A local Hausdorff-Young inequality on the classical compact Lie groups and related topics

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Abstract

Let G be a compact semisimple Lie group. The Hausdorff-Young inequality on G can be stated as follows

$$\|\widehat{f}\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} = \Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi \|\widehat{f}(\pi)\|_{\mathcal{S}_{p'}^{d_\pi}}^{p'} \Big)^{1/p'} \le \Big(\int_{\mathrm{G}} |f(g)|^p \, d\mu(g) \Big)^{1/p} = \|f\|_{L_p(\mathrm{G})}$$

where $1 \le p \le 2$ and p' denotes its conjugate exponent. Here $\widehat{\mathrm{G}}$ denotes the dual object of G, d_π is the degree of the irreducible representation $\pi : \mathrm{G} \to \mathcal{B}(\mathcal{H}_\pi)$, $\mathcal{S}_p^n$ stands for the Schatten p-class over the $n \times n$ matrices and μ denotes the normalized Haar measure on G. We are interested in the Hausdorff-Young quotients of central functions with arbitrary small support. In other words, if we define

$$\mathsf{hy}_p(\mathrm{G}, f) = \|\widehat{f}\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} / \|f\|_{L_p(\mathrm{G})}$$

and $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is any neighborhood basis around the identity $\mathbf{1}$ of G, we shall study the constant

$$\mathcal{K}(\mathrm{G}, p) = \inf_{n \ge 1} \sup \Big\{ \mathsf{hy}_p(\mathrm{G}, f) \,\Big|\, f \in L_p(\mathrm{G}), \ f \,\text{central}, \ \mathrm{supp} f \subset \mathcal{U}_n \Big\}.$$

The inequality $\mathcal{K}(\mathrm{G}, p) > 0$ for any $1 \le p \le 2$ is our main result and can be regarded as a local Hausdorff-Young type inequality for compact semisimple Lie groups. In particular, this includes the classical compact Lie groups. Our result extends to the non-commutative framework some related results (due to Andersson and Sjölin) for the torus $\mathbb{T}$. We shall also discuss the exact value of $\mathcal{K}(\mathrm{G}, p)$, which remains an open problem. As application, we shall obtain the sharp Fourier type exponents of (commutative and non commutative) Lebesgue spaces with respect to a compact group. In fact, the local Hausdorff-Young inequality was originally motivated by this problem in operator space theory.

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Introduction

Let $\mathbb{T}$ denote the quotient $\mathbb{R}/\mathbb{Z}$ with its natural group structure. Then, given any $1 \le p \le 2$ and any function $f : \mathbb{T} \to \mathbb{C}$ in $L_p(\mathbb{T})$, the classical Hausdorff-Young inequality claims that

$$\|\widehat{f}\|_{L_{p'}(\mathbb{Z})} = \Big(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^{p'} \Big)^{1/p'} \le \Big(\int_{\mathbb{T}} |f(t)|^p dt \Big)^{1/p} = \|f\|_{L_p(\mathbb{T})},$$

where p' denotes the conjugate exponent of p. This result was first proved by Young [45, 46] when p' is an even integer and extended by Hausdorff [15] to the general case. After that, Hardy and Littlewood proved that the only functions for which this inequality turns out to be an equality are the characters $\exp(2\pi int)$ with $n \in \mathbb{Z}$, see e.g. [47] for more on this. The Hausdorff-Young inequality naturally extends to any locally compact abelian group G. Namely, given $1 \le p \le 2$ and a function $f \in L_p(\mathrm{G})$, we have

$$\|\widehat{f}\|_{L_{p'}(\widehat{\mathrm{G}})} = \Big(\int_{\widehat{\mathrm{G}}} |\widehat{f}(\xi)|^{p'} d\mu_2(\xi) \Big)^{1/p'} \le \Big(\int_{\mathrm{G}} |f(g)|^p d\mu_1(g) \Big)^{1/p} = \|f\|_{L_p(\mathrm{G})},$$

where μ_1 and μ_2 are suitably chosen Haar measures on G and its dual group $\widehat{\mathrm{G}}$ respectively. In what follows, we shall write $\mathsf{hy}_p(\mathrm{G}, f)$ for the Hausdorff-Young quotient

$$\|\widehat{f}\|_{L_{p'}(\widehat{\mathrm{G}})} / \|f\|_{L_p(\mathrm{G})}.$$

Let us consider the constant

$$\mathcal{B}(\mathrm{G}, p) = \sup \Big\{ \mathsf{hy}_p(\mathrm{G}, f) \, \Big| \, f \in L_p(\mathrm{G}) \Big\}.$$

In 1924, Titchmarsh proved the Hausdorff-Young inequality $\mathsf{hy}_p(\mathbb{R}, f) \le 1$ for the real line. Then, the problem of finding the exact value of $\mathcal{B}(\mathbb{R}, p)$ came out. By analogy with the previous case, it is natural to guess that the maximizer should be invariant under the action of the Fourier transform. In particular, the Gaussian function $\exp(-\pi x^2)$ is a natural candidate. Babenko proved this result in [3] when p' is an even integer and deduced the identities

$$\mathcal{B}(\mathbb{R}, p) = \sqrt{p^{1/p} / p'^{1/p'}} \qquad \text{and} \qquad \mathcal{B}(\mathbb{R}^n, p) = \mathcal{B}(\mathbb{R}, p)^n$$

in that case. The validity of these identities in the general case $1 \le p \le 2$ was finally proved by Beckner in [4]. Moreover, Beckner extended this result to any locally compact abelian group G by using the factorization theorem for this class of groups ($\mathrm{G} = \mathbb{R}^{\mathrm{m(G)}} \times \mathrm{H_G}$ for some $\mathrm{m(G)} \ge 0$ and some locally compact abelian group $\mathrm{H_G}$ which contains an open compact subgroup)

$$\mathcal{B}(\mathrm{G}, p) = \mathcal{B}(\mathbb{R}, p)^{\mathrm{m(G)}} \qquad \text{for} \qquad 1 \le p \le 2.$$

The proof of this result uses that $\mathcal{B}(\mathrm{G}, p) = 1$ for any compact group G and any exponent $1 \leq p \leq 2$. After Beckner's result, the constant $\mathcal{B}(\mathbb{R}, p)$ is known in the literature as the Babenko-Beckner constant. We refer the reader to Lieb's paper [24] for more on this topic and to Russo's papers [36, 37, 38] for the study of $\mathcal{B}(\mathrm{G}, p)$ on more general classes of groups.

On the other hand, a local variant of the Hausdorff-Young inequality on $\mathbb{T}$ was considered by Andersson in his Ph.D. Thesis [1]. Note that, given any non-vanishing function $f \in L_p(\mathbb{T})$, we have $0 < \mathsf{hy}_p(\mathbb{T}, f) \leq 1$. Andersson's problem was to study the quotient $\mathsf{hy}_p(\mathbb{T}, f)$ for functions f with arbitrary small support. More concretely, after identifying $\mathbb{T}$ with the interval $[-\frac{1}{2}, \frac{1}{2})$ and by the translation invariance of the Haar measure, this problem reduces to study the value of the constant

$$\mathcal{K}(\mathbb{T}, p) = \inf_{n \geq 1} \sup \Big\{ \mathsf{hy}_p(\mathbb{T}, f) \, \Big| \, f \in L_p(\mathbb{T}), \ \mathrm{supp} f \subset (-\tfrac{1}{n}, \tfrac{1}{n}) \Big\}.$$

Note that any $f \in L_p(\mathbb{T})$ can be regarded as a function $f : \mathbb{R} \to \mathbb{C}$ with

$$\mathrm{supp}\, f \subset \Big\{ x \in \mathbb{R} \, \Big| \, -\tfrac{1}{2} \leq x \leq \tfrac{1}{2} \Big\}.$$

Then, the function $\varphi_k(t) = k^{1/p} f(kt)$ is supported on $[-1/2k, 1/2k]$ and we have

$$\begin{aligned}
\int_{\mathbb{R}} |\widehat{f}(\xi)|^{p'} d\xi &= \lim_{k \to \infty} \frac{1}{k} \sum_{n \in \mathbb{Z}} |\widehat{f}(n/k)|^{p'} \\
&= \lim_{k \to \infty} \frac{1}{k} \sum_{n \in \mathbb{Z}} \Big| \int_{\mathbb{R}} f(x) e^{-2\pi i n x/k} dx \Big|^{p'} \\
&= \lim_{k \to \infty} \sum_{n \in \mathbb{Z}} \Big| \int_{\mathbb{T}} k^{1/p} f(kt) e^{-2\pi i n t} dt \Big|^{p'} = \lim_{k \to \infty} \sum_{n \in \mathbb{Z}} |\widehat{\varphi}_k(n)|^{p'}.
\end{aligned}$$

Moreover, we clearly have

$$\int_{\mathbb{R}} |f(x)|^p \, dx = \int_{\mathbb{T}} |\varphi_k(t)|^p \, dt.$$

In summary, we can write

$$\frac{\|\widehat{f}\|_{L_{p'}(\mathbb{R})}}{\|f\|_{L_p(\mathbb{R})}} = \lim_{k \to \infty} \frac{\|\widehat{\varphi}_k\|_{L_{p'}(\mathbb{Z})}}{\|\varphi_k\|_{L_p(\mathbb{T})}}.$$

In other words, given any function $f \in L_p(\mathbb{T})$, there exists a family $\varphi_1, \varphi_2, \ldots$ of functions in $L_p(\mathbb{T})$ with $\mathrm{supp}\, \varphi_k \subset (-\frac{1}{k}, \frac{1}{k})$ and such that the Hausdorff-Young quotients converge to

$$\lim_{k \to \infty} \mathsf{hy}_p(\mathbb{T}, \varphi_k) = \mathsf{hy}_p(\mathbb{R}, f).$$

