OPERATOR SPACE EMBEDDING OF SCHATTEN p-CLASSES INTO VON NEUMANN ALGEBRA PREDUALS

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Abstract. Let X_1 and X_2 be subspaces of quotients of $R \oplus \mathrm{OH}$ and $C \oplus \mathrm{OH}$ respectively. We use new free probability techniques to construct a completely isomorphic embedding of the Haagerup tensor product $X_1 \otimes_h X_2$ into the predual of a sufficiently large QWEP von Neumann algebra. As an immediate application, given any $1 < q \leq 2$, our result produces a completely isomorphic embedding of ℓ_q (equipped with its natural operator space structure) into $L_1(\mathcal{A})$ with $\mathcal{A}$ a QWEP von Neumann algebra.

Introduction

The idea of replacing functions by linear operators, the process of quantization, goes back to the foundations of quantum mechanics and has a great impact in mathematics. This applies for instance to representation theory, noncommutative geometry, operator algebra, quantum and free probability or operator space theory among other fields. The quantization of measure theory leads to the theory of L_p spaces defined over general von Neumann algebras, so called *noncommutative L_p spaces*. This theory was initiated by Segal, Dixmier and Kunze in the fifties and continued years later by Haagerup, Fack, Kosaki and many others. We refer to the recent survey [28] for a complete exposition. In a series of papers beginning with this work, we will investigate noncommutative L_p spaces in the language of noncommutative Banach spaces, so called *operator spaces*.

The theory of operator spaces took off in 1988 with Ruan's work [30] and it has been developed since then by Blecher/Paulsen, Effros/Ruan and Pisier as a noncommutative generalization of Banach space theory [3, 20, 24]. In his book [23] on vector-valued noncommutative L_p spaces, Pisier considered a distinguished operator space structure on L_p. In fact, the right category when dealing with noncommutative L_p is in many aspects that of operator spaces. Indeed, this has become clear in the last years by recent results on noncommutative martingales and related topics. In this and forthcoming papers we shall prove a fundamental structure theorem of L_p spaces in the category of operator spaces, solving a problem formulated by Gilles Pisier. Our main contribution here will be the following.

Theorem A. *If $1 < q \leq 2$, there exists a* QWEP *von Neumann algebra $\mathcal{A}$ and a completely isomorphic embedding of the Schatten class S_q into $L_1(\mathcal{A})$, where both spaces are equipped with their natural operator space structures.*

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*Partially supported by the NSF DMS-0556120.
†Partially supported by 'Programa Ramón y Cajal 2005', Spain.
†Partially supported by Grants MTM2007-60952 and CCG06-UAM/ESP-0286, Spain.
2000 Mathematics Subject Classification: 46L07, 46L51, 46L52, 46L54.

In the category of operator spaces, the L_p embedding theory presents some significant differences, some of them already apply to OH. For example, in sharp contrast with the classical theory, it was proved in [6] that OH does not embed completely isomorphically into any L_p space for $2 < p < \infty$. Moreover, after [25] we know that there is no possible cb-embedding of OH into the predual of a semifinite von Neumann algebra. As it will be proved in a forthcoming paper, this also happens in our case and justifies the relevance of type III von Neumann algebras in the subject. In fact, our concern in this paper will be to prove a result which is *simple* enough to present a self-contained approach and *general* enough to serve as a guide for more general cb-embedding results. These latter results will appear in [13], except for the lack of semifiniteness announced above which will be considered independently in [14].

In order to put our result in the right context, let us stress the interaction between harmonic analysis, probability and Banach space theory carried out mostly in the 70's. Based on previous results by Beck, Grothendieck, Lévy, Orlicz... probabilistic methods in Banach spaces became the heart of the work developed by Kwapień, Maurey, Pisier, Rosenthal and many others. A fundamental motivation for this new field relied on the embedding theory of classical L_p spaces. This theory was born in 1966 with the seminal paper [1] of Bretagnolle, Dacunha-Castelle and Krivine. They constructed an isometric embedding of L_q into L_p for $1 \le p < q \le 2$, a Banach space version of our main result; see also [7] for the analogous result with noncommutative L_p spaces. The simplest form of such embedding was known to Lévy and is given by

$$(1) \qquad \Big(\sum_{k=1}^\infty |\alpha_k|^q \Big)^{\frac{1}{q}} = \Big\| \sum_{k=1}^\infty \alpha_k \, \theta_k \Big\|_{L_1(\Omega)},$$

for scalars $(\alpha_k)_{k\ge 1}$ and where $(\theta_k)_{k\ge1}$ is a suitable sequence of independent q-stable random variables in $L_1(\Omega)$ for some probability space (Ω, μ). In other words, we have the relation

$$\mathbb{E} \exp \big(i \sum\nolimits_k \alpha_k \theta_k \big) = \exp \big(- c_q \sum\nolimits_k |\alpha_k|^q \big).$$

More recently, it has been discovered a parallel connection between operator space theory and quantum probability. The operator space version of Grothendieck theorem by Pisier and Shlyakhtenko [27] and the embedding of OH [9] require tools from free probability. In this context we should replace the θ_k's by suitable operators so that (1) holds with matrix-valued coefficients $\alpha_1, \alpha_2, \ldots$. To that aim, we develop new tools in quantum probability and construct an operator space version of q-stable random variables. To formulate the quantized form of (1) we need some basic results of Pisier's theory [23]. The most natural operator space structure on ℓ_∞ comes from the diagonal embedding $\ell_\infty \hookrightarrow \mathcal{B}(\ell_2)$. The natural structure on ℓ_1 is given by operator space duality, while the spaces ℓ_p are defined by means of the complex interpolation method [22] for operator spaces. Let us denote by $(\delta_k)_{k\ge1}$ the unit vector basis of ℓ_q. If $\widehat{\otimes}$ denotes the operator space projective tensor product and S_p stands for the Schatten p-class over ℓ_2, it was shown in [23] that

$$\Big\| \sum_{k=1}^\infty a_k \otimes \delta_k \Big\|_{S_1 \widehat{\otimes} \ell_q} = \inf_{a_k = \alpha b_k \beta} \|\alpha\|_{S_{2q'}} \Big(\sum_{k=1}^\infty \|b_k\|_{S_q}^q \Big)^{\frac{1}{q}} \|\beta\|_{S_{2q'}}.$$

The answer to Pisier's problem for ℓ_q reads as follows.

Theorem B. *If $1 < q \le 2$, there exists a sufficiently large von Neumann algebra $\mathcal{A}$ and a sequence $(x_k)_{k\ge1}$ in $L_1(\mathcal{A})$ such that the equivalence below holds for any family $(a_k)_{k\ge1}$ of trace class operators*

$$\inf_{a_k=\alpha b_k \beta} \|\alpha\|_{2q'} \Big(\sum_{k=1}^\infty \|b_k\|_q^q \Big)^{\frac{1}{q}} \|\beta\|_{2q'} \sim_c \Big\| \sum_{k=1}^\infty a_k \otimes x_k \Big\|_{L_1(\mathcal{A} \bar{\otimes} \mathcal{B}(\ell_2))}.$$

This result provides a completely isomorphic embedding of ℓ_q into $L_1(\mathcal{A})$ and follows directly from Theorem A. Moreover, the sequence $x_1, x_2, \ldots$ provides an operator space version of a q-stable process and motivates a cb-embedding theory of L_p spaces. A particular case of Theorem B is the recent construction [9] of a cb-embedding of Pisier's operator Hilbert space OH into a von Neumann algebra predual. In other words, a complete embedding of ℓ_2 with its natural operator space structure into a noncommutative L_1 space, see also Pisier's paper [26] for a shorter proof and Xu's alternative construction [36]. Other related results appear in [11, 25, 27, 35], while semi-complete embeddings between vector-valued L_p spaces can be found in [12, 19]. All these papers will play a role in our analysis, either here or in the forthcoming papers.

The construction which leads to this operator space version of q-stable random variables is simpler than the one needed for more general cases and will serve as a model for the latter in [13]. Let us sketch the argument in some detail. A key ingredient in our proof is the notion of the Haagerup tensor product $\otimes_h$. We first note that ℓ_q is the diagonal subspace of the Schatten class S_q. According to [23], S_q can be written as the Haagerup tensor product of its first column and first row subspaces $S_q = C_q \otimes_h R_q$. Moreover, using a simple generalization of "Pisier's exercise" (see Exercise 7.9 in Pisier's book [24]) we have

$$\begin{array}{lll} C_q & \hookrightarrow_{cb} & \big(R \oplus \mathrm{OH}\big)/graph(\Lambda_1)^\perp, \\ R_q & \hookrightarrow_{cb} & \big(C \oplus \mathrm{OH}\big)/graph(\Lambda_2)^\perp, \end{array} \tag{2}$$

with $\Lambda_1 : C \to \mathrm{OH}$ and $\Lambda_2 : R \to \mathrm{OH}$ suitable injective, closed, densely-defined operators with dense range and where $\hookrightarrow_{cb}$ denotes a cb-embedding, see [9, 26, 35] for related results. By [9] and duality, it suffices to see that

$$graph(\Lambda_1) \otimes_h graph(\Lambda_2)$$

is cb-isomorphic to a cb-complemented subspace of $L_\infty(\mathcal{A}; \mathrm{OH})$. Before proceeding it is important to have a little discussion on this space. Namely, our methods in this paper will lead us to obtain such cb-embedding for a free product von Neumann algebra $\mathcal{A}$. In particular, $\mathcal{A}$ will not be hyperfinite and Pisier's theory [23] of vector valued noncommutative L_p spaces does not consider this space. Our definition of it is given by complex interpolation

$$L_\infty(\mathcal{A}; \mathrm{OH}_n) = \big[C_n(\mathcal{A}), R_n(\mathcal{A})\big]_{\frac{1}{2}},$$

where $R_n(\mathcal{A})$ and $C_n(\mathcal{A})$ are the row and column subspaces of

$$\mathrm{M}_n(\mathcal{A}) = \mathrm{M}_n \otimes_{\min} \mathcal{A} = C_n \otimes_h \mathcal{A} \otimes_h R_n.$$

In fact, given $1 \le p \le \infty$ and using instead the row and column subspaces of the vector-valued Schatten class $S_p^n(L_p(\mathcal{A}))$, we find a definition of $L_p(\mathcal{A}; \mathrm{OH}_n)$. On the other hand, we also know from [8] a definition of the spaces $L_p(\mathcal{A}; \ell_1^n)$ and $L_p(\mathcal{A}; \ell_\infty^n)$

for any von Neumann algebra $\mathcal{A}$. In particular, we might wonder whether or not our definition of $L_p(\mathcal{A}; \mathrm{OH}_n)$ satisfies the following complete isometry

$$L_p(\mathcal{A}; \mathrm{OH}_n) = \big[L_p(\mathcal{A}; \ell_\infty^n), L_p(\mathcal{A}; \ell_1^n) \big]_{\frac12}.$$

Fortunately this is the case. The proof follows combining results from [17] and [37].

Let us go on with the argument. By the injectivity of the Haagerup tensor product, $graph(\Lambda_1) \otimes_h graph(\Lambda_2)$ is an intersection of four spaces. Let us explain this in detail. By a standard discretization argument which will be given in Lemma 2.8 below, we may assume that $\Lambda_j = \mathsf{d}_{\lambda^j} = \sum_k \lambda_{jk} e_{kk}$ is a diagonal operator on ℓ_2 for $j = 1, 2$. Moreover, taking $\lambda_k = \lambda_{j[k+1/2]}$ for $k \equiv j \pmod 2$ with $[w]$ being the integer part of w, it is no restriction to assume that the eigenvalues of Λ_1 and Λ_2 are the same. In particular, our considerations allow us to rewrite $graph(\Lambda_1) \otimes_h graph(\Lambda_2)$ in the form below

$$\mathcal{J}_{\infty,2} = graph(\mathsf{d}_\lambda) \otimes_h graph(\mathsf{d}_\lambda) = \Big(C \cap \ell_2^{oh}(\lambda) \Big) \otimes_h \Big(R \cap \ell_2^{oh}(\lambda) \Big),$$

where $\ell_2^{oh}(\lambda)$ is a weighted form of OH according to the action of d_λ, so that

$$\begin{aligned} C \cap \ell_2^{oh}(\lambda) &= \Big\{ (e_{i1}, \lambda_i e_{i1}) \ \big| \ i \geq 1 \Big\} \subset C \oplus \mathrm{OH}, \\ R \cap \ell_2^{oh}(\lambda) &= \Big\{ (e_{1j}, \lambda_j e_{1j}) \ \big| \ j \geq 1 \Big\} \subset R \oplus \mathrm{OH}. \end{aligned}$$

The notation for $\mathcal{J}_{\infty,2}$ is in concordance with [13]. Namely, the symbol ∞ in $\mathcal{J}_{\infty,2}$ is used because we consider L_p versions of these spaces. The number 2 denotes that this space arises as a 'middle point' in the sense of interpolation theory between two related $\mathcal{J}$-spaces, see [13] for more details. Now, regarding $\mathsf{d}_\lambda^4 = \sum_k \lambda_k^4 e_{kk}$ as the density d_ψ of some normal strictly semifinite faithful weight ψ on $\mathcal{B}(\ell_2)$, the space $\mathcal{J}_{\infty,2}$ splits up into a 4-term intersection space. In other words, we find

$$\mathcal{J}_{\infty,2}(\psi) = \big(C \otimes_h R \big) \cap \big(C \otimes_h \mathrm{OH} \big) d_\psi^{\frac14} \cap d_\psi^{\frac14} \big(\mathrm{OH} \otimes_h R \big) \cap d_\psi^{\frac14} \big(\mathrm{OH} \otimes_h \mathrm{OH} \big) d_\psi^{\frac14}.$$

