

PSEUDO-LOCALIZATION OF SINGULAR INTEGRALS AND NONCOMMUTATIVE LITTLEWOOD-PALEY INEQUALITIES

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INTRODUCTION

Understood in a wide sense, square functions play a central role in classical Littlewood-Paley theory. This entails for instance dyadic type decompositions of Fourier series, Stein's theory for symmetric diffusion semigroups or Burkholder's martingale square function. All these topics provide a deep technique when dealing with quasi-orthogonality methods, sums of independent variables, Fourier multiplier estimates... The historical survey [34] is an excellent exposition. In a completely different setting, the rapid development of operator space theory and quantum probability has given rise to noncommutative analogs of several classical results in harmonic analysis. We find new results on Fourier/Schur multipliers, a settled theory of noncommutative martingale inequalities, an extension for semigroups on noncommutative L_p spaces of the Littlewood-Paley-Stein theory, a noncommutative ergodic theory and a germ for a noncommutative Calderón-Zygmund theory. We refer to [6, 8, 10, 11, 17, 24, 27] and the references therein.

The aim of this paper is to produce weak type inequalities for a large class of noncommutative square functions. In conjunction with BMO type estimates interpolation and duality, we will obtain the corresponding norm equivalences in the whole L_p scale. Apart from the results themselves, perhaps the main novelty relies on our approach. Indeed, emulating the classical theory, we shall develop a *row/column* valued theory of noncommutative martingale transforms and operator valued Calderón-Zygmund operators. This seems to be new in the noncommutative setting and may be regarded as a first step towards a noncommutative vector-valued theory. To illustrate it, let us state our result for noncommutative martingales.

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Theorem A1. *Let $(\mathcal{M}_n)_{n \geq 1}$ stand for a weak* dense increasing filtration in a semifinite von Neumann algebra (\mathcal{M}, τ) equipped with a normal semifinite faithful trace τ . Given $f = (f_n)_{n \geq 1}$ an $L_1(\mathcal{M})$ martingale, let*

$$T_m f = \sum_{k=1}^{\infty} \xi_{km} df_k \quad \text{with} \quad \sup_{k \geq 1} \sum_{m=1}^{\infty} |\xi_{km}|^2 \lesssim 1.$$

Then, there exists a decomposition $T_m f = A_m f + B_m f$, satisfying

$$\left\| \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right)^{\frac{1}{2}} \right\|_{1,\infty} + \left\| \left(\sum_{m=1}^{\infty} (B_m f)^*(B_m f) \right)^{\frac{1}{2}} \right\|_{1,\infty} \lesssim \sup_{n \geq 1} \|f_n\|_1.$$

In the statement above, df_k denotes the k -th martingale difference of f relative to the filtration $(\mathcal{M}_n)_{n \geq 1}$ and $\|\cdot\|_{1,\infty}$ refers to the norm on $L_{1,\infty}(\mathcal{M})$. In the result below, we also need to use the norm on $\text{BMO}(\mathcal{M})$ relative to our filtration as well as the norm on $L_p(\mathcal{M}; \ell_w^2)$. All these norms are standard in the noncommutative setting and we refer to Section 1 below for precise definitions. Moreover, in what follows δ_k and e_{ij} will stand for unit vectors of sequence spaces and matrix algebras respectively.

Theorem A2. *Let us set $\mathcal{R} = \mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)$. Assume that f is an $L_{\infty}(\mathcal{M})$ martingale relative to the filtration $(\mathcal{M}_n)_{n \geq 1}$ and define $T_m f$ with coefficients ξ_{km} satisfying the same condition above. Then, we have*

$$\left\| \sum_{m=1}^{\infty} T_m f \otimes e_{1m} \right\|_{\text{BMO}(\mathcal{R})} + \left\| \sum_{m=1}^{\infty} T_m f \otimes e_{m1} \right\|_{\text{BMO}(\mathcal{R})} \lesssim \sup_{n \geq 1} \|f_n\|_{\infty}.$$

Therefore, given $1 < p < \infty$ and $f \in L_p(\mathcal{M})$, we deduce

$$\left\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \right\|_{L_p(\mathcal{M}; \ell_w^2)} \leq c_p \|f\|_p.$$

Moreover, the reverse inequality also holds if $\sum_m |\xi_{km}|^2 \sim 1$ uniformly on k .

Let us briefly analyze Theorems A1 and A2. Taking ξ_{km} to be the Dirac delta on (k, m) , we find $T_m f = df_m$ and our results follow from the noncommutative Burkholder-Gundy inequalities [27, 30]. Moreover, taking $\xi_{km} = 0$ for $m > 1$ we simply obtain a martingale transform with scalar coefficients and our results follow from [29]. Other known examples appear by considering (ξ_{km}) of *diagonal-like shape*. For instance, taking an arbitrary partition

$$\mathbb{N} = \bigcup_{m \geq 1} \Omega_m \quad \text{and} \quad \xi_{km} = \begin{cases} 1 & \text{if } k \in \Omega_m, \\ 0 & \text{otherwise.} \end{cases}$$

It is apparent that $\sum_m |\xi_{km}|^2 = 1$ and Theorem A2 gives e.g. for $2 \leq p < \infty$

$$\left\| \left(\sum_{m=1}^{\infty} \left| \sum_{k \in \Omega_m} df_k \right|^2 \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum_{m=1}^{\infty} \left| \sum_{k \in \Omega_m} df_k^* \right|^2 \right)^{\frac{1}{2}} \right\|_p \sim c_p \|f\|_p.$$

Except for $p = 1$, this follows from the noncommutative Khintchine inequality in conjunction with the L_p boundedness of martingale transforms. The new examples appear when considering more general matrices (ξ_{km}) and will be further analyzed in the body of the paper.

In the framework of Theorems A1 and A2, the arguments in [27, 29, 30] are no longer valid. Instead, we think in our square functions as martingale transforms with row/column valued coefficients

$$\begin{aligned} \left(\sum_{m=1}^{\infty} (T_m f)(T_m f)^* \right)^{\frac{1}{2}} &\sim \sum_{k=1}^{\infty} (df_k \otimes e_{1,1}) \overbrace{\left(\sum_{m=1}^{\infty} \xi_{k,m} \mathbf{1}_{\mathcal{M}} \otimes e_{1m} \right)}^{\xi_k^r}, \\ \left(\sum_{m=1}^{\infty} (T_m f)^*(T_m f) \right)^{\frac{1}{2}} &\sim \sum_{k=1}^{\infty} \underbrace{\left(\sum_{m=1}^{\infty} \xi_{k,m} \mathbf{1}_{\mathcal{M}} \otimes e_{m1} \right)}_{\xi_k^c} (df_k \otimes e_{1,1}), \end{aligned}$$

where \sim means to have the same $L_{1,\infty}(\mathcal{M})$ or $L_p(\mathcal{M})$ norm. Tensorizing with the identity on $\mathcal{B}(\ell_2)$, we have $df_k \otimes e_{1,1} = d(f \otimes e_{1,1})_k$ and we find our row/column valued transforms. According to [29], we might expect

$$\left\| \sum_{k=1}^{\infty} \xi_k^r d(f \otimes e_{1,1})_k \right\|_p \leq c_p \sup_{k \geq 1} \|\xi_k^r\|_{\mathcal{B}(\ell_2)} \left\| \sum_{k=1}^{\infty} df_k \otimes e_{1,1} \right\|_p \lesssim c_p \|f\|_p$$

and the same estimate for the ξ_k^c 's. However, it is essential in [29] to have commuting coefficients $\xi_k \in \mathcal{R}_{k-1} \cap \mathcal{R}'_k$, where $\mathcal{R}_n = \mathcal{M}_n \bar{\otimes} \mathcal{B}(\ell_2)$ in our setting. This is not the case. In fact, the inequality above is false in general (e.g. take again $\xi_{km} = \delta_{(k,m)}$ with $1 < p < 2$) and Theorems A1 and A2 might be regarded as the right substitute. The same phenomenon will occur in the context of operator-valued Calderón-Zygmund operators below.

Our main tools to overcome it will be the noncommutative forms of Gundy's and Calderón-Zygmund decompositions [24, 26] for martingales transforms and singular integral operators respectively. As it was justified in [24], there exists nevertheless a substantial difference between both settings. Namely, martingale transforms are local operators while Calderón-Zygmund operators are only pseudo-local. In this paper we will illustrate this point by means of Rota's dilation theorem [33]. The pseudo-localization estimate that we need in this setting, to pass from martingale transforms to Calderón-Zygmund operators, is a Hilbert space valued version of that given in [24] and will be sketched in Appendix A.

Now we formulate our results for Calderón-Zygmund operators. Let Δ denote the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ and fix a Hilbert space \mathcal{H} . We will write in what follows T to denote an integral operator associated to a kernel $k : \mathbb{R}^{2n} \setminus \Delta \rightarrow \mathcal{H}$. This means that for any smooth test function f with compact support, we have

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy \quad \text{for all } x \notin \text{supp } f.$$

Given two points $x, y \in \mathbb{R}^n$, the distance $|x - y|$ between x and y will be taken for convenience with respect to the $\ell_{\infty}(n)$ metric. As usual, we impose size and smoothness conditions on the kernel:

a) If $x, y \in \mathbb{R}^n$, we have

$$\|k(x, y)\|_{\mathcal{H}} \lesssim \frac{1}{|x - y|^n}.$$

b) There exists $0 < \gamma \leq 1$ such that

$$\begin{aligned} \|k(x, y) - k(x', y)\|_{\mathcal{H}} &\lesssim \frac{|x - x'|^\gamma}{|x - y|^{n+\gamma}} \quad \text{if } |x - x'| \leq \frac{1}{2}|x - y|, \\ \|k(x, y) - k(x, y')\|_{\mathcal{H}} &\lesssim \frac{|y - y'|^\gamma}{|x - y|^{n+\gamma}} \quad \text{if } |y - y'| \leq \frac{1}{2}|x - y|. \end{aligned}$$

We will refer to this γ as the Lipschitz parameter of the kernel. The statement of our results below requires to consider appropriate \mathcal{H} -valued noncommutative function spaces as in [10]. Let us first consider the algebra \mathcal{A}_B of essentially bounded functions with values in \mathcal{M}

$$\mathcal{A}_B = \left\{ f : \mathbb{R}^n \rightarrow \mathcal{M} \mid f \text{ strongly measurable s.t. } \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \|f(x)\|_{\mathcal{M}} < \infty \right\},$$

equipped with the n.s.f. trace $\varphi(f) = \int_{\mathbb{R}^n} \tau(f(x)) dx$. The weak-operator closure \mathcal{A} of \mathcal{A}_B is a von Neumann algebra. Given a norm 1 element $e \in \mathcal{H}$, take p_e to be the orthogonal projection onto the one-dimensional subspace generated by e and define

$$\begin{aligned} L_p(\mathcal{A}; \mathcal{H}_r) &= (\mathbf{1}_{\mathcal{A}} \otimes p_e) L_p(\mathcal{A} \bar{\otimes} \mathcal{B}(\mathcal{H})), \\ L_p(\mathcal{A}; \mathcal{H}_c) &= L_p(\mathcal{A} \bar{\otimes} \mathcal{B}(\mathcal{H})) (\mathbf{1}_{\mathcal{A}} \otimes p_e). \end{aligned}$$

This definition is essentially independent of the choice of e . Indeed, given a function $f \in L_p(\mathcal{A}; \mathcal{H}_r)$ we may regard it as an element of $L_p(\mathcal{A} \bar{\otimes} \mathcal{B}(\mathcal{H}))$, so that the product ff^* belongs to $(\mathbf{1}_{\mathcal{A}} \otimes p_e) L_{p/2}(\mathcal{A} \bar{\otimes} \mathcal{B}(\mathcal{H})) (\mathbf{1}_{\mathcal{A}} \otimes p_e)$ which may be identified with $L_{p/2}(\mathcal{A})$. When $f \in L_p(\mathcal{A}; \mathcal{H}_c)$ the same holds for f^*f and we conclude

$$\|f\|_{L_p(\mathcal{A}; \mathcal{H}_r)} = \|(ff^*)^{\frac{1}{2}}\|_{L_p(\mathcal{A})} \quad \text{and} \quad \|f\|_{L_p(\mathcal{A}; \mathcal{H}_c)} = \|(f^*f)^{\frac{1}{2}}\|_{L_p(\mathcal{A})}.$$

Arguing as in [10, Chapter 2], we may use these identities to regard $L_p(\mathcal{A}) \otimes \mathcal{H}$ as a dense subspace of $L_p(\mathcal{A}; \mathcal{H}_r)$ and $L_p(\mathcal{A}; \mathcal{H}_c)$. More specifically, given a function $f = \sum_k g_k \otimes v_k \in L_p(\mathcal{A}) \otimes \mathcal{H}$, we have

$$\begin{aligned} \|f\|_{L_p(\mathcal{A}; \mathcal{H}_r)} &= \left\| \left(\sum_{i,j} \langle v_i, v_j \rangle g_i g_j^* \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{A})}, \\ \|f\|_{L_p(\mathcal{A}; \mathcal{H}_c)} &= \left\| \left(\sum_{i,j} \langle v_i, v_j \rangle g_i^* g_j \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{A})}. \end{aligned}$$

This procedure may also be used to define the spaces

$$L_{1,\infty}(\mathcal{A}; \mathcal{H}_r) \quad \text{and} \quad L_{1,\infty}(\mathcal{A}; \mathcal{H}_c).$$

It is clear that $L_2(\mathcal{M}; \mathcal{H}_r) = L_2(\mathcal{M}; \mathcal{H}_c)$ and we will denote it by $L_2(\mathcal{M}; \mathcal{H}_{oh})$.

Theorem B1. *Given $f \in L_1(\mathcal{A})$, define formally*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy$$

where the kernel $k : \mathbb{R}^{2n} \setminus \Delta \rightarrow \mathcal{H}$ satisfies the size/smoothness conditions imposed above. Assume further that T defines a bounded map $L_2(\mathcal{A}) \rightarrow L_2(\mathcal{A}; \mathcal{H}_{oh})$. Then we may find a decomposition $Tf = Af + Bf$, satisfying

$$\|Af\|_{L_{1,\infty}(\mathcal{A}; \mathcal{H}_r)} + \|Bf\|_{L_{1,\infty}(\mathcal{A}; \mathcal{H}_c)} \lesssim \|f\|_1.$$

To state the following result, we need to define the corresponding \mathcal{H} -valued BMO norm. Assume for simplicity that \mathcal{H} is separable and fix an orthonormal basis $(v_k)_{k \geq 1}$ in \mathcal{H} . Then, given a norm 1 element $e \in \mathcal{H}$, we may identify (as above) $f = \sum_k g_k \otimes v_k \in \text{BMO}(\mathcal{A}) \otimes \mathcal{H}$ with

$$\begin{aligned} ef &= \sum_k g_k \otimes (e \otimes v_k) = (\mathbf{1}_{\mathcal{A}} \otimes p_e) \left(\sum_k g_k \otimes (e \otimes v_k) \right), \\ f_e &= \sum_k g_k \otimes (v_k \otimes e) = \left(\sum_k g_k \otimes (e \otimes v_k) \right) (\mathbf{1}_{\mathcal{A}} \otimes p_e), \end{aligned}$$

where $e \otimes v_k$ is understood as the rank 1 operator $\xi \in \mathcal{H} \mapsto \langle v_k, \xi \rangle e$ and $v_k \otimes e$ stands for $\xi \in \mathcal{H} \mapsto \langle e, \xi \rangle v_k$. Then we define the spaces $\text{BMO}(\mathcal{A}; \mathcal{H}_r)$ and $\text{BMO}(\mathcal{A}; \mathcal{H}_c)$ as the closure of $\text{BMO}(\mathcal{A}) \otimes \mathcal{H}$ with respect to the norms

$$\|f\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_r)} = \|ef\|_{\text{BMO}(\mathcal{A} \otimes \mathcal{B}(\mathcal{H}))} \quad \text{and} \quad \|f\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_c)} = \|f_e\|_{\text{BMO}(\mathcal{A} \otimes \mathcal{B}(\mathcal{H}))}.$$

In the following result, we also use the standard terminology

$$L_p(\mathcal{A}; \mathcal{H}_{rc}) = \begin{cases} L_p(\mathcal{A}; \mathcal{H}_r) + L_p(\mathcal{A}; \mathcal{H}_c) & 1 \leq p \leq 2, \\ L_p(\mathcal{A}; \mathcal{H}_r) \cap L_p(\mathcal{A}; \mathcal{H}_c) & 2 \leq p \leq \infty. \end{cases}$$

Theorem B2. *If $f \in L_\infty(\mathcal{A})$, we also have*

$$\|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_r)} + \|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_c)} \lesssim \|f\|_\infty.$$

Therefore, given $1 < p < \infty$ and $f \in L_p(\mathcal{A})$, we deduce

$$\|Tf\|_{L_p(\mathcal{A}; \mathcal{H}_{rc})} \leq c_p \|f\|_p.$$

Moreover, the reverse inequality holds whenever $\|Tf\|_{L_2(\mathcal{A}; \mathcal{H}_{oh})} = \|f\|_{L_2(\mathcal{A})}$.

In Section 1 we prove Theorems A1 and A2. Then we study an specific example on ergodic averages as in [35]. In conjunction with Rota's theorem, this shows the relevance of pseudo-localization in the Calderón-Zygmund setting. We also find some multilinear and operator-valued forms of our results. Theorems B1 and B2 are proved in Section 2. The proof requires a Hilbert space valued pseudo-localization estimate adapted from [24] in Appendix A. After the proof, we list some examples and applications. Although most of the examples are *semicommutative*, we find several new square functions not considered in [10, 19] and find an application in the *fully* noncommutative setting which will be explored in [12]. Finally, following a referee's suggestion, we also include an additional Appendix B with some background on noncommutative L_p spaces, noncommutative martingales and a few examples for nonexpert readers.

