\gamma = \lim_\varepsilon (\mathrm{d}_\Theta^2 + \varepsilon \mathbf{1})^{\gamma/2}$ to avoid it. On the other hand, the kernel k —not its gradient— should be treated as a distribution since this allows certain Dirac deltas which do appear in the classical theory, see Paragraph 2.3.4 below for further details.

Proposition 2.15. *If T_k is a column CZO and $\varphi \in \mathcal{S}_\Theta$*

$$\left\| T_k(\varphi) \right\|_{\mathrm{BMO}_c(\mathcal{R}_\Theta)} \leq C_n(\alpha, \beta)(\mathrm{A}_1 + \mathrm{A}_2) \|\varphi\|_{\mathcal{R}_\Theta}.$$

Proof. We shall adapt our argument in the model case of Theorem 2.6. Given $\varphi = \lambda_\Theta(f) \in \mathcal{S}_\Theta$, this means that we need to control the operator-valued BMO norm of $\sigma_\Theta(T_k\varphi)$. According to Lemma 2.12, we have $\sigma_\Theta T_k = T_{k_\sigma}$ for $k_\sigma = (\sigma_\Theta \otimes id)(k)$ and we may decompose it as follows

$$k_\sigma = \underbrace{k_\sigma \bullet \pi_\Theta(\psi_{\mathrm{R}})}_{k_{\sigma 1}(\mathrm{R})} + \underbrace{k_\sigma \bullet \left(1 - \pi_\Theta(\psi_{\mathrm{R}})\right)}_{k_{\sigma 2}(\mathrm{R})},$$

where the decomposition uses Lemma 2.9 and Remark 2.11. Next, we need to show the validity of (2.1). To that end we follow the argument in Theorem 2.6 by recalling the crucial identity

$$T'_{k_\sigma}\big((\lambda_\Theta(f) \otimes \mathbf{1})\mathrm{flip}(\pi_\Theta(\psi))\big) = M^r_\psi(T_{k_\sigma})(\lambda_\Theta(f)),$$

which was justified in Lemma 2.12 for general kernels. This is the part of the proof which requires L_2-boundedness of T_k. Once we have completed our argument for $k_{\sigma_1}(\mathrm{R})$, we apply the Poincaré type inequality in Proposition 2.13 to the term associated to $k_{\sigma 2}(\mathrm{R})$. This gives

$$\left\|\left(\fint_{\mathrm{B_R}} \big|T_{k_{\sigma_2}(\mathrm{R})}\varphi - (T_{k_{\sigma_2}(\mathrm{R})}\varphi)_{\mathrm{B_R}}\big|^2 d\mu\right)^{\frac{1}{2}}\right\|_{\mathcal{R}_\Theta}$$

$$\lesssim \mathrm{R}\Big\|\underbrace{(\mathbf{1} \otimes q_\mathrm{R} \otimes \mathbf{1})(\nabla \otimes id_{\mathcal{R}_\Theta})(T_{k_{\sigma_2}(\mathrm{R})}\varphi)}_{T_{\mathbf{K}}(\varphi)}\Big\|_{\mathcal{L}(\mathbb{F}_n)\bar\otimes L_\infty(\mathbb{R}^n)\bar\otimes\mathcal{R}_\Theta}.$$

As in Theorem 2.6, the goal is to show that $\mathbf{K}$ is the distribution

$$\mathbf{K} = q_\mathrm{R}\nabla_\Theta^\sigma k \bullet (\mathbf{1} - \Psi_\mathrm{R}) \in \mathcal{L}(\mathbb{F}_n)\bar\otimes\big(\mathcal{S}(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta\big)'$$

with $q_\mathrm{R}\nabla_\Theta^\sigma k = (\mathbf{1} \otimes q_\mathrm{R} \otimes \mathbf{1}^{\otimes 2})(id \otimes \sigma_\Theta \otimes id)(\nabla_\Theta \otimes id)[k]$ and $\Psi_\mathrm{R} = \pi_\Theta(\psi_\mathrm{R})$. By density, it suffices to justify it for elementary kernels $k = \lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)$. This is possible via the identity $(\nabla\otimes id_{\mathcal{R}_\Theta})\circ\sigma_\Theta = (id_{\mathcal{L}(\mathbb{F}_n)}\otimes\sigma_\Theta)\circ\nabla_\Theta$ due to our extensions of the maps σ_Θ and ∇_Θ for tempered distributions above. We may now decompose $\mathbf{K}$ by means of Lemma 2.9 as follows

$$\mathbf{K} = \underbrace{q_\mathrm{R}\Psi_\mathrm{R} \bullet \nabla_\Theta^\sigma k \bullet (\mathbf{1} - \Psi_\mathrm{R})}_{\mathbf{K}_1} + \underbrace{q_\mathrm{R}(\mathbf{1} - \Psi_\mathrm{R}) \bullet \nabla_\Theta^\sigma k \bullet (\mathbf{1} - \Psi_\mathrm{R})}_{\mathbf{K}_2}.$$

At this point, the argument follows verbatim the proof of Theorem 2.6. Indeed, we further decompose $\mathbf{K}_j = \mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j$ as we did there and apply Lemma 2.2 —valid for affiliated kernels, as we assume for $\mathbf{K}_j$— to obtain

$$(2.3) \qquad \big\|T_{\mathbf{K}_j} : \mathcal{R}_\Theta \to \mathcal{B}_\Theta\big\|$$

$$\leq \big\|(id \otimes \tau_\Theta)(\mathbf{A}_j\mathbf{A}_j^*)\big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}} \big\|\mathbf{B}_j\big\|_{\mathcal{B}_\Theta\bar\otimes\mathcal{R}_\Theta^{\mathrm{op}}} \big\|(id \otimes \tau_\Theta)(\mathbf{C}_j^*\mathbf{C}_j)\big\|_{\mathcal{B}_\Theta}^{\frac{1}{2}}$$

with $\mathcal{B}_\Theta = \mathcal{L}(\mathbb{F}_n)\bar\otimes L_\infty(\mathbb{R}^n)\bar\otimes\mathcal{R}_\Theta$. The estimates for $\mathbf{A}_j, \mathbf{B}_j, \mathbf{C}_j$ also apply here. $\square$

Remark 2.16. Alternatively, if we do not want to assume that $\nabla_\Theta k$ is affiliated to $\mathcal{L}(\mathbb{F}_n)\bar\otimes\mathcal{R}_\Theta\bar\otimes\mathcal{R}_\Theta^{\mathrm{op}}$ and use the approximation argument indicated in Remark 2.14, we should be able to generalize the inequality (2.3) for tempered distributions. Recall that the norm of $T_{\mathbf{K}_j} : \mathcal{R}_\Theta \to \mathcal{B}_\Theta$ can be expressed as the supremum —over Schwartz elements φ, ϕ respectively in the unit ball of $\mathcal{R}_\Theta^{\mathrm{op}}$ and $L_1(\mathcal{B}_\Theta)$— of the linear brackets

$$\big|\langle T_{\mathbf{K}_j}\varphi, \phi\rangle\big| = \big|\langle\mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j, \phi \otimes \varphi\rangle\big|$$

$$\leq \big\|\mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j\big\|_{\mathcal{B}_\Theta\bar\otimes L_1(\mathcal{R}_\Theta^{\mathrm{op}})}\|\phi\|_{L_1(\mathcal{B}_\Theta)}\|\varphi\|_{\mathcal{R}_\Theta^{\mathrm{op}}}.$$

Now we use the following characterization of the norm in $\mathcal{M}\bar\otimes\mathrm{X}$

$$\big\|\mathrm{A}\big\|_{\mathcal{M}\bar\otimes\mathrm{X}} = \sup_{\alpha,\beta\in\mathrm{B}_1(L_2(\mathcal{M}))} \big\|(\alpha \otimes \mathbf{1})\mathrm{A}(\beta \otimes \mathbf{1})\big\|_{L_1(\mathcal{M};\mathrm{X})},$$

which is due to Pisier [59] when $\mathcal{M}$ is hyperfinite and X is any operator space. It is also well-known that Pisier's identity still holds for non-hyperfinite von Neumann

algebras —as in our case with $\mathcal{M} = \mathcal{B}_\Theta$— as long as X is a noncommutative L_p space. In fact, Pisier's identity generalizes to arbitrary mixed $L_p(L_q)$-norms. In our case

$$\left\| \mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j \right\|_{\mathcal{B}_\Theta \bar{\otimes} L_1(\mathcal{R}_\Theta^{\mathrm{op}})}$$
$$= \sup_{\mathrm{a,c} \in \mathrm{B}_1(L_2(\mathcal{B}_\Theta))} \left\| (\mathrm{a} \otimes \mathbf{1}) \bullet \mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j \bullet (\mathrm{c} \otimes \mathbf{1}) \right\|_{L_1(\mathcal{B}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}})}.$$

In particular, we find

$$\left\| (\mathrm{a} \otimes \mathbf{1}) \bullet \mathbf{A}_j \bullet \mathbf{B}_j \bullet \mathbf{C}_j \bullet (\mathrm{c} \otimes \mathbf{1}) \right\|_1 \leq \left\| (\mathrm{a} \otimes \mathbf{1}) \bullet \mathbf{A}_j \right\|_2 \left\| \mathbf{B}_j \right\|_\infty \left\| \mathbf{C}_j \bullet (\mathrm{c} \otimes \mathbf{1}) \right\|_2$$

and the elementary inequalities below complete the proof of (2.3)

$$\left\| (\mathrm{a} \otimes \mathbf{1}) \bullet \mathbf{A}_j \right\|_2 \leq \left\| \mathrm{a} \right\|_2 \left\| (id \otimes \tau_\Theta)(\mathbf{A}_j \mathbf{A}_j^*) \right\|_{\mathcal{B}_\Theta}^{\frac{1}{2}},$$
$$\left\| \mathbf{C}_j \bullet (\mathrm{c} \otimes \mathbf{1}) \right\|_2 \leq \left\| \mathrm{c} \right\|_2 \left\| (id \otimes \tau_\Theta)(\mathbf{C}_j^* \mathbf{C}_j) \right\|_{\mathcal{B}_\Theta}^{\frac{1}{2}}.$$

Proposition 2.17. *Every column CZO is normal. In particular*

$$\left\| T_k : \mathcal{R}_\Theta \to \mathrm{BMO}_c(\mathcal{R}_\Theta) \right\|_{\mathrm{cb}} \leq C_n(\alpha, \beta)\left(\mathrm{A}_1 + \mathrm{A}_2 \right).$$

Proof. Let $T_k^* : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ denote the adjoint of T_k, so that

$$\left| \tau_\Theta\left(T_k^*(\lambda_\Theta(f))\lambda_\Theta(g)^* \right) \right| = \left| \tau_\Theta\left(\lambda_\Theta(f) T_k(\lambda_\Theta(g))^* \right) \right|$$
$$\leq C_n(\alpha, \beta)\left(\mathrm{A}_1 + \mathrm{A}_2 \right) \left\| \lambda_\Theta(f) \right\|_{\mathrm{H}_c^1(\mathcal{R}_\Theta)} \left\| \lambda_\Theta(g) \right\|_{\mathcal{R}_\Theta}$$

for all $\lambda_\Theta(f), \lambda_\Theta(g) \in \mathcal{S}_\Theta$. Indeed, here $\mathrm{H}_c^1(\mathcal{R}_\Theta)$ denotes the predual of $\mathrm{BMO}_c(\mathcal{R}_\Theta)$ with respect to the antilinear duality bracket above, as described in Appendix B below. In particular, this inequality directly follows from Proposition 2.15. Now we claim that this implies

$$\left\| T_k^*(\varphi) \right\|_{L_1(\mathcal{R}_\Theta)} \leq C_n(\alpha, \beta)\left(\mathrm{A}_1 + \mathrm{A}_2 \right) \left\| \varphi \right\|_{\mathrm{H}_c^1(\mathcal{R}_\Theta)}$$

for all $\varphi = \lambda_\Theta(f) \in \mathcal{S}_\Theta$. Indeed, let us prove that

$$z = \frac{T_k^*(\varphi)}{C_n(\alpha, \beta)\left(\mathrm{A}_1 + \mathrm{A}_2 \right) \left\| \varphi \right\|_{\mathrm{H}_c^1(\mathcal{R}_\Theta)}}$$

belongs to the unit ball of $L_1(\mathcal{R}_\Theta)$. To that end, it clearly suffices to prove that $|\tau_\Theta(qzqa)| \leq 1$ for every contraction a in $\mathcal{R}_\Theta$ and every τ_Θ-finite projection q. Since $z \in L_2(\mathcal{R}_\Theta)$, we have $zq \in L_1(\mathcal{R}_\Theta)$ and Kaplansky density theorem provides a sequence $u_j \in \mathcal{S}_\Theta$ in the unit ball of $\mathcal{R}_\Theta$ so that

$$|\tau_\Theta(qzqa)| = \lim_{j \to \infty} |\tau_\Theta(u_j zq)|.$$

Moreover, since $u_j z \in L_1(\mathcal{R}_\Theta)$, we also find $v_k \in \mathcal{S}_\Theta$ in the unit ball of $\mathcal{R}_\Theta$ with

$$|\tau_\Theta(qzqa)| = \lim_{j \to \infty} |\tau_\Theta(u_j zq)| = \lim_{j \to \infty} \lim_{k \to \infty} |\tau_\Theta(u_j z v_k)|.$$

Finally, since $|\tau_\Theta(zw)| \leq 1$ for every $w \in \mathcal{S}_\Theta$ in the unit ball of $\mathcal{R}_\Theta$ —as we recalled at the beginning of the proof— and the Schwartz class $\mathcal{S}_\Theta$ is a $*$-algebra we obtain that $|\tau_\Theta(qzqa)| \leq 1$ as expected. This proves our claim. Next, we use the norm density of $\mathcal{S}_\Theta$ in $\mathrm{H}_c^1(\mathcal{R}_\Theta)$ from Corollary B.10 in Appendix B below to conclude that $T_k^* : \mathrm{H}_c^1(\mathcal{R}_\Theta) \to L_1(\mathcal{R}_\Theta)$ is bounded. The operator T_k^* is the antilinear adjoint corresponding to the duality

$$\overline{L_1(\mathcal{R}_\Theta)}^* = \mathcal{R}_\Theta$$

with respect to the antilinear duality bracket. Thus

$$T_k : \mathcal{R}_\Theta \to \overline{\mathrm{H}_c^1(\mathcal{R}_\Theta)}^* \simeq \mathrm{BMO}_c(\mathcal{R}_\Theta)$$

with the same constants, see Appendix B for further details on the duality $\mathrm{H}_1-\mathrm{BMO}$ in this setting. This proves the $L_\infty \to \mathrm{BMO}_c$ boundedness of T_k. As in the model case proved in Theorem 2.6, the cb-boundedness follows similarly and it just requires a more involved notation to incorporate matrix amplifications. $\qquad\square$

Once we have proved the complete $L_\infty \to \mathrm{BMO}_c$ boundedness of column CZOs, the general extrapolation theorem follows from additional assumptions of the same kind on the kernel, which makes them more symmetric. More precisely, we know that $T_k : \mathcal{R}_\Theta \to \mathrm{BMO}_r(\mathcal{R}_\Theta)$ is cb-bounded iff the operator

$$T_k^\dagger(\lambda_\Theta(f)) = T_k(\lambda_\Theta(f)^*)^*$$

defines a completely bounded map from $\mathcal{R}_\Theta \to \mathrm{BMO}_c(\mathcal{R}_\Theta)$. When this is the case we get a cb-map $T_k : \mathcal{R}_\Theta \to \mathrm{BMO}(\mathcal{R}_\Theta)$. Of course, the same assumptions for the adjoint T_k^* trivially imply that T_k also defines a cb-map $T_k : \mathrm{H}_1(\mathcal{R}_\Theta) \to L_1(\mathcal{R}_\Theta)$ and interpolation —see Appendix B for details— yields complete L_p-boundedness for $1 < p < \infty$. This means that we should impose that the maps $T_k^\dagger, T_k^*, T_k^{*\dagger}$ are column Calderón-Zygmund operators. It is clear that L_2-boundedness follows automatically from T_k. Therefore, we just need to impose new kernel smoothness conditions. We have

$$\mathrm{kernel}(T_k^\dagger) = k^*, \quad \mathrm{kernel}(T_k^*) = \mathrm{flip}(k)^*, \quad \mathrm{kernel}(T_k^{*\dagger}) = \mathrm{flip}(k).$$

Therefore, the results so far imply the following extrapolation theorem for CZOs.

Theorem 2.18. *Let $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ and assume*:

    i) Cancellation

$$\big\| T_k : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \big\| \leq \mathrm{A}_1.$$

    ii) Kernel smoothness. *There exists*

$$\alpha < \mathrm{K}_n - \frac{1}{2} < \beta < \mathrm{K}_n + \frac{1}{2} < \gamma$$

satisfying the gradient conditions below for $\rho = \alpha, \beta, \gamma$

$$\left| \mathrm{d}_\Theta^\rho \bullet (\nabla_\Theta \otimes id)(k) \bullet \mathrm{d}_\Theta^{n+1-\rho} \right| + \left| \mathrm{d}_\Theta^\rho \bullet (id \otimes \nabla_\Theta)(k) \bullet \mathrm{d}_\Theta^{n+1-\rho} \right| \leq \mathrm{A}_2.$$

Then, we find the following endpoint estimates for T_k

$$\begin{aligned}
\big\| T_k : \mathrm{H}_1(\mathcal{R}_\Theta) \to L_1(\mathcal{R}_\Theta) \big\|_{\mathrm{cb}} &\leq & C_n(\alpha, \beta, \gamma)\big(\mathrm{A}_1 + \mathrm{A}_2\big), \\
\big\| T_k : L_\infty(\mathcal{R}_\Theta) \to \mathrm{BMO}(\mathcal{R}_\Theta) \big\|_{\mathrm{cb}} &\leq & C_n(\alpha, \beta, \gamma)\big(\mathrm{A}_1 + \mathrm{A}_2\big).
\end{aligned}$$

In particular, $T_k : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$ is completely bounded for every $1 < p < \infty$.

In what follows, a *Calderón-Zygmund operator* over the quantum Euclidean space $\mathcal{R}_\Theta$ associated to the parameters $(\mathrm{A}_j, \alpha_j, \beta_j)$ will be any linear map $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$ satisfying the hypotheses in Theorem 2.18. The kernel considerations for the adjoint also appear in commutative Calderón-Zygmund theory, whereas the †-operation is standard and arises from noncommutativity.

2.3.4. *The principal value of kernel truncations.* As in classical Calderón-Zygmund theory, we want to understand how far is an operator $T_k \in \mathcal{B}(L_2(\mathcal{R}_\Theta))$ from the principal value singular integral determined by its kernel truncations. Our aim is to show that the difference is a left/right multiplier. Let us be more precise. Consider a smooth function $\psi \in \mathcal{S}(\mathbb{R}^n)$ which is identically 1 over $\mathrm{B}_1(0)$ and vanishes over $\mathbb{R}^n \setminus \mathrm{B}_2(0)$. Define

$$\Psi_{\Delta,\delta} = \pi_\Theta(\psi_{\Delta,\delta}) \quad \text{with} \quad \psi_{\Delta,\delta}(x) = \psi\Big(\frac{x}{\Delta}\Big) - \psi\Big(\frac{x}{\delta}\Big) = \psi_\Delta(x) - \psi_\delta(x)$$

for $0 < \delta << \Delta < \infty$. We shall study the kernel truncations $\Psi_{\Delta,\delta} \bullet k$ and $k \bullet \Psi_{\Delta,\delta}$ and how their limits are related to T_k. To that end, we introduce the notion of *admissible projection.* A projection $p \in \mathcal{R}_\Theta$ will be called admissible when the function $\mathbb{R}^n \to \mathrm{Proj}(\mathcal{R}_\Theta)$ defined as

$$\delta \longmapsto \bigvee_{s \in \mathrm{B}_\delta(0)} \sigma_\Theta^s(p)$$

is weak-$*$ continuous around $\delta = 0$. Here $\sigma_\Theta^s(\lambda_\Theta(\xi)) = \exp(2\pi i\langle s, \xi\rangle)\,\lambda_\Theta(\xi)$.

Remark 2.19. Even in the Euclidean setting with $\Theta = 0$, not all projections are admissible. In that case, the projection-valued function defined above associates a measurable set A with $\mathrm{B}_\delta[A]$, the union of all the balls of radius δ with center in A. If we take, for instance, a dense open subset of $[0,1]^n$ with measure strictly less than 1, we will have that $[0,1] \subset \mathrm{B}_\delta[A]$ for every $\delta > 0$, which poses an obstruction to admissibility. This can be easily fixed in the Euclidean setting by considering measurable sets which are closed up to a null set.

Lemma 2.20. *The bicommutant of admissible projections is the whole algebra $\mathcal{R}_\Theta$.*

Proof. It suffices to observe that one-dimensional spectral projections of the form $\chi_{[a,b]}(x_{\Theta,j})$ are admissible for each of the quantum variables $x_{\Theta,j}$ by Remark 2.19 and also that this family trivially generates $\mathcal{R}_\Theta$. This completes the proof. $\qquad\square$

Remark 2.21. Let us define a *closed projection* $p \in \mathcal{R}_\Theta$ as those whose complement $\mathbf{1} - p$ is the left support of certain element $\varphi \in \mathrm{E}_\Theta$ as defined at beginning of Section 1. By the $*$-stability of $\mathcal{S}_\Theta$ we could have replaced the left support $\ell(\varphi)$ by the right one $r(\varphi)$ or even by the full support $s(\varphi)$ of self-adjoint elements. We conjecture that all closed projections so defined are indeed admissible. At the time of this writing we have not been able to confirm this conjecture, but this will have no consequence in Theorem 2.23 below.

Lemma 2.22. *Given $\varphi \in \mathcal{R}_\Theta$, there exist projections p_δ, q_δ such that*

$$\begin{aligned}
\pi_\Theta(\psi_\delta) \bullet (\mathbf{1} \otimes \varphi) &= \pi_\Theta(\psi_\delta) \bullet (p_\delta \otimes \varphi), \\
\pi_\Theta(\psi_\delta) \bullet (\varphi \otimes \mathbf{1}) &= \pi_\Theta(\psi_\delta) \bullet (\varphi \otimes q_\delta).
\end{aligned}$$

If $r(\varphi)$ is admissible $\mathrm{w}^*\text{-}\lim\limits_{\delta\to 0} p_\delta = r(\varphi)$, if $\ell(\varphi)$ is admissible $\mathrm{w}^*\text{-}\lim\limits_{\delta\to 0} q_\delta = \ell(\varphi)$.

Proof. The assertions concerning q_δ follow from those for p_δ after applying the map $\mathrm{flip}^* : a \otimes b \mapsto b^* \otimes a^*$, details are left to the reader. Now, let us recall that the map $T : \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}} \to \mathcal{CB}(L_1(\mathcal{R}_\Theta), \mathcal{R}_\Theta)$ sending a kernel k to the corresponding map T_k is a complete isometry. Moreover, observe that

$$T_{k\bullet(\mathbf{1}\otimes\varphi)}(\phi) = T_k(\phi\varphi) \quad \text{and} \quad T_{k\bullet(\varphi\otimes\mathbf{1})}(\phi) = T_k(\phi)\varphi.$$

Since we clearly have

$$\|\pi_\Theta(\psi_\delta)\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \le \|\widehat{\psi}_\delta\|_{L_1(\mathbb{R}^n)} = \|\widehat{\psi}\|_{L_1(\mathbb{R}^n)} < \infty,$$

we know that $T_{\pi_\Theta(\psi_\delta)}$ is uniformly in $\mathcal{CB}(L_1(\mathcal{R}_\Theta), \mathcal{R}_\Theta)$. Let us define

$$\mathcal{N}_\delta = \overline{\mathrm{span}^{\mathrm{w}^*}}\big\{\varpi\, T_{\pi_\Theta(\psi_\delta)}(\phi\varphi) : \phi \in L_1(\mathcal{R}_\Theta),\, \varpi \in \mathcal{R}_\Theta\big\} \subset \mathcal{R}_\Theta.$$

Clearly $\mathcal{N}_\delta$ is a weak-$*$ closed left module. In particular, there must exist certain projection $p_\delta \in \mathcal{R}_\Theta$ satisfying $\mathcal{N}_\delta = \mathcal{R}_\Theta\, p_\delta$ and the following identity holds for every element $\phi \in L_1(\mathcal{R}_\Theta)$

$$T_{\pi_\Theta(\psi_\delta)\bullet(\mathbf{1}\otimes\varphi)}(\phi) = T_{\pi_\Theta(\psi_\delta)}(\phi\varphi) = T_{\pi_\Theta(\psi_\delta)}(\phi\varphi)\, p_\delta = T_{\pi_\Theta(\psi_\delta)\bullet(p_\delta\otimes\varphi)}(\phi).$$

Since T is (completely) isometric, $\pi_\Theta(\psi_\delta) \bullet (\mathbf{1} \otimes \varphi) = \pi_\Theta(\psi_\delta) \bullet (p_\delta \otimes \varphi)$. It remains to show that the projections p_δ so defined converges weakly to $r(\varphi)$ as $\delta \to 0^+$. Given any $\phi \in L_1(\mathcal{R}_\Theta)$, notice that

$$T_{\pi_\Theta(\psi_\delta)}(\phi\varphi) = \int_{\mathbb{R}^n} \widehat{\psi}_\delta(\xi)\, \tau_\Theta\big(\lambda_\Theta(\xi)^*\phi\varphi\big)\lambda_\Theta(\xi)\, d\xi = \int_{\mathrm{B}_{2\delta}(0)} \psi_\delta(s)\sigma_\Theta^s(\phi\varphi)\, ds.$$

Therefore, its right support satisfies

$$r\big(T_{\pi_\Theta(\psi_\delta)}(\phi\varphi)\big) \le \bigvee_{s\in \mathrm{B}_{2\delta}(0)} \sigma_\Theta^s(r(\varphi)) \quad \Rightarrow \quad p_\delta \le \bigvee_{s\in \mathrm{B}_{2\delta}(0)} \sigma_\Theta^s(r(\varphi)).$$

Hence, since $p_\delta \ge r(\varphi)$, we conclude by admissibility that $\mathrm{w}^*\text{-}\lim_{\delta\to 0} p_\delta = r(\varphi)$. $\qquad\square$

Given $T_k \in \mathcal{B}(L_2(\mathcal{R}_\Theta)) \subset \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$, we truncate it as follows

$$T_{\Delta,\delta}^\ell = M_{\Psi_{\Delta,\delta}}^\ell(T_k) \quad \text{and} \quad T_{\Delta,\delta}^r = M_{\Psi_{\Delta,\delta}}^r(T_k).$$

According to Remark 2.10 both truncations $T_{\Delta,\delta}^\dagger$ satisfy the L_2-estimate

$$\big\|T_{\Delta,\delta}^\dagger : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)\big\| \le 2\,\big\|\widehat{\psi}\big\|_1 \big\|T_k : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)\big\|.$$

In particular, the Banach-Alaoglu theorem confirms that certain subfamily of our truncations $T_{\Delta,\delta}^\dagger$ converges to some L_2-bounded operator $S_k^\dagger : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ for $\dagger \in \{\ell, r\}$. We shall assume for simplicity of notation that the whole family of truncations converges to $S_k^\dagger$ as $\Delta \to \infty$ and $\delta \to 0$.

Theorem 2.23. *There exist $z_\dagger \in \mathcal{R}_\Theta$ such that*

$$(T_k - S_k^\ell)(a) = a z_\ell \quad \text{and} \quad (T_k - S_k^r)(a) = z_r a.$$

Proof. Given an admissible projection p and by Remark 2.11

$$\Big\langle T_{\Delta,\delta}^r(\lambda_\Theta(f)p), \lambda_\Theta(g)\Big\rangle = \Big\langle k, \pi_\Theta(\Psi_{\Delta,\delta}) \bullet \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big)\Big\rangle.$$

Since $\pi_\Theta(\chi_{\mathrm{B}_\mathrm{R}(0)})$ converges to $\mathbf{1}$ in the strong operator topology, we can safely assume that $\lambda_\Theta(g) \otimes \lambda_\Theta(f)p = (\lambda_\Theta(g) \otimes p) \bullet (\mathbf{1} \otimes \lambda_\Theta(f))$ is left supported by $\pi_\Theta(\chi_{\mathrm{B}_\mathrm{R}(0)})$ for R large enough. Then we have

$$\Big\langle (T_k - T_{\Delta,\delta}^r)(\lambda_\Theta(f)p), \lambda_\Theta(g)\Big\rangle$$
$$= \Big\langle k, \pi_\Theta(\mathbf{1} - \psi_{\Delta,\delta}) \bullet \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big)\Big\rangle$$
$$= \Big\langle k, \pi_\Theta(\psi_\delta) \bullet \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big)\Big\rangle + \Big\langle k, \pi_\Theta(\mathbf{1} - \psi_\Delta) \bullet \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)\big)\Big\rangle.$$

Since $\ell(\lambda_\Theta(g) \otimes \lambda_\Theta(f)p) \leq \pi_\Theta(\psi_\Delta)$ for large Δ, the second term vanishes when Δ is large. The identity $\lambda_\Theta(g) \otimes \lambda_\Theta(f)p = (\mathbf{1} \otimes p) \bullet (\lambda_\Theta(g) \otimes \lambda_\Theta(f))$ allows to apply Lemma 2.22 to get

$$\pi_\Theta(\psi_\delta) \bullet \big(\lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big) = \pi_\Theta(\psi_\delta) \bullet \big(p_\delta \lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big)$$

for some projection p_δ converging to p in the weak$-*$ topology. This gives

$$\Big\langle (T_k - T_{\Delta,\delta}^r)(\lambda_\Theta(f)p), \lambda_\Theta(g) \Big\rangle$$
$$= \Big\langle k, \pi_\Theta(\psi_\delta) \bullet \big(p_\delta \lambda_\Theta(g) \otimes \lambda_\Theta(f)p\big) \Big\rangle = \Big\langle (T_k - T_{\Delta,\delta}^r)(\lambda_\Theta(f)p)p_\delta, \lambda_\Theta(g) \Big\rangle.$$

Taking limits in $\Delta \to \infty$ and $\delta \to 0$, we get $(T_k - S_k^r)(\lambda_\Theta(f)p) = (T_k - S_k^r)(\lambda_\Theta(f))p$ for any admissible projection $p \in \mathcal{R}_\Theta$. This readily implies that $T_k - S_k^r$ commutes with the von Neumann algebra generated by right multiplication with admissible projections and, by Lemma 2.20, we conclude that $T_k - S_k^r$ belongs to the commutant in $\mathcal{B}(L_2(\mathcal{R}_\Theta))$ of $\mathcal{R}_\Theta$ acting by right multiplication. Such algebra is given by $\mathcal{R}_\Theta$ acting on the left and so, there is a unique $z_r \in \mathcal{R}_\Theta$ such that $(T_k - S_k^r)(a) = z_r a$. A symmetric argument works for S_k^ℓ, which also satisfies the assertion. $\qquad\square$

Remark 2.24. We may also consider two-sided principal values $T_{\Delta,\delta}^\ell T_{\Delta',\delta'}^r$. Taking first a weak-$*$ accumulation point in (Δ, δ) and then another in (Δ', δ') gives an element S_k such that $S_k(\lambda_\Theta(f)) = z_r \lambda_\Theta(f) + \lambda_\Theta(f)z_\ell$, for certain $z_\dagger \in \mathcal{R}_\Theta$. This is the quantum analogue of a basic result in Calderón-Zygmund theory, further details can be found in [34, Proposition 8.1.11].