In particular, using that the maximizers for the Babenko-Beckner constant are given by Gaussians and applying a simple approximation argument, it is not difficult to check the inequality $\mathcal{K}(\mathbb{T},p) \geq \mathcal{B}(\mathbb{T},p)$. Thus, it is quite natural to wonder whether or not the equality holds. Andersson [1] gave an affirmative answer for p' an even integer and Sjölin proved the general case $1 \leq p \leq 2$ in [40]. Finally, Kamaly [19] generalized this result to the n-dimensional torus

$$\mathcal{K}(\mathbb{T}^n,p) = \mathcal{B}(\mathbb{R}^n,p).$$

The main purpose of this paper is the analysis of the constant $\mathcal{K}(\mathrm{G},p)$ for any compact semisimple Lie group G. A compact Lie group G is called semisimple when the corresponding Lie algebra $\mathfrak{g}$ has no proper subspaces $\mathfrak{h}$ included in the center of $\mathfrak{g}$. An excellent reference for the necessary background in this paper on compact semisimple Lie groups is Simon's book [39]. As we shall see, semisimplicity is an essential assumption in our arguments. Indeed, as is well-known any such group contains a family of maximal tori satisfying certain nice and deep properties such as the Weyl integration formula. In fact, Weyl's character and dimension formulas [42, 43, 44] will also play a very relevant role in the proof. Let us remember the reader some basic definitions in non-commutative harmonic analysis. Given a function $f:\mathrm{G}\to\mathbb{C}$ in $L_1(\mathrm{G})$ and a unitary irreducible representation $\pi:\mathrm{G}\to\mathcal{B}(\mathcal{H}_\pi)$, we define the Fourier coefficient of f at π as the operator-valued integral

$$\widehat{f}(\pi) = \int_{\mathrm{G}} f(g)\pi(g)^*\,d\mu(g),$$

where μ stands for the normalized Haar measure on G. Let $\widehat{\mathrm{G}}$ denote the dual object of G (i.e. the set of equivalence classes of unitary irreducible representations up to unitary equivalence). Then, given $1 \leq p < \infty$ we define the spaces

$$\begin{aligned}\mathcal{L}_p(\widehat{\mathrm{G}}) &= \Big\{\mathrm{A}\in\prod_{\pi\in\widehat{\mathrm{G}}}\mathrm{M}_{d_\pi} \ \big|\ \|\mathrm{A}\|_{\mathcal{L}_p(\widehat{\mathrm{G}})} = \Big(\sum_{\pi\in\widehat{\mathrm{G}}} d_\pi\|\mathrm{A}_\pi\|^p_{\mathcal{S}_p^{d_\pi}}\Big)^{1/p} < \infty\Big\},\\ \mathcal{L}_\infty(\widehat{\mathrm{G}}) &= \Big\{\mathrm{A}\in\prod_{\pi\in\widehat{\mathrm{G}}}\mathrm{M}_{d_\pi} \ \big|\ \|\mathrm{A}\|_{\mathcal{L}_\infty(\widehat{\mathrm{G}})} = \sup_{\pi\in\widehat{\mathrm{G}}}\|\mathrm{A}_\pi\|_{\mathcal{S}_\infty^{d_\pi}} < \infty\Big\}.\end{aligned}$$

Equipped with this norm, the space $\mathcal{L}_p(\widehat{\mathrm{G}})$ becomes a Banach space. Here d_π denotes the degree of π, we write M_n for the algebra of $n\times n$ matrices and $\mathcal{S}_p^n$ stands for the Schatten p-class over M_n. The Hausdorff-Young inequality on compact groups was proved by Kunze [22] and it says that for any $f\in L_p(\mathrm{G})$

$$\|\widehat{f}\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} = \Big(\sum_{\pi\in\widehat{\mathrm{G}}} d_\pi\|\widehat{f}(\pi)\|^{p'}_{\mathcal{S}_{p'}^{d_\pi}}\Big)^{1/p'} \leq \Big(\int_{\mathrm{G}}|f(g)|^p\,d\mu(g)\Big)^{1/p} = \|f\|_{L_p(\mathrm{G})},$$

where $1\leq p\leq 2$ and p' denotes its conjugate exponent. In other words, if we denote again the Hausdorff-Young quotient of f by $\mathsf{hy}_p(\mathrm{G},f)$, Kunze's result gives

$$\mathsf{hy}_p(\mathrm{G},f)\leq 1.$$

Moreover, this inequality can not be improved due to the compactness of G. Now we are ready to state the main result of this paper. A function $f : \mathrm{G} \to \mathbb{C}$ is called central if it is constant at the conjugacy classes of G

$$f(hgh^{-1}) = f(g) \quad \text{for all} \quad g, h \in \mathrm{G}.$$

Theorem A. *Let $1 \le p \le 2$ and let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a neighborhood basis around the identity $\mathbf{1}$ of a compact semisimple Lie group* G. *Then, we have*

$$\mathcal{K}(\mathrm{G}, p) = \inf_{n \ge 1} \sup \Big\{ \mathsf{hy}_p(\mathrm{G}, f) \, \Big| \, f \in L_p(\mathrm{G}), \ f \ \textit{central}, \ \mathrm{supp}(f) \subset \mathcal{U}_n \Big\} > 0.$$

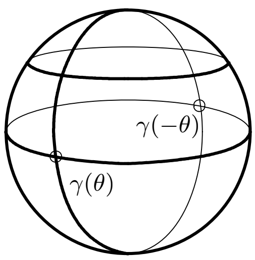
Note that the norm of f on $L_p(\mathrm{G})$ and the norm of $\widehat{f}$ on $\mathcal{L}_{p'}(\widehat{\mathrm{G}})$ are invariant under translations of f. Hence, the same result remains valid if $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a neighborhood basis of any other point in G. Thus, this result can be regarded as a local Hausdorff-Young inequality on compact semisimple Lie groups. We also recall that any function $f : \mathrm{G} \to \mathbb{C}$ is central when G is abelian. In particular, our constant $\mathcal{K}(\mathrm{G}, p)$ is a natural generalization of the constant $\mathcal{K}(\mathbb{T}, p)$ defined above.

For the convenience of the reader, we give a very brief sketch of the proof in the simplest case $\mathrm{G} = \mathrm{SU}(2)$. The special unitary group SU(2) is the group of unitary 2×2 matrices whose determinant is 1. In particular, it is not difficult to characterize SU(2) as the set of matrices

$$\mathsf{m}(a, b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \text{with} \quad a, b \in \mathbb{C} \quad \text{s.t.} \quad |a|^2 + |b|^2 = 1.$$

Now we give a classical geometric interpretation of SU(2). We refer the reader to Section 5.4 in [8] for further details. The correspondence $\mathsf{m}(a, b) \leftrightarrow (a, b)$ identifies SU(2) as a set with the unit sphere S_3 in the 4-dimensional $\mathbb{R}$-space $\mathbb{C}^2$ in such a way that the identity element is identified with the north pole $(1, 0)$. The 3-dimensional analog can be sketched as follows.

SU(2)



Parallels: Conjugacy classes

Meridian: Maximal torus $\mathbf{T}$

Weyl's Group: $\mathcal{W}_{\mathrm{SU}(2)} \simeq \mathbf{S}_2$

$f : \mathrm{G} \to \mathbb{C}$ central $\Leftrightarrow$ $f_{|\mathbf{T}} : \mathbf{T} \to \mathbb{C}$ is even

Figure I: Geometric interpretation of SU(2)

Let us consider the one-parameter subgroup $\gamma : \theta \in \mathbb{R} \mapsto \mathsf{m}(e^{i\theta}, 0) \in \mathrm{SU}(2)$. The image of γ can be regarded as a meridian joining the north and south poles. The subgroup induced by γ is a so-called maximal torus $\mathbf{T}$ (i.e. isomorphic to a torus of maximal dimension) in SU(2). Moreover, according to [8] the conjugacy classes of SU(2) are the 2-dimensional surfaces of constant latitude. In particular, any given function $f : \mathrm{SU}(2) \to \mathbb{C}$ is central iff is constant on the parallels in Figure I iff its restriction to $\mathbf{T}$ is an even function. In other words, a function f is central if and only if $f(\gamma(\theta)) = f(\gamma(-\theta))$ for all $\theta \in \mathbb{R}$. The transformation $\gamma(\theta) \mapsto \gamma(-\theta)$ is described as the action of the so-called Weyl group of SU(2), which in this case is the symmetric group $\mathbf{S}_2$ of permutations over a set with two elements.

The characterization of central functions just given is a general property in the class of compact semisimple Lie groups. That is, a function $f : \mathrm{G} \to \mathbb{C}$ is central if and only if its restriction to the maximal torus is invariant under the action of the Weyl group. This nice characterization and Weyl's integration formula will enable us to write the Fourier coefficients of a central function $f : \mathrm{G} \to \mathbb{C}$ in terms of the Fourier coefficients with respect to $\mathbf{T}$ of a function related to its restriction $f_{|\mathbf{T}} : \mathbf{T} \to \mathbb{C}$. This relation is a key point in the proof. Namely, once we have functions defined on $\mathbf{T}$ we can try to emulate Andersson's argument. The second and more difficult obstacle is the presence of the weights d_π in the Fourier series. It forces us to use the Weyl dimension formula in order to identify the Fourier series of f on G as another Fourier series on $\mathbf{T}$ of a fractional integral operator acting on f. This part of the proof is more technical so that we omit here the details.

At the time of this writing, the problem of finding the exact value of $\mathcal{K}(\mathrm{G}, p)$ remains open. However, after the proof of Theorem A, we shall show the reader which are the main difficulties. More concretely, it seems that the solution to this problem is equivalent to finding the best constant for a weighted Hausdorff-Young inequality of Pitt type, see below for the details.

Theorem A was proved for the first time in [11]. The original aim of that paper was to study the sharp Fourier type exponents of L_p spaces (either commutative or non-commutative) with respect to a compact semisimple Lie group. However, the techniques in [11] only provide a solution for the case $1 \leq p \leq 2$. After that, the complete solution to this problem was obtained by Junge and the author in [17] using techniques coming from non-commutative probability and operator algebra.

The study of the Hausdorff-Young inequality for vector-valued functions was initiated by Peetre [29] in 1969. Peetre considered functions $f : \mathbb{R} \to \mathrm{X}$ taking values in some Banach space X. In this case, the validity of the Hausdorff-Young inequality for some fixed p depends on the Banach space X. That paper lead to the notion of Fourier type of a Banach space with respect to a locally compact abelian group, introduced by Milman in [26]. The theory of Fourier type with respect to locally compact abelian groups was further developed in [2, 6, 10, 20].

The Hausdorff-Young inequality for vector-valued functions $f : \mathrm{G} \to \mathrm{X}$ defined on a non-commutative compact group G has been recently studied in some works summarized in [27]. Note that the Fourier coefficients of f are matrices with entries in X. Therefore, one has to be able to define norms for such matrices. By Ruan's theorem [35], this leads us to consider an operator space structure on X. The necessary background on operator spaces can be found in [7, 32]. Then, the right norms for our matrices are provided by Pisier's work [31] on non-commutative vector valued L_p spaces. In summary, in order to develop a theory of Fourier type in this context, we need to take values in operator spaces rather than Banach spaces. This crucial point is obviously at the root of the notion of Fourier type. We refer the reader to [12] for more on this topic and to [13, 28] for the notion of Rademacher type of an operator space.

Given an exponent $1 \le p \le 2$, an operator space X is said to have Fourier type p with respect to a compact group G if the vector-valued Fourier transform, defined as follows

$$f \otimes x \in L_p(\mathrm{G}) \otimes \mathrm{X} \mapsto \widehat{f} \otimes x \in \mathcal{L}_{p'}(\widehat{\mathrm{G}}) \otimes \mathrm{X},$$

extends to a completely bounded map

$$\mathcal{F}_{\mathrm{G},\mathrm{X}} : L_p(\mathrm{G}; \mathrm{X}) \to \mathcal{L}_{p'}(\widehat{\mathrm{G}}; \mathrm{X}).$$

In other words, a vector-valued Hausdorff-Young inequality of exponent p holds. Let $\mathsf{K}_p(\mathrm{X}, \mathrm{G})$ be the *cb* norm of $\mathcal{F}_{\mathrm{G},\mathrm{X}}$. It follows from [12] that the Fourier type becomes a stronger condition on (X, G) as p approaches the index 2. This gives rise to the notion of sharp Fourier type exponent. According to the commutative theory, the natural candidate for the sharp Fourier type of L_p is p. That is, we want to show that for $1 \le p < q \le 2$ we have $\mathsf{K}_q(L_p, \mathrm{G}) = \infty$. Of course, we have to require the group G not to be finite and the operator space L_p to be infinite-dimensional. Under such assumptions, we have

$$\mathsf{K}_q(L_p, \mathrm{G}) \ge \limsup_{n \to \infty} \mathsf{K}_q(\ell_p(n), \mathrm{G}).$$

Therefore the growth of $\mathsf{K}_q(\ell_p(n), \mathrm{G})$ is an even more interesting problem. Here is where the local Hausdorff-Young inequality stated in Theorem A helps to find a solution. Namely, we will prove the following result.

