# DIRECTIONAL MAXIMAL OPERATORS AND LACUNARITY IN HIGHER DIMENSIONS

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ABSTRACT. We introduce a notion of lacunarity in higher dimensions for which we can bound the associated directional maximal operators in  $L^p(\mathbb{R}^n)$ , with p > 1. In particular, we are able to treat the (almost disjoint) classes previously considered by Nagel–Stein–Wainger, Sjögren–Sjölin and Carbery. Closely related to this, we find a characterisation of the sets of directions which give rise to bounded maximal operators. The bounds enable Lebesgue-type differentiation of integrals in  $L^p_{loc}(\mathbb{R}^n)$ , replacing balls by tubes which point in these directions.

#### INTRODUCTION

For  $n \ge 2$  and a set of directions  $\Omega$  in the unit sphere  $\mathbb{S}^{n-1}$ , the directional maximal operator  $M_{\Omega}$  is defined, initially on Schwartz functions, by

$$M_{\Omega}f(x) = \sup_{\omega \in \Omega} \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - t\omega)| dt.$$

If  $\Omega$  consists of a single direction and p > 1, the boundedness of  $M_{\Omega}$  from  $L^{p}(\mathbb{R}^{n})$  to  $L^{p}(\mathbb{R}^{n})$  follows from the Hardy–Littlewood maximal theorem. If  $\Omega$  consists of many directions, two questions naturally arise:

- (i) When  $\Omega$  is an arbitrary finite subset of  $\mathbb{S}^{n-1}$ , a fundamental problem is to determine the best bounds for the  $L^p$ -operator norm of  $M_{\Omega}$  as a function of  $|\Omega|$  and p.
- (ii) When  $\Omega$  is an infinite subset of  $\mathbb{S}^{n-1}$ , one can also ask for conditions on the directions which ensure that  $M_{\Omega}$  is  $L^p$ -bounded.

In two dimensions, the questions have been answered with remarkable accuracy (see [7, 26, 16, 17] for the first question, [25, 10, 20, 23, 4] for the

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second question, or [2, 3, 1, 15] which address the two questions in a unified way), however much less is known in higher dimensions (see [27, 5, 19] for the first question and [20, 23, 6] for the second). We will prove a localization principle – subsets of the directions can be considered independently from the rest of the directions – from which we draw conclusions for the second question in three dimensions and more. A fundamental difference between the two-dimensional problem, with directions in  $\mathbb{S}^1$ , and that of higher dimensions is that we are no longer able to order the directions.

We partition the unit ball in such a way that it resembles a peeled orange with infinitely many segments. In three dimensions, we make three partitions, each time with a different axis of partition. The independent sets of our localisation principle will be contained in these segments. More precisely and more generally, for  $\sigma \in \Sigma$ , where

$$\Sigma \equiv \Sigma(n) = \{(j,k) : 1 \le j < k \le n\}$$

we consider  $\{\theta_{\sigma,i}\}_{i\in\mathbb{Z}}$  that satisfy  $0 < \theta_{\sigma,i+1} \leq \lambda_{\sigma} \theta_{\sigma,i}$  with lacunary constants  $0 < \lambda_{\sigma} < 1$ . Then, for an orthonormal basis  $(e_1, \ldots, e_n)$ , we divide the directions into the subsets  $\Omega_{\sigma,i}$  defined by

$$\Omega_{\sigma,i} = \left\{ \omega \in \Omega : \theta_{\sigma,i+1} < \left| \frac{\omega \cdot e_k}{\omega \cdot e_j} \right| \leqslant \theta_{\sigma,i} \right\}$$

(see Figure 1). Note that the segments become thinner as *i* converges to  $\pm \infty$  and the partition of  $\Omega$  is completed by including the set  $\Omega_{\sigma,\infty} = \Omega \cap (e_i^{\perp} \cup e_k^{\perp})$ .

Writing  $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$ , we prove the following localisation principle which recalls the separation of dyadic frequency scales provided by Littlewood– Paley theory (see also [21, 11] for another kind of one–dimensional localisation). A difference is that we have many lacunary partitions instead of one, however the result is sharp in the sense that the supremum over partitions must be taken over the whole of  $\Sigma$ . Nor could it be made more flexible by allowing the segments to 'accumulate' away from the hyperplanes perpendicular to the basis vectors.

