THE NORM OF SUMS OF INDEPENDENT NONCOMMUTATIVE RANDOM VARIABLES IN $L_p(\ell_1)$

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Abstract. We investigate the norm of sums of independent vector-valued random variables in noncommutative L_p spaces. This allows us to obtain a uniform family of complete embeddings of the Schatten class S_q^n in $S_p(\ell_q^m)$ with optimal order $m \sim n^2$. Using these embeddings we show the surprising fact that the sharp type (cotype) index in the sense of operator spaces for $L_p[0,1]$ is $\min(p,p')$ $(\max(p,p'))$. Similar techniques are used to show that the operator space notions of B-convexity and K-convexity are equivalent.

Introduction

Sums of independent random variables have a long tradition both in probability theory and Banach space geometry. More recently, the noncommutative analogs of these probabilistic results have been developed [9, 11, 25] and applied to operator space theory [8, 10, 24]. In this paper, we follow this line of research in studying type and cotype in the sense of operator spaces [19]. This theory is closely connected to the notions of B-convexity and K-convexity. Using embedding results we show that these notions remain equivalent in the category of operator spaces.

We recall from [16] that a Banach space X is called K-convex whenever the Gauss projection

$$\mathrm{P}_{\mathbf{G}} : f \in L_2(\Omega; \mathrm{X}) \longmapsto \sum_{k=1}^{\infty} \Big(\int_{\Omega} f(\omega) \overline{\mathrm{g}_k(\omega)} \, d\mu(\omega) \Big) \mathrm{g}_k \in L_2(\Omega; \mathrm{X})$$

is bounded. Here $\mathrm{g}_1, \mathrm{g}_2, \ldots$ are independent standard complex-valued Gaussian random variables defined over a probability space $(\Omega, \mathsf{A}, \mu)$. An operator space X is called OK-convex if $\mathrm{P}_{\mathbf{G}}$ is completely bounded or equivalently $S_2(\mathrm{X})$ is K-convex as a Banach space. Using standard tools from Banach space theory, we know that $S_2(\mathrm{X})$ is K-convex if and only if $S_p(\mathrm{X})$ is K-convex for some (any) $1 < p < \infty$. Therefore this notion does not depend on the parameter p. According to a deep theorem of Pisier [21] K-convexity is equivalent to B-convexity. Following Beck [1], a Banach space X is called B-convex if there exists $n \ge 1$ and $0 < \delta \le 1$ such that

$$\frac{1}{n} \inf_{|\alpha_k|=1} \Big\| \sum_{k=1}^{n} \alpha_k x_k \Big\| \le (1-\delta) \max_{1 \le k \le n} \|x_k\|$$

holds for any family $x_1, x_2, \ldots, x_n$ of vectors in X. Giesy proved in [5] that a Banach space X is B-convex if and only if X does not contain ℓ_1^n's uniformly. In the context

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of operator spaces, the noncommutative analogue of ℓ_1^n is the Schatten class S_1^n, the dual of $\mathcal{B}(\ell_2^n)$. More generally, one might consider arbitrary dual spaces

$$L_1(\mathcal{A}) = S_1^{n_1} \oplus S_1^{n_2} \oplus \cdots \oplus S_1^{n_m}$$

of finite dimensional C*-algebras. A priori, it is unclear which analogue of the notion of B-convexity is the right one for operator spaces. Namely, we could only exclude the ℓ_1^n's or all the $L_1(\mathcal{A})$'s. The following result clarifies this question.

Theorem 1. *Let* X *be an operator space and let* $(\mathcal{A}_n)$ *be a sequence of pairwise different finite dimensional* C**-algebras. The following are equivalent:*

i) X *is* OK-*convex.*
ii) $S_p(\mathrm{X})$ *does not contain* ℓ_1^n*'s uniformly for some* (*any*) $1 < p < \infty$.
iii) $S_p(\mathrm{X})$ *does not contain* $L_1(\mathcal{A}_n)$*'s uniformly for some* (*any*) $1 < p < \infty$.

In fact, we prove a stronger result. We shall say that the spaces $L_1(\mathcal{A}_n)$'s embed semi-completely uniformly in $S_p(\mathrm{X})$ when there exists a family of embeddings

$$\Lambda_n : L_1(\mathcal{A}_n) \to S_p(\mathrm{X})$$

satisfying $\|\Lambda_n\|_{cb}\|\Lambda_n^{-1}\| \le \mathrm{c}$ for some universal constant $\mathrm{c} > 1$. This notion came out naturally in the paper [18]. We refer to [17] for further applications of this concept. For the equivalence of ii) and iii), we prove that if ℓ_1^n embeds into $S_p(\mathrm{X})$ with constant c_n, then there is a map $u : S_1^n \to S_p(\mathrm{X})$ such that

$$\big\|u : S_1^n \to S_p(\mathrm{X})\big\|_{cb}\big\|u^{-1} : u(S_1^n) \to S_1^n\big\| \le \mathrm{C}\,\mathrm{c}_n.$$

This map is constructed using (noncommutative) probabilistic tools. Usually, estimates for sums of noncommutative random variables are motivated by classical probabilistic inequalities. Our probabilistic motivation here is given by the following result. Let us consider a finite collection $f_1, f_2, \ldots, f_n$ of independent random variables on a probability space $(\Omega, \mathsf{A}, \mu)$. Then, given $1 \le p < \infty$, the following equivalence of norms holds

$$(\Sigma_p) \qquad \Big(\int_\Omega \Big[\sum_{k=1}^n |f_k(\omega)|\Big]^p d\mu(\omega)\Big)^{1/p} \sim \max_{r\in\{1,p\}} \left\{ \Big(\sum_{k=1}^n \int_\Omega |f_k(\omega)|^r d\mu(\omega)\Big)^{1/r} \right\}.$$

We shall provide in this paper the natural analog of (Σ_p) for noncommutative random variables. This result requires the use of the so-called *asymmetric* L_p *spaces*, which will be defined below. Now, going back to the construction of the map $u : S_1^n \to S_p(\mathrm{X})$, we consider positive integers $m, n \ge 1$. Then, if $1 \le k \le m$ and τ stands for the normalized trace, let $\pi_k : L_p(\tau_n) \to L_p(\tau_{n^m})$ be the mapping defined by the relation $\pi_k(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$, where x is located at the k-th position and 1 stands for the identity of M_n. Then, the embedding u can be easily constructed by combining condition ii) with the following result, which might be of independent interest.

Theorem 2. *Let* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then, given* $n \ge 1$ *and* $m \ge n^2$, *the following map is a complete isomorphism onto a completely complemented subspace*

$$x \in S_q^n \longmapsto \frac{1}{n^{1/q}} \sum_{k=1}^m \delta_k \otimes \pi_k(x) \in L_p(\tau_{n^m}; \ell_q^m).$$

Moreover, the cb-distance constants are uniformly bounded on the dimensions.

Our proof requires $m \geq n^2$ which is different from the well-known commutative order $m \sim n$. Using type/cotype estimates we show that the order $m \sim n^2$ is best possible. The cotype for operator spaces is motivated by the Hausdorff-Young inequality for non-abelian compact groups. Let G be a noncommutative compact group and $\widehat{\mathrm{G}}$ its dual object. That is, a list of inequivalent irreducible unitary representations. Given $1 \leq p \leq 2$, an operator space X has Fourier type p with respect to G if the X-valued Hausdorff-Young inequality

$$\Big(\sum_{k=1}^n d_k \|A_k\|_{S_{p'}^{d_k}(\mathrm{X})}^{p'} \Big)^{1/p'} \leq_{cb} \mathcal{K}_p(\mathrm{X}, \widehat{\mathrm{G}}) \Big(\int_{\mathrm{G}} \Big\| \sum_{k=1}^n d_k \mathrm{tr}(A_k \pi_k(g)) \Big\|_{\mathrm{X}}^p d\mu(g) \Big)^{1/p}$$

holds for all finite sequence of matrices $A_1, A_2, \ldots, A_n$ with $A_k \in M_{d_{\pi_k}} \otimes \mathrm{X}$. Here μ is the normalized Haar measure and d_k denotes the degree of the irreducible representation $\pi_k : \mathrm{G} \to U(d_k)$. Moreover, here and in the following, the symbol $\leq_{cb}$ is used to indicate the corresponding linear map is indeed completely bounded. The notion of Fourier cotype is dual to the notion of Fourier type stated above. Following [22], we notice that this inequality forces us to consider an operator space structure on the vector space where we are taking values. In other words, we need to take values in operator spaces rather than Banach spaces.

Note that the span of the functions of the form $\mathrm{tr}(A_k \pi_k(g))$ is dense in $L_2(\mathrm{G})$. In the classical notion of cotype the right hand side is replaced by a suitable subset of characters. Since it is not entirely clear which will be such a canonical subset for arbitrary groups, we follow the approach of Marcus/Pisier [15] and consider random Fourier series of the form

$$\sum_{k=1}^n d_k \mathrm{tr}(A_k \pi_k(g) U_{\pi_k})$$

where the U_π's are random unitaries. However, the contraction principle allows us to eliminate the coefficients $\pi(g)$. Therefore, given such a family of random unitaries over a probability space $(\Omega, \mathsf{A}, \mu)$, a possible notion of cotype for operator spaces is given by the inequality

$$\Big(\sum_{k=1}^n d_k \|A_k\|_{S_{p'}^{d_k}(\mathrm{X})}^{p'} \Big)^{1/p'} \leq_{cb} \mathcal{K}_p(\mathrm{X}, \widehat{\mathrm{G}}) \Big(\int_{\Omega} \Big\| \sum_{k=1}^n d_k \mathrm{tr}(A_k U_{\pi_k}(\omega)) \Big\|_{\mathrm{X}}^p d\mu(\omega) \Big)^{1/p}.$$

Although this definition originated from compact groups, in this formulation only the degrees of the representations of the dual object and their multiplicity are kept. We may therefore consider this notion of cotype for arbitrary collections of random unitaries (U_σ) indexed by $\sigma \in \Sigma$ and where d_σ represents the dimension of U_σ. Examples for Σ's coming from groups are the *commutative* set of parameters ($\Sigma_0 = \mathbb{N}$ and $d_k = 1$ for all $k \geq 1$) which arises from any non-finite abelian compact group and the set $\Sigma_1 = \mathbb{N}$ with $d_k = k$ for $k \geq 1$, which comes from the classical Lie group $SU(2)$. As we shall see in this paper, these two sets of parameters are the most relevant ones in the theory. Let us mention that random unitaries can also be understood as a higher dimensional version of random signs or independent Steinhaus variables. It is rather surprising that in disproving cotype q larger matrices are not necessarily easier. In part because the sequence $(d_\sigma)_{\sigma \in \Sigma}$ provides a new normalization. Let us also note that Khintchine-Kahane inequalities are not available in the operator space setting because they even fail in the level of scalars [13, 14, 22]. Indeed, type (see Section 6 for a definition) and cotype turn out to be closer notions to Fourier type and cotype than in the classical theory.

Theorem 3. *Any infinite dimensional L_p space has:*

i) *Sharp Σ-type* $\min(p, p')$.
ii) *Sharp Σ-cotype* $\max(p, p')$.

The organization of the paper is as follows. Section 1 is devoted to describe the operator space structure and some basic properties of the asymmetric L_p spaces. These spaces provide an important tool in this paper. Section 2 contains some preliminary estimates that will be used in Section 3 to prove the analog of (Σ_p) for noncommutative random variables. In Section 4 we construct the embedding of S_q^n into $S_p(\ell_q^m)$ described in Theorem 2. Section 5 is devoted to prove the operator space version of Pisier's characterization of K-convexity. Finally, in Section 6 we find the sharp operator space type and cotype indices of L_p spaces.

1. Asymmetric L_p spaces

Throughout this paper, some basic notions of noncommutative L_p spaces and operator space theory will be assumed, see [22, 23] for a systematic treatment. We begin by studying some basic properties of the asymmetric L_p spaces, defined as follows. Let E be an operator space and let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a n.s.f. trace φ. Given a pair of exponents $2 \le r, s \le \infty$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, we define the **asymmetric L_p space** $L_{(r,s)}(\mathcal{M}, \varphi; E)$ as the completion of $L_p(\mathcal{M}, \varphi) \otimes E$ with respect to the following norm

$$\|x\|_{L_{(r,s)}(\mathcal{M},\varphi;E)} = \inf_{x=\alpha y \beta} \Big\{ \|\alpha\|_{L_r(\mathcal{M},\varphi)} \|y\|_{L_\infty(\mathcal{M},\varphi;E)} \|\beta\|_{L_s(\mathcal{M},\varphi)} \Big\},$$

where the infimum runs over all decompositions $x = \alpha y \beta$ with $\alpha \in L_r(\mathcal{M}, \varphi)$, $\beta \in L_s(\mathcal{M}, \varphi)$ and $y \in \mathcal{M} \otimes_{\min} E$. Recall that any noncommutative L_p space can be realized as $L_p(\mathcal{M}, \varphi; E) = L_{(2p,2p)}(\mathcal{M}, \varphi; E)$. In this paper, the von Neumann algebra $\mathcal{M}$ will always be a finite matrix algebra M_n so that the trace φ is unique up to a constant factor. In fact, we shall only work with the usual trace tr_n of M_n and its normalization $\tau_n = \frac{1}{n}\mathrm{tr}_n$. The spaces

$$S_{(r,s)}^n(E) = L_{(r,s)}(\mathrm{tr}_n; E)$$

can be regarded as the asymmetric version of Pisier's vector-valued Schatten classes. If R and C stand for the row and column operator Hilbert spaces, we shall denote by C_p and R_p the interpolation spaces $[C, R]_{1/p}$ and $[R, C]_{1/p}$. The superscript n will indicate the n-dimensional version. By using elementary properties of the Haagerup tensor product, it is not difficult to check that

$$S_{(r,s)}^n(E) = C_{r/2}^n \otimes_h E \otimes_h R_{s/2}^n$$

isometrically. This provides a *natural* operator space structure for $L_{(r,s)}(\tau_n; E)$.