Theorem B. *If $1 \le p < q \le 2$, we have*

$$\mathcal{K}(\mathrm{G}, q)\, n^{\frac{1}{p} - \frac{1}{q}} \le \mathsf{K}_q(\ell_p(n), \mathrm{G}) \le n^{\frac{1}{p} - \frac{1}{q}},$$

for any compact semisimple Lie group G *and any positive integer* $n \ge 1$.

Note that Theorem A implies that $\mathsf{K}_q(\ell_p(n), \mathrm{G})$ is arbitrary large as $n \to \infty$. Therefore, Theorem B provides the sharp Fourier type exponents of L_p and also the optimal growth of the finite-dimensional constants $\mathsf{K}_q(\ell_p(n), \mathrm{G})$. It is worthwhile to mention that the analog of Theorem B in the commutative theory is an absolutely trivial result, see e.g. [10]. In particular, Theorem B illustrates some of the extra difficulties that are intrinsic to the non-commutative theory.

On the other hand, it remains to study the sharp Fourier type exponent of $L_{p'}$ for $2 \le p' \le \infty$. Arguing as above, it suffices to show that the increasing sequence $\mathsf{K}_q(\ell_{p'}(n), \mathrm{G})$ diverges to infinity for any $1 \le p < q \le 2$. Our solution to this problem [17] uses techniques of non-commutative probability and it is out of the scope of this paper. The final result is the following.

Theorem C. *If* $1 \le p < q \le 2$, *we have*

$$\mathsf{K}_q(\ell_{p'}(n), \mathrm{G}) \simeq n^{\frac{1}{p}-\frac{1}{q}},$$

for any compact topological group G *and any integer* $n \ge 1$.

1 The Fourier coefficients of central functions

In this section we provide a simple expression for the Fourier coefficients of a central function $f : \mathrm{G} \to \mathbb{C}$ defined on a compact semisimple Lie group G. More concretely, we shall write them in terms of the Fourier coefficients (with respect the maximal torus $\mathbf{T}$) of a function $h_f : \mathbf{T} \to \mathbb{C}$ associated to f. To that aim, we shall apply some basic results from the representation theory of compact semisimple Lie groups. These algebraic preliminaries can be found in Simon's book [39] or alternatively in Fulton/Harris' book [9]. We summarize only here the main topics.

Let G be a compact semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. In what follows we choose once and for all an explicit maximal torus $\mathbf{T}$ in G while $\mathfrak{h}$ will stand for its Lie algebra. That is, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The rank r of G is the dimension of $\mathbf{T}$ so that $\mathbf{T} \simeq \mathbb{T}^r$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with its natural group structure. Also, as it is customary, we consider the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ with complex conjugates taken so that $\mathfrak{g}_{\mathbb{R}} = \big\{ \mathsf{Z} \in \mathfrak{g}_{\mathbb{C}} : \mathsf{Z} = \overline{\mathsf{Z}} \big\} = i\mathfrak{g}$ and similarly $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$. The bracket $\langle\ ,\ \rangle$ will denote the complex-valued inner product induced by the Killing form. We also recall that the Weyl group $\mathcal{W}_{\mathrm{G}}$ associated to G can be seen as a set of $r \times r$ unitary matrices W –isometries on $\mathfrak{h}_{\mathbb{R}}$– with integer entries and $\det \mathrm{W} = \pm 1$. In particular the set $\mathcal{W}_{\mathrm{G}}^* = \big\{ \mathrm{W}^{\mathrm{t}} : \mathrm{W} \in \mathcal{W}_{\mathrm{G}} \big\}$ becomes a set of isometries on $\mathfrak{h}_{\mathbb{R}}^*$. The symbol $\mathcal{R}$ will stand for the set of roots while, if we take $\mathsf{H}_0 \in \mathfrak{h}_{\mathbb{R}}$ such that $\alpha(\mathsf{H}_0) \neq 0$ for any root α, the symbol $\mathcal{R}^+ = \big\{ \alpha \in \mathcal{R} : \alpha(\mathsf{H}_0) > 0 \big\}$ denotes the set of positive roots. Finally we shall write Λ_{W} and Λ_{DW} for the weight lattice and the set of dominant weights respectively.

Let us consider a central function $f : \mathrm{G} \to \mathbb{C}$ and a given dominant weight $\lambda \in \Lambda_{\mathsf{DW}}$. By the dominant weight theorem there exists a unique $\pi_\lambda \in \widehat{\mathrm{G}}$ associated to λ and, since f is central, we can write by Schur's lemma

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \int_{\mathrm{G}} f(g) \overline{\chi_\lambda(g)} \, d\mu(g) \ 1_{d_\lambda}$$

where d_λ is the degree of π_λ, χ_λ is the character of π_λ and 1_n denotes the $n \times n$ identity matrix. We now recall the definition of the functions A_β which appear in the Weyl character formula. Given $\beta \in \mathfrak{h}_{\mathbb{R}}^*$, we define the functions $\exp_\beta : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ and $A_\beta : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ as follows

$$\begin{aligned} \exp_\beta(\mathsf{H}) &= e^{2\pi i \langle \beta, \mathsf{H} \rangle}, \\ A_\beta(\mathsf{H}) &= \sum_{\mathrm{W} \in \mathcal{W}_{\mathrm{G}}} \det \mathrm{W} \, \exp_\beta \big(\mathrm{W}(\mathsf{H}) \big). \end{aligned}$$

The maximal torus $\mathbf{T}$ is isomorphic via the exponential mapping to the quotient space $\mathfrak{h}_{\mathbb{R}}/\mathrm{L}_{\mathsf{W}}$, where L_{W} is the set of those $\mathsf{H} \in \mathfrak{h}_{\mathbb{R}}$ satisfying $\exp(2\pi i \mathsf{H}) = \mathbf{1}$. That is, L_{W} is the dual lattice of Λ_{W}. Therefore, the functions $\exp_\beta$ and A_β are well defined functions on $\mathbf{T}$ if and only if $\beta \in \Lambda_{\mathsf{W}}$. As it is well known, the integral form

$$\delta = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} \alpha$$

is not necessarily a weight and so the functions $\exp_\delta$ and A_δ could be not well defined on $\mathbf{T}$. To avoid this difficulty we assume for the moment that G is *simply connected*. This condition on G assures that $\delta \in \Lambda_{\mathsf{W}}$. Hence, applying consecutively the Weyl integration formula and the Weyl character formula, we obtain

$$\begin{aligned} \widehat{f}(\pi_\lambda) &= \frac{1}{d_\lambda |\mathcal{W}_{\mathrm{G}}|} \int_{\mathbf{T}} f(t) \overline{\chi_\lambda(t)} \, |A_\delta(t)|^2 \, dm(t) \ 1_{d_\lambda} \\ &= \frac{1}{d_\lambda |\mathcal{W}_{\mathrm{G}}|} \int_{\mathbf{T}} f(t) A_\delta(t) \, \overline{A_{\lambda+\delta}(t)} \, dm(t) \ 1_{d_\lambda}, \end{aligned}$$

where m denotes the Haar measure on $\mathbf{T}$ normalized so that $m(\mathbf{T}) = 1$. Now, if we write $A_{\lambda+\delta}$ as a linear combination of exponentials, we get

$$\begin{aligned} \widehat{f}(\pi_\lambda) &= \frac{1}{d_\lambda |\mathcal{W}_{\mathrm{G}}|} \sum_{\mathrm{W} \in \mathcal{W}_{\mathrm{G}}} \det \mathrm{W} \int_{\mathbf{T}} f(t) A_\delta(t) \ \exp_{-(\lambda+\delta)}(\mathrm{W}(t)) \, dm(t) \ 1_{d_\lambda} \\ &= \frac{1}{d_\lambda} \int_{\mathbf{T}} f(t) A_\delta(t) \ \exp_{-(\lambda+\delta)}(t) \, dm(t) \ 1_{d_\lambda} \end{aligned}$$

since $A_\delta(\mathrm{W}(t)) = \det \mathrm{W} A_\delta(t)$ and $f(\mathrm{W}(t)) = f(t)$. Note that, taking coordinates with respect to the basis $\{\omega_1, \omega_2, \ldots \omega_r\}$ of fundamental weights, any weight $\lambda \in \Lambda_{\mathsf{W}}$ has integer coordinates. Therefore, we can understand the last expression as the Fourier transform of $f A_\delta$ on the maximal torus $\mathbf{T}$ evaluated at $\lambda + \delta$. Hence

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \mathcal{F}_{\mathbf{T}}(f A_\delta)(\lambda + \delta) \ 1_{d_\lambda} \tag{1}$$

for $f : \mathrm{G} \to \mathbb{C}$ central and G any compact semisimple simply connected Lie group.

When G is *non-simply connected*, a more careful approach is needed. In that case we only know that $\mathrm{W}^{\mathrm{t}}(\delta) \pm \delta \in \Lambda_{\mathsf{W}}$ for all $\mathrm{W} \in \mathcal{W}_{\mathrm{G}}$. In particular, we note that the function

$$\exp_{\pm\delta} A_{\lambda+\delta} = \sum_{\mathrm{W} \in \mathcal{W}_{\mathrm{G}}} \det \mathrm{W} \exp_{\mathrm{W}^{\mathrm{t}}(\lambda+\delta) \pm \delta}$$

is a well-defined function on $\mathbf{T}$ for all $\lambda \in \Lambda_{\mathsf{DW}}$. This remark allows us to write $\overline{\chi_\lambda}\,|A_\delta|^2 = (\exp_\delta \overline{A_{\lambda+\delta}})\,(\exp_{-\delta} A_\delta)$ as a well-defined function on $\mathbf{T}$. Hence, applying again Schur's lemma, the Weyl integration and the Weyl character formulas, we get

$$\begin{aligned}\widehat{f}(\pi_\lambda) &= \frac{1}{|\mathcal{W}_{\mathrm{G}}|}\sum_{\mathrm{W}\in\mathcal{W}_{\mathrm{G}}}\frac{\det \mathrm{W}}{d_\lambda}\int_{\mathbf{T}} f(t)(\exp_{-\delta} A_\delta)(t)\exp_{\delta-\mathrm{W}^{\mathrm{t}}(\lambda+\delta)}(t)\,dm(t)\;1_{d_\lambda}\\ &= \frac{1}{d_\lambda}\int_{\mathbf{T}} f(t)(\exp_{-\delta} A_\delta)(t)\exp_{-\lambda}(t)\,dm(t)\;1_{d_\lambda},\end{aligned}$$

where the last equality follows from the change of variable $t \mapsto \mathrm{W}^{\mathrm{t}}(t)$. That is

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda}\,\mathcal{F}_{\mathbf{T}}(fB_\delta)(\lambda)\,1_{d_\lambda}, \tag{2}$$

where $B_\delta = \exp_{-\delta} A_\delta$. This expression is now valid for any compact semisimple Lie group and it coincides with (1) for simply connected ones. The expressions obtained in (1) and (2) will be crucial in the following section to prove the local Hausdorff-Young inequality for compact semisimple Lie groups.