The norm of x in $\mathcal{J}_{\infty,2}(\psi)$ is given by

$$\max \Big\{ \|x\|_{\mathcal{B}(\ell_2)}, \big\| x d_\psi^{\frac14} \big\|_{C \otimes_h \mathrm{OH}}, \big\| d_\psi^{\frac14} x \big\|_{\mathrm{OH} \otimes_h R}, \big\| d_\psi^{\frac14} x d_\psi^{\frac14} \big\|_{\mathrm{OH} \otimes_h \mathrm{OH}} \Big\}.$$

The two middle terms are not as unusual as it might seem

$$\begin{aligned} \big\| d_\psi^{\frac14} (x_{ij}) \big\|_{\mathrm{M}_m(\mathrm{OH} \otimes_h R)} &= \sup_{\|\alpha\|_{S_4^m} \leq 1} \Big\| d_\psi^{\frac14} \Big(\sum_{k=1}^m \alpha_{ik} x_{kj} \Big) \Big\|_{L_4(\mathrm{M}_m \otimes \mathcal{B}(\ell_2))}, \\ \big\| (x_{ij}) d_\psi^{\frac14} \big\|_{\mathrm{M}_m(C \otimes_h \mathrm{OH})} &= \sup_{\|\beta\|_{S_4^m} \leq 1} \Big\| \Big(\sum_{k=1}^m x_{ik} \beta_{kj} \Big) d_\psi^{\frac14} \Big\|_{L_4(\mathrm{M}_m \otimes \mathcal{B}(\ell_2))}. \end{aligned} \tag{3}$$

Let us now assume that we just try to embed the finite-dimensional space S_q^m. By approximation, it suffices to consider only finitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and according to the results from [11], we can take $n \sim m \log m$. In this case we rename ψ by ψ_n and we may easily assume that $\mathrm{tr}(d_{\psi_n}) = \sum_k \lambda_k^4 = \mathrm{k}_n$ is a positive integer. Therefore, we consider the following state on $\mathcal{B}(\ell_2^n)$

$$\varphi_n(x) = \frac{1}{\mathrm{k}_n} \sum_{k=1}^n \lambda_k^4 \, x_{kk}.$$

In this particular case, the space $\mathcal{J}_{\infty,2}(\psi_n)$ can be obtained using free probability.

Theorem C. *Let $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_{\mathrm{k}_n}$ be the reduced free product of k_n copies of $\mathcal{B}(\ell_2^n) \oplus \mathcal{B}(\ell_2^n)$ equipped with the state $\frac{1}{2}(\varphi_n \oplus \varphi_n)$. If $\pi_k : \mathsf{A}_j \to \mathcal{A}$ denotes the canonical embedding into the j-th component of $\mathcal{A}$, the mapping*

$$u_n : x \in \mathcal{J}_{\infty,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_j(x, -x) \otimes \delta_j \in L_\infty(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n})$$

is a cb-embedding with cb-complemented image and universal constants.

Theorem C and its generalization for arbitrary von Neumann algebras is a very recent result from [13]. However, since the result proved there covers a wider range of indices, the proof is rather long and quite technical. In order to be self-contained we provide a second proof of this particular case only using elementary tools from free probability. We think this approach is of independent interest. Now, by duality we obtain a cb-embedding

$$S_q^m \hookrightarrow_{cb} L_1(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n}).$$

Then, we conclude using the cb-embedding of OH from [9] and an ultraproduct procedure. In fact, what we shall prove is the following far reaching generalization of Theorem A, see Paragraph 2.1 below for details.

Theorem D. *Let X_1 be a subspace of a quotient of $R \oplus \mathrm{OH}$ and let X_2 be a subspace of a quotient of $C \oplus \mathrm{OH}$. Then, there exist some* QWEP *von Neumann algebra $\mathcal{A}$ and a cb-embedding*

$$\mathrm{X}_1 \otimes_h \mathrm{X}_2 \hookrightarrow_{cb} L_1(\mathcal{A}).$$

Background and notation. We shall assume that the reader is familiar with those branches of operator algebra related to the theories of operator spaces and noncommutative L_p spaces. The recent monographs [3] and [24] on operator spaces contain more than enough information for our purposes. We shall work over general von Neumann algebras so that we use Haagerup's definition [4] of L_p, see also Terp's excellent exposition of the subject [32]. The basics on von Neumann algebras and Tomita's modular theory to work with these notions appear in Kadison/Ringrose books [18]. We shall also assume certain familiarity with Pisier's vector-valued non-commutative L_p spaces [23] and Voiculescu's free probability theory [34].

We shall follow the standard notation in the subject. Anyway, let us say a few words on our terminology. The symbols (δ_k) and (e_{ij}) will denote the unit vector basis of ℓ_2 and $\mathcal{B}(\ell_2)$ respectively. The letters $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ are reserved to denote von Neumann algebras. Almost all the time, the inclusions $\mathcal{N} \subset \mathcal{M} \subset \mathcal{A}$ will hold. We shall use φ and ϕ to denote normal faithful (*n.f.* in short) states, while the letter ψ will be reserved for normal strictly semifinite faithful (*n.s.s.f.* in short) weights. Inner products and duality brackets will be anti-linear on the first component and linear on the second. Given $\gamma > 0$, we shall write $\gamma \mathrm{X}$ to denote the space X equipped with the norm $\|x\|_{\gamma \mathrm{X}} = \gamma \|x\|_{\mathrm{X}}$. In particular, if $\mathcal{M}$ is a finite von Neumann algebra equipped with a finite weight ψ, we shall usually write $\psi = \mathrm{k}\varphi$ with $\mathrm{k} = \psi(1)$ so that φ becomes a state on $\mathcal{M}$. In this situation, the associated L_p space will be denoted as $\mathrm{k}^{1/p} L_p(\mathcal{M})$, so that the L_p norm is calculated using the state φ. Note that this notation does not follow the tradition in interpolation theory, which denotes by $\gamma \mathrm{X}$ the Banach space with unit ball $\gamma \mathsf{B}_{\mathrm{X}}$ and so the norm is divided and not multiplied by γ. Given an operator space X, the X-valued Schatten p-class over the algebra $\mathcal{B}(\ell_2)$ will be denoted by $S_p(\mathrm{X})$.

Acknowledgement. The former version of this paper was much longer and also significantly more technical, since it almost contained the proof of our main result in full generality. This included all the results in this paper and also some of the results announced for [13]. We owe the referee to suggest this more accessible presentation which we hope will help the nonexpert reader.

1. Free harmonic analysis

We start with the proof of a generalized form of Theorem C for arbitrary von Neumann algebras. Although we just use in this paper its discrete version as stated in the Introduction, the general formulation does not present extra difficulties and will be instrumental when dealing with non-discrete algebras in [13]. Our starting point is a von Neumann algebra $\mathcal{M}$ equipped with a *n.f.* state φ and associated density d_φ. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. According to Takesaki [31], the existence and uniqueness of a *n.f.* conditional expectation $\mathsf{E} : \mathcal{M} \to \mathcal{N}$ is equivalent to the invariance of $\mathcal{N}$ under the action of the modular automorphism group σ_t^φ associated to $(\mathcal{M}, \varphi)$. In that case, E is φ-invariant and following Connes [2] we have $\mathsf{E} \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \mathsf{E}$. In what follows, we assume these properties in all subalgebras considered. Now we set $\mathsf{A}_k = \mathcal{M} \oplus \mathcal{M}$ and define $\mathcal{A}$ to be the reduced amalgamated free product von Neumann algebra $*_\mathcal{N} \mathsf{A}_k$ of $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ over $\mathcal{N}$, where the embedding of the subalgebra $\mathcal{N}$ into $\mathcal{M} \oplus \mathcal{M}$ is given by $x \mapsto (x, x)$. Note that our notation $*_\mathcal{N} \mathsf{A}_k$ for reduced amalgamated free products does not make explicit the dependence on the conditional expectations $\mathsf{E}_k : \mathsf{A}_k \to \mathcal{N}$, given by $\mathsf{E}_k(a, b) = \frac{1}{2}\mathsf{E}(a) + \frac{1}{2}\mathsf{E}(b)$. The following is the operator-valued version [9, 15] of Voiculescu inequality [33], for which we need to introduce the mean-zero subspaces

$$\mathring{\mathsf{A}}_k = \Big\{ x \in \mathsf{A}_k \ \big| \ \mathsf{E}_k(a_k) = 0 \Big\}.$$

Lemma 1.1. *If $a_k \in \mathring{\mathsf{A}}_k$ for $1 \le k \le n$ and $\mathsf{E}_\mathcal{N} : \mathcal{A} \to \mathcal{N}$ stands for the conditional expectation of $\mathcal{A}$ onto $\mathcal{N}$, the following equivalence of norms holds with universal constants*

$$\Big\| \sum_{k=1}^n a_k \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k a_k^*) \Big)^{\frac12} \Big\|_\mathcal{N} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k^* a_k) \Big)^{\frac12} \Big\|_\mathcal{N}.$$

Moreover, we also have

$$\Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k a_k^*) \Big)^{\frac12} \Big\|_\mathcal{N},$$

$$\Big\| \big(\sum_{k=1}^n a_k^* a_k \big)^{\frac12} \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k^* a_k) \Big)^{\frac12} \Big\|_\mathcal{N}.$$

Proof. For the first inequality we refer to [9]. The others can be proved in a similar way. Alternatively, both can be deduced from the first one. Indeed, using the identity

$$\Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_\mathcal{A} = \Big\| \sum_{k=1}^n a_k \otimes e_{1k} \Big\|_{\mathrm{M}_n(\mathcal{A})}$$

and recalling the isometric isomorphism

$$\mathrm{M}_n\big(*_\mathcal{N} \mathsf{A}_k \big) = *_{\mathrm{M}_n(\mathcal{N})} \mathrm{M}_n(\mathsf{A}_k),$$

we may apply Voiculescu's inequality over the triple

$$\Big(\mathrm{M}_n(\mathcal{A}), \mathrm{M}_n(\mathsf{A}_k), \mathrm{M}_n(\mathcal{N})\Big).$$

Taking $\widetilde{\mathsf{E}}_{\mathcal{N}} = id_{\mathrm{M}_n} \otimes \mathsf{E}_{\mathcal{N}}$, the last term disappears because

$$\Big\| \Big(\sum_{k=1}^n \widetilde{\mathsf{E}}_{\mathcal{N}}\big((a_k \otimes e_{1k})^*(a_k \otimes e_{1k})\big) \Big)^{\frac12} \Big\|_{\mathrm{M}_n(\mathcal{N})} = \sup_{1 \le k \le n} \big\| \mathsf{E}_{\mathcal{N}}(a_k^* a_k)^{\frac12} \big\|_{\mathcal{N}} \le \sup_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k}.$$

The third equivalence follows by taking adjoints. The proof is complete. □

Let $\pi_k : \mathsf{A}_k \to \mathcal{A}$ denote the embedding of A_k into the k-th component of $\mathcal{A}$. Given $x \in \mathcal{M}$, we shall write x_k as an abbreviation of $\pi_k(x, -x)$. Note that x_k is mean-zero. In the following we shall use with no further comment the identities $\mathsf{E}_{\mathcal{N}}(x_k x_k^*) = \mathsf{E}(xx^*)$ and $\mathsf{E}_{\mathcal{N}}(x_k^* x_k) = \mathsf{E}(x^* x)$. We will mostly work with identically distributed variables. In other words, given $x \in \mathcal{M}$ we shall work with the sequence $x_k = \pi_k(x, -x)$ for $1 \le k \le n$. In terms of the last equivalences in Lemma 1.1, we may consider the following norms

$$\begin{aligned} \|x\|_{\mathcal{R}^n_{\infty,1}} &= \max\Big\{ \|x\|_{\mathcal{M}}, \sqrt{n}\, \big\| \mathsf{E}(xx^*)^{\frac12} \big\|_{\mathcal{N}} \Big\}, \\ \|x\|_{\mathcal{C}^n_{\infty,1}} &= \max\Big\{ \|x\|_{\mathcal{M}}, \sqrt{n}\, \big\| \mathsf{E}(x^*x)^{\frac12} \big\|_{\mathcal{N}} \Big\}. \end{aligned}$$

Here the letters $\mathcal{R}$ and $\mathcal{C}$ stand for row and column according to Lemma 1.1. The symbol ∞ is motivated because in [13] we work with L_p versions of these spaces. The number 1 arises from interpolation theory, because we think of these spaces as endpoints in an interpolation scale. Finally, the norms on the right induce to introduce the spaces $L_\infty^r(\mathcal{M}, \mathsf{E})$ and $L_\infty^c(\mathcal{M}, \mathsf{E})$ as the closure of $\mathcal{M}$ with respect to the norms

$$\big\| \mathsf{E}(xx^*)^{\frac12} \big\|_{\mathcal{N}} \quad \text{and} \quad \big\| \mathsf{E}(x^*x)^{\frac12} \big\|_{\mathcal{N}}.$$

In this way, we obtain the spaces

$$\begin{aligned} \mathcal{R}^n_{\infty,1}(\mathcal{M}, \mathsf{E}) &= \mathcal{M} \cap \sqrt{n}\, L_\infty^r(\mathcal{M}, \mathsf{E}), \\ \mathcal{C}^n_{\infty,1}(\mathcal{M}, \mathsf{E}) &= \mathcal{M} \cap \sqrt{n}\, L_\infty^c(\mathcal{M}, \mathsf{E}). \end{aligned}$$

Remark 1.2. It is easily seen that

$$\begin{aligned} \big\| \mathsf{E}(xx^*)^{\frac12} \big\|_{\mathcal{N}} &= \sup\Big\{ \|\alpha x\|_{L_2(\mathcal{M})} \ \big| \ \|\alpha\|_{L_2(\mathcal{N})} \le 1 \Big\}, \\ \big\| \mathsf{E}(x^*x)^{\frac12} \big\|_{\mathcal{N}} &= \sup\Big\{ \|x\beta\|_{L_2(\mathcal{M})} \ \big| \ \|\beta\|_{L_2(\mathcal{N})} \le 1 \Big\}. \end{aligned}$$

This relation will be crucial in this paper and will be assumed in what follows.