Lemma 1.1. *Given $1 \le r, s, t \le \infty$ such that $\frac{1}{s} = \frac{1}{r} + \frac{1}{t}$, we have*

$$\|\alpha\|_{\mathcal{CB}(C_r^n, C_s^n)} = \|\alpha\|_{S_{2t}^n},$$
$$\|\beta\|_{\mathcal{CB}(R_r^n, R_s^n)} = \|\beta\|_{S_{2t}^n}.$$

Proof. Since the row case can be treated similarly, we just prove the first equality. When $r = s$ the result is trivial. Assume now that $r = \infty$. We begin by recalling the following well-known complete isometries

$$\mathcal{CB}(C_\infty^n, C_s^n) = R_\infty^n \otimes_{\min} C_s^n = C_s^n \otimes_h R_\infty^n.$$

Since the Haagerup tensor product commutes with complex interpolation, we obtain the following Banach space isometries

$$C_s^n \otimes_h R_\infty^n = [C_\infty^n \otimes_h R_\infty^n, R_\infty^n \otimes_h R_\infty^n]_{1/s} = [S_\infty^n, S_2^n]_{1/s} = S_{2s}^n.$$

Since $s = t$ when $r = \infty$, the identity holds. Now we take $s < r < \infty$. In that case, we use the fact that the complex interpolation space

$$S_{2t}^n = [S_{2s}^n, S_\infty^n]_{s/r} = [\mathcal{CB}(C_\infty^n, C_s^n), \mathcal{CB}(C_s^n, C_s^n)]_{s/r} \subset \mathcal{CB}(C_r^n, C_s^n)$$

is contractively included in $\mathcal{CB}(C_r^n, C_s^n)$. This gives $\|\alpha\|_{\mathcal{CB}(C_r^n, C_s^n)} \le \|\alpha\|_{S_{2t}^n}$. For the lower estimate, we consider the bilinear form

$$\mathcal{CB}(C_r^n, C_s^n) \times \mathcal{CB}(C_\infty^n, C_r^n) \longrightarrow \mathcal{CB}(C_\infty^n, C_s^n),$$

defined by $(\alpha, \beta) \mapsto \alpha \circ \beta$. Then, recalling that the Banach space $\mathcal{CB}(C_\infty^n, C_p^n)$ is isometrically isomorphic to the Schatten class S_{2p}^n (see above), we obtain the following inequality

$$\|\alpha\beta\|_{S_{2s}^n} \le \|\alpha\|_{\mathcal{CB}(C_r^n, C_s^n)} \|\beta\|_{S_{2r}^n}.$$

Taking the supremum over $\beta \in M_n$, we obtain $\|\alpha\|_{\mathcal{CB}(C_r^n, C_s^n)} \ge \|\alpha\|_{S_{2t}^n}$. □

In the following lemma, we state some basic properties of the asymmetric L_p spaces which naturally generalize some Pisier's results in Chapter 1 of [22].

Lemma 1.2. *The asymmetric Schatten classes satisfy the following properties:*

i) *Given* $2 \le p, q, r, s, u, v \le \infty$ *such that* $\frac{1}{p} = \frac{1}{r} + \frac{1}{u}$ *and* $\frac{1}{q} = \frac{1}{s} + \frac{1}{v}$, *we have*
$$\|\alpha x \beta\|_{S_{(p,q)}^n(E)} \le \|\alpha\|_{S_u^n} \|x\|_{S_{(r,s)}^n(E)} \|\beta\|_{S_v^n}.$$

ii) *Given* $x \in M_n \otimes E$ *and* $2 \le p, q \le \infty$, *we have*
$$\|x\|_{M_n(E)} = \sup \Big\{ \|\alpha x \beta\|_{S_{(p,q)}^n(E)} : \ \|\alpha\|_{S_p^n}, \|\beta\|_{S_q^n} \le 1 \Big\}.$$
Therefore, any linear map $u : E \to F$ *between operator spaces satisfies*
$$\|u\|_{cb} = \sup_{n \ge 1} \Big\| id \otimes u : S_{(p,q)}^n(E) \to S_{(p,q)}^n(F) \Big\|.$$

iii) *Any block-diagonal matrix* $\mathrm{D}_n(x) \in M_{mn} \otimes E$, *with blocks* $x_1, x_2, \ldots, x_n$ *in* $M_m \otimes E$, *satisfies the following identity*
$$\|\mathrm{D}_n(x)\|_{S_{(r,s)}^{mn}(E)} = \Big(\sum_{k=1}^n \|x_k\|_{S_{(r,s)}^m(E)}^p \Big)^{1/p}, \quad \textit{where} \quad \tfrac{1}{p} = \tfrac{1}{r} + \tfrac{1}{s}.$$

Proof. Let us define

$$\alpha \otimes_h id_E \otimes_h \beta^{\mathrm{t}} : \ C_{r/2}^n \otimes_h E \otimes_h R_{s/2}^n \longrightarrow C_{p/2}^n \otimes_h E \otimes_h R_{q/2}^n$$

to be the mapping $x \mapsto \alpha x \beta$. Then the inequality stated in (i) follows by Lemma 1.1 and the injectivity of the Haagerup tensor product. Let us prove the first identity in (ii). We point out that

$$\|x\|_{M_n(E)} \ge \sup \Big\{ \|\alpha x \beta\|_{S_{(p,q)}^n(E)} : \ \|\alpha\|_{S_p^n}, \|\beta\|_{S_q^n} \le 1 \Big\},$$

follows immediately from (i). On the other hand, following Lemma 1.7 of [22], we can write $\|x\|_{M_n(E)} = \|\alpha_0 x \beta_0\|_{S_1^n(E)}$ for some α_0, β_0 in the unit ball of S_2^n. Then, we consider decompositions $\alpha_0 = \alpha_1 \alpha$ and $\beta_0 = \beta \beta_1$ so that

$$\|\alpha_1\|_r = \|\alpha\|_p = 1 = \|\beta\|_q = \|\beta_1\|_s,$$

with $\frac{1}{p} + \frac{1}{r} = \frac{1}{2} = \frac{1}{q} + \frac{1}{s}$. Applying (i) one more time, we obtain

$$\|x\|_{M_n(E)} \le \sup \Big\{ \|\alpha x \beta\|_{S^n_{(p,q)}(E)} : \ \|\alpha\|_{S_p^n}, \|\beta\|_{S_q^n} \le 1 \Big\}.$$

The second identity in (ii) is immediate. Finally, we prove (iii). Let E_k be the subspace of E spanned by the entries of x_k. Since E_k is finite dimensional, we can find $\alpha_k, \beta_k \in M_m$ and $y_k \in M_m \otimes E_k$ satisfying $x_k = \alpha_k y_k \beta_k$ and

$$\|x_k\|_{S^m_{(r,s)}(E)} = \|\alpha_k\|_{S_r^m} \|\beta_k\|_{S_s^m}.$$

By homogeneity, we may assume that

$$\sum_{k=1}^n \|x_k\|^p_{S^m_{(r,s)}(E)} = \sum_{k=1}^n \|\alpha_k\|^r_{S_r^m} = \sum_{k=1}^n \|\beta_k\|^s_{S_s^m}.$$

Let us consider the block-diagonal matrices $\mathrm{D}_n(\alpha)$, $\mathrm{D}_n(y)$ and $\mathrm{D}_n(\beta)$, made up with blocks α_k, y_k and β_k $(1 \le k \le n)$ respectively. The upper estimate follows by considering the decomposition $\mathrm{D}_n(x) = \mathrm{D}_n(\alpha)\mathrm{D}_n(y)\mathrm{D}_n(\beta)$. In a similar way, taking $\rho = 2r/(r-2)$ and $\sigma = 2s/(s-2)$, the upper estimate also holds for the dual space $S^{nm}_{(r,s)}(E)^* = S^{nm}_{(\rho,\sigma)}(E^*)$. Therefore, the lower estimate follows by duality. □

2. Preliminary estimates

In this section we prove the main probabilistic estimates to study the norm of sums of independent noncommutative random variables. First a word of notation. Throughout this paper, e_{ij} and δ_k will denote the generic elements of the canonical basis of M_n and $\mathbb{C}^n$ respectively.

Lemma 2.1. *Let E be an operator space and let $\mathrm{D} : S_1^n(\ell_\infty^n(E)) \to \ell_1^n(E)$ be the mapping defined by*

$$\mathrm{D}\Big(\sum_{i,j=1}^n e_{ij} \otimes x_{ij} \Big) \longmapsto \sum_{k=1}^n \delta_k \otimes x_{kk}^k,$$

where $x_{ij}^k \in E$ stands for the k-th entry of x_{ij}. Then D is a complete contraction.

Proof. Let us consider the map $\mathrm{d} : \ell_\infty^{n^2} \to M_n$, defined by

$$\mathrm{d}\Big(\sum_{i,j=1}^n \lambda_{ij} (\delta_i \otimes \delta_j) \Big) = \sum_{k=1}^n \lambda_{kk} e_{kk}.$$

Since the diagonal projection of $\ell_\infty^{n^2}$ onto ℓ_∞^n is a complete contraction, we can use the completely isometric embedding of ℓ_∞^n into the subspace of diagonal matrices of M_n to deduce that d is a complete contraction. If $\mathcal{H}$ stands for a Hilbert space such that E embeds in $\mathcal{B}(\mathcal{H})$ completely isometrically, then we consider the mapping

$$w : \mathcal{CB}(\mathcal{B}(\mathcal{H}), \ell_\infty^n) \longrightarrow \mathcal{CB}(\ell_\infty^n(\mathcal{B}(\mathcal{H})), \ell_\infty^n(\ell_\infty^n))$$

defined by $w(\mathrm{T}) = id_{\ell_\infty^n} \otimes \mathrm{T}$. Clearly, w is a complete contraction. Therefore the linear map $v : \mathcal{CB}(\mathcal{B}(\mathcal{H}), \ell_\infty^n) \to \mathcal{CB}(\ell_\infty^n(\mathcal{B}(\mathcal{H})), M_n)$, given by $v(\mathrm{T}) = \mathrm{d} \circ w(\mathrm{T})$, is a complete contraction. Let $S_{\mathcal{H}}^1$ denote the predual of $\mathcal{B}(\mathcal{H})$. Recalling the completely isometric embeddings

$$\begin{array}{rcl} \ell_\infty^n(S_{\mathcal{H}}^1) & \hookrightarrow & \mathcal{CB}(\mathcal{B}(\mathcal{H}), \ell_\infty^n), \\ M_n(\ell_1^n(S_{\mathcal{H}}^1)) & \hookrightarrow & \mathcal{CB}(\ell_\infty^n(\mathcal{B}(\mathcal{H})), M_n), \end{array}$$

we deduce that $u : \ell_\infty^n(S_{\mathcal{H}}^1) \to M_n(\ell_1^n(S_{\mathcal{H}}^1))$, defined by the relation

$$u\Big(\sum_{k=1}^n \delta_k \otimes a_k\Big) = \sum_{k=1}^n e_{kk} \otimes (\delta_k \otimes a_k),$$

is a complete contraction. Namely, given $a \in \ell_\infty^n(S_{\mathcal{H}}^1)$ let

$$\mathrm{T}_a(x) = \sum_{k=1}^n \mathrm{tr}(a_k^{\mathrm{t}} x)\delta_k \in \mathcal{CB}(\mathcal{B}(\mathcal{H}), \ell_\infty^n).$$

Then, it can be easily checked that $u(a) = v(\mathrm{T}_a)$. Hence, u can be regarded as the restriction of v to $\ell_\infty^n(S_{\mathcal{H}}^1)$. This proves that u is a complete contraction. On the other hand, the original map D is now given by the restriction of the adjoint map $u^* : S_1^n(\ell_\infty^n(\mathcal{B}(\mathcal{H}))) \to \ell_1^n(\mathcal{B}(\mathcal{H}))$ to the subspace $S_1^n(\ell_\infty^n(E))$. □

In the following result we shall need the following description of the Haagerup tensor norm. Given an operator space E, we denote by $M_{p,q}(E)$ the space of $p \times q$ matrices with entries in E. The norm in $M_{p,q}(E)$ is given by embedding it into the upper left corner of $S_\infty^n(E)$ with $n = \max(p,q)$. Now, for any pair E_1, E_2 of operator spaces, let $x_1 \in M_{p,m}(E_1)$ and $x_2 \in M_{m,q}(E_2)$. We will denote by $x_1 \odot x_2$ the matrix x in $M_{p,q}(E_1 \otimes E_2)$ defined by

$$x(i,j) = \sum_{k=1}^m x_1(i,k) \otimes x_2(k,j).$$

Then, given a family $E_1, E_2, \ldots, E_n$ of operator spaces and given

$$x \in M_m \otimes \big(E_1 \otimes E_2 \otimes \cdots \otimes E_n\big),$$

we define the norm of x in the space $S_\infty^m\big(E_1 \otimes_h E_2 \otimes_h \cdots \otimes_h E_n\big)$ as follows

$$\|x\|_m = \inf \left\{ \prod_{k=1}^{n+1} \|x_k\|_{M_{p_k,p_{k+1}}(E_k)} \;\Big|\; p_1 = m = p_{n+1} \right\},$$

where the infimum runs over all possible decompositions

$$x = x_1 \odot x_2 \odot \cdots \odot x_n \qquad \text{with} \qquad x_k \in M_{p_k,p_{k+1}}(E_k).$$

Lemma 2.2. *Let E be an operator space and $x_1, x_2, \ldots, x_n \in M_m \otimes E$. If there are elements $a_k(r), b_k(s) \in M_m$ and $y_k(r,s) \in M_m \otimes E$ with $1 \le r \le \rho$ and $1 \le s \le \sigma$ such that*

$$x_k = \sum_{r,s} a_k(r) y_k(r,s) b_k(s)$$

holds for $1 \le k \le n$, then we have

$$\begin{aligned} &\Big\| \sum_{k=1}^n \delta_k \otimes x_k \Big\|_{L_p(\tau_m;\ell_1^n(E))} \\ &\le \Big\| \sum_{k,r} a_k(r) a_k(r)^* \Big\|_p^{1/2} \sup_k \Big\| \Big(y_k(r,s) \Big) \Big\|_\infty \Big\| \sum_{k,s} b_k(s)^* b_k(s) \Big\|_p^{1/2}. \end{aligned}$$

Proof. Let us consider the positive matrices $a, b \in M_m$ defined by

$$a = \Big(\sum_{k,r} a_k(r) a_k(r)^* \Big)^{1/2} \quad \text{and} \quad b = \Big(\sum_{k,s} b_k(s)^* b_k(s) \Big)^{1/2}.$$

Then, we can find matrices $\alpha_k(r)$ and $\beta_k(s)$ in M_m satisfying

$$\begin{array}{rcl} a_k(r) & = & a \alpha_k(r), \\ b_k(s) & = & \beta_k(s) b, \end{array}$$

and such that

$$\begin{array}{rcl} \Big\| \sum_{k,r} \alpha_k(r) \alpha_k(r)^* \Big\|_\infty & \leq & 1, \\ \Big\| \sum_{k,s} \beta_k(s)^* \beta_k(s) \Big\|_\infty & \leq & 1. \end{array}$$

Let us define, for $1 \le k \le n$, the matrices

$$z_k = \sum_{r,s} \alpha_k(r) y_k(r,s) \beta_k(s).$$

Then, by the definition of $L_p(\tau_m; \ell_1^n(E))$, we have

$$\Big\| \sum_{k=1}^n \delta_k \otimes x_k \Big\|_p \leq \Big\| \sum_{k,r} a_k(r) a_k(r)^* \Big\|_p^{1/2} \Big\| \sum_{k=1}^n \delta_k \otimes z_k \Big\|_\infty \Big\| \sum_{k,s} b_k(s)^* b_k(s) \Big\|_p^{1/2},$$

where $\| \ \|_\infty$ denotes here the norm on $S_\infty^m(\ell_1^n(E))$. Therefore, it suffices to estimate the middle term on the right. To that aim, we consider the matrices

$$\begin{array}{rcl} \alpha & = & \big(\cdots, \alpha_k(r) \otimes e_{1k}, \cdots \big) \ \in M_{m,mn\rho}(R_\infty^n), \\ \beta & = & \big(\cdots, \beta_k(s) \otimes e_{k1}, \cdots \big)^{\mathrm{t}} \in M_{mn\sigma,m}(C_\infty^n). \end{array}$$

Moreover, if $y_k = \big(y_k(r,s) \big) \in M_{m\rho,m\sigma} \otimes E$, we also consider the matrix

$$y = \sum_{k=1}^n e_{kk} \otimes (\delta_k \otimes y_k) \in M_{mn\rho,mn\sigma}(\ell_\infty^n(E)).$$

Finally, we notice that

$$\sum_{k=1}^n \delta_k \otimes z_k = (id \otimes \mathrm{D})(\alpha \odot y \odot \beta).$$

In particular, Lemma 2.1 gives

$$\Big\| \sum_{k=1}^n \delta_k \otimes z_k \Big\|_\infty \leq \Big\| \sum_{k,r} \alpha_k(r) \alpha_k(r)^* \Big\|_\infty^{1/2} \sup_k \Big\| \Big(y_k(r,s) \Big) \Big\|_\infty \Big\| \sum_{k,s} \beta_k(s)^* \beta_k(s) \Big\|_\infty^{1/2}.$$

This yields the assertion since the first and third terms on the right are ≤ 1. $\square$

Let us consider two positive integers l and n. Then, given $1 \le p \le \infty$, we define $\pi_k : L_p(\tau_l) \to L_p(\tau_{l^n})$ for each $1 \le k \le n$ to be the mapping defined by the relation

$$\pi_k(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1,$$

where x is located at the k-th position and 1 stands for the identity of M_l. These operators will appear quite frequently throughout this paper. In the following lemma, we give an upper estimate of the L_p norm of certain sums of positive matrices constructed by means of the mappings π_k.