1. MARTINGALE TRANSFORMS

In this section, we prove Theorems A1 and A2. As a preliminary, we recall the definition of some noncommutative function spaces and the statement of some auxiliary results. We shall assume that the reader is familiar with noncommutative L_p spaces. Given (\mathcal{M}, τ) a semifinite von Neumann algebra equipped with a n.s.f. trace, the noncommutative weak L_1 -space $L_{1,\infty}(\mathcal{M})$ is defined as the set of all τ -measurable operators f for which the quasi-norm

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda \tau \left\{ |f| > \lambda \right\}$$

is finite. In this case, we write $\tau\{|f| > \lambda\}$ to denote the trace of the spectral projection of $|f|$ associated to the interval (λ, ∞) . We find this terminology more intuitive, since it is reminiscent of the classical one. The space $L_{1,\infty}(\mathcal{M})$ satisfies a quasi-triangle inequality that will be used below with no further reference

$$\lambda \tau\{|f_1 + f_2| > \lambda\} \leq \lambda \tau\{|f_1| > \lambda/2\} + \lambda \tau\{|f_2| > \lambda/2\}.$$

We refer the reader to [4, 28] for a more in depth discussion on these notions.

Let us now define the space $\text{BMO}(\mathcal{M})$. Let $L_0(\mathcal{M})$ stand for the $*$ -algebra of τ -measurable operators affiliated to \mathcal{M} and fix a filtration $(\mathcal{M}_n)_{n \geq 1}$. Let us write $E_n : \mathcal{M} \rightarrow \mathcal{M}_n$ for the corresponding conditional expectation. Then we define $\text{BMO}_{\mathcal{M}}^r$ and $\text{BMO}_{\mathcal{M}}^c$ as the spaces of operators $f \in L_0(\mathcal{M})$ with norm (modulo multiples of $\mathbf{1}_{\mathcal{M}}$)

$$\begin{aligned} \|f\|_{\text{BMO}_{\mathcal{M}}^r} &= \sup_{n \geq 1} \left\| E_n \left((f - E_{n-1}(f))(f - E_{n-1}(f))^* \right) \right\|_{\mathcal{M}}^{\frac{1}{2}}, \\ \|f\|_{\text{BMO}_{\mathcal{M}}^c} &= \sup_{n \geq 1} \left\| E_n \left((f - E_{n-1}(f))^*(f - E_{n-1}(f)) \right) \right\|_{\mathcal{M}}^{\frac{1}{2}}. \end{aligned}$$

It is easily checked that we have the identities

$$\begin{aligned} \|f\|_{\text{BMO}_{\mathcal{M}}^r} &= \sup_{n \geq 1} \left\| E_n \left(\sum_{k \geq n} df_k df_k^* \right) \right\|_{\mathcal{M}}^{\frac{1}{2}}, \\ \|f\|_{\text{BMO}_{\mathcal{M}}^c} &= \sup_{n \geq 1} \left\| E_n \left(\sum_{k \geq n} df_k^* df_k \right) \right\|_{\mathcal{M}}^{\frac{1}{2}}. \end{aligned}$$

We define $\text{BMO}(\mathcal{M}) = \text{BMO}_{\mathcal{M}}^r \cap \text{BMO}_{\mathcal{M}}^c$ with norm given by

$$\|f\|_{\text{BMO}(\mathcal{M})} = \max \left\{ \|f\|_{\text{BMO}_{\mathcal{M}}^r}, \|f\|_{\text{BMO}_{\mathcal{M}}^c} \right\}.$$

Finally, the space $L_p(\mathcal{M}; \ell_{rc}^2)$ was already defined in the Introduction.

A key tool in proving weak type inequalities for noncommutative martingales is due to Cuculescu. It can be viewed as a noncommutative analogue of the weak type $(1, 1)$ boundedness of Doob's maximal function.

Cuculescu's construction [2]. *Let $f = (f_1, f_2, \dots)$ be a positive L_1 martingale relative to the filtration $(\mathcal{M}_n)_{n \geq 1}$ and let λ be a positive number. Then there exists a decreasing sequence of projections*

$$q(\lambda)_1, q(\lambda)_2, q(\lambda)_3, \dots$$

in \mathcal{M} satisfying the following properties

- i) $q(\lambda)_n$ commutes with $q(\lambda)_{n-1} f_n q(\lambda)_{n-1}$ for each $n \geq 1$.
- ii) $q(\lambda)_n$ belongs to \mathcal{M}_n for each $n \geq 1$ and $q(\lambda)_n f_n q(\lambda)_n \leq \lambda q(\lambda)_n$.
- iii) The following estimate holds

$$\tau \left(\mathbf{1}_{\mathcal{M}} - \bigwedge_{n \geq 1} q(\lambda)_n \right) \leq \frac{1}{\lambda} \sup_{n \geq 1} \|f_n\|_1.$$

Explicitly, we set $q(\lambda)_0 = \mathbf{1}_{\mathcal{M}}$ and define $q(\lambda)_n = \chi_{(0, \lambda]}(q(\lambda)_{n-1} f_n q(\lambda)_{n-1})$.

Another key tool for what follows is Gundy's decomposition for noncommutative martingales. We need a weak notion of support which is quite useful when dealing with weak type inequalities. For a non-necessarily self-adjoint $f \in \mathcal{M}$, the two sided null projection of f is the greatest projection q in \mathcal{M} satisfying $qfq = 0$. Then we define the *weak support projection* of f as

$$\text{supp}^* f = \mathbf{1}_{\mathcal{A}} - q.$$

It is clear that $\text{supp}^* f = \text{supp} f$ when \mathcal{M} is abelian. Moreover, this notion is weaker than the usual support projection in the sense that we have $\text{supp}^* f \leq \text{supp} f$ for any self-adjoint $f \in \mathcal{M}$ and $\text{supp}^* f$ is a subprojection of both the left and right supports in the non-self-adjoint case.

Gundy's decomposition [26]. *Let $f = (f_1, f_2, \dots)$ be a positive L_1 martingale relative to the filtration $(\mathcal{M}_n)_{n \geq 1}$ and let λ be a positive number. Then f can be decomposed $f = \alpha + \beta + \gamma$ as the sum of three martingales relative to the same filtration and satisfying*

$$\max \left\{ \frac{1}{\lambda} \sup_{n \geq 1} \|\alpha_n\|_2^2, \sum_{k=1}^{\infty} \|d\beta_k\|_1, \lambda \tau \left(\bigvee_{k \geq 1} \text{supp}^* d\gamma_k \right) \right\} \lesssim \sup_{n \geq 1} \|f_n\|_1.$$

We may write α, β and γ in terms of their martingale differences

$$\begin{aligned} d\alpha_k &= q_k(\lambda) df_k q_k(\lambda) - \mathbf{E}_{k-1}(q_k(\lambda) df_k q_k(\lambda)), \\ d\beta_k &= q_{k-1}(\lambda) df_k q_{k-1}(\lambda) - q_k(\lambda) df_k q_k(\lambda) + \mathbf{E}_{k-1}(q_k(\lambda) df_k q_k(\lambda)), \\ d\gamma_k &= df_k - q_{k-1}(\lambda) df_k q_{k-1}(\lambda). \end{aligned}$$

1.1. Weak type $(1, 1)$ boundedness. Here we prove Theorem A1. Let us begin with some harmless assumptions. First, we shall assume that \mathcal{M} is a finite von Neumann algebra with a normalized trace τ . The passage to the semifinite case is just technical. Moreover, we shall sketch it in Section 2 since the von Neumann algebra \mathcal{A} we shall work with can not be finite. Second, we may assume that the martingale f is positive and finite, so that we may use Cuculescu's construction and Gundy's decomposition for f and moreover we do not have to worry about convergence issues.

Now we provide the decomposition $T_m f = A_m f + B_m f$. If $(q_n(\lambda))_{n \geq 1}$ denotes the Cuculescu's projections associated to (f, λ) , let us write in what follows $q(\lambda)$ for the projection

$$q(\lambda) = \bigwedge_{n \geq 1} q_n(\lambda).$$

Then we define the projections

$$\pi_0 = \bigwedge_{s \geq 0} q(2^s) \quad \text{and} \quad \pi_k = \bigwedge_{s \geq k} q(2^s) - \bigwedge_{s \geq k-1} q(2^s)$$

for $k \geq 1$. Since $\sum_{k \geq 0} \pi_k = \mathbf{1}_{\mathcal{M}}$, we may write

$$df_k = \sum_{i \geq j} \pi_i df_k \pi_j + \sum_{i < j} \pi_i df_k \pi_j = \Delta_r(df_k) + \Delta_c(df_k).$$

Then, our decomposition $T_m f = A_m f + B_m f$ is given by

$$A_m f = \sum_{k=1}^{\infty} \xi_{km} \Delta_r(df_k) \quad \text{and} \quad B_m f = \sum_{k=1}^{\infty} \xi_{km} \Delta_c(df_k).$$

Since both terms can be handled in a similar way, we shall only prove that

$$\sup_{\lambda > 0} \lambda \tau \left\{ \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right)^{\frac{1}{2}} > \lambda \right\} \lesssim \sup_{n \geq 1} \|f\|_1.$$

By homogeneity, we may assume that the right hand side equals 1. This means in particular that we may also assume that $\lambda \geq 1$ since, by the finiteness of \mathcal{M} , the left hand side is bounded above by 1 for $0 < \lambda < 1$. Moreover, up to a constant 2 it suffices to prove the result for λ being a nonnegative power of 2. Let us fix a nonnegative integer ℓ , so that $\lambda = 2^\ell$ for the rest of the proof.

Let us define

$$w_\ell = \bigwedge_{s \geq \ell} q(2^s).$$

By the quasi-triangle inequality, we are reduced to estimate

$$\lambda \tau \left\{ w_\ell \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right) w_\ell > \lambda^2 \right\} + \lambda \tau(\mathbf{1}_{\mathcal{M}} - w_\ell) = A_1 + A_2.$$

According to Cuculescu's theorem, A_2 is dominated by

$$\lambda \sum_{s \geq \ell} \tau(\mathbf{1}_{\mathcal{M}} - q(2^s)) \leq 2^\ell \left(\sum_{s \geq \ell} \frac{1}{2^s} \right) \sup_{n \geq 1} \|f_n\|_1 \leq 2 \sup_{n \geq 1} \|f_n\|_1.$$

Let us now proceed with the term A_1 . We first notice that $w_\ell \pi_k = \pi_k w_\ell = 0$ for any integer $k > \ell$. Therefore, we find $w_\ell \Delta_r(df_k) = w_\ell (\sum_{j \leq i \leq \ell} \pi_i df_k \pi_j) = w_\ell \Delta_{r\ell}(df_k)$. Similarly, we have $\Delta_r(df_k)^* w_\ell = \Delta_{r\ell}(df_k)^* w_\ell$ and letting

$$A_m f = \sum_{k=1}^{\infty} \xi_{km} \Delta_{r\ell}(df_k),$$

we conclude

$$A_1 = \lambda \tau \left\{ w_\ell \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right) w_\ell > \lambda^2 \right\}.$$

Moreover, using the fact that the spectral projections $\chi_{(\lambda, \infty)}(xx^*)$ and $\chi_{(\lambda, \infty)}(x^*x)$ are Murray-von Neumann equivalent, we may kill the projection w_ℓ above and obtain the inequality $A_1 \leq \lambda \tau \left\{ \sum_m (A_m f)(A_m f)^* > \lambda^2 \right\}$. Now we use Gundy's decomposition for (f, λ) and quasi-triangle inequality to get

$$\begin{aligned} A_1 &\lesssim \lambda \tau \left\{ \sum_{m=1}^{\infty} (A_m \alpha)(A_m \alpha)^* > \lambda^2 \right\} \\ &+ \lambda \tau \left\{ \sum_{m=1}^{\infty} (A_m \beta)(A_m \beta)^* > \lambda^2 \right\} \\ &+ \lambda \tau \left\{ \sum_{m=1}^{\infty} (A_m \gamma)(A_m \gamma)^* > \lambda^2 \right\} = A_\alpha + A_\beta + A_\gamma. \end{aligned}$$

We claim that A_γ is identically 0. Indeed, note that

$$\Delta_{r\ell}(d\gamma_k) = \sum_{j \leq i \leq \ell} \pi_i \left(df_k - q_{k-1}(2^\ell) df_k q_{k-1}(2^\ell) \right) \pi_j = 0$$

since $\pi_i q_{k-1}(2^\ell) = \pi_i$ and $q_{k-1}(2^\ell) \pi_j = \pi_j$ for $i, j \leq \ell$. Therefore, it remains to control the terms A_α and A_β . Let us begin with A_α . Applying Fubini to the sum defining it, we obtain

$$\sum_{m=1}^{\infty} (A_{m\ell} \alpha) (A_{m\ell} \alpha)^* = \sum_{j,k=1}^{\infty} \left(\sum_{m=1}^{\infty} \xi_{jm} \overline{\xi_{km}} \right) \Delta_{r\ell}(d\alpha_j) \Delta_{r\ell}(d\alpha_k)^*.$$

It will be more convenient, to write this as follows

$$\begin{aligned} & \sum_{m=1}^{\infty} (A_{m\ell} \alpha) (A_{m\ell} \alpha)^* \\ &= \left(\sum_{j=1}^{\infty} \Delta_{r\ell}(d\alpha_j) e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \left[\sum_{m=1}^{\infty} \xi_{jm} \overline{\xi_{km}} \right] e_{jk} \right) \left(\sum_{k=1}^{\infty} \Delta_{r\ell}(d\alpha_k)^* e_{k1} \right) \\ &= \left(\sum_{j=1}^{\infty} \Delta_{r\ell}(d\alpha_j) e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right)^* \left(\sum_{k=1}^{\infty} \Delta_{r\ell}(d\alpha_k)^* e_{k1} \right). \end{aligned}$$

In particular, Chebychev's inequality gives

$$\begin{aligned} A_\alpha &\leq \frac{1}{\lambda} \left\| \left(\sum_{j=1}^{\infty} \Delta_{r\ell}(d\alpha_j) e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right) \right\|_{L_2(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))}^2 \\ &= \frac{1}{\lambda} \left\| \widehat{\Delta_{r\ell}} \left[\left(\sum_{j=1}^{\infty} d\alpha_j e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right) \right] \right\|_{L_2(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))}^2, \end{aligned}$$

with $\widehat{\Delta_{r\ell}} = \Delta_{r\ell} \otimes id_{\mathcal{B}(\ell_2)}$ a triangular truncation, bounded on $L_2(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))$. Thus, we get

$$\begin{aligned} A_\alpha &\lesssim \frac{1}{\lambda} \left\| \left(\sum_{j=1}^{\infty} d\alpha_j e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right) \right\|_{L_2(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))}^2 \\ &= \frac{1}{\lambda} \sum_{j,k=1}^{\infty} \left(\sum_{m=1}^{\infty} \xi_{jm} \overline{\xi_{km}} \right) \tau(d\alpha_j d\alpha_k^*) = \left(\sup_{k \geq 1} \sum_{m=1}^{\infty} |\xi_{km}|^2 \right) \frac{1}{\lambda} \sum_{k=1}^{\infty} \tau(d\alpha_k d\alpha_k^*). \end{aligned}$$

Therefore, the estimate for A_α follows from our hypothesis on the ξ_{km} 's and from the estimate for the α -term in Gundy's decomposition. Let us finally estimate the term A_β . Arguing as above, we clearly have

$$\begin{aligned} A_\beta &\leq \left\| \left(\sum_{m=1}^{\infty} (A_{m\ell} \beta) (A_{m\ell} \beta)^* \right)^{\frac{1}{2}} \right\|_{1,\infty} \\ &= \left\| \widehat{\Delta_{r\ell}} \left[\left(\sum_{j=1}^{\infty} d\beta_j e_{1j} \right) \left(\sum_{j,k=1}^{\infty} \xi_{jk} e_{jk} \right) \right] \right\|_{L_{1,\infty}(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))}. \end{aligned}$$

Then we use the weak type $(1, 1)$ boundedness of triangular truncations to get

$$\begin{aligned} A_\beta &\lesssim \left\| \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \xi_{jk} d\beta_j \right) \otimes e_{1k} \right\|_{L_1(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} \\ &\leq \left(\sup_{k \geq 1} \sum_{m=1}^{\infty} |\xi_{km}|^2 \right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \|d\beta_j\|_1 \lesssim \sup_{n \geq 1} \|f_n\|_1. \end{aligned}$$

The last inequality follows from our hypothesis and Gundy's decomposition. \square

1.2. BMO estimate, interpolation and duality. In this paragraph we prove Theorem A2. The key for the BMO estimate is certain commutation relation which can not be exploited in L_p for finite p . Namely, we have

$$\sum_{m=1}^{\infty} T_m f \otimes e_{1m} = \sum_{k=1}^{\infty} \underbrace{\left(\sum_{m=1}^{\infty} \xi_{km} \mathbf{1}_{\mathcal{M}} \otimes e_{1m} \right)}_{\xi_k^r} (df_k \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}) = \sum_{k=1}^{\infty} d(\xi_k^r(f \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}))_k$$

where the last martingale difference is considered with respect to the filtration $\mathcal{R}_n = \mathcal{M}_n \bar{\otimes} \mathcal{B}(\ell_2)$ of \mathcal{R} . Note that ξ_k^r commutes with $df_k \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}$ and therefore we find that

$$\begin{aligned} \left\| \sum_{m=1}^{\infty} T_m f \otimes e_{1m} \right\|_{\text{BMO}_{\mathcal{R}}^r}^2 &= \left\| \sup_{n \geq 1} \sum_{k \geq n} \mathbb{E}_n((df_k \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}) \xi_k^r \xi_k^{r*} (df_k \otimes \mathbf{1}_{\mathcal{B}(\ell_2)})^*) \right\|_{\mathcal{R}} \\ &\leq \left(\sup_{k \geq 1} \sum_{m=1}^{\infty} |\xi_{km}|^2 \right) \|f \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}\|_{\text{BMO}_{\mathcal{R}}^r}^2 \\ &\lesssim \|f \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}\|_{\text{BMO}_{\mathcal{R}}^r}^2 \leq \|f\|_{\text{BMO}_{\mathcal{M}}^r}^2 \leq \|f\|_{\infty}^2. \end{aligned}$$

Similarly, $\|\sum_m T_m f \otimes e_{1m}\|_{\text{BMO}_{\mathcal{R}}^c} \lesssim \|f\|_{\text{BMO}_{\mathcal{M}}^c} \leq \|f\|_{\infty}$ so that

$$\left\| \sum_{m=1}^{\infty} T_m f \otimes e_{1m} \right\|_{\text{BMO}(\mathcal{R})} \lesssim \|f\|_{\text{BMO}(\mathcal{M})} \leq \sup_{n \geq 1} \|f_n\|_{\infty}.$$

The estimate for $\sum_m T_m f \otimes e_{m1}$ is entirely analogous. This gives the L_{∞} – BMO estimate, or even better the BMO – BMO one. Note that we make crucial use of the identity $\|f \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}\|_{\infty} = \|f\|_{\infty}$! However, $\|f \otimes \mathbf{1}_{\mathcal{B}(\ell_2)}\|_{L_p(\mathcal{R})} \neq \|f\|_{L_p(\mathcal{M})}$ for p finite. That is why we can not reduce the L_p estimate to the commutative case.