The state φ induces the *n.f.* state $\phi = \varphi \circ \mathsf{E}_{\mathcal{N}}$ on $\mathcal{A}$. If $\mathcal{A}_{\oplus n}$ denotes the n-fold direct sum $\mathcal{A} \oplus \mathcal{A} \oplus \ldots \oplus \mathcal{A}$ (which we shall identify with $\mathcal{A} \otimes \mathbb{C}^n$ in the sequel) we consider the *n.f.* state $\phi_n : \mathcal{A}_{\oplus n} \to \mathbb{C}$ and the conditional expectation $\mathcal{E}_n : \mathcal{A}_{\oplus n} \to \mathcal{A}$ given by

$$\phi_n\big(\sum_{k=1}^n a_k \otimes \delta_k\big) = \frac{1}{n} \sum_{k=1}^n \phi(a_k) \qquad \text{and} \qquad \mathcal{E}_n\big(\sum_{k=1}^n a_k \otimes \delta_k\big) = \frac{1}{n} \sum_{k=1}^n a_k.$$

Let us consider the map

$$u : x \in \mathcal{M} \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in \mathcal{A}_{\oplus n} \quad \text{with} \quad x_k = \pi_k(x, -x). \tag{1.1}$$

Lemma 1.3. *The mappings*

$$\begin{array}{lll} u_r: & x \in \mathcal{R}^n_{\infty,1}(\mathcal{M}, \mathsf{E}) & \mapsto & \displaystyle\sum_{k=1}^n x_k \otimes e_{1k} \in R_n(\mathcal{A}), \\ u_c: & x \in \mathcal{C}^n_{\infty,1}(\mathcal{M}, \mathsf{E}) & \mapsto & \displaystyle\sum_{k=1}^n x_k \otimes e_{k1} \in C_n(\mathcal{A}), \end{array}$$

are isomorphisms onto complemented subspaces with universal constants.

Proof. Given $x \in \mathcal{R}^n_{\infty,1}(\mathcal{M}, \mathsf{E})$, Lemma 1.1 gives

$$\big\| u_r(x) \big\|_{R_n(\mathcal{A})} = \Big\| \big(\sum_{k=1}^n x_k x_k^* \big)^{\frac12} \Big\|_{\mathcal{A}} \sim \max_{1 \le k \le n} \|x_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}}.$$

In other words, we have

$$\big\| u_r(x) \big\|_{R_n(\mathcal{A})} \sim \|x\|_{\mathcal{M}} + \sqrt{n} \, \|x\|_{L_\infty^r(\mathcal{M}, \mathsf{E})} \sim \|x\|_{\mathcal{R}^n_{\infty,1}(\mathcal{M}, \mathsf{E})}.$$

Thus u_r is an isomorphism onto its image with constants independent of n. The same argument yields to the same conclusion for u_c. To prove the complementation let us construct the corresponding projection. Let $\mathsf{E}_{\mathsf{A}_k}: \mathcal{A} \to \mathsf{A}_k$ denote the conditional expectation onto A_k. Given $a_1, a_2, \ldots, a_n \in \mathcal{A}$, we can always write $\mathsf{E}_{\mathsf{A}_k}(a_k)$ as $\pi_k(\alpha_k^1, \alpha_k^2)$ for some $\alpha_k^1, \alpha_k^2 \in \mathcal{M}$. Then, if we set the elements

$$b_k = \pi_k \Big(\frac{\alpha_k^1 + \alpha_k^2}{2}, -\frac{\alpha_k^1 + \alpha_k^2}{2} \Big),$$

the projection onto the image of u_r is given by

$$\Pi : \sum_{k=1}^n a_k \otimes e_{1k} \in R_n(\mathcal{A}) \mapsto \sum_{k=1}^n b_k \otimes e_{1k} \in R_n(\mathcal{A}).$$

By means of the upper estimate in Lemma 1.1 we have

$$\Big\| \big(\sum_{k=1}^n b_k b_k^* \big)^{\frac12} \Big\|_{\mathcal{A}} \lesssim \max_{1 \le k \le n} \|b_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(b_k b_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}}.$$

Since it is clear that $\|b_k\|_{\mathsf{A}_k} \le \|\mathsf{E}_{\mathsf{A}_k}(a_k)\|_{\mathsf{A}_k} \le \big\| (\sum_k a_k a_k^*)^{\frac12} \big\|_{\mathcal{A}}$, it suffices to show

$$\Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(b_k b_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}} \lesssim \Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_{\mathcal{A}}.$$

However, using the identity

$$\begin{aligned} \mathsf{E}_{\mathcal{N}}(b_k b_k^*) &= \frac14 \mathsf{E}_{\mathcal{N}} \big((\alpha_k^1 + \alpha_k^2)(\alpha_k^1 + \alpha_k^2)^* \big) \\ &= \frac12 \mathsf{E}_{\mathcal{N}} \big(\mathsf{E}_{\mathsf{A}_k}(a_k) \mathsf{E}_{\mathsf{A}_k}(a_k)^* \big) + \frac14 \mathsf{E}_{\mathcal{N}} \big(\alpha_k^1 {\alpha_k^2}^* + \alpha_k^2 {\alpha_k^1}^* \big) \end{aligned}$$

combined with Hölder's inequality for conditional expectations [8], we get

$$\begin{aligned} \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(b_k b_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}} &\le \frac12 \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}} \big(\mathsf{E}_{\mathsf{A}_k}(a_k) \mathsf{E}_{\mathsf{A}_k}(a_k)^* \big) \Big)^{\frac12} \Big\|_{\mathcal{N}} \\ &+ \frac12 \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}} \big(\alpha_k^1 {\alpha_k^1}^* \big) \Big)^{\frac12} \Big\|_{\mathcal{N}}^{\frac12} \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}} \big(\alpha_k^2 {\alpha_k^2}^* \big) \Big)^{\frac12} \Big\|_{\mathcal{N}}^{\frac12} \end{aligned}$$

$$\leq \frac{1+\sqrt{2}}{2} \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}} \big(\mathsf{E}_{\mathsf{A}_k}(a_k) \mathsf{E}_{\mathsf{A}_k}(a_k)^* \big) \Big)^{\frac12} \Big\|_{\mathcal{N}}.$$

Kadison's inequality gives $\mathsf{E}_{\mathsf{A}_k}(a_k)\mathsf{E}_{\mathsf{A}_k}(a_k)^* \leq \mathsf{E}_{\mathsf{A}_k}(a_k a_k^*)$ so that we finally obtain

$$\Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(b_k b_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}} \leq \frac{1+\sqrt{2}}{2} \Big\| \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}(a_k a_k^*) \Big)^{\frac12} \Big\|_{\mathcal{N}} \leq \frac{1+\sqrt{2}}{2} \Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_{\mathcal{A}}$$

as we wanted. The column case follows in the same way. □

The careful reader will have observed that the projection maps $u_r w_r^*$ and $u_c w_c^*$ are algebraically the same map uw^*, modulo the identification of $R_n(\mathcal{A})$ and $C_n(\mathcal{A})$ with $\mathcal{A}_{\oplus n}$. If we consider the conditional expectation $\mathcal{E}_{\mathsf{A}_k} : \mathcal{A} \to \mathsf{A}_k$ and we set $\mathcal{E}_{\mathsf{A}_k}(z_k) = (\alpha_k, \beta_k)$, then

$$uw^* \Big(\sum_{k=1}^n z_k \otimes \delta_k \Big) = \sum_{k=1}^n \pi_k \Big(\frac{1}{2n} \sum_{j=1}^n \alpha_j - \beta_j, \frac{1}{2n} \sum_{j=1}^n \beta_j - \alpha_j \Big) \otimes \delta_k.$$

In particular, Lemma 1.3 allows us to identify the interpolation space

$$\mathrm{X}_{\frac12} = \big[\mathcal{C}_{\infty,1}^n(\mathcal{M}, \mathsf{E}), \mathcal{R}_{\infty,1}^n(\mathcal{M}, \mathsf{E}) \big]_{\frac12}$$

with a complemented subspace of $L_\infty(\mathcal{A}; \mathrm{OH}_n)$. This will be implicitly used below. However, the difficult part in proving Theorem C is to identify the norm of the space $\mathrm{X}_{\frac12}$. Of course, according to the fact that we are interpolating 2-term intersection spaces, we expect a 4-term maximum. This is the case and we define $\mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E})$ as the space of elements x in $\mathcal{M}$ equipped with the norm

$$\max_{u,v \in \{4,\infty\}} \Big\{ n^{\frac{1}{\xi(u,v)}} \sup \big\{ \|\alpha x \beta\|_{L_{\xi(u,v)}(\mathcal{M})} \, \big| \, \|\alpha\|_{L_u(\mathcal{N})}, \|\beta\|_{L_v(\mathcal{N})} \leq 1 \big\} \Big\},$$

where $\xi(u,v)$ is given by $\frac{1}{\xi(u,v)} = \frac1u + \frac1v$. Obviously, multiplying by elements α, β in the unit ball of $L_\infty(\mathcal{N})$ and taking suprema does not contribute to the corresponding $L_{\xi(u,v)}(\mathcal{M})$ term. In other words, we may rewrite the norm of x in $\mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E})$ as

$$\|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})} = \max \Big\{ \|x\|_{\Lambda_{(u,v)}^n} \, \Big| \, u,v \in \{4,\infty\} \Big\}$$

where the $\Lambda_{(u,v)}^n$ norms are given by

$$\begin{aligned} \|x\|_{\Lambda_{(\infty,\infty)}^n} &= \|x\|_{\mathcal{M}}, \\ \|x\|_{\Lambda_{(\infty,4)}^n} &= n^{\frac14} \sup \Big\{ \|x\beta\|_{L_4(\mathcal{M})} \, \Big| \, \|\beta\|_{L_4(\mathcal{N})} \leq 1 \Big\}, \\ \|x\|_{\Lambda_{(4,\infty)}^n} &= n^{\frac14} \sup \Big\{ \|\alpha x\|_{L_4(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_4(\mathcal{N})} \leq 1 \Big\}, \\ \|x\|_{\Lambda_{(4,4)}^n} &= n^{\frac12} \sup \Big\{ \|\alpha x \beta\|_{L_2(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_4(\mathcal{N})}, \|\beta\|_{L_4(\mathcal{N})} \leq 1 \Big\}. \end{aligned}$$

These norms arise as particular cases of the so-called conditional L_p spaces, which are defined and will be further exploited in [13]. Before identifying the norm of $\mathrm{X}_{\frac12}$, we need some information on interpolation spaces.

Lemma 1.4. *If $(1/u, 1/v) = (\theta/2, (1-\theta)/2)$, we have for $x \in \mathcal{M}$*

$$\begin{aligned} \|x\|_{[\mathcal{M}, L_\infty^r(\mathcal{M},\mathsf{E})]_\theta} &= \sup \Big\{ \|\alpha x\|_{L_u(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_u(\mathcal{N})} \leq 1 \Big\}, \\ \|x\|_{[L_\infty^c(\mathcal{M},\mathsf{E}), \mathcal{M}]_\theta} &= \sup \Big\{ \|x\beta\|_{L_v(\mathcal{M})} \, \Big| \, \|\beta\|_{L_v(\mathcal{N})} \leq 1 \Big\}, \end{aligned}$$

$$\|x\|_{[L_\infty^c(\mathcal{M},\mathsf{E}),L_\infty^r(\mathcal{M},\mathsf{E})]_\theta} = \sup\Big\{\|\alpha x\beta\|_{L_2(\mathcal{M})} \ \big| \ \|\alpha\|_{L_u(\mathcal{N})}, \|\beta\|_{L_v(\mathcal{N})} \le 1\Big\}.$$

The proof can be found in [13]. In the finite setting, this result follows from a well-known application of Helson/Lowdenslager, Wiener/Masani type results on the existence of operator-valued analytic functions. This kind of applications has been used extensively by Pisier in his theory of vector-valued L_p spaces. We shall also need an explicit expression for the norm of $L_\infty(\mathcal{A};\mathrm{OH}_n)$. The formula below was proved by Pisier in [21] for semifinite von Neumann algebras. The general case is due to Haagerup, see also [13, 37] for more general statements.

Lemma 1.5. *We have*

$$\Big\|\sum_{k=1}^n a_k\otimes\delta_k\Big\|_{L_\infty(\mathcal{A};\mathrm{OH}_n)} = \sup\left\{\Big\|\sum_{k=1}^n a_k^*\alpha a_k\Big\|_{L_2(\mathcal{A})}^{\frac12} \ \big| \ \alpha\ge 0, \ \|\alpha\|_2\le 1\right\}.$$

Now we are ready for the main result in this section.

Theorem 1.6. *We have isomorphically*

$$\big[\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}),\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})\big]_{\frac12} \simeq \mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E}).$$

Moreover, the constants in these isomorphisms are uniformly bounded on n.