Lemma 2.3. *Let $a^1, a^2, \ldots, a^n \in M_{ml}$ be a collection of positive matrices and let $1 \le p < \infty$. Then, we have*

$$\Big\| \sum_{k=1}^n \pi_k(a^k) \Big\|_{L_p(\tau_{ml^n})} \le cp \, \max \left\{ \Big\| \frac{1}{l} \sum_{k=1}^n \sum_{i=1}^l a_{ii}^k \Big\|_{L_p(\tau_m)}, \Big(\sum_{k=1}^n \|a^k\|_{L_p(\tau_{ml})}^p \Big)^{1/p} \right\}.$$

Here c denotes an absolute constant not depending on the dimensions.

Proof. By homogeneity, we may assume that the maximum on the right is 1. Let $\mathcal{E}$ be the conditional expectation onto $L_p(\tau_m)$, regarded as a subspace of $L_p(\tau_{ml^n})$. Let b_k stand for $\pi_k(a^k)$, by the triangle inequality

$$\Big\| \sum_{k=1}^n \pi_k(a^k) \Big\|_p \le \Big\| \sum_{k=1}^n \mathcal{E}(b_k) \Big\|_{L_p(\tau_m)} + \Big\| \sum_{k=1}^n b_k - \mathcal{E}(b_k) \Big\|_p = \mathrm{A} + \mathrm{B}.$$

Let us observe that

$$\mathcal{E}(b_k) = \tau_l(a^k) = \frac{1}{l} \sum_{i=1}^l a_{ii}^k.$$

Therefore, by assumption, the term A may be estimated by 1. To estimate B, we first assume that $p > 2$. Then, applying the Burkholder inequality given in [11] for noncommutative martingales, we obtain

$$\begin{aligned} \mathrm{B} &\le cp \, \max \left\{ \Big\| \sum_{k=1}^n \mathcal{E}(b_k^2) - \mathcal{E}(b_k)^2 \Big\|_{L_{p/2}(\tau_m)}^{1/2}, \Big(\sum_{k=1}^n \|b_k - \mathcal{E}(b_k)\|_p^p \Big)^{1/p} \right\} \\ &\le cp \, \max \left\{ \Big\| \sum_{k=1}^n \mathcal{E}(b_k^2) \Big\|_{L_{p/2}(\tau_m)}^{1/2}, \; 2 \Big(\sum_{k=1}^n \|a^k\|_{L_p(\tau_{ml})}^p \Big)^{1/p} \right\}. \end{aligned}$$

On the other hand, since $q = p/2 > 1$, we invoke Lemma 5.2 of [11] to obtain

$$\begin{aligned} \Big\| \sum_{k=1}^n \mathcal{E}(b_k^2) \Big\|_{L_{p/2}(\tau_m)} &\le \Big\| \sum_{k=1}^n \mathcal{E}(b_k) \Big\|_{L_{2q}(\tau_m)}^{\frac{2(q-1)}{2q-1}} \Big(\sum_{k=1}^n \|b_k\|_{2q}^{2q} \Big)^{\frac{1}{2q-1}} \\ &= \Big\| \sum_{k=1}^n \mathcal{E}(b_k) \Big\|_{L_{2q}(\tau_m)}^{\frac{2(q-1)}{2q-1}} \Big(\sum_{k=1}^n \|a^k\|_{L_p(\tau_{ml})}^p \Big)^{\frac{1}{2q-1}} \end{aligned}$$

The first factor on the right is a power of A. By assumption, the second factor may be estimated by 1. When $1 \le p \le 2$, we proceed in a different way. Given a subset Γ of $\{1, 2, \ldots, n\}$ with cardinal $|\Gamma|$, let us consider the conditional expectation $\mathcal{E}_\Gamma$ onto $L_p(\tau_{ml^{|\Gamma|}})$ given by

$$\mathcal{E}_\Gamma \Big(z_0 \otimes \bigotimes_{k=1}^n z_k \Big) = \prod_{k \notin \Gamma} \tau_l(z_k) \Big(z_0 \otimes \bigotimes_{k \in \Gamma} z_k \Big),$$

with $z_0 \in M_m$ and $z_k \in M_l$ for $1 \le k \le n$. Therefore, since $x_k = b_k - \mathcal{E}(b_k)$ are independent mean 0 random variables, we can estimate B as follows. For any family of signs $\varepsilon_k = \pm 1$ with $1 \le k \le n$, let $\Gamma = \{k : \varepsilon_k = 1\}$. Then we have

$$\Big\| \sum_{k=1}^n x_k \Big\|_p \le \Big\| \mathcal{E}_\Gamma \Big(\sum_{k=1}^n \varepsilon_k x_k \Big) \Big\|_p + \Big\| \mathcal{E}_{\Gamma^c} \Big(\sum_{k=1}^n \varepsilon_k x_k \Big) \Big\|_p \le 2 \Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\|_p.$$

Then, if we write $\mathrm{r}_1, \mathrm{r}_2, \ldots$ for the classical Rademacher variables, we use the fact that $L_p(\tau_{ml^n})$ has Rademacher type p to obtain

$$\begin{aligned} \mathrm{B} = \Big\| \sum_{k=1}^n x_k \Big\|_p & \le 2 \Big(\int_0^1 \Big\| \sum_{k=1}^n \mathrm{r}_k(t) x_k \Big\|_p^2 dt \Big)^{1/2} \\ & \le 2 \Big(\sum_{k=1}^n \|x_k\|_p^p \Big)^{1/p} \\ & \le 4 \Big(\sum_{k=1}^n \|a^k\|_{L_p(\tau_{ml})}^p \Big)^{1/p}. \end{aligned}$$

This yields the assertion for $1 \le p \le 2$. Therefore, the proof is completed. □

3. Proof of noncommutative (Σ_p)

In this section, we prove the noncommutative analog of the equivalence of norms (Σ_p) described in the Introduction. However, before that we need to set some notation. Let E and F be operator spaces such that (E, F) is a compatible pair for interpolation. In what follows, we shall denote by $J_t(E, F)$ and $K_t(E, F)$ the J and K functionals on (E, F) endowed with their natural operator space structures as defined in [27]. Moreover, given $2 \le r, s \le \infty$, we shall write $\ell_{(r,s)}^n(E)$ to denote the linear space E^n endowed with the operator space structure which arises from the natural identification with the diagonal matrices of $S_{(r,s)}^n(E)$. Then, given $1 \le p, q \le \infty$, we use these spaces to define the operator space

$$J_{p,q}^n(M_l; E) = \bigcap_{r,s \in \{2p, 2q\}} \ell_{(r,s)}^n \big(L_{(r,s)}(\tau_l; E) \big).$$

Lemma 3.1. *Let $1 \le p, q \le \infty$ and $\lambda_p = l^{-1/p}$. Then, given $t = l^{\frac{1}{2p} - \frac{1}{2q}}$, we consider the mapping $u : J_{p,q}^n(M_l; E) \to J_t(C_p^{nl}, C_q^{nl}) \otimes_h E \otimes_h J_t(R_p^{nl}, R_q^{nl})$, defined by the relation*

$$u \Big(\sum_{k=1}^n \delta_k \otimes x_k \Big) = \lambda_p \sum_{k=1}^n e_{kk} \otimes x_k.$$

Then, u is a complete isometry with completely contractively complemented image.

Proof. By the injectivity of the Haagerup tensor product, it can be checked that

$$J_t(C_p^{nl}, C_q^{nl}) \otimes_h E \otimes_h J_t(R_p^{nl}, R_q^{nl}) = \bigcap_{r,s \in \{2p, 2q\}} \lambda_p^{-1} L_{(r,s)}(\mathrm{tr}_n \otimes \tau_l; E).$$

Taking diagonals at both sides, the first assertion follows. In order to see that the diagonal is completely contractively complemented, we use the standard diagonal projection

$$\mathrm{P}((x_{ij})) = \int_{\{-1,1\}^n} (\varepsilon_i x_{ij} \varepsilon_j) \, d\mu(\varepsilon),$$

where μ is the normalized counting measure on $\{-1, 1\}^n$. Here $x = (x_{ij})$ is a $n \times n$ matrix with entries in $M_l \otimes E$. By means of the second identity of Lemma 1.2 (ii), it clearly suffices to check that P is a contraction on $L_{(r,s)}(\mathrm{tr}_n \otimes \tau_l; E)$. For each $\varepsilon \in \{-1, 1\}^n$, we consider the matrix $a_\varepsilon = (\varepsilon_i \delta_{ij})$. Then we have

$$\|\mathrm{P}(x)\|_{(r,s)} = \Big\| 2^{-n} \sum_\varepsilon a_\varepsilon x a_\varepsilon \Big\|_{(r,s)} \le 2^{-n} \sum_\varepsilon \|a_\varepsilon x a_\varepsilon\|_{(r,s)} = \|x\|_{(r,s)},$$

since a_ε is unitary for any $\varepsilon \in \{-1,1\}^n$. This completes the proof. □

Proposition 3.2. *Let E be an operator space and let $1 \le p < \infty$. Then*

$$\Lambda_{p1} : \sum_{k=1}^n \delta_k \otimes x_k \in J_{p,1}^n(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{l^n}; \ell_1^n(E))$$

is a completely bounded map with $\|\Lambda_{p1}\|_{cb} \le cp$, where c is independent of l and n.

Proof. Given $t = l^{\frac{1}{2p}-\frac{1}{2}}$, if we regard $J_t(C_p^{nl}, R_\infty^{nl}) \otimes_h E \otimes_h J_t(R_p^{nl}, C_\infty^{nl})$ as a space of $n \times n$ matrices with entries in $M_l \otimes E$, then we consider its diagonal subspace $\mathcal{J}_{p,1}^n(M_l; E)$. By Lemma 3.1, it suffices to check that the mapping

$$\widetilde{\Lambda}_{p1} : \sum_{k=1}^n e_{kk} \otimes x_k \in \mathcal{J}_{p,1}^n(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{l^n}; \ell_1^n(E)),$$

satisfies $\|\widetilde{\Lambda}_{p1}\|_{cb} \le cp\,\lambda_p$. Given $m \ge 1$, let us consider $x \in S_p^m(\mathcal{J}_{p,1}^n(M_l; E))$ of norm less than one. Let us consider the spaces

$$\begin{aligned} F_1 &= C_p^m \otimes_h J_t(C_p^{nl}, R_\infty^{nl}) \\ F_2 &= J_t(R_p^{nl}, C_\infty^{nl}) \otimes_h R_p^m. \end{aligned}$$

Since the space $S_p^m(\mathcal{J}_{p,1}^n(M_l; E))$ embeds completely isometrically in $F_1 \otimes_h E \otimes_h F_2$, we can write $x = a \odot y \odot b$, with $a \in M_{1,mln}(F_1)$, $y \in M_{mln}(E)$, $b \in M_{mln,1}(F_2)$ so that $\|y\|_{M_{mnl}(E)} < 1$ and

$$\begin{aligned} \max\Big\{ \|a\|_{S_{2p}^{mln}},\, t\|a\|_{C_p^m \otimes_h R_\infty^{ml^2n^2}} \Big\} &< 1 \\ \max\Big\{ \|b\|_{S_{2p}^{mln}},\, t\|b\|_{C_\infty^{ml^2n^2} \otimes_h R_p^m} \Big\} &< 1. \end{aligned}$$

Now, if we write $a = (a_{ij})$, $y = (y_{ij})$ and $b = (b_{ij})$ as $n \times n$ matrices of $ml \times ml$ matrices, we have

$$x_k = \sum_{i,j=1}^n a_{ki} y_{ij} b_{jk} \quad \text{where} \quad x = \sum_{k=1}^n e_{kk} \otimes x_k.$$

Therefore, we have

$$\widetilde{\Lambda}_{p1}(x) = \sum_{k=1}^n \sum_{i,j=1}^n \delta_k \otimes \pi_k(a_{ki})\pi_k(y_{ij})\pi_k(b_{jk}).$$

According to Lemma 2.2, we deduce

$$\big\|\widetilde{\Lambda}_{p1}(x)\big\|_p \le \Big\| \sum_{k,i} \pi_k(a_{ki}a_{ki}^*) \Big\|_p^{1/2} \sup_k \Big\| \Big(\pi_k(y_{ij}) \Big) \Big\|_\infty \Big\| \sum_{k,j} \pi_k(b_{jk}^* b_{jk}) \Big\|_p^{1/2},$$

where $\|\widetilde{\Lambda}_{p1}(x)\|_p$ stands for the norm of $\widetilde{\Lambda}_{p1}(x)$ in $S_p^m(L_p(\tau_{l^n}; \ell_1^n(E)))$. As we know, the middle term on the right is bounded above by 1. On the other hand, Lemma 2.3 allows us to write

$$\begin{aligned} &\Big\| \sum_{k,i} \pi_k(a_{ki}a_{ki}^*) \Big\|_{L_p(\mathrm{tr}_m \otimes \tau_{l^n})} \\ &\le cp \max\left\{ \Big\| \sum_{k,i} \mathcal{E}(a_{ki}a_{ki}^*) \Big\|_{S_p^m},\, \Big(\sum_{k=1}^n \Big\| \sum_{i=1}^n a_{ki}a_{ki}^* \Big\|_{L_p(\mathrm{tr}_m \otimes \tau_l)}^p \Big)^{1/p} \right\} \end{aligned}$$

$$\leq \quad cp \max \left\{ \frac{1}{l} \Big\| \sum_{r,s=1}^{l} \sum_{k,i} a_{ki}(r,s) a_{ki}(r,s)^* \Big\|_{S_p^m}, \ \lambda_p \|aa^*\|_{S_p^{nml}} \right\}.$$

The last inequality follows from the fact that the projection onto block diagonal matrices is completely contractive, see Corollary 1.3 of [22]. Now, recalling that

$$\Big\| \sum_{r,s=1}^{l} \sum_{k,i} a_{ki}(r,s) a_{ki}(r,s)^* \Big\|_{S_p^m} = \|a\|^2_{C_p^m \otimes_h R_\infty^{ml^2n^2}},$$

we obtain

$$\Big\| \sum_{k,i} \pi_k (a_{ki} a_{ki}^*) \Big\|_{L_p(\mathrm{tr}_m \otimes \tau_{l^n})} \leq cp\lambda_p.$$

Since the same estimate holds for b, we get the desired estimate for $\|\widetilde{\Lambda}_{p1}\|_{cb}$. □

Proposition 3.2 provides an upper estimate for the norm of sums of independent noncommutative random variables in $L_p(\ell_1^n(E))$. Now, we are interested on the dual version of this result. Hence, it is natural to consider the operator spaces $K_{p,q}^n(M_l;E)$, which arise when replacing intersections by sums in $J_{p,q}^n(M_l;E)$. That is, we define

$$K_{p,q}^n(M_l;E) = \sum_{r,s \in \{2p,2q\}} \ell_{(r,s)}^n \big(L_{(r,s)}(\tau_l;E) \big).$$

Remark 3.3. Arguing as in Lemma 3.1, we can regard $K_{p,q}^n(M_l;E)$ as the diagonal in the Haagerup tensor product $K_t(C_p^{nl}, C_q^{nl}) \otimes_h E \otimes_h K_t(R_p^{nl}, R_q^{nl})$ normalized by $\lambda_p = l^{-1/p}$. The projection P onto the diagonal is also a complete contraction.