3. Pseudodifferential L_p calculus

The aim of this section is to establish sufficient smoothness conditions on a given symbol $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ for the L_p-boundedness of the pseudodifferential operator Ψ_a associated to it. This is the content of Theorem B in the Introduction. Sobolev p-estimates naturally follow from this analysis. Before that, subtle transference methods will be needed to extend the classical composition and adjoint formulae to the context of quantum Euclidean spaces. The proof of Theorem B is divided into several blocks. We begin with an analysis of L_2-boundedness, which includes the quantum forms of Calderón-Vaillancourt theorem and Bourdaud's condition stated in Theorem B i) and ii) respectively. Theorem B iii) follows from it and Theorem A, once we prove that Ψ_a is a Calderón-Zygmund operator.

3.1. **Adjoint and product formulae.** Recall that a symbol over $\mathcal{R}_\Theta$ must be understood as a smooth function $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ whose associated pseudodifferential operator takes the form

$$\Psi_a(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} a(\xi)f(\xi)\lambda_\Theta(\xi)\, d\xi.$$

Given $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, the Hörmander classes $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ are

$$S_{\rho,\delta}^m(\mathcal{R}_\Theta) = \Big\{ a : \mathbb{R}^n \to \mathcal{R}_\Theta \ : \ \big|\partial_\Theta^\beta \partial_\xi^\alpha a(\xi)\big| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \text{ for all } \alpha, \beta \in \mathbb{Z}_+^n \Big\}.$$

Here we follow standard notation $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. Pseudodifferential operators are formally generated by Fourier multipliers and left multiplication operators. It is easy to see that these families of operators generate in turn the whole $\mathcal{B}(L_2(\mathcal{R}_\Theta))$

as a von Neumann algebra. It is therefore reasonable to think that adjoints and composition of pseudodifferential operators are pseudodifferential operators. Our first goal is to develop asymptotic formulae for adjoints and compositions to justify that the adjoint of a regular ($\delta < \rho$) pseudodifferential operator of degree m is again a pseudodifferential operator of degree m and that the composition of operators of degrees m_1 and m_2 yields a pseudodifferential operator of degree $m_1 + m_2$.

We start by defining Ψ_a in the distributional sense. First, we have $\Psi_a : \mathcal{S}_\Theta \to \mathcal{S}_\Theta$ continuously whenever $a \in \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta)$ is a Schwartz function itself. Indeed, recall that $a \in \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta)$ means that

$$(3.1) \qquad a(\xi) = \int_{\mathbb{R}^n} \widehat{a}(z, \xi) \lambda_\Theta(z) \, dz$$

for some $\widehat{a} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. This immediately gives

$$\Psi_a(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} \widehat{a}(z - \xi, \xi) f(\xi) e^{2\pi i \langle z - \xi, \Theta_\downarrow \xi \rangle} d\xi}_{F(z)} \lambda_\Theta(z) \, dz$$

with $F \in \mathcal{S}(\mathbb{R}^n)$, which implies the assertion. The following lemma refines it.

Lemma 3.1. *Given any* $a \in S_{\rho, \delta}^m(\mathcal{R}_\Theta)$*, we have that* $\Psi_a : \mathcal{S}_\Theta \to \mathcal{S}_\Theta$ *continuously.*

Proof. Note that

$$
\begin{aligned}
\left\| \Psi_a(\lambda_\Theta(f)) \right\|_{\mathcal{R}_\Theta} &= \left\| \int_{\mathbb{R}^n} a(\xi) \langle \xi \rangle^{n+1} f(\xi) \lambda_\Theta(\xi) \frac{d\xi}{\langle \xi \rangle^{n+1}} \right\|_{\mathcal{R}_\Theta} \\
&\lesssim \sup_{\xi \in \mathbb{R}^n} \left\{ \left\| \langle \xi \rangle^{-m} a(\xi) \right\|_{\mathcal{R}_\Theta} \left| \langle \xi \rangle^{n+m+1} f(\xi) \right| \right\}.
\end{aligned}
$$

The first term is bounded by the Hörmander condition with $\alpha = \beta = 0$ and the second one since $f \in \mathcal{S}(\mathbb{R}^n)$. According to Remark 1.10 and Lemma 2.8, it suffices to see that the operators

$$P(x_\Theta) \partial_\Theta^\beta \Psi_a(\lambda_\Theta(f)) Q(x_\Theta)$$

satisfy similar inequalities for arbitrary monomials P, Q and $\beta \in \mathbb{Z}_+^n$. Recall that

$$\partial_\Theta^j \Psi_a(\lambda_\Theta(f)) = \Psi_{\partial_\Theta^j a}(\lambda_\Theta(f)) + \Psi_a(\partial_\Theta^j(\lambda_\Theta(f))),$$

but $\partial_\Theta^j a \in S_{\rho, \delta}^{m+\delta}(\mathcal{R}_\Theta)$ and $\partial_\Theta^j(\lambda_\Theta(f)) = \lambda_\Theta(2\pi i \xi_j f)$. In particular, $\partial_\Theta^\beta \Psi_a(\lambda_\Theta(f))$ behaves as $\Psi_a(\lambda_\Theta(f))$ and we may ignore β. Thus, it will be enough to illustrate the argument for $(P, Q, \beta) = (1, x_{\Theta,j}, 0)$ and $(P, Q, \beta) = (x_{\Theta,j}, 1, 0)$. In the first case, since our pseudodifferential operators act by left multiplication of the symbol a, the exact same argument given in the proof of Proposition 1.9 gives the identity below, even for a taking values in $\mathcal{R}_\Theta$ as it is the case

$$
\begin{aligned}
\Psi_a(\lambda_\Theta(f)) x_{\Theta,j} &= \int_{\mathbb{R}^n} D_{\Theta,j}^r(af)(\xi) \lambda_\Theta(\xi) \, d\xi \\
&= \int_{\mathbb{R}^n} \left(a(\xi) D_{\Theta,j}^r(f)(\xi) - \frac{1}{2\pi i} \partial_\xi^j a(\xi) f(\xi) \right) \lambda_\Theta(\xi) \, d\xi.
\end{aligned}
$$

Clearly $D_{\Theta,j}^r f \in \mathcal{S}(\mathbb{R}^n)$ and $\partial_\xi^j a \in S_{\rho, \delta}^{m-\rho}(\mathcal{R}_\Theta)$, so we may proceed as above. We need a similar expression when $x_{\Theta,j}$ acts by left multiplication. In this second case

we need to be a bit more careful

$$x_{\Theta,j}\Psi_a(\lambda_\Theta(f))$$
$$= x_{\Theta,j}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\widehat{a}(z,\xi)\lambda_\Theta(z)\,dz\Big)f(\xi)\lambda_\Theta(\xi)\,d\xi$$
$$= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widehat{a}(z,\xi)f(\xi)\lambda_\Theta(se_j)\lambda_\Theta(z)\lambda_\Theta(\xi)\,dzd\xi$$
$$= \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n}\underbrace{\Big[\int_{\mathbb{R}^n}\widehat{a}(z,\xi)e^{2\pi is\langle e_j,\Theta z\rangle}\lambda_\Theta(z)dz\Big]}_{a_{js}(\xi)}f(\xi)e^{2\pi is\langle e_j,\Theta_\downarrow\xi\rangle}\lambda_\Theta(\xi+se_j)d\xi.$$

Equivalently, we may write it as follows

$$x_{\Theta,j}\Psi_a(\lambda_\Theta(f)) = \frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=0}\int_{\mathbb{R}^n}a_{js}(\xi-se_j)f(\xi-se_j)e^{2\pi is\langle e_j,\Theta_\downarrow\xi\rangle}\lambda_\Theta(\xi)d\xi.$$

In particular, Leibniz rule and the argument in Proposition 1.9 give

$$x_{\Theta,j}\Psi_a(\lambda_\Theta(f)) = \frac{1}{2\pi i}\int_{\mathbb{R}^n}\frac{d}{ds}\Big|_{s=0}a_{js}(\xi)f(\xi)\lambda_\Theta(\xi)d\xi$$
$$+ \frac{1}{2\pi i}\int_{\mathbb{R}^n}\frac{d}{ds}\Big|_{s=0}\Big(a(\xi-se_j)f(\xi-se_j)e^{2\pi is\langle e_j,\Theta_\downarrow\xi\rangle}\Big)\lambda_\Theta(\xi)d\xi$$
$$= \frac{1}{2\pi i}\int_{\mathbb{R}^n}\Big\{\Big(\sum_{k=1}^n\Theta_{jk}\partial_\Theta^k a(\xi)\Big)f(\xi)+2\pi iD_{\Theta,j}^\ell(af)(\xi)\Big\}\lambda_\Theta(\xi)d\xi.$$

Since the new terms $\partial_\Theta^k a\in S_{\rho,\delta}^{m+\delta}(\mathcal{R}_\Theta)$, we may proceed as above once more. $\qquad\square$

Consider a pair of symbols $a_1,a_2:\mathbb{R}^n\to\mathcal{R}_\Theta$. In order to properly identify Ψ_{a_j} with a_j, we need to confirm that $\Psi_{a_1}=\Psi_{a_2}$ implies that $a_1=a_2$. This is the case when the symbols a_j are of polynomial growth —there exists $k\geq 0$ such that $|a_j(\xi)|\leq C_j\langle\xi\rangle^k$— and $\Psi_{a_1}=\Psi_{a_2}$ holds as operators in $\mathcal{B}(\mathcal{S}_\Theta,\mathcal{S}_\Theta)$. This result will be enough for our purposes and it follows by an elementary application of Fourier inversion for distributions, which we omit.

Lemma 3.2. *Given* $a,a_1,a_2\in\mathcal{S}(\mathbb{R}^n;\mathcal{R}_\Theta)$, *we find*:

    i) $\Psi_a^*=\Psi_{a_\dagger^*}$ *where*

$$a_\dagger(\xi)=\int_{\mathbb{R}^n}\widehat{a}(z,\xi-z)\lambda_\Theta(z)\,dz.$$

    ii) $\Psi_{a_1}\circ\Psi_{a_2}=\Psi_{a_1\diamond a_2}$ *where*

$$(a_1\diamond a_2)(\xi)=\int_{\mathbb{R}^n}a_1(z)\widehat{a}_2(z-\xi,\xi)\lambda_\Theta(z-\xi)\,dz.$$

Proof. By Lemma 2.2 i) $\Psi_a^*=T_{k_a}^*=T_{\mathrm{flip}(k_a)^*}$. By (3.1)

$$\mathrm{flip}(k_a)^* = \int_{\mathbb{R}^n}\lambda_\Theta(\xi)\otimes\lambda_\Theta(\xi)^*a(\xi)^*\,d\xi$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\overline{\widehat{a}(z,\xi)}\,e^{-2\pi i\langle z,\Theta_\downarrow\xi\rangle}\lambda_\Theta(\xi)\otimes\lambda_\Theta(z+\xi)^*\,dz\,d\xi$$
$$= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\widehat{a}(z,\xi-z)\lambda_\Theta(z)dz\Big)^*\lambda_\Theta(\xi)\otimes\lambda_\Theta(\xi)^*\,d\xi=k_{a_\dagger^*},$$

which implies $\Psi_a^* = \Psi_{a_\dagger^*}$. The composition formula is obtained similarly. $\qquad\square$

The formulas above are difficult to treat directly. Following the classical setting we introduce double pseudodifferential operators. Namely, if $A : \mathbb{R}^n \to \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ is a double symbol, its associated operator is given by

$$\mathcal{D}_A(\varphi) = (id \otimes \tau_\Theta)\Big\{ \Big(\int_{\mathbb{R}^n} A(\xi) \bullet \big(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \big) d\xi \Big) (\mathbf{1} \otimes \varphi) \Big\}.$$

Observe that if the double symbol is of the form $A(\xi) = a(\xi) \otimes \mathbf{1}$ then $\mathcal{D}_A = \Psi_a$. The advantage of the above class of operators is that they admit simpler expressions for adjoints $\mathcal{D}_{a \otimes \mathbf{1}}^* = \mathcal{D}_{\mathbf{1} \otimes a^*}$ and products $\mathcal{D}_{a_1 \otimes \mathbf{1}} \circ \mathcal{D}_{\mathbf{1} \otimes a_2} = \mathcal{D}_{a_1 \otimes a_2}$. We now introduce extended Hörmander classes for double symbols. To that end, we recall the definition of the Haagerup tensor product. Given $z \in \mathcal{R}_\Theta \otimes_{\mathrm{alg}} \mathcal{R}_\Theta$, let us define

$$\|z\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta} = \inf \Big\{ \Big\| \sum_j x_j x_j^* \Big\|_{\mathcal{R}_\Theta}^{\frac{1}{2}} \Big\| \sum_j y_j^* y_j \Big\|_{\mathcal{R}_\Theta}^{\frac{1}{2}} : z = \sum_j x_j \otimes y_j \Big\}.$$

The Haagerup tensor product $\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta$ is defined by completion and also admits a natural operator space structure [60]. We will say that $A : \mathbb{R}^n \to \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ belongs to $S_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$ when

$$\Big\| (\partial_\Theta^{\beta_1} \otimes \partial_\Theta^{\beta_2}) \partial_\xi^\alpha A(\xi) \Big\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta} \leq C_{\alpha,\beta_1,\beta_2} \langle \xi \rangle^{m - \rho|\alpha| + \delta_1|\beta_1| + \delta_2|\beta_2|},$$

for all multiindices α, β_1, β_2. Our next result provides a compression map

$$\mathfrak{B} : \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta) \to \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta),$$

$$\mathfrak{B}(A)(\xi) = (id \otimes \tau_\Theta) \int_{\mathbb{R}^n} A(\eta) \bullet (\lambda_\Theta(\eta) \otimes \lambda_\Theta(\eta)^*) \bullet (\lambda_\Theta(\xi)^* \otimes \lambda_\Theta(\xi)) \, d\eta,$$

which sends double symbols into symbols inducing the same operators. This map involves in turn the map $m : \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta \to \mathcal{S}_\Theta$ defined by linear extension of $\varphi_1 \otimes \varphi_2 \mapsto \varphi_1 \varphi_2$. If $\Theta = 0$, m is the restriction to the diagonal $\varphi(x, y) \mapsto \varphi(x, x)$ which extends to a positive preserving contraction with the C*-norm. This fails in general for nonabelian algebras. Instead, the Haagerup tensor product can be understood as the smallest (operator space) tensor product making the operation m continuous. This justifies our use of the Haagerup tensor product in the above definition of double Hörmander classes. Define

$$L_{\Theta,\xi} = \exp \Big(\frac{1}{2\pi i} \sum_{j=1}^n \partial_\xi^j \otimes id_{\mathcal{R}_\Theta} \otimes \partial_\Theta^j \Big) \in \mathcal{B}\big(\mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta) \big).$$

Theorem 3.3. *The compression map* $\mathfrak{B} : \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta \otimes_\pi \mathcal{S}_\Theta) \to \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta)$ *above satisfies the identity* $\mathcal{D}_A = \Psi_{\mathfrak{B}(A)}$ *as operators in* $\mathcal{B}(\mathcal{S}_\Theta, \mathcal{S}_\Theta')$. *In addition, the following identities hold*

$$\begin{aligned} \mathfrak{B}(A)(\xi) &= \mathcal{D}_A(\lambda_\Theta(\xi))\lambda_\Theta(\xi)^* \\ &= m\big(L_{\Theta,\xi} A(\xi) \big) \sim \sum_{\gamma \in \mathbb{Z}_+^n} \frac{m\big((\partial_\xi^\gamma \otimes id_{\mathcal{R}_\Theta} \otimes \partial_\Theta^\gamma) A(\xi) \big)}{(2\pi i)^{|\gamma|} \gamma!}. \end{aligned}$$

Moreover, given $A \in S_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$ *with* $\delta_2 < \rho$ *and* $\mathrm{N} \in \mathbb{Z}_+$ *large we find*

$$\Big\| \mathfrak{B}(A)(\xi) - \sum_{|\gamma| < \mathrm{N}} \frac{m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma) A(\xi) \big)}{(2\pi i)^{|\gamma|} \gamma!} \Big\|_{\mathcal{R}_\Theta} \leq C_\mathrm{N} \langle \xi \rangle^{m+n-(\rho-\delta_2)\mathrm{N}}.$$

In particular, $\mathfrak{B} : S^m_{\rho,\delta_1,\delta_2}(\mathcal{R}_\Theta) \to S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ for $\delta = \max\{\delta_1, \delta_2\}$ whenever $\delta_2 < \rho$.

Proof. The proof is divided into three blocks:

A. Expressions for $\mathfrak{B}(A)$. Clearly $\mathfrak{B}(A)(\xi) = \mathcal{D}_A(\lambda_\Theta(\xi))\lambda_\Theta(\xi)^*$, so

$$\mathcal{D}_A(\varphi) = \mathcal{D}_A\Big(\int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\lambda_\Theta(\xi)\,d\xi\Big) = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\mathfrak{B}(A)(\xi)\lambda_\Theta(\xi)\,d\xi = \Psi_{\mathfrak{B}(A)}(\varphi)$$

for any $\varphi \in \mathcal{S}_\Theta$. To prove the identity $\mathfrak{B}(A)(\xi) = m(L_{\Theta,\xi}A(\xi))$, we write

$$A(\xi) = \int_{\mathbb{R}^n} \widehat{A}(u)e^{2\pi i \langle u,\xi \rangle}\,du = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \widetilde{A}(u,v) \otimes \lambda_\Theta(v)\,dv\Big)e^{2\pi i \langle u,\xi \rangle}\,du$$

where $\widehat{A}$ is the Euclidean Fourier transform of $A : \mathbb{R}^n \to \mathcal{R}_\Theta \otimes \mathcal{R}_\Theta^{\mathrm{op}}$ and $\widetilde{A}$ is the quantum (partial) Fourier transform of it in the second tensor. In other words, we have

$$\begin{aligned} \widetilde{A}(u,v) &= (id \otimes \tau_\Theta)\Big(\widehat{A}(u) \bullet (\mathbf{1} \otimes \lambda_\Theta(v)^*)\Big) \\ &= \int_{\mathbb{R}^n} (id \otimes \tau_\Theta)\Big(A(s) \bullet (\mathbf{1} \otimes \lambda_\Theta(v)^*)\Big)e^{-2\pi i \langle u,s \rangle}\,ds. \end{aligned}$$

Now, using the Taylor series expansion

$$L_{\Theta,\xi} = \sum_{k=0}^{\infty} \frac{(2\pi i)^k}{k!} \sum_{j_1,j_2,\ldots,j_k=1}^{n} \Big(\prod_{s=1}^{k} \frac{\partial_\xi^{j_s}}{2\pi i}\Big) \otimes id \otimes \Big(\prod_{s=1}^{k} \frac{\partial_\Theta^{j_s}}{2\pi i}\Big)$$

we easily get the following identity for $L_{\Theta,\xi}A$

$$L_{\Theta,\xi}A = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \widetilde{A}(u,v) \otimes \lambda_\Theta(v)\,dv\Big)e^{2\pi i \langle u,\xi+v \rangle}\,du.$$

Applying m to this expression and writing $\widetilde{A}$ in terms of A, we get

$$\begin{aligned} m(L_{\Theta,\xi}A) &= (id \otimes \tau_\Theta)\int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} A(s) \bullet \pi_\Theta(\exp_v)\,e^{2\pi i \langle u,\xi+v-s \rangle}\,dsdudv \\ &= (id \otimes \tau_\Theta)\int_{\mathbb{R}^n \times \mathbb{R}^n} A(s) \bullet \pi_\Theta(\exp_v)\,\delta_\xi(s-v)\,dsdv \\ &= (id \otimes \tau_\Theta)\int_{\mathbb{R}^n} A(s) \bullet \pi_\Theta(\exp_{s-\xi})\,ds. \end{aligned}$$

This proves $\mathfrak{B}(A) = m(L_{\Theta,\xi}A)$. On the other hand

$$L_{\Theta,\xi} = \sum_{\gamma \in \mathbb{Z}_+^n} \frac{\partial_\xi^\gamma \otimes id_{\mathcal{R}_\Theta} \otimes \partial_\Theta^\gamma}{(2\pi i)^{|\gamma|}\gamma!}$$

by standard modification of the Taylor series. This gives the formal series expansion.

B. Estimate for the remainder. Thus, our next goal is to justify the Taylor remainder estimate in the statement. This requires yet another expression for $\mathfrak{B}(A)$. We begin by noticing that

$$\begin{aligned} \mathfrak{B}(A)(\xi) &= m\Big\{(id \otimes \tau_\Theta)\Big(\int_{\mathbb{R}^n} A(\eta) \bullet \pi_\Theta(\exp_{\eta-\xi})\,d\eta\Big) \otimes \mathbf{1}\Big\} \\ &= m\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (id \otimes \sigma_\Theta^z)\big(A(\eta) \bullet \pi_\Theta(\exp_{\eta-\xi})\big)\,d\eta dz\Big\} \end{aligned}$$

$$= m\Big\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (id \otimes \sigma^z_\Theta)\big(A(\eta)\big) \bullet \pi_\Theta(\exp_{\eta-\xi})e^{-2\pi i\langle z,\eta-\xi\rangle}\, d\eta dz \Big\}.$$

The first identity follows from **A** above. The second identity reduces to

$$\int_{\mathbb{R}^n} \sigma^z_\Theta(\varphi)\, dz = \tau_\Theta(\varphi)\, \mathbf{1} \quad \text{with} \quad \sigma^z_\Theta(\lambda_\Theta(\zeta)) = e^{2\pi i\langle z,\zeta\rangle}\lambda_\Theta(\zeta)$$

and the last one since σ^z_Θ is a $*$-homomorphism. Using $m(\mathsf{A} \bullet \pi_\Theta(\exp_\zeta)) = m(\mathsf{A})$ we get

$$(3.2) \qquad \mathfrak{B}(A)(\xi) = \int_{\mathbb{R}^n} \underbrace{\Big(\int_{\mathbb{R}^n} m\big((id \otimes \sigma^z_\Theta)A(\xi+\eta)\big)e^{-2\pi i\langle z,\eta\rangle}\, dz \Big)}_{\Omega_\eta(\xi+\eta)}\, d\eta.$$

On the other hand, we use $\partial_y^\gamma \sigma^y_\Theta = \sigma^y_\Theta \partial_\Theta^\gamma$ to deduce

$$m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma)A(\xi)\big)$$

$$= \partial_\xi^\gamma \partial_y^\gamma\Big|_{y=0} m\big((id \otimes \sigma^y_\Theta)A(\xi)\big)$$

$$= \partial_\xi^\gamma \partial_y^\gamma\Big|_{y=0} \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m\big((id \otimes \sigma^z_\Theta)A(\xi)\big)e^{2\pi i\langle y-z,\eta\rangle}\, dz d\eta \Big)$$

$$= \partial_\xi^\gamma \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m\big((id \otimes \sigma^z_\Theta)A(\xi)\big)e^{-2\pi i\langle z,\eta\rangle}(2\pi i\eta)^\gamma\, dz d\eta \Big)$$

$$= \partial_\xi^\gamma \int_{\mathbb{R}^n} \Omega_\eta(\xi)(2\pi i\eta)^\gamma\, d\eta.$$

This implies that

$$\mathfrak{B}(A)(\xi) - \sum_{|\gamma|<\mathrm{N}} \frac{m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma)A(\xi)\big)}{(2\pi i)^{|\gamma|}\gamma!} = \int_{\mathbb{R}^n} \underbrace{\Omega_\eta(\xi+\eta) - \sum_{|\gamma|<\mathrm{N}} \frac{1}{\gamma!}\partial_\xi^\gamma \Omega_\eta(\xi)\eta^\gamma}_{\mathrm{R}_\xi(\eta)}\, d\eta.$$

By Taylor remainder formula

$$\mathrm{R}_\xi(\eta) = \sum_{|\gamma|=\mathrm{N}} \frac{\mathrm{N}}{\gamma!}\Big(\int_0^1 (1-t)^{\mathrm{N}-1}\partial_s^\gamma\Big|_{s=\eta+t\xi}\Omega_\eta(s)\, dt \Big)\eta^\gamma.$$

In particular, we obtain the following estimate

$$\Big\| \int_{\mathbb{R}^n} \mathrm{R}_\xi(\eta)\, d\eta \Big\|_{\mathcal{R}_\Theta} \le C_\mathrm{N} \sup_{\substack{|\gamma|=\mathrm{N}\\0\le t\le 1}} \Big\| \int_{\mathbb{R}^n} \eta^\gamma \partial_s^\gamma \Omega_\eta(\eta+t\xi)\, d\eta \Big\|_{\mathcal{R}_\Theta}.$$

Since ∂_ξ^γ commutes with m, we get the identity

$$\partial_s^\gamma \Omega_\eta(s) = \int_{\mathbb{R}^n} m\big((id \otimes \sigma^z_\Theta)\partial_s^\gamma A(s)\big)e^{-2\pi i\langle z,\eta\rangle}\, dz.$$

Next, we use the standard oscillatory integral trick

$$e^{-2\pi i\langle z,\eta\rangle} = \frac{(-\Delta_\eta)^n}{(4\pi^2|z|^2)^n}e^{-2\pi i\langle z,\eta\rangle},$$

$$e^{-2\pi i\langle z,\eta\rangle} = \frac{(\mathbf{1}-\Delta_z)^{\frac{\mathrm{N}}{2}}}{(1+4\pi^2|\eta|^2)^{\frac{\mathrm{N}}{2}}}e^{-2\pi i\langle z,\eta\rangle}.$$

Taking $M(\xi, \eta, z, t) = m\big((id \otimes \sigma_\Theta^z)\partial_s^\gamma A(\eta + t\xi)\big)$ and integrating by parts

$$\int_{\mathbb{R}^n} \eta^\gamma \partial_s^\gamma \Omega_\eta(\eta + t\xi)\, d\eta$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\eta^\gamma(\mathbf{1} - \Delta_z)^{\frac{N}{2}}}{(1 + 4\pi^2|\eta|^2)^{\frac{N}{2}}}\Big(M(\xi,\eta,z,t)\Big)e^{-2\pi i\langle z,\eta\rangle}\, dz d\eta$$

$$= \int_{\mathrm{B}_1(0)}\Big(\int_{\mathbb{R}^n} \frac{\eta^\gamma(\mathbf{1} - \Delta_z)^{\frac{N}{2}}}{(1 + 4\pi^2|\eta|^2)^{\frac{N}{2}}}\Big(M(\xi,\eta,z,t)\Big)e^{-2\pi i\langle z,\eta\rangle}\, d\eta\Big)\, dz$$

$$+ \int_{\mathrm{B}_1^c(0)}\Big(\int_{\mathbb{R}^n} \frac{(-\Delta_\eta)^n}{(4\pi^2|z|^2)^n}\Big[\frac{\eta^\gamma(\mathbf{1} - \Delta_z)^{\frac{N}{2}}}{(1 + 4\pi^2|\eta|^2)^{\frac{N}{2}}}\Big(M(\xi,\eta,z,t)\Big)\Big]e^{-2\pi i\langle z,\eta\rangle}\, d\eta\Big)\, dz.$$

Let us write Π_1 and Π_2 for the two terms in the right hand side. Then, we use one more time the identity $\partial_z^\gamma \sigma_\Theta^z = \sigma_\Theta^z \partial_\Theta^\gamma$ together with the contractivity of the map $m : \mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta \to \mathcal{R}_\Theta$ and the $S_{\rho,\delta_1,\delta_2}^m$– condition. This yields the following inequality for any $|\gamma| = \mathrm{N}$ and $0 \le t \le 1$

$$\|\Pi_1\|_{\mathcal{R}_\Theta} \lesssim \int_{\mathbb{R}^n} \Big\|\big(id \otimes (\mathbf{1} - \Delta_\Theta)^{\frac{N}{2}}\big)\partial_s^\gamma A(\eta + t\xi)\Big\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta}\, d\eta$$

$$\lesssim \int_{\mathbb{R}^n} \max\big\{\langle\xi\rangle, \langle\eta\rangle\big\}^{m - (\rho - \delta_2)\mathrm{N}}\, d\eta \lesssim \langle\xi\rangle^{m + n - (\rho - \delta_2)\mathrm{N}}.$$

Similarly, Π_2 is dominated by

$$\sum_{|\nu_1 + \nu_2| = 2n} \int_{\mathrm{B}_1^c(0)\times\mathbb{R}^n} \Big|\partial_\eta^{\nu_1}\Big(\frac{\eta^\gamma}{\langle\eta\rangle^{\mathrm{N}}}\Big)\Big| \Big\|\big(id \otimes (\mathbf{1} - \Delta_\Theta)^{\frac{N}{2}}\big)\partial_s^{\gamma + \nu_2} A(\eta + t\xi)\Big\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta} \frac{dz d\eta}{|z|^{2n}}$$

which is bounded by $\langle\xi\rangle^{m + n - (\rho - \delta_2)\mathrm{N}}$. This completes the estimate of the remainder.