2 The local Hausdorff-Young inequality

This section is devoted to the proof of the local variant of the Hausdorff-Young inequality described in the Introduction. We begin by proving Theorem A for simply connected compact semisimple Lie groups. The proof for non-simply connected groups will be outlined after that. It follows essentially the same ideas. However, some points will have to be slightly modified. Finally, we give a brief discussion on the exact value of the constant $\mathcal{K}(\mathrm{G}, p)$, which remains open.

2.1 Simply connected groups

Before the proof of Theorem A, we need some auxiliary results. Let us assume that G is simply connected and let $f : \mathrm{G} \to \mathbb{C}$ be a central function. A quick look at relation (1) given above, allows us to write

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda}\,\det \mathrm{W}\,\mathcal{F}_{\mathbf{T}}(fA_\delta)(\mathrm{W}^{\mathrm{t}}(\lambda+\delta))\;1_{d_\lambda} \tag{3}$$

for all $\mathrm{W} \in \mathcal{W}_{\mathrm{G}}$. On the other hand, let us denote by P_α the hyperplane of $\mathfrak{h}_{\mathbb{R}}^*$ orthogonal to α with respect to the complex inner product given by the Killing form. The infinitesimal Cartan-Stiefel diagram is then given by the expression

$$\mathrm{P} = \bigcup_{\alpha\in\mathcal{R}} P_\alpha.$$

Lemma 2.1 *Let* G *be a compact semisimple simply connected Lie group. Then we have* $\left\{\mathrm{W}^{\mathrm{t}}(\lambda+\delta) \mid \mathrm{W} \in \mathcal{W}_{\mathrm{G}},\, \lambda \in \Lambda_{\mathsf{DW}}\right\} = \Lambda_{\mathsf{W}} \setminus \mathrm{P}$. *Moreover, the following mapping is injective*

$$(\mathrm{W}, \lambda) \in \mathcal{W}_{\mathrm{G}} \times \Lambda_{\mathsf{DW}} \mapsto \mathrm{W}^{\mathrm{t}}(\lambda+\delta) \in \Lambda_{\mathsf{W}} \setminus \mathrm{P}.$$

Proof. Since G is simply connected we have that

$$\left\{\lambda+\delta \;\Big|\; \lambda \in \Lambda_{\mathsf{DW}}\right\} = \Lambda_{\mathsf{W}} \cap \mathbf{C}^{\mathrm{int}}.$$

Here $\mathbf{C}$ denotes the fundamental Weyl chamber and $\mathbf{C}^{\mathrm{int}}$ its interior. Now, since P and Λ_{W} are invariant under the action of $\mathcal{W}_{\mathrm{G}}^{*}$ and for any Weyl chamber C there exists a unique $\mathrm{W} \in \mathcal{W}_{\mathrm{G}}$ with $\mathrm{W}^{\mathrm{t}}(\mathbf{C}) = \mathrm{C}$, we obtain the desired equality. Finally, the injectivity follows from the uniqueness mentioned above. ∎

Lemma 2.2 *Let* G *be a compact semisimple simply connected Lie group and let* $f : \mathrm{G} \to \mathbb{C}$ *be a central function. Then there exists a constant* $\mathcal{A}(\mathrm{G}, p)$ *depending on* G *and* p*, such that*

$$\|\widehat{f}\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} = \mathcal{A}(\mathrm{G}, p)\Bigg(\sum_{\lambda \in \Lambda_{\mathsf{W}} \setminus \mathrm{P}} \frac{\big|\mathcal{F}_{\mathbf{T}}(fA_\delta)(\lambda)\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda\rangle|^{p'-2}}\Bigg)^{1/p'}.$$

Proof. Since f is central and G is simply connected, (3) gives

$$\begin{aligned}
\|\widehat{f}\,\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} &= \Bigg(\sum_{\lambda \in \Lambda_{\mathsf{DW}}} d_\lambda \big\|\widehat{f}(\pi_\lambda)\big\|_{\mathcal{S}_{p'}^{d_\lambda}}^{p'}\Bigg)^{1/p'} \\
&= \Bigg(\frac{1}{|\mathcal{W}_{\mathrm{G}}|} \sum_{\mathrm{W} \in \mathcal{W}_{\mathrm{G}}} \sum_{\lambda \in \Lambda_{\mathsf{DW}}} d_\lambda \Big|\frac{1}{d_\lambda}\mathcal{F}_{\mathbf{T}}(fA_\delta)(\mathrm{W}^{\mathrm{t}}(\lambda+\delta))\Big|^{p'} \big\|1_{d_\lambda}\big\|_{\mathcal{S}_{p'}^{d_\lambda}}^{p'}\Bigg)^{1/p'}
\end{aligned}$$

By the Weyl dimension formula for d_λ, the norm of $\widehat{f}$ in $\mathcal{L}_{p'}(\widehat{\mathrm{G}})$ equals

$$\Bigg(\frac{1}{|\mathcal{W}_{\mathrm{G}}|} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta\rangle|^{p'-2} \sum_{\mathrm{W} \in \mathcal{W}_{\mathrm{G}}} \sum_{\lambda \in \Lambda_{\mathsf{DW}}} \frac{\big|\mathcal{F}_{\mathbf{T}}(fA_\delta)(\mathrm{W}^{\mathrm{t}}(\lambda+\delta))\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \lambda+\delta\rangle\big|^{p'-2}}\Bigg)^{1/p'}$$

Finally we observe that

$$\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \lambda+\delta\rangle\big| = \prod_{\alpha \in \mathcal{R}} \big|\langle \mathrm{W}(\alpha), \lambda+\delta\rangle\big|^{1/2} = \prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \mathrm{W}^{\mathrm{t}}(\lambda+\delta)\rangle\big|$$

since any $\mathrm{W} \in \mathcal{W}_{\mathrm{G}}$ permutes the root set $\mathcal{R}$. Therefore, by Lemma 2.1 we have

$$\|\widehat{f}\,\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} = \Bigg(\frac{1}{|\mathcal{W}_{\mathrm{G}}|} \prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \delta\rangle\big|^{p'-2} \sum_{\lambda \in \Lambda_{\mathsf{W}} \setminus \mathrm{P}} \frac{\big|\mathcal{F}_{\mathbf{T}}(fA_\delta)(\lambda)\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \lambda\rangle\big|^{p'-2}}\Bigg)^{1/p'}.$$

The proof is completed by taking $\mathcal{A}(\mathrm{G}, q) = \Big(\frac{1}{|\mathcal{W}_{\mathrm{G}}|} \prod_{\alpha \in \mathcal{R}^+} \big| \langle \alpha, \delta \rangle \big|^{p'-2} \Big)^{1/p'}$. ■

We are now ready to give the proof of Theorem A for simply connected groups. Let $\{\mathsf{H}_1, \mathsf{H}_2, \dots, \mathsf{H}_r\}$ be the predual basis of the fundamental weights, any element of L_{W} can be written as a linear combination of $\mathsf{H}_1, \mathsf{H}_2, \dots, \mathsf{H}_r$ with integer coefficients. Then, since $\mathbf{T} \simeq \mathfrak{h}_{\mathbb{R}}/\mathrm{L}_{\mathsf{W}}$, we can regard $\mathbf{T}$ as the subset of $\mathfrak{h}_{\mathbb{R}}$

$$\mathfrak{T} = \Big\{ \sum_{k=1}^{r} x_k \mathsf{H}_k \, \Big| \, -\tfrac{1}{2} \leq x_k < \tfrac{1}{2} \Big\}.$$

On the other hand, let us fix a bounded central function $f_0 : \mathrm{G} \to \mathbb{C}$. Then f_0 can be understood as a function on $\mathbf{T}$ invariant under the action of $\mathcal{W}_{\mathrm{G}}$. Now, since the Weyl group is generated by a set of reflections in $\mathfrak{h}_{\mathbb{R}}$, f_0 can be regarded as a complex-valued function on $\mathfrak{h}_{\mathbb{R}}$, supported in $\mathfrak{T}$ and symmetric under such reflections. Let us recall that $\{\omega_1, \omega_2, \dots \omega_r\}$ stands for the basis of fundamental weights. Let $\tau = 1 - 2/p'$, the way we have interpreted the function f_0 allows us to define the function

$$\widehat{I_\tau(f_0 A_\delta)} : \mathfrak{h}_{\mathbb{R}}^* \longrightarrow \mathbb{C} \qquad \text{as}$$

$$\widehat{I_\tau(f_0 A_\delta)}(\xi) = \frac{1}{\displaystyle\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^{\tau}} \; \mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)(\xi) \qquad \text{where} \quad \xi = \sum_{k=1}^{r} \xi_k \omega_k.$$

Remark 2.3 The motivation for the notation employed is that in a classical group such as SU(2), the function just defined is nothing but the Fourier transform of the fractional integral operator

$$I_\tau(f)(x) = \frac{1}{\Gamma(\tau)} \int_{-\infty}^{x} f(y)(x-y)^{\tau-1} dy$$

acting on $f_0 A_\delta$. Here lies the main difference with the commutative case [1] since the presence of the degrees d_λ (as a product in Lemma 2.2 by the Weyl dimension formula) requires the presence of a factor of $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)$. This does not happen in the commutative case since $d_\lambda = 1$ for all $\lambda \in \Lambda_{\mathsf{DW}}$.

Lemma 2.4 *Let* G *be a compact semisimple simply connected Lie group and let* $f : \mathrm{G} \to \mathbb{C}$ *be a central function. Then we have*

$$\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f A_\delta)(\xi) = 0 \quad \textit{for all} \quad \xi \in \mathrm{P}.$$

Proof. If $\xi \in \mathrm{P}$, there exists a root α such that $\xi \in P_\alpha$. Let S_α be the reflection in P_α so that $S_\alpha(\xi) = \xi$. Then, as is well known $S_\alpha \in \mathcal{W}_{\mathrm{G}}^*$ and therefore we have

$$\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f A_\delta)(\xi) = \det S_\alpha \, \mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f A_\delta)(\xi) = -\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f A_\delta)(\xi). \qquad \square$$

The function $\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)$ is analytic since $f_0 A_\delta$ has compact support and, by Lemma 2.4, it vanishes at

$$\mathrm{P} = \Big\{ \xi \in \mathfrak{h}_\mathbb{R}^* : \prod_{\alpha \in \mathcal{R}^+} \langle \alpha, \xi \rangle = 0 \Big\}.$$