Lemma 3.4. *Let $1 \leq p \leq q \leq \infty$ and let E be an operator space. Then, given a positive integer $n \geq 1$, the following identity maps are complete contractions*

$$id : \ell_{(2p,2q)}^n(E) \longrightarrow \ell_q^n(E)$$
$$id : \ell_{(2q,2p)}^n(E) \longrightarrow \ell_q^n(E).$$

Proof. Given $2 \leq r,s \leq \infty$, we can argue as in Lemma 3.1 to see that the diagonal projection $\mathrm{P} : S_{(r,s)}^n(E) \to \ell_{(r,s)}^n(E)$ is a complete contraction. Therefore, the complex interpolation space between the diagonals of two asymmetric Schatten classes is the diagonal of the interpolated asymmetric Schatten class. That is, we have the following complete isometries

$$\ell_{(2p,2q)}^n(E) = \big[\ell_q^n(E), \ell_{(2,2q)}^n(E) \big]_\theta \qquad \ell_{(2,2q)}^n(E) = \big[\ell_{(2,\infty)}^n(E), \ell_1^n(E) \big]_\gamma$$
$$\ell_{(2q,2p)}^n(E) = \big[\ell_q^n(E), \ell_{(2q,2)}^n(E) \big]_\theta \qquad \ell_{(2q,2)}^n(E) = \big[\ell_{(\infty,2)}^n(E), \ell_1^n(E) \big]_\gamma$$

completely isometrically for $\frac{1}{2p} = \frac{1-\theta}{2q} + \frac{\theta}{2}$ and $\gamma = 1/q$. Hence, it suffices to show the result for $p = 1$ and $q = \infty$. That is, we have to see that the identity mappings

$$id : \ell_{(2,\infty)}^n(E) \longrightarrow \ell_\infty^n(E)$$
$$id : \ell_{(\infty,2)}^n(E) \longrightarrow \ell_\infty^n(E),$$

are complete contractions. In other words, we have to consider the diagonals of $R_\infty^n \otimes_h E \otimes_h R_\infty^n$ and $C_\infty^n \otimes_h E \otimes_h C_\infty^n$. However, we recall the completely isometric isomorphisms $E \otimes_h R_\infty^n = E \otimes_{\min} R_\infty^n$ and $C_\infty^n \otimes_h E = C_\infty^n \otimes_{\min} E$ and the complete contractions

$$R_\infty^n \otimes_h (E \otimes_{\min} R_\infty^n) \longrightarrow R_\infty^n \otimes_{\min} (E \otimes_{\min} R_\infty^n),$$
$$(C_\infty^n \otimes_{\min} E) \otimes_h C_\infty^n \longrightarrow (C_\infty^n \otimes_{\min} E) \otimes_{\min} C_\infty^n.$$

Hence, it suffices to show our claim for the diagonals of $R_\infty^{n^2} \otimes_{\min} E$ and $C_\infty^{n^2} \otimes_{\min} E$. In the first case the diagonal is $R_\infty^n \otimes_{\min} E$ while in the second case is $C_\infty^n \otimes_{\min} E$. By the injectivity of the minimal tensor product and since ℓ_∞^n carries the minimal operator space structure, the assertion follows. This completes the proof. □

The following result can be regarded as the dual version of Proposition 3.2, where the spaces $J_{p,q}^n(M_l; E)$ are replaced by the spaces $K_{p,q}^n(M_l; E)$. Here we skip the assumption that $q = 1$ and we work in the range $1 \le p \le q \le \infty$.

Proposition 3.5. *Let E be an operator space and let $1 \le p \le q \le \infty$. Then, the following map is a complete contraction*

$$\Lambda_{pq} : \sum_{k=1}^n \delta_k \otimes x_k \in K_{p,q}^n(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{l^n}; \ell_q^n(E)).$$

Proof. Let $t = l^{\frac{1}{2p} - \frac{1}{2q}}$, regarding again $K_t(C_p^{nl}, C_q^{nl}) \otimes_h E \otimes_h K_t(R_p^{nl}, R_q^{nl})$ as a space of $n \times n$ matrices with entries in $M_l \otimes E$, we consider its diagonal subspace $\mathcal{K}_{p,q}^n(M_l; E)$. By Remark 3.3, it suffices to check that the mapping

$$\widetilde{\Lambda}_{pq} : \sum_{k=1}^n e_{kk} \otimes x_k \in \mathcal{K}_{p,q}^n(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{l^n}; \ell_q^n(E)),$$

satisfies $\|\widetilde{\Lambda}_{pq}\|_{cb} \le \lambda_p$. Since the diagonal projection P is a complete contraction, it suffices to prove this estimate for the diagonal in each of the following spaces

$$C_p^{nl} \otimes_h E \otimes_h R_p^{nl} \ , \ tC_p^{nl} \otimes_h E \otimes_h R_q^{nl} \ , \ tC_q^{nl} \otimes_h E \otimes_h R_p^{nl} \ , \ t^2 C_q^{nl} \otimes_h E \otimes_h R_q^{nl}.$$

Notice that, given a scalar γ and an operator space F, we denote by γF the operator space with operator space structure given by

$$\|f\|_{M_m \otimes_{\min} \gamma F} = \gamma \|f\|_{M_m \otimes_{\min} F}.$$

The first one is

$$\big\|\widetilde{\Lambda}_{pq} : \ell_p^n(S_p^l(E)) \to L_p(\tau_{l^n}; \ell_q^n(E))\big\|_{cb} \le \lambda_p.$$

This estimate obviously holds for $p = q$ and, since the identity map $\ell_p^n(E) \to \ell_q^n(E)$ is a complete contraction, the desired estimate follows. For the last one, we note that

$$\big\|\widetilde{\Lambda}_{pq} : t^2 \ell_q^n(S_q^l(E)) \to L_q(\tau_{l^n}; \ell_q^n(E))\big\|_{cb} \le \lambda_p.$$

Moreover, since we are using a probability measure, we know that the identity map $L_q(\tau_m; F) \to L_p(\tau_m; F)$ is a complete contraction. Therefore, the desired estimate for the last case holds. For the second and third terms, we use a similar trick. We claim that the identity mappings

$$\begin{array}{rcl} L_{(2p,2q)}(\tau_m; F_1) & \longrightarrow & L_p(\tau_m; F_1) \\ L_{(2q,2p)}(\tau_m; F_2) & \longrightarrow & L_p(\tau_m; F_2) \end{array}$$

are complete contractions. Namely, by complex interpolation it reduces to the case $p = 1$ and $q = \infty$. However, if we rescale these mappings to replace τ_m by tr_m, this case follows easily by the injectivity of the Haagerup tensor product and the well-know estimates

$$\big\|id : R_\infty^m \to C_\infty^m\big\|_{cb} \le \sqrt{m} \quad \text{and} \quad \big\|id : C_\infty^m \to R_\infty^m\big\|_{cb} \le \sqrt{m}.$$

We take $m = l^n$ and the operator spaces $F_1 = \ell^n_{(2p,2q)}(E)$ and $F_2 = \ell^n_{(2q,2p)}(E)$. According to Lemma 3.4, it suffices to prove the following estimates

$$\begin{aligned}\big\|\widetilde{\Lambda}_{pq} : t\,\ell^n_{(2p,2q)}\big(S^l_{(2p,2q)}(E)\big) \to L_{(2p,2q)}(\tau_{l^n}; \ell^n_{(2p,2q)}(E))\big\|_{cb} &\le \lambda_p \\ \big\|\widetilde{\Lambda}_{pq} : t\,\ell^n_{(2q,2p)}\big(S^l_{(2q,2p)}(E)\big) \to L_{(2q,2p)}(\tau_{l^n}; \ell^n_{(2q,2p)}(E))\big\|_{cb} &\le \lambda_p.\end{aligned}$$

Since both estimates can be treated in a similar way, we just prove the first one. Given a positive integer m, let us consider a diagonal matrix

$$x = \sum_{k=1}^n e_{kk} \otimes x_k \in M_n \otimes S^{ml}_{(2p,2q)}(E).$$

According to Lemma 1.2 (iii), the following identities hold for $\frac{1}{r} = \frac{1}{2p} + \frac{1}{2q}$

$$\begin{aligned}\big\|\widetilde{\Lambda}_{pq}(x)\big\|_{(2p,2q)} &= \Big(\sum_{k=1}^n \|\pi_k(x_k)\|^r_{L_{(2p,2q)}(\mathrm{tr}_m\otimes\tau_{l^n};E)}\Big)^{1/r} \\ &= \Big(\sum_{k=1}^n \|x_k\|^r_{L_{(2p,2q)}(\mathrm{tr}_m\otimes\tau_l;E)}\Big)^{1/r} \\ &= \lambda_p\, t\|x\|_{\ell^n_{(2p,2q)}(S^{ml}_{(2p,2q)}(E))},\end{aligned}$$

where $\| \;\|_{(2p,2q)}$ denotes the norm on the space $L_{(2p,2q)}(\mathrm{tr}_m \otimes \tau_{l^n}; \ell^n_{(2p,2q)}(E))$. Thus, applying the second identity of Lemma 1.2 (ii), the assertion follows. $\square$

Once we have seen the estimates for intersections and sums given in Propositions 3.2 and 3.5, we are in position to prove the complete equivalence of norms (Σ_p) for sums of independent noncommutative random variables in $L_p(\ell_1(E))$.

Theorem 3.6. *Let E be an operator space and let $1 \le p < \infty$. Then, the map*

$$\Lambda_{p1} : \sum_{k=1}^n \delta_k \otimes x_k \in J^n_{p,1}(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_p(\tau_{l^n}; \ell^n_1(E))$$

is a complete isomorphism onto a completely complemented subspace. Similarly, the same holds for the map

$$\Lambda_{p'\infty} : \sum_{k=1}^n \delta_k \otimes x_k \in K^n_{p',\infty}(M_l; E) \longmapsto \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \in L_{p'}(\tau_{l^n}; \ell^n_\infty(E))$$

for $1 < p' \le \infty$. Moreover, the cb-distance constants are independent of l and n.

Proof. It is clear that we can assume E to be a finite-dimensional operator space. In particular, all the spaces we shall consider along the proof will be of finite dimension and hence reflexive. Now the duality theory for the Haagerup tensor product, see for instance the Chapter 5 of [23], provides a complete isometry

$$\mathrm{S} : \big(J_t(C^{nl}_p, C^{nl}_1) \otimes_h E \otimes_h J_t(R^{nl}_p, R^{nl}_1)\big)^* \to K_{t^{-1}}(C^{nl}_{p'}, C^{nl}_\infty) \otimes_h E^* \otimes_h K_{t^{-1}}(R^{nl}_{p'}, R^{nl}_\infty).$$

On the other hand, according to Lemma 3.1 and Remark 3.3, the projection onto the diagonal is always a complete contraction. Therefore, we obtain the following completely isometric isomorphism

$$J^n_{p,1}(M_l; E)^* = K^n_{p',\infty}(M_l; E^*).$$

Indeed, if P denotes the diagonal projection and $\mathrm{T} = u^{-1} \circ \mathrm{P}$ where u stands for the linear mapping considered in Lemma 3.1, then the mapping

$$\mathrm{T} \circ \mathrm{S} \circ \mathrm{T}^* : J_{p,1}^n(M_l; E)^* \longrightarrow K_{p',\infty}^n(M_l; E^*)$$

is a completely isometric isomorphism. Here, the duality is given by

$$\langle a, b \rangle = \Big\langle \sum_{k=1}^n \delta_k \otimes (a_k \otimes e_k), \sum_{k=1}^n \delta_k \otimes (b_k \otimes e_k^*) \Big\rangle = \sum_{k=1}^n \tau_l(a_k^{\mathrm{t}} b_k) \langle e_k, e_k^* \rangle.$$

Thus, we obviously have

$$\langle \Lambda_{p1}(a), \Lambda_{p'\infty}(b) \rangle = \langle a, b \rangle \qquad \forall \ a \in J_{p,1}^n(M_l; E), \ b \in K_{p',\infty}^n(M_l; E^*).$$

Consequently, the map $\Lambda_{p'\infty}^* \Lambda_{p1}$ is the identity on $J_{p,1}^n(M_l; E)$. In particular, by Propositions 3.2 and 3.5, Λ_{p1} becomes a complete isomorphism with constants not depending on the dimensions. Moreover, its image is a completely complemented subspace since $\Lambda_{p1} \Lambda_{p'\infty}^*$ is a completely bounded projection with $\|\Lambda_{p1} \Lambda_{p'\infty}^*\|_{cb} \le cp$. This proves the assertions for Λ_{p1}, but the arguments for $\Lambda_{p'\infty}$ are similar. □

Remark 3.7. Let us state Theorem 3.6 in a more explicit way. To that aim, we introduce some notation. If $\frac{1}{\gamma_{rs}} = \frac{1}{r} + \frac{1}{s}$, we define

$$\|x\|_{p,q}^{\cap} = \max_{r,s \in \{2p,2q\}} \Big\{ \Big(\sum_{k=1}^n \|x_k\|_{L_{(r,s)}(\tau_l;E)}^{\gamma_{rs}} \Big)^{1/\gamma_{rs}} \Big\},$$

$$\|x\|_{p,q}^{\Sigma} = \inf_{x = \sum_{r,s} x_{rs}} \Big\{ \sum_{r,s} \Big(\sum_{k=1}^n \|x_{rs}^k\|_{L_{(r,s)}(\tau_l;E)}^{\gamma_{rs}} \Big)^{1/\gamma_{rs}} \ \Big| \ r,s \in \{2p,2q\} \Big\}.$$

Then, recalling the meaning of $\le_{cb}$ from the Introduction, we have

- Given $1 \le p < \infty$, we have
$$\|x\|_{p,1}^{\cap} \le_{cb} \Big\| \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \Big\|_{L_p(\ell_1^n(E))} \le_{cb} cp \ \|x\|_{p,1}^{\cap}.$$
- Given $1 < p' \le \infty$, we have
$$\frac{1}{cp} \ \|x\|_{p',\infty}^{\Sigma} \le_{cb} \Big\| \sum_{k=1}^n \delta_k \otimes \pi_k(x_k) \Big\|_{L_{p'}(\ell_\infty^n(E))} \le_{cb} \|x\|_{p',\infty}^{\Sigma}.$$

4. A cb embedding of S_q^n into $S_p(\ell_q^m)$

We begin by stating a complementation result for the subspace of $J_{p,q}^n(M_l; E)$ given by constant diagonal matrices. As we shall see immediately, this result plays a relevant role in the embeddings we want to consider.