**Theorem A.** Let  $n \ge 2$  and p > 1. Then

$$\|M_{\Omega}\|_{p \to p} \leqslant C \sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}^*} \|M_{\Omega_{\sigma,i}}\|_{p \to p},$$

where C depends only on n, p and the lacunary constants  $\lambda_{\sigma}$  for  $\sigma \in \Sigma$ .

As with the almost orthogonality principle of Alfonseca, Soria and Vargas in two dimensions [2, 3, 1], we recover the previously known results for question (ii) in higher dimensions. Nagel, Stein and Wainger [20] proved the  $L^p$ -boundedness of the maximal operator associated to the directions

$$\{(\vartheta_i^{a_1},\ldots,\vartheta_i^{a_n})\}_{i\geqslant 1},$$



**Figure 1.** A partition for  $\sigma = (1, 2)$  of a three-dimensional dissection.

where  $0 < a_1 < \ldots < a_n$  and  $0 < \vartheta_{i+1} \leq \lambda \vartheta_i$  with lacunary constant  $0 < \lambda < 1$ . We can apply Theorem A with  $\theta_{\sigma,i} = \vartheta_i^{a_k - a_j}$  and  $\lambda_{\sigma} = \lambda^{a_k - a_j}$ , where  $\sigma = (j, k)$ , reducing the problem to that of a single direction. Note that it makes no difference if the directions are normalised to live on the unit sphere or not. On the other hand, Carbery [6] proved that the maximal operator associated to the directions

(1) 
$$\{(2^{k_1},\ldots,2^{k_n})\}_{k_1,\ldots,k_n\in\mathbb{Z}}$$

is  $L^p$ -bounded with p > 1. Taking  $\theta_{\sigma,i} = 2^{-i}$ , the resulting sets of directions  $\Omega_{\sigma,i}$  are restricted to (n-1)-dimensional hyperplanes, so that by choosing a suitable basis and applying Fubini's theorem, we reduce to the (n-1)-dimensional problem. Iterating the process we eventually end up with isolated directions as before.

It is not sufficient to constrain the angles between an infinite number of directions if they are to give rise to a bounded maximal operator in higher dimensions, even if the directions live inside certain (indeed most) smooth curves. However Theorem A suggests a definition of lacunarity that gives rise to bounded maximal operators in general. Given  $\Omega \subset \mathbb{S}^{n-1}$ , an orthonormal basis of span $(\Omega) = \mathbb{R}^d$  with  $d \leq n$ , and lacunary sequences  $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ , define partitions  $\{\Omega_{\sigma,i}\}_{i \in \mathbb{Z}^*}$  for each  $\sigma \in \Sigma(d)$ . We call such a choice of  $\frac{1}{2}d(d-1)$  partitions a dissection. We say that  $\Omega$  is

- *lacunary of order* 0 if it consists of a single direction
- *lacunary of order* L if there is a dissection for which the sets  $\Omega_{\sigma,i}$  are lacunary of order  $\leq L-1$  for all  $i \in \mathbb{Z}^*$  and  $\sigma \in \Sigma(d)$ , with uniformly bounded lacunary constants.

We say that a set of directions is *lacunary* if it is a finite union of sets which are lacunary of finite order and we denote the class of such sets by Lac(n). Note that the class contains sets of directions which are confined to lower dimensional subspaces.

According to this definition, the Nagel–Stein–Wainger directions are lacunary of order 1 and the Carbery directions are lacunary of order n-1. By repeatedly applying Theorem A as before, if  $\Omega$  is lacunary (of finite order), then  $M_{\Omega}$  is  $L^p(\mathbb{R}^n)$ –bounded with p > 1. This extends the two–dimensional result due to Sjögren–Sjölin [23] (the union of K sets of directions of lacunary order L with respect to their definition, is lacunary of order 2KL + 1with respect to ours). We have broken with the two–dimensional tradition (which is one–dimensional in the sense that the directions are contained in a circle), whereby the label 'lacunary' is reserved for the sets which are lacunary of order one. This is because such sets are less special in higher dimensions (they cannot necessarily be represented as a sequence for example) and at first glance the set of directions defined in (1) seems as deserving of the label 'lacunary' as any other.

As a corollary we obtain a generalisation of the Fundamental Theorem of Calculus. After a suitably fine finite splitting of the directions, the operator  $M_{\Omega}$  can be composed with one-dimensional Hardy-Littlewood maximal operators to dominate a constant multiple of the maximal operator  $\mathcal{M}_{\Omega}$ defined by

$$\mathcal{M}_{\Omega}f(x) = \sup_{x \in T \in \mathcal{T}_{\Omega}} \frac{1}{|T|} \int_{T} |f(y)| \, dy.$$

Here,  $\mathcal{T}_{\Omega}$  denotes the family of tubes which point in a direction of  $\Omega$ . Standard density arguments yield the following Lebesgue type differentiation result.