C. $\mathfrak{B}$ respects the Hörmander classes. It remains to show that $\mathfrak{B}(A)$ belongs to the Hörmander class $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ for $\delta = \max\{\delta_1, \delta_2\}$ whenever $A \in S_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$ and $\delta_2 < \rho$. Since we have

$$\partial_\Theta^\beta \circ m = \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1!\beta_2!}\, m \circ \big(\partial_\Theta^{\beta_1} \otimes \partial_\Theta^{\beta_2}\big),$$

it turns out that the following inequality holds for any $\gamma \in \mathbb{Z}_+^n$

$$\Big\|\partial_\Theta^\beta \partial_\xi^\alpha m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma)A(\xi)\big)\Big\|_{\mathcal{R}_\Theta}$$

$$\le \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1!\beta_2!}\Big\|\big(\partial_\Theta^{\beta_1} \otimes \partial_\Theta^{\gamma + \beta_2}\big)\partial_\xi^{\gamma + \alpha} A(\xi)\Big\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta}.$$

Since the Hörmander classes are nested in the degree m, this implies that

$$\sum_{|\gamma| < \mathrm{N}} m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma)A(\xi)\big) \in \bigcup_{|\gamma| < \mathrm{N}} S_{\rho,\delta}^{m - (\rho - \delta_2)|\gamma|}(\mathcal{R}_\Theta) = S_{\rho,\delta}^m(\mathcal{R}_\Theta)$$

as a consequence of $A \in S_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$, $\delta = \max\{\delta_1, \delta_2\}$ and $\delta_2 < \rho$. Therefore, the inclusion $\mathfrak{B}(A) \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ will follow if there exists a large enough $\mathrm{N} \in \mathbb{Z}_+$ satisfying the inequality

$$\Big\|\partial_\Theta^\beta \partial_\xi^\alpha\Big(\mathfrak{B}(A)(\xi) - \sum_{|\gamma| < \mathrm{N}} m\big((\partial_\xi^\gamma \otimes id \otimes \partial_\Theta^\gamma)A(\xi)\big)\Big)\Big\|_{\mathcal{R}_\Theta} \le C_{\mathrm{N},\alpha,\beta}\langle\xi\rangle^{m - \rho|\alpha| + \delta|\beta|}.$$

Our estimate for the Taylor remainder above shows that this is indeed the case when $\alpha = \beta = 0$. Using $\partial_\xi^\alpha m = m \partial_\Theta^\alpha$ and the commutation formula for $\partial_\Theta^\beta \circ m$ given above, the exact same argument applies for general α, β. This gives that any $N \geq n/(\rho - \delta_2)$ works, details are left to the reader. $\qquad\square$

Corollary 3.4. *The following stability results hold*:

   i) *If $a \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ and $\rho < \delta$, then $\Psi_a^* = \Psi_{a_\dagger^*}$ with*

$$a_\dagger^* \sim \sum_{\gamma \in \mathbb{Z}_+^n} \frac{\partial_\Theta^\gamma \partial_\xi^\gamma a^*(\xi)}{(2\pi i)^{|\gamma|}\gamma!} \in S_{\rho,\delta}^m(\mathcal{R}_\Theta).$$

   ii) *If $a_j \in S_{\rho_j,\delta_j}^{m_j}$, then $\Psi_{a_1} \circ \Psi_{a_2} = \Psi_{a_1 \diamond a_2}$ with*

$$a_1 \diamond a_2 \sim \sum_{\gamma \in \mathbb{Z}_+^n} \frac{\partial_\xi^\gamma a_1(\xi)\partial_\Theta^\gamma a_2(\xi)}{(2\pi i)^{|\gamma|}\gamma!} \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$$

   for $m = m_1 + m_2$, $\rho = \min\{\rho_1,\rho_2\}$ and $\delta = \max\{\delta_1,\delta_2\}$ when $\delta_2 < \rho$.

Proof. Recall that
$$\Psi_a^* = \mathcal{D}_{a\otimes \mathbf{1}}^* = \mathcal{D}_{\mathbf{1}\otimes a^*} = \Psi_{\mathfrak{B}(\mathbf{1}\otimes a^*)}.$$
If $a \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ then $a \otimes \mathbf{1} \in S_{\rho,\delta,0}^m(\mathcal{R}_\Theta)$ and $\mathbf{1} \otimes a^* \in S_{\rho,0,\delta}^m(\mathcal{R}_\Theta)$. By Theorem 3.3 we have that $a_\dagger^* = \mathfrak{B}(\mathbf{1} \otimes a^*) \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$. The second assertion follows similarly by recalling that

$$\begin{aligned}
\Psi_{a_1 \diamond a_2} &= \Psi_{a_1} \circ \Psi_{a_2}^{**} = \Psi_{a_1} \circ \Psi_{a_{2\dagger}^*}^* \\
&= \mathcal{D}_{a_1 \otimes \mathbf{1}} \circ \mathcal{D}_{\mathbf{1}\otimes a_{2\dagger}} = \mathcal{D}_{a_1 \otimes a_{2\dagger}} = \Psi_{\mathfrak{B}(a_1 \otimes a_{2\dagger})}.
\end{aligned}$$

Indeed, according to the first assertion, we know that $a_1 \otimes a_{2\dagger} \in S_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$. The asymptotic expansions also follow easily from Theorem 3.3 using the identities $a_\dagger^* = \mathfrak{B}(\mathbf{1} \otimes a^*)$ and $a_1 \diamond a_2 = \mathfrak{B}(a_1 \otimes a_{2\dagger})$, see e.g. [69] for a similar approach. $\quad\square$

Remark 3.5. A natural question is whether the classes $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ are closed under products and adjoints for $\delta < \rho$. This question is still open. Indeed, proceeding as for $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ we may define a new class $\Sigma_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta)$ of mixed double symbols $A : \mathbb{R}^n \to \mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta$ satisfying the condition

$$\left\| (\partial_\Theta^{\beta_1} \otimes \partial_\Theta^{\beta_2}) \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} A(\xi) \right\|_{\mathcal{R}_\Theta \otimes_h \mathcal{R}_\Theta} \leq C_{\alpha_1,\alpha_2,\beta_1,\beta_2} \langle\xi\rangle^{m - \rho|\alpha_1+\alpha_2|+\delta_1|\beta_1|+\delta_2|\beta_2|},$$

where, abusing of notation, $\partial_{\Theta,\xi}^j$ acts on $\mathcal{S}(\mathbb{R}^n; S(\mathcal{R}_\Theta) \otimes_\pi \mathcal{S}(\mathcal{R}_\Theta))$ as follows

$$\begin{aligned}
(\partial_{\Theta,\xi}^j A)(\xi) &= \partial_\xi^j A(\xi) + 2\pi i \big[A(\xi), d_{\Theta,j}\big] \\
&= \pi_\Theta(\exp_\xi)^* \bullet \partial_\xi^j \big\{\pi_\Theta(\exp_\xi) \bullet A(\xi) \bullet \pi_\Theta(\exp_\xi)^*\big\} \bullet \pi_\Theta(\exp_\xi).
\end{aligned}$$

The operator $d_{\Theta,j}$ is just $x_{\Theta,j} \otimes \mathbf{1} - \mathbf{1} \otimes x_{\Theta,j}$. We shall identify the first term with $x_{\Theta,j}$ and the second with $y_{\Theta,j}$. Of course, we expect that our contraction map satisfies $\mathfrak{B} : \Sigma_{\rho,\delta_1,\delta_2}^m(\mathcal{R}_\Theta) \to \Sigma_{\rho,\delta_1 \vee \delta_2}^m(\mathcal{R}_\Theta)$ for $\delta_2 < \rho$. Unfortunately, our argument above does not admit a direct generalization. The problem arises since the automorphism σ_Θ in the oscillatory integral (3.2) for $\mathfrak{B}$ does not commute with $\partial_{\Theta,\xi}^j$. We refer to Lemmas 4.1 and 4.2 and Remark 4.4 for the calculus of parametrices in this setting. On the other hand, a minimum stability for products —necessary for our Sobolev

p-estimates, see the proof of Corollary 3.26— does hold. Namely, if $a_1 \in \Sigma_{\rho_1,0}^{m_1}(\mathcal{R}_\Theta)$ takes values in $\mathbb{C}\mathbf{1}$ or, more generally, in the center of $\mathcal{R}_\Theta$, we have that

$$a_1 \diamond a_2 \in \Sigma_{\rho_1 \wedge \rho_2,\delta}^{m_1+m_2}(\mathcal{R}_\Theta) \quad \text{whenever} \quad a_2 \in \Sigma_{\rho_2,\delta}^{m_2}(\mathcal{R}_\Theta).$$

In particular, composition with polynomials of ∂_Θ^j's transforms degrees as expected.

3.2. L_2-boundedness: Sufficient conditions.

We now explore L_2-boundedness of pseudodifferential operators in $S_{\rho,\delta}^0(\mathcal{R}_\Theta)$. Since $S_{\rho,\delta}^0(\mathcal{R}_\Theta) \subset S_{\delta,\delta}^0(\mathcal{R}_\Theta) \cap S_{\rho,\rho}^0(\mathcal{R}_\Theta)$ it suffices to study L_2-boundedness for exotic $0 \le \delta = \rho < 1$ and forbidden $\delta = \rho = 1$ symbols. The first case $\rho < 1$ requires a quantum analogue of the celebrated Calderón-Vaillancourt theorem [9]. The second one also requires an additional assumption extending Bourdaud's condition [7], which can be regarded as a form of the $T(1)$ theorem for pseudodifferential operators.

3.2.1. *The Calderón-Vaillancourt theorem in $\mathcal{R}_\Theta$.* As in the Euclidean setting, the hardest part in proving a quantum form of Calderón-Vaillancourt theorem is still the case $\rho = 0$. Our argument follows from a combination of [24, 69] adapted to $\mathcal{R}_\Theta$ which demands a careful argument due to the presence of a Θ-phase. Given $a \in S_{0,0}^0(\mathcal{R}_\Theta)$, the first step consists in decomposing the symbol as follows. The double Fourier transform of a in the quantum and classical variables (x_Θ, ξ) is given by

$$
\begin{aligned}
\widehat{\widehat{a}}(z,\zeta) &= \int_{\mathbb{R}^n} \tau_\Theta\big(a(\xi)\lambda_\Theta(z)^*\big)e^{-2\pi i\langle\xi,\zeta\rangle}d\xi \\
&= \underbrace{\big(1+4\pi^2|z|^2\big)^{\mathrm{N}}\big(1+4\pi^2|\zeta|^2\big)^{\mathrm{N}}\widehat{\widehat{a}}(z,\zeta)}_{\widehat{\widehat{b}}(z,\zeta)} \underbrace{\big(1+4\pi^2|z|^2\big)^{-\mathrm{N}}\big(1+4\pi^2|\zeta|^2\big)^{-\mathrm{N}}}_{\widehat{\widehat{g}}(z,\zeta)}.
\end{aligned}
$$

Here we fix N large enough. We shall also use the terminology

$$\widehat{a}(z,\xi) = \tau_\Theta\big(a(\xi)\lambda_\Theta(z)^*\big) = \int_{\mathbb{R}^n} \widehat{\widehat{a}}(z,\zeta)e^{2\pi i\langle\xi,\zeta\rangle}d\zeta$$

for a, b and g. In order to express Ψ_a in terms of b and g we need to introduce two auxiliary maps. The first one is a left-module extension $\Pi_\Theta : \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}} \to \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ of the $*$-homomorphism π_Θ defined as follows

$$\Pi_\Theta\big(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)\big) = \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*\lambda_\Theta(\eta) = \big(\mathbf{1} \otimes \lambda_\Theta(\eta)\big) \bullet \pi_\Theta(\exp_\xi).$$

$\Pi_\Theta^*\big(\lambda_\Theta(\xi) \otimes \lambda_\Theta(\eta)\big) = \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)\lambda_\Theta(\eta) = (\mathbf{1} \otimes \lambda_\Theta(\eta)) \bullet (\lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi))$ gives the adjoint with respect to the module bracket $\langle\langle\alpha,\beta\rangle\rangle = (\tau_\Theta \otimes id)(\alpha \bullet \beta^*)$. The second one is the left-modulation map $M_\eta(\varphi_1 \otimes \varphi_2) = \lambda_\Theta(\eta)\varphi_1 \otimes \varphi_2$ with adjoint $M_\eta^*(\varphi_1 \otimes \varphi_2) = \lambda_\Theta(\eta)^*\varphi_1 \otimes \varphi_2$ with respect to the same bracket above. In the next result we shall use the following symbol

$$
\begin{aligned}
g_\eta(\xi) &= \int_{\mathbb{R}^n} \widehat{g}_\eta(z,\xi)\lambda_\Theta(z)\,dz \\
&= \int_{\mathbb{R}^n} \widehat{g}(z,\xi)e^{-2\pi i\langle\xi,\Theta_\downarrow z\rangle}e^{2\pi i\langle\Theta\eta,z\rangle}\lambda_\Theta(z)\,dz.
\end{aligned}
$$

Lemma 3.6. *If $\Phi_\eta = \Pi_\Theta^* \circ M_\eta^*$, the following identity holds*

$$\Psi_a(\varphi) = (id \otimes \tau_\Theta)\int_{\mathbb{R}^n} (\mathbf{1} \otimes b(\eta))\Big(\Phi_\eta^* \circ \big(\Psi_{g_\eta} \otimes id\big) \circ \Phi_\eta\Big)(\varphi \otimes \mathbf{1})\,d\eta.$$

Proof. We first claim that

$$a(\xi) = (id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (\mathbf{1} \otimes b(\eta)) \Big(\underbrace{\int_{\mathbb{R}^n} \widehat{g}(z, \xi - \eta) \pi_\Theta(\exp_z) \, dz}_{\Gamma(\xi - \eta)} \Big) \, d\eta.$$

Indeed, writing the symbol a in terms of b and g we obtain

$$\begin{aligned} a(\xi) &=& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{\widehat{b}}(z, \zeta) \widehat{g}(z, \zeta) e^{2\pi i \langle \xi, \zeta \rangle} \lambda_\Theta(z) \, dz d\zeta \\ &=& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{b}(z, \eta) \widehat{g}(z, \xi - \eta) \lambda_\Theta(z) \, dz d\eta. \end{aligned}$$

Now the claim follows from the quantum form of convolution via the identity

$$\int_{\mathbb{R}^n} f_1(z) f_2(z) \lambda_\Theta(z) \, dz = (id \otimes \tau_\Theta) \Big\{ \Big(\mathbf{1} \otimes \lambda_\Theta(f_1) \Big) \Big(\int_{\mathbb{R}^n} f_2(z) \pi_\Theta(\exp_z) \, dz \Big) \Big\}.$$

Next we use the claim to produce an expression for $\Psi_a(\varphi)$. Namely, we have

$$\Psi_a(\varphi) = (id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (a(\xi) \otimes \varphi) \pi_\Theta(\exp_\xi) \, d\xi$$

$$= (id \otimes \tau_\Theta \otimes \tau_\Theta) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[(\mathbf{1} \otimes b(\eta)) \Gamma(\xi - \eta) \otimes \varphi \Big] \pi_\Theta(\exp_\xi)_{[13]} \, d\xi d\eta$$

$$= (id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (\mathbf{1} \otimes b(\eta)) \Big\{ (id \otimes id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (\Gamma(\xi - \eta) \otimes \varphi) \pi_\Theta(\exp_\xi)_{[13]} \, d\xi \Big\} \, d\eta$$

with $(a \otimes b)_{[13]} = a \otimes \mathbf{1} \otimes b$. The assertion reduces to prove the following identity

$$A := (id \otimes id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (\Gamma(\xi - \eta) \otimes \varphi) \pi_\Theta(\exp_\xi)_{[13]} \, d\xi = \Big(\Phi_\eta^* \circ (\Psi_{g_\eta} \otimes id) \circ \Phi_\eta \Big)(\varphi \otimes \mathbf{1}) =: B.$$

Expanding $\Gamma(\xi - \eta)$ it is clear that

$$A = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) \widehat{g}(z, \xi - \eta) \lambda_\Theta(z) \lambda_\Theta(\xi) \otimes \lambda_\Theta(z)^* \, dz d\xi.$$

On the other hand, we have the identity

$$B = (\lambda_\Theta(\eta) \otimes \mathbf{1}) \Pi_\Theta \Big\{ \int_{\mathbb{R}^n} \Big(g_\eta(\xi) \otimes \underbrace{(\tau_\Theta \otimes id) (\Phi_\eta(\varphi \otimes \mathbf{1}) (\lambda_\Theta(\xi)^* \otimes \mathbf{1}))}_{\beta_\eta(\xi)} \Big) (\lambda_\Theta(\xi) \otimes \mathbf{1}) \, d\xi \Big\}$$

where it is easily checked that

$$\begin{aligned} \Phi_\eta(\varphi \otimes \mathbf{1}) &=& \Pi_\Theta^* \Big(\int_{\mathbb{R}^n} \widehat{\varphi}(s) \lambda_\Theta(\eta)^* \lambda_\Theta(s) \otimes \mathbf{1} \, ds \Big) \\ &=& \int_{\mathbb{R}^n} \widehat{\varphi}(s) e^{2\pi i \langle \eta, \Theta_\downarrow(\eta - s) \rangle} \lambda_\Theta(s - \eta) \otimes \lambda_\Theta(s - \eta) \, ds, \end{aligned}$$

so that $\beta_\eta(\xi) = \widehat{\varphi}(\xi + \eta) e^{-2\pi i \langle \eta, \Theta_\downarrow \xi \rangle} \lambda_\Theta(\xi)$. This yields

$$B = (\lambda_\Theta(\eta) \otimes \mathbf{1}) \Pi_\Theta \Big\{ \int_{\mathbb{R}^n} \Big(g_\eta(\xi) \otimes \beta_\eta(\xi) \Big) (\lambda_\Theta(\xi) \otimes \mathbf{1}) \, d\xi \Big\}$$

$$= (\lambda_\Theta(\eta) \otimes \mathbf{1}) \Pi_\Theta \Big\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big(\widehat{g}_\eta(z, \xi) \lambda_\Theta(z) \otimes \beta_\eta(\xi) \Big) (\lambda_\Theta(\xi) \otimes \mathbf{1}) \, dz d\xi \Big\}$$

$$= (\lambda_\Theta(\eta) \otimes \mathbf{1}) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i \langle z, \Theta_\downarrow \xi \rangle} \widehat{g}_\eta(z, \xi) \lambda_\Theta(z + \xi) \otimes \lambda_\Theta(z + \xi)^* \beta_\eta(\xi) \, dz d\xi$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi + \eta) e^{2\pi i \langle z - \eta, \Theta_\downarrow \xi \rangle} \widehat{g}_\eta(z, \xi) \lambda_\Theta(\eta) \lambda_\Theta(z + \xi) \otimes \lambda_\Theta(z + \xi)^* \lambda_\Theta(\xi) \, dz d\xi.$$

Rearranging and using $\widehat{g}_\eta(z, \xi) = \widehat{g}(z, \xi) e^{-2\pi i \langle \xi, \Theta_\downarrow z \rangle} e^{2\pi i \langle \Theta \eta, z \rangle}$ yields A = B. $\qquad \square$

Remark 3.7. The above lemma may be regarded as the quantum analogue of the identity in [69, Lemma XIII.1.1], whose Euclidean proof is trivial. Unfortunately the quantum analogue gives an extra Θ-phase which vanishes for $\Theta = 0$. It is this phase what forces us to be very careful in adapting Cordes argument [24] below.

Lemma 3.8. Ψ_{g_η} *admits the factorization*

$$\Psi_{g_\eta} = A_\eta^* \circ B \circ A_\eta \quad \text{with} \quad \sup_{\eta \in \mathbb{R}^n} \big\| A_\eta : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \big\| \, \big\| B \big\|_{S_1(L_2(\mathcal{R}_\Theta))} < \infty.$$

Proof. Let $w_\eta(z, \xi) = e^{-2\pi i \langle \xi, \Theta_\downarrow z \rangle} e^{2\pi i \langle \Theta \eta, z \rangle}$, so that

$$
\begin{aligned}
\Psi_{g_\eta}(\varphi) &= \int_{\mathbb{R}^n} g_\eta(\xi) \widehat{\varphi}(\xi) \lambda_\Theta(\xi) \, d\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{g}(z, \xi) w_\eta(z, \xi) \widehat{\varphi}(\xi) \lambda_\Theta(z) \lambda_\Theta(\xi) \, dz d\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{\widehat{g}}(z, \zeta) w_\eta(z, \xi) e^{2\pi i \langle \xi, \zeta \rangle} \widehat{\varphi}(\xi) \lambda_\Theta(z) \lambda_\Theta(\xi) \, dz d\zeta d\xi.
\end{aligned}
$$

Let us define $j_{\xi\eta} : \mathbb{R}^n \to \mathbb{C}$ and $m_N : \mathbb{R}^n \to \mathbb{C}$ as follows

$$
\begin{aligned}
j_{\xi\eta}(z) &= \frac{w_\eta(z, \xi)}{(1 + 4\pi^2 |z|^2)^N}, \\
m_N(\xi) &= \int_{\mathbb{R}^n} \frac{e^{2\pi i \langle \xi, \zeta \rangle}}{(1 + 4\pi^2 |\zeta|^2)^N} \, d\zeta = \widehat{j}_{00}(\xi),
\end{aligned}
$$

where $\widehat{j}_{00}$ stands for the Euclidean Fourier transform of $j_{\xi\eta}$ when $(\xi, \eta) = (0, 0)$. Inserting our definition of $\widehat{\widehat{g}}(z, \zeta)$, we finally end up with the following factorization

$$
\begin{aligned}
\Psi_{g_\eta}(\varphi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} j_{\xi\eta}(z) m_N(\xi) \widehat{\varphi}(\xi) \lambda_\Theta(z) \lambda_\Theta(\xi) \, dz d\xi \\
&= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} j_{\xi\eta}(z) m_N(\xi - z) \widehat{\varphi}(\xi - z) e^{2\pi i \langle z, \Theta_\downarrow(\xi - z) \rangle} \, dz \Big) \lambda_\Theta(\xi) \, d\xi \\
&= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \underbrace{\big(j_{\xi\eta}(\xi - z) e^{2\pi i \langle \xi - z, \Theta_\downarrow z \rangle} \big) m_N(z)}_{k_\eta(\xi, z)} \widehat{\varphi}(z) \, dz \Big) \lambda_\Theta(\xi) \, d\xi.
\end{aligned}
$$

This gives $\Psi_{g_\eta} = \lambda_\Theta \circ T_{k_\eta} \circ \lambda_\Theta^{-1}$, which reduces our goal to justify the assertion for T_{k_η} instead of Ψ_{g_η}. Indeed, assume $T_{k_\eta} = \mathsf{A}_\eta^* \circ \mathsf{B} \circ \mathsf{A}_\eta$ with A_η uniformly bounded in $\mathcal{B}(L_2(\mathbb{R}^n))$ and B a trace class operator on the Hilbert space $L_2(\mathbb{R}^n)$. Then we consider the maps

$$A_\eta = \lambda_\Theta \circ \mathsf{A}_\eta \circ \lambda_\Theta^{-1} \quad \text{and} \quad B = \lambda_\Theta \circ \mathsf{B} \circ \lambda_\Theta^{-1},$$

which factorize Ψ_{g_η} and satisfy

$$
\begin{aligned}
\|A_\eta\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} &= \|\mathsf{A}_\eta\|_{\mathcal{B}(L_2(\mathbb{R}^n))}, \\
\|B\|_{S_1(L_2(\mathcal{R}_\Theta))} &= \|\mathsf{B}\|_{S_1(L_2(\mathbb{R}^n))}.
\end{aligned}
$$

The kernel k_η can be written as follows

$$k_\eta(x, y) = e^{2\pi i \langle x - y, \Theta \eta + \Theta_\downarrow y + \Theta_\uparrow x \rangle} j_{00}(x - y) \, m_N(y) = e^{2\pi i \langle x - y, \Theta \eta \rangle} k(x, y).$$

If $\mathsf{A}_\eta f(x) = e^{-2\pi i \langle x, \Theta\eta \rangle} f(x)$, we see that $T_{k_\eta} = \mathsf{A}_\eta^* \circ \mathsf{B} \circ \mathsf{A}_\eta$ with $\mathsf{B} = T_k$ and A_η unitaries. Thus, it suffices to show that B is trace class on $L_2(\mathbb{R}^n)$. Composing it with the Euclidean Fourier transform $\mathcal{F} = \lambda_0^{-1}$ as in the proof of [24, Lemma 1] we end up with $L = \mathcal{F} \circ T_k$, whose kernel is given by

$$\ell(x, y) = e^{-2\pi i \langle x, y \rangle} \widehat{\alpha}(x - \Theta y) m_{\mathrm{N}}(y),$$

where $\widehat{\alpha}$ is the Euclidean Fourier transform of $\alpha(z) = j_{00}(z) e^{-2\pi i \langle z, \Theta_\downarrow z \rangle} = j_{z0}(z)$. This is very similar to the kernel in [24, Lemma 1 - (1.25)], in fact we recover the same kernel for $\Theta = 0$. Unfortunately, due to the Θ-phase we are carrying, we do not have separated variables as in [24]. However, a detailed analysis of Cordes argument shows that what really matters is that the x-factor of the kernel —$\psi_\tau(x)$ in [24]— yields a pointwise multiplier by m_{N}. We only have that in the y-variable. Taking the adjoint $L^* = T_{k_\eta}^* \circ \mathcal{F}^{-1}$ we get the kernel

$$\ell^*(x, y) = \overline{\ell(y, x)} = e^{2\pi i \langle x, y \rangle} \overline{m_{\mathrm{N}}(x) \widehat{\alpha}(y - \Theta x)}.$$

Then, Cordes factorization $m_{\mathrm{N}}(x) = \zeta(x) \kappa(x)$ with $\zeta(x) = \exp(-\frac{1}{2}\langle x \rangle)$ implies in turn that $L^* = R \circ S$ where their respective kernels $r(x, z)$ and $s(z, y)$ are given by

$$r(x, z) \;=\; \frac{\overline{\kappa(x)}}{(1 + 4\pi^2 |z|^2)^{\mathrm{M}}} \int_{\mathbb{R}^n} \frac{e^{2\pi i \langle x - z, s \rangle}}{(1 + 4\pi^2 |s|^2)^{\mathrm{M}}} \, ds,$$

$$s(z, y) \;=\; (1 + 4\pi^2 |z|^2)^{\mathrm{M}} (\mathbf{1} - \Delta_z)^{\mathrm{M}} \Big(\zeta(z) e^{2\pi i \langle z, y \rangle} \overline{\widehat{\alpha}(y - \Theta z)} \Big).$$

By [24], R is Hilbert-Schmidt for M large enough. Since $\widehat{\alpha}$ is as smooth as $\widehat{j_{00}}$, it is $\mathcal{C}^k(\mathbb{R}^n)$ for $\mathrm{N} > \frac{1}{2}(n + k)$ and exponentially decreasing at ∞. In particular, S is also Hilbert-Schmidt for N large enough. Thus $\mathsf{B} = \mathcal{F}^{-1} S^* R^* \in S_1(L_2(\mathbb{R}^n))$. $\square$

Theorem 3.9. *If $a \in S_{0,0}^0(\mathcal{R}_\Theta)$, then $\Psi_a \colon L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ is bounded.*

Proof. According to Lemmas 3.6 and 3.8 we find

$$\Psi_a(\varphi) = (id \otimes \tau_\Theta) \int_{\mathbb{R}^n} (\mathbf{1} \otimes b(\eta)) \Big(\Phi_\eta^* \big(\underbrace{A_\eta^* B A_\eta \otimes id}_{\mathbf{A}_\eta^* \mathbf{B} \mathbf{A}_\eta} \big) \Phi_\eta \Big) (\varphi \otimes \mathbf{1}) \, d\eta.$$

Given φ_1, φ_2 in the unit ball of $L_2(\mathcal{R}_\Theta)$, it suffices to get a uniform bound for

$$\big\langle \Psi_a(\varphi_1), \varphi_2 \big\rangle \;=\; \int_{\mathbb{R}^n} (\tau_\Theta \otimes \tau_\Theta) \Big\{ \Phi_\eta^* \mathbf{A}_\eta^* \mathbf{B} \mathbf{A}_\eta \Phi_\eta (\varphi_1 \otimes \mathbf{1}) (\varphi_2 \otimes b(\eta)^*)^* \Big\} \, d\eta$$

$$=\; \int_{\mathbb{R}^n} (\tau_\Theta \otimes \tau_\Theta) \Big\{ \mathbf{B}_1 \mathbf{A}_\eta \Phi_\eta (\varphi_1 \otimes \mathbf{1}) \mathbf{B}_2 \mathbf{A}_\eta \Phi_\eta (\varphi_2 \otimes b(\eta)^*)^* \Big\} \, d\eta,$$

where $\mathbf{B} = (u|\mathbf{B}|^{\frac{1}{2}}) |\mathbf{B}|^{\frac{1}{2}} = \mathbf{B}_2^* \mathbf{B}_1$ from polar decomposition. By Cauchy-Schwarz

$$\big| \big\langle \Psi_a(\varphi_1), \varphi_2 \big\rangle \big| \leq \Big(\int_{\mathbb{R}^n} (\tau_\Theta \otimes \tau_\Theta) \Big\{ |\mathbf{B}_1|^2 \mathbf{A}_\eta \Phi_\eta (\varphi_1 \otimes \mathbf{1}) \mathbf{A}_\eta \Phi_\eta (\varphi_1 \otimes \mathbf{1})^* \Big\} \, d\eta \Big)^{\frac{1}{2}}$$

$$\times \;\; \Big(\int_{\mathbb{R}^n} (\tau_\Theta \otimes \tau_\Theta) \Big\{ |\mathbf{B}_2|^2 \mathbf{A}_\eta \Phi_\eta (\varphi_2 \otimes b(\eta)^*) \mathbf{A}_\eta \Phi_\eta (\varphi_2 \otimes b(\eta)^*)^* \Big\} \, d\eta \Big)^{\frac{1}{2}} = \alpha\beta.$$

Writing $\mathbf{B}_2 = B_2 \otimes id$, we claim that the second term above β is dominated by

$$\sup_{\eta \in \mathbb{R}^n} \big\| b(\eta) : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta) \big\| \, \big\| |B_2|^2 \big\|_{S_1(L_2(\mathcal{R}_\Theta))}^{\frac{1}{2}}.$$