Lemma 4.1. *Let $1 \le p, q \le \infty$ and let $t = \big(\frac{n}{l}\big)^{\frac{1}{2p} - \frac{1}{2q}}$ with l and n positive integers. Then, the map*

$$\mathrm{T} : J_t(C_q^l, C_p^l) \otimes_h E \otimes_h J_t(R_q^l, R_p^l) \longrightarrow J_{p,q}^n(M_l; E)$$

defined by

$$\mathrm{T}(x) = \Big(\frac{n}{l}\Big)^{-1/q} \Big(\sum_{k=1}^n e_{kk} \otimes x \Big)$$

is a complete isometry. The image of T *is completely contractively complemented.*

Proof. To see that the image of T is completely contractively complemented in $J_{p,q}^n(M_l;E)$, we consider the following projection

$$\mathrm{P}(x_1,x_2,\ldots,x_n)=\Big(\frac{1}{n}\sum_{k=1}^n x_k,\frac{1}{n}\sum_{k=1}^n x_k,\ldots,\frac{1}{n}\sum_{k=1}^n x_k\Big).$$

Then, it suffices to see that P is a complete contraction in

$$\ell_{(r,s)}^n(L_{(r,s)}(\tau_l;E))$$

whenever $r,s\in\{2p,2q\}$. It is clear that, given any operator space E, the projection P is contractive in these four spaces. Then, the complete contractivity follows easily from Lemma 1.2 (ii) and the obvious Fubini type results. Now, given $r,s\in\{2p,2q\}$, let $\xi_{rs}=\delta_{r,2p}+\delta_{s,2p}$. To see that T is a complete isometry, it suffices to check that

$$\mathrm{T}:t^{\xi_{rs}}S_{(r,s)}^l(E)\longrightarrow\ell_{(r,s)}^n(L_{(r,s)}(\tau_l;E))$$

is a complete isometry for any $r,s\in\{2p,2q\}$. However, this follows one more time as a consequence of Lemma 1.2 (ii) and (iii). □

The following theorem provides an embedding of the Schatten class $S_q^n(E)$ into $L_p(\mathcal{M},\tau;\ell_q^m(E))$ with uniformly bounded *cb*-distance constants.

Theorem 4.2. *Let $1\le q\le p<\infty$. Then, given any positive integer $n\ge 1$ and any operator space E, the following mapping is a complete isomorphism onto a completely complemented subspace*

$$\Phi_{pq}:x\in S_q^n(E)\longmapsto\frac{1}{n^{1/q}}\sum_{k=1}^{n^2}\delta_k\otimes\pi_k(x)\in L_p(\tau_{n^{n^2}};\ell_q^{n^2}(E)).$$

Moreover, $\|\Phi_{pq}\|_{cb}\le cp$ while the inverse mapping Φ_{pq}^{-1} is completely contractive.

Proof. By Lemma 1.1, $J_t(C_1^n,C_p^n)=R_\infty^n$ and $J_t(R_1^n,R_p^n)=C_\infty^n$ for $t=n^{\frac{1}{2p}-\frac{1}{2}}$. In particular, we can write

$$S_1^n(E)=J_t(C_1^n,C_p^n)\otimes_h E\otimes_h J_t(R_1^n,R_p^n).$$

Then, Proposition 3.2 and Lemma 4.1 give that

$$\Phi_{p1}:S_1^n(E)\longrightarrow L_p(\tau_{n^{n^2}};\ell_1^{n^2}(E))$$

is a *cb* embedding with $\|\Phi_{p1}\|_{cb}\le cp$. That is, the upper estimate holds for $q=1$. On the other hand, the map

$$\Phi_{pp}:S_p^n(E)\longrightarrow L_p(\tau_{n^{n^2}};\ell_p^{n^2}(E))$$

is clearly a complete isometry. Hence, for the general case, the upper estimate follows by complex interpolation. In order to see that the image of the mapping Φ_{pq} is completely complemented and Φ_{pq}^{-1} is completely contractive, we observe again that, by elementary properties of the local theory, we have

$$S_{q'}^n(E^*)=\big(J_t(C_q^n,C_p^n)\otimes_h E\otimes_h J_t(R_q^n,R_p^n)\big)^*$$

for $t=n^{\frac{1}{2p}-\frac{1}{2q}}$. Thus, if $\mathcal{C}_{p',q'}^{n^2}(M_n;E^*)$ stands for the subspace of $K_{p',q'}^{n^2}(M_n;E^*)$ of constant diagonals, Lemma 4.1 and duality give

$$S_{q'}^n(E^*)=\frac{1}{n^{1/q}}\,\mathcal{C}_{p',q'}^{n^2}(M_n;E^*).$$

In particular, Proposition 3.5 gives that

$$\Phi_{p'q'} : S_{q'}^n(E^*) \longrightarrow L_{p'}(\tau_{n^{n^2}}; \ell_{q'}^{n^2}(E^*))$$

is completely contractive. Finally, we observe that

$$\big\langle \Phi_{pq}(a \otimes e), \Phi_{p'q'}(b \otimes e^*) \big\rangle = \frac{1}{n} \sum_{k=1}^{n^2} \tau_n(a^t b) \langle e, e^* \rangle = \langle a \otimes e, b \otimes e^* \rangle.$$

Hence, since $\Phi_{p'q'}^* \Phi_{pq}$ is the identity and $\Phi_{pq} \Phi_{p'q'}^*$ is a projection, we are done. □

Remark 4.3. By simple dual arguments, it is not difficult to check that Theorem 4.2 holds for $1 < p \le q \le \infty$, with Φ_{pq} completely contractive and $\|\Phi_{pq}^{-1}\|_{cb} \le cp$. Namely, we first recall that

$$S_\infty^n(E) = K_t(C_\infty^n, C_p^n) \otimes_h E \otimes_h K_t(R_\infty^n, R_p^n) \qquad \text{for} \quad t = n^{\frac{1}{2p}}.$$

Then, by Theorem 3.6 and the dual version of Lemma 4.1 for the K functional, the complete contractivity of $\Phi_{p\infty}$ holds. Finally, we end by interpolation and duality.

Remark 4.4. Rescaling Theorem 4.2, we get an embedding $\Psi_{pq} : S_q^n \to S_p(\ell_q^m)$. In fact, we have taken m to be n^2. As we shall see in Section 6, when seeking for *cb*-embeddings with uniformly bounded constants, the choice $m = n^2$ is optimal.

5. K-convex operator spaces

The theory of type and cotype is essential to study some geometric properties of Banach spaces. The operator space analog of that theory has been recently initiated in some works summarized in [19]. The aim of this section is to explore the relation between B-convexity and K-convexity in the category of operator spaces.

5.1. **A variant of the embedding theorem.** In this paragraph, we study the inverse of Φ_{pq} when we impose on ℓ_q its minimal operator space structure. The resulting mapping will the key in the operator space analog of Pisier's equivalence between B-convex and K-convex spaces.

Lemma 5.1. *The following map extends to an anti-linear isometry*

$$\mathrm{T} : \sum_{k=1}^n a_k \otimes e_k \in L_p(\tau_n; \min(E)) \longmapsto \sum_{k=1}^n a_k^* \otimes \overline{e}_k \in L_p(\tau_n; \overline{\min(E)}).$$

Here, $\overline{\min(E)}$ *stands for the complex conjugate operator space as defined in* [23].

Proof. Since $\min(E)$ embeds completely isometrically in ℓ_∞, we take E to be ℓ_∞. Under this assumption, the result is clear for $p = \infty$. Namely, given $x = (x_n)_{n \ge 1}$ in $L_\infty(\tau_n; \ell_\infty)$, we have

$$\|\mathrm{T}(x)\| = \sup_{n \ge 1} \|x_n^*\| = \|x\|.$$

Now, if $x \in L_p(\tau_n; \ell_\infty)$, there exist $a, b \in L_{2p}(\tau_n)$ and $y \in L_\infty(\tau_n; \ell_\infty)$ such that $x = ayb$ and

$$\|a\|_{2p} \|y\|_\infty \|b\|_{2p} < (1 + \varepsilon) \|x\|.$$

Therefore

$$\|\mathrm{T}(x)\| \le \|b^*\|_{2p} \|y\|_\infty \|a^*\|_{2p} < (1 + \varepsilon) \|x\|.$$

Since $\varepsilon > 0$ is arbitrary, the assertion follows easily. This completes the proof. □

Let us consider the operator space $\widetilde{\mathsf{F}}_{pq}^n$ defined as the image of S_q^n under the map Φ_{pq}, with the operator space structure inherited from

$$L_p(\tau_{n^2}; \min(\ell_q^{n^2})).$$

Proposition 5.2. *The estimate* $\|\Phi_{pq}^{-1}\|_{\mathcal{B}(\widetilde{\mathsf{F}}_{pq}^n, S_q^n)} \le 2$ *holds for any* $n \ge 1$.

Proof. We first consider a self-adjoint matrix x in S_q^n. Then taking $m = n^2$, the sequence $\pi_1(x), \pi_2(x), \ldots, \pi_m(x)$ lies in a commutative subalgebra of M_{n^m}. In fact, using the spectral theorem, we can write $x = u^* \mathrm{d}_\lambda u$ where d_λ stands for the matrix of eigenvalues of x and u is unitary. In particular, after multiplication by $u^{\otimes m}$ from the left and by $(u^*)^{\otimes m}$ from the right, we may assume that

$$\sum_{k=1}^m \delta_k \otimes \pi_k(x)$$

is a diagonal matrix. In that case, we may apply Corollary 1.3 of [22] to obtain

$$\Big\| \sum_{k=1}^m \delta_k \otimes \pi_k(x) \Big\|_{L_p(\tau_{n^m}; \min(\ell_q^m))} = \Big\| \sum_{k=1}^m \delta_k \otimes \pi_k(x) \Big\|_{L_p(\tau_{n^m}; \ell_q^m)}.$$

Therefore, Theorem 4.2 gives

$$\|x\|_{S_q^n} \le \|\Phi_{pq}(x)\|_{\widetilde{\mathsf{F}}_{pq}^n}.$$

For arbitrary x, we consider its decomposition into self-adjoint elements

$$a = \frac{1}{2}(x + x^*) \quad \text{and} \quad b = \frac{1}{2i}(x - x^*).$$

Then, we deduce from Lemma 5.1 that

$$\|\Phi_{pq}(a)\|_{\widetilde{\mathsf{F}}_{pq}^n} \le \frac{1}{2}\|\Phi_{pq}(x)\|_{\widetilde{\mathsf{F}}_{pq}^n} + \frac{1}{2}\|\Phi_{pq}(x)^*\|_{\widetilde{\mathsf{F}}_{pq}^n} \le \|\Phi_{pq}(x)\|_{\widetilde{\mathsf{F}}_{pq}^n}.$$

Obviously, the same estimate holds for b. Thus, we obtain the desired estimate. □

Let us consider an infinite dimensional operator space E and a family of finite dimensional operator spaces $\mathcal{A} = \{A_n \mid n \ge 1\}$. We shall say that the family $\mathcal{A}$ *embeds semi-completely uniformly* in E, and we shall write $\mathcal{A} \prec E$, when there exists a constant c and embeddings $\Lambda_n : A_n \to E$ such that

$$\|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \le \mathrm{c} \qquad \text{for all} \qquad n \ge 1.$$

Corollary 5.3. *Let* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then, we have*

$$\Big\{ \ell_q^n \;\Big|\; n \ge 1 \Big\} \prec S_p(E) \Rightarrow \Big\{ S_q^n \;\Big|\; n \ge 1 \Big\} \prec S_p(E).$$

Proof. By hypothesis, there exist $\mathrm{c}_1 > 1$ and embeddings $\Lambda_n : \ell_q^n \to S_p(E)$ such that $\|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \le \mathrm{c}_1$ for each positive integer n. Let F_n denote the image $\Lambda_n(\ell_q^n)$ of Λ_n in $S_p(E)$. On the other hand, according to Theorem 4.2, we know how to construct linear isomorphisms

$$\Phi_n : S_q^n \to \mathsf{F}_{pq}^n \subset S_p(\ell_q^{n^2}) \qquad \text{such that} \qquad \|\Phi_n\|_{cb} \le \mathrm{c}_2$$

for some constant c_2 independent of n. Moreover, let $\widetilde{\mathsf{F}}_{pq}^n$ be the image of Φ_n endowed with the operator space structure inherited from

$$S_p(\min(\ell_q^{n^2})).$$

Then, if $\Psi_n : \widetilde{\mathsf{F}}_{pq}^n \to S_q^n$ stands for Φ_n^{-1}, Proposition 5.2 gives $\|\Psi_n\| \le \mathrm{c}_3$ for some constant c_3 independent of n. Let us define

$$\widetilde{\Lambda}_n : S_q^n \to S_p(E) \qquad \text{by} \qquad \widetilde{\Lambda}_n = \big(id \otimes \Lambda_{n^2}\big) \circ \Phi_n.$$

Then we have

$$\begin{aligned} \|\widetilde{\Lambda}_n\|_{cb}\|\widetilde{\Lambda}_n^{-1}\| & \le \|\Lambda_{n^2}\|_{cb}\|\Phi_n\|_{cb}\|\Psi_n\|\|\Lambda_{n^2}^{-1}\|_{\mathcal{CB}(F_{n^2}, \min(\ell_q^{n^2}))} \\ & = \|\Lambda_{n^2}\|_{cb}\|\Phi_n\|_{cb}\|\Psi_n\|\|\Lambda_{n^2}^{-1}\| \le \mathrm{c}_1\mathrm{c}_2\mathrm{c}_3. \end{aligned}$$

Since the constant $\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3$ does not depend on n, the assertion follows. □

5.2. OB**-convexity and** OK**-convexity.** Let us start by defining the notion of OB-convex operator space. Following [18], let us fix a family $\mathbf{d}_\Sigma = \big\{d_\sigma : \sigma \in \Sigma\big\}$ of positive integers indexed by an infinite set Σ and, given a finite subset Γ of Σ, let

$$\Delta_\Gamma = \sum_{\sigma \in \Gamma} d_\sigma^2.$$

An operator space E is called OB$_\Sigma$**-convex** if there exists a finite subset Γ of Σ and certain $0 < \delta \le 1$ such that, for any family

$$\Big\{A^\sigma \in M_{d_\sigma} \otimes S_2(E)\Big\}_{\sigma \in \Gamma},$$

we have

$$\frac{1}{\Delta_\Gamma} \inf_{B^\sigma\, unitary} \Big\| \sum_{\sigma \in \Gamma} d_\sigma \mathrm{tr}(A^\sigma B^\sigma) \Big\|_{S_2(E)} \le (1-\delta) \max_{\sigma \in \Gamma} \|A^\sigma\|_{M_{d_\sigma}(S_2(E))}.$$

If we replace the Schatten class $S_2(E)$ above by $S_p(E)$ we get an equivalent notion whenever $1 < p < \infty$, see [18]. This definition is inspired by Beck's original notion for Banach spaces, which corresponds to the *commutative set of parameters* $\Sigma_0 = \mathbb{N}$ with $d_\sigma = 1$ for all $\sigma \in \Sigma_0$. Our definition depends a priori on the set of parameters $(\Sigma, \mathbf{d}_\Sigma)$. However, we shall see below that there is no dependence on Σ. On the other hand, we also need to provide an operator space analog of the property of containing (uniformly) finite dimensional L_1 spaces. However, this time we need to allow the noncommutative L_1's to appear in the definition. Given an operator space E, a set of parameters $(\Sigma, \mathbf{d}_\Sigma)$ and $1 \le p < \infty$, we define the spaces

$$\mathcal{L}_p(\Sigma; E) = \left\{ A \in \prod_{\sigma \in \Sigma} M_{d_\sigma} \otimes E : \Big(\sum_{\sigma \in \Sigma} d_\sigma \|A^\sigma\|_{S_p^{d_\sigma}(E)}^p \Big)^{1/p} < \infty \right\}.$$

We impose on $\mathcal{L}_p(\Sigma; E)$ its natural operator space structure, see Chapter 2 of [22] for the details. We shall write $\mathcal{L}_p(\Sigma)$ for the scalar-valued case. We shall say that $S_p(E)$ **contains** $\mathcal{L}_1(\Gamma)$**'s semi-completely** λ**-uniformly** if, for each finite subset Γ of Σ, there exists a linear embedding $\Lambda_\Gamma : \mathcal{L}_1(\Gamma) \to S_p(E)$ such that

$$\|\Lambda_\Gamma\|_{cb}\|\Lambda_\Gamma^{-1}\| \le \lambda.$$

In other words, if

$$\Big\{\mathcal{L}_1(\Gamma) \,\Big|\, \Gamma \text{ finite}\Big\} \prec S_p(E).$$

The following is the analog of a well-known result for Banach spaces, see [18].