With the weak type $(1, 1)$ and the BMO estimates in hand, we may follow by interpolation. Namely, since the case $p = 2$ is trivial, we interpolate for $1 < p < 2$ following Randrianantoanina's argument [30] and for $2 < p < \infty$ following Junge and Musat [13, 21]. This gives rise to

$$\left\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \right\|_{L_p(\mathcal{M}; \ell_{rc}^2)} \leq c_p \|f\|_p$$

for all $1 < p < \infty$. Assuming further that $\sum_m |\xi_{km}|^2 = \gamma_k \sim 1$, we find

$$\begin{aligned} \|f\|_p &= \sup_{\|g\|_{p'} \leq 1} \sum_{k=1}^{\infty} \langle df_k, dg_k \rangle \\ &= \sup_{\|g\|_{p'} \leq 1} \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \sum_{m=1}^{\infty} \langle \xi_{km} df_k, \xi_{km} dg_k \rangle \\ &= \sup_{\|g\|_{p'} \leq 1} \sum_{m=1}^{\infty} \left\langle \sum_{k=1}^{\infty} \xi_{km} df_k, \sum_{k=1}^{\infty} \frac{\xi_{km}}{\gamma_k} dg_k \right\rangle \\ &\leq \left\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \right\|_{L_p(\mathcal{M}; \ell_{rc}^2)} \sup_{\|g\|_{p'} \leq 1} \left\| \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_{km}}{\gamma_k} dg_k \otimes \delta_m \right\|_{L_{p'}(\mathcal{M}; \ell_{rc}^2)}. \end{aligned}$$

Therefore, the reverse inequality follows by duality whenever $\sum_m |\xi_{km}|^2 \sim 1$. \square

Remark 1.1. The aim of Theorem A2 is

$$\left\| \left(\sum_{m=1}^{\infty} (T_m f)(T_m f)^* \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum_{m=1}^{\infty} (T_m f)^* (T_m f) \right)^{\frac{1}{2}} \right\|_p \lesssim c_p \|f\|_p$$

for $2 < p < \infty$, since the remaining inequalities follow from it and Theorem A1. Nevertheless, a direct argument (not including the BMO estimate) is also available for $p \in 2\mathbb{Z}$. Indeed, by Khintchine and Burkholder-Gundy inequalities

$$\begin{aligned} \left\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \right\|_{L_p(\mathcal{M}; \ell_{rc}^2)}^p &\sim \int_{\Omega} \left\| \sum_{k=1}^{\infty} \underbrace{\left(\sum_{m=1}^{\infty} \xi_{km} r_m(w) \right)}_{\xi_k(w)} df_k \right\|_{L_p(\mathcal{M})}^p d\mu(w) \\ &\sim \int_{\Omega} \left\| \sum_{k=1}^{\infty} \xi_k(w) df_k \otimes \delta_k \right\|_{L_p(\mathcal{M}; \ell_{rc}^2)}^p d\mu(w). \end{aligned}$$

If $p = 2j$, we use Hölder and again Khintchine + Burkholder-Gundy to obtain

$$\begin{aligned} &\tau \int_{\Omega} \left(\sum_{k=1}^{\infty} |\xi_k|^2 |df_k|^2 \right)^{\frac{p}{2}} d\mu \\ &= \sum_{k_1, k_2, \dots, k_j=1}^{\infty} \left(\int_{\Omega} \prod_{s=1}^j |\xi_{k_s}|^2 d\mu \right) \tau \left[|df_{k_1}|^2 |df_{k_2}|^2 \cdots |df_{k_j}|^2 \right] \\ &\leq \sum_{k_1, k_2, \dots, k_j=1}^{\infty} \prod_{s=1}^j \left(\int_{\Omega} |\xi_{k_s}|^{2j} d\mu \right)^{\frac{1}{j}} \tau \left[|df_{k_1}|^2 |df_{k_2}|^2 \cdots |df_{k_j}|^2 \right] \\ &\lesssim \left(\sup_{k \geq 1} \sum_{m=1}^{\infty} |\xi_{km}|^2 \right)^{\frac{1}{2}} \tau \left(\sum_{k=1}^{\infty} |df_k|^2 \right)^{\frac{p}{2}} \leq c_p^p \|f\|_p^p. \end{aligned}$$

The row term is estimated in the same way. This completes the argument. \square

Remark 1.2. Our results so far and the implications for semigroups explored in the next paragraph can be regarded as an alternative argument in producing Littlewood-Paley inequalities from martingale inequalities. Namely, the key result is Gundy's decomposition while in [35, Chapter IV] the main ideas are based on Stein's inequality for martingales, which is not necessary from our viewpoint.

Remark 1.3. The adjoint of $\sum_m T_m \otimes \delta_m$ is

$$\widehat{g} = \sum_{m=1}^{\infty} g^m \otimes \delta_m \in L_p(\mathcal{M}; \ell_{rc}^2) \mapsto \sum_{m=1}^{\infty} T_m g^m \in L_p(\mathcal{M}).$$

Letting $\widehat{\xi}_k = \sum_m \xi_{km} \otimes \delta_m$, this mapping can be formally written as

$$\sum_{m=1}^{\infty} T_m g^m = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \xi_{km} dg_k^m = \sum_{k=1}^{\infty} \langle \widehat{\xi}_k, dg_k \rangle.$$

In other words, we find a martingale transform with row/column valued coefficients and martingale differences. In this case, we have again noncommuting coefficients and Theorem A2 gives the right L_p estimate.

1.3. Rota's dilation theorem and pseudo-localization. We show that the key new ingredient to produce weak type inequalities for square functions associated to a large family of semigroups is a certain pseudo-localization estimate. This applies in particular for semicommutative Calderón-Zygmund operators and illustrates the difference between Theorems A1 and B1. Our argument uses Rota's theorem [33] in conjunction with Stein's ergodic averages [35], we find it quite transparent.

According to [10], we will say that a bounded operator $T : \mathcal{M} \rightarrow \mathcal{M}$ satisfies *Rota's dilation property* if there exists a von Neumann algebra \mathcal{R} equipped with a normalized trace, a normal unital faithful $*$ -representation $\pi : \mathcal{M} \rightarrow \mathcal{R}$ which preserves trace, and a decreasing sequence $(\mathcal{R}_m)_{m \geq 1}$ of von Neumann subalgebras of \mathcal{R} such that $T^m = \mathbb{E} \circ \mathcal{E}_m \circ \pi$ for any $m \geq 1$ and where $\mathcal{E}_m : \mathcal{R} \rightarrow \mathcal{R}_m$ is the canonical conditional expectation and $\mathbb{E} : \mathcal{R} \rightarrow \mathcal{M}$ is the conditional expectation associated with π . By Rota's theorem, T^2 satisfies it whenever \mathcal{M} is commutative and $T : \mathcal{M} \rightarrow \mathcal{M}$ is a normal unital positive self-adjoint operator. This was used by Stein [35] for averages of discretized symmetric diffusion commutative semigroups and also applies in the semicommutative setting of [19]. We also know from [10] that the noncommutative Poisson semigroup on the free group fits in.

The problem that we want to study is the following. Assume that $T : \mathcal{M} \rightarrow \mathcal{M}$ is a normal unital completely positive self-adjoint operator satisfying Rota's dilation property. Given $f \in L_1(\mathcal{M})$ and $m \geq 0$, set

$$\Sigma_m f = \frac{1}{m+1} \sum_{k=0}^m T^k f \quad \text{and} \quad \Gamma_m f = \Sigma_m f - \Sigma_{m-1} f.$$

What can we say about the inequality

$$\left\| \sum_{m=1}^{\infty} \sqrt{m} \Gamma_m f \otimes \delta_m \right\|_{L_{1,\infty}(\mathcal{M}; \ell_{rc}^2)} \lesssim \|f\|_1?$$

Of course, the norm in $L_{1,\infty}(\mathcal{M}; \ell_{rc}^2)$ denotes the one used in Theorem A1.

This might be related to weak type estimates for general symmetric diffusion semigroups satisfying Rota's dilation property. It seems though that the classical method [35] only works for $p > 1$, see [10, 17] for noncommutative forms of Stein's fractional averages. Nevertheless there are concrete cases, like the noncommutative Poisson semigroup on the free group, where more information is available. The idea is to prove first its martingale analog and apply then Rota's property to recognize pseudo-localization as the key new ingredient. The martingale inequality below is the weak type (1,1) extension of [10, Proposition 10.8].

Corollary 1.4. *Let $(\mathcal{M}_n)_{n \geq 1}$ be either an increasing or decreasing filtration of the von Neumann algebra \mathcal{M} . Given $f \in L_1(\mathcal{M})$, we set $f_n = \mathcal{E}_n(f)$ and define the following operators associated to f for $m \geq 0$*

$$\sigma_m f = \frac{1}{m+1} \sum_{k=0}^m f_k \quad \text{and} \quad \gamma_m f = \sigma_m f - \sigma_{m-1} f.$$

Then, there exists a decomposition $\sqrt{m} \gamma_m f = \alpha_m f + \beta_m f$ such that

$$\left\| \left(\sum_{m=1}^{\infty} (\alpha_m f)(\alpha_m f)^* \right)^{\frac{1}{2}} \right\|_{1,\infty} + \left\| \left(\sum_{m=1}^{\infty} (\alpha_m f)^*(\alpha_m f) \right)^{\frac{1}{2}} \right\|_{1,\infty} \lesssim \|f\|_1.$$

Proof. Theorem A1 still holds for decreasing filtrations $(\mathcal{M}_n)_{n \geq 1}$. Indeed, it suffices to observe that any finite (reverse) martingale $(f_1, f_2, \dots, f_n, f_n, f_n, \dots)$ may be rewritten as a finite ordinary martingale by reversing the order, and that this procedure does not affect the arguments in the proof of Theorem A1, details are left to the reader. On the other hand, as in [10, 35] it is easily checked that

$$\sqrt{m} \gamma_m f = \sum_{k=1}^m \frac{k}{\sqrt{m}(m+1)} df_k.$$

Therefore, the result follows from Theorem A1 since $\xi_{km} = \delta_{k \leq m} \frac{k}{\sqrt{m}(m+1)}$. \square

Now we are in a position to study the norm of $\sum_m \sqrt{m} \Gamma_m f \otimes \delta_m$ in $L_{1,\infty}(\mathcal{M}; \ell_{rc}^2)$. Namely, since we have assumed that T satisfies Rota's dilation property, we have $\sqrt{m} \Gamma_m f = \sqrt{m} \mathbb{E} \circ \gamma_m \circ \pi(f)$. The presence of π is harmless since it just means that we are working in a bigger algebra. Moreover, when working in L_p the conditional expectation \mathbb{E} is a contraction so that we may eliminate it. However, this is not the case in $L_{1,\infty}$ and we have to review the arguments in the proof of Theorem A1 for the operator

$$T_m f = \sqrt{m} \mathbb{E} \circ \gamma_m f = \sum_{k=1}^m \frac{k}{\sqrt{m}(m+1)} \mathbb{E} df_k = \sum_{k=1}^{\infty} \xi_{km} \mathbb{E} df_k.$$

We consider the decomposition $T_m f = A_m f + B_m f$ with

$$A_m f = \sum_{k=1}^{\infty} \xi_{km} \Delta_r(\mathbb{E} df_k) \quad \text{and} \quad B_m f = \sum_{k=1}^{\infty} \xi_{km} \Delta_c(\mathbb{E} df_k).$$

Following the proof of Theorem A1, we are reduced to estimate the three square functions associated to the terms α, β and γ in Gundy's decomposition. The α and β terms are estimated in the same way, since the weak L_1 and the L_2 boundedness of $\Delta_{r\ell} \otimes id_{\mathcal{B}(\ell_2)}$ is also satisfied by $\Delta_{r\ell} \mathbb{E} \otimes id_{\mathcal{B}(\ell_2)}$.

Conclusion. The only term that can not be estimated from the argument in Theorem A1 is the γ -term. Nevertheless, the key to estimate that term is the fact that it is supported by a sufficiently small projection. More specifically, it is easily checked that we have $\text{supp}^* df_k \leq \mathbf{1}_{\mathcal{M}} - w_\ell$ for $k \geq 1$ and $\lambda \tau(\mathbf{1}_{\mathcal{M}} - w_\ell) \leq 2\|f\|_1$. According to [24, Remark 5.1], this gives

$$df_k = (\mathbf{1}_{\mathcal{M}} - w_\ell) df_k + df_k (\mathbf{1}_{\mathcal{M}} - w_\ell) - (\mathbf{1}_{\mathcal{M}} - w_\ell) df_k (\mathbf{1}_{\mathcal{M}} - w_\ell).$$

Moreover, since $w_\ell \pi_k = \pi_k w_\ell = \pi_k$ for $k \leq \ell$, we find

$$\sum_{k=1}^{\infty} \xi_{km} \Delta_{r\ell}(\mathbb{E} df_k) = \Delta_{r\ell} \left[w_\ell \mathbb{E} \left(w_\ell^\perp \Lambda_m f + \Lambda_m f w_\ell^\perp - w_\ell^\perp \Lambda_m f w_\ell^\perp \right) w_\ell \right],$$

where $\Lambda_m f = \sum_k \xi_{km} df_k$ and $w_\ell^\perp = \mathbf{1}_{\mathcal{M}} - w_\ell$. That is why we obtain a zero term in the martingale case, with \mathbb{E} being the identity map. In the general case, this indicates that we shall be able to obtain satisfactory inequalities whenever \mathbb{E} *almost* behaves as a local map, in the sense that *respects* the supports. As we shall see in the next section, when dealing with operator-valued Calderón-Zygmund operators, the role of γ will be played by the off-diagonal terms of the good and bad parts of Calderón-Zygmund decomposition. The pseudo-localization estimate needed for the bad part is standard, while the one for the good part requires some almost-orthogonality methods described in Appendix A.

As far as we know, this pseudo-localization property is unknown for all the *fully* noncommutative Calderón-Zygmund-type operators in the literature. Particularly it would be very interesting to know the behavior of the noncommutative Poisson semigroup on the free group. This leads us to formulate it as a problem for the interested reader.

Problem. Let us consider the Poisson semigroup

$$T_t(\lambda(g)) = e^{-t|g|}\lambda(g)$$

given by the length function in the free group von Neumann algebra. It is known [10, 17] that square and maximal functions are bounded maps on L_p . Are these mappings of type $(1, 1)$?

1.4. Further comments. We have seen so far several particular cases of Theorems A1 and A2. Namely, in the Introduction we mentioned noncommutative martingale transforms and square functions as well as *shuffled* square functions where the martingale differences are grouped according to an arbitrary partition of \mathbb{N} . In the previous paragraph we have seen that Stein's ergodic averages, in connection with semigroups satisfying Rota's property, also fall in the framework of Theorems A1 and A2. In this paragraph, we indicate further applications and generalizations of Theorems A1 and A2.

A. Multi-indexed coefficients/martingales. Replacing df_k in our main results by Rademacher variables or free generators clearly gives rise to new Khintchine type inequalities. The iteration of Khintchine inequalities was the basis in [25] for some multilinear generalizations that were further explored in [14, 32]. Although a detailed analysis of these methods in the context of our results is out of the scope of this paper, let us mention two immediate consequences.

In the iteration of Khintchine type inequalities, it is sometime quite interesting to being able to dominate the cross terms that appear by the row/column terms and thereby reduce it to a standard Khintchine type inequality. Our result is *flexible* enough to produce such estimates.