Note that the same estimate applies to the first term with $b(\eta) = \mathbf{1}$ and $(\varphi_2, \mathbf{B}_2)$ replaced by $(\varphi_1, \mathbf{B}_1)$. Moreover, since $|B_j|^2 \leq |B| + u|B|u^*$ and B is trace class, it suffices to check that $b(\xi) = (\mathbf{1} - \Delta_\Theta)^\mathrm{N}(\mathbf{1} - \Delta_\xi)^\mathrm{N} a(\xi)$ is uniformly bounded in $\mathcal{R}_\Theta$ which follows from the fact that $a \in S^0_{00}(\mathcal{R}_\Theta)$. Therefore, it only remains to justify our claim above. Since $|B_2|^2$ is trace class, let s_j denote its singular numbers and consider the corresponding set u_j of unit eigenvectors. This gives

$$|\mathbf{B}_2|^2(h) = \sum_j s_j (\tau_\Theta \otimes id)\big(h(u_j \otimes \mathbf{1})^*\big)(u_j \otimes \mathbf{1}).$$

In particular, using the module bracket $\langle\langle h_1, h_2 \rangle\rangle = (\tau_\Theta \otimes id)(h_1 \bullet h_2^*)$, we get

$$
\begin{aligned}
\beta^2 &= \sum_j s_j \int_{\mathbb{R}^n} \tau_\Theta \Big\{ \big|\langle\langle \mathbf{A}_\eta \Phi_\eta(\varphi_2 \otimes b(\eta)^*), u_j \otimes \mathbf{1} \rangle\rangle\big|^2 \Big\} \, d\eta \\
&= \sum_j s_j \int_{\mathbb{R}^n} \tau_\Theta \Big\{ \big|\langle\langle M_\eta^*(\varphi_2 \otimes b(\eta)^*), \Pi_\Theta \mathbf{A}_\eta^*(u_j \otimes \mathbf{1}) \rangle\rangle\big|^2 \Big\} \, d\eta.
\end{aligned}
$$

Now, recalling that $\mathbf{A}_\eta^*(u_j \otimes \mathbf{1}) = A_\eta^*(u_j) \otimes \mathbf{1} = \lambda_\Theta \circ \mathsf{A}_\eta^* \circ \lambda_\Theta^{-1}(u_j) \otimes \mathbf{1}$, we get

$$
\Big\langle\Big\langle M_\eta^*(\varphi_2 \otimes b(\eta)^*), \Pi_\Theta \mathbf{A}_\eta^*(u_j \otimes \mathbf{1}) \Big\rangle\Big\rangle
$$

$$
\begin{aligned}
&= \Big\langle\Big\langle \lambda_\Theta(\eta)^* \varphi_2 \otimes b(\eta)^*, \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, \Theta \eta \rangle} \widehat{u}_j(\xi) \pi_\Theta(\exp_\xi) \, d\xi \Big\rangle\Big\rangle \\
&= \Big(\int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, \Theta \eta \rangle} \overline{\widehat{u}_j(\xi)} (\tau_\Theta \otimes id)\Big(\big(\lambda_\Theta(\eta)^* \varphi_2 \otimes \mathbf{1}\big) \pi_\Theta(\exp_\xi)^* \Big) \, d\xi \Big) b(\eta)^* \\
&= \Big(\int_{\mathbb{R}^n} \underbrace{e^{-2\pi i \langle \xi, \Theta \eta \rangle} e^{-2\pi i \langle \eta, \Theta_\downarrow \xi \rangle} \overline{\widehat{u}_j(\xi)} \widehat{\varphi}_2(\xi + \eta)}_{\widehat{\phi}_{\eta j}(\xi)} \, \lambda_\Theta(\xi) \, d\xi \Big) b(\eta)^*.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\beta^2 &\leq \sup_{\eta \in \mathbb{R}^n} \big\| b(\eta) \big\|_{\mathcal{R}_\Theta}^2 \sum_j s_j \int_{\mathbb{R}^n} \big\| \lambda_\Theta(\phi_{\eta j}) \big\|_{L_2(\mathcal{R}_\Theta)}^2 \, d\eta \\
&\leq \sup_{\eta \in \mathbb{R}^n} \big\| b(\eta) \big\|_{\mathcal{R}_\Theta}^2 \sum_j s_j \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big| \widehat{u}_j(\xi) \widehat{\varphi}_2(\xi + \eta) \big|^2 \, d\eta d\xi,
\end{aligned}
$$

which is exactly the estimate we were looking for. This completes the proof. $\qquad \square$

Remark 3.10. A careful analysis of the function $\widehat{\alpha}$ in the proof of Lemma 3.8 could lead as in [24] to the sharp condition $\mathrm{N} > n/4$. This would imply that Theorem 3.9 holds under the optimal assumption

$$\big| \partial_\Theta^\beta \partial_\xi^\alpha a(\xi) \big| \leq C_{\alpha\beta} \quad for \quad |\alpha|, |\beta| \leq \Big[\frac{n}{2}\Big] + 1.$$

Now we are ready to study the L_2-boundedness for exotic symbols in $S^0_{\rho,\rho}(\mathcal{R}_\Theta)$ with $0 < \rho < 1$. A weak form of Cotlar's almost orthogonality lemma naturally plays a crucial role. Namely, given a family of operators $(T_j)_{j \geq 0} \subset \mathcal{B}(\mathcal{H})$ and a summable sequence $(c_j)_{j \geq 0} \subset \mathbb{R}_+$ we find

$$\Big\| \sum_{j \geq 0} T_j \Big\|_{\mathcal{B}(\mathcal{H})} \lesssim \sum_{j \geq 0} c_j$$

provided that the following conditions hold for $j \neq k$

$$\sup_{j \geq 0} \big\| T_j \big\|_{\mathcal{B}(\mathcal{H})} < \infty, \qquad \big\| T_j T_k^* \big\|_{\mathcal{B}(\mathcal{H})} = 0, \qquad \big\| T_j^* T_k \big\|_{\mathcal{B}(\mathcal{H})} \leq c_j c_k.$$

The other ingredient is a dilation argument among different deformations $\mathcal{R}_\Theta$.

Lemma 3.11. *Given* $\mathrm{R} > 0$, *the map*

$$\mathrm{D}_\mathrm{R} : \mathcal{R}_\Theta \ni \lambda_\Theta(\xi) \mapsto \lambda_{\mathrm{R}^2\Theta}\Big(\frac{\xi}{\mathrm{R}}\Big) \in \mathcal{R}_{\mathrm{R}^2\Theta}$$

is a $*$-*homomorphism. Moreover,* $\Psi_a = \mathrm{D}_\mathrm{R}^{-1}\Psi_{\widetilde{a}_\mathrm{R}}\mathrm{D}_\mathrm{R}$ *for*

$$a : \mathbb{R}^n \to \mathcal{R}_\Theta \quad and \quad \widetilde{a}_\mathrm{R}(\xi) = \int_{\mathbb{R}^n} \widehat{a}(z, \mathrm{R}\xi)\lambda_{\mathrm{R}^2\Theta}(z/\mathrm{R})\, dz = \mathrm{D}_\mathrm{R}(a(\mathrm{R}\xi)) \in \mathcal{R}_{\mathrm{R}^2\Theta}.$$

Proof. To prove that D_R is a $*$-homomorphism is straightforward. Now

$$\mathrm{D}_\mathrm{R}\Psi_a\mathrm{D}_\mathrm{R}^{-1}(\varphi) = \int_{\mathbb{R}^n} (\widehat{\Psi_a\mathrm{D}_\mathrm{R}^{-1}\varphi})(\xi)\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\, d\xi$$

$$= \int_{\mathbb{R}^n} \tau_\Theta\Big\{ \Big(\int_{\mathbb{R}^n} a(\eta)\widehat{\mathrm{D}_\mathrm{R}^{-1}\varphi}(\eta)\lambda_\Theta(\eta)\, d\eta \Big)\lambda_\Theta(\xi)^* \Big\}\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\, d\xi$$

$$= \int_{\mathbb{R}^n} \tau_\Theta\Big\{ \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{a}(z, \eta)\lambda_\Theta(z)\mathrm{R}^{-n}\widehat{\varphi}(\eta/\mathrm{R})\lambda_\Theta(\eta)\, dzd\eta \Big)\lambda_\Theta(\xi)^* \Big\}\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\, d\xi$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\langle\xi-\eta,\Theta_\downarrow\eta\rangle}\widehat{a}(\xi - \eta, \eta)\mathrm{R}^{-n}\widehat{\varphi}(\eta/\mathrm{R})\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\, d\eta d\xi$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{a}(\xi, \eta)\mathrm{R}^{-n}\widehat{\varphi}(\eta/\mathrm{R})\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\lambda_{\mathrm{R}^2\Theta}(\eta/\mathrm{R})\, d\eta d\xi$$

$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \widehat{a}(\xi, \mathrm{R}\eta)\lambda_{\mathrm{R}^2\Theta}(\xi/\mathrm{R})\, d\xi \Big)\widehat{\varphi}(\eta)\lambda_{\mathrm{R}^2\Theta}(\eta)\, d\eta$$

$$= \int_{\mathbb{R}^n} \widetilde{a}_\mathrm{R}(\eta)\widehat{\varphi}(\eta)\lambda_{\mathrm{R}^2\Theta}(\eta)\, d\eta \;=\; \Psi_{\widetilde{a}_\mathrm{R}}(\varphi). \qquad \qquad \square$$

Theorem 3.12. *If* $a \in S^0_{\rho,\rho}(\mathcal{R}_\Theta)$ *and* $\rho < 1$, $\Psi_a: L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ *is bounded.*

Proof. Let $\phi_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$ radial, identically 1 in $\mathrm{B}_1(0)$ and zero outside $\mathrm{B}_2(0)$. Using the partition of unity $\phi_0 + \sum_{j\geq 1}\phi_j \equiv 1$ with $\phi_j(\xi) = \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi)$ we decompose Ψ_a as follows

$$\Psi_a = \sum_{j=0}^\infty \Psi_{a_j} = \sum_{j=0}^\infty \Psi_{a_{2j}} + \sum_{j=0}^\infty \Psi_{a_{2j+1}} = \Psi_\mathrm{even} + \Psi_\mathrm{odd},$$

where $a_j(\xi) = a(\xi)\phi_j(\xi)$. We shall only bound the even part, since both are treated in a similar way. To do so, we apply Cotlar's lemma as stated above. Given j, k distinct even numbers, we clearly have $\Psi_{a_j}\Psi_{a_k}^* = 0$ since ϕ_j and ϕ_k have disjoint supports. Therefore, it suffices to prove that

   i) $\displaystyle\sup_{j\geq 0} \big\|\Psi_{a_j}\big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} < \infty,$

   ii) $\big\|\Psi_{a_j}^*\Psi_{a_k}\big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \leq c_j c_k,$

for some summable sequence $(c_j)_{j\geq 0} \subset \mathbb{R}_+$ and any pair of distinct even integers j, k. The first condition follows from our form of Calderón-Vaillancourt theorem in $S^0_{00}(\mathcal{R}_\Theta)$ and Lemma 3.11. Indeed, pick $\mathrm{R}_j = 2^{j\rho}$ and let

$$a_{[j]} = \widetilde{(a_j)}_{\mathrm{R}_j} = \mathrm{D}_{\mathrm{R}_j}\big(a_j(\mathrm{R}_j\,\cdot)\big).$$

Then

$$\big\|\Psi_{a_j}\big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \leq \big\|\Psi_{a_{[j]}}\big\|_{\mathcal{B}(L_2(\mathcal{R}_{\mathrm{R}_j^2\Theta}))}$$

since $\Psi_{a_j} = \mathrm{D}_{\mathrm{R}_j}^{-1}\Psi_{a_{[j]}}\mathrm{D}_{\mathrm{R}_j}$ and

$$\big\|\mathrm{D}_{\mathrm{R}_j}^{-1}\big\|_{\mathcal{B}(L_2(\mathcal{R}_{\mathrm{R}_j^2\Theta}),L_2(\mathcal{R}_\Theta))}\big\|\mathrm{D}_{\mathrm{R}_j}\big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta),L_2(\mathcal{R}_{\mathrm{R}_j^2\Theta}))} = 1.$$

The L_2-boundedness of $\Psi_{a_{[j]}}$ follows from Theorem 3.9 since $a_{[j]} \in S_{00}^0(\mathcal{R}_{\mathrm{R}_j^2\Theta})$. The proof of this fact follows essentially as in the Euclidean setting. Indeed, write $a_{[j]}$ in terms of $\widehat{a}_j$ —Lemma 3.11— and use that $\mathrm{D}_{\mathrm{R}_j}$ is a $*$-homomorphism. In conjunction with the ξ-localization of a_j in the annulus of radii $\sim 2^j$, this easily gives $a_{[j]} \in S_{00}^0(\mathcal{R}_{\mathrm{R}_j^2\Theta})$, we leave details to the reader. It remains to estimate the norm of $\Psi_{a_j}^*\Psi_{a_k}$ for even $j \neq k$. After a calculation we obtain that the kernel k_{jk} of such operator is given by

$$k_{jk} = (id\otimes\tau_\Theta\otimes id)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\pi_\Theta(\exp_\xi)\otimes\mathbf{1}\right)\left(\mathbf{1}\otimes a_j(\xi)^*a_k(\eta)\otimes\mathbf{1}\right)\left(\mathbf{1}\otimes\pi_\Theta(\exp_\eta)\right)d\xi d\eta,$$

where, in an abuse of notation, the element $\pi_\Theta(\exp_\xi) = \lambda_\Theta(\xi)\otimes\lambda_\Theta(\xi)^*$ is seen as belonging to $\mathcal{R}_\Theta\bar{\otimes}\mathcal{R}_\Theta$ instead of $\mathcal{R}_\Theta\bar{\otimes}\mathcal{R}_\Theta^{\mathrm{op}}$. We are also going to shorten $a\otimes b\otimes\mathbf{1}$ by $(a\otimes b)_{[12]}$ where the leg numbers just mean that the tensor components are placed in the first and second places respectively. Now, we use

$$\frac{(\mathbf{1}-\Delta_\Theta)^{\mathrm{N}}}{(1+4\pi^2|\xi-\eta|^2)^{\mathrm{N}}}\,\lambda_\Theta(\eta)\lambda_\Theta(\xi)^* \quad = \quad \lambda_\Theta(\eta)\lambda_\Theta(\xi)^*,$$

$$\pi_\Theta\left(\frac{1}{(1+4\pi^2\mathrm{d}^2)^n}\right)(\mathbf{1}-\Delta_\eta)^n\,\lambda_\Theta(\eta)\otimes\lambda_\Theta(\eta)^* \quad = \quad \lambda_\Theta(\eta)\otimes\lambda_\Theta(\eta)^*,$$

$$\pi_\Theta\left(\frac{1}{(1+4\pi^2\mathrm{d}^2)^n}\right)(\mathbf{1}-\Delta_\xi)^n\,\lambda_\Theta(\xi)\otimes\lambda_\Theta(\xi)^* \quad = \quad \lambda_\Theta(\xi)\otimes\lambda_\Theta(\xi)^*,$$

where $\mathrm{d}(x) = |x|$ is the Euclidean distance. Integration by parts yields

$$k_{jk} = (id\otimes\tau_\Theta\otimes id)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\pi_\Theta(\varphi_\xi)_{[12]}B(\xi,\eta)_{[2]}\pi_\Theta(\varphi_\eta)_{[23]}\,d\eta d\xi,$$

where

$$B(\xi,\eta) = (\mathbf{1}-\Delta_\eta)^n(\mathbf{1}-\Delta_\xi)^n\left\{\frac{(\mathbf{1}-\Delta_\Theta)^{\mathrm{N}}}{(1+4\pi^2|\eta-\xi|^2)^{\mathrm{N}}}a_j(\xi)^*a_k(\eta)\right\}$$

and the function φ_ζ is given by $\exp_\zeta(1+4\pi^2\mathrm{d}^2)^{-n}$. After expanding the derivatives using the Leibniz rule, we obtain that B is a finite sum of simple terms of the form

$$B_s(\xi,\eta) = \partial_\xi^{\alpha_1}\partial_\eta^{\beta_1}\left(\frac{1}{(1+4\pi^2|\xi-\eta|^2)^{\mathrm{N}}}\right)\underbrace{\partial_\xi^{\alpha_2}\partial_\Theta^{\sigma_1}a_j(\xi)^*}_{b_j^s(\xi)^*}\,\underbrace{\partial_\eta^{\beta_2}\partial_\Theta^{\sigma_2}a_k(\eta)}_{b_k^s(\eta)},$$

where $\alpha_i,\beta_i,\sigma_i \in \mathbb{Z}_+^n$ satisfy $|\alpha_1+\alpha_2| \leq 2n$, $|\beta_1+\beta_2| \leq 2n$, $|\sigma_1+\sigma_2| \leq 2\mathrm{N}$ and s is the combination of the involved multiindices. We can bound each of the above summands in s independently

$$\big\|\Psi_{a_j}^*\Psi_{a_k}\big\| \leq \sum_s\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}T_{(id\otimes\tau_\Theta\otimes id)\{\pi_\Theta(\varphi_\xi)_{[12]}B_s(\xi,\eta)_{[2]}\pi_\Theta(\varphi_\eta)_{[23]}\}}d\eta d\xi\right\| = \sum_s A_s.$$

Using $\left| \partial_\xi^{\alpha_1} \partial_\eta^{\beta_1} \langle \xi - \eta \rangle^{-2N} \right| \lesssim \langle \xi - \eta \rangle^{-2N}$ we obtain

$$A_s \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \xi - \eta \rangle^{-2N} \big\| \underbrace{T_{(id \otimes \tau_\Theta \otimes id)\{\pi_\Theta(\varphi_\xi)_{[12]} b_j^s(\xi)_{[2]}^* b_k^s(\eta)_{[2]} \pi_\Theta(\varphi_\eta)_{[23]}\}}}_{T_{j\xi k\eta}} \big\| \, d\eta d\xi.$$

$T_{j\xi k\eta}$ can be factorized as $T_{(b_j^s \otimes \mathbf{1}) \bullet \pi_\Theta(\varphi_\xi)} \circ T_{(b_k^s(\eta) \otimes \mathbf{1}) \bullet \pi_\Theta(\varphi_\eta)}$, so that

$$\big\| T_{j\xi k\eta} \big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \leq \| b_j^s(\xi) \|_{\mathcal{R}_\Theta} \| b_k^s(\eta) \|_{\mathcal{R}_\Theta} \| T_{\pi_\Theta(\varphi_\xi)} \|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \| T_{\pi_\Theta(\varphi_\eta)} \|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))}.$$

Recall that $\| T_{\pi_\Theta(\varphi_\zeta)} \|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \leq \| \varphi_\zeta \|_{L_1(\mathbb{R}^n)} \lesssim 1$. Moreover, using that $\xi \sim 2^j$ and $\eta \sim 2^k$ from the supports of $a_j(\xi)$ and $a_k(\eta)$ as well as the Hörmander condition for a, we deduce the following bound

$$A_s \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\| b_j^s(\xi) \|_{\mathcal{R}_\Theta} \| b_k^s(\eta) \|_{\mathcal{R}_\Theta}}{\langle \xi - \eta \rangle^{2N}} \, d\xi d\eta \lesssim 2^{-2N \max\{j,k\}} 2^{2N\rho \max\{j,k\}} 2^{n(j+k)}.$$

Summing all the terms indexed by s we obtain

$$\big\| \Psi_{a_j}^* \Psi_{a_k} \big\|_{\mathcal{B}(L_2(\mathcal{R}_\Theta))} \lesssim 4^{-\max\{j,k\}((1-\rho)N-n)} \leq 2^{-j((1-\rho)N-n)} 2^{-k((1-\rho)N-n)} = c_j c_k,$$

which arises from $(c_j)_{j \geq 0}$ summable for N large enough. The proof is complete. □

Remark 3.13. Let $0 \leq \delta \leq \rho \leq 1$, since

$$S_{\rho,\delta}^0(\mathcal{R}_\Theta) \subset S_{\delta,\delta}^0(\mathcal{R}_\Theta) \cap S_{\rho,\rho}^0(\mathcal{R}_\Theta),$$

we deduce $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ for $a \in S_{\rho,\delta}^0(\mathcal{R}_\Theta)$ as long as $(\rho, \delta) \neq (1,1)$.

Remark 3.14. A standard (nonoptimal) proof of Calderón-Vaillancourt theorem for S_{00}^0 in the Euclidean setting [66] follows from a suitable partition of unity in the variables $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with no known analogue for $x_\Theta \in \mathcal{R}_\Theta$ and $\xi \in \mathbb{R}^n$. An alternative way to proceed is the following. Given $a \in S_{00}^0(\mathcal{R}_\Theta)$, let

$$\mathbf{a}(x, \xi) = \sigma_\Theta^x a(\xi) = \int_{\mathbb{R}^n} \widehat{a}(z, \xi) \sigma_\Theta^x(\lambda_\Theta(z)) \, dz = \int_{\mathbb{R}^n} \widehat{a}(z, \xi) e^{2\pi i \langle x, z \rangle} \lambda_\Theta(z) \, dz.$$

Using the intertwining identity $\Psi_{\mathbf{a}} \circ \sigma_\Theta = \sigma_\Theta \circ \Psi_a$ and recalling from Appendix B that $\sigma_\Theta : L_2^c(\mathcal{R}_\Theta) \to L_2^c(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ is a complete isometry, it turns our that the L_2-boundedness of Ψ_a is equivalent to the boundedness of the operator-valued map $\Psi_{\mathbf{a}} : L_2^c(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta \to L_2^c(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$. Now, since $\Psi_{\mathbf{a}}$ is a right $\mathcal{R}_\Theta$-module map, it follows from [42, Remark 2.4] that this will hold as long as $\Psi_{\mathbf{a}}$ is bounded over the Hilbert space $L_2(\mathbb{R}^n; L_2(\mathcal{R}_\Theta))$. $\Psi_{\mathbf{a}}$ comes equipped with an operator-valued kernel acting by left multiplication. This kind of maps are generally bad behaved [37] but we know from our proof above that L_2-boundedness must hold in this case. Thus this also opens the door to prove Calderón-Vaillancourt using a partition of unity in the x-component, which mirrors the behavior of its quantum analogue x_Θ.

3.2.2. *Bourdaud's condition for forbidden symbols in* $\mathcal{R}_\Theta$. We have justified that all symbols in $S_{\rho,\delta}^0(\mathcal{R}_\Theta)$ yield L_2-bounded pseudodifferential operators except for the class of so-called forbidden symbols with $\rho = \delta = 1$, which is known to fail it even in the Euclidean setting. Bourdaud established a sufficient condition in [7] playing the role of the $T(1)$-theorem for pseudodifferential operators and which we now study in $\mathcal{R}_\Theta$. Given $p \geq 1$ and $s \in \mathbb{R}$, let $W_{2,s}(\mathcal{R}_\Theta)$ be the Sobolev space defined as the closure of $\mathcal{S}_\Theta$ with respect to the norm

$$\| \varphi \|_{W_{2,s}(\mathcal{R}_\Theta)} = \big\| (\mathbf{1} - \Delta_\Theta)^{\frac{s}{2}} \varphi \big\|_2.$$

Lemma 3.15. *Let $\phi : \mathbb{R}^n \to \mathbb{R}_+$ be a radial smooth function identically 1 in $\mathrm{B}_1(0)$ and vanishing outside $\mathrm{B}_2(0)$. Let $\psi_j(\xi) = \phi(2^{-j}\xi) - \phi(2^{-j+1}\xi)$ for any integer $j \in \mathbb{Z}$. Then, we have a norm equivalence*

$$\left\| \lambda_\Theta(f) \right\|_{2,s}^2 \sim \sum_{j \in \mathbb{Z}} 2^{2sj} \left\| \lambda_\Theta(\psi_j f) \right\|_2^2 = \sum_{j \in \mathbb{Z}} 2^{2sj} \left\| \psi_j f \right\|_2^2.$$

In particular, the following properties hold:

    i) $\mathrm{W}_{2,-s}(\mathcal{R}_\Theta)^* = \mathrm{W}_{2,s}(\mathcal{R}_\Theta)$ *under the pairing* $\langle x, y \rangle = \tau_\Theta(x^* y)$.
    ii) $\left[\mathrm{W}_{2,s}(\mathcal{R}_\Theta), \mathrm{W}_{2,-s}(\mathcal{R}_\Theta) \right]_{\frac{1}{2}} = L_2(\mathcal{R}_\Theta)$ *by complex interpolation.*

The proofs of all assertions above are straightforward. Properties i) and ii) above hold isomorphically from the first assertion, but also isometrically. We shall use the following terminology for the rest of this section. Let us consider a function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ which is identically 1 in $\mathrm{B}_{1/8}(0)$ and vanishing outside $\mathrm{B}_{1/4}(0)$. Set $\psi_0 = \phi$ and construct $\psi_j(\xi) = \phi(2^{-j}\xi) - \phi(2^{-j+1}\xi)$ for $j \geq 1$. We shall also use the partition of unity $\rho_0 = \psi_0 + \psi_1$ and $\rho_j = \psi_{j-1} + \psi_j + \psi_{j+1}$ for $j \geq 1$, so that

- $\displaystyle\sum_{j \geq 0} \psi_j(\xi) = 1$,
- $\displaystyle\sum_{j \geq 0} \rho_j(\xi) = 3 - \phi(\xi)$,
- $\rho_j(\xi)\psi_j(\xi) = \psi_j(\xi)$ for $j \geq 0$.

Lemma 3.16. *If $a \in S_{1,1}^0(\mathcal{R}_\Theta)$ and $\mathrm{N} > n$, we have*

$$a(\xi) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-\mathrm{N}} \sum_{j \geq 0} c_{j,k} \rho_j(\xi) e^{2\pi i \langle 2^{-j}\xi, k \rangle}$$

where the coefficients $c_{j,k} \in \mathcal{R}_\Theta$ satisfy the following estimate

$$\sup_{j \geq 0} \sup_{k \in \mathbb{Z}^n} \left(\| c_{j,k} \|_{\mathcal{R}_\Theta} + 2^{-j} \left\| \Big(\sum_{i=1}^n |\partial_\Theta^i c_{j,k}|^2 \Big)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} \right) < \infty.$$

Proof. Let $a_j(\xi) = a(\xi)\psi_j(\xi)$ and $b_j(\xi) = a_j(2^j\xi)$, so that

$$a(\xi) = \sum_{j \geq 0} a_j(\xi) = \sum_{j \geq 0} b_j(2^{-j}\xi) = \sum_{j \geq 0} b_j(2^{-j}\xi)\rho_j(\xi).$$

According to this and recalling that $\operatorname{supp} \rho_j(2^j\xi) \subset [-\frac{1}{2}, \frac{1}{2}]^n = Q$, it suffices to see that $b_j(\xi) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-\mathrm{N}} c_{j,k} e^{2\pi i \langle \xi, k \rangle} \chi_Q(\xi)$ for some $c_{j,k}$ satisfying the estimates in the statement. Now, since b_j is also supported by Q, we find that $b_j(\xi) = d_j(\xi)\chi_Q(\xi)$ where d_j is the $\mathbb{Z}^n$-periodization of b_j. This gives rise to the identity

$$
\begin{aligned}
b_j(\xi) &= \sum_{k \in \mathbb{Z}^n} \widehat{b}_j(k) e^{2\pi i \langle \xi, k \rangle} \chi_Q(\xi) \\
&= \sum_{k \in \mathbb{Z}^n} \Big(\int_{\mathbb{T}^n} b_j(s) e^{-2\pi i \langle s, k \rangle} ds \Big) e^{2\pi i \langle \xi, k \rangle} \chi_Q(\xi).
\end{aligned}
$$

Integrating by parts, we obtain

$$\widehat{b}_j(k) = \frac{1}{(1 + 4\pi^2 |k|^2)^{\frac{\mathrm{N}}{2}}} \int_{\mathbb{T}^n} (\mathbf{1} - \Delta_s)^{\frac{\mathrm{N}}{2}} \big(b_j(s) \big) e^{-2\pi i \langle s, k \rangle} ds = \langle k \rangle^{-\mathrm{N}} c_{j,k}.$$

To estimate $(\mathbf{1} - \Delta_\xi)^{\frac{N}{2}} b_j(\xi)$ we notice that $|\xi| \sim 1$, so

$$\left\| \partial_\xi^\alpha b_j(\xi) \right\|_{\mathcal{R}_\Theta} = 2^{j|\alpha|} \left\| (\partial_\xi^\alpha a_j)(2^j \xi) \right\|_{\mathcal{R}_\Theta} \lesssim 2^{j|\alpha|} \langle 2^j \xi \rangle^{-|\alpha|} \lesssim 1$$

by the Hörmander condition and therefore $(\mathbf{1} - \Delta_\xi)^{\frac{N}{2}} b_j(\xi)$ is uniformly bounded in norm. The second inequality uses a similar calculation for $\nabla_\Theta (\mathbf{1} - \Delta_\xi)^{\frac{N}{2}} b(\xi)$. $\quad\square$

Lemma 3.17. *We have*

$$\left\| \lambda_\Theta(\psi_j f) \right\|_{\mathcal{R}_\Theta} \lesssim 2^{-j} \left\| \Big(\sum_{k=1}^n \big| \partial_\Theta^k \lambda_\Theta(f) \big|^2 \Big)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} \quad for \quad j > 0.$$

Proof. Given $g \in \mathcal{S}(\mathbb{R}^n)$

$$\big(\sigma_\Theta \big(\lambda_\Theta(f) \big) * g \big)(x) = \int_{\mathbb{R}^n} \sigma_\Theta^{x-y} \big(\lambda_\Theta(f) \big) g(y) \, dy = \sigma_\Theta^x \big(\lambda_\Theta(\widehat{g} f) \big).$$

We also have $\left\| \lambda_\Theta(\psi_j f) \right\|_{\mathcal{R}_\Theta} = \left\| \sigma_\Theta \lambda_\Theta(\psi_j f) \right\|_{\mathcal{R}_\Theta \bar{\otimes} L_\infty(\mathbb{R}^n)}$ and

$$
\begin{aligned}
\sigma_\Theta \lambda_\Theta(\psi_j f) &= \sigma_\Theta \lambda_\Theta(f) * \widehat{\psi}_j \\
&= \int_{\mathbb{R}^n} \sigma_\Theta^{x-y} \big(\lambda_\Theta(f) \big) \widehat{\psi}_j(y) \, dy \\
&= \int_{\mathbb{R}^n} \Big(\sigma_\Theta^{x-y} \big(\lambda_\Theta(f) \big) - \sigma_\Theta^x \big(\lambda_\Theta(f) \big) \Big) \widehat{\psi}_j(y) \, dy \\
&= \int_{\mathbb{R}^n} \Big(\int_0^1 \sum_{k=1}^n y_k \sigma_\Theta^{x-ty} \big(\partial_\Theta^k \lambda_\Theta(f) \big) \, dt \Big) \widehat{\psi}_j(y) \, dy.
\end{aligned}
$$