Proof. We have a contractive inclusion

$$\mathrm{X}_{\frac12}\subset\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E}).$$

Indeed, by elementary properties of interpolation spaces we find

$$\mathrm{X}_{\frac12}\subset[\mathcal{M},\mathcal{M}]_{\frac12}\cap[\sqrt{n}L_\infty^c,\mathcal{M}]_{\frac12}\cap[\mathcal{M},\sqrt{n}L_\infty^r]_{\frac12}\cap[\sqrt{n}L_\infty^c,\sqrt{n}L_\infty^r]_{\frac12},$$

where L_∞^r and L_∞^c are abbreviations for $L_\infty^r(\mathcal{M},\mathsf{E})$ and $L_\infty^c(\mathcal{M},\mathsf{E})$ respectively. Using the obvious identity $[\lambda_0\mathrm{X}_0,\lambda_1\mathrm{X}_1]_\theta=\lambda_0^{1-\theta}\lambda_1^\theta\,\mathrm{X}_\theta$ and applying Lemma 1.4 we rediscover the norm of the space $\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})$ on the right hand side. This gives that the inclusion $\mathrm{X}_{\frac12}\subset\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})$ holds contractively.

To prove the reverse inclusion, we note from Lemmas 1.3 and 1.5 that

$$\begin{aligned}\|x\|_{\mathrm{X}_{\frac12}} &\sim \big\|u(x)\big\|_{[C_n(\mathcal{A}),R_n(\mathcal{A})]_{\frac12}}\\ &= \Big\|\sum_{k=1}^n x_k\otimes\delta_k\Big\|_{L_\infty(\mathcal{A};\mathrm{OH}_n)}\\ &= \sup\left\{\Big\|\sum_{k=1}^n x_k^*ax_k\Big\|_{L_2(\mathcal{A})}^{\frac12} \ \big| \ a\ge0, \ \|a\|_2\le1\right\}=\mathrm{A}.\end{aligned}$$

Thus, it remains to see that

$$\mathrm{A} \lesssim \max\Big\{\|x\|_{\Lambda_{(u,v)}^n} \ \big| \ u,v\in\{4,\infty\}\Big\} = \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}. \tag{1.2}$$

In order to justify (1.2), we introduce the orthogonal projections L_k and R_k on $L_2(\mathcal{A})$ defined as follows. Given $a\in L_2(\mathcal{A})$, the vector $\mathsf{L}_k(a)$ (resp. $\mathsf{R}_k(a)$) collects the reduced words in a starting (resp. ending) with a letter in A_k. In other words, following standard terminology in free probability, we have

$$\mathsf{L}_k: L_2(\mathcal{A}) \longrightarrow L_2\Big(\big[\bigoplus_{m\ge1}\bigoplus_{j_1=k\ne j_2\ne\cdots\ne j_m}\mathring{\mathsf{A}}_{j_1}\mathring{\mathsf{A}}_{j_2}\cdots\mathring{\mathsf{A}}_{j_m}\big]''\Big),$$

$$\mathsf{R}_k : L_2(\mathcal{A}) \quad \longrightarrow \quad L_2\Big(\big[\bigoplus_{m\ge 1}\bigoplus_{j_1\neq j_2\neq\cdots\neq k=j_m} \mathring{\mathsf{A}}_{j_1}\mathring{\mathsf{A}}_{j_2}\cdots\mathring{\mathsf{A}}_{j_m}\big]''\Big).$$

Now, given a positive operator a in $L_2(\mathcal{A})$ and a fixed integer $1 \le k \le n$, we consider the following way to decompose a in terms of the projections L_k and R_k and the conditional expectation $\mathsf{E}_\mathcal{N} : \mathcal{A} \to \mathcal{N}$

$$a = \mathsf{E}_\mathcal{N}(a) + \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k\mathsf{L}_k(a)\big) + \gamma_k(a), \tag{1.3}$$

where the term $\gamma_k(a)$ has the following form

$$\gamma_k(a) = a - \mathsf{E}_\mathcal{N}(a) - \mathsf{L}_k(a) - \mathsf{R}_k(a - \mathsf{L}_k(a)).$$

The triangle inequality gives $\mathrm{A}^2 \le \mathrm{A}_1^2 + \mathrm{A}_2^2 + \mathrm{A}_3^2$, where the terms A_j are the result of replacing a in A by the j-th term in the decomposition (1.3). The terms in the bracket of (1.3) are understood as one term which gives rise to A_2. For the first term we use

$$\begin{aligned}
\mathrm{A}_1^2 &= \Big\| \sum_{k=1}^n x_k^* \mathsf{E}_\mathcal{N}(a) x_k \Big\|_{L_2(\mathcal{A})} \\
&\le \Big\| \sum_{k=1}^n \mathsf{E}_\mathcal{N}\big(x_k^* \mathsf{E}_\mathcal{N}(a) x_k\big) \Big\|_{L_2(\mathcal{N})} \\
&+ \Big\| \sum_{k=1}^n x_k^* \mathsf{E}_\mathcal{N}(a) x_k - \mathsf{E}_\mathcal{N}\big(x_k^* \mathsf{E}_\mathcal{N}(a) x_k\big) \Big\|_{L_2(\mathcal{A})} = \mathrm{A}_{11}^2 + \mathrm{A}_{12}^2.
\end{aligned}$$

Since $\mathsf{E}_\mathcal{N}\big(x_k^* \mathsf{E}_\mathcal{N}(a) x_k\big) = \mathsf{E}\big(x^* \mathsf{E}_\mathcal{N}(a) x\big)$ and $a \in \mathsf{B}_{L_2(\mathcal{A})}^+$, we obtain

$$\begin{aligned}
\mathrm{A}_{11} &= n^{\frac12} \sup\Big\{ \mathrm{tr}_\mathcal{N}\Big(\beta^* \mathsf{E}\big(x^* \mathsf{E}_\mathcal{N}(a) x\big)\beta\Big)^{\frac12} \ \big| \ \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\} \\
&\le n^{\frac12} \sup\Big\{ \mathrm{tr}_\mathcal{M}\big(\beta^* x^* \alpha^* \alpha x \beta\big)^{\frac12} \ \big| \ \|\alpha\|_{L_4(\mathcal{N})}, \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\}.
\end{aligned}$$

This gives $\mathrm{A}_{11} \le \|x\|_{\Lambda_{(4,4)}^n} \le \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}$. On the other hand, by freeness

$$\begin{aligned}
\mathrm{A}_{12}^2 &= \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{E}_\mathcal{N}(a) x_k - \mathsf{E}_\mathcal{N}\big(x_k^* \mathsf{E}_\mathcal{N}(a) x_k\big) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \\
&\le 2 \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{E}_\mathcal{N}(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} = 2\, n^{\frac12} \, \big\| x^* \mathsf{E}_\mathcal{N}(a) x \big\|_{L_2(\mathcal{M})}.
\end{aligned}$$

Then positivity gives

$$\mathrm{A}_{12} \le \sqrt{2}\, \|x\|_{\Lambda_{(4,\infty)}^n} \le \sqrt{2}\, \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}.$$

Let us now estimate the term A_2

$$\begin{aligned}
&\Big\| \sum_{k=1}^n x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k\mathsf{L}_k(a)\big) x_k \Big\|_{L_2(\mathcal{A})} \\
&= \sup\Big\{ \sum_{k=1}^n \mathrm{tr}_\mathcal{A}\Big(b x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k\mathsf{L}_k(a)\big) x_k \Big) \ \big| \ \|b\|_{L_2(\mathcal{A})} \le 1 \Big\} \\
&= \sup\Big\{ \sum_{k=1}^n \mathrm{tr}_\mathcal{A}\Big(x_k b x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k\mathsf{L}_k(a)\big) \Big) \ \big| \ \|b\|_{L_2(\mathcal{A})} \le 1 \Big\}
\end{aligned}$$

$$\leq \sup_{\|b\|_{L_2(\mathcal{A})}\leq 1} \Big(\sum_{k=1}^n \big\| x_k b x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \Big(\sum_{k=1}^n \big\| \mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k \mathsf{L}_k(a) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12}.$$

The second factor on the right is estimated by orthogonality

$$\Big(\sum_{k=1}^n \big\| \mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k \mathsf{L}_k(a) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \leq 3 \, \|a\|_{L_2(\mathcal{A})} \leq 3.$$

For the first factor, we write b as a linear combination $(b_1 - b_2) + i(b_3 - b_4)$ of four positive operators. Therefore, all these terms are covered by the following estimate, to be proved below.

Claim. Given $a \in \mathsf{B}_{L_2(\mathcal{A})}^+$, we have

$$\Big(\sum_{k=1}^n \big\| x_k a x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \lesssim \max \Big\{ \|x\|_{\mathcal{M}}, \, n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})}\leq 1} \|x\beta\|_{L_4(\mathcal{M})} \Big\}. \tag{1.4}$$

Before justifying our claim, we complete the proof. It remains to estimate the term A_3 associated to $\gamma_k(a)$. We first observe that $\gamma_k(a)$ is a mean-zero element of $L_2(\mathcal{A})$ made up of reduced words not starting nor ending with a letter in A_k. Indeed, note that $\mathsf{E}_{\mathcal{N}}(\gamma_k(a)) = 0$ and that we first eliminate the words starting with a letter in A_k by subtracting $\mathsf{L}_k(a)$ and, after that, we eliminate the remaining words which end with a letter in A_k by subtracting $\mathsf{R}_k(a - \mathsf{L}_k(a))$. Therefore, it turns out that the family of random variables $x_1^* \gamma_1(a) x_1, x_2^* \gamma_2(a) x_2, \ldots, x_n^* \gamma_n(a) x_n$ is free over $\mathcal{N}$. In particular, by orthogonality

$$\Big\| \sum_{k=1}^n x_k^* \gamma_k(a) x_k \Big\|_{L_2(\mathcal{A})}^{\frac12} = \Big(\sum_{k=1}^n \big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14}.$$

However, recalling that $\gamma_k(a)$ is a mean-zero element made up of words not starting nor ending with a letter in A_k, the following identities hold for the conditional expectation $\mathcal{E}_{\mathsf{A}_k} : L_1(\mathcal{A}) \to L_1(\mathsf{A}_k)$

$$\begin{aligned} \mathcal{E}_{\mathsf{A}_k} \Big(\gamma_k(a)^* \big(x_k x_k^* - \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \gamma_k(a) \Big) &= 0, \\ \mathcal{E}_{\mathsf{A}_k} \Big(\gamma_k(a) \big(x_k x_k^* - \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \gamma_k(a)^* \Big) &= 0. \end{aligned} \tag{1.5}$$

Using property (1.5) we find

$$\begin{aligned} \big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 &= \mathrm{tr}_{\mathcal{A}} \big(x_k^* \gamma_k(a)^* x_k x_k^* \gamma_k(a) x_k \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(x_k^* \mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a)^* x_k x_k^* \gamma_k(a) \big) x_k \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(x_k^* \mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a) \big) x_k \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(\gamma_k(a) x_k x_k^* \gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(\mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a) x_k x_k^* \gamma_k(a)^* \big) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(\mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a)^* \big) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ &= \mathrm{tr}_{\mathcal{A}} \big(\mathsf{E}_{\mathcal{N}}(x_k x_k^*)^{\frac12} \gamma_k(a) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*)^{\frac12} \big). \end{aligned}$$

In conjunction with $\|\gamma_k(a)\|_2 \leq 5 \, \|a\|_2$ and Hölder's inequality, this yields

$$\big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 = \big\| \mathsf{E}_{\mathcal{N}}(xx^*)^{\frac12} \gamma_k(a) \mathsf{E}_{\mathcal{N}}(xx^*)^{\frac12} \big\|_{L_2(\mathcal{A})}^2 \leq 25 \, \|x\|_{L_\infty^r(\mathcal{M}, \mathsf{E})}^4.$$

It follows from Remark 1.2 that

$$\|x\|_{L_\infty^r(\mathcal{M},\mathsf{E})} \le \sup \Big\{ \|\alpha x\|_{L_4(\mathcal{M})} \ \big| \ \|\alpha\|_{L_4(\mathcal{N})} \le 1 \Big\}.$$

The inequalities proved so far give rise to the following estimate

$$\Big\| \sum_{k=1}^n x_k^* \gamma_k(a) x_k \Big\|_{L_2(\mathcal{A})}^{\frac12} \le \sqrt{5}\, \|x\|_{\Lambda_{(4,\infty)}^n} \le \sqrt{5}\, \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}.$$

Therefore, it remains to prove the claim. We proceed in a similar way. According to the decomposition (1.3), we may use the triangle inequality and decompose the left hand side of (1.4) into three terms $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3$. For the first term, we deduce from positivity that

$$\Big(\sum_{k=1}^n \big\| x_k \mathsf{E}_\mathcal{N}(a) x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} = n^{\frac14} \big\| x \mathsf{E}_\mathcal{N}(a) x^* \big\|_{L_2(\mathcal{M})}^{\frac12} \le n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})} \le 1} \|x\beta\|_{L_4(\mathcal{M})}.$$

On the other hand, it is elementary that

$$\begin{aligned} & \Big(\sum_{k=1}^n \big\| x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k \mathsf{L}_k(a) \big) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \\ & \qquad \le \ \|x\|_\mathcal{M} \Big(\sum_{k=1}^n \big\| x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k \mathsf{L}_k(a) \big) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \le 3\, \|x\|_\mathcal{M}. \end{aligned}$$

This leaves us with the term B_3. Arguing as above

$$\Big(\sum_{k=1}^n \big\| x_k \gamma_k(a) x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \le \sqrt{5}\ n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})} \le 1} \|x\beta\|_{L_4(\mathcal{M})}.$$

Therefore, the claim holds and the proof is complete. □

Remark 1.7. The arguments in Theorem 1.6 also give

$$\begin{aligned} \|x\|_{[\mathcal{M}, \mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})]_{\frac12}} & \sim \max \Big\{ \|x\|_\mathcal{M}, \|x\|_{\Lambda_{(4,\infty)}^n} \Big\}, \\ \|x\|_{[\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E})\,,\,\mathcal{M}]_{\frac12}} & \sim \max \Big\{ \|x\|_\mathcal{M}, \|x\|_{\Lambda_{(\infty,4)}^n} \Big\}. \end{aligned}$$

Now we show how the space $\mathrm{X}_{1/2}$ is related to Theorem C. The idea follows from a well-known argument in which complete boundedness arises as a particular case of amalgamation. More precisely, if $L_2^r(\mathcal{M})/L_2^c(\mathcal{M})$ denote the row/column quantizations of $L_2(\mathcal{M})$ and $2 \le q \le \infty$, the row/column operator space structures on $L_q(\mathcal{M})$ are defined as follows

$$\begin{aligned} L_q^r(\mathcal{M}) & = \big[\mathcal{M}, L_2^r(\mathcal{M}) \big]_{\frac2q}, \\ L_q^c(\mathcal{M}) & = \big[\mathcal{M}, L_2^c(\mathcal{M}) \big]_{\frac2q}. \end{aligned} \tag{1.6}$$

The following result from [13] is a generalized form of (3) in the Introduction.