**Corollary B.** Let  $n \ge 2$  and  $\Omega \in Lac(n)$ . Then

$$\lim_{\substack{x \in T \in \mathcal{T}_{\Omega} \\ \operatorname{liam}(T) \to 0}} \frac{1}{|T|} \int_{T} f(y) \, dy = f(x), \quad a.e. \ x \in \mathbb{R}^{n},$$

for all  $f \in L^p_{loc}(\mathbb{R}^n)$  with p > 1.

Sets of directions which give rise to unbounded  $M_{\Omega}$  and  $\mathcal{M}_{\Omega}$  can be considered if we place further restrictions on the tubes  $\mathcal{T}_{\Omega}$ . Most commonly the eccentricity (length/width) of the tubes is fixed, especially when treating question (i) above. For question (ii), Córdoba [8] proved that the associated maximal operator is bounded, with a logarithmic dependency on the eccentricity, if the directions are restricted to a curve which intersects the hyperplanes of  $\mathbb{R}^n$  no more than a uniformly bounded number of times.

We turn now to the question of characterising the sets of directions  $\Omega$ for which  $M_{\Omega}$  is bounded from  $L^{p}(\mathbb{R}^{n})$  to  $L^{p}(\mathbb{R}^{n})$ . We denote the class of such sets by  $Max_p(n)$ . Bateman [4] proved that  $Max_p(2) \subset Lac(2)$  which, combined with the result of Sjögren–Sjölin, yields the equivalence

$$Max_p(2) \equiv Lac(2), \quad 1$$

We do not know if this is true in higher dimensions, however we characterise  $\mathsf{Max}_{\mathsf{p}}(n)$  using a formally larger class. For this we take advantage of a quantitive version of Bateman's theorem via projections onto two-dimensional subspaces  $\Pi \subset \mathbb{R}^n$ . Given a set of directions  $\Omega$ , we define the *shadow of*  $\Omega$  on  $\Pi$  to be the normalised orthogonal projection onto  $\Pi$ , so that the shadow lives in a copy of  $\mathbb{S}^1 \subset \Pi$ . We denote by  $\mathsf{Lsh}(n)$  the class of sets of directions whose shadows are all lacunary of finite order, where both the lacunary orders and lacunary constants are uniformly bounded above.

In order to define our characterising class, we fix an auxiliary  $\varepsilon > 0$  and say that a set  $\Omega_{\tilde{\sigma},i_{\tilde{\sigma}}}$  of a dissection is *dominating* if it satisfies

$$\|M_{\Omega_{\sigma,i}}\|_{p\to p} \leqslant \|M_{\Omega_{\tilde{\sigma},i\sim}}\|_{p\to p} + \varepsilon \quad \text{for all} \quad i \in \mathbb{Z}^*, \ \sigma \in \Sigma(d).$$

Similarly to before, we say that  $\Omega$  is

- p-lacunary of order  $\theta$  if it consists of a single direction
- *p*-lacunary of order L if there is a dissection with a dominating set which is *p*-lacunary of order  $\leq L 1$ .

We say that a set of directions is *p*-lacunary if it is *p*-lacunary of finite order and we denote the class of such sets by  $Lac_p(n, \varepsilon)$ . Finally we write

$$\mathsf{Lac}_{\mathsf{p}}(n) := \bigcap_{\varepsilon > 0} \mathsf{Lac}_{\mathsf{p}}(n, \varepsilon).$$

In the following equivalence we see that the directions which give rise to bounded maximal operators can be no worse, loosely speaking, than directions that can be divided into isolated directions by a finite number of lacunary dissections.

# **Theorem C.** Let $n \ge 2$ and 1 . Then

$$Lac(n) \subset Max_p(n) \equiv Lac_p(n) \subset Lsh(n).$$

With n = 2, these classes coincide of course, and so  $Max_p(2)$  is the same for all 1 . It is tempting to suppose that this is also true in higherdimensions – it seems reasonable to expect that a member of <math>Lsh(n) could be dissected into isolated directions – however it may also be that  $Max_p(n)$ grows with p. In any case, given the nature of the definitions of Lac(n) and Lsh(n), we see that  $Max_p(n)$  is not so far from a purely two-dimensional concept. This should be compared with [18], where Kakeya sets in  $\mathbb{R}^3$  with near minimal dimension were shown to have a 'planiness' property.