Corollary 1.5. *Assume that*

$$\sum_{m=1}^{\infty} |\rho_{km}|^2 \sim 1 \sim \sum_{n=1}^{\infty} |\eta_{kn}|^2 \quad \text{for all } k \geq 1.$$

Let $T_{mn}f = \sum_k \rho_{km} \eta_{kn} df_k$. Then, if $2 < p < \infty$, we have

$$\left\| \sum_{m,n=1}^{\infty} T_{mn}f \otimes e_{m,n} \right\|_p \lesssim \left\| \sum_{m,n=1}^{\infty} T_{mn}f \otimes e_{1,mn} \right\|_p + \left\| \sum_{m,n=1}^{\infty} T_{mn}f \otimes e_{mn,1} \right\|_p.$$

We may also replace $e_{m,n}$ by $e_{n,m}$. If $1 < p < 2$, certain dual inequalities hold.

Proof. Let us consider the spaces

$$\begin{aligned} \mathcal{K}_p^1 &= R_p \otimes_h R_p + C_p \otimes_h C_p, \\ \mathcal{J}_p^1 &= R_p \otimes_h R_p \cap C_p \otimes_h C_p, \\ \mathcal{K}_p^2 &= R_p \otimes_h R_p + R_p \otimes_h C_p + C_p \otimes_h R_p + C_p \otimes_h C_p, \\ \mathcal{J}_p^2 &= R_p \otimes_h R_p \cap R_p \otimes_h C_p \cap C_p \otimes_h R_p \cap C_p \otimes_h C_p. \end{aligned}$$

The assertion for $2 < p < \infty$ gives that the norm of $(T_{mn}f)$ in $L_p(\mathcal{M}; \mathcal{J}_p^2)$ is bounded above by the norm in $L_p(\mathcal{M}; \mathcal{J}_p^1)$. The dual formulation just means that

$$\|(T_{mn}f)_{m,n \geq 1}\|_{L_p(\mathcal{M}; \mathcal{K}_p^1)} \lesssim \|(T_{mn}f)_{m,n \geq 1}\|_{L_p(\mathcal{M}; \mathcal{K}_p^2)}.$$

The proofs of both inequalities are similar, thus we can assume $2 < p < \infty$. Since

$$\sum_{m,n=1}^{\infty} |\rho_{km} \eta_{kn}|^2 \sim 1,$$

we may apply Theorem A2 for $\xi_{k,(m,n)} = \rho_{km} \eta_{kn}$ and obtain that the norm of $(T_{mn}f)$ in $L_p(\mathcal{M}; \mathcal{J}_p^1)$ is equivalent to the norm of f in $L_p(\mathcal{M})$. We claim that the same equivalence holds for $L_p(\mathcal{M}; \mathcal{J}_p^2)$. Indeed, applying Theorem A2 first in the variable m and then in n , we obtain

$$\begin{aligned} & \left\| \sum_{m,n=1}^{\infty} T_{mn}f \otimes e_{m,n} \right\|_p \\ &= \left\| \sum_{m=1}^{\infty} \left[\sum_{k=1}^{\infty} \rho_{km} \left(\sum_{n=1}^{\infty} \eta_{kn} e_{1,n} \right) \otimes df_k \right] \otimes e_{m,1} \right\|_p \\ &\lesssim \left\| \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \eta_{kn} df_k \right) \otimes e_{1,n} \right\|_p \lesssim \|f\|_p. \end{aligned}$$

The same argument for $e_{n,m}$ instead of $e_{m,n}$ applies and the result follows. \square

Note that Corollary 1.5 also applies for d indices. In this case all the terms are dominated by the row/column terms $e_{1,m_1 \dots m_d}$ and $e_{m_1 \dots m_d, 1}$. Another multilinear form of Theorem A2 is given by considering multi-indexed martingales. We refer to [23, Section 4.2] for the definition of a multi-indexed martingale. Like it was pointed in [23], successive iterations of Theorem A2 give rise to a generalization of it for multi-indexed martingales.

B. Commuting operator coefficients. A natural question is whether Theorems A1 and A2 still hold when the coefficients ξ_{km} are operators instead of scalars. In view of [29], we must impose certain commuting condition of the coefficients ξ_{km} , we refer to [24, Section 6] for more details on this topic. We state below the precise statement for operator coefficients. It is not difficult to check that the same arguments apply to this more general case, details are left to the reader.

Corollary 1.6. *Assume that $\xi_{km} \in \mathcal{M}_{k-1} \cap \mathcal{M}'$ and*

$$\sup_{k \geq 1} \left\| \sum_{m=1}^{\infty} \xi_{km} \xi_{km}^* \right\|_{\mathcal{M}} + \left\| \sum_{m=1}^{\infty} \xi_{km}^* \xi_{km} \right\|_{\mathcal{M}} \lesssim 1,$$

where \mathcal{M}' denotes the commutant of \mathcal{M} . Then, Theorems A1 and A2 still hold.

Remark 1.7. It is finally worth mentioning that Junge and Köstler have recently developed an H_p theory of noncommutative martingales for continuous filtrations [9]. Our results could also be studied in such setting, although it is again out of the scope of the paper.

2. CALDERÓN-ZYGMUND OPERATORS

In this section, we prove Theorems B1 and B2. We recall the definition of the von Neumann algebra (\mathcal{A}, φ) from the Introduction. Note that $L_p(\mathcal{A})$ is the Bochner space of L_p functions on \mathbb{R}^n with values in $L_p(\mathcal{M})$. We shall need some additional terminology. The size of a cube Q in \mathbb{R}^n is the length $\ell(Q)$ of its edges. Given $k \in \mathbb{Z}$, we use \mathcal{Q}_k for the set of dyadic cubes of size $1/2^k$. If Q is a dyadic cube and $f : \mathbb{R}^n \rightarrow L_p(\mathcal{M})$, we set

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy.$$

Let $(E_k)_{k \in \mathbb{Z}}$ denote this time the family of conditional expectations associated to the classical dyadic filtration on \mathbb{R}^n . E_k will also stand for the tensor product $E_k \otimes id_{\mathcal{M}}$ acting on \mathcal{A} . If $1 \leq p < \infty$ and $f \in L_p(\mathcal{A})$,

$$E_k(f) = f_k = \sum_{Q \in \mathcal{Q}_k} f_Q 1_Q.$$

$(\mathcal{A}_k)_{k \in \mathbb{Z}}$ denotes the corresponding filtration $\mathcal{A}_k = E_k(\mathcal{A})$. We use \widehat{Q} for the dyadic father of a dyadic cube Q , the dyadic cube containing Q with double size. Given $\delta > 1$, the δ -concentric father δQ of Q is the cube concentric with Q satisfying $\ell(\delta Q) = \delta \ell(Q)$. Given $f : \mathbb{R}^n \rightarrow \mathbb{C}$, let df_k denote the k -th martingale difference with respect to the dyadic filtration. That is,

$$df_k = \sum_{Q \in \mathcal{Q}_k} (f_Q - f_{\widehat{Q}}) 1_Q.$$

Let \mathcal{R}_k be the class of sets in \mathbb{R}^n being the union of a family of cubes in \mathcal{Q}_k . Given such an \mathcal{R}_k -set $\Omega = \bigcup_j Q_j$, we shall work with the dilations $9\Omega = \bigcup_j 9Q_j$, where $9Q$ denotes the 9-concentric father of Q .

The right substitute of Gundy's decomposition in our new setting will be the noncommutative form of Calderón-Zygmund decomposition. Again, the main tool is Cuculescu's construction associated (this time) to the filtration $(\mathcal{A}_k)_{k \in \mathbb{Z}}$. Let us consider the dense subspace

$$\mathcal{A}_{c,+} = L_1(\mathcal{A}) \cap \left\{ f : \mathbb{R}^n \rightarrow \mathcal{M} \mid f \in \mathcal{A}_+, \overrightarrow{\text{supp}} f \text{ is compact} \right\} \subset L_1(\mathcal{A}).$$

Here $\overrightarrow{\text{supp}}$ means the support of f as a vector-valued function in \mathbb{R}^n . In other words, we have $\overrightarrow{\text{supp}} f = \text{supp} \|f\|_{\mathcal{M}}$. We employ this terminology to distinguish from $\text{supp } f$, the support of f as an operator in \mathcal{A} . Any function $f \in \mathcal{A}_{c,+}$ gives rise to a martingale $(f_n)_{n \in \mathbb{Z}}$ with respect to the dyadic filtration and we may consider the Cuculescu's sequence $(q_k(\lambda))_{k \in \mathbb{Z}}$ associated to (f, λ) for any $\lambda > 0$. Since λ will be fixed most of the time, we will shorten the notation by q_k and only write $q_k(\lambda)$ when needed. Define the sequence $(p_k)_{k \in \mathbb{Z}}$ of disjoint projections

$$p_k = q_{k-1} - q_k.$$

As noted in [24], we have $q_k = \mathbf{1}_{\mathcal{A}}$ for k small enough and

$$\sum_{k \in \mathbb{Z}} p_k = \mathbf{1}_{\mathcal{A}} - q \quad \text{with} \quad q = \bigwedge_{k \in \mathbb{Z}} q_k.$$

Calderón-Zygmund decomposition [24]. *Let $f \in \mathcal{A}_{c,+}$ and let λ be a positive number. Then, f can be decomposed $f = g_d + g_{off} + b_d + b_{off}$ as the sum of four functions*

$$\begin{aligned} g_d &= qf q + \sum_{k \in \mathbb{Z}} p_k f_k p_k, & g_{off} &= \sum_{i \neq j} p_i f_{i \vee j} p_j + qf q^\perp + q^\perp f q, \\ b_d &= \sum_{k \in \mathbb{Z}} p_k (f - f_k) p_k, & b_{off} &= \sum_{i \neq j} p_i (f - f_{i \vee j}) p_j, \end{aligned}$$

where $i \vee j = \max(i, j)$ and $q^\perp = \mathbf{1}_{\mathcal{A}} - q$. Moreover, we have the diagonal estimates

$$\left\| qf q + \sum_{k \in \mathbb{Z}} p_k f_k p_k \right\|_2^2 \leq 2^n \lambda \|f\|_1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \|p_k (f - f_k) p_k\|_1 \leq 2 \|f\|_1.$$

Remark 2.1. Some comments:

- Let m_λ be the larger integer with $q_{m_\lambda}(\lambda) = \mathbf{1}_{\mathcal{A}}$.
- There exist weaker off-diagonal estimates for g and b , see [24, Appendix B].

Remark 2.2. We have

$$g_{off} = \sum_{s=1}^{\infty} \sum_{k=m_\lambda+1}^{\infty} p_k df_{k+s} q_{k+s-1} + q_{k+s-1} df_{k+s} p_k = \sum_{s=1}^{\infty} \sum_{k=m_\lambda+1}^{\infty} g_{k,s} = \sum_{s=1}^{\infty} g(s).$$

Moreover, it is easily checked that

$$\sup_{s \geq 1} \|g(s)\|_2^2 = \sup_{s \geq 1} \sum_{k=m_\lambda+1}^{\infty} \|g_{k,s}\|_2^2 \lesssim \lambda \|f\|_1$$

and that $\text{supp}^* dg(s)_{k+s} = \text{supp}^* g_{k,s} \leq p_k \leq \mathbf{1}_{\mathcal{A}} - q_k$, see [24] for further details.

We need one more preliminary result. Given $\lambda > 0$, we adopt the terminology from [24] and write $q_k(\lambda) = \sum_{Q \in \mathcal{Q}_k} \xi_Q \mathbf{1}_Q$ with ξ_Q projections in \mathcal{M} . Thus, since we are assuming that $q_{m_\lambda}(\lambda) = \mathbf{1}_{\mathcal{A}}$, we may write

$$\mathbf{1}_{\mathcal{A}} - q_k(\lambda) = \sum_{s=m_\lambda+1}^k \sum_{Q \in \mathcal{Q}_s} (\xi_{\widehat{Q}} - \xi_Q) \mathbf{1}_Q.$$

An \mathbb{R}^n -dilated version (by a factor 9) of it is given by

$$\text{supp } \psi_k(\lambda) \quad \text{with} \quad \psi_k(\lambda) = \sum_{s=m_\lambda+1}^k \sum_{Q \in \mathcal{Q}_s} (\xi_{\widehat{Q}} - \xi_Q) \mathbf{1}_{9Q},$$

the support projection of $\psi_k(\lambda)$. The result below is proved in [24, Lemma 4.2].

Lemma 2.3. *Let us set*

$$\zeta(\lambda) = \bigwedge_{k \in \mathbb{Z}} \zeta_k(\lambda) \quad \text{with} \quad \zeta_k(\lambda) = \mathbf{1}_{\mathcal{A}} - \text{supp } \psi_k(\lambda).$$

Then, $\zeta(\lambda)$ is a projection in \mathcal{A} and we have

- $\lambda \varphi(\mathbf{1}_{\mathcal{A}} - \zeta(\lambda)) \leq 9^n \|f\|_1$.
- If Q_0 is any dyadic cube, then we have

$$\zeta(\lambda)(\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{9Q_0}) \leq (\mathbf{1}_{\mathcal{M}} - \xi_{\widehat{Q}_0} + \xi_{Q_0}) \otimes \mathbf{1}_{9Q_0}.$$

In particular, it can be deduced that $\zeta(\lambda)(\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{9Q_0}) \leq \xi_{Q_0} \otimes \mathbf{1}_{9Q_0}$.

2.1. A pseudo-localization result. Now we show how to transfer the result in Appendix A to the noncommutative setting. We shall need the weak notion supp^* of support projection introduced before the statement of Gundy's decomposition above. It is easily seen that $\text{supp}^* f$ is the smallest projection p in \mathcal{A} satisfying the identity $f = pf + fp - pfp$. We shall prove the following result.

Theorem 2.4. *Given a Hilbert space \mathcal{H} , consider a kernel $k : \mathbb{R}^{2n} \setminus \Delta \rightarrow \mathcal{H}$ which satisfies the size/smoothness conditions imposed in the Introduction and formally defines the Calderón-Zygmund operator*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy.$$

Assume further that $T : L_2(\mathcal{A}) \rightarrow L_2(\mathcal{A}; \mathcal{H}_{oh})$ is of norm 1 and fix a positive integer s . Given $f \in L_2(\mathcal{A})$ and $k \in \mathbb{Z}$, let us consider any projection q_k in \mathcal{A}_k satisfying that $\mathbf{1}_{\mathcal{A}} - q_k$ contains $\text{supp}^ df_{k+s}$ as a subprojection. If we write*

$$q_k = \sum_{Q \in \mathcal{Q}_k} \xi_Q \mathbf{1}_Q$$

with ξ_Q projections, we may further consider the projection

$$\zeta_{f,s} = \bigwedge_{k \in \mathbb{Z}} \left(\mathbf{1}_{\mathcal{A}} - \bigvee_{Q \in \mathcal{Q}_k} (\mathbf{1}_{\mathcal{M}} - \xi_Q) \mathbf{1}_Q \right).$$

Then we have the following localization estimate in $L_2(\mathcal{A}; \mathcal{H}_{oh})$

$$\left(\int_{\mathbb{R}^n} \| [\zeta_{f,s} T f \zeta_{f,s}](x) \|_{L_2(\mathcal{M}; \mathcal{H}_{oh})}^2 dx \right)^{\frac{1}{2}} \leq c_{n,\gamma} s 2^{-\gamma s/2} \left(\int_{\mathbb{R}^n} \tau(|f(x)|^2) dx \right)^{\frac{1}{2}}.$$

Proof. We shall reduce this result to its commutative counterpart in Appendix A below. According to the shift condition imposed $\text{supp}^* df_{k+s} \prec \mathbf{1}_{\mathcal{A}} - q_k$, we have

$$df_{k+s} = q_k^\perp df_{k+s} + df_{k+s} q_k^\perp - q_k^\perp df_{k+s} q_k^\perp$$

where we write $q_k^\perp = \mathbf{1}_{\mathcal{A}} - q_k$ for convenience. On the other hand, let

$$(2.1) \quad \zeta_k = \mathbf{1}_{\mathcal{A}} - \bigvee_{Q \in \mathcal{Q}_k} (\mathbf{1}_{\mathcal{M}} - \xi_Q) \mathbf{1}_Q,$$

so that $\zeta_{f,s} = \bigwedge_k \zeta_k$. As in Lemma 2.3, it is easily seen that $\mathbf{1}_{\mathcal{A}} - \zeta_k$ represents the \mathbb{R}^n -dilated projection associated to $\mathbf{1}_{\mathcal{A}} - q_k$ with a factor 9. Let \mathcal{L}_a and \mathcal{R}_a denote the left and right multiplication maps by the operator a . Let also \mathcal{LR}_a stand for $\mathcal{L}_a + \mathcal{R}_a - \mathcal{L}_a \mathcal{R}_a$. Then our considerations so far and the fact that $\mathcal{L}_{\zeta_k}, \mathcal{R}_{\zeta_k}$ and $\mathcal{LR}_{q_k^\perp}$ commute with E_j for $j \geq k$ give

$$\begin{aligned} & \zeta_{f,s} T f \zeta_{f,s} \\ &= \mathcal{L}_{\zeta_{f,s}} \mathcal{R}_{\zeta_{f,s}} \left(\sum_k E_k T \Delta_{k+s} \mathcal{LR}_{q_k^\perp} + \sum_k (id - E_k) \mathcal{L}_{\zeta_k} \mathcal{R}_{\zeta_k} T \mathcal{LR}_{q_k^\perp} \Delta_{k+s} \right) (f). \end{aligned}$$

Now we claim that

$$\mathcal{L}_{\zeta_k} \mathcal{R}_{\zeta_k} T \mathcal{LR}_{q_k^\perp} = \mathcal{L}_{\zeta_k} \mathcal{R}_{\zeta_k} T_{4 \cdot 2^{-k}} \mathcal{LR}_{q_k^\perp}.$$