In particular, since $0 \leq \tau < 1$, $I_\tau \widehat{(f_0 A_\delta)}$ is continuous and takes the value 0 on P. Now we write the norm of this function in terms of a Riemann sum

$$\big\| I_\tau \widehat{(f_0 A_\delta)} \big\|_{L_{p'}(\mathfrak{h}_\mathbb{R}^*)} = \lim_{k \to \infty} \Bigg(\sum_{\lambda \in \Lambda_\mathrm{W}} \frac{\mathrm{V_G}}{k^r} \frac{\big| \mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)(k^{-1}\lambda) \big|^{p'}}{\displaystyle\prod_{\alpha \in \mathcal{R}^+} \big| \langle \alpha, k^{-1}\lambda \rangle \big|^{\tau p'}} \Bigg)^{1/p'},$$

with $\mathrm{V_G}$ being the volume of a cell of Λ_W. Moreover $\phi_k(x) = k^\sigma f_0(kx) A_\delta(kx)$ is supported in $\mathfrak{T}$ and the relation $\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)(k^{-1}\lambda) = k^{r-\sigma} \mathcal{F}_\mathbf{T}(\phi_k)(\lambda)$ holds for all $\lambda \in \Lambda_\mathrm{W}$. Taking $\sigma = \tau |\mathcal{R}^+| + r/p$, we obtain

$$\big\| I_\tau \widehat{(f_0 A_\delta)} \big\|_{L_{p'}(\mathfrak{h}_\mathbb{R}^*)} = \mathrm{V_G}^{1/p'} \lim_{k \to \infty} \Bigg(\sum_{\lambda \in \Lambda_\mathrm{W} \setminus \mathrm{P}} \frac{\big| \mathcal{F}_\mathbf{T}(\phi_k)(\lambda) \big|^{p'}}{\displaystyle\prod_{\alpha \in \mathcal{R}^+} \big| \langle \alpha, \lambda \rangle \big|^{\tau p'}} \Bigg)^{1/p'},$$

since we know that for $\lambda \in \mathrm{P}$ we get nothing. Finally, let us define $\varphi_k : \mathfrak{h}_\mathbb{R} \to \mathbb{C}$ by the relation $\phi_k = \varphi_k A_\delta$. The function φ_k satisfies $\varphi_k(\mathrm{W}(x)) = \varphi_k(x)$ for all $\mathrm{W} \in \mathcal{W}_\mathrm{G}$ and is supported in $k^{-1}\mathfrak{T}$. Hence we can understand φ_k as a central function on G. We can also say that, as a consequence of the well known relation

$$A_\delta = \exp_{-\delta} \prod_{\alpha \in \mathcal{R}^+} (\exp_\alpha - 1), \tag{4}$$

φ_k has no singularities. Therefore Lemma 2.2 provides the following relation for some constant $\mathcal{B}(\mathrm{G}, p)$ depending on G and p

$$\big\| I_\tau \widehat{(f_0 A_\delta)} \big\|_{L_{p'}(\mathfrak{h}_\mathbb{R}^*)} = \mathcal{B}(\mathrm{G}, p) \lim_{k \to \infty} \|\widehat{\varphi}_k\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})}. \tag{5}$$

On the other hand, since φ_k can be seen as a central function on G, we can estimate the norm of φ_k on $L_p(\mathrm{G})$. By the Weyl integration formula we get

$$\begin{aligned} \|\varphi_k\|_{L_p(\mathrm{G})} &= \Big(\frac{1}{|\mathcal{W}_\mathrm{G}|} \int_\mathbf{T} |\varphi_k A_\delta(t)|^p |A_\delta(t)|^{2-p} dm(t) \Big)^{1/p} \\ &= \Big(\frac{k^{\sigma p}}{|\mathcal{W}_\mathrm{G}|} \int_\mathfrak{T} |f_0 A_\delta(kx)|^p |A_\delta(x)|^{2-p} dx \Big)^{1/p} \\ &\leq \Big(\frac{(2\pi)^{(2-p)|\mathcal{R}^+|}}{|\mathcal{W}_\mathrm{G}|} \; k^{\sigma p} \int_\mathfrak{T} |f_0 A_\delta(kx)|^p \prod_{\alpha \in \mathcal{R}^+} |\alpha(x)|^{2-p} dx \Big)^{1/p}, \end{aligned}$$

where the last inequality follows from (4). Now, under the change of variable $y = kx$ and taking $\mathcal{C}(\mathrm{G}, q) = (2\pi)^{\tau|\mathcal{R}^+|}|\mathcal{W}_\mathrm{G}|^{-1/p}$, we obtain

$$\|\varphi_k\|_{L_p(\mathrm{G})} \leq \mathcal{C}(\mathrm{G}, p)\, k^{\sigma-\tau|\mathcal{R}^+|-r/p} \Big(\int_{\mathfrak{T}} |f_0 A_\delta(y)|^p \prod_{\alpha\in\mathcal{R}^+} |\alpha(y)|^{\tau p} dy \Big)^{1/p}.$$

Recall that $\mathrm{supp}(f_0 A_\delta) \subset \mathfrak{T}$. Thus, the integral over $k\mathfrak{T}$ (the domain of integration after the change of variable) reduces to the same integral over $\mathfrak{T}$. However,

$$\sigma - \tau|\mathcal{R}^+| - r/p = 0$$

and the product inside the integral is bounded over $\mathfrak{T}$, say by $\mathrm{M_G}$. Therefore

$$\|\varphi_k\|_{L_p(\mathrm{G})} \leq \mathcal{C}(\mathrm{G}, p)\, \mathrm{M_G}\, \|f_0 A_\delta\|_{L_p(\mathfrak{h}_\mathbb{R})}. \tag{6}$$

In summary, by (5) and (6), we know there exists a constant $\mathcal{D}(\mathrm{G}, q)$ depending on G and p such that

$$0 < \mathcal{D}(\mathrm{G}, p)\, \frac{\big\|I_\tau \widehat{(f_0 A_\delta)}\big\|_{L_{p'}(\mathfrak{h}_\mathbb{R}^*)}}{\|f_0 A_\delta\|_{L_p(\mathfrak{h}_\mathbb{R})}} \leq \liminf_{k\to\infty} \frac{\|\widehat{\varphi}_k\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})}}{\|\varphi_k\|_{L_p(\mathrm{G})}} \leq 1.$$

Since f_0 is bounded we easily obtain that $f_0 A_\delta \in L_p(\mathfrak{h}_\mathbb{R})$, $I_\tau \widehat{(f_0 A_\delta)} \in L_{p'}(\mathfrak{h}_\mathbb{R}^*)$ and $\mathcal{D}(\mathrm{G}, p) > 0$. Therefore, we have found a family $\varphi_1, \varphi_2, \ldots$ of central functions on G whose supports are eventually in $\mathcal{U}_n$ for any positive integer $n \geq 1$ and such that their Hausdorff-Young quotient $\mathsf{hy}_p(\mathrm{G}, \varphi_k)$ of exponent p is bounded below by a positive constant. This concludes the proof of Theorem A for compact semisimple simply connected Lie groups.

2.2 Non-simply-connected groups

If G is not simply connected, some extra comments have to be made. In any case we shall not give complete proofs of any of them, the details are left to the reader.

i) Generalization (3) of formula (1) has no meaning here, but we can generalize formula (2) as

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \det \mathrm{W}\, \mathcal{F}_\mathbf{T}(f B_\delta)(\mathrm{W}^t(\lambda+\delta) - \delta)\ 1_{d_\lambda}.$$

This provides a couple of results parallel to Lemmas 2.1 and 2.4. Namely,

- We have

$$\Lambda_\mathsf{W} \setminus (\mathrm{P} - \delta) = \Big\{ \mathrm{W}^t(\lambda+\delta) - \delta \,\Big|\, \mathrm{W} \in \mathcal{W}_\mathrm{G},\, \lambda \in \Lambda_\mathsf{DW} \Big\}.$$

  Moreover, the following mapping is injective

$$(\mathrm{W}, \lambda) \in \mathcal{W}_\mathrm{G} \times \Lambda_\mathsf{DW} \mapsto \mathrm{W}^t(\lambda+\delta) - \delta \in \Lambda_\mathsf{W} \setminus (\mathrm{P} - \delta).$$

- If $f : \mathrm{G} \to \mathbb{C}$ is central, then $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(fB_\delta)(\xi) = 0$ for all $\xi \in \mathrm{P} - \delta$.

ii) Lemma 2.2 is now replaced by the following identity, valid for central functions $f : \mathrm{G} \to \mathbb{C}$

$$\|\widehat{f}\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})} = \mathcal{A}(G, q) \Big(\sum_{\lambda \in \Lambda_{\mathsf{W}} \setminus (\mathrm{P} - \delta)} \frac{\big|\mathcal{F}_{\mathbf{T}}(fB_\delta)(\lambda)\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \lambda + \delta \rangle\big|^{p'-2}} \Big)^{1/p'}.$$

iii) The bases of $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}$ respectively which generate Λ_{W} and L_{W} with integer coefficients are no longer the basis of fundamental weights and its predual. In fact, the fundamental weights generate the weight lattice of the universal covering group of G, which is a lattice containing Λ_{W} and strictly bigger than it. Therefore we need to define $\{\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_r\}$ and $\{\omega_1, \omega_2, \ldots, \omega_r\}$ just as the bases of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$ respectively for which L_{W} and Λ_{W} have integer coefficients. Once we have clarified this point, we can define $\mathfrak{T}$ in the same way and regard f_0 as a bounded complex-valued function on $\mathfrak{h}_{\mathbb{R}}$, supported in $\mathfrak{T}$ and symmetric under the reflections that generate $\mathcal{W}_{\mathrm{G}}$.

iv) Let us recall that if $\delta \notin \Lambda_{\mathsf{W}}$, the function A_δ is not well-defined on $\mathbf{T}$. But A_δ is originally defined on $\mathfrak{h}_{\mathbb{R}}$ and $\delta \notin \Lambda_{\mathsf{W}}$ is not an obstacle to work with A_δ as a function defined on $\mathfrak{h}_{\mathbb{R}}$. On the other hand, ii) leads us to consider (in the same spirit as in the proof given for simply connected groups) the function

$$\widehat{\widetilde{I}_\tau(f_0B_\delta)}(\xi) = \frac{1}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi + \delta \rangle|^\tau} \; \mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0B_\delta)(\xi).$$

Now, the remark given about A_δ shows that

$$\widehat{\widetilde{I}_\tau(f_0B_\delta)}(\xi) = \widehat{I_\tau(f_0A_\delta)}(\xi + \delta).$$

Hence we can proceed as before expressing the norm of this function in $L_{p'}(\mathfrak{h}_{\mathbb{R}}^*)$ as a Riemann sum, but this time we take the lattice $\Lambda_{\mathsf{W}} + \delta$ instead of Λ_{W}

$$\big\|\widehat{\widetilde{I}_\tau(f_0B_\delta)}\big\|_{L_{p'}(\mathfrak{h}_{\mathbb{R}}^*)} = \lim_{k \to \infty} \Big(\sum_{\lambda \in \Lambda_{\mathsf{W}} + \delta} \frac{\mathrm{V}_{\mathrm{G}}}{k^r} \frac{\big|\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0A_\delta)(k^{-1}\lambda)\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, k^{-1}\lambda \rangle\big|^{\tau p'}} \Big)^{1/p'}.$$

v) It is not difficult to check that $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0A_\delta)(k^{-1}\lambda) = k^{r-\sigma}\mathcal{F}_{\mathbf{T}}(\varphi_kB_\delta)(\lambda - \delta)$, where φ_k is defined as we did above. Hence we get

$$\begin{aligned} \big\|\widehat{\widetilde{I}_\tau(f_0B_\delta)}\big\|_{L_{p'}(\mathfrak{h}_{\mathbb{R}}^*)} &= \mathrm{V}_{\mathrm{G}}{}^{1/p'} \lim_{k \to \infty} \Big(\sum_{\lambda \in \Lambda_{\mathsf{W}} \setminus (\mathrm{P} - \delta)} \frac{\big|\mathcal{F}_{\mathbf{T}}(\varphi_kB_\delta)(\lambda)\big|^{p'}}{\prod_{\alpha \in \mathcal{R}^+} \big|\langle \alpha, \lambda + \delta \rangle\big|^{\tau p'}} \Big)^{1/p'} \\ &= \mathcal{B}(\mathrm{G}, p) \lim_{k \to \infty} \|\widehat{\varphi}_k\|_{\mathcal{L}_{p'}(\widehat{\mathrm{G}})}. \end{aligned}$$

The norm of φ_k is estimated as above. This completes the proof of Theorem A.