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In the following section, we prove Theorem A which implies the first inclusion of Theorem C. The key ingredient is a nonlinear and nonpositive partition of a hyperplane in which we compensate for the points which are covered more than once by removing smaller sets. That is to say, our partition of unity is more like a covering (normally completely inadequate on the frequency side), but by adding and subtracting enough times we are able to partition the hyperplane with intersections of tensor products of twodimensional cones. These give rise to a priori frightening nonlinear terms, however they are dealt with later in a reasonably trivial fashion. In the second section we prove the equivalence and the final inclusion of Theorem C. Unusually in this context, this follows by a topological argument. In the third section, we justify a number of remarks from above by constructing sets of apparently well-behaved directions for which the associated maximal operators are unbounded. In the final section, we provide a corollary for the maximal directional Hilbert transform. Some of these results were announced in [22].

### 1. Proof of Theorem A

By a finite splitting we can suppose that the directions  $\Omega$  are contained in the first open 'octant' of the unit sphere  $\mathbb{S}^{n-1} \cap \mathbb{R}^n_+$ . We consider intersections of the segments to obtain cells of directions

$$\Omega_{\mathbf{i}} = \bigcap_{\sigma \in \Sigma} \Omega_{\sigma, i_{\sigma}} \quad \text{for each} \quad \mathbf{i} = (i_{\sigma})_{\sigma \in \Sigma} \in \mathbb{Z}^{\Sigma}.$$

This yields a finer partition than those of the introduction;

$$\Omega = \bigcup_{\mathbf{i} \in \mathbb{Z}^{\Sigma}} \Omega_{\mathbf{i}} \quad \text{so that} \quad M_{\Omega} = \sup_{\mathbf{i} \in \mathbb{Z}^{\Sigma}} M_{\Omega_{\mathbf{i}}}.$$

Note that many of the cells are empty, however we will see that this overdetermination is somehow unavoidable. Let  $K_{\sigma,i}$  denote the convolution operator associated to a Fourier multiplier  $\psi_{\sigma,i}$ , smooth on  $\mathbb{R}^n \setminus \{0\}$ , equal to one on

$$\Psi_{\sigma,i} = \left\{ \xi \in \mathbb{R}^n : \frac{1}{n} \theta_{\sigma,i+1} < -\frac{\xi_j}{\xi_k} \leqslant n \theta_{\sigma,i} \right\},\,$$

and supported in a similar cone with n replaced by n + 1.

The key geometric fact used in the proof of the following lemma is that the hyperplane perpendicular to  $\omega$  is contained in  $\bigcup_{\sigma \in \Sigma} \Psi_{\sigma,i_{\sigma}}$  for all  $\omega \in \Omega_{\mathbf{i}}$ . This is no longer true if, in the definition of the cones, n is replaced by a constant strictly less than n-1. At this point we do not use that the dividing sequences are lacunary.

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Lemma 1.1. Let p > 1. Then

$$\|M_{\Omega}\|_{p\to p} \leqslant C \sup_{\emptyset \neq \Gamma \subset \Sigma} \|\sup_{\mathbf{i} \in \mathbb{Z}^{\Sigma}} M_{\Omega_{\mathbf{i}}} \prod_{\sigma \in \Gamma} K_{\sigma, i_{\sigma}} \|_{p\to p},$$

where C depends only on n and p.

*Proof.* Fix a nonnegative, even, smooth function  $m_0^{\vee}$  which is positive on [-1,1] and with sufficient decay so that, for positive functions,  $M_{\Omega}f$  is pointwise equivalent to

$$\sup_{\omega \in \Omega} \sup_{r>0} \left| \frac{1}{r} \int m_{o}^{\vee}(\frac{t}{r}) f(\cdot - t\omega) dt \right|.$$

As the operator norm of  $M_{\Omega}$  can be realised by testing on positive functions, we can work with the maximal operator  $\widetilde{M}_{\Omega}$  defined by

$$f \mapsto \sup_{\omega \in \Omega} \sup_{r > 0} \left| \frac{1}{r} \int m_{o}^{\vee}(\frac{t}{r}) f(\cdot - t\omega) \, dt \right|,$$

which is more amenable to Fourier analysis. Throughout, ^ and  $^\vee$  denote the Fourier transform and inverse transform, respectively. One can calculate that

$$\left(\frac{1}{r}\int m_{\rm o}^{\vee}(\frac{t}{r})f(\cdot-t\omega)\,dt\right)^{\wedge}(\xi)=m_{\rm o}(r\omega\cdot\xi)f^{\wedge}(\xi).$$