Indeed, this clearly reduces to see

$$\mathcal{L}_{\zeta_k} T \mathcal{LR}_{q_k^\perp} = \mathcal{L}_{\zeta_k} T_{4 \cdot 2^{-k}} \mathcal{LR}_{q_k^\perp} \quad \text{and} \quad \mathcal{R}_{\zeta_k} T \mathcal{LR}_{q_k^\perp} = \mathcal{R}_{\zeta_k} T_{4 \cdot 2^{-k}} \mathcal{LR}_{q_k^\perp}.$$

By symmetry, we just prove the first identity

$$\mathcal{L}_{\zeta_k} T \mathcal{L}_{q_k^\perp} f(x) = \sum_{Q \in \mathcal{Q}_k} \zeta_k(x) (\mathbf{1}_{\mathcal{M}} - \xi_Q) \int_Q k(x, y) f(y) dy.$$

Assume that $x \in 9Q$ for some $Q \in \mathcal{Q}_k$, then it follows as in Lemma 2.3 (see the definition of the projection ζ_k) that $\zeta_k(x) \leq \xi_Q$. In particular, we deduce from the expression above that for each $y \in Q$ we must have $x \in \mathbb{R}^n \setminus 9Q$. This implies $|x - y| \geq 4 \cdot 2^{-k}$ as desired. Finally, the operators \mathcal{L}, \mathcal{R} and $\mathcal{L}\mathcal{R}$ inside the bracket are clearly absorbed by f and $\zeta_{f,s}$. Thus, we obtain the identity below

$$\zeta_{f,s} T f \zeta_{f,s} = \mathcal{L}_{\zeta_{f,s}} \mathcal{R}_{\zeta_{f,s}} \left(\sum_k \mathbf{E}_k T \Delta_{k+s} + \sum_k (id - \mathbf{E}_k) T_{4 \cdot 2^{-k}} \Delta_{k+s} \right) (f).$$

Assume that $T^*1 = 0$. Our shifted quasi-orthogonal decomposition in Appendix A asserts that the operator inside the brackets has norm in $\mathcal{B}(L_2, L_2(\mathcal{H}))$ controlled by $c_{n,\gamma} s 2^{-\gamma s/2}$. In particular, the same happens when we tensor with the identity on $L_2(\mathcal{M})$, which is the case. When $T^*1 \neq 0$, we may follow verbatim the paraproduct argument given in Appendix A by noting that $\zeta_{f,s} q_k^\perp = q_k^\perp \zeta_{f,s} = 0$. \square

Remark 2.5. The projections

$$(\mathbf{1}_{\mathcal{A}} - q_k, \zeta_{f,s}, \zeta_k)$$

represent the sets $(\Omega_k, \mathbb{R}^n \setminus \Sigma_{f,s}, \mathbb{R}^n \setminus 9\Omega_k)$ in the commutative formulation.

2.2. Weak type (1,1) boundedness. We prove Theorem B1 in this section. As usual, we may take $f \in \mathcal{A}_{c,+}$. To provide the decomposition $Tf = Af + Bf$, we use the projections $\zeta(\lambda)$ in Lemma 2.3. Define

$$\pi_k = \bigwedge_{s \geq k} \zeta(2^s) - \bigwedge_{s \geq k-1} \zeta(2^s) \quad \text{for } k \in \mathbb{Z}.$$

If we set $\lambda_f = \|f\|_\infty$ (which is finite since $f \in \mathcal{A}_{c,+}$), we have $\|f_k\|_\infty \leq \lambda_f$ for all integers k . In particular, given any $\lambda > \lambda_f$, it is easily checked that $q_k(\lambda) = \mathbf{1}_{\mathcal{A}}$ for all $k \in \mathbb{Z}$ and we deduce that $\zeta(\lambda) = \mathbf{1}_{\mathcal{A}}$ for all $\lambda > \lambda_f$. This gives rise to

$$\sum_{k \in \mathbb{Z}} \pi_k = \lim_{k_1, k_2 \rightarrow \infty} \left[\bigwedge_{s \geq k_1} \zeta(2^s) - \bigwedge_{s \geq -k_2} \zeta(2^s) \right] = \mathbf{1}_{\mathcal{A}} - \bigwedge_{k \in \mathbb{Z}} \zeta(2^k) = \mathbf{1}_{\mathcal{A}} - \psi.$$

Our decomposition for Tf is the following

$$\begin{aligned} Tf &= \psi Tf \psi \\ &+ (\mathbf{1}_{\mathcal{A}} - \psi) Tf \psi + \sum_{j \leq i} \pi_i Tf \pi_j \\ &+ \psi Tf (\mathbf{1}_{\mathcal{A}} - \psi) + \sum_{i < j} \pi_i Tf \pi_j = \psi Tf \psi + Af + Bf. \end{aligned}$$

We claim that $\|Af\|_{L_{1,\infty}(\mathcal{A}; \mathcal{H}_r)} + \|Bf\|_{L_{1,\infty}(\mathcal{A}; \mathcal{H}_c)} \lesssim \|f\|_1$. As we shall see at the end of the proof, the term $\psi Tf \psi$ is even easier to handle. Assume for simplicity that \mathcal{H} is separable and fix an orthonormal basis $(u_m)_{m \geq 1}$ of \mathcal{H} . If $k_m(x, y) = \langle u_m, k(x, y) \rangle$, we denote by $T_m f$ the Calderón-Zygmund operator associated to the kernel k_m , while $A_m f$ and $B_m f$ stand for the corresponding parts of $T_m f$. Then, we have

$$\|Af\|_{L_{1,\infty}(\mathcal{A}; \mathcal{H}_r)} = \sup_{\lambda > 0} \lambda \varphi \left\{ \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right)^{\frac{1}{2}} > \lambda \right\}.$$

As in the martingale case, we may assume that $\lambda = 2^\ell$ for some integer ℓ . Note that we do not impose $\ell \geq 0$ since \mathcal{A} is no longer finite and we also need to consider the case $0 < \lambda < 1$. Let us define

$$w_\ell = \bigwedge_{s \geq \ell} \zeta(2^s).$$

By the quasi-triangle inequality, we majorize again by

$$\lambda \varphi \left\{ w_\ell \left(\sum_{m=1}^{\infty} (A_m f)(A_m f)^* \right) w_\ell > \lambda^2 \right\} + \lambda \varphi(\mathbf{1}_{\mathcal{A}} - w_\ell) = \mathbf{A}_1 + \mathbf{A}_2.$$

Then we apply Lemma 2.3 to estimate \mathbf{A}_2 and obtain

$$\mathbf{A}_2 \leq 2^\ell \sum_{s \geq \ell} \varphi(\mathbf{1}_{\mathcal{A}} - \zeta(2^s)) \leq 9^n 2^\ell \sum_{s \geq \ell} 2^{-s} \|f\|_1 \leq 2 \cdot 9^n \|f\|_1.$$

We will use that $\psi w_\ell = w_\ell \psi = \psi$ and

$$\pi_k w_\ell = w_\ell \pi_k = \begin{cases} \pi_k & \text{if } k \leq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, defining $\rho_k = \psi + \sum_{j \leq k} \pi_j$, we have

$$w_\ell A_m f = \sum_{i \leq \ell} \pi_i (w_\ell T_m f w_\ell) \rho_i = A_m f.$$

Note that $w_\ell T_m f \neq w_\ell T_m f w_\ell$ and that is where we need to break $T_m f$ into row/column terms. On the other hand, by the Calderón-Zygmund decomposition of (f, λ) , we conclude

$$\begin{aligned} \mathbf{A}_1 &\lesssim \lambda \varphi \left\{ \sum_{m=1}^{\infty} (A_m f g_d)(A_m f g_d)^* > \lambda^2 \right\} \\ &+ \lambda \varphi \left\{ \sum_{m=1}^{\infty} (A_m f b_d)(A_m f b_d)^* > \lambda^2 \right\} \\ &+ \lambda \varphi \left\{ \sum_{m=1}^{\infty} (A_m f g_{\text{off}})(A_m f g_{\text{off}})^* > \lambda^2 \right\} \\ &+ \lambda \varphi \left\{ \sum_{m=1}^{\infty} (A_m f b_{\text{off}})(A_m f b_{\text{off}})^* > \lambda^2 \right\} = \mathbf{A}_{g,d} + \mathbf{A}_{b,d} + \mathbf{A}_{g,\text{off}} + \mathbf{A}_{b,\text{off}}. \end{aligned}$$

The term $\mathbf{A}_{g,d}$. Chebychev's inequality gives for $\mathbf{A}_{g,d}$

$$\begin{aligned} \mathbf{A}_{g,d} &\leq \frac{1}{\lambda} \sum_{m=1}^{\infty} \sum_{i \leq \ell} \varphi(\pi_i (w_\ell T_m g_d w_\ell) \rho_i (w_\ell T_m g_d w_\ell)^* \pi_i) \\ &\leq \frac{1}{\lambda} \sum_{m=1}^{\infty} \varphi((w_\ell T_m g_d w_\ell) (w_\ell T_m g_d w_\ell)^*) \\ &\leq \frac{1}{\lambda} \sum_{m=1}^{\infty} \varphi((T_m g_d) (T_m g_d)^*) = \frac{1}{\lambda} \|T g_d\|_{L_2(\mathcal{A}; \mathcal{H}_{oh})}^2 \leq \frac{1}{\lambda} \|g_d\|_2^2 \leq 2^n \|f\|_1. \end{aligned}$$

Last inequality is part of the statement of Calderón-Zygmund decomposition above.

The term $A_{g, \text{off}}$. Arguing as for g_d we find

$$\begin{aligned} A_{g, \text{off}} &\leq \frac{1}{\lambda} \sum_{m=1}^{\infty} \varphi((w_{\ell} T_m g_{\text{off}} w_{\ell})(w_{\ell} T_m g_{\text{off}} w_{\ell})^*) \\ &= \frac{1}{\lambda} \int_{\mathbb{R}^n} \|[w_{\ell} T g_{\text{off}} w_{\ell}](x)\|_{L_2(\mathcal{M}; \mathcal{H}_{oh})}^2 dx. \end{aligned}$$

On the other hand, we have

$$\text{supp}^* dg_{(s)}_{k+s} \leq \mathbf{1}_{\mathcal{A}} - q_k(2^{\ell})$$

from Remark 2.2 and we claim $w_{\ell} \leq \zeta_{g_{(s)}, s}$. Indeed, clearly $w_{\ell} \leq \zeta(2^{\ell})$ and

$$\begin{aligned} \bigvee_{Q \in \mathcal{Q}_k} (\mathbf{1}_{\mathcal{M}} - \xi_Q) 1_{9Q} &= \text{supp} \sum_{Q \in \mathcal{Q}_k} (\mathbf{1}_{\mathcal{M}} - \xi_Q) 1_{9Q} \\ &\leq \text{supp} \sum_{s=m_{\lambda}+1}^k \sum_{Q \in \mathcal{Q}_s} (\xi_{\widehat{Q}} - \xi_Q) 1_{9Q} = \text{supp} \psi_k(2^{\ell}). \end{aligned}$$

Therefore, we conclude

$$w_{\ell} \leq \zeta(2^{\ell}) = \bigwedge_{k \in \mathbb{Z}} (\mathbf{1}_{\mathcal{A}} - \text{supp} \psi_k(2^{\ell})) \leq \bigwedge_{k \in \mathbb{Z}} (\mathbf{1}_{\mathcal{A}} - \bigvee_{Q \in \mathcal{Q}_k} (\mathbf{1}_{\mathcal{M}} - \xi_Q) 1_{9Q}) = \zeta_{g_{(s)}, s}.$$

Using this inequality and pseudo-localization (Theorem 2.4), we obtain

$$\begin{aligned} A_{g, \text{off}} &\leq \frac{1}{\lambda} \int_{\mathbb{R}^n} \|\zeta_{g_{(s)}, s} T g_{\text{off}} \zeta_{g_{(s)}, s}(x)\|_{L_2(\mathcal{M}; \mathcal{H}_{oh})}^2 dx \\ &\leq \frac{1}{\lambda} \left(\sum_{s=1}^{\infty} \left[\int_{\mathbb{R}^n} \|\zeta_{g_{(s)}, s} T g_{(s)} \zeta_{g_{(s)}, s}(x)\|_{L_2(\mathcal{M}; \mathcal{H}_{oh})}^2 dx \right]^{\frac{1}{2}} \right)^2 \\ &\leq \frac{c_{n, \gamma}}{\lambda} \left(\sum_{s=1}^{\infty} s e^{-\gamma s/2} \left[\sum_{k=m_{\lambda}+1}^{\infty} \|g_{k, s}\|_2^2 \right]^{\frac{1}{2}} \right)^2 \lesssim c_{n, \gamma} \|f\|_1, \end{aligned}$$

where the last estimate follows from the inequality given in Remark 2.2.

The term $A_{b, d}$. We have

$$A_{m\ell} b_d = \sum_{j \leq i \leq \ell} \pi_i (w_{\ell} T_m b_d w_{\ell}) \pi_j + \left[\sum_{i \leq \ell} \pi_i \right] (w_{\ell} T_m b_d w_{\ell}) \psi.$$

Since $\sum_{j \leq i \leq \ell} \pi_i \cdot \pi_j$ and $[\sum_{i \leq \ell} \pi_i] \cdot \psi$ are of weak type $(1, 1)$

$$A_{b, d} \leq \left\| \sum_{m=1}^{\infty} A_{m\ell} b_d \otimes e_{1m} \right\|_{1, \infty} \lesssim \left\| \sum_{m=1}^{\infty} (w_{\ell} T_m b_d w_{\ell}) \otimes e_{1m} \right\|_1.$$

Letting $b_{d, k} = p_k(f - f_k)p_k$, we find that $A_{b, d}$ is dominated by

$$\sum_{k=m_{\lambda}+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \tau \otimes \int_{\mathbb{R}^n} \left(\sum_{m=1}^{\infty} \left| \int_Q k_m(x, y) (w_{\ell}(x) b_{d, k}(y) w_{\ell}(x)) dy \right|^* \right)^2 dx.$$

Given $y \in Q \in \mathcal{Q}_k$, we have $p_k(y) = \xi_{\widehat{Q}} - \xi_Q$ and

$$w_{\ell}(x) b_{d, k}(y) w_{\ell}(x) = 0 \quad \text{for } (x, y) \in 9Q \times Q$$

by Lemma 2.3. This means that

$$\begin{aligned} \mathbf{A}_{b,d} &\lesssim \sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \left\| w_\ell \left(\sum_{m=1}^{\infty} 1_{\mathbb{R}^n \setminus 9Q} T_m(b_{d,k} 1_Q) \otimes e_{1m} \right) w_\ell \right\|_1 \\ &\leq \sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \tau \otimes \int_{\mathbb{R}^n \setminus 9Q} \left(\sum_{m=1}^{\infty} \left| \int_Q k_m(x, y) b_{d,k}(y) dy \right|^2 \right)^{\frac{1}{2}} dx. \end{aligned}$$

By the mean zero of $b_{d,k}$ on Q , we rewrite our term as follows

$$\sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \tau \otimes \int_{\mathbb{R}^n \setminus 9Q} \left(\sum_{m=1}^{\infty} \left| \int_Q (k_m(x, y) - k_m(x, c_Q)) b_{d,k}(y) dy \right|^2 \right)^{\frac{1}{2}} dx,$$

where c_Q denotes the center of Q . We see it as an $L_1(\mathcal{A}; \mathcal{H}_r)$ norm of

$$\sum_{m=1}^{\infty} \left(\int_Q (k_m(x, y) - k_m(x, c_Q)) b_{d,k}(y) dy \right) \otimes u_m.$$

Putting the $L_1(\mathcal{A}, \mathcal{H}_r)$ norm into the integral \int_Q gives us a larger term

$$\begin{aligned} &\sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \tau \otimes \int_{\mathbb{R}^n \setminus 9Q} \int_Q \left(\sum_{m=1}^{\infty} \left| b_{d,k}^*(y) (k_m^*(x, y) - k_m^*(x, c_Q)) \right|^2 \right)^{\frac{1}{2}} dy dx \\ &= \sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \tau \otimes \int_{\mathbb{R}^n \setminus 9Q} \int_Q \left(\sum_{m=1}^{\infty} |k_m^*(x, y) - k_m^*(x, c_Q)|^2 \right)^{\frac{1}{2}} |b_{d,k}^*(y)| dy dx \\ &\lesssim \sum_{k=m_\lambda+1}^{\infty} \sum_{Q \in \mathcal{Q}_k} \left[\int_{\mathbb{R}^n \setminus 9Q} \frac{|\ell(Q)|^\gamma}{|x - c_Q|^{n+\gamma}} dx \right] \left[\tau \otimes \int_Q |b_{d,k}^*(y)| dy \right] \\ &\lesssim \sum_{k=m_\lambda+1}^{\infty} \tau \otimes \int_{\mathbb{R}^n} |b_{d,k}^*(y)| dy \leq 2 \|f\|_1. \end{aligned}$$

The last inequality uses the diagonal estimate in Calderón-Zygmund decomposition.

The term $\mathbf{A}_{b,off}$. Although it is more technical, the estimate for $\mathbf{A}_{b,off}$ is very similar in nature to that for $\mathbf{A}_{b,d}$. Indeed, the only significant difference relies on the fact that we have to show

$$\sum_{k=m_\lambda+1}^{\infty} \left\| \sum_{m=1}^{\infty} (w_\ell T_m b_{k,s} w_\ell) \otimes e_{1m} \right\|_1 \lesssim 2^{-\gamma s} \|f\|_1$$

for each $s \geq 1$. Namely, here we write $b_{k,s}$ for the sum of the k -th terms in the upper and lower s -th diagonals of b_{off} , and the estimate above resembles the same procedure that we used for g_{off} in comparison with g_d . That is, there exists a geometric *almost diagonal* phenomenon. It is nevertheless straightforward to check that the arguments above for $\mathbf{A}_{b,d}$ also apply for $\mathbf{A}_{b,off}$ using the ideas in [24] for the off-diagonal term of the bad part in Calderón-Zygmund decomposition.