We have used that the integral of $\widehat{\psi}_j$ is 0 for any $j > 0$. Taking norms gives

$$\left\| \sigma_\Theta \lambda_\Theta(\psi_j f) \right\|_{\mathcal{R}_\Theta \bar{\otimes} L_\infty(\mathbb{R}^n)}$$

$$\leq \int_{\mathbb{R}^n} \left\| \sum_{k=1}^n y_k \big(\partial_\Theta^k \lambda_\Theta(f) \big) \right\|_{\mathcal{R}_\Theta} |\widehat{\psi}_j(y)| \, dy$$

$$\leq \Big(\int_{\mathbb{R}^n} |y| |\widehat{\psi}_j(y)| \, dy \Big) \left\| \Big(\sum_{k=1}^n \big| \partial_\Theta^k \lambda_\Theta(f) \big|^2 \Big)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta}$$

$$\lesssim 2^{-j} \int_{\mathbb{R}^n} |y| |\widehat{\phi}(y)| \, dy \left\| \Big(\sum_{k=1}^n \big| \partial_\Theta^k \lambda_\Theta(f) \big|^2 \Big)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} \lesssim 2^{-j} \left\| \Big(\sum_{k=1}^n \big| \partial_\Theta^k \lambda_\Theta(f) \big|^2 \Big)^{\frac{1}{2}} \right\|_{\mathcal{R}_\Theta} \square$$

Theorem 3.18. *If $a \in S_{1,1}^0(\mathcal{R}_\Theta)$*

$$\Psi_a : W_{2,s}(\mathcal{R}_\Theta) \to W_{2,s}(\mathcal{R}_\Theta) \quad is \ bounded \ for \quad 0 < s < 1.$$

Proof. By Lemma 3.16 we have that

$$a(\xi) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-N} \underbrace{\sum_{j \geq 0} c_{j,k} \rho_j(\xi) e^{2\pi i \langle 2^{-j} \xi, k \rangle}}_{a_k(\xi)}.$$

By taking $N > n$ we obtain that the symbol a is just a summable combination of terms a_k and we can concentrate on studying such terms. If $\lambda_\Theta(f) \in W_{2,s}(\mathcal{R}_\Theta)$,

we have that

$$\Psi_{a_k}(\lambda_\Theta(f)) = \sum_{j\geq 0} c_{j,k}\lambda_\Theta(\rho_j \exp_{2^{-j}k} f) = \sum_{j\geq 0} c_{j,k}b_{j,k}.$$

Taking another partition of unity $(\psi_\ell)_{\ell\geq 0}$ we get

$$c_{j,k} = \sum_{\ell\geq 0}\int_{\mathbb{R}^n}\psi_\ell(\xi)\widehat{c}_{j,k}(\xi)\lambda_\Theta(\xi)\,d\xi = \sum_{\ell\geq 0} c_{j,k}^\ell$$

and Lemmas 3.16 and 3.17 give $\|c_{j,k}^\ell\|_{\mathcal{R}_\Theta} \lesssim 2^{j-\ell}$ for $\ell > 0$. Now decompose

$$\Psi_{a_k}(\lambda_\Theta(f)) = \sum_{\ell\leq j-4} c_{j,k}^\ell b_{j,k} + \sum_{j-4<\ell<j+4} c_{j,k}^\ell b_{j,k} + \sum_{\ell\geq j+4} c_{j,k}^\ell b_{j,k} = \mathrm{L} + \mathrm{D} + \mathrm{U}.$$

Let us begin with the estimate of the upper term U. The Fourier support of $\lambda_\Theta(f)\lambda_\Theta(g) = \lambda_\Theta(f *_\Theta g)$ is contained in the sum of the Fourier supports of $\lambda_\Theta(f)$ and $\lambda_\Theta(g)$ respectively. In particular, the Fourier support of $c_{j,k}^\ell b_{j,k}$ is contained in $\operatorname{supp}\psi_\ell + \operatorname{supp}\rho_j \subset [\mathrm{B}_{2^{\ell-2}}(0)\setminus \mathrm{B}_{2^{\ell-4}}(0)] + \mathrm{B}_{2^{j-1}}(0) \subset \mathrm{B}_{2^{\ell-1}}(0)\setminus \mathrm{B}_{2^{\ell-5}}(0)$. Now we apply Lemma 3.15 to obtain

$$\|\mathrm{U}\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2 \lesssim \sum_{\ell\geq 4} 2^{2\ell s}\Big\|\sum_{j\leq\ell-4} c_{j,k}^\ell b_{j,k}\Big\|_2^2 \lesssim \sum_{\ell\geq 4} 2^{2\ell s}\Big(\sum_{j\leq\ell-4} \|c_{j,k}^\ell\|_{\mathcal{R}_\Theta}\|b_{j,k}\|_2\Big)^2$$

$$\lesssim \sum_{\ell\geq 4} 2^{2\ell s}\Big(\sum_{j\leq\ell-4} 2^{j-\ell}\|b_{j,k}\|_2\Big)^2 \lesssim \sum_{\ell\geq 4} 2^{2\ell(s-1)}\Big(\sum_{j\leq\ell-4} 2^j\|b_{j,k}\|_2\Big)^2.$$

On the other hand, given $0 < \delta < 1-s$ we have that

$$\Big(\sum_{j\leq\ell-4} 2^j\|b_{j,k}\|_2\Big)^2 \leq C_\delta\, 2^{\ell\delta}\sum_{j\leq\ell-4} 4^{j(1-\delta)}\|b_{j,k}\|_2^2.$$

In particular, we finally obtain the expected estimate

$$\|\mathrm{U}\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2 \lesssim \sum_{\ell\geq 0}\sum_{j\leq\ell-4} 2^{2\ell(s-1+\delta)}4^{j(1-\delta)}\|b_{j,k}\|_2^2$$

$$= \sum_{j\geq 0}\Big(\sum_{\ell\geq j+4} 2^{2\ell(s-1+\delta)}\Big)4^{j(1-\delta)}\|b_{j,k}\|_2^2$$

$$\lesssim \sum_{j\geq 0} 4^{j(s-1+\delta)}4^{j(1-\delta)}\|b_{j,k}\|_2^2 \lesssim \|\lambda_\Theta(f)\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2.$$

For the lower part L, a similar argument yields that the Fourier support of $c_{j,k}^\ell b_{j,k}$ is contained inside $\mathrm{B}_{2^j}(0)\setminus \mathrm{B}_{2^{j-6}}(0)$. Then we can apply the same principle so that

$$\|\mathrm{L}\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2 \lesssim \sum_{j\geq 4} 2^{2js}\Big\|\sum_{\ell\leq j-4} c_{j,k}^\ell\, b_{j,k}\Big\|_2^2$$

$$\leq \sum_{j\geq 4} 2^{2js}\Big\|\sum_{\ell\leq j-4} c_{j,k}^\ell\Big\|_{\mathcal{R}_\Theta}\|b_{j,k}\|_2^2$$

$$\lesssim \sum_{j\geq 4} 2^{2js}\|b_{j,k}\|_2^2 \lesssim \|\lambda_\Theta(f)\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2.$$

In the third inequality we have used that $\sum_\ell c_{j,k}^\ell = c_{j,k}$ and therefore

$$\Big\| \sum_{\ell \leq j-4} c_{j,k}^\ell \Big\|_{\mathcal{R}_\Theta} = \Big\| c_{j,k} - \sum_{\ell=j-3}^\infty c_{j,k}^\ell \Big\|_{\mathcal{R}_\Theta} = \|c_{j,k}\|_{\mathcal{R}_\Theta} + \sum_{\ell=j-3}^\infty \|c_{j,k}^\ell\|_{\mathcal{R}_\Theta} \lesssim 1.$$

The diagonal part D is easier to bound. Assume for simplicity that $j = \ell$. The Fourier support of $c_{j,k}^j b_{j,k}$ is comparable this time to a fixed dilation of $\mathrm{B}_{2^j}(0)$, not an annulus. Nevertheless, although we do not have a norm equivalence, the norm in $\mathrm{W}_{2,s}(\mathcal{R}_\Theta)$ is still dominated by the corresponding weighted L_2-sum, and we get

$$\|\mathrm{D}\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2 \leq \sum_{j\geq 0} 2^{2js} \|c_{j,k}^j b_{j,k}\|_2^2 \lesssim \sum_{j\geq 0} 2^{2js} \|b_{j,k}\|_2^2 \sim \|\lambda_\Theta(f)\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}^2. \qquad \square$$

Theorem 3.19. *If* $a, a_\dagger^* \in S_{1,1}^0(\mathcal{R}_\Theta)$, *then* $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$ *is bounded.*

Proof. By Theorem 3.18, Ψ_a and its adjoint are bounded in $\mathrm{W}_{2,s}(\mathcal{R}_\Theta)$. By Lemma 3.15, taking duals gives $\Psi_a : \mathrm{W}_{2,-s}(\mathcal{R}_\Theta) \to \mathrm{W}_{2,-s}(\mathcal{R}_\Theta)$ and interpolating both inequalities for Ψ_a yields the assertion. $\qquad \square$

Remark 3.20. A careful examination yields that

$$\big\| \partial_\Theta^\beta \partial_\xi^\alpha a(\xi) \big\|_{\mathcal{R}_\Theta} \lesssim \langle \xi \rangle^{-|\alpha|+|\beta|} \quad \text{for} \quad |\alpha| \leq n+1 \quad \text{and} \quad |\beta| \leq 1$$

for a and its dual symbol $a_\dagger^*$ suffices to deduce the L_2-boundedness of Ψ_a.

3.3. L_p-boundedness and Sobolev p-estimates. The L_2-boundedness results above together with our Calderón-Zygmund theory for $\mathcal{R}_\Theta$ are the tools to find sufficient smoothness conditions on a given symbol for the L_p-boundedness of its pseudodifferential operator. As pointed in the Introduction, this naturally requires to work with a different quantum form of the Hörmander classes, which is more demanding, but still recovers the classical definition for $\Theta = 0$. Given $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ we say that it belongs to $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ when

$$\big| \partial_\Theta^\beta \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} a(\xi) \big| \leq C_{\alpha_1,\alpha_2,\beta} \langle \xi \rangle^{m-\rho|\alpha_1+\alpha_2|+\delta|\beta|}$$

for all $\alpha_1, \alpha_2, \beta \in \mathbb{Z}_+^n$. Here are some trivial, albeit important, properties:

   i) $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta) \subset S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ since one condition reduces to the other when $\alpha_1 = 0$.

  ii) All of the three derivatives involved in the definition of $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ commute with each other. In particular, the order considered is completely irrelevant.

 iii) Fix (ρ, δ, m) and set $|a|_{\alpha,\beta}^S$ and $|a|_{\alpha_1,\alpha_2,\beta}^\Sigma$ for the seminorms given by the optimal constant in the defining inequalities of $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$ with parameters (α, β) or $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ with parameters $(\alpha_1, \alpha_2, \beta)$ respectively. Then, we have

$$\lim_{\Theta \to 0} |a|_{\alpha_1,\alpha_2,\beta}^\Sigma = \lim_{\Theta \to 0} |a|_{\alpha_1+\alpha_2,\beta}^S.$$

Given $a \in \Sigma_{1,1}^0(\mathcal{R}_\Theta) \subset S_{1,1}^0(\mathcal{R}_\Theta)$, we will now prove that the integral kernel k_a associated with Ψ_a satisfies the Calderón-Zygmund kernel conditions in Theorem A. In conjunction with our Bourdaud type condition in Theorem 3.19, it will give the complete L_p-boundedness of Ψ_a stated in Theorem B iii). Composition results further yield Sobolev p-estimates

$$\big\| \Psi_a : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-m}(\mathcal{R}_\Theta) \big\|_{\mathrm{cb}} < \infty,$$

for many symbols of degree m, with $1 < p < \infty$ and $\|\varphi\|_{p,s} = \|(1 - \Delta_\Theta)^{s/2}\varphi\|_p$.

Lemma 3.21. *Given $a \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$, let*

$$k_1 = (\nabla_\Theta \otimes id)(k_a) \quad and \quad k_2 = (id \otimes \nabla_\Theta)(k_a).$$

Then, there exist $b_1, b_2 \in \Sigma_{\rho,\delta}^{m+1}(\mathcal{R}_\Theta)$ satisfying that $k_{b_1} = k_1$ and $k_{b_2} = k_2$.

Proof. It is easily checked that

$$(\nabla_\Theta \otimes id)(k_a) = \sum_{j=1}^n s(e_j) \otimes \int_{\mathbb{R}^n} \partial_\Theta^j[a(\xi)\lambda_\Theta(\xi)] \otimes \lambda_\Theta(\xi)^* \, d\xi = k_{b_1}$$

where $b_1(\xi) = \nabla_\Theta(a) + 2\pi i s(\xi) \otimes a(\xi)$ takes values in $\mathcal{L}(\mathbb{F}_n)\bar{\otimes}\mathcal{R}_\Theta$, we omit the extra tensor component just to simplify our notation. A simple calculation also gives that $b_2(\xi) = -2\pi i s(\xi) \otimes a(\xi)$. It is clear that $\nabla_\Theta(a) \in \Sigma_{\rho,\delta}^{m+\delta}(\mathcal{R}_\Theta) \subset \Sigma_{\rho,\delta}^{m+1}(\mathcal{R}_\Theta)$ while the inclusion for $s(\xi) \otimes a(\xi)$ follows by Leibniz rule and $\partial_{\Theta,\xi}^j\xi = \partial_\xi^j\xi$. □

Lemma 3.22. *Given $a \in \mathcal{S}(\mathbb{R}^n; \mathcal{S}_\Theta)$, let k_a be the kernel of Ψ_a. Recall that $\pi_\Theta(P)$ is a distribution in $\mathcal{S}_{\Theta\oplus\Theta}$ for any polynomial P in $\xi_1, ..., \xi_n$. Then, the following identities hold in the sense of distributions for all $\alpha \in \mathbb{Z}_+^n$*

$$\begin{aligned} k_a \bullet \pi_\Theta((2\pi i z)^\alpha) &= k_{\partial_\xi^\alpha a}, \\ \pi_\Theta((2\pi i z)^\alpha) \bullet k_a &= k_{\partial_{\Theta,\xi}^\alpha a}. \end{aligned}$$

Proof. Note that

$$k_a \bullet \pi_\Theta(\exp_\zeta) = \left(\int_{\mathbb{R}^n} (a(\xi) \otimes 1) \bullet \pi_\Theta(\exp_{\xi+\zeta}) \, d\xi \right) = k_{a(\cdot - \zeta)}.$$

Taking derivatives formally gives

$$k_a \bullet \pi_\Theta(2\pi i z_j) = k_a \bullet \frac{d}{ds}\Big|_{s=0} \pi_\Theta(\exp_{se_j}) = k_{\partial_\xi^j a}.$$

This symbolic calculation can be justified in the distributional sense. For the second identity, we recall the identity $(\partial_{\Theta,\xi}^j a)(\xi) = \lambda_\Theta(\xi)\partial_\xi^j\{\lambda_\Theta(\xi)^*a(\xi)\lambda_\Theta(\xi)\}\lambda_\Theta(\xi)^*$ and notice that

$$\int_{\mathbb{R}^n} (a(\xi) \otimes 1) \bullet \pi_\Theta(\exp_\xi) \, d\xi = \int_{\mathbb{R}^n} \pi_\Theta(\exp_\xi) \bullet (\lambda_\Theta(\xi)^*a(\xi)\lambda_\Theta(\xi) \otimes 1) \, d\xi.$$

Therefore, arguing as above, we obtain the identity for left multiplication. □

Lemma 3.23. *Let $\psi_j(\xi) = \phi(2^{-j}\xi) - \phi(2^{-j+1}\xi)$ be a standard partition of unity in $\mathbb{R}^n$ from a smooth, radial and compactly supported ϕ. If we let $a_j(\xi) = a(\xi)\psi_j(\xi)$ for $a \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ and $\ell_1, \ell_2 \geq 0$, we have*

$$\left\| d_\Theta^{\ell_1} \bullet k_{a_j} \bullet d_\Theta^{\ell_2} \right\|_{\mathcal{R}_\Theta\bar{\otimes}\mathcal{R}_\Theta^{op}} \leq C_{\ell_1,\ell_2} \, 2^{j(n+m-\rho(\ell_1+\ell_2))}.$$

Proof. It is clear that

$$\partial_{\Theta,\xi}^\alpha a_j = \sum_{\beta+\gamma=\alpha} a(\partial_\xi^\beta\psi_j) + (\partial_{\Theta,\xi}^\gamma a)\psi_j.$$

Since $|\partial_\xi^\beta \psi_j(\xi)| \lesssim 2^{-j|\beta|}$ when $\langle\xi\rangle \sim 2^j$, the a_j's are in $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ with constants independent of j. Assume first that ℓ_1, ℓ_2 are even numbers, $\ell_k = 2N_k$. Then the ℓ_k-th power of $|z|$ is a polynomial of the form

$$|z|^{\ell_k} = \sum_{|\alpha|=N_k} z^{2\alpha}.$$

Applying Lemma 3.22 gives that

$$\left\| d_\Theta^{\ell_1} \bullet k_{a_j} \bullet d_\Theta^{\ell_2} \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} = \frac{1}{(2\pi)^{\ell_1+\ell_2}} \left\| \sum_{|\alpha_1|=N_1} \sum_{|\alpha_2|=N_2} k_{\partial_{\Theta,\xi}^{2\alpha_1} \partial_\xi^{2\alpha_2} a_j} \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}}$$

and for each of the terms we have the estimate

$$\begin{aligned}
\left\| k_{\partial_{\Theta,\xi}^{2\alpha_1} \partial_\xi^{2\alpha_2} a_j} \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} &= \left\| \int_{\mathbb{R}^n} \left(\partial_{\Theta,\xi}^{2\alpha_1} \partial_\xi^{2\alpha_2} a_j(\xi) \otimes \mathbf{1} \right) \bullet \pi_\Theta(\exp_\xi) \, d\xi \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \\
&\lesssim 2^{jn} \sup_{|\xi|\lesssim 2^j} \left\| (\partial_\xi^{2\alpha_1} \partial_{\Theta,\xi}^{2\alpha_2} a_j)(\xi) \right\|_{\mathcal{R}_\Theta} \lesssim 2^{j(n+m-\rho(\ell_1+\ell_2))}.
\end{aligned}$$

For general (noneven) ℓ_1, ℓ_2 we proceed by interpolation. Note that the norm of k_a is not altered under left/right multiplication by d_Θ^{is} for any $s \in \mathbb{R}$. Therefore we have a bounded and holomorphic function $\zeta \mapsto \pi_\Theta(|z|^{\ell_1+2\zeta}) \bullet k_a \bullet \pi_\Theta(|z|^{\ell_2})$ defined in the band $0 < \Re(\zeta) < 1$. An application of the three lines lemma gives the bound for any ℓ_1, the same follows for ℓ_2. This completes the proof. $\square$

Proposition 3.24. *Given $a \in \Sigma_{\rho,\rho}^m(\mathcal{R}_\Theta)$ and $m_1, m_2 > 0$*

$$\left\| d_\Theta^{m_1} \bullet k_a \bullet d_\Theta^{m_2} \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim 1$$

provided $\rho(m_1 + m_2) > n + m$ for $\rho < 1$ or $\rho(m_1 + m_2) \geq n + m$ for $\rho = 1$.

Proof. Let

$$\rho_j(\xi) = \phi(2^{-j}\xi) - \phi(2^{-j+1}\xi)$$

be another partition of unity —this time for $j \in \mathbb{Z}$— and set $b_j = \pi_\Theta(\rho_j)$. Then

$$d_\Theta^{m_1} \bullet k_a \bullet d_\Theta^{m_2} = \sum_{j,k\geq 0} + \sum_{j,k\leq 0} + \sum_{j\cdot k<0} \left(b_j \bullet d_\Theta^{m_1} \bullet k_a \bullet d_\Theta^{m_2} \bullet b_k \right) = A_+ + A_- + A_\pm.$$

Estimate of A_+. Letting $a_\ell = a\psi_\ell$ as in Lemma 3.23

$$\|A_+\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \leq \sum_{j,k\geq 0} \sum_{\ell\geq 0} \left\| b_j \bullet d_\Theta^{m_1} \bullet k_{a_\ell} \bullet d_\Theta^{m_2} \bullet b_k \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} = \sum_{j,k\geq 0} \sum_{\ell\geq 0} A_+(j,k,\ell).$$

Pick ℓ_1 and ℓ_2 large enough (see below) and use Lemma 3.23 to estimate $A_+(j,k,\ell)$

$$\begin{aligned}
A_+(j,k,\ell) &\leq \left\| d_\Theta^{\ell_1} \bullet k_{a_\ell} \bullet d_\Theta^{\ell_2} \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \\
&\quad \times \left\| \pi_\Theta(\rho_j |z|^{(m_1-\ell_1)}) \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \left\| \pi_\Theta(\rho_k |z|^{(m_2-\ell_2)}) \right\|_{\mathcal{R}_\Theta \bar\otimes \mathcal{R}_\Theta^{\mathrm{op}}} \\
&\lesssim 2^{\ell(n+m-\rho(\ell_1+\ell_2))} 2^{j(m_1-\ell_1)} 2^{k(m_2-\ell_2)}.
\end{aligned}$$

Taking $\ell_1 > m_1$, $\ell_2 > m_2$ and $\rho(\ell_1 + \ell_2) > n + m$ we may sum over $j, k, \ell \geq 0$.

Estimate of A_-. Letting $a_\ell = a\psi_\ell$ once more, we get

$$A_- = \sum_{j,k\leq 0} \left\{ \sum_{\ell>|j|} + \sum_{\ell\leq|j|} \right\} \left(b_j \bullet d_\Theta^{m_1} \bullet k_{a_\ell} \bullet d_\Theta^{m_2} \bullet b_k \right) = \sum_{j,k\leq 0} A_-^1(j,k) + A_-^2(j,k)$$

$$= \underbrace{\sum_{j \le k \le 0} A_-^1(j,k)}_{A_-^{11}} + \underbrace{\sum_{k < j \le 0} A_-^1(j,k)}_{A_-^{12}} + \underbrace{\sum_{j \le k \le 0} A_-^2(j,k)}_{A_-^{21}} + \underbrace{\sum_{k < j \le 0} A_-^2(j,k)}_{A_-^{22}}.$$

First, we may bound $A_-^1(j,k)$ and $A_-^2(j,k)$ in norm via Lemma 3.23

$$\left\| A_-^1(j,k) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim \sum_{\ell > |j|} 2^{j(m_1 - \ell_1)} 2^{\ell((n+m) - \rho(\ell_1 + \ell_2))} 2^{k(m_2 - \ell_2)},$$

$$\left\| A_-^2(j,k) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim \sum_{\ell \le |j|} 2^{j(m_1 - \ell_1)} 2^{\ell((n+m) - \rho(\ell_1 + \ell_2))} 2^{k(m_2 - \ell_2)}.$$

A_-^{11}) Taking $\rho(\ell_1 + \ell_2) > n + m$ and $r = k - j \ge 0$, we get

$$\left\| A_-^1(j,k) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim 2^{j(m_1 - \ell_1)} 2^{|j|(n+m-\rho(\ell_1 + \ell_2))} 2^{k(m_2 - \ell_2)}$$

$$= 2^{r(m_2 - \ell_2)} 2^{|j|(n+m+(1-\rho)(\ell_1 + \ell_2) - (m_1 + m_2))}.$$

If $\rho < 1$, our condition $\rho(m_1 + m_2) > n + m$ allows us to pick ℓ_1, ℓ_2 satisfying $\ell_1 + \ell_2 = m_1 + m_2$ and $\ell_2 > m_2$. If $\rho = 1$ we pick $\ell_j > m_j$ for $j = 1, 2$. In both cases we get the inequalities $n + m + (1 - \rho)(\ell_1 + \ell_2) - (m_1 + m_2) \le 0$ and $m_2 - \ell_2 > 0$. This gives

$$\left\| A_-^{11} \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \le \sum_{r \ge 0} \left\| \sum_{j \le 0} A_-^1(j, j+r) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}}$$

$$\le \sum_{r \ge 0} r \sup_{j \le 0} \left\| A_-^1(j, j+r) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim \sum_{r \ge 0} r 2^{r(m_2 - \ell_2)} \lesssim 1.$$

By almost orthogonality of b_j's the sum inside the norm is a r-th diagonal operator, dominated by r times the supremum of the norms of its entries.

A_-^{21}) Letting $r = k - j \ge 0$, we get

$$\left\| A_-^2(j,k) \right\|_{\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}} \lesssim 2^{j(m_1 - \ell_1)} \max \left\{ 1, 2^{|j|(n+m-\rho(\ell_1 + \ell_2))} \right\} 2^{k(m_2 - \ell_2)}$$

$$= 2^{r(m_2 - \ell_2)} \max \left\{ 2^{|j|((\ell_1 + \ell_2) - (m_1 + m_2))}, 2^{|j|(n+m+(1-\rho)(\ell_1 + \ell_2) - (m_1 + m_2))} \right\}.$$

Then any choice with $\ell_1 + \ell_2 = m_1 + m_2$ and $\ell_2 > m_2$ gives $\|A_-^{21}\| \lesssim 1$.

A_-^{12}) We may write

$$A_-^{12} = \sum_{k < j \le 0} \left\{ \sum_{\ell > |k|} + \sum_{|j| < \ell \le |k|} \right\} \left(b_j \bullet \mathrm{d}_\Theta^{m_1} \bullet k_{a_\ell} \bullet \mathrm{d}_\Theta^{m_2} \bullet b_k \right) = A_-^{121} + A_-^{122}.$$

Then, A_-^{121} is estimated exactly as A_-11. On the other hand, the estimate of A_-^{122} is very much similar to that of A_-^{21}, we leave the details to the reader.

A_-^{22}) Interchanging roles of (j,k), A_-^{22} is estimated as A_-^{21} above (even simpler).

Estimate of $A_\pm$. Since the conditions on m_1, m_2 are symmetric in the statement and the sum $\sum_{i \cdot j < 0}$ splits into $\sum_{j < 0 < k} + \sum_{k < 0 < j}$, it suffices to estimate one of these two sums. Arguing as above, if we pick ℓ_1, ℓ_2 so that $\rho(\ell_1 + \ell_2) > n + m$, the problem reduces to estimate

$$\sum_{j < 0 < k} \left\{ \sum_{\ell > k} + \sum_{\ell \le k} \right\} 2^{j(m_1 - \ell_1)} 2^{\ell(n+m-\rho(\ell_1 + \ell_2))} 2^{k(m_2 - \ell_2)} = A + B.$$

Again as above, we pick $r = k - j > 0$ and obtain

$$A \lesssim \sum_{r>0} \sum_{k>0} 2^{-r(m_1-\ell_1)} 2^{k(n+m-(1+\rho)(\ell_1+\ell_2)+m_1+m_2)},$$

$$B \lesssim \sum_{r>0} \sum_{k>0} 2^{-r(m_1-\ell_1)} \max\left\{ 2^{k((m_1+m_2)-(\ell_1+\ell_2))}, 2^{k(n+m-(1+\rho)(\ell_1+\ell_2)+m_1+m_2)} \right\}.$$

Since $n + m \leq \rho(m_1 + m_2)$, it suffices to take $\ell_1 > m_1$ and $\ell_1 + \ell_2 > m_1 + m_2$. □

Theorem 3.25. *If $a, a_\dagger^* \in \Sigma_{1,1}^0(\mathcal{R}_\Theta)$, we have*

$$\left\| \Psi_a : \mathrm{H}_1(\mathcal{R}_\Theta) \to L_1(\mathcal{R}_\Theta) \right\|_{\mathrm{cb}} < \infty,$$

$$\left\| \Psi_a : L_\infty(\mathcal{R}_\Theta) \to \mathrm{BMO}(\mathcal{R}_\Theta) \right\|_{\mathrm{cb}} < \infty.$$

In particular, $\Psi_a : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$ is completely bounded when $1 < p < \infty$.

Proof. According to our Calderón-Zygmund extrapolation in Theorem 2.18, it suffices to see that Ψ_a is L_2-bounded and its kernel k_a satisfies the CZ conditions there. The L_2-boundedness follows from the quantum form of Bourdaud's condition in Theorem 3.19. On the other hand, according to Lemma 3.21, both $(\nabla_\Theta \otimes id)(k_a)$ and $(id \otimes \nabla_\Theta)(k_a)$ belong to $\Sigma_{1,1}^1(\mathcal{R}_\Theta)$. In particular, Proposition 3.24 yields the CZ kernel conditions which we need for $m_1 + m_1 = n + 1$.   □

Corollary 3.26. *If $a, a_\dagger^* \in \Sigma_{1,1}^m(\mathcal{R}_\Theta)$, we have*

$$\left\| \Psi_a : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-m}(\mathcal{R}_\Theta) \right\|_{\mathrm{cb}} < \infty \quad \text{for every} \quad 1 < p < \infty.$$

Proof. We have that

where $(\mathbf{1} - \Delta_\Theta)^{u/2} : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-u}(\mathcal{R}_\Theta)$ are complete isometries. On the other hand, the complete L_p-boundedness of $(\mathbf{1}-\Delta_\Theta)^{(s-m)/2} \Psi_a (\mathbf{1}-\Delta_\Theta)^{-s/2}$ follows from Theorem 3.25 once we observe that this pseudodifferential operator and its adjoint are associated to symbols in $\Sigma_{1,1}^0(\mathcal{R}_\Theta)$, which in turn follows from the composition rules for $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ established in Remark 3.5.   □

Remark 3.27. The assertions in Theorem 3.25 and Corollary 3.26 remain valid for symbols $a \in \Sigma_{1,\delta}^0(\mathcal{R}_\Theta)$ with $0 \leq \delta < 1$. Indeed, $\Sigma_{1,\delta}^0(\mathcal{R}_\Theta) \subset S_{1,\delta}^0(\mathcal{R}_\Theta) \subset S_{1,1}^0(\mathcal{R}_\Theta)$ and the middle class is stable under adjoints. Thus, we may apply our Bourdaud's condition as we did in the proof of Theorem 3.25. In addition, $\Sigma_{1,\delta}^0(\mathcal{R}_\Theta) \subset \Sigma_{1,1}^0(\mathcal{R}_\Theta)$ so that Proposition 3.24 applies. The argument for Sobolev spaces is similar.