Remark 5.4. Given an operator space E, the following are equivalent:

i) $S_p(E)$ contains $\mathcal{L}_1(\Gamma)$'s semi-completely λ-uniformly for any $\lambda > 1$.
ii) $S_p(E)$ contains $\mathcal{L}_1(\Gamma)$'s semi-completely λ-uniformly for some $\lambda > 1$.

Finally we recall, as have already done in the Introduction, that an operator space E will be considered OK-**convex** whenever the vector-valued Schatten class $S_2(E)$ is K-convex when regarded as a Banach space.

Remark 5.5. The given definition of OK-convexity is a bit more flexible. Indeed, an operator space E is OK-convex if and only if $S_p(E)$ is a K-convex Banach space for some (any) $1 < p < \infty$. This follows from the fact that, given $1 < p < \infty$, the Schatten class $S_p(E)$ is K-convex if and only $S_2(E)$ is K-convex. Indeed, it follows from [20, 21] that Banach space K-convexity is stable by complex interpolation assuming only that one of the endpoint spaces is K-convex. Now assume that $S_2(E)$ is K-convex and let $1 < p < \infty$. If $p < 2$ (resp. $p > 2$) we have

$$S_p(E) = [S_2(E), S_1(E)]_\theta \qquad \big(\text{resp. } S_p(E) = [S_2(E), S_\infty(E)]_\theta\big)$$

for some $0 < \theta < 1$. Therefore, we find by complex interpolation that $S_p(E)$ is also a K-convex Banach space. A similar argument shows that $S_2(E)$ is a K-convex Banach space whenever $S_p(E)$ is also K-convex. Thus our claim follows.

Remark 5.6. In [18] it was given an a priori more general notion of K-convexity for operator spaces. Namely, let $(\Omega, \mathsf{A}, \mu)$ be a probability space with no atoms. Then, following [15] we define the *quantized Gauss system* associated to $(\Sigma, \mathbf{d}_\Sigma)$ as a collection of matrix-valued functions

$$\mathbf{G}_\Sigma = \Big\{ \gamma^\sigma : \Omega \to M_{d_\sigma} \Big\}_{\sigma \in \Sigma} \quad \text{where} \quad \gamma^\sigma = \frac{1}{\sqrt{d_\sigma}} \Big(\ \mathrm{g}_{ij}^\sigma \ \Big).$$

Here, the functions $\mathrm{g}_{ij}^\sigma : \Omega \to \mathbb{C}$ form a family, indexed by $1 \le i, j \le d_\sigma$ and $\sigma \in \Sigma$, of independent standard complex-valued gaussian random variables. Given a function $f \in L_2(\Omega; E)$, we can consider the Fourier coefficients of f with respect to the quantized Gauss system

$$\widehat{f}_{\mathbf{G}}(\sigma) = \int_\Omega f(\omega) \gamma^\sigma(\omega)^* d\mu(\omega).$$

This gives rise to the Gauss projection defined below

$$\mathrm{P}_{\mathbf{G}} : f \in L_2(\Omega; E) \longmapsto \sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(\widehat{f}_{\mathbf{G}}(\sigma) \gamma^\sigma) \in L_2(\Omega; E).$$

An operator space E is called OK_Σ-*convex* if the Gauss projection associated to the parameters $(\Sigma, \mathbf{d}_\Sigma)$ is a completely bounded map. However, recalling the definition of the quantized Gauss system, we can write

$$\sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(\widehat{f}_{\mathbf{G}}(\sigma) \gamma^\sigma) = \sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} \int_\Omega f(\omega) \overline{\mathrm{g}_{ij}^\sigma(\omega)} d\mu(\omega) \, \mathrm{g}_{ij}^\sigma.$$

Therefore, since now the right hand side can be regarded as the classical Gauss projection, it turns out that the notion of OK_Σ-convexity does not depend on the set $(\Sigma, \mathbf{d}_\Sigma)$, so that we shall simply use in the sequel the term OK-convex, without any explicit reference to the set of parameters $(\Sigma, \mathbf{d}_\Sigma)$.

Remark 5.7. We can replace $L_2(\Omega; E)$ above by $L_p(\Omega; E)$ for any $1 < p < \infty$.

Theorem 5.8. *Given an operator space E, the following are equivalent:*

i) *E is* OK*-convex.*
ii) *E is* OB_Σ*-convex for some* (*any*) *set of parameters* $(\Sigma, \mathbf{d}_\Sigma)$.
iii) *$S_p(E)$ does not contain ℓ_1^n's uniformly for some* (*any*) $1 < p < \infty$.

iv) $S_p(E)$ *does not contain* $\mathcal{L}_1(\Gamma)$*'s semi-completely for some* (*any*) $1 < p < \infty$.

Proof. By definition, E is OK-convex if and only if $S_p(E)$ is a K-convex Banach space for some (any) $1 < p < \infty$, see Remark 5.5 above. Then, applying Pisier's characterization [21] of K-convexity, conditions i) and iii) are equivalent. Now we prove the equivalence between iii) and iv). To that aim we can fix $1 < p < \infty$ without lost of generality (note that iii) is independent of the index $p \in (1,\infty)$ by its equivalence with i) and Remark 5.5). The implication iii) $\Rightarrow$ iv) is trivial. Reciprocally, let us assume that $S_p(E)$ contains ℓ_1^n's uniformly. Note that, since ℓ_1^n carries the maximal operator space structure, any Banach space embedding of ℓ_1^n is automatically a semi-complete embedding with the same constants. Then, Corollary 5.3 claims that the family

$$\Big\{ S_1^n \ \Big| \ n \geq 1 \Big\}$$

also embeds semi-completely uniformly in $S_p(E)$. That is, there exists c > 1 and embeddings $\Lambda_n : S_1^n \to S_p(E)$ such that

$$\|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \leq \mathrm{c}.$$

Now, given a finite subset Γ of Σ, we also consider the map

$$\mathrm{S}_\Gamma : A \in \mathcal{L}_1(\Gamma) \longmapsto \bigoplus_{\sigma \in \Gamma} d_\sigma A^\sigma \in S_1^{\mathrm{N}} \qquad \text{for} \qquad \mathrm{N} = \sum_{\sigma\in\Gamma} d_\sigma.$$

Finally, let $\mathrm{R}_\Gamma : \mathcal{L}_1(\Gamma) \to S_p(E)$ stand for $\Lambda_{\mathrm{N}} \circ \mathrm{S}_\Gamma$. Then we have

$$\|\mathrm{R}_\Gamma\|_{cb} \|\mathrm{R}_\Gamma^{-1}\| \leq \|\Lambda_{\mathrm{N}}\|_{cb} \|\Lambda_{\mathrm{N}}^{-1}\| \leq \mathrm{c},$$

since S_Γ is a complete isometry. In summary, the $\mathcal{L}_1(\Gamma)$'s embed semi-completely uniformly in $S_p(E)$. This proves the implication iv) $\Rightarrow$ iii). It remains to see that ii) is equivalent to some (any) of the other conditions. As in the commutative case, the implication ii) $\Rightarrow$ iv) follows from Remark 5.4 and by plugging in the 'right unit vectors', for details see [18]. The converse implication iv) $\Rightarrow$ ii) (a bit more technical) is the main result in [18]. This completes the proof. □

Remark 5.9. Theorem 5.8 implies the Σ-independence of OB_Σ-convexity.

Remark 5.10. We have already mentioned that semi-complete and Banach space embeddings of ℓ_1^n's are the same since ℓ_1^n carries the maximal o.s.s. It is worthy of mention that, although $\mathcal{L}_1(\Gamma)$'s are not longer equipped with the maximal operator space structure, a similar property holds for the latter spaces. Indeed, it is clear that if $\mathcal{L}_1(\Gamma)$'s are uniformly contained in $S_p(E)$ in the Banach space sense, then $S_p(E)$ also contains ℓ_1^n's uniformly. Finally, by Theorem 5.8 we see that $\mathcal{L}_1(\Gamma)$'s embed semicompletely uniformly in $S_p(E)$. The converse is trivial.

6. Operator space type and cotype

The notions of Fourier type and cotype of an operator space with respect to a noncommutative compact group were already defined in the Introduction. These are particular cases of a more general notion of type and cotype for operator spaces introduced in [4]. In that paper, the (*uniformly bounded*) *quantized orthonormal systems* play the same role of the uniformly bounded orthonormal systems in the classical theory. Some relevant examples of this notion are the dual object of a noncommutative compact group and the quantized analog of the Steinhaus system introduced in [15]. Before introducing the notions of type and cotype for operator spaces, let us recover the classical notions. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of random signs or independent Steinhaus variables over a probability space $(\Omega, \mathsf{A}, \mu)$. Given $1 \le p \le 2$, a Banach space X is called of type p when there exists a constant $\mathsf{T}_p(\mathrm{X})$ such that

$$\Big(\int_\Omega \Big\| \sum_{k=1}^n x_k \varepsilon_k(\omega) \Big\|_{\mathrm{X}}^{p'} d\mu(\omega) \Big)^{1/p'} \le \mathsf{T}_p(\mathrm{X}) \Big(\sum_{k=1}^n \|x_k\|_{\mathrm{X}}^p \Big)^{1/p}$$

for any finite family $x_1, x_2, \ldots, x_n$ in X. As we mentioned in the Introduction, the basic idea is to replace the random variables (ε_k) by a sequence $U_1, U_2, \ldots$ of independent random unitaries. That is, each $U_k : \Omega \to U(d_k)$ is a random unitary $d_k \times d_k$ matrix uniformly distributed in the unitary group $U(d_k)$ with respect to the normalized Haar measure. In this setting, we might define the following notion of type

$$\Big(\int_\Omega \Big\| \sum_{k=1}^n d_k \sum_{i,j=1}^n A_k(i,j) U_k(j,i) \Big\|_{\mathrm{X}}^{p'} d\mu(\omega) \Big)^{1/p'} \le \widetilde{\mathsf{T}}_p(\mathrm{X}) \Big(\sum_{k=1}^n d_k \|A_k\|_{S_p^{d_k}(\mathrm{X})}^p \Big)^{1/p}.$$

We want to point out that the right hand side is only well-defined for operator spaces. Moreover, this notion depends on the dimension d_k and their multiplicity. Note that the presence of d_k's in the inequality stated above is quite natural in view of the Peter-Weyl theorem and the connection (explained in the Introduction) with the Hausdorff-Young inequality for non-abelian compact groups. Let us give the precise definitions. The *quantized Steinhaus system* associated to $(\Sigma, \mathbf{d}_\Sigma)$ is defined as a collection

$$\mathbf{S}_\Sigma = \Big\{ \zeta^\sigma : \Omega \to U(d_\sigma) \Big\}_{\sigma \in \Sigma}$$

of independent uniformly distributed random unitaries with respect to the set of parameters $(\Sigma, \mathbf{d}_\Sigma)$. Given an operator space E and a function $f \in L_2(\Omega; E)$, we can consider the Fourier coefficients of f with respect to the quantized Steinhaus system

$$\widehat{f}_{\mathbf{S}}(\sigma) = \int_\Omega f(\omega) \zeta^\sigma(\omega)^* d\mu(\omega).$$

Let $\mathbf{St}_p(\Sigma; E)$ be the closure in $L_p(\Omega; E)$ of the subspace given by functions

$$f_\Gamma = \sum_{\sigma \in \Gamma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \qquad \text{with} \qquad A^\sigma \in M_{d_\sigma} \otimes E$$

and Γ a finite subset of Σ. We shall write $\mathbf{St}_p(\Sigma)$ for the scalar-valued case. Then, given $1 \le p \le 2$, we say that the operator space E has **Σ-type** p when the following inequality holds for any function $f \in \mathbf{St}_{p'}(\Sigma; E)$

$$\Big(\int_\Omega \|f(\omega)\|_E^{p'} \, d\mu(\omega) \Big)^{1/p'} \le_{cb} \mathcal{K}_p^1(E, \mathbf{S}_\Sigma) \Big(\sum_{\sigma \in \Sigma} d_\sigma \|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_p^{d_\sigma}(E)}^p \Big)^{1/p}.$$

In a similar way, **Σ-cotype** p' means that any $f \in \mathbf{St}_p(\Sigma; E)$ satisfies

$$\Big(\sum_{\sigma \in \Sigma} d_\sigma \|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_{p'}^{d_\sigma}(E)}^{p'} \Big)^{1/p'} \le_{cb} \mathcal{K}_{p'}^2(E, \mathbf{S}_\Sigma) \Big(\int_\Omega \|f(\omega)\|_E^p \, d\mu(\omega) \Big)^{1/p}.$$

Recall that the symbol $\le_{cb}$ means the complete boundedness of the corresponding linear map. The best constants $\mathcal{K}_p^1(E, \mathbf{S}_\Sigma)$ and $\mathcal{K}_{p'}^2(E, \mathbf{S}_\Sigma)$ in the inequalities stated above are called the Σ-type p and Σ-cotype p' constants of E. More concretely, using the spaces $\mathcal{L}_p(\Sigma; E)$ introduced in Section 5, the given definitions of Σ-type and Σ-cotype can be rephrased be requiring the complete boundedness of the following operators

$$\mathrm{T}_p : A \in \mathcal{L}_p(\Sigma; E) \longmapsto \sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \in \mathbf{St}_{p'}(\Sigma; E),$$

$$\mathrm{C}_{p'} : \sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \in \mathbf{St}_p(\Sigma; E) \longmapsto A \in \mathcal{L}_{p'}(\Sigma; E).$$

Remark 6.1. Let us recall that Σ_0 stands for the commutative set of parameters defined in Section 5. The classical Khintchine inequalities can be rephrased by saying that the norm of $\mathbf{St}_p(\Sigma_0)$, regarded as a Banach space, is equivalent to that of $\mathbf{St}_q(\Sigma_0)$ whenever $1 \le p \neq q < \infty$. On the other hand, by means of the noncommutative Khintchine inequalities [13, 14], it turns out that the norm of $\mathbf{St}_p(\Sigma_0)$ is *not* completely equivalent to that of $\mathbf{St}_q(\Sigma_0)$. That is, the operator spaces $\mathbf{St}_p(\Sigma_0)$ and $\mathbf{St}_q(\Sigma_0)$ are isomorphic but not completely isomorphic. More generally, $\mathbf{St}_p(\Sigma)$ is Banach isomorphic but not completely isomorphic to $\mathbf{St}_q(\Sigma)$, see [15] for the details. Therefore, each space $\mathbf{St}_q(\Sigma)$ in the definition of Σ-type p and Σ-cotype p' gives a priori a different notion!