Conclusion. By symmetry, the same arguments apply to estimate the norm of the column term Bf . It therefore remains to consider the term $\psi T f \psi$. However, note that $\psi \leq w_\ell$ and we know from the arguments above how to estimate the term $w_\ell T f w_\ell$. The proof is completed. \square

2.3. BMO estimate, interpolation and duality. We now prove Theorem B2. Let us briefly recall the definition of the noncommutative analogue of function-BMO spaces. According to [19], we define the norms (modulo constants)

$$\begin{aligned} \|f\|_{\text{BMO}_{\mathcal{A}}^r} &= \sup_{Q \text{ cube} \subset \mathbb{R}^n} \left\| \left(\frac{1}{|Q|} \int_Q (f(x) - f_Q)(f(x) - f_Q)^* dx \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}, \\ \|f\|_{\text{BMO}_{\mathcal{A}}^c} &= \sup_{Q \text{ cube} \subset \mathbb{R}^n} \left\| \left(\frac{1}{|Q|} \int_Q (f(x) - f_Q)^*(f(x) - f_Q) dx \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}. \end{aligned}$$

We also set

$$\|f\|_{\text{BMO}(\mathcal{A})} = \max \left\{ \|f\|_{\text{BMO}_{\mathcal{A}}^r}, \|f\|_{\text{BMO}_{\mathcal{A}}^c} \right\}.$$

The spaces $\text{BMO}(\mathcal{A}; \mathcal{H}_r)$ and $\text{BMO}(\mathcal{A}; \mathcal{H}_c)$ were defined in the Introduction. As in the previous paragraph, we fix an orthonormal basis $(u_m)_{m \geq 1}$ of \mathcal{H} and define $k_m(x, y)$ and $T_m f$ accordingly. Letting $\mathcal{R} = \mathcal{A} \bar{\otimes} \mathcal{B}(\ell_2)$, we have

$$\begin{aligned} \|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_r)} &= \left\| \sum_{m=1}^{\infty} T_m f \otimes e_{1m} \right\|_{\text{BMO}(\mathcal{R})}, \\ \|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_c)} &= \left\| \sum_{m=1}^{\infty} T_m f \otimes e_{m1} \right\|_{\text{BMO}(\mathcal{R})}. \end{aligned}$$

By symmetry, we just prove $\|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_c)} \lesssim \|f\|_{\infty}$. As usual, in the definition of the BMO norm of a function f , we may replace the averages f_Q by any other operator α_Q depending on Q . Fix a cube Q in \mathbb{R}^n and let

$$\alpha_Q = \sum_{m=1}^{\infty} \alpha_{Q,m} \otimes e_{m1} = \sum_{m=1}^{\infty} \frac{1}{|Q|} \int_Q \left(\int_{\mathbb{R}^n \setminus 2Q} k_m(z, y) f(y) dy \right) dz \otimes e_{m1}.$$

We have

$$\begin{aligned} [T_m f - \alpha_{Q,m}](x) &= \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n \setminus 2Q} (k_m(x, y) - k_m(z, y)) f(y) dy dz + \int_{2Q} k_m(x, y) f(y) dy \\ &= B_{m1} f(x) + B_{m2} f(x). \end{aligned}$$

For $B_1 f = \sum_m B_{m1} f \otimes e_{m1}$, we have

$$\begin{aligned} \sup_{x \in Q} \|B_1 f(x)\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} &\leq \sup_{x, z \in Q} \left\| \sum_{m=1}^{\infty} \int_{\mathbb{R}^n \setminus 2Q} (k_m(x, y) - k_m(z, y)) f(y) dy \otimes e_{m1} \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\ &= \sup_{x, z \in Q} \left\| \sum_{m=1}^{\infty} \left| \int_{\mathbb{R}^n \setminus 2Q} (k_m(x, y) - k_m(z, y)) f(y) dy \right|^2 \right\|_{\mathcal{M}}^{\frac{1}{2}} \\ &\leq \sup_{x, z \in Q} \left(\sum_{m=1}^{\infty} \left[\int_{\mathbb{R}^n \setminus 2Q} |k_m(x, y) - k_m(z, y)|^2 dy \right]^{\frac{1}{2}} \right) \|f\|_{\infty} \\ &\leq \sup_{x, z \in Q} \left[\int_{\mathbb{R}^n \setminus 2Q} \left(\sum_{m=1}^{\infty} |k_m(x, y) - k_m(z, y)|^2 \right)^{\frac{1}{2}} dy \right] \|f\|_{\infty} \\ &\leq \sup_{x, z \in Q} \left[\int_{\mathbb{R}^n \setminus 2Q} \frac{\ell(Q)^{\gamma}}{|x - y|^{n+\gamma}} dy \right] \|f\|_{\infty} \lesssim \|f\|_{\infty}. \end{aligned}$$

For $B_2f = \sum_m B_{m2}f \otimes e_{m1}$, we have

$$\begin{aligned}
& \frac{1}{|Q|} \left\| \int_Q (B_2f(x))^* (B_2f(x)) dx \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\
&= \frac{1}{|Q|} \left\| \int_Q \left| \sum_{m=1}^{\infty} \int_{2Q} k_m(x, y) f(y) dy \otimes e_{m1} \right|^2 dx \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\
&= \frac{1}{|Q|} \left\| \int_Q \sum_{m=1}^{\infty} \left| \int_{2Q} k_m(x, y) f(y) dy \right|^2 dx \right\|_{\mathcal{M}} \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \tau \otimes \int_Q \sum_{m=1}^{\infty} \left| \int_{2Q} k_m(x, y) f(y) a dy \right|^2 dx \\
&\leq \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \tau \otimes \int_{\mathbb{R}^n} \sum_{m=1}^{\infty} \left| \int_{\mathbb{R}^n} k_m(x, y) f(y) a 1_{2Q}(y) dy \right|^2 dx \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \|T(fa 1_{2Q})\|_{L_2(\mathcal{A}; \mathcal{H}_{oh})}^2 \\
&\lesssim \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \|fa 1_{2Q}\|_{L_2(\mathcal{A})}^2 \lesssim \|f\|_{\infty}^2.
\end{aligned}$$

On the other hand, if $\beta_m f(x) = \int_{2Q} k_m(x, y) f(y) dy$, we also have

$$\begin{aligned}
& \frac{1}{|Q|} \left\| \int_Q (B_2f(x))(B_2f(x))^* dx \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\
&= \frac{1}{|Q|} \left\| \sum_{m_1, m_2=1}^{\infty} \left[\int_Q \beta_{m_1} f(x) \beta_{m_2}^* f(x) dx \right] \otimes e_{m_1, m_2} \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\
&= \frac{1}{|Q|} \|\Lambda\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)}.
\end{aligned}$$

Since Λ is a positive operator acting on $L_2(\mathcal{M}; \ell_2)$

$$\begin{aligned}
& \frac{1}{|Q|} \left\| \int_Q (B_2f(x))(B_2f(x))^* dx \right\|_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)} \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1} \langle \Lambda a, a \rangle \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1} \tau \left(\left[\sum_{m_1=1}^{\infty} a_{m_1}^* \otimes e_{1m_1} \right] \Lambda \left[\sum_{m_2=1}^{\infty} a_{m_2} \otimes e_{m_2 1} \right] \right) \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1} \tau \otimes \int_Q \left| \sum_{m=1}^{\infty} \beta_m^* f(x) a_m \right|^2 dx \\
&= \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1} \tau \otimes \int_Q \left| \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} \overline{k_m(x, y)} f(y)^* a_m 1_{2Q}(y) dy \right|^2 dx \\
&\leq \frac{1}{|Q|} \sup_{\substack{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1 \\ \|g\|_{L_2(\mathcal{A})} \leq 1}} \left[\tau \otimes \int_{\mathbb{R}^n} \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} \overline{k_m(x, y)} f(y)^* a_m 1_{2Q}(y) dy g(x) dx \right]^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|Q|} \sup_{\substack{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1 \\ \|g\|_{L_2(\mathcal{A})} \leq 1}} \left[\tau \otimes \int_{\mathbb{R}^n} \sum_{m=1}^{\infty} \left(\int_{\mathbb{R}^n} \overline{k_m(x, y)} g(x) dx \right) f(y)^* a_m 1_{2Q}(y) dy \right]^2 \\
&\leq \frac{1}{|Q|} \sup_{\substack{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1 \\ \|g\|_{L_2(\mathcal{A})} \leq 1}} \|Tg^*\|_{L_2(\mathcal{A}; \mathcal{H}_{oh})}^2 \left(\tau \otimes \int_{\mathbb{R}^n} \sum_{m=1}^{\infty} |f(y)^* a_m|^2 1_{2Q}(y) dy \right) \\
&\leq \frac{1}{|Q|} \sup_{\substack{\|a\|_{L_2(\mathcal{M}; \ell_2)} \leq 1 \\ \|g\|_{L_2(\mathcal{A})} \leq 1}} \|Tg^*\|_{L_2(\mathcal{A}; \mathcal{H}_{oh})}^2 \tau \left(\sum_{m=1}^{\infty} |a_m|^2 \right) |2Q| \|f\|_{\infty}^2 \lesssim \|f\|_{\infty}^2.
\end{aligned}$$

The estimates so far and their row analogues give rise to

$$\max \left\{ \|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_r)}, \|Tf\|_{\text{BMO}(\mathcal{A}; \mathcal{H}_c)} \right\} \lesssim \|f\|_{\infty}.$$

This completes the BMO estimate. By interpolation as in Section 1, we get

$$\|Tf\|_{L_p(\mathcal{A}; \mathcal{H}_{rc})} \leq c_p \|f\|_p$$

for $1 < p < \infty$. Finally, if T is an isometry $L_2(\mathcal{A}) \rightarrow L_2(\mathcal{A}; \mathcal{H}_{oh})$, polarization gives

$$\|f\|_{L_p(\mathcal{A})} = \sup_{\|g\|_{L_{p'}(\mathcal{A})} \leq 1} \langle Tf, Tg \rangle \lesssim \|Tf\|_{L_p(\mathcal{A}; \mathcal{H}_{rc})}. \quad \square$$

2.4. Examples and further comments. The first examples that come to mind are the scalar-valued Calderón-Zygmund operators studied in [24], given by \mathcal{H} one dimensional. On the other hand, as in [24] or in Section 1 above, we might also consider commuting operator-valued coefficients in Theorems B1 and B2. We omit the formal statement of such result. Let us study some more examples/applications of our results.

A. Lusin square functions and g -functions. A row/column form of the Lusin area function and the Littlewood-Paley g -function for the Poisson kernel was given in [19]. It is standard that both square functions are associated to Calderón-Zygmund kernels satisfying our size/smoothness conditions. Moreover, the L_2 boundedness of these operators is straightforward. Therefore, Theorems B1 and B2 apply and we obtain the weak L_1 , strong L_p and L_{∞} -BMO boundedness of these operators in the operator valued setting. The strong L_p inequalities for Lusin square functions and g -functions were one of the main results in [19], while the weak L_1 and L_{∞} -BMO estimates are new. In a similar way, we may consider any other symmetric diffusion semigroup as far as it satisfies our kernel assumptions. Note that, when dealing with strong L_p estimates, more general families of semigroups were considered in [10]. It seems however that the approach in [10] does not permit to work with general Calderón-Zygmund operators (not coming from semigroups) or providing weak L_1 and L_{∞} -BMO estimates.

B. Operator-valued Littlewood-Paley theorem (with better constants). Consider

$$\widehat{M_k f} = 1_{\Delta_k} \widehat{f} \quad \text{with} \quad \Delta_k = (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}).$$

An immediate application of Theorem B2 is a generalization to the operator valued setting of the Littlewood-Paley theorem, which assets in the commutative case that

for any $1 < p < \infty$ we have

$$\|f\|_p \sim_{c_p} \left\| \left(\sum_k |M_k f|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Indeed, Theorem B2 easily gives the L_∞ -BMO estimate by the boundedness of Riesz transforms on BMO and the classical “shift-truncation” argument of smoothing multipliers. An \mathcal{H}_1 -weak (1,1) estimate can be also obtained by considering atomic decomposition [31] and a dual version of Theorem B2. However, using the fact that $L_p(\mathcal{M})$ is a UMD Banach space, this also follows from Bourgain’s vector-valued Littlewood-Paley inequality for UMD spaces [1]

$$\|f\|_{L_p(\mathbb{X})} \sim_{C_p} \left(\int_{\Omega} \left\| \sum_k M_k f r_k(w) \right\|_{L_p(\mathbb{X})}^p dw \right)^{\frac{1}{p}}$$

combined with the noncommutative Khintchine inequality. In conclusion, Theorem B2 provides a new proof of Bourgain’s result for $L_p(\mathcal{M})$ -valued functions. The advantage is that we will get the optimal constants $c_p \sim \frac{1}{p-1}, p$ as $p \rightarrow 1, \infty$.[†]

C. Beyond the Lebesgue measure on \mathbb{R}^n . Let us point some possible generalizations of our results for future research. First, following Han and Sawyer [5], we may consider operator-valued Littlewood-Paley inequalities on homogeneous spaces. We are confident these results should hold. Second, in a less obvious way, we might work in the nondoubling setting. Recall that Littlewood-Paley theory for nondoubling measures was a corner stone through the $T1$ theorem by Nazarov, Treil and Volberg [22] and Tolsa [38]. Tolsa’s technique seems reasonable as far as we know how to prove the weak type (1,1) inequality. The Calderón-Zygmund decomposition in [37] looks like the most difficult step and an interesting problem.

D. An application to the fully noncommutative setting. In a forthcoming paper [12], we are going to apply our results in this paper to Fourier multipliers on group von Neumann algebras $VN(G)$. The idea is to embed $VN(G)$ into $L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{B}(\ell_2(G))$ and to reduce the boundedness of Fourier multipliers on $VN(G)$ to the boundedness of singular integrals studied here.

APPENDIX A. HILBERT SPACE VALUED PSEUDO-LOCALIZATION

Let us sketch the modifications from the argument in [24] needed to extend the pseudo-localization result there to the context of Hilbert space valued kernels. We adopt the terminology from Section 2. Besides, we shall write just L_p to refer to the commutative L_p space on \mathbb{R}^n equipped with the Lebesgue measure dx and $L_p(\mathcal{H})$ for its \mathcal{H} -valued extension.

Hilbert space valued pseudo-localization. *Given a Hilbert space \mathcal{H} , let us consider a kernel $k : \mathbb{R}^{2n} \setminus \Delta \rightarrow \mathcal{H}$ satisfying the size/smoothness conditions imposed in the Introduction, which formally defines*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy,$$

a Calderón-Zygmund operator. Assume further that $T : L_2 \rightarrow L_2(\mathcal{H})$ is of norm 1. Let us fix a positive integer s . Given a function f in L_2 and any integer k , we

[†] Musat’s constants were improved to $\sim p$ for $p > 2$ after Randrianantoanina’s work [30].

define Ω_k to be the smallest \mathcal{R}_k -set containing the support of df_{k+s} . If we further consider the set

$$\Sigma_{f,s} = \bigcup_{k \in \mathbb{Z}} 9\Omega_k,$$

then we have the localization estimate

$$\left(\int_{\mathbb{R}^n \setminus \Sigma_{f,s}} \|Tf(x)\|_{\mathcal{H}}^2 dx \right)^{\frac{1}{2}} \leq c_{n,\gamma} s 2^{-\gamma s/2} \left(\int_{\mathbb{R}^n} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

This appendix is devoted to sketch the proof of the result stated above.

A.1. Three auxiliary results. As in [24], Cotlar and Schur lemmas as well as the localization estimate from [20] are the building blocks of the argument. We shall need some Hilbert space valued forms of these results. The exact statements are given below. The proofs are simple generalizations.