Remark 3.28. A careful analysis of our proof for Theorem 3.25 and Corollary 3.26 yields that the condition $\Sigma_{1,1}^0(\mathcal{R}_\Theta)$ can be replaced by the weaker condition below

$$\left| \partial_\Theta^\beta \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} a(\xi) \right| + \left| \partial_\Theta^\beta \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} a_\dagger^*(\xi) \right| \leq C_{\alpha_1,\alpha_2,\beta} \langle \xi \rangle^{-|\alpha_1+\alpha_2|+|\beta|}$$

for $|\alpha_1 + \alpha_2| \leq n + 2$ and $|\beta| \leq 1$. Indeed, according to Remark 3.20, Bourdaud's condition in Theorem 3.19 can be weakened to $|\alpha_1 + \alpha_2| \leq n + 1$ and $|\beta| \leq 1$. Moreover, our proof of Proposition 3.24 for $m_1+m_2 = n+1$ requires $|\alpha_1+\alpha_2| \leq n+2$.

The L_p-theory for exotic symbols $\Sigma_{\rho,\rho}^m(\mathcal{R}_\Theta)$ ($\rho < 1$) is only possible due to the regularizing effect of a negative degree m. Fefferman proved in [29] the L_p bounds for the critical index $m = -(1 - \rho)\frac{n}{2}$. The noncritical range was obtained by Hirschman and Wainger [36, 73] (constant coefficients) and Hörmander [38] (general symbols). Standard interpolation arguments yield even more general statements [66, VII 5.12]. Now we shall prove (non optimal) inequalities of this kind in $\mathcal{R}_\Theta$ with applications below for L_p-regularity of elliptic PDEs. Namely, in what follows we shall write N for the best possible constant in Remark 3.10. As explained there we suspect that any $N > n/4$ is valid and this would be optimal, as it is the case in the Euclidean theory. Consider the index

$$\Lambda_{\rho,n} = -(1 - \rho)\max\big\{2N, n + 2\big\}.$$

It follows from arguments in [69] that 2N is (at least) less or equal than $3n + 2$.

Corollary 3.29. *Let $a \in \Sigma_{\rho,\rho}^m(\mathcal{R}_\Theta)$ be a symbol satisfying $m \leq \Lambda_{\rho,n}$ for some $\rho < 1$. Then, the pseudodifferential operator Ψ_a satisfies the following estimates for $1 < p < \infty$*

$$\big\|\Psi_a : \mathrm{H}_1(\mathcal{R}_\Theta) \to L_1(\mathcal{R}_\Theta)\big\|_{\mathrm{cb}} < \infty,$$
$$\big\|\Psi_a : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)\big\|_{\mathrm{cb}} < \infty,$$
$$\big\|\Psi_a : L_\infty(\mathcal{R}_\Theta) \to \mathrm{BMO}(\mathcal{R}_\Theta)\big\|_{\mathrm{cb}} < \infty.$$

Moreover, if m is any real number and $\ell = m - \Lambda_{\rho,n}$

$$\big\|\Psi_a : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-\ell}(\mathcal{R}_\Theta)\big\|_{\mathrm{cb}} < \infty.$$

Proof. Since $a \in \Sigma_{\rho,\rho}^m(\mathcal{R}_\Theta)$ and $m = \ell + \Lambda_{\rho,n}$

$$\langle\xi\rangle^{m-\rho|\alpha_1+\alpha_2|+\rho|\beta|} = \langle\xi\rangle^{m+(1-\rho)|\alpha_1+\alpha_2|-|\alpha_1+\alpha_2|+\rho|\beta|}$$
$$\leq \langle\xi\rangle^{m+(1-\rho)\max\{2N,n+2\}-|\alpha_1+\alpha_2|+\rho|\beta|} \leq \langle\xi\rangle^{\ell-|\alpha_1+\alpha_2|+\rho|\beta|}$$

as long as $|\alpha_1 + \alpha_2| \leq \max\{2N, n + 2\}$. This means that a satisfies the Hörmander condition $\Sigma_{1,\rho}^\ell(\mathcal{R}_\Theta)$ for $\partial_{\Theta,\xi}$, ∂_ξ of order up to $|\alpha_1 + \alpha_2| \leq \max\{2N, n + 2\}$. For the first assertion we apply Theorem A. The L_2-boundedness is guaranteed by our Calderón-Valillancourt theorem since $\Sigma_{1,\rho}^0(\mathcal{R}_\Theta) \subset S_{\rho,\rho}^0(\mathcal{R}_\Theta)$ and 2N ξ-derivatives suffice, according to Remark 3.10. Next, inclusion $\Sigma_{1,\rho}^0(\mathcal{R}_\Theta) \subset \Sigma_{1,1}^0(\mathcal{R}_\Theta)$ together with the fact that Proposition 3.24 only requires $|\alpha_1 + \alpha_2| \leq n + 2$ —see Remark 3.28— imply that the CZ kernel conditions also hold. This proves that the first assertion follows from Theorem A. Then, the second assertion follows by adapting the argument in the proof of Corollary 3.26. Indeed, arguing as in Remark 3.5 we deduce that

$$a \in \Sigma_{1,\rho}^\ell(\mathcal{R}_\Theta) \ \Rightarrow \ (\mathbf{1}-\Delta_\Theta)^{\frac{s-\ell}{2}}\Psi_a(\mathbf{1}-\Delta_\Theta)^{-\frac{s}{2}} = \Psi_b \quad \text{for some} \quad b \in \Sigma_{1,\rho}^0(\mathcal{R}_\Theta).$$

In fact, the same holds limiting the conditions above to a prescribed number of derivatives $\partial_{\Theta,\xi}$ and ∂_ξ. Hence, we apply Theorem A once more to conclude. $\square$

Remark 3.30. It is very tempting to claim that Corollary 3.29 holds for the index $\Lambda_{\rho,n} = -(1 - \rho)(n + 2)$ since it is reasonable to think that the above result follows from a direct combination of Remarks 3.27 and 3.28 above. However, at the time of this writing, we are not able to circumvent the adjoint stability used in Remark 3.27, since we need it for Hörmander conditions limited to a prescribed number of derivatives. The product stability used above is straightforward instead.

Remark 3.31. L_p-boundedness up to the critical index $m=-(1-\rho)\frac{n}{2}$ is still open.

4. L_p regularity for elliptic PDEs

In this section, we illustrate our results with a basic application to elliptic PDEs in quantum Euclidean spaces. Given $0 \le \delta \le \rho \le 1$ and $m \in \mathbb{R}$, a symbol $a \in S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ is called elliptic of order m when there exist constants $C, R > 0$ for which the following inequality holds

$$\big|a(\xi)\big| \ge C|\xi|^m \quad \text{for all} \quad |\xi| \ge R.$$

A prototypical example of elliptic symbol of order 2 is given by $a(\xi) = \xi^{\mathrm{t}} A\xi$ for some uniformly positive definite $A \in M_n(\mathcal{R}_\Theta)$. We shall be interested in the elliptic PDE

$$\Psi_a(u) = \varphi$$

with data φ in the Sobolev space $\mathrm{W}_{p,s}(\mathcal{R}_\Theta)$ and $a \in \Sigma^m_{1,\delta}(\mathcal{R}_\Theta)$. L_p-regularity means that, no matter which a priori regularity do we have in a given solution u, it must belong at least to the Sobolev space $\mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)$. When the regularity gained is smaller than m we speak about hypoellipticity. In the Euclidean case, elliptic regularity arises naturally for $(\rho, \delta) = (1,0)$ and still holds for $\rho = 1$, whereas the case $\rho < 1$ leads to hypoelliptic scenarios [69]. As we shall see, this is also the case in the quantum setting. Equipped with our results so far, the main obstruction we shall need to overcome will be to construct suitable parametrices for symbols in Σ-classes, for which we can not use product stability in that class. Our first step yields Sobolev p-estimates $\mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-\ell}(\mathcal{R}_\Theta)$ for symbols in $S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ instead of $\Sigma^\ell_{\rho,\delta}(\mathcal{R}_\Theta)$, provided the order m is small enough.

Lemma 4.1. *Given* $s, \ell \in \mathbb{R}$, *we have*

$$\Psi_a : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s-\ell}(\mathcal{R}_\Theta)$$

provided $a \in S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ *with degree* $m + (1+\delta) \max\big\{2\mathrm{N}, n+2\big\} \le \ell$.

Proof. Arguing as in the proof of Corollary 3.29, it suffices to see that a satisfies the $\Sigma^\ell_{1,\delta}(\mathcal{R}_\Theta)$-condition for $\partial_{\Theta,\xi}$ and ∂_ξ of order up to $\max\{2\mathrm{N}, n+2\}$. Then recalling that

$$\partial^j_{\Theta,\xi} = \partial^j_\xi + \frac{1}{2\pi i} \sum_{k=1}^n \Theta_{jk} \partial^k_\Theta,$$

we easily get the following estimate

$$\big\|\partial^\beta_\Theta \partial^{\alpha_1}_{\Theta,\xi} \partial^{\alpha_2}_\xi a(\xi)\big\|_{\mathcal{R}_\Theta} \quad \lesssim \quad \sum_{\alpha_{11}+\alpha_{12}=\alpha_1} \sum_{|\gamma|=|\alpha_{11}|} \big\|\partial^{\beta+\gamma}_\Theta \partial^{\alpha_{12}+\alpha_2}_\xi a(\xi)\big\|_{\mathcal{R}_\Theta}$$

$$\lesssim \quad \sum_{\alpha_{11}+\alpha_{12}=\alpha_1} \sum_{|\gamma|=|\alpha_{11}|} \langle\xi\rangle^{m-\rho|\alpha_{12}+\alpha_2|+\delta|\beta+\gamma|}.$$

When $|\alpha_1 + \alpha_2| \le \max\{2\mathrm{N}, n+2\}$, we use

$$\begin{aligned}
m &- \rho|\alpha_{12} + \alpha_2| + \delta|\beta + \gamma| \\
&= \quad m + (1-\rho)|\alpha_{12} + \alpha_2| + (1+\delta)|\gamma| - |\alpha_1 + \alpha_2| + \delta|\beta| \\
&\le \quad m + (1+\delta)|\alpha_1 + \alpha_2| - |\alpha_1 + \alpha_2| + \delta|\beta| \ \le \ \ell - |\alpha_1 + \alpha_2| + \delta|\beta|
\end{aligned}$$

since $m + \max\{2\mathrm{N}, n+2\} \le \ell$ by hypothesis. This completes the proof. $\qquad\square$

Lemma 4.2. *Let $a \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ be an elliptic symbol for some $0 \leq \delta < \rho \leq 1$ and degree m. Let $\ell = m + \Lambda_{\rho,n}$. Then, for every $k \in \mathbb{N}$, there exist symbols b_k and c_k satisfying the following properties:*

   i) $\Psi_{b_k}\Psi_a = id - \Psi_{c_k}$,
   ii) $c_k \in S_{\rho,\delta}^{k\gamma}(\mathcal{R}_\Theta)$ *with* $\gamma = \delta - \rho < 0$,
   iii) *If $\rho = 1$, then $\Psi_{b_k} : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)$ for all $s \in \mathbb{R}$.*
   iv) *If $\rho < 1$, then $\Psi_{b_k} : \mathrm{W}_{2,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{2,\,s+\ell}(\mathcal{R}_\Theta)$ for all $s \in \mathbb{R}$.*

In fact, the last assertion holds under the weaker assumption that $a \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$.

Proof. Let
$$b_1(\xi) = \big(1 - \phi(\xi)\big)a^{-1}(\xi)$$
where ϕ is a smooth function which is identically 1 in $\mathrm{B}_R(0)$ and vanishes outside $\mathrm{B}_{R+1}(0)$. Here R is determined by the ellipticity of a, so that $|a(\xi)| \geq C|\xi|^m$ for $|\xi| \geq R$. We claim that

   A) $b_1 \in \Sigma_{\rho,\delta}^{-m}(\mathcal{R}_\Theta)$,
   B) $\Psi_{b_1}\Psi_a = id - \Psi_{c_1}$ for some $c_1 \in S_{\rho,\delta}^\gamma(\mathcal{R}_\Theta)$.

Assuming the claim, let b_k and c_k be determined by

$$\Psi_{b_k} = \sum_{j=0}^{k-1} \Psi_{c_1}^j \Psi_{b_1} \quad \text{and} \quad \Psi_{c_k} = \Psi_{c_1}^k.$$

   i) $\Psi_{b_k}\Psi_a = \sum_{j=0}^{k-1} \Psi_{c_1}^j \Psi_{b_1}\Psi_a = \sum_{j=0}^{k-1} \Psi_{c_1}^j \big(id - \Psi_{c_1}\big) = id - \Psi_{c_1}^k = id - \Psi_{c_k}.$
   ii) $c_k \in S_{\rho,\delta}^{k\gamma}(\mathcal{R}_\Theta)$ with $\gamma = \delta - \rho < 0$ follows from Corollary 3.4, since $\delta < \rho$.

   iii) We may not use our results directly since we ignore whether or not b_k belongs to the right Σ-class, due to the lack (so far) of stability results for the product of symbols in these classes. However, when $\rho = 1$ we know from claim A) above and Remark 3.27 that

   $$\Psi_{b_1} : \mathrm{W}_{p,s}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s+m}(\mathcal{R}_\Theta).$$

   Let us note in passing that Corollary 3.29 would also do the job here for $\rho < 1$ and ℓ in place of m. Next, it suffices to show that $\Psi_{c_1}^j$ takes the Sobolev space $\mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)$ to itself. This is clear for the identity map with $j = 0$. On the other hand, the boundedness for $j > 0$ trivially follows from the case $j = 1$. Since $(a,b_1) \in \Sigma_{1,\delta}^m(\mathcal{R}_\Theta)\times \in \Sigma_{1,\delta}^{-m}(\mathcal{R}_\Theta)$, the boundedness of $\Psi_{c_1} = id - \Psi_{b_1}\Psi_a$ follows again from Remark 3.27.

   iv) By the product stability of S-classes from Section 3.1 and according to claims A) and B) we know that $b_k \in S_{\rho,\delta}^{-m}(\mathcal{R}_\Theta)$ and the result follows from the argument in Corollary 3.26. Indeed, it works in L_2 when Σ-classes are replaced by S-classes since we just need to apply Calderón-Vaillancourt and composition with the right powers of $\mathbf{1} - \Delta_\Theta$ in that case.

Once we have proved the assertion, it remains to justify our claim. Point A) follows easily once we express the involved derivatives of $a^{-1}(\xi)$ in terms of those

for a. It is clear that $\partial^j(a^{-1})(\xi) + a^{-1}(\xi)\partial^j(a)(\xi)a^{-1}(\xi) = 0$ for the derivations $\partial \in \{\partial_\xi, \partial_{\Theta,\xi}, \partial_\Theta\}$. By ellipticity we obtain the estimates below for $|\xi| \geq R$

$$
\begin{aligned}
\left\|\partial_\xi^j(a^{-1})(\xi)\right\|_{\mathcal{R}_\Theta} &\lesssim \langle\xi\rangle^{-m-\rho}, \\
\left\|\partial_\Theta^j(a^{-1})(\xi)\right\|_{\mathcal{R}_\Theta} &\lesssim \langle\xi\rangle^{-m+\delta}, \\
\left\|\partial_{\Theta,\xi}^j(a^{-1})(\xi)\right\|_{\mathcal{R}_\Theta} &\lesssim \langle\xi\rangle^{-m-\rho}.
\end{aligned}
$$

By Leibniz rule and induction we get $b_1 \in \Sigma_{\rho,\delta}^{-m}(\mathcal{R}_\Theta)$ which proves A). Then B) follows from the product stability in Corollary 3.4 as in [69, Theorem III.1.3]. $\quad\square$

Remark 4.3. The above result for $p = 2$ is still open for $1 < p < \infty$. According to point ii) and Lemma 4.1, we know that $\Psi_{c_1}^j$ is bounded on $\mathrm{W}_{p,s+\ell}(\mathcal{R}_\Theta)$ for j large enough. It would be tempting to deduce the result by complex interpolation with $j = 0$. However, imaginary powers of Ψ_{c_1} are generally unbounded in L_p since the same happens for Ψ_{c_1}, due to Fefferman's critical index $-(1-\rho)n/2$. Indeed, Ψ_{c_1} will not be bounded in L_p or $\mathrm{W}_{p,s}$ when $|\gamma|$ is small enough and $\rho < 1$.

Remark 4.4. In the absence of stability for products of symbols in Σ-classes —left open in Section 3.1— Lemmas 4.1 and 4.2 give together a good substitute for many applications. Lemma 4.2 provides a parametrix Ψ_{b_k} which, despite we ignore for the moment whether or not it lives in the right Σ-class, it does send $\mathrm{W}_{p,s}(\mathcal{R}_\Theta)$ to the correct Sobolev space. Moreover, we know from Lemma 4.1 that the same holds for the error term Ψ_{c_k} provided k is large enough, since $\gamma < 0$.

Theorem 4.5. *Given $0 \leq \delta < \rho \leq 1$, consider $a \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ an elliptic symbol for some $m \in \mathbb{R}$ and let $\ell = m + \Lambda_{\rho,n}$. Given $1 < p < \infty$ and $r, s \in \mathbb{R}$, assume $\varphi \in \mathrm{W}_{p,s}(\mathcal{R}_\Theta)$ and let u solve*

$$\Psi_a(u) = \varphi$$

for some $u \in \mathrm{W}_{p,r}(\mathcal{R}_\Theta)$. Then, the following estimates hold:

   i) *If $\rho = 1$, we get $\|u\|_{\mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)} \lesssim \|u\|_{\mathrm{W}_{p,r}(\mathcal{R}_\Theta)} + \|\varphi\|_{\mathrm{W}_{p,s}(\mathcal{R}_\Theta)}.$*
   ii) *If $\rho < 1$ and $p = 2$, we get $\|u\|_{\mathrm{W}_{2,s+\ell}(\mathcal{R}_\Theta)} \lesssim \|u\|_{\mathrm{W}_{2,r}(\mathcal{R}_\Theta)} + \|\varphi\|_{\mathrm{W}_{2,s}(\mathcal{R}_\Theta)}.$*

Proof. According to Lemma 4.2

$$u - \Psi_{c_k}(u) = \Psi_{b_k}\Psi_a(u) = \Psi_{b_k}(\varphi)$$

for any $k \geq 0$. This gives in particular

   i) If $\rho = 1$

$$
\begin{aligned}
\|u\|_{p,s+m} &\leq \|\Psi_{b_k}(\varphi)\|_{p,s+m} + \|\Psi_{c_k}(u)\|_{p,s+m} \\
&\lesssim \|\varphi\|_{p,s} + \|\Psi_{c_k}(u)\|_{p,s+m}.
\end{aligned}
$$

   ii) If $\rho < 1$ and $p = 2$

$$
\begin{aligned}
\|u\|_{2,s+\ell} &\leq \|\Psi_{b_k}(\varphi)\|_{2,s+\ell} + \|\Psi_{c_k}(u)\|_{2,s+\ell} \\
&\lesssim \|\varphi\|_{2,s} + \|\Psi_{c_k}(u)\|_{2,s+\ell}.
\end{aligned}
$$

Next, Lemma 4.1 gives that $\Psi_{c_k} : \mathrm{W}_{p,r}(\mathcal{R}_\Theta) \to \mathrm{W}_{p,s+\ell}(\mathcal{R}_\Theta)$ for k large enough. $\quad\square$

Remark 4.6. As in the Euclidean setting [69], Theorem 4.5 above gives elliptic L_p-regularity in the Hörmander class $\Sigma_{1,\delta}^m(\mathcal{R}_\Theta)$ and hypoelliptic L_2-regularity in $\Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$ when $\rho < 1$. The latter result remains open for other values of $p \neq 2$.

Compared to [69] our result for $p = 2$ quantifies the loss of regularity in terms of ρ and it holds in the larger class $S_{\rho,\delta}^m(\mathcal{R}_\Theta)$.

Appendix A. Noncommutative tori

Given any $n \times n$ anti-symmetric $\mathbb{R}$-matrix Θ, the subalgebra of $\mathcal{R}_\Theta$ generated by $w_j = u_j(1)$ is the rotation algebra $\mathcal{A}_\Theta$ —also known as quantum or noncommutative torus— and we have

$$w_j w_k = \exp(2\pi i \Theta_{jk}) w_k w_j.$$

$\mathcal{A}_\Theta$ can also be described as the $\mathbb{Z}^n$-periodic subalgebra

$$\mathcal{A}_\Theta \ = \ \big\langle \lambda_\Theta(\mathrm{k}) : \mathrm{k} \in \mathbb{Z}^n \big\rangle'' \ = \ \Big\{ \varphi \in \mathcal{R}_\Theta \,\big|\, \sigma_\Theta^{\mathrm{k}}(\varphi) = \varphi \text{ for all } \mathrm{k} \in \mathbb{Z}^n \Big\}.$$

The extension of our results for pseudodifferential operators to noncommutative tori $\mathcal{A}_\Theta$ follows by a combination of well-known transference arguments, which we recall now. Given a symbol $a : \mathbb{Z}^n \to \mathcal{A}_\Theta$ we shall say that

- $a \in S_{\rho,\delta}^m(\mathcal{A}_\Theta)$ when

$$\big| \partial_\Theta^\beta \partial_{\mathrm{k}}^\alpha a(\mathrm{k}) \big| \ \leq \ C_{\alpha,\beta} \, \langle \mathrm{k} \rangle^{m - \rho|\alpha| + \delta|\beta|}.$$

- $a \in \Sigma_{\rho,\delta}^m(\mathcal{A}_\Theta)$ when

$$\big| \partial_\Theta^\beta \partial_{\Theta,\mathrm{k}}^{\alpha_1} \partial_{\mathrm{k}}^{\alpha_2} a(\mathrm{k}) \big| \ \leq \ C_{\alpha_1,\alpha_2,\beta} \, \langle \mathrm{k} \rangle^{m - \rho|\alpha_1 + \alpha_2| + \delta|\beta|}.$$

In the above definitions, ∂_Θ remains the same differential operator as in $\mathcal{R}_\Theta$ whereas ∂_{k} is the difference operator $(\partial_{\mathrm{k}}^j a)(\mathrm{k}) = a(\mathrm{k} + e_j) - a(\mathrm{k})$. The mixed derivatives $\partial_{\Theta,\mathrm{k}}$ are again Θ-deformations of ∂_{k} by ∂_Θ's

$$\partial_{\Theta,\mathrm{k}}^j a(\mathrm{k}) \ = \ \partial_{\mathrm{k}}^j a(\mathrm{k}) + 2\pi i \big[x_{\Theta,j}, a(\mathrm{k}) \big] \ = \ \partial_{\mathrm{k}}^j a(\mathrm{k}) + \frac{1}{2\pi i} \sum_{\ell=1}^n \Theta_{j\ell} \, \partial_\Theta^\ell a(\mathrm{k}).$$

The associated pseudodifferential operator is

$$\Psi_a(\varphi) \ = \ \sum_{\mathrm{k} \in \mathbb{Z}^n} a(\mathrm{k}) \widehat{\varphi}(\mathrm{k}) \lambda_\Theta(\mathrm{k}) \quad \text{for} \quad \varphi = \sum_{\mathrm{k} \in \mathbb{Z}^n} \widehat{\varphi}(\mathrm{k}) \lambda_\Theta(\mathrm{k}).$$

We say that $\widetilde{a} : \mathbb{R}^n \to \mathcal{A}_\Theta$ is a Euclidean lifting of a when its restriction to $\mathbb{Z}^n$ coincides with the original symbol $a : \mathbb{Z}^n \to \mathcal{A}_\Theta$. Recall that we impose the lifting to take values in the periodic subalgebra $\mathcal{A}_\Theta$, not just in $\mathcal{R}_\Theta$. The extension/restriction theorem below provides a useful characterization of the quantum Hörmander classes in $\mathcal{A}_\Theta$ defined above since it relates them with their siblings in $\mathcal{R}_\Theta$.

Theorem A.1. *Assume $\rho > 0$:*

   i) *$a \in S_{\rho,\delta}^m(\mathcal{A}_\Theta)$ iff it admits a lifting $\widetilde{a} \in S_{\rho,\delta}^m(\mathcal{R}_\Theta)$.*
   ii) *$a \in \Sigma_{\rho,\delta}^m(\mathcal{A}_\Theta)$ iff it admits a lifting $\widetilde{a} \in \Sigma_{\rho,\delta}^m(\mathcal{R}_\Theta)$.*

In fact, the lifting $\widetilde{a} : \mathbb{R}^n \to \mathcal{A}_\Theta$ has the form

$$\widetilde{a}(\xi) \ = \ \sum_{\mathrm{k} \in \mathbb{Z}^n} \phi(\xi - \mathrm{k}) a(\mathrm{k})$$

for certain Schwartz function $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\phi(\mathrm{k}) = \delta_{\mathrm{k},0}$ for $\mathrm{k} \in \mathbb{Z}^n$.

The proof follows verbatim [64, Theorem 4.5.3] since the argument only affects the classical variables $k \in \mathbb{Z}^n$ and $\xi \in \mathbb{R}^n$. In particular, the exact same argument applies when we take values in $\mathcal{A}_\Theta$. In fact, the same extension procedure applies when the Hörmander condition is only required for finitely many derivatives in the line of Remark 3.28, see [64, Corollary 4.5.7]. The equality of the associated pseudodifferential operators is also proved in [64, Theorem 4.6.12 and Corollary 4.6.13]. Namely, the class of pseudodifferential operators associated to $S^m_{\rho,\delta}(\mathcal{A}_\Theta)$ or $\Sigma^m_{\rho,\delta}(\mathcal{A}_\Theta)$ can be identified with the corresponding Hörmander classes in $\mathcal{R}_\Theta$ for periodic symbols —that is, taking values in $\mathcal{A}_\Theta$— when acting on periodic elements $\varphi = \sigma^k_\Theta(\varphi)$ for $k \in \mathbb{Z}^n$. Finally, it is also worth mentioning that the extension above also respects ellipticity, as shown in [64, Theorem 4.9.15].

Theorem A.2. *Let $a : \mathbb{Z}^n \to \mathcal{A}_\Theta$ and $1 < p < \infty$:*

i) *If $a \in S^0_{\rho,\rho}(\mathcal{A}_\Theta)$ with $0 \le \rho < 1$, $\Psi_a : L_2(\mathcal{A}_\Theta) \to L_2(\mathcal{A}_\Theta)$.*
ii) *If $a \in S^0_{1,1}(\mathcal{A}_\Theta) \cap S^0_{1,1}(\mathcal{A}_\Theta)^*$, then $\Psi_a : L_2(\mathcal{A}_\Theta) \to L_2(\mathcal{A}_\Theta)$.*
iii) *If $a \in \Sigma^0_{1,1}(\mathcal{A}_\Theta) \cap \Sigma^0_{1,1}(\mathcal{A}_\Theta)^*$, then $\Psi_a : L_p(\mathcal{A}_\Theta) \to L_p(\mathcal{A}_\Theta)$.*

Proof of Theorem A.2 i) **and** ii)**.** Let $b_j = \mathrm{B}_{2^{-j}}(0)$ and

$$h_j = \frac{1}{\sqrt{|b_j|}} \lambda_\Theta(1_{b_j}) \quad \text{and} \quad \Lambda_j : \lambda_\Theta(k) \mapsto \lambda_\Theta(k) h_j.$$

Observe that $\Lambda_j : L_2(\mathcal{A}_\Theta) \to L_2(\mathcal{R}_\Theta)$ is an isometry for all $j \ge 1$. Indeed

$$
\begin{aligned}
\big\|\Lambda_j(\varphi)\big\|^2_{L_2(\mathcal{R}_\Theta)} &= \frac{1}{|b_j|} \Big\| \sum_{k \in \mathbb{Z}^n} \widehat{\varphi}(k) \lambda_\Theta(k) \lambda_\Theta(1_{b_j}) \Big\|^2_{L_2(\mathcal{R}_\Theta)} \\
&= \frac{1}{|b_j|} \Big\| \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} \widehat{\varphi}(k) 1_{b_j}(\xi - k) e^{2\pi i \langle k, \Theta_\downarrow \xi - k \rangle} \lambda_\Theta(\xi) \, d\xi \Big\|^2_{L_2(\mathcal{R}_\Theta)}.
\end{aligned}
$$

By Plancherel theorem and using that $b_j + k$ are pairwise disjoint, we get

$$
\begin{aligned}
\big\|\Lambda_j(\varphi)\big\|^2_{L_2(\mathcal{R}_\Theta)} &= \frac{1}{|b_j|} \int_{\mathbb{R}^n} \Big| \sum_{k \in \mathbb{Z}^n} \widehat{\varphi}(k) 1_{b_j}(\xi - k) e^{2\pi i \langle k, \Theta_\downarrow \xi - k \rangle} \Big|^2 \, d\xi \\
&= \frac{1}{|b_j|} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} \big| \widehat{\varphi}(k) \big|^2 1_{b_j}(\xi - k) \, d\xi = \sum_{k \in \mathbb{Z}^n} \big| \widehat{\varphi}(k) \big|^2 = \|\varphi\|^2_{L_2(\mathcal{A}_\Theta)}.
\end{aligned}
$$

Then, the assertion follows from the following claim

$$\lim_{j \to \infty} \Big\| \Lambda_j\big(\Psi_a(\varphi)\big) - \Psi_{\widetilde{a}}\big(\Lambda_j(\varphi)\big) \Big\|_{L_2(\mathcal{R}_\Theta)} = 0$$

for any trigonometric polynomial φ. In other words, for finite linear combinations of the $\lambda_\Theta(k)$'s. Indeed, assume the limit above vanishes, then Ψ_a is L_2-bounded since trigonometric polynomials are dense and

$$
\begin{aligned}
\big\|\Psi_a(\varphi)\big\|_{L_2(\mathcal{A}_\Theta)} &= \lim_{j \to \infty} \big\|\Lambda_j\big(\Psi_a(\varphi)\big)\big\|_{L_2(\mathcal{R}_\Theta)} \\
&= \lim_{j \to \infty} \big\|\Psi_{\widetilde{a}}\big(\Lambda_j(\varphi)\big)\big\|_{L_2(\mathcal{R}_\Theta)} \\
&\le \lim_{j \to \infty} \big\|\Lambda_j(\varphi)\big\|_{L_2(\mathcal{R}_\Theta)} = \|\varphi\|_{L_2(\mathcal{A}_\Theta)}.
\end{aligned}
$$