Lemma 1.8. *If $\mathcal{M}_m = \mathrm{M}_m(\mathcal{M})$, we have*

$$\begin{aligned} \big\| d_\varphi^{\frac14} (x_{ij}) \big\|_{\mathrm{M}_m(L_4^r(\mathcal{M}))} & = \sup_{\|\alpha\|_{S_4^m} \le 1} \Big\| d_\varphi^{\frac14} \Big(\sum_{k=1}^m \alpha_{ik} x_{kj} \Big) \Big\|_{L_4(\mathcal{M}_m)}, \\ \big\| (x_{ij}) d_\varphi^{\frac14} \big\|_{\mathrm{M}_m(L_4^c(\mathcal{M}))} & = \sup_{\|\beta\|_{S_4^m} \le 1} \Big\| \Big(\sum_{k=1}^m x_{ik} \beta_{kj} \Big) d_\varphi^{\frac14} \Big\|_{L_4(\mathcal{M}_m)}. \end{aligned}$$

The proof follows from

$$\begin{aligned} \big\| d_\varphi^{\frac12} (x_{ij}) \big\|_{\mathrm{M}_m(L_2^r(\mathcal{M}))} & = \sup_{\|\alpha\|_{S_2^m} \le 1} \Big\| d_\varphi^{\frac12} \Big(\sum_{k=1}^m \alpha_{ik} x_{kj} \Big) \Big\|_{L_2(\mathcal{M}_m)}, \\ \big\| (x_{ij}) d_\varphi^{\frac12} \big\|_{\mathrm{M}_m(L_2^c(\mathcal{M}))} & = \sup_{\|\beta\|_{S_2^m} \le 1} \Big\| \Big(\sum_{k=1}^m x_{ik} \beta_{kj} \Big) d_\varphi^{\frac12} \Big\|_{L_2(\mathcal{M}_m)}, \end{aligned} \tag{1.7}$$

and some complex interpolation formulas developed in [13]. The identity (1.7) from which we interpolate is a well-known expression in operator space theory, see e.g. p.56 in [3]. Now we define the space $\mathcal{J}_{\infty,2}^n(\mathcal{M})$ as follows

$$\mathcal{J}_{\infty,2}^n(\mathcal{M}) = \mathcal{M} \cap n^{\frac14} L_4^c(\mathcal{M}) \cap n^{\frac14} L_4^r(\mathcal{M}) \cap n^{\frac12} L_2(\mathcal{M}).$$

Lemma 1.8 determines the operator space structure of the cross terms in $\mathcal{J}_{\infty,2}^n(\mathcal{M})$. On the other hand, according to Lemma 1.5 it is easily seen that the following identity holds

$$\big\| d_\varphi^{\frac14} (x_{ij}) d_\varphi^{\frac14} \big\|_{\mathrm{M}_m(L_2(\mathcal{M}))} = \sup_{\|\alpha\|_{S_4^m}, \|\beta\|_{S_4^m} \le 1} \Big\| d_\varphi^{\frac14} \Big(\sum_{k,l=1}^m \alpha_{ik} x_{kl} \beta_{lj} \Big) d_\varphi^{\frac14} \Big\|_{L_2(\mathcal{M}_m)}.$$

In other words, the o.s.s. of $\mathcal{J}_{\infty,2}^n(\mathcal{M})$ is described by the isometry

$$\mathrm{M}_m\big(\mathcal{J}_{\infty,2}^n(\mathcal{M})\big) = \mathcal{J}_{\infty,2}^n(\mathcal{M}_m, \mathsf{E}_m), \tag{1.8}$$

where $\mathcal{M}_m = \mathrm{M}_m(\mathcal{M})$ and $\mathsf{E}_m = id_{\mathrm{M}_m} \otimes \varphi : \mathcal{M}_m \to \mathrm{M}_m$ for $m \ge 1$. This means that the *vector-valued* spaces $\mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E})$ describe the o.s.s. of the *scalar-valued* spaces $\mathcal{J}_{\infty,2}^n(\mathcal{M})$. In the result below we prove the operator space/free analogue of a form of Rosenthal's inequality in the limit case $p \to \infty$, see [13] for more details. This result does not have a commutative counterpart. The particular case for $\mathcal{M} = \mathcal{B}(\ell_2^n)$ recovers Theorem C. Given a von Neumann algebra $\mathcal{M}$, we set as above $\mathsf{A}_k = \mathcal{M} \oplus \mathcal{M}$.

Corollary 1.9. *If $\mathcal{A}_\mathcal{N} = *_\mathcal{N} \mathsf{A}_k$, the map*

$$u : x \in \mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}_\mathcal{N}; \mathrm{OH}_n)$$

is an isomorphism with complemented image and universal constants. In particular, replacing $(\mathcal{M}, \mathcal{N}, \mathsf{E})$ by $(\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ and replacing $\mathcal{A}_\mathcal{N}$ by the non-amalgamated algebra $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_n$, we obtain a cb-isomorphism with cb-complemented image and universal constants

$$\sigma : x \in \mathcal{J}_{\infty,2}^n(\mathcal{M}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}; \mathrm{OH}_n).$$

Proof. The first assertion follows from Lemma 1.3 and Theorem 1.6. To prove the second assertion we choose the triple $(\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ and apply (1.8). This provides us with an isomorphic embedding

$$\sigma_m : x \in \mathrm{M}_m\big(\mathcal{J}_{\infty,2}^n(\mathcal{M})\big) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}_m; \mathrm{OH}_n),$$

where the von Neumann algebra $\mathcal{A}_m$ is given by

$$\mathcal{A}_m = \mathrm{M}_m(\mathcal{A}) = \mathrm{M}_m(\mathsf{A}_1) *_{\mathrm{M}_m} \mathrm{M}_m(\mathsf{A}_2) *_{\mathrm{M}_m} \cdots *_{\mathrm{M}_m} \mathrm{M}_m(\mathsf{A}_n).$$

The last isometry is well-known, see e.g. [9]. In particular

$$L_\infty(\mathcal{A}_m; \mathrm{OH}_n) = \mathrm{M}_m\big(L_\infty(\mathcal{A}; \mathrm{OH}_n)\big)$$

and it turns out that the map $\sigma_m = id_{\mathrm{M}_m} \otimes \sigma$. This completes the proof. □

2. Complete embedding of Schatten classes

Given $1 < q \le 2$, we construct a completely isomorphic embedding of S_q into the predual of a QWEP von Neumann algebra. In fact, we prove Theorem D and deduce the cb-embedding stated in Theorem A via Lemma 2.1 below. Theorem B and the subsequent family of operator space q-stable random variables arise by injecting the space ℓ_q into the diagonal of the Schatten class S_q.

2.1. **On "Pisier's exercise".** Given a Hilbert space $\mathcal{H}$, we shall write in what follows $\mathcal{H}_r = \mathcal{B}(\mathcal{H}, \mathbb{C})$ and $\mathcal{H}_c = \mathcal{B}(\mathbb{C}, \mathcal{H})$ for the row and column quantizations on $\mathcal{H}$. Moreover, given $1 \le p \le \infty$ we shall use the following terminology

$$\mathcal{H}_{r_p} = \big[\mathcal{H}_r, \mathcal{H}_c\big]_{\frac{1}{p}} \quad \text{and} \quad \mathcal{H}_{c_p} = \big[\mathcal{H}_c, \mathcal{H}_r\big]_{\frac{1}{p}}.$$

When $\mathcal{H} = \ell_2$, we shall use (R, C, R_p, C_p) instead. In the same fashion, $\mathcal{H}_{oh}$ stands for the operator Hilbert space structure on $\mathcal{H}$. Given two operator spaces X_1 and X_2, the expression $\mathrm{X}_1 \simeq_{cb} \mathrm{X}_2$ means that there exists a complete isomorphism between them. We shall write $\mathrm{X}_1 \in \mathcal{QS}(\mathrm{X}_2)$ to denote that X_1 is completely isomorphic to a quotient of a subspace of X_2. Let us recall a generalization of Exercise 7.9 in [24]. This result became popular after Pisier applied it in [26] to obtain a simpler way to cb-embed OH into the predual of a von Neumann algebra. In fact, the argument in [35] for a similar result is easily adaptable to our setting and we will omit details.

Lemma 2.1. *If $1 \le p < q \le 2$, we have*

$$R_q \in \mathcal{QS}\big(R_p \oplus_2 \mathrm{OH}\big) \quad \textit{and} \quad C_q \in \mathcal{QS}\big(C_p \oplus_2 \mathrm{OH}\big).$$

Remark 2.2. A few comments are in order:

i) The argument in [35] also gives

$$R_p \in \mathcal{QS}\big(R_{p_0} \oplus_2 R_{p_1}\big) \quad \text{and} \quad C_p \in \mathcal{QS}\big(C_{p_0} \oplus_2 C_{p_1}\big)$$

for any indices p, p_0, p_1 satisfying $1 \le \min(p_0, p_1) \le p \le \max(p_0, p_1) \le \infty$.

ii) We shall apply Lemma 2.1 for $p = 1$. In [13] we will use it in full generality.

iii) Let $\mathcal{S}$ denote the strip

$$\mathcal{S} = \Big\{ z \in \mathbb{C} \ \big| \ 0 < \mathrm{Re}(z) < 1 \Big\}$$

and let $\partial\mathcal{S} = \partial_0 \cup \partial_1$ be the partition of its boundary into

$$\partial_0 = \Big\{ z \in \mathbb{C} \ \big| \ \mathrm{Re}(z) = 0 \Big\} \quad \text{and} \quad \partial_1 = \Big\{ z \in \mathbb{C} \ \big| \ \mathrm{Re}(z) = 1 \Big\}.$$

Given $0 < \theta < 1$, let μ_θ be the harmonic measure of the point $z = \theta$. This is a probability measure on $\partial\mathcal{S}$ (with density given by the Poisson kernel in the strip) that can be written as $\mu_\theta = (1-\theta)\mu_0 + \theta\mu_1$, with μ_j being probability measures supported by ∂_j and such that $f(\theta) = \int_{\partial\mathcal{S}} f d\mu_\theta$ for any bounded analytic function $f : \mathcal{S} \to \mathbb{C}$ which is extended non-tangentially to $\partial\mathcal{S}$. Let us consider the following pair of Hilbert spaces

$$\mathcal{H} = (1-\theta)^{\frac{1}{2}} L_2(\partial_0, \mu_0; \ell_2) \quad \text{and} \quad \mathcal{K} = \theta^{\frac{1}{2}} L_2(\partial_1, \mu_1; \ell_2).$$

The space of bounded analytic functions $f : \mathcal{S} \to R_p + \mathrm{OH}$ will be denoted by $\mathcal{F}(R_p, \mathrm{OH})$ and we will equip it with the operator space structure of $\mathcal{H}_{r_p} \oplus_2 \mathcal{K}_{oh}$. Similarly, we set $\mathcal{F}(C_p, \mathrm{OH}) = \mathcal{H}_{c_p} \oplus_2 \mathcal{K}_{oh}$. Then, the argument in [35] shows that we have

$$R_q \simeq_{cb} \mathcal{F}(R_p, \mathrm{OH}) / \ker \mathcal{Q} \in \mathcal{QS}(R_p \oplus_2 \mathrm{OH}),$$

$$C_q \simeq_{cb} \mathcal{F}(C_p, \mathrm{OH}) / \ker \mathcal{Q} \in \mathcal{QS}(C_p \oplus_2 \mathrm{OH}),$$

for the mapping $\mathcal{Q} : f \mapsto f(\theta)$. However, it will be convenient in the sequel to observe that $\ker \mathcal{Q}$ can be regarded in both cases as the annihilator of the graph of certain linear operator, see (2). Recall that for a linear map between Hilbert spaces $\Lambda : \mathcal{K}_1 \to \mathcal{K}_2$ with domain $\mathsf{dom}(\Lambda)$

$$graph(\Lambda) = \Big\{ (x_1, x_2) \in \mathcal{K}_1 \oplus_2 \mathcal{K}_2 \;\big|\; x_1 \in \mathsf{dom}(\Lambda) \text{ and } x_2 = \Lambda(x_1) \Big\}.$$