Cotlar lemma. *Given Hilbert spaces $\mathcal{K}_1, \mathcal{K}_2$, let us consider a family $(T_k)_{k \in \mathbb{Z}}$ of bounded operators $\mathcal{K}_1 \rightarrow \mathcal{K}_2$ with finitely many non-zero T_k 's. Assume that there exists a sumable sequence $(\alpha_k)_{k \in \mathbb{Z}}$ such that*

$$\max \left\{ \|T_i^* T_j\|_{\mathcal{B}(\mathcal{K}_1)}, \|T_i T_j^*\|_{\mathcal{B}(\mathcal{K}_2)} \right\} \leq \alpha_{i-j}^2$$

for all $i, j \in \mathbb{Z}$. Then we automatically have

$$\left\| \sum_k T_k \right\|_{\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)} \leq \sum_k \alpha_k.$$

Schur lemma. *Let T be given by*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy.$$

Let us define the Schur integrals associated to k

$$\mathcal{S}_1(x) = \int_{\mathbb{R}^n} \|k(x, y)\|_{\mathcal{H}} dy \quad \text{and} \quad \mathcal{S}_2(y) = \int_{\mathbb{R}^n} \|k(x, y)\|_{\mathcal{H}} dx.$$

If $\mathcal{S}_1, \mathcal{S}_2 \in L_\infty$, then T is bounded on L_2 and we have

$$\|T\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \leq \sqrt{\|\mathcal{S}_1\|_\infty \|\mathcal{S}_2\|_\infty}.$$

A localization estimate. *Assume that*

$$\|k(x, y)\|_{\mathcal{H}} \lesssim \frac{1}{|x - y|^n} \quad \text{for all } x, y \in \mathbb{R}^n,$$

and let T be a Calderón-Zygmund operator associated to the kernel k . Assume further that $T : L_2 \rightarrow L_2(\mathcal{H})$ is of norm 1. Then, given $x_0 \in \mathbb{R}^n$ and $r_1, r_2 \in \mathbb{R}_+$ with $r_2 > 2r_1$, the estimate below holds for any pair f, g of bounded scalar-valued functions respectively supported by $\mathcal{B}_{r_1}(x_0)$ and $\mathcal{B}_{r_2}(x_0)$

$$\left\| \int_{\mathbb{R}^n} Tf(x)g(x) dx \right\|_{\mathcal{H}} \leq c_n r_1^n \log(r_2/r_1) \|f\|_\infty \|g\|_\infty.$$

A.2. A quasi-orthogonal decomposition. According to the conditions imposed on T , it is clear that its adjoint $T^* : L_2(\mathcal{H}^*) \rightarrow L_2$ is a norm 1 operator with kernel given by $k^*(x, y) = \langle k(y, x), \cdot \rangle$. Indeed, we have

$$\begin{aligned} \langle Tf, g \rangle &= \int_{\mathbb{R}^n} \left\langle \int_{\mathbb{R}^n} k(x, y) f(y) dy, g(x) \right\rangle dx \\ &= \int_{\mathbb{R}^n} \overline{f(y)} \int_{\mathbb{R}^n} \langle k(x, y), g(x) \rangle dx dy = \langle f, T^*g \rangle, \end{aligned}$$

for any $g \in L_2(\mathcal{H}^*)$. In particular, the kernel $k^*(x, y)$ may be regarded as a linear functional $\mathcal{H}^* \rightarrow \mathbb{C}$ or, equivalently, an element in \mathcal{H} . Thus, it satisfies the same size and smoothness estimates as $k(x, y)$ and we may and shall view T^* as an operator $L_2 \rightarrow L_2(\mathcal{H})$ which formally maps $g \in L_2$ to

$$T^*g(x) = \int_{\mathbb{R}^n} k^*(x, y) g(y) dy \in L_2(\mathcal{H}).$$

We shall also use the terminology $\langle Tf, g \rangle = \langle f, T^*g \rangle$ to denote the continuous bilinear form $(f, g) \in L_2 \times L_2 \rightarrow \mathcal{H}$. In fact, we may also define T^*1 in a weak sense as in the scalar-valued case. Moreover, the condition $T^*1 = 0$ which we shall assume at some points in this section, implies as usual that the relation below holds for any $f \in H_1$

$$(A1) \quad \int_{\mathbb{R}^n} Tf(x) dx = 0.$$

Indeed, we formally have $\langle Tf, 1 \rangle = \langle f, T^*1 \rangle = 0$. Nevertheless, we refer to Hytönen and Weis [7] for a more in depth explanation of all the identifications we have done so far. Let us go back to our problem. As usual, let \mathbf{E}_k be the k -th dyadic conditional expectation and fix Δ_k for the martingale difference $\mathbf{E}_k - \mathbf{E}_{k-1}$, so that $\mathbf{E}_k(f) = f_k$ and $\Delta_k(f) = df_k$. Then we consider the following decomposition

$$1_{\mathbb{R}^n \setminus \Sigma_{f,s}} Tf = 1_{\mathbb{R}^n \setminus \Sigma_{f,s}} \left(\sum_{k \in \mathbb{Z}} \mathbf{E}_k T \Delta_{k+s} + \sum_{k \in \mathbb{Z}} (id - \mathbf{E}_k) T_{4 \cdot 2^{-k}} \Delta_{k+s} \right) (f),$$

where T_ε denotes the truncated singular integral

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x, y) f(y) dy.$$

We refer to [24] more details. Our first step towards the proof is the following.

Shifted quasi-orthogonal decomposition. *Let $T : L_2 \rightarrow L_2(\mathcal{H})$ be a normalized Calderón-Zygmund operator with Lipschitz parameter γ , as defined above. Assume further that $T^*1 = 0$, so that*

$$\int_{\mathbb{R}^n} Tf(x) dx = 0$$

for any $f \in H_1$. Then, we have

$$\|\Phi_s\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} = \left\| \sum_k \mathbf{E}_k T \Delta_{k+s} \right\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \leq c_{n,\gamma} s 2^{-\gamma s/2}.$$

Moreover, regardless the value of T^*1 , we also have

$$\|\Psi_s\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} = \left\| \sum_k (id - \mathbf{E}_k) T_{4 \cdot 2^{-k}} \Delta_{k+s} \right\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \leq c_{n,\gamma} 2^{-\gamma s/2}.$$

A.2.1. *The norm of Φ_s .* Define

$$\begin{aligned}\phi_{R_x}(w) &= \frac{1}{|R_x|} 1_{R_x}(w), \\ \psi_{\widehat{Q}_y}(z) &= \frac{1}{|\widehat{Q}_y|} \sum_{j=2}^{2^n} 1_{Q_y}(z) - 1_{Q_j}(z).\end{aligned}$$

where R_x is the only cube in \mathcal{Q}_k containing x and Q_y is the only cube in \mathcal{Q}_{k+s} containing y . Moreover, the cubes Q_2, Q_3, \dots, Q_{2^n} represent the remaining cubes in \mathcal{Q}_{k+s} sharing dyadic father with Q_y . Arguing as in [24], it is easily checked that the kernel $k_{s,k}(x, y)$ of $E_k T \Delta_{k+s}$ has the form

$$(A2) \quad k_{s,k}(x, y) = \left\langle T(\psi_{\widehat{Q}_y}), \phi_{R_x} \right\rangle.$$

Lemma A.1. *If $T^*1 = 0$, the following estimates hold:*

a) *If $y \in \mathbb{R}^n \setminus 3R_x$, we have*

$$\|k_{s,k}(x, y)\|_{\mathcal{H}} \leq c_n 2^{-\gamma(k+s)} \frac{1}{|x - y|^{n+\gamma}}.$$

b) *If $y \in 3R_x \setminus R_x$, we have*

$$\|k_{s,k}(x, y)\|_{\mathcal{H}} \leq c_{n,\gamma} 2^{-\gamma(k+s)} 2^{nk} \min \left\{ \int_{R_x} \frac{dw}{|w - c_y|^{n+\gamma}}, s 2^{\gamma(k+s)} \right\}.$$

c) *Similarly, if $y \in R_x$ we have*

$$\|k_{s,k}(x, y)\|_{\mathcal{H}} \leq c_{n,\gamma} 2^{-\gamma(k+s)} 2^{nk} \min \left\{ \int_{\mathbb{R}^n \setminus R_x} \frac{dw}{|w - c_y|^{n+\gamma}}, s 2^{\gamma(k+s)} \right\}.$$

The constant $c_{n,\gamma}$ only depends on n and γ ; c_y denotes the center of the cube \widehat{Q}_y .

The main ingredients of the proof are the size/smoothness estimates imposed on the kernel, plus the cancellation condition (A1) and the localization lemma given above. In particular, the proof in [24, Lemma 2.3] translates verbatim to the Hilbert space valued context. Moreover, the result below is a direct consequence of Lemma A.1 and some calculations provided in [24].

Lemma A.2. *Let us define*

$$\begin{aligned}\mathcal{S}_{s,k}^1(x) &= \int_{\mathbb{R}^n} \|k_{s,k}(x, y)\|_{\mathcal{H}} dy, \\ \mathcal{S}_{s,k}^2(y) &= \int_{\mathbb{R}^n} \|k_{s,k}(x, y)\|_{\mathcal{H}} dx.\end{aligned}$$

Then, there exists a constant $c_{n,\gamma}$ depending only on n, γ such that

$$\begin{aligned}\mathcal{S}_{s,k}^1(x) &\leq \frac{c_{n,\gamma} s}{2^{\gamma s}} \quad \text{for all } (x, k) \in \mathbb{R}^n \times \mathbb{Z}, \\ \mathcal{S}_{s,k}^2(y) &\leq c_{n,\gamma} s \quad \text{for all } (y, k) \in \mathbb{R}^n \times \mathbb{Z}.\end{aligned}$$

Now we are in position to complete our estimate for Φ_s . In fact, the argument we are giving greatly simplifies the one provided in [24]. Namely, let us write $\Lambda_{s,k}$ for $E_k T \Delta_{k+s}$. Then, Lemma A.2 in conjunction with Schur lemma provides the

estimate $\|\Lambda_{s,k}\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \leq c_{n,\gamma} s 2^{-\gamma s/2}$. On the other hand, according to Cotlar lemma, it remains to check that

$$\max \left\{ \|\Lambda_{s,i}^* \Lambda_{s,j}\|_{\mathcal{B}(L_2)}, \|\Lambda_{s,i} \Lambda_{s,j}^*\|_{\mathcal{B}(L_2(\mathcal{H}))} \right\} \leq c_{n,\gamma} s^2 e^{-\gamma s} \alpha_{i-j}^2$$

for some sumable sequence $(\alpha_k)_{k \in \mathbb{Z}}$. Now, by the orthogonality of martingale differences, it suffices to estimate the mappings $\Lambda_{s,i}^* \Lambda_{s,j}$ in $\mathcal{B}(L_2)$. Indeed, the only nonzero mapping $\Lambda_{s,i} \Lambda_{s,j}^*$ is the one given by $i = j$ and

$$\|\Lambda_{s,k} \Lambda_{s,k}^*\|_{\mathcal{B}(L_2(\mathcal{H}))} \leq \|\Lambda_{s,k}\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))}^2 \leq c_{n,\gamma} s^2 e^{-\gamma s},$$

by the estimate given above. To estimate the norm of $\Lambda_{s,i}^* \Lambda_{s,j}$, we assume (with no loss of generality) that $i \geq j$. The martingale property then gives $\mathbb{E}_i \mathbb{E}_j = \mathbb{E}_j$, so that $\Lambda_{s,i}^* \Lambda_{s,j} = \Delta_{i+s} T^* \mathbb{E}_j T \Delta_{j+s}$. If we combine this with the estimate deduced from Lemma A.2, we get

$$\begin{aligned} \|\Lambda_{s,i}^* \Lambda_{s,j}\|_{\mathcal{B}(L_2)} &\leq \|\Lambda_{s+i-j,j}\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \|\Lambda_{s,j}\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \\ &\leq c_{n,\gamma} s(s + |i - j|) e^{-\gamma s} e^{-\gamma |i-j|/2}. \end{aligned}$$

A.2.2. The norm of Ψ_s . The arguments in this case follows a similar pattern to those used for Φ_s . The main differences are two. First, we can not use (A1) anymore but this is solved by the cancellation produced by the term $(id - \mathbb{E}_k)$. Second, the simplification with respect to the original argument in [24] given above –using the martingale property– does not work anymore in this setting and we have to follow the complete argument in [24] for this case. Nevertheless, the translation of the original proof to the present setting is again straightforward. We leave the details to the interested reader.

A.3. The paraproduct argument. To complete the proof, we need to provide an alternative argument for those Calderón-Zygmund operators failing the cancellation condition (A1). Going back to our original decomposition of $1_{\mathbb{R}^n \setminus \Sigma_{f,s}} T f$, we may write $1_{\mathbb{R}^n \setminus \Sigma_{f,s}} T f = 1_{\mathbb{R}^n \setminus \Sigma_{f,s}} (\Phi_s f + \Psi_s f)$. The second term $\Psi_s f$ is fine because the quasi-orthogonal methods applied to it do not require condition (A1). For the first term we need to use the dyadic paraproduct associated to $\rho = T^* 1$

$$\Pi_\rho(f) = \sum_{j=-\infty}^{\infty} \Delta_j(\rho) \mathbb{E}_{j-1}(f).$$

Lemma A.3. *If $T : L_2 \rightarrow L_2(\mathcal{H})$ is bounded, then so is Π_ρ .*

Proof. We have

$$\|\Pi_\rho(f)\|_{L_2(\mathcal{H})}^2 = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \|d_j \rho(x)\|_{\mathcal{H}}^2 |f_{j-1}(x)|^2 dx.$$

Since $\mathbb{E}_{j-1}(\|d_j \rho(x)\|_{\mathcal{H}}) = \|d_j \rho(x)\|_{\mathcal{H}}$, we may define

$$R = \sum_{j=-\infty}^{\infty} \|d_j \rho(x)\|_{\mathcal{H}} r_j$$

with r_j the j -th Rademacher function. Then, it is clear that

$$\|\Pi_\rho(f)\|_{L_2(\mathcal{H})} = \|\Pi_R(f)\|_2 \leq \|R\|_{\text{BMO}_d(\mathbb{R}^n)} \|f\|_2.$$

On the other hand, it is not difficult to check that the identities below hold

$$\begin{aligned}
\|R\|_{\text{BMO}_d(\mathbb{R}^n)} &= \sup_{j \in \mathbb{Z}} \left\| \mathbb{E}_j \sum_{k > j} |dR_k|^2 \right\|_{\infty}^{\frac{1}{2}} \\
&= \sup_{j \in \mathbb{Z}} \left\| \mathbb{E}_j \sum_{k > j} \|d\rho_k\|_{\mathcal{H}}^2 \right\|_{\infty}^{\frac{1}{2}} \\
&= \sup_{j \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_j} \left[\frac{1}{|Q|} \int_Q \|\rho(x) - \rho_Q\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}} \\
&\sim \sup_{j \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_j} \inf_{\alpha_Q \in \mathcal{H}} \left[\frac{1}{|Q|} \int_Q \|\rho(x) - \alpha_Q\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}}.
\end{aligned}$$

Then, we take $\alpha_Q = T^* 1_{\mathbb{R}^n \setminus 2Q}(c_Q)$ with c_Q the center of Q and obtain

$$\begin{aligned}
&\left[\frac{1}{|Q|} \int_Q \|\rho(x) - \alpha_Q\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}} \\
&\leq \left[\frac{1}{|Q|} \int_Q \|T^* 1_{2Q}(x)\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}} \\
&+ \left[\frac{1}{|Q|} \int_Q \|T^* 1_{\mathbb{R}^n \setminus 2Q}(x) - T^* 1_{\mathbb{R}^n \setminus 2Q}(c_Q)\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}} = \text{A} + \text{B}.
\end{aligned}$$

The fact that $T^* : L_2 \rightarrow L_2(\mathcal{H})$ is bounded yields

$$\text{A} \leq \left[\frac{1}{|Q|} \int_{\mathbb{R}^n} \|T^* 1_{2Q}(x)\|_{\mathcal{H}}^2 dx \right]^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{|Q|}} \|1_{2Q}\|_2 \leq 2^{\frac{n}{2}}.$$

On the other hand, Lipschitz smoothness gives for $x \in Q$

$$\|T^* 1_{\mathbb{R}^n \setminus 2Q}(x) - T^* 1_{\mathbb{R}^n \setminus 2Q}(c_Q)\|_{\mathcal{H}} \leq \int_{\mathbb{R}^n \setminus 2Q} \|k^*(x, y) - k^*(c_Q, y)\|_{\mathcal{H}} dy \leq c_n.$$

This automatically gives an absolute bound for B and the result follows. \square

According to Lemma A.3, the dyadic paraproduct $\Pi_{\rho} : L_2 \rightarrow L_2(\mathcal{H})$ defines a bounded operator. Therefore, regarding its adjoint as a mapping $L_2 \rightarrow L_2(\mathcal{H})$ via the identifications explained above, we find a bounded map

$$\Pi_{\rho}^*(f) = \sum_{j=-\infty}^{\infty} \mathbb{E}_{j-1}(\langle \Delta_j(\rho), \cdot \rangle f).$$

This allows us to consider the decomposition

$$T = T_0 + \Pi_{\rho}^*.$$

Now we go back to our estimate. Following [24], we have

$$1_{\mathbb{R}^n \setminus \Sigma_{f,s}} \sum_k \mathbb{E}_k \Pi_{\rho}^* \Delta_{k+s} f = 0.$$

Therefore, it suffices to see that T_0 satisfies the following estimate

$$\left\| \sum_k \mathbb{E}_k T_0 \Delta_{k+s} \right\|_{\mathcal{B}(L_2, L_2(\mathcal{H}))} \leq c_{n,\gamma} s e^{-\gamma s/2}.$$

It is clear that $T_0^* 1 = 0$, so that condition (A1) holds for T_0 . Moreover, according to our previous considerations, $T_0 : L_2 \rightarrow L_2(\mathcal{H})$ is bounded and its kernel satisfies the size estimate since the same properties hold for T and Π_{ρ}^* . In particular,

the only problem to apply the quasi-orthogonal argument to T_0 is to verify that its kernel satisfies Lipschitz smoothness estimates. We know by hypothesis that T does. However, the dyadic paraproduct only satisfies dyadic analogues. It is nevertheless enough. Indeed, since Lemma A.2 follows automatically from Lemma A.1, it suffices to check the latter. However, following the argument in [24], it is easily seen that the instances in Lemma A.1 where the smoothness of the kernel is used equally work (giving rise to 0) in the dyadic setting.

APPENDIX B. BACKGROUND ON NONCOMMUTATIVE INTEGRATION

We end this article with a brief survey on noncommutative L_p spaces and related topics that have been used along the paper. Most of these results are well-known to experts in the field. The right framework for a noncommutative analog of the classical measure theory and integrations is the theory of von Neumann algebras. We refer to [18, 36] for a systematic study of von Neumann algebras and to the recent survey by Pisier/Xu [28] for a detailed exposition of noncommutative L_p spaces.