The inequality above follows by application of Theorem A.1 in conjunction with Theorems 3.9, 3.12 and 3.19. Let us then justify our claim above. It clearly suffices to prove it with $\varphi = \lambda_\Theta(k)$ for any $k \in \mathbb{Z}^n$. Given an arbitrary $\varepsilon > 0$, we shall

prove that the quantity $\|\Lambda_j(\Psi_a(\lambda_\Theta(k))) - \Psi_{\widetilde{a}}(\Lambda_j(\lambda_\Theta(k)))\|_2 < C\varepsilon$ for some absolute constant C independent of (k, ε) and j large enough. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be the function used in Theorem A.1 for the construction of the lifting. Since ϕ is a Schwartz function and $\phi(k) = \delta_{k,0}$ for $k \in \mathbb{Z}^n$, there must exists a $\delta > 0$ satisfying

$$|\xi - k| < \delta \ \Rightarrow \ \max\left\{\left|\phi(\xi - k) - 1\right|, \sup_{j \neq k}|\phi(\xi - j)|\right\} < \varepsilon R_\varepsilon^{-n}$$

where R_ε is large enough to satisfy

$$\sum_{|k| > R_\varepsilon} \frac{1}{|k|^{n+1}} < \varepsilon.$$

Next, consider the Fourier multiplier $M_{b_{k\delta}}(\varphi) = \int_{\mathbb{R}^n} 1_{b_{k\delta}}(\xi)\widehat{\varphi}(\xi)\lambda_\Theta(\xi)d\xi$ where we write $b_{k\delta}$ for $B_\delta(k)$. Then, we decompose the L_2-norm into three terms as follows

$$\begin{aligned}
&\left\|\Lambda_j\big(\Psi_a(\lambda_\Theta(k))\big) - \Psi_{\widetilde{a}}(\Lambda_j(\lambda_\Theta(k)))\right\|_2 \\
&= \ \left\|a(k)\lambda_\Theta(k)h_j - \Psi_{\widetilde{a}}\big(\lambda_\Theta(k)h_j\big)\right\|_2 \\
&\leq \ \left\|a(k)\big(\lambda_\Theta(k)h_j - M_{b_{k\delta}}(\lambda_\Theta(k)h_j)\big)\right\|_2 \\
&+ \ \left\|a(k)M_{b_{k\delta}}(\lambda_\Theta(k)h_j) - \Psi_{\widetilde{a}}\big(M_{b_{k\delta}}(\lambda_\Theta(k)h_j)\big)\right\|_2 \\
&+ \ \left\|\Psi_{\widetilde{a}}\big(M_{b_{k\delta}}(\lambda_\Theta(k)h_j)\big) - \Psi_{\widetilde{a}}\big(\lambda_\Theta(k)h_j\big)\right\|_2 \ = \ \mathrm{A} + \mathrm{B} + \mathrm{C}.
\end{aligned}$$

We recall one more time from Theorem A.1 and Theorem B in the Introduction that $\Psi_{\widetilde{a}} : L_2(\mathcal{R}_\Theta)) \to L_2(\mathcal{R}_\Theta)$ is a bounded map. Moreover, we also know that $a \in \ell_\infty(\mathbb{Z}^n; \mathcal{A}_\Theta)$ since it has degree 0. In particular

$$\mathrm{A} + \mathrm{C} \leq \Big(\sup_{k \in \mathbb{Z}^n}\|a(k)\|_{\mathcal{A}_\Theta} + \big\|\Psi_{\widetilde{a}}\big\|_{L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)}\Big)\big\|\lambda_\Theta(k)h_j - M_{b_{k\delta}}(\lambda_\Theta(k)h_j))\big\|_2.$$

The L_2-norm above can be estimated with Plancherel theorem

$$\begin{aligned}
&\left\|\lambda_\Theta(k)h_j - M_{b_{k\delta}}(\lambda_\Theta(k)h_j))\right\|_2 \\
&= \ \frac{1}{\sqrt{|b_j|}}\left\|\int_{\mathbb{R}^n}(1 - 1_{b_{k\delta}}(\xi))1_{b_j}(\xi - k)e^{2\pi i\langle k, \Theta_\downarrow\xi - k\rangle}\lambda_\Theta(\xi)\,d\xi\right\|_2 \\
&= \ \Big(\frac{1}{|b_j|}\int_{k+b_j}\big|1 - 1_{b_{k\delta}}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}} \longrightarrow \ \big|1 - b_{k\delta}(k)\big| \ = \ 0
\end{aligned}$$

as $j \to \infty$. Therefore, it remains to estimate the term B. Letting

$$\begin{aligned}
\mathbf{a}_k(\xi) \ &= \ \big(\widetilde{a}(\xi) - a(k)\big)1_{b_{k\delta}(\xi)} \\
\\
&= \ a(k)\big(\phi(\xi - k) - 1\big)1_{b_{k\delta}(\xi)} \\
\\
&+ \ \sum_{\substack{j \neq k \\ |j-k| \leq R_\varepsilon}} a(j)\phi(\xi - j)1_{b_{k\delta}(\xi)} \\
\\
&+ \ \sum_{|j-k| > R_\varepsilon} a(j)\phi(\xi - j)1_{b_{k\delta}(\xi)} \ = \ \mathbf{a}_{1k}(\xi) + \mathbf{a}_{2k}(\xi) + \mathbf{a}_{3k}(\xi),
\end{aligned}$$

we clearly have

$$\mathrm{B} \ = \ \big\|\Psi_{\mathbf{a}_k}(\lambda_\Theta(k)h_j)\big\|_2 \ \leq \ \sum_{j=1}^{3}\big\|\Psi_{\mathbf{a}_{jk}} : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)\big\|$$

since $\lambda_\Theta(\mathrm{k})h_j$ is a unit vector in $L_2(\mathcal{R}_\Theta)$. This gives

$$
\begin{aligned}
\mathrm{B} \;\leq\; & \Big(\sup_{\mathrm{j}\in\mathbb{Z}^n} \|a(\mathrm{j})\|_{\mathcal{A}_\Theta} \Big) \Big(\sup_{|\xi-\mathrm{k}|<\delta} \big|\phi(\xi-\mathrm{k})-1\big| \Big) \\[2mm]
& + \; \Big(\sup_{\mathrm{j}\in\mathbb{Z}^n} \|a(\mathrm{j})\|_{\mathcal{A}_\Theta} \Big) \Big(\sum_{\substack{\mathrm{j}\neq\mathrm{k}\\ |\mathrm{j}-\mathrm{k}|\le R_\varepsilon}} \sup_{|\xi-\mathrm{k}|<\delta} \big|\phi(\xi-\mathrm{j})\big| \Big) \\[2mm]
& + \; \Big(\sup_{\mathrm{j}\in\mathbb{Z}^n} \|a(\mathrm{j})\|_{\mathcal{A}_\Theta} \Big) \big\||\xi|^{n+1}\phi(\xi)\big\| \Big(\sum_{|\mathrm{j}-\mathrm{k}|>R_\varepsilon} \frac{1}{|\mathrm{j}-\mathrm{k}|^{n+1}} \Big) \;\lesssim\; 3\varepsilon.
\end{aligned}
$$

Then, letting $\varepsilon \to 0^+$ this completes the proof of the claim. $\qquad\square$

Remark A.3. The above argument was inspired by the proof of [11, Theorem A].

Proof of Theorem A.2 iii)**.** The next ingredient we need is the natural BMO space in $\mathcal{A}_\Theta$. Define $\mathrm{BMO}_c(\mathcal{A}_\Theta)$ as the column BMO space associated to the transferred heat semigroup $\varphi \mapsto \sum_{\mathrm{k}} \widehat{\varphi}(\mathrm{k})\exp(-t|\mathrm{k}|^2)\lambda_\Theta(\mathrm{k})$. As in Section 1.2.1 it can be regarded as the weak-$*$ closure of $\sigma_\Theta(\mathcal{A}_\Theta)$ with respect to the pair $(\mathrm{H}_1^c(\mathcal{Q}_\Theta), \mathrm{BMO}_c(\mathcal{Q}_\Theta))$. In other words, we find

$$
\|a\|_{\mathrm{BMO}_c(\mathcal{A}_\Theta)} \;\sim\; \sup_{Q\in\mathcal{Q}} \left\| \Big(\fint_Q \big|\sigma_\Theta(a)-\sigma_\Theta(a)_Q\big|^2 \, d\mu \Big)^{\frac{1}{2}} \right\|_{\mathcal{A}_\Theta},
$$

where $\mathcal{Q}$ is the set of all Euclidean cubes in $\mathbb{R}^n$ with sides parallel to the axes, μ stands for the Lebesgue measure and $\sigma_\Theta(a)_Q$ is the average of $\sigma_\Theta(a)$ over the cube Q. Up to absolute constants, it is not difficult to recover an equivalent norm when restricting to cubes Q of side length $\ell(Q) \in (0,1)\cup\mathbb{N}$. Moreover, since a is spanned by $\lambda_\Theta(\mathrm{k})$ for $\mathrm{k}\in\mathbb{Z}^n$, it is clear that $\sigma_\Theta(a)$ is $\mathbb{Z}^n$-periodic. In particular, the quantity above for $\ell(Q)\in\mathbb{N}$ coincides with the same quantity for $Q=[0,1]\times\ldots\times[0,1]$, so that we may assume in addition $\ell(Q)\le 1$. We have proved

$$
\|a\|_{\mathrm{BMO}_c(\mathcal{A}_\Theta)} \;\sim\; \sup_{Q\in\mathbb{T}^n} \left\| \Big(\fint_Q \big|\sigma_\Theta(a)-\sigma_\Theta(a)_Q\big|^2 \, d\mu \Big)^{\frac{1}{2}} \right\|_{\mathcal{A}_\Theta}.
$$

In other words, $\mathrm{BMO}_c(\mathcal{A}_\Theta)$ embeds into $\mathrm{BMO}_c(\mathbb{T}^n;\mathcal{A}_\Theta)$ using Mei's terminology [51]. The interpolation behavior and other natural properties which we explore for $\mathrm{BMO}(\mathcal{R}_\Theta)$ in Appendix B are well-known in this case [41], due to the finiteness of $\mathcal{A}_\Theta$. Note that, according to our definition of $\mathrm{BMO}(\mathcal{A}_\Theta)$, the natural inclusion map $\mathcal{A}_\Theta \to \mathcal{R}_\Theta$ extends to an embedding $\mathrm{BMO}(\mathcal{A}_\Theta) \to \mathrm{BMO}(\mathcal{R}_\Theta)$. In other words, $\mathrm{BMO}(\mathcal{A}_\Theta)$ is the subspace of periodic elements in $\mathrm{BMO}(\mathcal{R}_\Theta)$. Now, recalling that $\Psi_{\widetilde{a}}$ sends periodic elements into periodic elements, this makes the following a commutative diagram

$$
\begin{array}{ccc}
L_\infty(\mathcal{A}_\Theta) & \xrightarrow{\;\;id\;\;} & L_\infty(\mathcal{R}_\Theta) \\
{\scriptstyle \Psi_a}\downarrow & & \downarrow{\scriptstyle \Psi_{\widetilde{a}}} \\
\mathrm{BMO}(\mathcal{A}_\Theta) & \xrightarrow{\;\;id\;\;} & \mathrm{BMO}(\mathcal{R}_\Theta)
\end{array}
$$

The assertion follows from it and Theorem A.2 ii) by interpolation and duality. $\quad\square$

Remark A.4. Of course, our L_p-inequalities also hold in the category of operator spaces and admit the endpoint estimates $\mathrm{H}_1 \to L_1$ and $L_\infty \to \mathrm{BMO}$, as in the

quantum Euclidean setting. Besides, the natural analogues of Remarks 3.27 and 3.28 as well as Corollary 3.29 concerning L_p-estimates still apply. On the other hand, the Sobolev p-estimates in Corollary 3.26 and the L_p-regularity for elliptic PDEs require in addition analogues of the product stability of Hörmander classes in Section 3.1, which seems to be straightforward but we shall not generalize it here.

Appendix B. BMO space theory in $\mathcal{R}_\Theta$

The theory of BMO spaces was developed as a natural endpoint class for singular integral operators. In particular, the natural requirements for a reasonable BMO space are:

1) Interpolation endpoint for the L_p-scale.
2) John-Nirenberg inequalities and $\mathrm{H}_1 - \mathrm{BMO}$ duality.
3) $L_\infty \to \mathrm{BMO}$ boundedness for Calderón-Zygmund operators.

BMO spaces over von Neumann algebras were introduced by Pisier and Xu in [61] and have been investigated since then. The theory when averages over balls or martingale filtrations are replaced by the action of a Markovian semigroup has been addressed for finite von Neumann algebras in [41]. Interpolation requires a different approach over $\mathcal{R}_\Theta$ —less intricate than the general semifinite case— which we present here. Duality was developed by Mei [51, 52] and endpoint estimates for imaginary powers A^{is} of infinitesimal generators, noncommutative Riesz transforms or more general Fourier multipliers have been studied in [13, 40, 41, 42, 63]. In the setting of $\mathcal{A}_\Theta$ and $\mathcal{R}_\Theta$, Theorems A and B include many more singular integrals.

B.1. *Operator space structures on* BMO *and* H_1. Let us recall the definitions of several natural operator space structures —o.s.s. in short— for $\mathrm{BMO}(\mathbb{R}^n)$ and its predual. We define the column operator space structure by the family of matrix norms on $f = [f_{ij}] \in M_m[\mathrm{BMO}_c(\mathbb{R}^n)]$ given by

$$\|f\|_{M_m[\mathrm{BMO}_c(\mathbb{R}^n)]} = \sup_{Q \in \mathcal{Q}} \left\| \fint_Q (f - f_Q)^* (f - f_Q) \, d\mu \right\|_{M_m}^{\frac{1}{2}},$$

where f_Q is the average of f over Q and $\mathcal{Q}$ stands for the set of all the Euclidean balls. We will denote the resulting operator space by $\mathrm{BMO}_c(\mathbb{R}^n)$. Similarly, we can define the row o.s.s. by $\|f\|_{M_m[\mathrm{BMO}_r(\mathbb{R}^n)]} = \|f^*\|_{M_m[\mathrm{BMO}_c(\mathbb{R}^n)]}$. We shall also denote by $\mathrm{BMO}(\mathbb{R}^n)$ —sometimes $\mathrm{BMO}_{r \wedge c}(\mathbb{R}^n)$ for convenience— the operator space structure

$$\|f\|_{M_m[\mathrm{BMO}(\mathbb{R}^n)]} = \max \left\{ \|f\|_{M_m[\mathrm{BMO}_c(\mathbb{R}^n)]}, \|f\|_{M_m[\mathrm{BMO}_r(\mathbb{R}^n)]} \right\}.$$

These are dual operator spaces, with preduals $\mathrm{H}_1^\dagger(\mathbb{R}^n)^* = \mathrm{BMO}_\dagger(\mathbb{R}^n)$ given by

$$\|f\|_{\mathrm{H}_1^c(\mathbb{R}^n)} = \left\| \left(\int_{\mathbb{R}_+} \left| s(\nabla + \partial_s^2) P_s f \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^n)},$$

$$\|f\|_{\mathrm{H}_1^r(\mathbb{R}^n)} = \left\| \left(\int_{\mathbb{R}_+} \left| s(\nabla + \partial_s^2) P_s f^* \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^n)},$$

where P_s is the Poisson semigroup. The quantities above are just pseudonorms. A natural way of turning them into norms is working with 0-integral functions, in a way dual to the quotient of constants taken in the definition of BMO. Comparable

norms can be defined by removing the ∂_s^2 inside the square function and by using the semigroup analogue of Lusin area integral, given by

$$\|f\|_{\mathrm{H}_1^c(\mathbb{R}^n)} \sim \left\| \left(\int_{\Gamma_x} \left| (\nabla + \partial_s^2) P_s f(y) \right|^2 dsdy \right)^{\frac{1}{2}} \right\|_{L_1(\mathbb{R}^n)},$$

where $\Gamma_x = \{(y,s) \in \mathbb{R}^n \times \mathbb{R}_+ : |y-x| \leq s\}$ is the cone centered at x. The row case can be expressed analogously. The o.s.s. of Hardy spaces can be easily described by taking matrix-valued functions $f = [f_{ij}]$ in the expression above and taking norms in $S_1^m \widehat{\otimes} L_1(\mathbb{R}^n) = L_1(\mathbb{R}^n; S_1^m)$. That will give a family of matrix norms which describes the operator space structure. Indeed, using [59, Lemma 1.7] and the well-known relation

$$M_m[\mathrm{H}_1^\dagger(\mathbb{R}^n)] = \mathcal{CB}(S_1^m, \mathrm{H}_1^\dagger(\mathbb{R}^n))$$

see e.g. [60, Theorem 4.1], we can easily express the norm of $M_m[\mathrm{H}_1^\dagger(\mathbb{R}^n)]$ in terms of the known norms. The operator space predual $\mathrm{H}_1(\mathbb{R}^n)$ of $\mathrm{BMO}(\mathbb{R}^n)$ is given by the sum $\mathrm{H}_1^c(\mathbb{R}^n) + \mathrm{H}_1^r(\mathbb{R}^n)$, whose norm is

$$\|f\|_{S_1^m \widehat{\otimes} \mathrm{H}_1(\mathbb{R}^n)} = \inf \left\{ \|g\|_{S_1^m \widehat{\otimes} \mathrm{H}_1^r(\mathbb{R}^n)} + \|h\|_{S_1^m \widehat{\otimes} \mathrm{H}_1^c(\mathbb{R}^n)} : f = g + h \right\}.$$

Let us note that, by computations in Section 1.2.1 we have that σ_Θ gives an isomorphic embedding $\mathrm{BMO}_\dagger(\mathcal{R}_\Theta) \to \mathrm{BMO}_\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$. In particular, since $\mathcal{R}_\Theta$ is hyperfinite, we may equip $\mathrm{BMO}_\dagger(\mathcal{R}_\Theta)$ with an o.s.s. naturally inherited from $\mathrm{BMO}_\dagger(\mathbb{R}^n)$. Following Mei [52, 53], the definition of $\mathrm{H}_1^\dagger(\mathcal{R}_\Theta)$ will be given by completion on the 0-trace functions with respect to

$$\|\varphi\|_{\mathrm{H}_1^c(\mathcal{R}_\Theta)} = \left\| \left(\int_{\mathbb{R}_+} S_{\Theta,t} \left| \nabla_\Theta S_{\Theta,t} \varphi \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{R}_\Theta)},$$

$$\|\varphi\|_{\mathrm{H}_1^r(\mathcal{R}_\Theta)} = \left\| \left(\int_{\mathbb{R}_+} S_{\Theta,t} \left| \nabla_\Theta S_{\Theta,t} \varphi^* \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{R}_\Theta)}.$$

The operator space structures of such spaces are defined in the same way as the operator space structures of the classical ones, which could also have been defined with this square function instead of the given one yielding an equivalent norm. Note also that when $\Theta = 0$, the semigroup $S_{\Theta,t}$ behaves (intuitively) like an average over balls of radius $\sqrt{t}$ and a calculation gives that the quantities above are comparable to the Lusin integral and therefore recover the classical $\mathrm{H}_1(\mathbb{R}^n)$. We will write $\mathrm{H}_1(\mathcal{R}_\Theta)$ or $\mathrm{H}_1^{r+c}(\mathcal{R}_\Theta)$ for the sum

$$\mathrm{H}_1(\mathcal{R}_\Theta) = \mathrm{H}_1^r(\mathcal{R}_\Theta) + \mathrm{H}_1^c(\mathcal{R}_\Theta).$$

B.2. *The* H_1-BMO *duality.* The von Neumann algebra analogue of the celebrated $\mathrm{H}_1 - \mathrm{BMO}$ duality [30] has been carefully studied in our semigroup setting by Tao Mei. By [53, Theorem 0.2], the duality between $\mathrm{H}_1(\mathcal{R}_\Theta)$ and $\mathrm{BMO}(\mathcal{R}_\Theta)$ can be deduced after verifying that the associated heat semigroup $(S_{\Theta,t})_{t\geq 0}$ satisfies the following conditions:

i) Bakry's $\Gamma_2 \geq 0$ condition.

ii) For all $\varepsilon, t > 0$ and $\varphi \in L_1(\mathcal{R}_\Theta)$

$$\left\| (S_{\Theta,(1+\epsilon)t} - S_{\Theta,t})\varphi \right\|_{L_1(\mathcal{R}_\Theta)} \lesssim \varepsilon^r \|\varphi\|_{L_1(\mathcal{R}_\Theta)}.$$

iii) For every $t > 0$ and $\varphi \in L_1(\mathcal{R}_\Theta)$

$$\sup_{t>0} \left\| \left(\fint_0^{8t} S_{\Theta,s}(|S_{\Theta,t}(\varphi)|^2)\, ds \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{R}_\Theta)} \lesssim \|\varphi\|_{L_1(\mathcal{R}_\Theta)}.$$

Verifying such identities is relatively easy for the heat semigroup $(S_{\Theta,t})_{t \geq 0}$ after noting that it can be presented as an integrable convolution with respect to the z-variable $\sigma_\Theta^z(\varphi)$ and using bounds in $L_{1/2}(\mathcal{R}_\Theta)$. In particular, we obtain the expected duality theorem.

Theorem B.1. *We have*

$$\mathrm{H}_1^\dagger(\mathcal{R}_\Theta)^* = \mathrm{BMO}_\dagger(\mathcal{R}_\Theta)$$

in the category of operator spaces for $\dagger \in \{r, c\}$. Also $\mathrm{H}_1(\mathcal{R}_\Theta)^* = \mathrm{BMO}(\mathcal{R}_\Theta)$.

B.3. *Complex interpolation.* We are now interested in proving the generalization of the classical interpolation identities between L_p, BMO and H_1. According to Wolff's interpolation theorem [74], this can be easily reduced to justifying the complete isomorphism $[L_2(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta)]_\theta = L_p(\mathcal{R}_\Theta)$ for $p = \frac{2}{1-\theta}$ which in turn will be reduced, via suitable complemented subspaces, to the same result in $\mathbb{R}^n$ but with operator values in certain hyperfinite von Neumann algebra.

Let us recall a few standard definitions from interpolation theory. Given $\mathrm{X}_0, \mathrm{X}_1$ Banach spaces, assume that they embed inside a topological vector space with dense intersection, so that we can define $\mathrm{X}_0 \cap \mathrm{X}_1$ and $\mathrm{X}_0 + \mathrm{X}_1$ with their natural norms. Let us write $\mathcal{F}(\mathrm{X}_0, \mathrm{X}_1)$ for the space of $(\mathrm{X}_0 + \mathrm{X}_1)$-valued holomorphic functions in the strip $0 < \Re(z) < 1$ which admit a continuous extension to the boundary, with X_j-values at ∂_j for $j = 1, 2$. Such space is a Banach space with respect to the norm given by

$$\|f\|_{\mathcal{F}(\mathrm{X}_0,\mathrm{X}_1)} = \max \left\{ \sup_{s \in \mathbb{R}} \|f(is)\|_{\mathrm{X}_0}, \sup_{s \in \mathbb{R}} \|f(1+is)\|_{\mathrm{X}_1} \right\}.$$

The interpolated space with parameter $0 < \theta < 1$ is

$$\left[X_0, X_1 \right]_\theta = \mathcal{F}(\mathrm{X}_0, \mathrm{X}_1)/\mathfrak{N}_\theta,$$

where $\mathfrak{N}_\theta$ is the subspace of functions with $f(\theta) = 0$. We can also define a larger interpolation functor $[\mathrm{X}_0, \mathrm{X}_1]^\theta$ that contains $[\mathrm{X}_0, \mathrm{X}_1]_\theta$ isometrically by changing $\mathcal{F}(\mathrm{X}_0, \mathrm{X}_1)$ by a la larger space $\mathcal{F}_*(\mathrm{X}_0, \mathrm{X}_1)$ of holomorphic functions in which $f|_{\partial_j}$ is a more general X_j-valued distribution. These interpolation functors satisfy that $[\mathrm{X}_0, \mathrm{X}_1]_\theta^* = [\mathrm{X}_0^*, \mathrm{X}_1^*]^\theta$ and both coincide if any of the spaces involved $\mathrm{X}_0, \mathrm{X}_1$ is reflexive [5, Corollary 4.5.2] and [60, Theorem 2.7.4]. If X_j are operator spaces, the o.s.s. of $[\mathrm{X}_0, \mathrm{X}_1]_\theta$ is given by the identification

$$M_m\big([\mathrm{X}_0, \mathrm{X}_1]_\theta\big) = \big[M_m(\mathrm{X}_0), M_m(\mathrm{X}_1)\big]_\theta.$$

We first need an auxiliary result concerning complex interpolation of tensor products against hyperfinite von Neumann algebras. This result is a consequence of the interpolation identity $[\mathcal{M}_* \widehat{\otimes} \mathrm{X}_0, \mathcal{M}_* \widehat{\otimes} \mathrm{X}_1]_\theta = \mathcal{M}_* \widehat{\otimes} \mathrm{X}_\theta$ which can be found in [59, page 40]. We prove it for completeness.

Lemma B.2. *We have*

$$\big[\mathcal{M} \bar{\otimes} \mathrm{X}_0, \mathcal{M} \bar{\otimes} \mathrm{X}_1\big]^\theta = \mathcal{M} \bar{\otimes} [\mathrm{X}_0, \mathrm{X}_1]^\theta$$

for any hyperfinite algebra $\mathcal{M}$ and any pair of dual operator spaces X_0, X_1.

Proof. According to [26] the spaces involved are dual operator spaces. Indeed, von Neumann algebra preduals have the OAP so $\mathcal{M}\bar{\otimes}X^* = (\mathcal{M}_*\widehat{\otimes}X)^*$. Now, since hyperfiniteness and semidiscreteness are equivalent $id : \mathcal{M} \to \mathcal{M}$ is approximable in the pointwise weak-$*$ topology by a net $i_\alpha = \psi_\alpha\phi_\alpha$ where $\phi_\alpha : \mathcal{M} \to M_{m_\alpha}(\mathbb{C})$ and $\psi_\alpha : M_{m_\alpha}(\mathbb{C}) \to \mathcal{M}$ are ucp. We have

$$
\begin{array}{ccc}
\mathcal{M}\bar{\otimes}X^\theta & \xrightarrow{\quad i_\alpha\otimes id \quad} & \big[\mathcal{M}\bar{\otimes}X_0, \mathcal{M}\bar{\otimes}X_1\big]^\theta \\[2mm]
{\scriptstyle \psi_\alpha\otimes id}\searrow & & \nearrow{\scriptstyle \phi_\alpha\otimes id} \\[2mm]
& M_{m_\alpha}(X^\theta) \xrightarrow{\ \simeq\ } \big[M_{m_\alpha}(X_0), M_{m_\alpha}(X_1)\big]^\theta &
\end{array}
$$

a commutative diagram for $X^\theta = [X_0, X_1]^\theta$. The maps i_α approximate the identity and taking a weak-$*$ accumulation point in $\mathcal{CB}(\mathcal{M}\bar{\otimes}X\theta, [\mathcal{M}\bar{\otimes}X_0, \mathcal{M}\bar{\otimes}X_1]^\theta)$, which is a dual space since $\mathcal{CB}(X, Y^*) = (X\widehat{\otimes}Y)^*$, we obtain a complete isomorphism. $\square$

A key point in our interpolation argument will be to show that the co-action $\sigma_\Theta : \mathcal{R}_\Theta \to L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ also carries other $\mathcal{R}_\Theta$-spaces —L_p and BMO— into their $\mathcal{R}_\Theta$-valued Euclidean counterparts.

Proposition B.3. *We have complete contractions*:

    i) $\sigma_\Theta : L_2^\dagger(\mathcal{R}_\Theta) \longrightarrow L_2^\dagger(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ *for* $\dagger \in \{r, c\}$,

    ii) $\sigma_\Theta : L_p(\mathcal{R}_\Theta) \longrightarrow L_p(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ *for any* $2 \le p \le \infty$,

    iii) $\sigma_\Theta : \mathrm{BMO}_\dagger(\mathcal{R}_\Theta) \longrightarrow \mathrm{BMO}_\dagger(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ *for* $\dagger \in \{r, c, r\wedge c\}$.

Proof. Let us recall the Fubini-type identity

$$\mathbf{1}\tau_\Theta(\varphi) = \int_{\mathbb{R}^n} \sigma_\Theta^z(\varphi)\,dz.$$

In particular, given $\varphi = [\varphi_{ij}] \in M_m[L_2^c(\mathcal{R}_\Theta)]$ we obtain

$$
\begin{aligned}
\|\varphi\|_{M_m[L_2^c(\mathcal{R}_\Theta)]}^2 &= \big\|(id \otimes \tau_\Theta)(\varphi^*\varphi)\big\|_{M_m} = \big\|(id \otimes \mathbf{1}\tau_\Theta)(\varphi^*\varphi)\big\|_{M_m[\mathcal{R}_\Theta]} \\
&= \Big\|\int_{\mathbb{R}^n} \sigma_\Theta^z(\varphi)^*\sigma_\Theta^z(\varphi)\,dz\Big\|_{M_m[\mathcal{R}_\Theta]} = \big\|\sigma_\Theta(\varphi)\big\|_{M_m[L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta]}^2.
\end{aligned}
$$

The same follows in the row case. In fact, σ_Θ is a complete isometry in case i) and also in case iii) by construction of $\mathrm{BMO}(\mathcal{R}_\Theta)$. Assertion ii) follows by interpolation from Lemma B.2. Indeed, since $L_2(\mathbb{R}^n) = [L_2^c(\mathbb{R}^n), L_2^r(\mathbb{R}^n)]_{1/2}$ in the category of operator spaces and all spaces involved are reflexive, we obtain from i) that σ_Θ is a complete contraction from $L_2(\mathcal{R}_\Theta)$ to $L_2(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$. The case $p > 2$ also follows by complex interpolation, using the reflexivity of L_2, since the contractivity of the other endpoint for $p = \infty$ was already justified in Corollary 1.4. $\square$

Remark B.4. It is interesting to know whether an analogue of Proposition B.3 holds for $p = 1$. Note that $L_1(\mathbb{R}^n)$ is not a dual space, so that we can not use the weak-$*$ closed tensor product. Instead, we shall consider the mixed-norm space $L_\infty(\mathcal{R}_\Theta; L_1(\mathbb{R}^n))$ as introduced in [39, 45]

$$L_\infty\big(\mathcal{R}_\Theta; L_1(\mathbb{R}^n)\big) = \big(L_2^r(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta\big)\big(L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta\big),$$

where the operator space structure for $\omega \in M_m(L_\infty(\mathcal{R}_\Theta; L_1(\mathbb{R}^n)))$ is

$$\inf \left\{ \left\| \sum_k \alpha_k \otimes e_{1k} \right\|_{M_m(L_2^r(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta \otimes R)} \left\| \sum_k \beta_k \otimes e_{k1} \right\|_{M_m(L_2^c(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta \otimes C)} \right\}$$

where the infimum runs over all possible factorizations $\omega = \sum_k \alpha_k \beta_k$. Now, the contractivity of $\sigma_\Theta : L_1(\mathbb{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta; L_1(\mathbb{R}^n))$ follows easily from Proposition B.3 i). Particular cases of this kind of spaces —over finite von Neumann algebras or discrete ℓ_1 spaces— have been proved to interpolate in the expected way with the corresponding L_p scale [39, 45]. The lack of an available argument in the literature for the general case has led us to avoid the case $1 < p < 2$ in Proposition B.3. This contractivity result is unnecessary for our goals.