Remark 6.2. As in the classical theory, every operator space has Σ-type 1 and Σ-cotype ∞. An operator space E has *non-trivial Σ-type* whenever it has Σ-type p for some $1 < p \le 2$. According to [18] and in contrast with the commutative theory, OK-convexity is not equivalent to having non-trivial Σ-type. Indeed, the operator Hilbert spaces R and C fail this equivalence since both are OK-convex operator spaces but do not have Σ-type for any $1 < p \le 2$. This constitutes an important difference between the classical and the noncommutative contexts. Namely, it turns out that we can not expect an operator space version of the Maurey-Pisier theorem [16] since the simplest form of this result asserts that the property of having non-trivial type is equivalent to K-convexity.

Remark 6.3. It is not clear whether or not the notions of Σ-type and Σ-cotype depend on $(\Sigma, \mathbf{d}_\Sigma)$. Moreover, if we replace the quantized Steinhaus system by the dual object of a noncommutative compact group G, we can ask ourselves the same question for the notions of Fourier type and cotype. Note that this *group independence* is an open problem even in the commutative theory. The reader is referred to the paper [6] for more information on this problem.

The Σ-type (resp. Σ-cotype) becomes a stronger condition on any operator space as the exponent p (resp. p') approaches 2. In particular, given an operator space E we consider (as in the Banach space context) the notions of *sharp* Σ-type of E (i.e. the supremum over all $1 \le p \le 2$ for which E has Σ-type p) as well as *sharp* Σ-cotype of E (i.e. the infimum over all $2 \le p' \le \infty$ for which E has Σ-type p'). The aim of this section is to investigate the sharp Σ-type and Σ-cotype indices of Lebesgue spaces, either commutative or not. However, as we shall see below, some other related problems will be solved with the same techniques.

6.1. **Sharp Σ-type of L_p for $1 \le p \le 2$.** We begin with the finite dimensional Σ-type constants for any *bounded* set of parameters $(\Sigma, \mathbf{d}_\Sigma)$. More concretely, let us consider a set of parameters $(\Sigma, \mathbf{d}_\Sigma)$ with $\mathbf{d}_\Sigma$ bounded. Then, given a finite subset Γ of Σ, we shall write $\ell_p(\Gamma)$ to denote the space of functions $\xi : \Gamma \to \mathbb{C}$ endowed with the customary norm

$$\|\xi\|_{\ell_p(\Gamma)} = \Big(\sum_{\sigma \in \Gamma} |\xi(\sigma)|^p \Big)^{1/p}.$$

Let us consider the function $f : \Omega \to \ell_p(\Gamma)$ defined by

$$f = \sum_{\sigma \in \Gamma} d_\sigma \mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma) \zeta^\sigma) \qquad \text{with} \qquad \widehat{f}_{\mathbf{S}}(\sigma) = e_{11} \otimes \delta_\sigma \in M_{d_\sigma} \otimes \ell_p(\Gamma).$$

Then we recall that

$$\Big(\int_\Omega |\sqrt{d_\sigma} \zeta_{11}^\sigma|^q d\mu \Big)^{\frac{1}{q}} \sim \Big(\int_\Omega |\sqrt{d_\sigma} \zeta_{11}^\sigma|^2 d\mu \Big)^{\frac{1}{2}} = 1, \qquad \text{for any} \quad 1 \le q < \infty.$$

Indeed, the norm equivalence follows from the analog of the Khintchine-Kahane inequalities for the quantized Steinhaus system, proved in [15]. The last equality follows from the definition of $\mathbf{S}_\Sigma$. In particular, if we use the symbol $\lesssim$ to denote an inequality up to a universal positive constant, then we have the following estimate for any $1 \le p < q \le 2$

$$\begin{aligned} |\Gamma|^{1/p} & \lesssim \Big(\int_\Omega \sum_{\sigma \in \Gamma} |d_\sigma \zeta_{11}^\sigma|^p d\mu \Big)^{1/p} \\ & \le \Big(\int_\Omega \Big\| \sum_{\sigma \in \Gamma} d_\sigma \mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma) \zeta^\sigma) \Big\|_{\ell_p(\Gamma)}^{q'} d\mu \Big)^{1/q'} \\ & \le \mathcal{K}_q^1(\ell_p(\Gamma), \mathbf{S}_\Sigma) \Big(\sum_{\sigma \in \Gamma} d_\sigma \|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_q^{d_\sigma}(\ell_p(\Gamma))}^q \Big)^{1/q} \lesssim \mathcal{K}_q^1(\ell_p(\Gamma), \mathbf{S}_\Sigma) \, |\Gamma|^{1/q}. \end{aligned}$$

In other words

$$c \, |\Gamma|^{1/p - 1/q} \le \mathcal{K}_q^1(\ell_p(\Gamma), \mathbf{S}_\Sigma) \le |\Gamma|^{1/p-1/q},$$

for some constant $0 < c \le 1$. The upper estimate is much simpler and it can be found in [3]. Therefore, since any infinite dimensional (either commutative or noncommutative) L_p space contains completely isometric copies of $\ell_p(\Gamma)$ for any finite subset Γ of Σ, we deduce that any infinite dimensional L_p space has sharp Σ-type p for any bounded set of parameters $(\Sigma, \mathbf{d}_\Sigma)$. However, it is evident that our argument doesn't work for unbounded sets of parameters. This case requires to find the right matrices which give the optimal constants. In the following theorem we compute the finite dimensional constants for the Schatten classes.

Theorem 6.4. *If $1 \le p < q \le 2$, the estimate*

$$\mathcal{K}_q^1(S_p^{d_\sigma}, \mathbf{S}_\Sigma) \ge d_\sigma^{2(1/p-1/q)}$$

holds for any unbounded set of parameters $(\Sigma, \mathbf{d}_\Sigma)$ and any element σ of Σ.

Proof. Let us take $f : \Omega \to S_p^{d_\sigma}$ so that $\widehat{f}_{\mathbf{S}}(\xi) = 0$ if $\xi \in \Sigma \setminus \{\sigma\}$ and

$$\widehat{f}_{\mathbf{S}}(\sigma) = \Big(\sum_{i=1}^{d_\sigma} e_{i1} \otimes e_{i1} \Big) \otimes \Big(\sum_{j=1}^{d_\sigma} e_{1j} \otimes e_{1j} \Big) \in C_q^{d_\sigma} \otimes_h C_p^{d_\sigma} \otimes_h R_p^{d_\sigma} \otimes_h R_q^{d_\sigma} = S_q^{d_\sigma}(S_p^{d_\sigma}).$$

Then, the following estimate holds by definition of Σ-type

$$\Big(\int_\Omega \Big\| d_\sigma \mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma) \zeta^\sigma) \Big\|_{S_p^{d_\sigma}}^{q'} d\mu \Big)^{1/q'} \le \mathcal{K}_q^1(S_p^{d_\sigma}, \mathbf{S}_\Sigma) \, d_\sigma^{1/q} \|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_q^{d_\sigma}(S_p^{d_\sigma})}.$$

Note that we have

$$\mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma)\zeta^\sigma) = \sum_{i,j=1}^{d_\sigma} e_{ij} \otimes \zeta_{ji}^\sigma = (\zeta^\sigma)^{\mathrm{t}}.$$

Thus, since the ζ^σ's are unitary, the left hand side of the inequality above is $d_\sigma^{1+\frac{1}{p}}$. On the other hand, we need to compute the norm of $\widehat{f}_{\mathbf{S}}(\sigma)$ in $S_q^{d_\sigma}(S_p^{d_\sigma})$. Since the Haagerup tensor product commutes with complex interpolation, it is not difficult to check that the following natural identifications are Banach space isometries

$$C_q^{d_\sigma} \otimes_h C_p^{d_\sigma} = S_r^{d_\sigma} = R_p^{d_\sigma} \otimes_h R_q^{d_\sigma} \qquad \text{with} \qquad \frac{1}{r} = \frac{1}{2}\Big(1 - \frac{1}{p} + \frac{1}{q}\Big).$$

For instance,

$$C_\infty^{d_\sigma} \otimes_h C_p^{d_\sigma} = [C_\infty^{d_\sigma} \otimes_h C_\infty^{d_\sigma}, C_\infty^{d_\sigma} \otimes_h C_1^{d_\sigma}]_{1/p} = [S_2^{d_\sigma}, S_\infty^{d_\sigma}]_{1/p} = S_{2p'}^{d_\sigma},$$

$$C_q^{d_\sigma} \otimes_h C_p^{d_\sigma} = [C_\infty^{d_\sigma} \otimes_h C_p^{d_\sigma}, C_p^{d_\sigma} \otimes_h C_p^{d_\sigma}]_{p/q} = [S_{2p'}^{d_\sigma}, S_2^{d_\sigma}]_{p/q} = S_r^{d_\sigma}.$$

In particular, due to our choice of $\widehat{f}_{\mathbf{S}}(\sigma)$, we can write

$$\|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_q^{d_\sigma}(S_p^{d_\sigma})} = \|1_{M_{d_\sigma}}\|_{S_r^{d_\sigma}}^2 = d_\sigma^{2/r}.$$

Combining our previous results, we obtain the desired estimate. □

Corollary 6.5. *If $1 \le p < q \le 2$, the estimate*

$$\mathcal{K}_q^1(\ell_p^{d_\sigma^2}, \mathbf{S}_\Sigma) \gtrsim d_\sigma^{2(1/p-1/q)}$$

holds for any unbounded set of parameters $(\Sigma, \mathbf{d}_\Sigma)$ and any element σ of Σ.

Proof. By Theorem 4.2, we have

$$\mathcal{K}_q^1(S_p^{d_\sigma}, \mathbf{S}_\Sigma) \lesssim \mathcal{K}_q^1(S_q(\ell_p^{d_\sigma^2}), \mathbf{S}_\Sigma) \le \mathcal{K}_q^1(\ell_p^{d_\sigma^2}, \mathbf{S}_\Sigma).$$

The last inequality follows by Minkowski inequality for operator spaces, see [3]. □

Remark 6.6. The arguments applied up to now also provide the finite dimensional estimates for the Σ-cotype constants when $2 \le q' < p' \le \infty$. Namely, the following estimates hold

$$\mathcal{K}_{q'}^2(S_{p'}^{d_\sigma}, \mathbf{S}_\Sigma) \ge d_\sigma^{2(1/q'-1/p')} \qquad \text{and} \qquad \mathcal{K}_{q'}^2(\ell_{p'}^{d_\sigma^2}, \mathbf{S}_\Sigma) \gtrsim d_\sigma^{2(1/q'-1/p')}.$$

Remark 6.7. By a simple result of [3], we have $\mathcal{K}_q^1(S_p^{d_\sigma}, \mathbf{S}_\Sigma) \le d_{cb}(S_p^{d_\sigma}, S_q^{d_\sigma})$. In particular, in Theorem 6.4 we actually have equality

$$\mathcal{K}_q^1(S_p^{d_\sigma}, \mathbf{S}_\Sigma) = d_\sigma^{2(1/p-1/q)}.$$

A similar argument applies to Corollary 6.5. In summary, our estimates provide the *exact* order of growth of the Σ-type (resp. Σ-cotype by Remark 6.6) constants of the corresponding finite dimensional Lebesgue spaces considered above. Moreover, now we can prove the claim given in Remark 4.4. Namely, let us consider the set $\Sigma = \mathbb{N}$ with $d_k = k$ for all $k \ge 1$. Then, if $\Psi_{pq} : S_q^n \to S_p(\ell_q^m)$ is a *cb* embedding with constants not depending on the dimensions n and m, Corollary 6.5 provides the following estimate

$$\begin{aligned} \mathcal{K}_q^1(S_p^n, \mathbf{S}_\Sigma) &\le \|\Psi_{pq}\|_{cb}\|\Psi_{pq}^{-1}\|_{cb}\, \mathcal{K}_q^1(S_q(\ell_p^m), \mathbf{S}_\Sigma) \\ &\le \|\Psi_{pq}\|_{cb}\|\Psi_{pq}^{-1}\|_{cb}\, \mathcal{K}_q^1(\ell_p^m, \mathbf{S}_\Sigma) \\ &\le \|\Psi_{pq}\|_{cb}\|\Psi_{pq}^{-1}\|_{cb}\, m^{1/p-1/q}. \end{aligned}$$

Therefore, since $\mathcal{K}_q^1(S_p^n, \mathbf{S}_\Sigma) = n^{2(1/p-1/q)}$, we conclude by taking n arbitrary large.

Remark 6.8. The main topic of [2] is the sharp Fourier type and cotype of L_p spaces. Given $1 \le p \le 2$, it is showed that L_p has sharp Fourier type p with respect to any compact semisimple Lie group. The arguments employed are very different. Namely, the key point is a Hausdorff-Young type inequality for functions defined on a compact semisimple Lie group with arbitrary small support. However, the sharp Fourier cotype of L_p for $1 \le p \le 2$ is left open in [2]. Now we can solve it by using Corollary 6.5 and the following inequality

$$\mathcal{K}_{q'}^2(L_p, \widehat{\mathrm{G}}) \ge \mathcal{K}_q^1(L_p, \mathbf{S}_{\widehat{\mathrm{G}}}).$$

Here $\mathcal{K}_{q'}^2(L_p, \widehat{\mathrm{G}})$ denotes the Fourier cotype q' constant of L_p with respect to G and $\mathbf{S}_{\widehat{\mathrm{G}}}$ stands for the quantized Steinhaus system with the parameters given by the degrees of the irreducible representations of G. That inequality is a particular case of the noncommutative version of the contraction principle given in [15]. This solves the problem posed in [2] not only for compact semisimple Lie groups, but for any non-finite topological compact group.