Let us consider for instance the case of R_q. We first note that $\mathcal{F}(R_p, \mathrm{OH})$ is the graph of an injective, closed, densely-defined operator Λ with dense range. This operator is given by $\Lambda(f_{|\partial_0}) = f_{|\partial_1}$ for all $f \in \mathcal{F}(R_p, \mathrm{OH})$. Λ is well defined and injective by Poisson integration due to the analyticity of elements in $\mathcal{F}(R_p, \mathrm{OH})$. On the other hand, $\ker \mathcal{Q}$ is the subspace of $\mathcal{F}(R_p, \mathrm{OH})$ composed of functions f vanishing at $z = \theta$. Then, it easily follows from the identity $f(\theta) = \int_{\partial \mathcal{S}} f d\mu_\theta$ that $\ker \mathcal{Q}$ is the annihilator of $\mathcal{F}(R_{p'}, \mathrm{OH}) = graph(\Lambda)$ regarded as a subspace of

$$(1-\theta)^{\frac{1}{2}} L_2(\partial_0; \ell_2)_{r_{p'}} \oplus_2 \theta^{\frac{1}{2}} L_2(\partial_1; \ell_2)_{oh}.$$

2.2. **Embedding of $\mathcal{K}$-spaces.** We fix $\mathcal{M} = \mathcal{B}(\ell_2)$ and consider a family $\gamma_1, \gamma_2, \ldots$ of strictly positive numbers. Then we consider the diagonal operator on ℓ_2 defined by $\mathsf{d}_\gamma = \sum_k \gamma_k e_{kk}$. This operator can be regarded as the density d_ψ associated to a normal strictly semifinite faithful (*n.s.s.f.* in short) weight ψ on $\mathcal{B}(\ell_2)$. Let us set q_n to be the projection $\sum_{k \le n} e_{kk}$ and let us consider the restriction of ψ to the subalgebra $q_n \mathcal{B}(\ell_2) q_n$

$$\psi_n \Big(q_n \big(\sum\nolimits_{ij} x_{ij} e_{ij} \big) q_n \Big) = \sum_{k=1}^n \gamma_k x_{kk}.$$

Note that if we set $\mathrm{k}_n = \psi_n(q_n)$, we obtain $\psi_n = \mathrm{k}_n \varphi_n$ for some state φ_n on $q_n \mathcal{B}(\ell_2) q_n$. If d_{ψ_n} denotes the density on $q_n \mathcal{B}(\ell_2) q_n$ associated to the weight ψ_n, we define the space $\mathcal{J}_{\infty,2}(\psi_n)$ as the subspace

$$\Big\{ \big(z, z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14} \big) \;\big|\; z \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the direct sum

$$\mathcal{L}_\infty^n = \big(C_n \otimes_h R_n \big) \oplus_2 \big(C_n \otimes_h \mathrm{OH}_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h R_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h \mathrm{OH}_n \big).$$

In other words, we obtain the intersection space considered in the Introduction

$$(C_n \otimes_h R_n) \cap (C_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14} \cap d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h R_n) \cap d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}.$$

Lemma 2.3. *Let us consider*

$$\mathcal{K}_{1,2}(\psi_n) = \mathcal{J}_{\infty,2}(\psi_n)^*.$$

Assume that $\mathrm{k}_n = \sum_{k=1}^n \gamma_k$ *is an integer and define* $\mathcal{A}_n$ *to be the* k_n*-fold reduced free product of* $q_n\mathcal{B}(\ell_2)q_n \oplus q_n\mathcal{B}(\ell_2)q_n$*. If* $\pi_j : q_n\mathcal{B}(\ell_2)q_n \oplus q_n\mathcal{B}(\ell_2)q_n \to \mathcal{A}_n$ *is the natural embedding into the* j*-th component of* $\mathcal{A}_n$ *and* $x_j = \pi_j(x,-x)$

$$\omega : x \in \mathcal{K}_{1,2}(\psi_n) \mapsto \frac{1}{\mathrm{k}_n}\sum_{j=1}^{\mathrm{k}_n} x_j \otimes \delta_j \in L_1(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n})$$

is a cb-embedding with cb-complemented image and universal constants.

Proof. We claim that

$$\mathcal{J}_{\infty,2}(\psi_n) = \mathcal{J}_{\infty,2}^{\mathrm{k}_n}(q_n\mathcal{B}(\ell_2)q_n)$$

completely isometrically. Indeed, by (1.6)

$$\begin{aligned}
\mathrm{k}_n^{\frac14} L_4^r(q_n\mathcal{B}(\ell_2)q_n, \varphi_n) &= \mathrm{k}_n^{\frac14}\big[\mathcal{B}(\ell_2^n), L_2^r(\mathcal{B}(\ell_2^n), \varphi_n)\big]_{\frac12} \\
&= \mathrm{k}_n^{\frac14}\big[\mathcal{B}(\ell_2^n), d_{\varphi_n}^{\frac12} L_2^r(\mathcal{B}(\ell_2^n), \mathrm{tr}_n)\big]_{\frac12} \\
&= \mathrm{k}_n^{\frac14} d_{\varphi_n}^{\frac14}\big[C_n \otimes_h R_n, R_n \otimes_h R_n\big]_{\frac12} = d_{\psi_n}^{\frac14}(\mathrm{OH}_n \otimes_h R_n).
\end{aligned}$$

We can treat the other terms similarly and obtain

$$\begin{aligned}
\mathrm{k}_n^{\frac14} L_4^c(q_n\mathcal{B}(\ell_2)q_n, \varphi_n) &= (C_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}, \\
\mathrm{k}_n^{\frac12} L_2(q_n\mathcal{B}(\ell_2)q_n, \varphi_n) &= d_{\psi_n}^{\frac14}(\mathrm{OH}_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}.
\end{aligned}$$

In particular, Corollary 1.9 provides a cb-isomorphism

$$\sigma : x \in \mathcal{J}_{\infty,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} x_j \otimes \delta_j \in L_\infty(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n})$$

onto a cb-complemented subspace with constants independent of n and

$$\big\langle \sigma(x), \omega(y)\big\rangle = \frac{1}{\mathrm{k}_n}\sum_{j=1}^{\mathrm{k}_n} \mathrm{tr}_{\mathcal{A}_n}(x_j^* y_j) = \mathrm{tr}_n(x^* y) = \langle x, y\rangle.$$

Therefore, the stated properties of ω follow from those of the mapping σ. □

Now we give a more explicit description of $\mathcal{K}_{1,2}(\psi_n)$. Using the terminology introduced before Lemma 2.3, the dual of the space $\mathcal{L}_\infty^n$ is given by the following direct sum

$$\mathcal{L}_1^n = \big(R_n \otimes_h C_n\big) \oplus_2 \big(R_n \otimes_h \mathrm{OH}_n\big) \oplus_2 \big(\mathrm{OH}_n \otimes_h C_n\big) \oplus_2 \big(\mathrm{OH}_n \otimes_h \mathrm{OH}_n\big).$$

Thus, we may consider the map

$$\Psi_n : \mathcal{L}_1^n \to L_1(q_n\mathcal{B}(\ell_2)q_n)$$

given by

$$\Psi_n(x_1, x_2, x_3, x_4) = x_1 + x_2 d_{\psi_n}^{\frac14} + d_{\psi_n}^{\frac14} x_3 + d_{\psi_n}^{\frac14} x_4 d_{\psi_n}^{\frac14}.$$

Then it is easily checked that $\ker \Psi_n = \mathcal{J}_{\infty,2}(\psi_n)^\perp$ with respect to the anti-linear duality bracket and we deduce $\mathcal{K}_{1,2}(\psi_n) = \mathcal{L}_1^n / \ker \Psi_n$. The finite-dimensional spaces defined so far allow us to take direct limits

$$\mathcal{J}_{\infty,2}(\psi) = \overline{\bigcup_{n\ge1} \mathcal{J}_{\infty,2}(\psi_n)} \quad \text{and} \quad \mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n\ge1} \mathcal{K}_{1,2}(\psi_n)}.$$

Lemma 2.4. *Let $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$ be a sequence of strictly positive numbers and define the diagonal operator $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$ on ℓ_2. Let us equip the space graph(d_λ) with the following operator space structures*

$$\begin{array}{rcccl} R \cap \ell_2^{oh}(\lambda) & = & \mathit{graph}(\mathsf{d}_\lambda) & \subset & R \oplus_2 \mathrm{OH}, \\ C \cap \ell_2^{oh}(\lambda) & = & \mathit{graph}(\mathsf{d}_\lambda) & \subset & C \oplus_2 \mathrm{OH}. \end{array}$$

Then, if we consider the dual spaces

$$\begin{array}{rcl} C + \ell_2^{oh}(\lambda) & = & \big(C \oplus_2 \mathrm{OH}\big) / \big(R \cap \ell_2^{oh}(\lambda)\big)^\perp, \\ R + \ell_2^{oh}(\lambda) & = & \big(R \oplus_2 \mathrm{OH}\big) / \big(C \cap \ell_2^{oh}(\lambda)\big)^\perp, \end{array}$$

there exists a n.s.s.f. weight ψ on $\mathcal{B}(\ell_2)$ such that

$$\big(R + \ell_2^{oh}(\lambda)\big) \otimes_h \big(C + \ell_2^{oh}(\lambda)\big) = \mathcal{K}_{1,2}(\psi).$$

Proof. Let q_n be the projection

$$\sum_{k=1}^\infty \alpha_k \, \delta_k \in \ell_2 \mapsto \sum_{k=1}^n \alpha_k \, \delta_k \in \ell_2^n.$$

If $\widehat{q}_n = q_n \oplus q_n$, we define the subspaces

$$\begin{array}{l} q_n\big(C + \ell_2^{oh}(\lambda)\big) = \Big\{ \widehat{q}_n(a,b) + \big(R \cap \ell_2^{oh}(\lambda)\big)^\perp \ \big| \ (a,b) \in C \oplus_2 \mathrm{OH} \Big\} \subset C + \ell_2^{oh}(\lambda), \\ q_n\big(R + \ell_2^{oh}(\lambda)\big) = \Big\{ \widehat{q}_n(a,b) + \big(C \cap \ell_2^{oh}(\lambda)\big)^\perp \ \big| \ (a,b) \in R \oplus_2 \mathrm{OH} \Big\} \subset R + \ell_2^{oh}(\lambda). \end{array}$$

Note that, since the corresponding annihilators are q_n-invariant, these are quotients of $C_n \oplus_2 \mathrm{OH}_n$ and $R_n \oplus_n \mathrm{OH}_n$ respectively. Moreover, recalling that $q_n(x) \to x$ as $n \to \infty$ in the norms of R, OH, C, it is not difficult to see that we may write the Haagerup tensor product $\big(R + \ell_2^{oh}(\lambda)\big) \otimes_h \big(C + \ell_2^{oh}(\lambda)\big)$ as the direct limit

$$\overline{\bigcup_{n \ge 1} q_n(R + \ell_2^{oh}(\lambda)) \otimes_h q_n(C + \ell_2^{oh}(\lambda))}.$$

Therefore, it suffices to show that

$$q_n(R + \ell_2^{oh}(\lambda)) \otimes_h q_n(C + \ell_2^{oh}(\lambda)) = \mathcal{K}_{1,2}(\psi_n),$$

where ψ_n denotes the restriction to $q_n \mathcal{B}(\ell_2) q_n$ of some *n.s.s.f.* weight ψ. However, by duality this is equivalent to see that $q_n(C \cap \ell_2^{oh}(\lambda)) \otimes_h q_n(R \cap \ell_2^{oh}(\lambda)) = \mathcal{J}_{\infty,2}(\psi_n)$ where the spaces $q_n(R \cap \ell_2^{oh}(\lambda))$ and $q_n(C \cap \ell_2^{oh}(\lambda))$ are the span of

$$\Big\{ (\delta_k, \lambda_k \delta_k) \ \big| \ 1 \le k \le n \Big\}$$

in $R_n \oplus_2 \mathrm{OH}_n$ and $C_n \oplus_2 \mathrm{OH}_n$ respectively. Indeed, we have

$$\begin{array}{rcl} q_n\big(C + \ell_2^{oh}(\lambda)\big) & = & \big(C_n \oplus_2 \mathrm{OH}_n\big) / q_n(R \cap \ell_2^{oh}(\lambda))^\perp, \\ q_n\big(R + \ell_2^{oh}(\lambda)\big) & = & \big(R_n \oplus_2 \mathrm{OH}_n\big) / q_n(C \cap \ell_2^{oh}(\lambda))^\perp, \end{array}$$

completely isometrically. Using row/column terminology in terms of matrix units

$$\begin{array}{rcl} q_n\big(C \cap \ell_2^{oh}(\lambda)\big) & = & \mathrm{span}\Big\{ (e_{i1}, \lambda_i e_{i1}) \in C_n \oplus_2 \mathrm{OH}_n \Big\}, \\ q_n\big(R \cap \ell_2^{oh}(\lambda)\big) & = & \mathrm{span}\Big\{ (e_{1j}, \lambda_j e_{1j}) \in R_n \oplus_2 \mathrm{OH}_n \Big\}. \end{array}$$

Therefore, the space $q_n(C \cap \ell_2^{oh}(\lambda)) \otimes_h q_n(R \cap \ell_2^{oh}(\lambda))$ is the subspace

$$\mathrm{span}\Big\{ (e_{ij}, \lambda_j e_{ij}, \lambda_i e_{ij}, \lambda_i \lambda_j e_{ij}) \Big\} = \Big\{ (z, z\mathsf{d}_\lambda, \mathsf{d}_\lambda z, \mathsf{d}_\lambda z \mathsf{d}_\lambda) \ \big| \ z \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the space $\mathcal{L}_\infty^n$ defined above. Then, we define $\gamma_k \in \mathbb{R}_+$ by the relation $\lambda_k = \gamma_k^{\frac{1}{4}}$ and consider the *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$ induced by d_γ. In particular, we immediately obtain

$$q_n\big(C \cap \ell_2^{oh}(\lambda)\big) \otimes_h q_n\big(R \cap \ell_2^{oh}(\lambda)\big) = \Big\{\big(z, zd_{\psi_n}^{\frac{1}{4}}, d_{\psi_n}^{\frac{1}{4}} z, d_{\psi_n}^{\frac{1}{4}} zd_{\psi_n}^{\frac{1}{4}}\big)\Big\}.$$

The space on the right is by definition $\mathcal{J}_{\infty,2}(\psi_n)$. This completes the proof. □

2.3. **Proof of Theorem D.** In this last paragraph, we prove Theorem D. Let us begin with some preliminary results. The next lemma has been known to Xu and the first-named author for quite some time. We refer to Xu's paper [35] for an even more general statement than the result presented below.