2.3 On the exact value of $\mathcal{K}(\mathrm{G}, p)$

Let us recall that, given $1 \le p \le 2$, the constant $\mathcal{K}(\mathrm{G}, p)$ is defined as follows

$$\mathcal{K}(\mathrm{G}, p) = \inf_{n \ge 1} \sup \Big\{ \mathsf{hy}_p(\mathrm{G}, f) \, \Big| \, f \in L_p(\mathrm{G}), \ f \text{ central}, \ \mathrm{supp}(f) \subset \mathcal{U}_n \Big\},$$

where $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a neighborhood basis of the identity $\mathbf{1}$ of G. This constant does not depend on the chosen basis and Theorem A states that $0 < \mathcal{K}(\mathrm{G}, p) \le 1$ for any $1 \le p \le 2$ and any compact semisimple Lie group. However, it would be extremely interesting to find the exact value of that constant. Sharp constants for the Hausdorff-Young inequality were investigated in [3], [4] or [36, 37, 38]. As it was pointed out in the Introduction, in the local case if

$$\mathcal{B}(\mathbb{R}, p) = \sqrt{p^{1/p} / p'^{1/p'}}$$

stands for the Babenko-Beckner constant, it is already known that

$$\mathcal{K}(\mathbb{T}^n, p) = \mathcal{B}(\mathbb{R}^n, p).$$

Also it is obvious that $\mathcal{K}(\mathrm{G}, 1) = \mathcal{K}(\mathrm{G}, 2) = 1$ for any compact group G. In the general case, a detailed look at the proof of Theorem A gives that the constant $\mathcal{K}(\mathrm{G}, p)$ is the supremum of

$$\frac{|\mathcal{W}_\mathrm{G}|^\tau}{\mathrm{V}_\mathrm{G}^{1/p'}} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^\tau \lim_{k \to \infty} \frac{\Big(\int_{\mathfrak{h}_\mathbb{R}^*} \big| \mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta(\xi)) \big|^{p'} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^{2-p'} d\xi \Big)^{1/p'}}{\Big(\int_{\mathfrak{h}_\mathbb{R}} |f_0 A_\delta(x)|^p \, \big| k^{|\mathcal{R}^+|} A_\delta(x/k) \big|^{2-p} dx \Big)^{1/p}}$$

for $1 < p \le 2$, where the supremum runs over the family of functions $f_0 : \mathfrak{h}_\mathbb{R} \to \mathbb{C}$, supported in $\mathfrak{T}$ and symmetric under the reflections generating the Weyl group of G. If $\mathcal{K}_{f_0}(\mathrm{G}, p)$ denotes the expression given above, then one easily gets that $\mathcal{K}_{f_0}(\mathrm{G}, p)$ equals

$$\frac{|\mathcal{W}_\mathrm{G}|^\tau}{(2\pi)^{\tau |\mathcal{R}^+|} \mathrm{V}_\mathrm{G}^{1/p'}} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^\tau \frac{\Big(\int_{\mathfrak{h}_\mathbb{R}^*} \big| \mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta(\xi)) \big|^{p'} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^{2-p'} d\xi \Big)^{1/p'}}{\Big(\int_{\mathfrak{h}_\mathbb{R}} |f_0 A_\delta(x)|^p \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2-p} dx \Big)^{1/p}}.$$

Moreover, taking $p = 2$ and by Plancherel theorem on compact groups, it follows that $\mathrm{V}_\mathrm{G} = 1$. The boundedness of this expression can be regarded as a weighted Hausdorff-Young inequality of Pitt type, see [5] for more on this topic.

3 Sharp Fourier type exponents

We conclude this paper by applying the local Hausdorff-Young inequality in the study of the sharp Fourier type exponents of commutative and non-commutative L_p spaces. In what follows, we shall assume the reader is familiar with some basic notions from operator space theory and vector-valued Schatten classes. The reader is referred to [7, 32] for the necessary background on operator spaces while Pisier's theory of vector-valued non-commutative L_p spaces can be found in [31]. In [12] there is also a condensed summary of results according to our needs.

Let G be a compact group equipped with its normalized Haar measure μ and let X be an operator space. Then, given an irreducible representation $\pi : \mathrm{G} \to \mathcal{B}(\mathcal{H}_\pi)$ of degree d_π and an integrable function $f : \mathrm{G} \to \mathrm{X}$, we define the Fourier coefficient of f at π as follows

$$\widehat{f}(\pi) = \int_{\mathrm{G}} f(g)\pi(g)^* \, d\mu(g).$$

Note that this is a $d_\pi \times d_\pi$ matrix with entries in X. According to [31] and using the operator space structure of X, we can consider the norm of $\widehat{f}(\pi)$ in the Schatten class $\mathcal{S}_p^{d_\pi}(\mathrm{X})$. This gives rise to the vector-valued analog of $\mathcal{L}_p(\widehat{\mathrm{G}})$

$$\mathcal{L}_p(\widehat{\mathrm{G}};\mathrm{X}) = \Big\{ \mathrm{A} \in \prod_{\pi \in \widehat{\mathrm{G}}} \mathrm{M}_{d_\pi} \otimes \mathrm{X} \ \big| \ \|\mathrm{A}\|_{\mathcal{L}_p(\widehat{\mathrm{G}};\mathrm{X})} = \Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi \|\mathrm{A}_\pi\|_{\mathcal{S}_p^{d_\pi}(\mathrm{X})}^p \Big)^{1/p} < \infty \Big\},$$

with the obvious modifications for $p = \infty$. Given $1 \le p \le 2$, we shall say that the operator space X has Fourier type p with respect to the compact group G if every $f \in L_p(\mathrm{G};\mathrm{X})$ satisfies the X-valued Hausdorff-Young inequality on G

$$\Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi \|\widehat{f}(\pi)\|_{\mathcal{S}_{p'}^{d_\pi}(\mathrm{X})}^{p'} \Big)^{1/p'} \le_{cb} \mathsf{K}_p(\mathrm{X},\mathrm{G}) \Big(\int_{\mathrm{G}} \|f(g)\|_{\mathrm{X}}^p \, d\mu(g) \Big)^{1/p}.$$

Here the symbol $\le_{cb}$ is used to indicate that the associated linear map $f \mapsto \widehat{f}$ is indeed completely bounded with *cb* norm $\mathsf{K}_p(\mathrm{X},\mathrm{G})$. Note that this does not make any difference in the scalar-valued case since Kunze's Hausdorff-Young inequality provides not only a bounded but a completely bounded map. The proof of this is a triviality when using some basic results from operator space theory, see [12] for a detailed explanation. The dual notion to Fourier type is called Fourier cotype. Roughly speaking, X has Fourier cotype p' whenever the inverse of the Fourier transform $\widehat{f} \in \mathcal{L}_{p'}(\widehat{\mathrm{G}};\mathrm{X}) \mapsto f \in L_p(\mathrm{G};\mathrm{X})$ is a completely bounded map. According to [12], X has Fourier type p iff X^* has Fourier cotype p' and X^* has Fourier type p iff X has Fourier cotype p'. In particular, all the forthcoming results could be stated in the language of Fourier cotype and obtain in such a way an equivalent formulation of the problem. We shall omit these equivalent formulations here.

3.1 Statement of the problem

According to [12] and similar to the commutative theory, every operator space X has Fourier type 1 with respect to any compact group G. Moreover, the *cb* norm $\mathsf{K}_1(\mathrm{X},\mathrm{G})$ is always 1. In particular, the complex interpolation method for operator spaces [30] provides the following inequality for $1 \leq p_1 \leq p_2 \leq 2$

$$\mathsf{K}_{p_1}(\mathrm{X},\mathrm{G}) \leq \mathsf{K}_1(\mathrm{X},\mathrm{G})^{1-\theta}\mathsf{K}_{p_2}(\mathrm{X},\mathrm{G})^{\theta} = \mathsf{K}_{p_2}(\mathrm{X},\mathrm{G})^{\theta} \quad \text{with} \quad \theta = p_2'/p_1'.$$

This means that the Fourier type becomes a stronger condition on the pair (X,G) as the exponent p tends to 2. The sharp Fourier type exponent of an operator space X with respect to a given compact group G is defined by

$$\mathsf{Sft}(\mathrm{X},\mathrm{G}) = \sup \Big\{ 1 \leq p \leq 2 \, \Big| \, \mathsf{K}_p(\mathrm{X},\mathrm{G}) < \infty \Big\}.$$

If X has Fourier type $\mathsf{Sft}(\mathrm{X},\mathrm{G})$ with respect to G we shall say that X has sharp Fourier type $\mathsf{Sft}(\mathrm{X},\mathrm{G})$. In the rest of this paper, we want to study the sharp Fourier type of commutative and non-commutative L_p spaces.

The analog of Hilbert spaces in the category of operator spaces are the so-called OH operator Hilbert spaces. The reader is referred to [30, 32] for more on this topic. Any commutative or non-commutative L_2 space will be equipped in this paper with its (natural) OH operator space structure. According to [12], any OH operator space satisfies $\mathsf{K}_2(\mathrm{OH},\mathrm{G}) = 1$ with respect to any compact group G. In particular, applying again the complex interpolation method for operator spaces, we obtain

$$\mathsf{K}_p(L_p(\mathcal{M}),\mathrm{G}) = 1 = \mathsf{K}_p(L_{p'}(\mathcal{M}),\mathrm{G}), \tag{7}$$

for any compact group G and any $1 \leq p \leq 2$. Here L_p denotes its most general notion. That is, the space $L_p(\mathcal{M})$ associated to a general von Neumann algebra $\mathcal{M}$. We point out that the classical L_p spaces arise by considering commutative von Neumann algebras. Note also that the complex interpolation of L_p spaces in this general framework is studied in Kosaki's paper [21], see also [14] and [34].

According to (7), given any exponent $1 \leq s \leq \infty$, the space $L_s(\mathcal{M})$ always has Fourier type $\min(s,s')$. In the commutative theory (i.e. when dealing with locally compact abelian groups) this is exactly the sharp Fourier type exponent of $L_s(\mathcal{M})$ regarded some natural exceptional cases are excluded. Therefore, the natural guess is that (7) should be the best Fourier type we can expect after excluding the exceptional cases. In other words, given $1 \leq p < q \leq 2$, we would like to find out conditions on G and $L_p(\mathcal{M})$ / $L_{p'}(\mathcal{M})$ under which

$$\mathsf{K}_q(L_p(\mathcal{M}),\mathrm{G}) = \infty \qquad \text{and} \qquad \mathsf{K}_q(L_{p'}(\mathcal{M}),\mathrm{G}) = \infty. \tag{8}$$

3.1.1 Necessary conditions

Before any other consideration on the identities given in (8) we need to exclude from our study the exceptional cases mentioned above. First we have to exclude *finite groups* from our treatment since, as we shall see immediately, every operator space X has sharp Fourier type 2 with respect to any finite group. Anyway the next result is a bit more accurate.