B.1. Noncommutative L_p . A *von Neumann algebra* is a weak-operator closed C^* -algebra. By the Gelfand-Naimark-Segal theorem, any von Neumann algebra \mathcal{M} can be embedded in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on some Hilbert space \mathcal{H} . In what follows we will identify \mathcal{M} with a subalgebra of $\mathcal{B}(\mathcal{H})$. The positive cone \mathcal{M}_+ is the set of positive operators in \mathcal{M} . A *trace* $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$ on \mathcal{M} is a linear map satisfying the tracial property $\tau(a^*a) = \tau(aa^*)$. A trace τ is *normal* if $\sup_\alpha \tau(a_\alpha) = \tau(\sup_\alpha a_\alpha)$ for any bounded increasing net (a_α) in \mathcal{M}_+ ; it is *semifinite* if for any non-zero $a \in \mathcal{M}_+$, there exists $0 < a' \leq a$ such that $\tau(a') < \infty$ and it is *faithful* if $\tau(a) = 0$ implies $a = 0$. Taking into account that τ plays the role of the integral in measure theory, all these properties are quite familiar. A von Neumann algebra \mathcal{M} is called *semifinite* whenever it admits a normal semifinite faithful (*n.s.f.* in short) trace τ . Recalling that any operator a can be written as a linear combination $a_1 - a_2 + ia_3 - ia_4$ of four positive operators, we can extend τ to the whole algebra \mathcal{M} . Then, the tracial property can be restated in the familiar way $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{M}$.

According to the GNS construction, it is easily seen that the noncommutative analogs of measurable sets (or equivalently characteristic functions of those sets) are orthogonal projections. Given $a \in \mathcal{M}_+$, the support projection of a is defined as the least projection q in \mathcal{M} such that $qa = a = aq$ and will be denoted by $\text{supp } a$. Let \mathcal{S}_+ be the set of all $a \in \mathcal{M}_+$ such that $\tau(\text{supp } a) < \infty$ and set \mathcal{S} to be the linear span of \mathcal{S}_+ . If we write $|x|$ for the operator $(x^*x)^{\frac{1}{2}}$, we can use the spectral measure $\gamma_{|x|} : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{H})$ of the operator $|x|$ to define

$$|x|^p = \int_{\mathbb{R}_+} s^p d\gamma_{|x|}(s) \quad \text{for } 0 < p < \infty.$$

We have $x \in \mathcal{S} \Rightarrow |x|^p \in \mathcal{S}_+ \Rightarrow \tau(|x|^p) < \infty$. If we set $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$, it turns out that $\|\cdot\|_p$ is a norm in \mathcal{S} for $1 \leq p < \infty$ and a p -norm for $0 < p < 1$. Using that \mathcal{S} is a w^* -dense $*$ -subalgebra of \mathcal{M} , we define the *noncommutative L_p space* $L_p(\mathcal{M})$ associated to the pair (\mathcal{M}, τ) as the completion of $(\mathcal{S}, \|\cdot\|_p)$. On the other hand, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. Many fundamental

properties of classical L_p spaces, like duality, real and complex interpolation... have been transferred to this setting. The most important properties for our purposes are

- Hölder inequality. If $1/r = 1/p + 1/q$, we have $\|ab\|_r \leq \|a\|_p \|b\|_q$.
- The trace τ extends to a continuous functional on $L_1(\mathcal{M})$: $|\tau(x)| \leq \|x\|_1$.

See [28] for L_p spaces over type III algebras. Let us recall a few examples:

- (a) **Commutative L_p spaces.** Let \mathcal{M} be commutative and semifinite. Then there exists a semifinite measure space (Ω, Σ, μ) for which $\mathcal{M} = L_\infty(\Omega)$ and $L_p(\mathcal{M}) = L_p(\Omega)$ with the *n.s.f.* trace τ determined by

$$\tau(f) = \int_{\Omega} f(\omega) d\mu(\omega).$$

- (b) **Semicommutative L_p spaces.** Given a measure space (Ω, Σ, μ) and a semifinite von Neumann algebra (\mathcal{N}, τ) . We consider the von Neumann algebra

$$(\mathcal{M}, \mu \otimes \tau) = \left(L_\infty(\Omega) \bar{\otimes} \mathcal{N}, \int_{\Omega} \tau(\cdot) d\mu \right).$$

In this case,

$$L_p(\mathcal{M}) = L_p(\Omega; L_p(\mathcal{N})),$$

the L_p -space of $L_p(\mathcal{N})$ -valued Bochner-integrable functions, for all $p < \infty$.

- (c) **Schatten p -classes.** Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with the standard trace

$$\text{tr}(x) = \sum_{\lambda} \langle x e_{\lambda}, e_{\lambda} \rangle_{\mathcal{H}},$$

where $(e_{\lambda})_{\lambda}$ is any orthonormal basis of \mathcal{H} . Then, the associated L_p space is called the Schatten p -class $\mathcal{S}_p(\mathcal{H})$. When \mathcal{H} is separable, the Schatten p -class is the noncommutative analog of ℓ_p , which embeds isometrically into the diagonal of \mathcal{S}_p .

- (d) **Hyperfinitesimal II_1 factor.** Let M_2 be the algebra of 2×2 matrices equipped with the normalized trace $\sigma = \frac{1}{2} \text{tr}$. A description of the so-called hyperfinite II_1 factor \mathcal{R} is by the following von Neumann algebra tensor product

$$(\mathcal{R}, \tau) = \overline{\bigotimes_{n \geq 1} (M_2, \sigma)}.$$

That is, \mathcal{R} is the von Neumann algebra generated by all elementary tensors $x_1 \otimes \cdots \otimes x_n \otimes \mathbf{1} \otimes \mathbf{1} \cdots$ and the trace τ is the unique normalized trace on \mathcal{R} which is determined by

$$\tau \left(x_1 \otimes \cdots \otimes x_n \otimes \mathbf{1} \otimes \mathbf{1} \cdots \right) = \prod_{k=1}^n \sigma(x_k).$$

$L_p(\mathcal{R})$ may be regarded as a noncommutative analog of $L_p[0, 1]$.

There are many other nice examples which we are omitting, like free product von Neumann algebras, q -deformed algebras, group von Neumann algebras... We refer to [40] for a more detailed explanation.

Remark B.1. Assume that the Hilbert space \mathcal{H} has a countable orthonormal basis $(e_k)_{k \in \mathbb{Z}}$ and consider the associated unit vectors $e_{m,n}$ of $\mathcal{B}(\mathcal{H})$ which are determined by the relation $e_{m,n}(x) = \langle x, e_n \rangle e_m$. Let T_+ and T_- be the projections from $\mathcal{B}(\mathcal{H})$ onto the subspaces

$$\Lambda_+ = \text{span}\{e_{m,n} \mid m > n\} \quad \text{and} \quad \Lambda_- = \text{span}\{e_{m,n} \mid m < n\}$$

respectively. T_+ and T_- are the first examples of noncommutative singular integrals and $T_+ - T_-$ is a noncommutative analog of the classical Hilbert transform. In fact, when $\mathcal{H} = L_2(\mathbb{T})$ is the space of all L_2 -integrable functions on the unit circle and $e_k = \exp(ik \cdot)$, we embed $L_\infty(\mathbb{T})$ into $\mathcal{B}(\mathcal{H})$ via the map

$$\Psi : f \in L_\infty(\mathbb{T}) \mapsto \Psi(f) \in \mathcal{B}(\mathcal{H}) \quad \text{with} \quad \Psi(f)[g] = \bar{f}g,$$

for any $g \in L_2(\mathbb{T})$. Then $\Psi(e_k) = \sum_m e_{m,m-k}$ and

$$(T_+ - T_-)(\Psi(f)) = -i\Psi(Hf),$$

for any $f \in L_2(\mathbb{T}) \cap L_\infty(\mathbb{T})$. Here H denotes the classical Hilbert transform on \mathbb{T} .

B.2. Noncommutative symmetric spaces. Let

$$\mathcal{M}' = \left\{ b \in \mathcal{B}(\mathcal{H}) \mid ab = ba \text{ for all } a \in \mathcal{M} \right\}$$

be the commutant of \mathcal{M} . A closed densely-defined operator on \mathcal{H} is *affiliated* with \mathcal{M} when it commutes with every unitary u in the commutant \mathcal{M}' . Recall that $\mathcal{M} = \mathcal{M}''$ and this implies that every $a \in \mathcal{M}$ is affiliated with \mathcal{M} . The converse fails in general since we may find unbounded operators. If a is a densely defined self-adjoint operator on \mathcal{H} and $a = \int_{\mathbb{R}} s d\gamma_a(s)$ is its spectral decomposition, the spectral projection $\int_{\mathcal{R}} d\gamma_a(s)$ will be denoted by $\chi_{\mathcal{R}}(a)$. An operator a affiliated with \mathcal{M} is τ -measurable if there exists $s > 0$ such that

$$\tau\{|a| > s\} = \tau(\chi_{(s,\infty)}(|a|)) < \infty.$$

The *generalized singular-value* $\mu(a) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$\mu_t(a) = \inf \left\{ s > 0 \mid \tau\{|x| > s\} \leq t \right\}.$$

This provides us with a noncommutative analogue of the so-called non-increasing rearrangement of a given function. We refer to [4] for a detailed exposition of the function $\mu(a)$ and the corresponding notion of convergence in measure. If $L_0(\mathcal{M})$ denotes the $*$ -algebra of τ -measurable operators, we have the following equivalent definition of L_p

$$L_p(\mathcal{M}) = \left\{ a \in L_0(\mathcal{M}) \mid \left(\int_{\mathbb{R}_+} \mu_t(a)^p dt \right)^{\frac{1}{p}} < \infty \right\}.$$

The same procedure applies to symmetric spaces. Given the pair (\mathcal{M}, τ) , let X be a rearrangement invariant quasi-Banach function space on the interval $(0, \tau(\mathbf{1}_{\mathcal{M}}))$. The *noncommutative symmetric space* $X(\mathcal{M})$ is defined by

$$X(\mathcal{M}) = \left\{ a \in L_0(\mathcal{M}) \mid \mu(a) \in X \right\} \quad \text{with} \quad \|a\|_{X(\mathcal{M})} = \|\mu(a)\|_X.$$

It is known that $X(\mathcal{M})$ is a Banach (resp. quasi-Banach) space whenever X is a Banach (resp. quasi-Banach) function space. We refer the reader to [3, 39] for more in depth discussion of this construction. Our interest in this paper is restricted to noncommutative L_p -spaces and *noncommutative weak L_1 -spaces*. Following the

construction of symmetric spaces of measurable operators, the noncommutative weak L_1 -space $L_{1,\infty}(\mathcal{M})$, is defined as the set of all a in $L_0(\mathcal{M})$ for which the quasi-norm

$$\|a\|_{1,\infty} = \sup_{t>0} t \mu_t(x) = \sup_{\lambda>0} \lambda \tau\{|x| > \lambda\}$$

is finite. As in the commutative case, the noncommutative weak L_1 space satisfies a quasi-triangle inequality that will be used below with no further reference. Indeed, the following inequality holds for $a_1, a_2 \in L_{1,\infty}(\mathcal{M})$

$$\lambda \tau\{|a_1 + a_2| > \lambda\} \leq \lambda \tau\{|a_1| > \lambda/2\} + \lambda \tau\{|a_2| > \lambda/2\}.$$

B.3. Noncommutative martingales. Consider a von Neumann subalgebra (a weak* closed *-subalgebra) \mathcal{N} of a semifinite von Neumann algebra (\mathcal{M}, τ) . A *conditional expectation* $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ is a positive contractive projection from \mathcal{M} onto \mathcal{N} . The conditional expectation \mathcal{E} is called *normal* if the adjoint map \mathcal{E}^* satisfies $\mathcal{E}^*(\mathcal{M}_*) \subset \mathcal{N}_*$. In this case, there is a map $\mathcal{E}_* : \mathcal{M}_* \rightarrow \mathcal{N}_*$ whose adjoint is \mathcal{E} . Such normal conditional expectation exists if and only if the restriction of τ to the von Neumann subalgebra \mathcal{N} remains semifinite, see e.g. Theorem 3.4 in [36]. This is always the case when $\tau(\mathbf{1}_{\mathcal{M}}) < \infty$. Any such conditional expectation is trace preserving (i.e. $\tau \circ \mathcal{E} = \tau$) and satisfies the bimodule property

$$\mathcal{E}(a_1 b a_2) = a_1 \mathcal{E}(b) a_2 \quad \text{for all } a_1, a_2 \in \mathcal{N} \text{ and } b \in \mathcal{M}.$$

Let $(\mathcal{M}_k)_{k \geq 1}$ be an increasing sequence of von Neumann subalgebras of \mathcal{M} such that the union of the \mathcal{M}_k 's is weak* dense in \mathcal{M} . Assume that for every $k \geq 1$, there is a normal conditional expectation $\mathcal{E}_k : \mathcal{M} \rightarrow \mathcal{M}_k$. Note that for every $1 \leq p < \infty$ and $k \geq 1$, \mathcal{E}_k extends to a positive contraction $\mathcal{E}_k : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}_k)$. A *noncommutative martingale* with respect to the filtration $(\mathcal{M}_k)_{k \geq 1}$ is a sequence $a = (a_k)_{k \geq 1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_j(a_k) = a_j \quad \text{for all } 1 \leq j \leq k < \infty.$$

If additionally $a \in L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$ and $\|a\|_p = \sup_{k \geq 1} \|a_k\|_p < \infty$, then a is called an *L_p -bounded martingale*. Given a martingale $a = (a_k)_{k \geq 1}$, we assume the convention that $a_0 = 0$. Then, the martingale difference sequence $da = (da_k)_{k \geq 1}$ associated to x is defined by $da_k = a_k - a_{k-1}$.

Let us now comment some examples of noncommutative martingales:

- (a) **Classical martingales.** Given a commutative finite von Neumann algebra \mathcal{M} equipped with a normalized trace τ and a filtration $(\mathcal{M}_n)_{n \geq 1}$, there exists a probability space (Ω, Σ, μ) and an increasing sequence $(\Sigma_n)_{n \geq 1}$ of σ -subalgebras satisfying

$$L_p(\mathcal{M}) = L_p(\Omega, \Sigma, \mu) \quad \text{and} \quad L_p(\mathcal{M}_n) = L_p(\Omega, \Sigma_n, \mu).$$

Thus, classical martingales are a form of noncommutative martingales.

- (b) **Semicommutative martingales.** Let (Ω, Σ, μ) be a probability space and (\mathcal{N}, τ) be a semifinite von Neumann algebra. Given $(\Sigma_n)_{n \geq 1}$ an increasing filtration of σ -subalgebras of Σ , we consider the filtration

$$(\mathcal{M}_n, \tau) = \left(L_\infty(\Omega, \Sigma_n, \mu) \bar{\otimes} \mathcal{N}, \int_\Omega \tau(\cdot) d\mu \right),$$

of

$$(\mathcal{M}, \tau) = \left(L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{N}, \int_\Omega \tau(\cdot) d\mu \right).$$

In this case, conditional expectations are given by $E_n = \mathbb{E}_n \otimes id_{\mathcal{N}}$ where \mathbb{E}_n denotes the conditional expectation $(\Omega, \Sigma) \rightarrow (\Omega, \Sigma_n)$. Once more (in this particular setting) noncommutative martingales can be viewed as vector valued commutative martingales.

- (c) **Finite martingales.** When dealing with Schatten p -classes, there is no natural finite trace unless we work in the finite-dimensional case $\mathcal{S}_p(n)$ where we consider the normalized trace $\sigma = \frac{1}{n} \text{tr}$. A natural filtration is obtained taking (\mathcal{M}_k, σ) to be the subalgebra of $k \times k$ matrices (i.e. with vanishing entries in the last $n - k$ rows and columns). This choice is useful to obtain certain counterexamples, see [16].
- (d) **Dyadic martingales.** Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \dots$ be a collection of independent ± 1 Bernoullis. Classical dyadic martingales are constructed over the filtration $\Sigma_n = \sigma(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. The noncommutative analog consists of a filtration $(\mathcal{R}_n)_n$ in the hyperfinite II_1 factor as follows

$$(\mathcal{R}_n, \tau) = \overline{\bigotimes_{1 \leq m \leq n} (M_2, \sigma)}.$$

\mathcal{R}_n embeds into \mathcal{R} by means of $a_1 \otimes \dots \otimes a_n \mapsto a_1 \otimes \dots \otimes a_n \otimes \mathbf{1} \otimes \mathbf{1} \dots$. Moreover, given $1 \leq n \leq r$, the conditional expectation $E_n : \mathcal{R} \rightarrow \mathcal{R}_n$ is determined by

$$E_n(a_1 \otimes \dots \otimes a_r \otimes \mathbf{1} \otimes \mathbf{1} \dots) = \left(\prod_{k=n+1}^r \sigma(a_k) \right) a_1 \otimes \dots \otimes a_n \otimes \mathbf{1} \otimes \mathbf{1} \dots$$

As we did with noncommutative L_p spaces, we omit some standard examples like free martingales, q -deformed martingales or noncommutative martingales on the group algebra of a discrete group. We refer again to [40] for a more in depth exposition.

The theory of noncommutative martingales has achieved considerable progress in recent years. The renewed interest on this topic started from the fundamental paper of Pisier and Xu [27], where they introduced a new functional analytic approach to study Hardy spaces and the Burkholder-Gundy inequalities for noncommutative martingales. Shortly after, many classical inequalities have been transferred to the noncommutative setting. A noncommutative analogue of Doob's maximal function [8], the noncommutative John-Nirenberg theorem [11], extensions of Burkholder inequalities for conditioned square functions [15] and related weak type inequalities [29, 30]; see [26] for a simpler approach to some of them.

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