Lemma 2.5. *Given $1 \le p \le \infty$ and a closed subspace X of $R_p \oplus_2 \mathrm{OH}$, there exist closed subspaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ of ℓ_2 and an injective closed densely-defined operator $\Lambda : \mathcal{K}_1 \to \mathcal{K}_2$ with dense range such that*

$$\mathrm{X} \simeq_{cb} \mathcal{H}_{1,r_p} \oplus_2 \mathcal{H}_{2,oh} \oplus_2 graph(\Lambda),$$

where the graph of Λ is regarded as a subspace of $\mathcal{K}_{1,r_p} \oplus_2 \mathcal{K}_{2,oh}$ and the relevant constants in the complete isomorphism above do not depend on the subspace X. Moreover, since $R_p = C_{p'}$ the same result can be written in terms of column spaces.

In the following, we shall also need to recognize Pisier's operator Hilbert space OH as the graph of certain diagonal operator on ℓ_2. More precisely, the following result will be used below.

Lemma 2.6. *Given $1 \le p \le \infty$, there exists a sequence $\lambda_1, \lambda_2, \ldots$ in $\mathbb{R}_+$ for which the associated diagonal map $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk} : R_p \to \mathrm{OH}$ satisfies the following complete isomorphism*

$$\mathrm{OH} \simeq_{cb} graph(\mathsf{d}_\lambda).$$

Proof. Let us define

$$u : \delta_k \in \mathrm{OH} \mapsto (\lambda_k^{-1}\delta_k, \delta_k) \in graph(\mathsf{d}_\lambda).$$

The mapping u establishes a linear isomorphism between OH and $graph(\mathsf{d}_\lambda)$. The inverse map of u is the coordinate projection into the second component, which is clearly a complete contraction. Regarding the cb-norm of u, since $graph(\mathsf{d}_\lambda)$ is equipped with the o.s.s. of $R_p \oplus_2 \mathrm{OH}$, we have

$$\|u\|_{cb} = \sqrt{1+\xi^2}$$

with ξ standing for the cb-norm of $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to R_p$. We claim that

$$\xi \le \Big(\sum\nolimits_k |\lambda_k^{-1}|^4\Big)^{\frac{1}{4}},$$

so that it suffices to take $\lambda_1, \lambda_2, \ldots$ large enough to deduce the assertion. Indeed, it is well-known that the inequality above holds for the map $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to R$ and also for $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to C$. Therefore, our claim follows by complex interpolation. □

Remark 2.7. The constants in Lemma 2.6 are uniformly bounded on p.

Before beginning with the proof, we need a bit more preparation. The following discretization result might be also well-known. Nevertheless, since we are not aware of any reference for it, we include the proof for the sake of completeness.

Lemma 2.8. *Given $1 \le p \le \infty$ and a closed densely-defined operator $\Lambda : R_p \to \mathrm{OH}$ with dense range in* OH, *there exists a diagonal operator $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$ on ℓ_2 such that, when regarded as a map $\mathsf{d}_\lambda : R_p \to \mathrm{OH}$, we obtain*

$$graph(\mathsf{d}_\lambda) \simeq_{cb} graph(\Lambda).$$

Moreover, the relevant constants in the cb-isomorphism above do not depend on Λ.

Proof. Let us first assume that Λ is positive. Then, since R_p is separable we deduce from spectral calculus [18] that there exists a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ for which Λ is unitarily equivalent to a multiplication operator $M_f : L_2(\Omega) \to L_2(\Omega)$. Thus we may assume $\Lambda = M_f$. Now, we employ a standard procedure to create a diagonal operator. Given $\delta > 0$, we may approximate the function f by

$$g = \sum\nolimits_k (k\delta) 1_{k\delta < f \le (k+1)\delta}.$$

This yields a $1+\delta$ cb-isomorphism

$$graph(M_f) \simeq_{cb} graph(M_g).$$

Therefore, defining the measurable sets

$$\Omega_k = \Big\{ w \in \Omega \, \big| \, k\delta < f(w) \le (k+1)\delta \Big\},$$

we have that $L_2(\Omega_k)$ is isomorphic to $\ell_2(n_k)$ with $0 \le n_k = \dim L_2(\Omega_k) \le \infty$. Choosing an orthonormal basis for $L_2(\Omega_k)$, we find that M_g is similar to d_λ where $\lambda_k = k\delta$ with multiplicity n_k. This gives the assertion for positive operators. If Λ is not positive, we consider the polar decomposition $\Lambda = u|\Lambda|$. By extension we may assume that u is a unitary. Thus, we get a cb-isometry $graph(\Lambda) \simeq_{cb} graph(|\Lambda|)$ and the general case can be reduced to the case of positive operators. □

Lemma 2.9. *Let $\mathcal{M}$ be a von Neumann algebra. Then*

$$L_1(\mathcal{M}; \mathrm{OH}) = [L_1(\mathcal{M}; C), L_1(\mathcal{M}, R)]_{\frac12}$$

completely embeds into $L_1(\mathcal{A})$ for some von Neumann algebra $\mathcal{A}$. Moreover, we have

- **i)** *If $\mathcal{M}$ is* QWEP, *we can choose $\mathcal{A}$ to be* QWEP.
- **ii)** *If $\mathcal{M}$ is hyperfinite, we can choose $\mathcal{A}$ to be hyperfinite.*

Proof. We recall from Pisier's theorem [21] (see also [13, 37]) that

$$\Big\| \sum\nolimits_k x_k \otimes \delta_k \Big\|_{L_1(\mathcal{M};\mathrm{OH})} = \inf_{x_k = a y_k b} \|a\|_4 \, \big(\sum\nolimits_k \|y_k\|_2^2 \big)^{\frac12} \, \|b\|_4.$$

The first part of the proof recaptures Pisier's argument in [26]. Pisier's exercise with endpoints $L_1(\mathcal{M}; C)$ and $L_1(\mathcal{M}; R)$ implies that $L_1(\mathcal{M}; \mathrm{OH})$ is completely isomorphic to the quotient of $\mathcal{F}(L_1(\mathcal{M}; C), L_1(\mathcal{M}; R))$ by the kernel of the mapping

$$\mathcal{Q} : f \in \mathcal{F}(L_1(\mathcal{M}; C), L_1(\mathcal{M}; R)) \mapsto f(1/2) \in L_1(\mathcal{M}; \mathrm{OH}).$$

Here $\mathcal{F}(L_1(\mathcal{M}; C), L_1(\mathcal{M}; R))$ is viewed as a subspace of

$$L_1(\mathcal{M}; L_2^c(\partial_0; \ell_2)) \oplus_1 L_1(\mathcal{M}; L_2^r(\partial_0; \ell_2))$$

and we find a completely isomorphic embedding

$$L_1(\mathcal{M}; \mathrm{OH}) \subset L_1(\mathcal{M}; L_2^c(\partial_0, \ell_2)) \oplus_1 L_1(\mathcal{M}; L_2^r(\partial_1, \ell_2)) / \ker \mathcal{Q}.$$

It is worth mentioning that formally we might need a finite $\mathcal{M}$ here to make the interpolation argument work. However, this is no restriction in view of Haagerup's reduction procedure [5]. Write $\ker \mathcal{Q}$ as a tensor product $L_1(\mathcal{M}) \otimes \ker q$ with

$$q : f \in \mathcal{F}(C,R) \mapsto f(1/2) \in \mathrm{OH}$$

and diagonalize $\ker q$ as in Lemma 2.5. Hence, it suffices to consider the quotient space

$$\begin{aligned} & \Big\| \sum\nolimits_k x_k \otimes \delta_k \Big\|_{L_1(\mathcal{M};Q(\lambda,\mu))} \\ & \quad = \inf_{x_k = a_{kj} + b_{kj}} \Big\| \Big(\sum\nolimits_{k,j} \lambda_j \, a_{kj}^* a_{kj} \Big)^{\frac12} \Big\|_1 + \Big\| \Big(\sum\nolimits_{k,j} \mu_j \, b_{kj} b_{kj}^* \Big)^{\frac12} \Big\|_1 . \end{aligned}$$

Indeed, a continuous version of this formula has been obtained in [9] and the proof there generalizes to arbitrary von Neumann algebras. Now, we may apply Pisier's approach [26] and find an embedding of $L_1(\mathcal{M}; Q(\lambda,\mu))$ in $L_1(\mathcal{M} \otimes \Gamma(\lambda,\mu))$ where $\Gamma(\lambda,\mu)$ is the free quasi-free factor introduced by Shlyakhtenko.

To preserve hyperfiniteness, something that will not be applied in this paper but in [13], we have to refer to [11]. The results there are also stated in the operator space setting. However, the isomorphism is based on the central limit procedure and a Khintchine type inequality which holds in full generality. For the central limit procedure we shall first consider finitely many coordinates $x_1, x_2, \ldots, x_m$ and finite sequences

$$\lambda(n) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \text{and} \quad \mu(n) = (\mu_1, \mu_2, \ldots, \mu_n).$$

Let $R(\lambda,\mu)$ denote the Araki-Woods factor constructed from (λ,μ) with quasi-free state ϕ and density d. Let $R(\lambda(n),\mu(n))$ be the corresponding finite dimensional matrix algebras and

$$\mathcal{R}_{m,n} = \bigotimes_{1 \le k \le m} R(\lambda(n), \mu(n)).$$

This yields elements $\xi_k \in L_1(\mathcal{R}_{m,n})$ such that

$$\Big\| \sum\nolimits_{k \le m} x_k \otimes \delta_k \Big\|_{L_1(\mathcal{M};Q(\lambda(n),\mu(n)))} \sim_c \Big\| \sum\nolimits_{k \le m} x_k \otimes \xi_k(\lambda_n,\mu_n) \Big\|_{L_1(\mathcal{M} \bar{\otimes} \mathcal{R}_{m,n})}.$$

Then we take first $n \to \infty$ and then $m \to \infty$. Here we use the isometric embedding of $L_1(\mathcal{M}, Q(\lambda,\mu))$ in its bidual and the fact that finite sequences are dense in $L_1(\mathcal{M}; \mathrm{OH})$. We obtain an infinite tensor product $\mathcal{R} = \otimes_{k \ge 1} R(\lambda,\mu)$. This yields an embedding in $L_1(\mathcal{M} \bar{\otimes} \mathcal{R})$ where $\mathcal{R}$ is an Araki-Woods factor. Indeed, the III_1 factor will do. By replacing $\mathcal{M}$ with $M_m \otimes \mathcal{M}$ we see that this embedding is automatically a complete isomorphism. Thus for $\mathcal{M}$ hyperfinite (resp. QWEP) the tensor product $\mathcal{M} \bar{\otimes} \mathcal{R}$ has the same property. □

Proof of Theorem D. By injectivity of the Haagerup tensor product, we may assume that $(\mathrm{X}_1, \mathrm{X}_2) \in \mathcal{Q}(R \oplus_2 \mathrm{OH}) \times \mathcal{Q}(C \oplus_2 \mathrm{OH})$. In particular, the duals X_1^* and X_2^* are subspaces of $C \oplus_2 \mathrm{OH}$ and $R \oplus_2 \mathrm{OH}$ respectively. Therefore, according to Lemma 2.5 above, we may find Hilbert spaces $\mathcal{H}_{ij}$ and $\mathcal{K}_{ij}$ for $i,j = 1,2$ such that

$$\begin{aligned} \mathrm{X}_1^* & \simeq_{cb} & \mathcal{H}_{11,c} \oplus_2 \mathcal{H}_{12,oh} \oplus_2 \mathit{graph}(\Lambda_1), \\ \mathrm{X}_2^* & \simeq_{cb} & \mathcal{H}_{21,r} \oplus_2 \mathcal{H}_{22,oh} \oplus_2 \mathit{graph}(\Lambda_2), \end{aligned}$$

where the operators $\Lambda_1 : \mathcal{K}_{11,c} \to \mathcal{K}_{12,oh}$ and $\Lambda_2 : \mathcal{K}_{21,r} \to \mathcal{K}_{22,oh}$ are injective, closed, densely-defined with dense range. On the other hand, using the complete isometries $\mathcal{H}_r^* = \mathcal{H}_c$ and $\mathcal{H}_c^* = \mathcal{H}_r$, we easily obtain the cb-isomorphisms

$$\begin{aligned} \mathrm{X}_1 & \simeq_{cb} \mathcal{H}_{11,r} \oplus_2 \mathcal{H}_{12,oh} \oplus_2 \Big(\big(\mathcal{K}_{11,r} \oplus_2 \mathcal{K}_{12,oh}\big) / graph(\Lambda_1)^\perp \Big), \\ \mathrm{X}_2 & \simeq_{cb} \mathcal{H}_{21,c} \oplus_2 \mathcal{H}_{22,oh} \oplus_2 \Big(\big(\mathcal{K}_{21,c} \oplus_2 \mathcal{K}_{22,oh}\big) / graph(\Lambda_2)^\perp \Big). \end{aligned}$$