Lemma 3.1 *If* G_0 *is a finite group and* $1 \le p \le 2$, *every operator space* X *satisfies*

$$\mathsf{K}_p(\mathrm{X}, \mathrm{G}_0) \le |\mathrm{G}_0|^{1/p'}.$$

Proof. By complex interpolation it suffices to see that $\mathsf{K}_2(\mathrm{X}, \mathrm{G}_0) \le |\mathrm{G}_0|^{1/2}$ holds for any operator space X. According to the definition of Fourier type and to the notion of complete boundedness, we have to show that for all $m \ge 1$ and any family of functions

$$\Big\{ f_{ij} : \mathrm{G}_0 \to \mathrm{X} \, \Big| \, 1 \le i, j \le m \Big\}$$

the following inequality holds

$$\Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi \Big\| \Big(\; \widehat{f}_{ij}(\pi) \; \Big) \Big\|_{\mathcal{S}_2^{d_\pi m}(\mathrm{X})}^2 \Big)^{1/2} \le \sqrt{|\mathrm{G}_0|} \; \Big\| \Big(\; f_{ij} \; \Big) \Big\|_{\mathcal{S}_2^m(L_2(\mathrm{G}_0;\mathrm{X}))}.$$

However, if $\mathrm{G}_0 = \big\{ g_1, g_2, \ldots, g_n \big\}$ we have

$$\begin{aligned} \Big\| \Big(\; \widehat{f}_{ij}(\pi) \; \Big) \Big\|_{\mathcal{S}_2^{d_\pi m}(\mathrm{X})} &= \Big\| \Big(\; \frac{1}{n} \sum_{k=1}^n f_{ij}(g_k) \pi(g_k)^* \; \Big) \Big\|_{\mathcal{S}_2^{d_\pi m}(\mathrm{X})} \\ &\le \frac{1}{n} \sum_{k=1}^n \| \pi(g_k)^* \|_{\mathcal{S}_2^{d_\pi}} \Big\| \Big(\; f_{ij}(g_k) \; \Big) \Big\|_{\mathcal{S}_2^m(\mathrm{X})} \\ &\le \sqrt{d_\pi} \; \Big\| \Big(\; f_{ij} \; \Big) \Big\|_{\mathcal{S}_2^m(L_2(\mathrm{G}_0;\mathrm{X}))} \end{aligned}$$

Therefore we obtain

$$\Big\| \Big(\; \widehat{f}_{ij} \; \Big) \Big\|_{\mathcal{S}_2^m(\mathcal{L}_E^2(\widehat{\mathrm{G}}))} \le \sqrt{\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi^2} \; \Big\| \Big(\; f_{ij} \; \Big) \Big\|_{\mathcal{S}_2^m(L_2(\mathrm{G}_0;\mathrm{X}))}$$

and, since $\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi^2 = |\mathrm{G}_0|$ by the Peter-Weyl theorem, the proof is completed. ■

Next we show that we can not work with *finite-dimensional* L_p *spaces.* To that aim we need to define the *cb* distance between two operator spaces. This notion is due to Pisier and it provides the analog of the Banach-Mazur distance between two Banach spaces in the context of operator space theory. Let X_1 and X_2 be operator spaces. We define their *cb* distance by the relation

$$d_{cb}(\mathrm{X}_1, \mathrm{X}_2) = \inf \Big\{ \|\Lambda\|_{cb(\mathrm{X}_1,\mathrm{X}_2)} \|\Lambda^{-1}\|_{cb(\mathrm{X}_2,\mathrm{X}_1)} \Big\}$$

where the infimum runs over all complete isomorphisms $\Lambda : \mathrm{X}_1 \to \mathrm{X}_2$. The following inequality, also extracted from [12], relates the Fourier type of two operator spaces X_1 and X_2 with their *cb* distance

$$\mathsf{K}_p(\mathrm{X}_2, \mathrm{G}) \le d_{cb}(\mathrm{X}_1, \mathrm{X}_2)\, \mathsf{K}_p(\mathrm{X}_1, \mathsf{G}). \tag{9}$$

In particular, given $1 \le p \le 2$ we deduce from (7) that

$$\begin{array}{rcl} \mathsf{K}_2(L_p(\mathcal{M}), \mathrm{G}) & \le & d_{cb}(L_p(\mathcal{M}), L_2(\mathcal{M}))\ \mathsf{K}_2(L_2(\mathcal{M}), \mathrm{G}) = d_{cb}(L_p(\mathcal{M}), L_2(\mathcal{M})), \\ \mathsf{K}_2(L_{p'}(\mathcal{M}), \mathrm{G}) & \le & d_{cb}(L_{p'}(\mathcal{M}), L_2(\mathcal{M}))\, \mathsf{K}_2(L_2(\mathcal{M}), \mathrm{G}) = d_{cb}(L_{p'}(\mathcal{M}), L_2(\mathcal{M})). \end{array}$$

Thus, since the *cb* distance between any two finite-dimensional L_p spaces (defined over the same finite-dimensional von Neumann algebra $\mathcal{M}$) is finite, we conclude that any finite-dimensional L_p space has Fourier type 2. Therefore, we shall exclude the finite-dimensional L_p spaces from our study in what follows.

3.1.2 Sufficient conditions

For the moment we have imposed two necessary conditions. Namely, we have to exclude finite groups and finite-dimensional L_p spaces. Now we concentrate on a sufficient condition. Indeed, given any infinite-dimensional space $L_s(\mathcal{M})$ for some $1 \le s \le \infty$, it is clear that $\ell_s(n)$ embeds isometrically in $L_s(\mathcal{M})$ for any $n \ge 1$. Hence, we obtain the following lower bounds for the constants in (8)

$$\mathsf{K}_q(L_s(\mathcal{M}), \mathrm{G}) \ge \lim_{n \ge 1} \mathsf{K}_q(\ell_s(n), \mathrm{G}). \tag{10}$$

Remark 3.2 Note that the inequality above is implied by the fact that, given a closed subspace Y of an operator space X, we always have $\mathsf{K}_p(\mathrm{Y}, \mathrm{G}) \le \mathsf{K}_p(\mathrm{X}, \mathrm{G})$ for any $1 \le p \le 2$ and any compact group G. The proof can be found in [12]. Note also that the limit above always exists since, using one more time $\mathsf{K}_p(\mathrm{Y}, \mathrm{G}) \le \mathsf{K}_p(\mathrm{X}, \mathrm{G})$, we deduce that the sequence $\mathsf{K}_q(\ell_s(n), \mathrm{G})$ is non-decreasing.

In summary, given exponents $1 \le p < q \le 2$ and going back to our problem, it suffices to show that the sequences $\mathsf{K}_q(\ell_p(n), \mathrm{G})$ and $\mathsf{K}_q(\ell_{p'}(n), \mathrm{G})$ diverge to infinity as $n \to \infty$. Therefore, in what follows we shall study the growth of the constants $\mathsf{K}_q(\ell_p(n), \mathrm{G})$ and $\mathsf{K}_q(\ell_{p'}(n), \mathrm{G})$. Note that this problem is thereby more interesting that the sharp Fourier type exponents of L_p spaces. The first remark concerning these constants is that the following upper bounds follows from (9)

$$\begin{array}{rcll} \mathsf{K}_q(\ell_p(n), \mathrm{G}) & \le & d_{cb}(\ell_p(n), \ell_q(n))\ \mathsf{K}_q(\ell_q(n), \mathrm{G}) & = \ n^{1/p-1/q}, \\ \mathsf{K}_q(\ell_{p'}(n), \mathrm{G}) & \le & d_{cb}(\ell_{p'}(n), \ell_{q'}(n))\, \mathsf{K}_q(\ell_{q'}(n), \mathrm{G}) & = \ n^{1/q'-1/p'}. \end{array} \tag{11}$$

Note that $1/p - 1/q = 1/q' - 1/p'$ so that we obtain the same upper bound for both constants. The rest of this paper is devoted to show that this is exactly the order of growth of the constants $\mathsf{K}_q(\ell_p(n),\mathrm{G})$ and $\mathsf{K}_q(\ell_{p'}(n),\mathrm{G})$. We begin by studying the growth of the first constants for compact semisimple Lie groups. The local Hausdorff-Young inequality will be essential in our arguments. After that, we give two different approaches which provides a complete solution of the problem.

Remark 3.3 Given an operator space X, it is also natural to consider the sharp Fourier type exponent of the vector-valued space $L_s(\mathcal{M};\mathrm{X})$. According to Remark 3.2, we have

$$\mathsf{K}_q(L_s(\mathcal{M};\mathrm{X}),\mathrm{G}) \geq \max\Big\{\mathsf{K}_q(L_s(\mathcal{M}),\mathrm{G}),\mathsf{K}_q(\mathrm{X},\mathrm{G})\Big\}.$$

In particular, it is not difficult to check that (see [11, 12])

$$\mathsf{Sft}(L_s(\mathcal{M};\mathrm{X}),\mathrm{G}) = \min\Big\{\mathsf{Sft}(L_s(\mathcal{M}),\mathrm{G}),\mathsf{Sft}(\mathrm{X},\mathrm{G})\Big\}.$$

3.2 On the growth of $\mathsf{K}_q(\ell_p(n),\mathrm{G})$

In this paragraph we study the growth of $\mathsf{K}_q(\ell_p(n),\mathrm{G})$ for any compact semisimple Lie group G. Semisimplicity is an essential assumption since eventually we shall need to apply the local Hausdorff-young inequality. More concretely, we are proving Theorem B in the Introduction. The first consequence we shall need from the semisimplicity of G is the existence of a maximal torus $\mathbf{T}$. Namely, it allows us to consider a countable family $g_1, g_2, \ldots$ of pairwise commuting elements of G, just take $g_k \in \mathbf{T}$ for all $k \geq 1$. On the other hand, given $n \geq 1$ we take $\mathcal{U}_n$ to be a neighborhood of the identity $\mathbf{1}$ of G satisfying

$$g_j^{-1}\mathcal{U}_n \cap g_k^{-1}\mathcal{U}_n = \emptyset \quad \text{for} \quad 1 \leq j,k \leq n \quad \text{and} \quad j \neq k.$$

Note that we can always consider a central function $f_n \in L_q(\mathrm{G})$ supported in $\mathcal{U}_n$, for example take $\mathcal{U}_n$ to be invariant under conjugations (*cf.* Lemma (5.24) in [8]) and $f_n = 1_{\mathcal{U}_n}$ where $1_{\mathcal{U}}$ denotes the characteristic function of $\mathcal{U}$. Henceforth, the function $f_n : \mathrm{G} \to \mathbb{C}$ will be a central function in $L_q(\mathrm{G})$ supported in $\mathcal{U}_n$, to be fixed later. Then we define $\Phi_n : \mathrm{G} \to \mathbb{C}^n$ by

$$\Phi_n(g) = (f_n(g_1 g), f_n(g_2 g), \ldots, f_n(g_n g)).$$

We obviously have the estimate

$$\mathsf{K}_q(\ell_p(n),\mathrm{G}) \geq \mathsf{hy}_q\big(\mathrm{G},\Phi_n\big) = \frac{\|\widehat{\Phi}_n\|_{\mathcal{L}_{q'}(\widehat{\mathrm{G}};\ell_p(n))}}{\|\Phi_n\|_{L_q(\mathrm{G};\ell_p(n))}}. \tag{12}$$

Let us point out that (11) provides the upper estimate in Theorem B. Therefore, it suffices to show that the quotient in (12) is bounded below by $\mathcal{K}(\mathrm{G},q)\, n^{1/p-1/q}$. The following result will be needed for that purpose.