Observe that

$$\sigma_\Theta(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} f(\xi) \, (\exp_\xi \otimes \lambda_\Theta(\xi)) \, d\xi \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Clearly such element is invariant under the group of trace preserving automorphisms β_z given by $\beta_z = \sigma_0^{-z} \otimes \sigma_\Theta^z$. Let us denote by $(X \bar{\otimes} \mathcal{R}_\Theta)^\beta$ the β-invariant part of the $X \bar{\otimes} \mathcal{R}_\Theta$ with X any of the Euclidean function spaces in Proposition B.3. We need to see that $(X \bar{\otimes} \mathcal{R}_\Theta)^\beta$ coincides with the image of σ_Θ and that the β-invariant subspace is complemented. Let us start with the complementation.

Proposition B.5. *The following subspaces*

i) $(L_2^\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta)^\beta \subset L_2^\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *for* $\dagger \in \{r, c\}$,

ii) $(L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta)^\beta \subset L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *for any* $2 \le p \le \infty$,

iii) $(\mathrm{BMO}_\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta)^\beta \subset \mathrm{BMO}_\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *for* $\dagger \in \{c, r, r \wedge c\}$,

are completely complemented as operator spaces in the respective ambient spaces.

Proof. By amenability of $\mathbb{R}^n$, let $m \in L_\infty(\mathbb{R}^n)^*$ be an invariant mean and let m_α be a sequence of probability measures in $L_1(\mathbb{R}^n)$ which approximate m. Given ω in $L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$, the function $z \mapsto \beta_z \omega$ sits in the space $L_\infty(\mathbb{R}^n) \bar{\otimes} L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$, so

$$P_\alpha(\omega) = (m_\alpha \otimes id \otimes id)(\beta_z \omega)$$

defines a family of completely positive operators

$$P_\alpha : L_\infty(\mathbb{R}^n) \bar{\otimes} L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta \to L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta.$$

Since the image is in a dual space, we use $\mathcal{CB}(X, Y^*) = (X \widehat{\otimes} Y)^*$ and Banach-Alaoglu theorem. Let P be an accumulation point of $(P_\alpha \circ \beta)_\alpha$ in the weak-$*$ topology. P gives a cb-bounded projection into the β-invariant part. We have only used that $L_p(\mathbb{R}^n)$ is a dual space for weak-$*$ compactness. Therefore, the same proof applies to $\mathrm{BMO}_\dagger(\mathbb{R}^n)$. The projections $P : L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta \to L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ form compatible family: they are restrictions of a map defined in the sum of the above spaces. $\square$

Proposition B.6. *We have*

i) $\sigma_\Theta(L_2^\dagger(\mathcal{R}_\Theta)) = (L_2^\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta)^\beta$ *for* $\dagger \in \{r, c\}$,

ii) $\sigma_\Theta(L_p(\mathcal{R}_\Theta)) = (L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta)^\beta$ *for any* $2 \le p \le \infty$,

iii) $\sigma_\Theta(\mathrm{BMO}_\dagger(\mathcal{R}_\Theta)) = (\mathrm{BMO}(\mathbb{R}^n)_\dagger \bar{\otimes} \mathcal{R}_\Theta)^\beta$ *for* $\dagger \in \{r, c, r \wedge c\}$.

Proof. Since the spaces $(L_p(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta)^\beta$ are complemented subspaces, it is enough to prove the identity for $p = 2$ and $p = \infty$ and interpolation will yield the result for $2 < p < \infty$ since the maps $\sigma_\Theta : L_p(\mathcal{R}_\Theta) \to L_p(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ are compatible. The same argument gives that the L_2 case follows by interpolation between $L_2^c(\mathcal{R}_\Theta)$ and $L_2^r(\mathcal{R}_\Theta)$. This reduces the proof to the row/column cases, the case $p = \infty$ and $\mathrm{BMO}(\mathcal{R}_\Theta)$. We shall only prove it for columns and for $p = \infty$, since the argument is similar in BMO. Let us define the map $\mathcal{W} : L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta \to L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ by extension of $\exp_\xi \otimes\lambda_\Theta(\eta) \mapsto \exp_\xi \otimes\lambda_\Theta(\xi)\lambda_\Theta(\eta)$. A calculation easily yields that $\mathcal{W}$ is a complete isometry. The same follows in the row case if one takes the map $\exp_\xi \otimes\lambda_\Theta(\eta) \mapsto \exp_\xi \otimes\lambda_\Theta(\eta)\lambda_\Theta(\xi)$ instead. We have that $\mathcal{W}$ gives an isomorphism between $L_2^c(\mathbb{R}^n) \otimes \mathbf{1}$ and $\sigma_\Theta[\lambda_\Theta[L_2(\mathbb{R}^n)]]$. We also have that $\mathcal{W}$ intertwines the action β_z as follows

$$
\begin{array}{ccc}
L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta & \xrightarrow{\ \ \mathcal{W}\ \ } & L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta \\
\Big\downarrow{\sigma_\Theta^z} & & \Big\downarrow{\beta_z} \\
L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta & \xrightarrow{\ \ \mathcal{W}\ \ } & L_2^c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta.
\end{array}
$$

Therefore, the subspace fixed by β corresponds under $\mathcal{W}$ with the subspace fixed by $id \otimes \sigma_\Theta$. But evaluating such space against every $\varphi \otimes id$, with $\varphi \in L_2^r(\mathbb{R}^n)$, gives that the fixed subspace of $id \otimes \sigma_\Theta$ is $L_2^c(\mathbb{R}^n)$ tensored with the subspace fixed by σ_Θ. Such subspace is $\mathbb{C}\mathbf{1}$. Indeed, if $\varphi \in \mathcal{R}_\Theta$ is invariant under σ_Θ we obtain that $\varphi = \lambda_\Theta(\psi)$, where $\psi \in \mathcal{S}(\mathbb{R}^n)'$ is a distribution supported on $\{0\}$. But such distribution is a linear combination of distributions of the form $\langle\psi, f\rangle = f^{(k)}(0)$, where $f \in \mathcal{S}(\mathbb{R}^n)$. The derivatives with $k > 0$ give rise to unbounded elements and so we obtain that ψ has to be a multiple of δ_0 or, equivalently, that $\varphi \in \mathbb{C}\mathbf{1}$.

The case of $p = \infty$ follows similarly. We first define a normal $*$-homomorphim $\mathcal{U} : L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta \to L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ by extension of $\exp_\xi \otimes\lambda_\Theta(\eta) \mapsto \exp_{\eta+\xi} \otimes\lambda_\Theta(\xi)$. To prove that such map is a $*$-homomorphism we can implement it spatially with techniques analogous to that of Corollary 1.4. We have that $\mathcal{U}$ carries $\mathbf{1} \otimes \mathcal{R}_\Theta$ in $\sigma_\Theta[\mathcal{R}_\Theta]$ and that it intertwines the actions in the expected way. Proceeding like in the case $p = 2$ we can conclude.

The case of $\mathrm{BMO}_\dagger$ can be deduced from a similar result for mixed spaces. First we note that the result for $\mathrm{BMO}_{r\wedge c}$ follows from the corresponding ones for BMO_r and BMO_c, we shall only prove it for BMO_c. Fix a Euclidean ball $\mathrm{B} \subset \mathbb{R}^n$ and consider the following operator-valued inner product

$$
\langle f, f\rangle_\mathrm{B} = \fint_\mathrm{B} |f_s|^2\, ds - \Big|\fint_\mathrm{B} f_s\, ds\Big|^2 \quad \text{for} \quad f \in L_\infty(\mathbb{R}^n) \otimes_{\mathrm{alg}} \mathcal{R}_\Theta.
$$

Let $\mathcal{H}_\Theta^c(\mathrm{B})$ denote the corresponding Hilbert module over $\mathcal{R}_\Theta$ and let $\mathcal{H}_\Theta^c$ be the direct sum, in the ℓ_∞-sense, of $\mathcal{H}_\Theta^c(\mathrm{B})$ over all balls B. Clearly $\mathrm{BMO}_c(\mathcal{R}_\Theta)$ embeds in $\mathcal{H}_\Theta^c$ and we have that

$$
\begin{array}{ccc}
\mathcal{H}_\Theta^c & \xrightarrow{\ \ id\otimes\sigma_\Theta\ \ } & \mathcal{H}_\Xi^c \\
\Big\uparrow & & \Big\uparrow \\
\mathrm{BMO}_c(\mathcal{R}_\Theta) & \xrightarrow{\ \ \sigma_\Theta\ \ } & \mathrm{BMO}_c(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta
\end{array}
$$

where Ξ is the $2n \times 2n$-matrix $\Xi = 0 \otimes \Theta$. Now, we can define a map preserving the operator-valued inner product (and thus an isometry) $\mathcal{W}_c : \mathcal{H}_\Theta^c \to \mathcal{H}_\Xi^c$ by extension of $\exp_{\xi_1} \otimes \exp_{\xi_2} \otimes \lambda_\Theta(\eta) \mapsto \exp_{\xi_1} \otimes \exp_{\xi_2+\eta} \otimes \lambda_\Theta(\eta)$ for every ball B. Such map carries the copy of $\mathcal{H}_\Theta^c \otimes \mathbf{1}$ that lives in the first and third tensor components into $\sigma_\Theta[\mathcal{H}_\Theta^c]$ and proceeding like in the previous cases we get that $\sigma_\Theta[\mathcal{H}_\Theta^c]$ coincides with the subspace of $\mathcal{H}_\Xi^c$ invariant under the group of automorphisms $\beta_z = id \otimes \sigma_0^{-z} \otimes \sigma_\Theta^z$. That result restricts to BMO$_c$. $\qquad\square$

Proposition B.7. *We have complete isometries*:

i) $\sigma_\Theta : L_2^\dagger(\mathcal{R}_\Theta) \longrightarrow L_2^\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *for* $\dagger \in \{r, c\}$,

ii) $\sigma_\Theta : L_p(\mathcal{R}_\Theta) \longrightarrow L_p(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$ *for any* $2 \le p \le \infty$,

iii) $\sigma_\Theta : \mathrm{BMO}_\dagger(\mathcal{R}_\Theta) \longrightarrow \mathrm{BMO}_\dagger(\mathcal{R}_\Theta) \bar{\otimes} \mathcal{R}_\Theta$ *for* $\dagger \in \{r, c, r \wedge c\}$.

Proof. Assertions i) and iii) were proved in the proof of Proposition B.3. Assertion ii) for $p = \infty$ was already justified in Corollary 1.4. The rest of the cases trivially follow by complementation and complex interpolation from our results above. $\qquad\square$

Theorem B.8. *We have*

$$\big[\mathrm{H}_1(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta)\big]_\theta = \big[L_1^\circ(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta)\big]_\theta$$
$$= \big[\mathrm{H}_1(\mathcal{R}_\Theta), L_\infty(\mathcal{R}_\Theta)\big]_\theta = L_p(\mathcal{R}_\Theta)$$

for $p = \frac{1}{1-\theta}$. *All isomorphisms above hold in the category of operator spaces.*

Proof. Since $L_2(\mathcal{R}_\Theta)$ is reflexive

$$\big[L_2(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta)\big]_\theta = \big[L_2(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta)\big]^\theta$$
$$= \big[\sigma_\Theta(L_2(\mathcal{R}_\Theta)), \sigma_\Theta(\mathrm{BMO}(\mathcal{R}_\Theta))\big]^\theta$$
$$= \big[P(L_2(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta), P(\mathrm{BMO}(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta)\big]^\theta$$
$$= P\big(\big[L_2(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta, \mathrm{BMO}(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta\big]^\theta\big)$$
$$= P\big(\big[L_2(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n)\big]^\theta\bar{\otimes}\mathcal{R}_\Theta\big)$$
$$= P\big(L_p(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta\big) = \sigma_\Theta\big(L_p(\mathcal{R}_\Theta)\big) = L_p(\mathcal{R}_\Theta)$$

for $p = \frac{2}{1-\theta}$. Indeed, the second identity follows from Proposition B.7, which gives $\mathrm{X} = \sigma_\Theta(\mathrm{X})$ completely isomorphic for $\mathrm{X} = L_2(\mathcal{R}_\Theta)$ and $\mathrm{X} = \mathrm{BMO}(\mathcal{R}_\Theta)$. The third and fourth identities follow from Proposition B.5, which shows that $\sigma_\Theta(\mathrm{X})$ can be identified with $P(\mathrm{Z})$ where Z is the ambient space of operator-valued functions in $\mathbb{R}^n$ associated to X. Moreover, since P is a bounded projection, it commutes with the complex interpolation functor by complementation. The fifth identity follows from Lemma B.2 and the sixth one from Mei's interpolation theorem [53]. The last two identities apply from Propositions B.5 and B.7 again. Once this is known we use the reflexivity of $L_2(\mathcal{R}_\Theta)$ and duality $\mathrm{H}_1(\mathcal{R}_\Theta)^* = \mathrm{BMO}(\mathcal{R}_\Theta)$ to obtain

$$\big[\mathrm{H}_1(\mathcal{R}_\Theta), L_2(\mathcal{R}_\Theta)\big]_\theta^* = \big[\mathrm{BMO}(\mathcal{R}_\Theta), L_2(\mathcal{R}_\Theta)\big]^\theta = L_{q'}(\mathcal{R}_\Theta)$$

for $q = \frac{2}{2-\theta}$. This shows that $[\mathrm{H}_1(\mathcal{R}_\Theta), L_2(\mathcal{R}_\Theta)]_\theta$ must be reflexive and we get

$$\big[\mathrm{H}_1(\mathcal{R}_\Theta), L_2(\mathcal{R}_\Theta)\big]_\theta = L_q(\mathcal{R}_\Theta).$$

The interpolation results in the statement follow from Wolff's theorem [74], which states that if X_1, X_2, X_3, X_4 are spaces with $X_1 \cap X_4$ dense inside both X_2 and X_3, then

$$X_2 = [X_1, X_3]_{\theta_1} \text{ and } X_3 = [X_2, X_4]_{\theta_2} \Rightarrow X_2 = [X_1, X_4]_{\vartheta_1} \text{ and } X_3 = [X_1, X_4]_{\vartheta_2}$$

where $\vartheta_1 = \theta_1 \theta_2 / (1 - \theta_1 + \theta_1 \theta_2)$ and $\vartheta_2 = \theta_2 / (1 - \theta_1 + \theta_1 \theta_2)$. Taking

$$\begin{aligned}
X_1 &= H_1(\mathcal{R}_\Theta), & X_2 &= L_{\frac{4}{3}}(\mathcal{R}_\Theta), & X_3 &= L_2(\mathcal{R}_\Theta), & X_4 &= L_4(\mathcal{R}_\Theta), \\
Z_1 &= H_1(\mathcal{R}_\Theta), & Z_2 &= L_2(\mathcal{R}_\Theta), & Z_3 &= L_4(\mathcal{R}_\Theta), & Z_4 &= \mathrm{BMO}(\mathcal{R}_\Theta),
\end{aligned}$$

we first obtain, using X_j-spaces and the interpolation of $H_1(\mathcal{R}_\Theta)$ with $L_2(\mathcal{R}_\Theta)$, that $H_1(\mathcal{R}_\Theta)$ and $L_4(\mathcal{R}_\Theta)$ interpolate in the expected way. Then, using the same procedure with the Z_j-spaces and the interpolation of $L_2(\mathcal{R}_\Theta)$ with $\mathrm{BMO}(\mathcal{R}_\Theta)$, we finally get the expected result for the bracket

$$\left[H_1(\mathcal{R}_\Theta), \mathrm{BMO}(\mathcal{R}_\Theta) \right]_\theta.$$

The other two brackets in the statement can be treated analogously. $\square$

Remark B.9. It is worth mentioning that our techniques have at least another potential application in the abelian case. Let (X, μ) be a G-space with a G-invariant measure. In that case, we can identify X and $H \times G/H$ as measure spaces, where H is the stabilizer and we have the following Fubini-type identity

$$\int_G f(g^{-1} x) \, d\mu_G(g) = \int_{G/H = X} \int_H f(h^{-1} g^{-1} x) \, d\mu_H(h) \, d\mu(x),$$

see [31, Chapter 2]. If the stabilizer is compact we can exchange integration in X and integration in G in a way analogous to the Fubini-type identity which relates τ_Θ and σ_Θ. If there is a natural definition of $\mathrm{BMO}(G)$, either with averages over the balls of an invariant measure or with translation-invariant semigroups, and that BMO interpolates, then we can transfer the interpolation to $\mathrm{BMO}(X)$ provided that G is amenable. This seems to be a very direct approach for proving interpolation of G-invariant BMO-spaces over $X = G/K$, where G is a solvable and unimodular Lie group and K is a compact subgroup.

B.4. *An auxiliary density result.* Let us write in what follows $\mathcal{S}_\Theta^\circ$ for the kernel of the trace functional $\tau_\Theta : \mathcal{S}_\Theta \to \mathbb{C}$, which is of course continuous over $\mathcal{S}_\Theta$. It is trivial that $\mathcal{S}_\Theta^\circ \subset H_1^\dagger(\mathcal{R}_\Theta)$. We are going to see that it is in fact dense. It will be an easy consequence of the fact that $\sigma_\Theta : \mathrm{BMO}_\dagger(\mathcal{R}_\Theta) \to \mathrm{BMO}_\dagger(\mathbb{R}^n) \bar{\otimes} \mathcal{R}_\Theta$, for $\dagger \in \{r, c, r \wedge c\}$ are normal and complete isometries. Taking preduals we obtain a complete and surjective projection

$$(\sigma_\Theta)_* : H_1^\dagger(\mathbb{R}^n) \widehat{\otimes} (\mathcal{R}_\Theta)_* \longrightarrow H_1^\dagger(\mathcal{R}_\Theta),$$

for $\dagger \in \{r, c, r + c\}$. We are just going to need that such map carries $\mathcal{S}^\circ(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta$ into $\mathcal{S}_\Theta^\circ$ but indeed much more is true and the map $(\sigma_\Theta)_*$ can be explicitly described as a diagonal restriction multiplier. That is, it satisfies the following commutative diagram, where $\mathcal{S}_0(\mathbb{R}^n)$ is the subclass of Schwartz functions with $f(0) = 0$

$$
\begin{array}{ccc}
H_1^\dagger(\mathbb{R}^n) \widehat{\otimes} (\mathcal{R}_\Theta)_* & \xrightarrow{\;(\sigma_\Theta)_*\;} & H_1^\dagger(\mathcal{R}_\Theta) \\
{\scriptstyle \lambda_0 \otimes \lambda_\Theta} \big\uparrow & & \big\downarrow {\scriptstyle \lambda_\Theta^{-1}} \\
\mathcal{S}_0(\mathbb{R}^n) \otimes_\pi \mathcal{S}(\mathbb{R}^n) & \xrightarrow{\;f \mapsto f|_\Delta\;} & \mathcal{S}_0(\mathbb{R}^n)
\end{array}
$$

Now, the proof of the density is immediate.

Corollary B.10. $\mathcal{S}_\Theta^\circ$ *is dense inside* $\mathrm{H}_1^\dagger(\mathcal{R}_\Theta)$ *for* $\dagger \in \{r, c, r + c\}$.

Proof. We just have to use that $\mathcal{S}^\circ(\mathbb{R}^n) \otimes_\pi \mathcal{S}_\Theta \subset \mathrm{H}_1^\dagger(\mathbb{R}^n) \widehat{\otimes}(\mathcal{R}_\Theta)_*$ is a dense subset. Since $(\sigma_\Theta)_*(\mathcal{S}^\circ(\mathbb{R}^n)) \subset \mathcal{S}_\Theta^\circ$ and the image under a projection of a dense set is a dense set we conclude. $\qquad\square$

References

1. S. Baaj, Calcul pseudo-différentiel et produits croisés de C*-algèbres. I. C. R. Acad. Sci. Paris **307** (1988), 581-586.
2. S. Baaj, Calcul pseudo-différentiel et produits croisés de C*-algèbres. II. C. R. Acad. Sci. Paris **307** (1988), 663-666.
3. D. Bakry, Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée. Sém. Probabilités XXI, 137-172. Lecture Notes in Math. 1247. Springer, 1987.
4. R. Beals and C. Fefferman, On local solvability of linear partial differential equations. Ann. of Math. **97** (1973), 482-498.
5. J. Bergh and J. Lofstrom, Interpolation spaces: an introduction. Springer, 2012.
6. T.A. Bhuyain and M. Marcolli, The Ricci flow on noncommutative two-tori. Lett. Math. Phys. **101** (2012), 173-194.
7. G. Bourdaud, Une algèbre maximale d'opérateurs pseudo-differentiels. Comm. PDEs **13** (1988), 1059-1083.
8. N.P. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations. Grad. Stud. in Math. **88**. Amer. Math. Soc. 2008.
9. A.P. Calderón and R. Vaillancourt, A class of bounded pseudo-differential operators. Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 1185-1187.
10. A.P. Calderón and A. Zygmund, On the existence of certain singular integrals. Acta Math. **88** (1952), 85-139.
11. M. Caspers, J. Parcet, M. Perrin and É. Ricard, Noncommutative de Leeuw theorems. Forum Math. Σ **3** (2015), e21.
12. M. Caspers, D. Potapov, F. Sukochev and D. Zanin, Weak type commutator and Lipschitz estimates: resolution of the Nazarov-Peller conjecture. Amer. J. Math. To appear.
13. Z. Chen, Q. Xu and Z. Yin, Harmonic analysis on quantum tori. Comm. Math. Phys. **322** (2013), 755-805.
14. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. **83** (1977), 569-645.
15. J.M. Conde-Alonso, T. Mei and J. Parcet, Large BMO spaces vs interpolation. Anal. & PDE **8** (2015), 713-746
16. J.M. Conde-Alonso and J. Parcet, Nondoubling Calderón-Zygmund theory — A dyadic approach. Preprint 2016.
17. A. Connes, Noncommutative Geometry. Academic Press Inc. 1994.
18. A. Connes, Classification of injective factors. Ann. Math. **104** (1976), 73-115.
19. A. Connes, C*-algèbres et géométrie différentielle. C. R. Acad. Sci. Paris **290** (1980), 599-604.
20. A. Connes, M. Douglas and A. Schwarz, Noncommutative geometry and matrix theory: The compactification on tori. J. High Energy Physics **02** (1998) 3.

21. A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations. Comm. Math. Phys. **221** (2001), 141-159.
22. A. Connes and H. Moscovici, Modular curvature for noncommutative two-tori. J. Amer. Math. Soc. **27** (2014), 639-684.
23. A. Connes and P. Tretkoff, The Gauss-Bonnet theorem for the noncommutative two torus. Noncommutative geometry and related topics. Johns Hopkins Univ. Press. 2011, 141-158.
24. H.O. Cordes, On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators. J. Funct. Anal. **18** (1975), 115-131.
25. E.G. Effros and Z.J. Ruan, Operator Spaces. Oxford University Press, 2000.
26. E.G. Effros and Z.J. Ruan, Operator space tensor products and Hopf convolution algebras. J. Operator Th. **50** (2003), 131-156.
27. G.A. Elliott and H. Li, Morita equivalence of smooth noncommutative tori. Acta Math. **199** (2007), 1-27.
28. F. Fathizadeh and M. Khalkhali, The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. J. Noncommut. Geom. **6** (2012), 457-480.
29. C. Fefferman, L^p bounds for pseudo-differential operators. Israel J. Math. **14** (1973), 413-417.
30. C. Fefferman and E.M. Stein, H^p spaces of several variables. Acta Math. **129** (1972), 137-195.
31. G.B. Folland, A course in abstract harmonic analysis. CRC Press, 2016.
32. A.M. González-Pérez, M. Junge and J. Parcet, Smooth Fourier multipliers in group algebras via Sobolev dimension. Ann. Sci. École Norm. Sup. To appear.
33. L. Grafakos, Classical Fourier Analysis. Springer, 2008.
34. L. Grafakos, Modern Fourier Analysis. Springer, 2008.
35. U. Haagerup and M. Rørdam. Perturbations of the rotation C^*-algebras and of the Heisenberg commutation relation. Duke Math. J. **77** (1995), 627-656.
36. I.I. Hirschman, On multiplier transformations. Duke Math. J **26** (1959), 221-242.
37. G. Hong, L.D. López-Sánchez, J.M. Martell and J. Parcet, Calderón-Zygmund operators associated to matrix-valued kernels. Int. Math. Res. Not. **14** (2014), 1221-1252.
38. L. Hörmander, Pseudo-differential operators. Comm. Pure Appl. Math. **18** (1965), 501-517.
39. M. Junge, Doob inequality for non-commutative martingales. J. reine angew. Math. **549** (2002), 149-190.
40. M. Junge and T. Mei, Noncommutative Riesz transforms — A probabilistic approach. Amer. J. Math. **132** (2010), 611-680.
41. M. Junge and T. Mei, BMO spaces associated with semigroups of operators. Math. Ann. **352** (2012), 691-743
42. M. Junge, T. Mei and J. Parcet, Smooth Fourier multipliers on group von Neumann algebras. Geom. Funct. Anal. **24** (2014), 1913-1980.
43. M. Junge, T. Mei and J. Parcet, Noncommutative Riesz transforms – Dimension free bounds and Fourier multipliers. J. Eur. Math. Soc. To appear.
44. M. Junge, T. Mei and J. Parcet, Algebraic Calderón-Zygmund theory. In progress.
45. M. Junge and J. Parcet, Mixed-norm inequalities and operator space L_p embedding theory. Mem. Amer. Math. Soc. **952**, 2010.
46. M. Junge, J. Parcet and Q. Xu, Rosenthal type inequalities for free chaos. Ann. Probab. **35** (2007), 1374-1437.
47. T. de Laat and M. de la Salle, Approximation properties for noncommutative L_p-spaces of high rank lattices and nonembeddability of expanders. J. Reine Angew. Math. To appear.
48. V. Lafforgue and M. de la Salle, Noncommutative L_p-spaces without the completely bounded approximation property. Duke. Math. J. **160** (2011), 71-116.
49. E.C. Lance, Hilbert C*-modules. Cambridge University Press, 1995.
50. M. Lesch and H. Moscovici, Modular curvature and Morita equivalence. Geom. Funct. Anal. **26** (2016), 818-873.
51. T. Mei, Operator-valued Hardy spaces. Mem. Amer. Math. Soc. **188**, 2007.
52. T. Mei, Tent spaces associated with semigroups of operators. J. Funct. Anal. **255** (2008), 3356-3406.
53. T. Mei, An H1-BMO duality theory for semigroups of operators. ArXiv:1204.5082.
54. T. Mei and J. Parcet, Pseudo-localization of singular integrals and noncommutative Littlewood-Paley inequalities. Int. Math. Res. Not. **9** (2009), 1433-1487.
55. N. Nekrasov and A. Schwarz, Instantons in noncommutative $\mathbb{R}^4$ and $(2, 0)$ superconformal six dimensional theory. Comm. Math. Phys. **198** (1998), 689-703.

56. J. Parcet, Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory. J. Funct. Anal. **256** (2009), 509-593.
57. J. Parcet and K.M. Rogers, Twisted Hilbert transforms vs Kakeya sets of directions. J. Reine Angew. Math. **710** (2016), 137-172.
58. G. Pisier, The operator Hilbert space OH, complex interpolation and tensor norms. Mem. Amer. Math. Soc. **585**, 1996.
59. G. Pisier, Non-commutative vector valued L_p-spaces and completely p-summing maps. Astérisque **247**, 1998.
60. G. Pisier, Introduction to Operator Space Theory. Cambridge University Press, 2003.
61. G. Pisier and Q. Xu, Non-commutative martingale inequalities. Comm. Math. Phys. **189** (1997), 667-698.
62. G. Pisier and Q. Xu, Non-commutative L_p-spaces. Handbook of the Geometry of Banach Spaces II. North-Holland (2003), 1459-1517.
63. É. Ricard, L_p-multipliers on quantum tori. J. Funct. Anal. **270** (2016), 4604-4613.
64. M. Ruzhansky and V. Turunen, Pseudo-Differential Operators and Symmetries. Birkhäuser, 2010.
65. D. Sarason, Functions of vanishing mean oscillation. Trans. Amer. Math. Soc. **207** (1975), 391-405.
66. E.M. Stein, Harmonic Analysis. Princeton Univ. Press. 1993.
67. N. Seiberg and E. Witten, String theory and noncommutative geometry. J. High Energy Physics **09** (1999) 32.
68. M. Takesaki, Theory of operator algebras I, II and III. Encyclopedia of Mathematical Sciences **124**, **125** and **127**. Springer-Verlag, Berlin, 2003.
69. M.E. Taylor, Pseudodifferential operators. Princeton Univ. Press, 1981.
70. M.E. Taylor, Pseudodifferential operators and Nonlinear PDE. Birkhäuser, 1991.
71. D.V. Voiculescu, A strengthened asymptotic freeness result for random matrices with applications to free entropy. Int. Math. Res. Not. **1** (1998), 41-63.
72. D.V. Voiculescu, K. Dykema and A. Nica, Free random variables. CRM Monograph Series **1**. American Mathematical Society, 1992.
73. S. Wainger, Special trigonometric series in k-dimensions. Mem. Amer. Math. Soc. **59**, 1965.
74. T.H. Wolff, A note on interpolation spaces. Harmonic analysis. Springer (1982), 199-204.
75. R. Xia, X. Xiong, Q. Xu, Characterizations of operator-valued Hardy spaces and applications to harmonic analysis on quantum tori. Adv. Math. **291** (2016), 183-227.
76. G. Zeller-Meier, Produits croisés d'une C*-algèbre par un groupe d'automorphismes. J. Math. Pures Appl. **47** (1968), 101-239.

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