6.2. **Sharp Σ-cotype of L_p for $1 \le p \le 2$.** Given any σ-finite measure space $(\widetilde{\Omega}, \mathsf{B}, \nu)$, any set of parameters $(\Sigma, \mathbf{d}_\Sigma)$ and any finite subset Γ of Σ, let us consider a family of matrices

$$\mathbf{A} = \Big\{ A^\sigma \in M_{d_\sigma} \otimes L_p(\widetilde{\Omega}) \Big\}_{\sigma\in\Gamma}.$$

Then, we can estimate the norm of $\mathbf{A}$ in $\mathcal{L}_2(\Sigma; L_p(\widetilde{\Omega}))$ for any $1 \le p \le 2$ as follows. First, Minkowski inequality and Plancherel theorem give

$$\begin{aligned}\Big(\sum_{\sigma\in\Gamma} d_\sigma \|A^\sigma\|^2_{S_2^{d_\sigma}(L_p(\widetilde{\Omega}))} \Big)^{1/2} &\le \Big(\int_{\widetilde{\Omega}} \Big[\sum_{\sigma\in\Gamma} d_\sigma \|A^\sigma(x)\|^2_{S_2^{d_\sigma}} \Big]^{p/2} d\nu(x) \Big)^{1/p} \\ &= \Big(\int_{\widetilde{\Omega}} \Big[\int_\Omega \Big| \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \Big|^2 d\mu \Big]^{p/2} d\nu \Big)^{1/p}\end{aligned}$$

Second, by the analog given in [15] of Khintchine-Kahane inequalities for $\mathbf{S}_\Sigma$

$$\Big(\int_{\widetilde{\Omega}} \Big[\int_\Omega \Big| \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \Big|^2 d\mu \Big]^{\frac{p}{2}} d\nu \Big)^{\frac{1}{p}} \sim \Big(\int_\Omega \Big[\int_{\widetilde{\Omega}} \Big| \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma) \Big|^p d\nu \Big]^{\frac{2}{p}} d\mu \Big)^{\frac{1}{2}}.$$

Therefore, there exists some constant c such that

$$\Big(\sum_{\sigma\in\Gamma} d_\sigma \|A^\sigma\|^2_{S_2^{d_\sigma}(L_p(\widetilde{\Omega}))} \Big)^{1/2} \le \mathrm{c} \Big(\int_\Omega \Big\| \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A^\sigma \zeta^\sigma(\omega)) \Big\|^2_{L_p(\widetilde{\Omega})} d\mu(\omega) \Big)^{1/2}$$

for any family of matrices $\mathbf{A}$. In other words, we have proved that the mapping C_2 defined above is bounded when we take values in $L_p(\widetilde{\Omega})$. However, we can not claim Σ-cotype 2 unless we prove that the same operator C_2 is not only bounded, but completely bounded. Now, looking at Remark 6.1, we realize that our arguments do not work to show the complete boundedness. In this paragraph we study this problem. We begin by computing the sharp cotype of $S_q(S_p)$ as a Banach space. This will be the key to find the sharp Σ-cotype indices of L_p spaces. We want to point out that this fact was independently discovered by Lee in [12].

Lemma 6.9. *The Schatten class $S_q(S_p)$ has sharp Banach cotype r with*
$$\frac{1}{r} = \frac{1}{2}\Big(1 - \frac{1}{p} + \frac{1}{q}\Big) \qquad \textit{whenever} \qquad 1 \le p \le 2 \quad \textit{and} \quad p \le q \le p'.$$

Proof. First, we show that $S_q(S_p)$ has cotype r. The case $p > 1$ is simple. Indeed, we just need to check that the predual $S_{q'}(S_{p'})$ has Banach type r'. To that aim we observe that
$$S_{q'}(S_{p'}) = [S_p(S_{p'}), S_{p'}(S_{p'})]_\theta \qquad \text{with} \qquad 1 - \frac{1}{q} = \frac{1-\theta}{p} + \theta\Big(1 - \frac{1}{p}\Big).$$
Moreover, we have
$$S_p(S_{p'}) = [S_2(S_2), S_1(S_\infty)]_\eta \qquad \text{with} \qquad \frac{1}{p} = \frac{1-\eta}{2} + \frac{\eta}{1}.$$
Hence $S_p(S_{p'})$ has type p and, since $S_{p'}(S_{p'})$ has type 2, $S_{q'}(S_{p'})$ has type s with
$$\frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{2} = 1 - \frac{1}{q} + \theta\Big(\frac{1}{p} - \frac{1}{2}\Big) = 1 - \frac{1}{q} + \frac{1}{2}\Big(\frac{1}{p} - 1 + \frac{1}{q}\Big) = 1 - \frac{1}{r}.$$
It remains to see that $S_q(S_1)$ has cotype $2q$. Let us denote by $\mathcal{R}_p$ the subspace generated in $L_p(\Omega)$ by the sequence $\mathrm{r}_1, \mathrm{r}_2, \ldots$ of Rademacher functions. Then, if $\mathcal{R}_p(E)$ stands for the closure of the tensor product $\mathcal{R}_p \otimes E$ in $L_p(\Omega; E)$, we need to see that the following mapping is bounded
$$\mathrm{C}_{2q} : \sum_{k=1}^n \mathrm{r}_k \otimes x_k \in \mathcal{R}_2(S_q(S_1)) \longmapsto \sum_{k=1}^n \delta_k \otimes x_k \in \ell_{2q}(S_q(S_1)).$$
First we recall that, according to Khintchine-Kahane and Minkowski inequalities, the following natural map is contractive
$$\mathcal{R}_2(S_q(S_1)) \simeq S_q(\mathcal{R}_q(S_1)) \to S_q(\mathcal{R}_1(S_1)).$$
By the well-known complete isomorphism $\mathcal{R}_1 \simeq R + C$, which follows from the noncommutative Khintchine inequalities (see [14, 22]), we can write $S_q(\mathcal{R}_1(S_1))$ as the sum $S_q(S_1(R)) + S_q(S_1(C))$. Therefore, it suffices to see that the following natural mappings
$$\begin{array}{rccc} \mathrm{S}: & S_q(S_1(R)) & \to & \ell_{2q}(S_q(S_1)) \\ \mathrm{T}: & S_q(S_1(C)) & \to & \ell_{2q}(S_q(S_1)), \end{array}$$
which send the canonical basis of R or C to the canonical basis of ℓ_{2q}, are bounded. Since both cases are similar, we only prove the boundedness of T. To that aim we recall that, since $S_q(S_1(C)) = [S_\infty(S_1(C)), S_1(S_1(C))]_{1/q}$, it suffices to prove the boundedness of
$$\begin{array}{rccc} \mathrm{T}_0: & S_\infty(S_1(C)) & \to & \ell_\infty(S_\infty(S_1)) \\ \mathrm{T}_1: & S_1(S_1(C)) & \to & \ell_2(S_1(S_1)). \end{array}$$
If we observe that T_0 factors through $S_\infty(S_1(\ell_\infty))$, it is clear that T_0 is even contractive. To show that T_1 is bounded, let us consider a finite family $x_1, x_2, .., x_n$ of elements in $S_1(S_1)$. Then, since $S_1(S_1(C))$ embeds completely isometrically in $S_1(\mathbb{N}^3)$, we know from [26] that it has Banach cotype 2 so that we get
$$\begin{aligned} \Big\| \sum_{k=1}^n \delta_k \otimes x_k \Big\|_{\ell_2(S_1(S_1))} &= \Big(\sum_{k=1}^n \|x_k \otimes e_{k1}\|_{S_1(S_1(C))}^2 \Big)^{1/2} \\ &\le \mathrm{c} \int_0^1 \Big\| \sum_{k=1}^n \mathrm{r}_k(t)(x_k \otimes e_{k1}) \Big\|_{S_1(S_1(C))} dt \end{aligned}$$

$$= \quad \mathrm{c} \, \Big\| \sum_{k=1}^n x_k \otimes e_{k1} \Big\|_{S_1(S_1(C))}.$$

The last equality follows since

$$\Big\| \sum_{k=1}^n \mathrm{r}_k(t)(x_k \otimes e_{k1}) \Big\|_{S_1(S_1(C))} = \Big\| \Big(\sum_{k=1}^n (\mathrm{r}_k(t) x_k)^* (\mathrm{r}_k(t) x_k) \Big)^{1/2} \Big\|_{S_1(S_1)}.$$

This gives the boundedness of T_1 and consequently the map C_{2q} is also bounded. In summary, we have seen that $S_q(S_p)$ has Banach cotype r in the range of parameters considered. To complete the proof, we need to see that this exponent is sharp. However, recalling that

$$S_q(S_p) = C_q \otimes_h C_p \otimes_h R_p \otimes_h R_q,$$

we can regard $C_q \otimes_h C_p$ as a subspace of $S_q(S_p)$. Now, since the Haagerup tensor product commutes with complex interpolation, we obtain the following Banach space isometries

$$C_q \otimes_h C_p = [C_{p'} \otimes_h C_p, C_p \otimes_h C_p]_\theta = [S_{p'}, S_2]_\theta \qquad \text{with} \qquad \frac{1}{q} = 1 - \frac{1}{p} + \theta \Big(\frac{2}{p} - 1 \Big).$$

This gives that $C_q \otimes_h C_p = S_r$ as a Banach space, we leave the details to the reader. Therefore, $S_q(S_p)$ can not have better cotype than r. This completes the proof. □

Let us recall that the commutative set of parameters $(\Sigma_0, \mathbf{d}_{\Sigma_0})$ is the given by $\Sigma_0 = \mathbb{N}$ where we take $d_\sigma = 1$ for all $\sigma \in \Sigma_0$. In the following result we show that, in contrast with the Banach space situation, any infinite dimensional (commutative or noncommutative) L_p space with $p \neq 2$ fails to have Σ-cotype 2.

Theorem 6.10. *Any infinite dimensional L_p space has sharp Σ-cotype* $\max(p, p')$.

Proof for $\mathbf{d}_\Sigma$ bounded. As it was pointed out in [4], it is obvious the any L_p space has Σ-cotype $\max(p, p')$ with respect to any set of parameters Σ. Let us see that this exponent is sharp when $\mathbf{d}_\Sigma$ is bounded. In this particular case, it clearly suffices to consider the commutative set of parameters Σ_0. We also assume that $1 \le p \le 2$ since the case $2 \le p \le \infty$ has been considered in Remark 6.6. Moreover, since any infinite dimensional L_p space contains a completely isometric copy of ℓ_p, it suffices to check it for ℓ_p. Now, let us assume that ℓ_p has Σ_0-cotype q', for some $q' < p'$. Then we can argue as in Corollary 6.5. Namely, combining Theorem 4.2 with Minkowski inequality for operator spaces, we have

$$\mathcal{K}_{q'}^2(S_{q'}(S_p); \mathbf{S}_{\Sigma_0}) \lesssim \mathcal{K}_{q'}^2(S_{q'}(\mathbb{N}^2; \ell_p); \mathbf{S}_{\Sigma_0}) \le \mathcal{K}_{q'}^2(\ell_p; \mathbf{S}_{\Sigma_0}).$$

Now, by Lemma 6.9, the best Σ_0-cotype we can expect to have is r where

$$\frac{1}{r} = \frac{1}{2p'} + \frac{1}{2q'} < \frac{1}{q'}.$$

Therefore, we deduce that $r > q'$ and the result follows by contradiction. □

Proof for $\mathbf{d}_\Sigma$ unbounded. Arguing as in the previous case, it suffices to see that ℓ_p has sharp Σ-cotype p' for $1 \le p \le 2$. Let us assume that ℓ_p has Σ-cotype q' for some $q' < p'$. Then, again by Theorem 4.2 and Minkowski inequality, the space $S_{q'}(S_p)$ should have Σ-cotype q'. However, recalling that

$$S_{q'}(S_p) = C_{q'} \otimes_h C_p \otimes_h R_p \otimes_h R_{q'},$$

we conclude that the subspace $\mathcal{C}(p,q) = C_{q'} \otimes_h C_p = C_{q'} \otimes_h R_{p'}$ of $S_{q'}(S_p)$ must also have Σ-cotype q'. Then, we proceed as in Theorem 6.4. Namely, let us consider a function $f : \Omega \to \mathcal{C}(p,q)$ so that

$$\widehat{f}_{\mathbf{S}}(\xi) = 0 \qquad \text{for} \qquad \xi \in \Sigma \setminus \{\sigma\}$$

and such that

$$\widehat{f}_{\mathbf{S}}(\sigma) = \sum_{i,j=1}^{d_\sigma} e_{i1} \otimes e_{i1} \otimes e_{1j} \otimes e_{1j} \in C_{q'}^{d_\sigma} \otimes_h \mathcal{C}(p,q) \otimes_h R_{q'}^{d_\sigma}.$$

By the definition of Σ-cotype we have

$$\|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_{q'}^{d_\sigma}(\mathcal{C}(p,q))} \leq \mathcal{K}_{q'}^2(\mathcal{C}(p,q), \mathbf{S}_\Sigma) \Big(\int_\Omega \Big\| d_\sigma^{1/q} \mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma)\zeta^\sigma) \Big\|_{\mathcal{C}(p,q)}^q d\mu \Big)^{1/q}.$$

Now using the Banach space isometry

$$S_s^n = C_u^n \otimes_h R_v^n \qquad \text{for} \qquad \frac{1}{s} = \frac{1}{2u} + \frac{1}{2v},$$

which follows easily by complex interpolation, we obtain

$$\|\widehat{f}_{\mathbf{S}}(\sigma)\|_{S_{q'}^{d_\sigma}(\mathcal{C}(p,q))} = \Big\| \sum_{k=1}^n e_{kk} \Big\|_{C_{q'}^{d_\sigma} \otimes_h C_{q'}^{d_\sigma}} \Big\| \sum_{k=1}^n e_{kk} \Big\|_{C_p^{d_\sigma} \otimes_h R_{q'}^{d_\sigma}} = d_\sigma^{\frac{1}{2} + \frac{1}{2p} + \frac{1}{2q'}}.$$

Moreover, since $\mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma)\zeta^\sigma(\omega)) = \sum_{i,j=1}^{d_\sigma} e_{ij} \otimes \zeta_{ji}^\sigma(\omega) = (\zeta^\sigma(\omega))^{\mathrm{t}}$, we have

$$\Big(\int_\Omega \Big\| d_\sigma^{1/q} \mathrm{tr}(\widehat{f}_{\mathbf{S}}(\sigma)\zeta^\sigma(\omega)) \Big\|_{\mathcal{C}(p,q)}^q d\mu(\omega) \Big)^{1/q} = d_\sigma^{\frac{1}{q} + \frac{1}{2q'} + \frac{1}{2p'}}.$$

Combining our previous results, we obtain $q' \geq p'$. This completes the proof. $\square$

Remark 6.11. By duality, the sharp Σ-type index of L_p is $\min(p,p')$.

Remark 6.12. By a standard argument using the contraction principle, our results for sharp Σ-type and Σ-cotype also hold for *any* uniformly bounded quantized orthonormal system. The reader is referred to [4] for further details.

Remark 6.13. As it was recalled in Remark 6.2, it seems that there is no analog of the Maurey-Pisier theorem for operator spaces. Theorem 3 clearly reinforces that idea. Finally, the reader is referred to Section 4.2 of [7] for an unrelated notion of operator space cotype 2 for which L_p has cotype 2 whenever $1 \leq p \leq 2$.

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