Lemma 3.4 *Let $\pi \in \widehat{\mathrm{G}}$ and $n \geq 1$. Consider the matrix-valued vector*

$$\mathrm{A}_{\pi,n} = (\pi(g_1), \pi(g_2), \ldots, \pi(g_n)).$$

Then, given $1 \leq p_1, p_2 \leq \infty$, we have

$$\|\mathrm{A}_{\pi,n}\|_{\ell_{p_1}(n;\mathcal{S}_{p_2}^{d_\pi})} = \|\mathrm{A}_{\pi,n}\|_{\mathcal{S}_{p_2}^{d_\pi}(\ell_{p_1}(n))} = n^{1/p_1} d_\pi^{1/p_2}.$$

Proof. Since $g_1, g_2, \ldots, g_n$ are pairwise commuting, there exists a basis of $\mathbb{C}^{d_\pi}$ made up of common eigenvectors of $\pi(g_1), \pi(g_2), \ldots, \pi(g_n)$. Therefore, in that basis, all these matrices are diagonal

$$\pi(g_k) = \begin{pmatrix} \rho_{1,k} & & \\ & \ddots & \\ & & \rho_{d_\pi,k} \end{pmatrix}.$$

Moreover, the unitarity of $\pi(g_k)$ gives $|\rho_{j,k}| = 1$ for $1 \leq j \leq d_\pi$. Hence, applying the complete isometry between $\ell_{p_2}(d_\pi; \mathrm{X})$ and the subspace of diagonal matrices of $\mathcal{S}_{p_2}^{d_\pi}(\mathrm{X})$ (*cf.* Corollary (1.3) of [31]), we easily obtain the desired equality. ∎

The norm of $\widehat{\Phi}_n$ in $\mathcal{L}_{q'}(\widehat{\mathrm{G}}; \ell_p(n))$. We begin by recalling that, since f_n is central,

$$\widehat{f_n}(\pi) = \frac{1}{d_\pi} \int_G f_n(g) \overline{\chi_\pi(g)}\, d\mu(g)\ 1_{d_\pi} = \gamma_{\pi,n} 1_{d_\pi}$$

by Schur's lemma. Again, χ_π denotes the irreducible character associated to π and 1_m stands for the $m \times m$ identity matrix. On the other hand $f_n(g_k \cdot)$ is the translation by g_k of f_n, therefore

$$\widehat{\Phi}_n(\pi) = \frac{1}{d_\pi} \int_G f_n(g) \overline{\chi_\pi(g)}\, d\mu(g)\ (\pi(g_1), \pi(g_2), \ldots, \pi(g_n)) = \gamma_{\pi,n} \mathrm{A}_{\pi,n}.$$

Therefore, Lemma 3.4 gives

$$\begin{aligned} \|\widehat{\Phi}_n\|_{\mathcal{L}_{q'}(\widehat{\mathrm{G}};\ell_p(n))} &= \Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi |\gamma_{\pi,n}|^{q'} \|\mathrm{A}_{\pi,n}\|_{\mathcal{S}_{q'}^{d_\pi}(\ell_p(n))}^{q'} \Big)^{1/q'} \\ &= n^{1/p} \Big(\sum_{\pi \in \widehat{\mathrm{G}}} d_\pi^2 |\gamma_{\pi,n}|^{q'} \Big)^{1/q'} = n^{1/p}\, \|\widehat{f_n}\|_{\mathcal{L}_{q'}(\widehat{\mathrm{G}})}. \end{aligned}$$

The norm of Φ_n in $L_q(\mathrm{G}; \ell_p(n))$. We have

$$\begin{aligned} \|\Phi_n\|_{L_q(\mathrm{G};\ell_p(n))} &= \Big(\int_G \Big(\sum_{k=1}^n |f_n(g_k g)|^p \Big)^{q/p} d\mu(g) \Big)^{1/q} \\ &= \Big(\sum_{k=1}^n \|f_n(g_k \cdot)\|_{L_q(\mathrm{G})}^q \Big)^{1/q} = n^{1/q}\, \|f_n\|_{L_q(\mathrm{G})}. \end{aligned}$$

Note that we are using $\operatorname{supp} f_n(g_k \,\cdot) = g_k^{-1}\mathcal{U}_n$ so that

$$\operatorname{supp} f_n(g_j \,\cdot) \cap \operatorname{supp} f_n(g_k \,\cdot) = \emptyset \quad \text{for} \quad 1 \le j,k \le n \quad \text{and} \quad j \neq k.$$

In summary, we have obtained that $\mathsf{K}_q(\ell_p(n), \mathrm{G}) \ge \mathcal{K}(\mathrm{G}, q, n)\, n^{1/p-1/q}$ where the constant $\mathcal{K}(\mathrm{G}, q, n)$ is given by

$$\mathcal{K}(\mathrm{G}, q, n) = \mathsf{hy}_q(\mathrm{G}, f_n) = \frac{\|\widehat{f}_n\|_{\mathcal{L}_{q'}(\widehat{\mathrm{G}})}}{\|f_n\|_{L_q(\mathrm{G})}}.$$

In particular,

$$\mathcal{J}(\mathrm{G}, q)\, n^{\frac{1}{p}-\frac{1}{q}} = \Big(\inf_{n\ge 1} \mathcal{K}(\mathrm{G}, q, n)\Big)\, n^{\frac{1}{p}-\frac{1}{q}} \le \mathsf{K}_q(\ell_p(n), \mathrm{G}).$$

Therefore, since the f_n's are not fixed yet, we can consider any central function $f_n : \mathrm{G} \to \mathbb{C}$ in $L_q(\mathrm{G})$ supported in $\mathcal{U}_n$ and Theorem B follows from Theorem A. Indeed, we have

$$\mathcal{J}(\mathrm{G}, q) = \inf_{n\ge 1} \sup \Big\{ \mathsf{hy}_q(\mathrm{G}, f) \;\Big|\; f \in L_q(\mathrm{G}),\ f \text{ central},\ \operatorname{supp}(f) \subset \mathcal{U}_n \Big\} = \mathcal{K}(\mathrm{G}, q) > 0.$$

3.3 On the growth of $\mathcal{K}_q(\ell_{p'}(n), \mathrm{G})$

As we recalled in the Introduction, the growth of $\mathsf{K}_q(\ell_{p'}(n), \mathrm{G})$ can not be obtained by applying the local Hausdorff-Young inequality. The techniques employed in [17] for the solution to this problem como from non-commutative probability and are out the scope of this paper. However, we can at least describe the main ideas.

To explain the main arguments in [17], we consider a probability space $(\Omega, \mathcal{M}, \mu)$, an infinite index set Σ and a family $\{d_\sigma \mid \sigma \in \Sigma\}$ of positive integers. Then, the quantized Rademacher system associated to Σ is defined by a collection

$$\mathcal{R}_\Sigma = \Big\{ \rho_\sigma : \Omega \to O(d_\sigma) \;\Big|\; \sigma \in \Sigma \Big\}$$

of independent random orthogonal matrices, uniformly distributed on the orthogonal group $O(d_\sigma)$. This kind of systems were defined by Marcus and Pisier in [25] and provide a non-commutative counterpart of the classical Rademacher variables. In [13], we defined the notions of $\mathcal{R}_\Sigma$-type and $\mathcal{R}_\Sigma$-cotype of an operator space X. These notions may be considered as an operator space analog of the classical notions of Rademacher type and cotype. Here we shall only use the notion of Rademacher cotype. Given $1 \le p \le 2$, an operator space X is said to have $\mathcal{R}_\Sigma$-cotype p' if there exists an absolute constant $\mathsf{R}_{p'}(\mathrm{X}, \Sigma)$ such that the inequality

$$\Big(\sum_{\sigma\in\Gamma} d_\sigma \|A_\sigma\|_{\mathcal{S}_{p'}^{d_\sigma}(\mathrm{X})}^{p'} \Big)^{1/p'} \le_{cb} \mathsf{R}_{p'}(\mathrm{X}, \Sigma) \Big(\int_\Omega \Big\| \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A_\sigma \rho_\sigma(\omega)) \Big\|_{\mathrm{X}}^p d\mu(\omega) \Big)^{1/p}$$

holds for any finite subset Γ of Σ and any collection

$$\Big\{A_\sigma \in \mathrm{M}_{d_\sigma} \ \Big| \ \sigma \in \Gamma\Big\}.$$

Applying from [25] a matrix-valued version of the contraction principle, we proved in [13] that Fourier type p with respect to a compact group G implies $\mathcal{R}_\Sigma$-cotype p' whenever we take $\Sigma = \widehat{\mathrm{G}}$. More concretely, we have

$$\mathsf{R}_{p'}(\mathrm{X}, \widehat{\mathrm{G}}) \le \mathsf{K}_p(\mathrm{X}, \mathrm{G})$$

for any $1 \le p \le 2$, any compact group G and any operator space X. Clearly, this reduces the problem to find the sharp Rademacher cotype exponents of L_p. Then, the stochastic independence of the Rademacher variables allows us to use different tools, such as the non-commutative martingale inequalities [16, 18, 33] and other techniques from non-commutative probability. These techniques give rise to the following result, see [17] for more on this topic.

Theorem D. If $1 \le p < q \le 2$, the following holds for any compact group G

$$\begin{array}{lcll} \mathsf{R}_{q'}(\ell_p(n), \widehat{\mathrm{G}}) & \longrightarrow & \infty & \text{as} \quad n \to \infty, \\ \mathsf{R}_{q'}(\ell_{p'}(n), \widehat{\mathrm{G}}) & \longrightarrow & \infty & \text{as} \quad n \to \infty. \end{array}$$

Note that Theorem D provides the sharp Fourier type exponent of L_p for any compact group G when $1 \le p \le 2$. This case is not considered in Theorem B. However, the constants obtained in [17] are

$$\mathsf{R}_{q'}(\ell_p(n), \widehat{\mathrm{G}}) \gtrsim n^{1/2p - 1/2q} \quad \text{and} \quad \mathsf{R}_{q'}(\ell_{p'}(n), \widehat{\mathrm{G}}) \simeq n^{1/p-1/q}.$$

The first one is obviously worse than the one provided by the local Hausdorff-Young inequality. The second one is the one given in Theorem C. Finally, we also point out that the argument given above (based on the contraction principle) applies to any uniformly bounded quantized orthonormal system (*cf.* [13] for the definition). In particular, Theorem D solves the problem of sharp type/cotype exponents of L_p with respect to any such system.

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