Let us set for the sequel

$$\begin{aligned} \mathcal{Z}_1 & = \big(\mathcal{K}_{11,r} \oplus_2 \mathcal{K}_{12,oh}\big) / graph(\Lambda_1)^\perp, \\ \mathcal{Z}_2 & = \big(\mathcal{K}_{21,c} \oplus_2 \mathcal{K}_{22,oh}\big) / graph(\Lambda_2)^\perp. \end{aligned}$$

Then, we have the following cb-isometric inclusion

$$\mathrm{X}_1 \otimes_h \mathrm{X}_2 \subset \mathrm{A}_1 \oplus_2 \mathrm{A}_2 \oplus_2 \mathrm{A}_3 \oplus_2 \mathrm{A}_4 \oplus_2 \mathrm{A}_5 \oplus_2 \mathrm{A}_6, \tag{2.1}$$

where the A_j's are given by

$$\begin{aligned} \mathrm{A}_1 & = \mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \\ \mathrm{A}_2 & = \mathcal{H}_{11,r} \otimes_h \mathrm{X}_2 \\ \mathrm{A}_3 & = \mathrm{X}_1 \otimes_h \mathcal{H}_{21,c} \\ \mathrm{A}_4 & = \mathcal{H}_{12,oh} \otimes_h \mathcal{Z}_2 \\ \mathrm{A}_5 & = \mathcal{Z}_1 \otimes_h \mathcal{H}_{22,oh} \\ \mathrm{A}_6 & = \mathcal{H}_{12,oh} \otimes_h \mathcal{H}_{22,oh}. \end{aligned}$$

We now reduce the proof to the construction of a cb-embedding $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \to L_1(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Indeed, according to [9] we know that OH cb-embeds in $L_1(\mathcal{A})$ for some QWEP type III factor $\mathcal{A}$. Hence, the term A_6 automatically satisfies the assertion. A similar argument works for the terms A_2 and A_3. Indeed, they clearly embed into the vector-valued Schatten classes $S_1(\mathrm{X}_2)$ and $S_1(\mathrm{X}_1)$ cb-isometrically. On the other hand, since $\mathrm{OH} \in \mathcal{QS}(C \oplus R)$ by "Pisier's exercise" and we have by hypothesis

$$\mathrm{X}_1 \in \mathcal{QS}(R \oplus_2 \mathrm{OH}) \quad \text{and} \quad \mathrm{X}_2 \in \mathcal{QS}(C \oplus_2 \mathrm{OH}),$$

both X_1 and X_2 are cb-isomorphic to an element in $\mathcal{QS}(C \oplus R)$. According to [9] one more time, we know that any operator space in $\mathcal{QS}(C \oplus R)$ cb-embeds into $L_1(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Thus, the spaces $S_1(\mathrm{X}_1)$ and $S_1(\mathrm{X}_2)$ also satisfy the assertion. Finally, for A_4 and A_5 we apply Lemma 2.6 and write OH as the graph of a diagonal operator on ℓ_2. By the self-duality of OH we conclude that these terms can be regarded as particular cases of A_1.

It remains to see that the term $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$ satisfies the assertion. According to the discretization Lemma 2.8, we may assume that the graphs appearing in the terms $\mathcal{Z}_1$ and $\mathcal{Z}_2$ above are graphs of diagonal operators d_{λ_1} and d_{λ_2}. In fact, by polar decomposition as in the proof of Lemma 2.8, we may also assume that both diagonal operators are positive. Moreover, by adding a perturbation term we can take the eigenvalues $\lambda_{1k}, \lambda_{2k} \in \mathbb{R}_+$ strictly positive. Indeed, if we replace λ_{jk} by $\xi_{jk} = \lambda_{jk} + \varepsilon_k$ for $j = 1, 2$, the new diagonal operators d_{ξ_1} and d_{ξ_2} satisfy the cb-isomorphisms

$$graph(\mathsf{d}_{\lambda_j}) \simeq_{cb} graph(\mathsf{d}_{\xi_j}) \quad \text{for} \quad j = 1, 2$$

where (arguing as in Lemma 2.6 above) the cb-norms are controlled by

$$\Big(\sum_k |\varepsilon_k|^4\Big)^{\frac{1}{4}}.$$

Therefore, taking the ε_k's small enough, we may write

$$\begin{aligned} \mathcal{Z}_1 &= \big(R \oplus_2 \mathrm{OH}\big)/\big(C \cap \ell_2^{oh}(\lambda_1)\big)^{\perp} = R + \ell_2^{oh}(\lambda_1) \quad \text{with} \quad \mathsf{d}_{\lambda_1} : C \to \mathrm{OH}, \\ \mathcal{Z}_2 &= \big(C \oplus_2 \mathrm{OH}\big)/\big(R \cap \ell_2^{oh}(\lambda_2)\big)^{\perp} = C + \ell_2^{oh}(\lambda_2) \quad \text{with} \quad \mathsf{d}_{\lambda_2} : R \to \mathrm{OH}, \end{aligned}$$

where $\mathsf{d}_{\lambda_1}, \mathsf{d}_{\lambda_2}$ are positive and invertible. Now we set

$$\lambda_k = \lambda^j_{[k+1/2]} \quad \text{for} \quad k \equiv j \ (\mathrm{mod}\, 2).$$

If we define $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$, we find a complete embedding

$$\mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \subset \big(R + \ell_2^{oh}(\lambda)\big) \otimes_h \big(C + \ell_2^{oh}(\lambda)\big).$$

According to Lemma 2.4, we can regard $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$ as a subspace of

$$\mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}_{1,2}(\psi_n)},$$

for some *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$. It remains to construct a completely isomorphic embedding from $\mathcal{K}_{1,2}(\psi)$ into $L_1(\mathcal{A})$ for some QWEP algebra $\mathcal{A}$. To that aim we assume that the numbers $\mathrm{k}_n = \psi_n(q_n)$ are non-decreasing positive integers. This can always be achieved by the same perturbation argument used above. This will allow us to apply Lemma 2.3. Now, in order to cb-embed $\mathcal{K}_{1,2}(\psi)$ into $L_1(\mathcal{A})$ it suffices to construct a cb-embedding of $\mathcal{K}_{1,2}(\psi_n)$ into $L_1(\mathcal{A}_n')$ for some $\mathcal{A}_n'$ being QWEP and with relevant constants independent of n. Indeed, if so we may consider an ultrafilter $\mathcal{U}$ containing all the intervals (n, ∞), so that we have a completely isometric embedding

$$\mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}_{1,2}(\psi_n)} \to \prod_{n,\mathcal{U}} \mathcal{K}_{1,2}(\psi_n).$$

Then, according to [29], our assumption provides a cb-embedding

$$\mathcal{K}_{1,2}(\psi) \to L_1(\mathcal{A}) \quad \text{with} \quad \mathcal{A} = \Big(\prod_{n,\mathcal{U}} \mathcal{A}_{n*}'\Big)^{*}.$$

Moreover, we know from [10] that $\mathcal{A}$ is QWEP provided the $\mathcal{A}_n'$'s are. Thus, it remains to construct the cb-embeddings $\mathcal{K}_{1,2}(\psi_n) \to L_1(\mathcal{A}_n')$. According to Lemma 2.3, we have a cb-embedding

$$\mathcal{K}_{1,2}(\psi_n) \to L_1(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n}).$$

Now we observe that $\mathcal{A}_n$ is QWEP since it is the free product of k_n copies of $\mathrm{M}_n \oplus \mathrm{M}_n$ and we know from [9] that the QWEP is stable under reduced free products. Hence Lemma 2.9 implies that $L_1(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n}) \subset L_1(\mathcal{A}_n')$ such that $\mathcal{A}_n'$ is also QWEP. □

Proof of Theorem A. Use $S_q = C_q \otimes_h R_q$, Lemma 2.1 and Theorem D. □

References

1. J. Bretagnolle, D. Dacunha-Castelle and J.L. Krivine, Lois stable et space L^p. Ann. Inst. H. Poincaré **2** (1966), 231-259.
2. A. Connes, Une classification des facteurs de type III. Ann. Sci. École Norm. Sup. **6** (1973), 133-252.
3. E.G. Effros and Z.J. Ruan, Operator Spaces. London Math. Soc. Monogr. **23**, Oxford University Press, 2000.
4. U. Haagerup, L_p spaces associated with an arbitrary von Neumann algebra. Algèbres d'opérateurs et leurs applications en physique mathématique, CNRS (1979), 175-184.
5. U. Haagerup, Non-commutative integration theory. Unpublished manuscript (1978). See also Haagerup's Lecture given at the Symposium in Pure Mathematics of the Amer. Math. Soc. Queens University, Kingston, Ontario, 1980.
6. M. Junge, Factorization theory for spaces of operators. Habilitation thesis. Kiel university, 1996.
7. M. Junge, Embeddings of non-commutative L_p-spaces into non-commutative L_1-spaces, $1 < p < 2$. Geom. Funct. Anal. **10** (2000), 389-406.
8. M. Junge, Doob's inequality for non-commutative martingales. J. reine angew. Math. **549** (2002), 149-190.
9. M. Junge, Embedding of the operator space OH and the logarithmic 'little Grothendieck inequality'. Invent. Math. **161** (2005), 225-286.
10. M. Junge, Fubini's theorem for ultraproducts of noncommutative L_p-spaces. Canad. J. Math. **56** (2004), 983-1021.
11. M. Junge, Operator spaces and Araki-Woods factors: A quantum probabilistic approach. Int. Math. Res. Pap. 2006.
12. M. Junge and J. Parcet, The norm of sums of independent noncommutative random variables in $L_p(\ell_1)$. J. Funct. Anal. **221** (2005), 366-406.
13. M. Junge and J. Parcet, Mixed-norm inequalities and operator space L_p embedding theory. Submitted to Mem. Amer. Math. Soc.
14. M. Junge and J. Parcet. Paper in progress.
15. M. Junge, J. Parcet and Q. Xu, Rosenthal type inequalities for free chaos. Ann. Probab. **35** (2007), 1374-1437.
16. M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities. Ann. Probab. **31** (2003), 948-995.
17. M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities II: Applications. To appear in Israel J. Math.
18. R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras I and II. Grad. Stud. Math., **15** & **16**, American Mathematical Society, 1997.
19. J. Parcet, B-convex operator spaces. Proc. Edinburgh Math. Soc. **46** (2003), 649-668.
20. V. Paulsen, Completely bounded maps and operator algebras. Cambridge Studies in Advanced Mathematics **78**. Cambridge University Press, 2002.
21. G. Pisier, Projections from a von Neumann algebra onto a subalgebra. Bull. Soc. Math. France **123** (1995), 139-153.
22. G. Pisier, The Operator Hilbert Space OH, Complex Interpolation and Tensor Norms. Mem. Amer. Math. Soc. **122** (1996).
23. G. Pisier, Non-Commutative Vector Valued L_p-Spaces and Completely p-Summing Maps. Astérisque **247** (1998).
24. G. Pisier, Introduction to Operator Space Theory. Cambridge University Press, 2003.
25. G. Pisier, The operator Hilbert space OH and type III von Neumann algebras. Bull. London Math. Soc. **36** (2004), 455-459.
26. G. Pisier, Completely bounded maps into certain Hilbertian operator spaces. Internat. Math. Res. Notices **74** (2004), 3983-4018.
27. G. Pisier and D. Shlyakhtenko, Grothendieck's theorem for operator spaces. Invent. Math. **150** (2002), 185-217.
28. G. Pisier and Q. Xu, Non-commutative L_p-spaces. Handbook of the Geometry of Banach Spaces II (Ed. W.B. Johnson and J. Lindenstrauss) North-Holland (2003), 1459-1517.
29. Y. Raynaud, On ultrapowers of non-commutative L_p spaces. J. Operator Theory **48** (2002), 41-68.

30. Z.J. Ruan, Subspaces of C^*-algebras. J. Funct. Anal. **76** (1988), 217-230.
31. M. Takesaki, Conditional expectations in von Neumann algebras. J. Func. Anal. **9** (1972), 306-321.
32. M. Terp, L_p spaces associated with von Neumann algebras. Math. Institute Copenhagen University, 1981.
33. D.V. Voiculescu, Addition of certain noncommuting random variables. J. Funct. Anal. **66** (1986), 323-346.
34. D.V. Voiculescu, K. Dykema and A. Nica, Free random variables. CRM Monograph Series **1**, American Mathematical Society, 1992.
35. Q. Xu, Embedding of C_q and R_q into noncommutative L_p-spaces, $1 \leq p < q \leq 2$. Math. Ann. **335** (2006), 109-131.
36. Q. Xu, Real interpolation approach to Junge's works on embedding of OH and the little Grothendieck inequality. Preprint.
37. Q. Xu, A description of $(C_p[L_p(M)], R_p[L_p(M)])_\theta$. Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), 1073-1083.

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