It will simplify things to take  $m_0$  supported in [-1, 1], which can be arranged by choosing  $m_0 = \phi_0 * \phi_0$  where  $\phi_0$  is an even, smooth function supported in [-1/2, 1/2]. We also fix a smooth function  $\eta_0$ , supported in the ball of radius  $4n^2$ , centred at the origin and equal to one on the concentric ball of radius  $2n^2$ , and consider the operator

$$f \mapsto \sup_{\omega \in \Omega} \sup_{r > 0} |S_{r,\omega}f|,$$

where  $(S_{r,\omega}f)^{\wedge}(\xi) = \eta_0(r(\omega_1\xi_1,\ldots,\omega_n\xi_n))m_0(r\omega \cdot \xi)f^{\wedge}(\xi)$ . This is pointwise dominated by a constant multiple of the strong maximal operator  $\mathcal{M}_{\text{str}}$ , which can be bounded by iterated applications of the one-dimensional Hardy-Littlewood maximal theorem. Defining m by

$$m(\xi) = (1 - \eta_{\rm o})(\xi)m_{\rm o}(\mathbf{1} \cdot \xi)$$

with  $\mathbf{1} = (1, \ldots, 1)$ , we are left with the maximal operator  $T_{\Omega}$  defined by

$$f \mapsto \sup_{\omega \in \Omega} \sup_{r > 0} |T_{r,\omega}f|,$$

where  $(T_{r,\omega}f)^{\wedge}(\xi) = m(r(\omega_1\xi_1,\ldots,\omega_n\xi_n))f^{\wedge}(\xi)$ . A variant of this reduction was originally employed by Nagel, Stein and Wainger [20].

It will suffice to prove the pointwise estimate

(2) 
$$T_{\Omega}f \leq \sum_{\emptyset \neq \Gamma \subset \Sigma} \sup_{\mathbf{i} \in \mathbb{Z}^{\Sigma}} T_{\Omega_{\mathbf{i}}} \Big[ \prod_{\sigma \in \Gamma} K_{\sigma, i_{\sigma}} \Big] f.$$

The desired  $L^p$ -estimate then follows by combining with the inequalities

$$M_{\Omega}f \leqslant C(\mathcal{M}_{\mathrm{str}}f + T_{\Omega}f), \quad T_{\Omega_{\mathbf{i}}}f \leqslant C(\mathcal{M}_{\mathrm{str}}f + M_{\Omega_{\mathbf{i}}}f),$$

and, when  $\Omega_{\mathbf{i}} \neq \emptyset$ ,

$$\left|\sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}}\mathcal{M}_{\mathrm{str}}\prod_{\sigma\in\Gamma}K_{\sigma,i_{\sigma}}\right\|_{p\to p}\leqslant C\left\|\sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}}M_{\Omega_{\mathbf{i}}}\prod_{\sigma\in\Gamma}K_{\sigma,i_{\sigma}}\right\|_{p\to p}$$

The final inequality is a trivial consequence of the boundedness of the strong maximal operator, combined with the fact that  $|f| \leq M_{\Omega_i} f$ .

Before proving (2), we motivate why it is reasonable to hope that it should be true. As suggested earlier, the frequency support of  $T_{r,\omega}f$  is contained in the union of  $\Psi_{\sigma,i_{\sigma}}$  whenever  $\omega \in \Omega_{\mathbf{i}}$  with  $\mathbf{i} = (i_{\sigma})_{\sigma \in \Sigma}$ . If this covering were in fact a partition, we would obtain

$$T_{r,\omega}f = \sum_{\sigma\in\Sigma} T_{r,\omega}K_{\sigma,i\sigma}f, \qquad \omega\in\Omega_{\mathbf{i}},$$

and so, recalling that  $T_{\Omega}f = \sup_{i \in \mathbb{Z}^{\Sigma}} T_{\Omega_i}f$ , a simplified version of (2), with less terms on the right-hand side, would follow easily. Now the conic supports do not form a partition and so to compensate we remove the pairwise intersections of the cones and then add back the intersections of each triple of cones, and so on, until we obtain a partition. Some of these intersections may in fact be empty, but we ignore this as there is no advantage for us to have less terms in the sum. Indeed, we will see that for our purposes there is no difference between the earlier simplified version and the following complicated looking formula. In three dimensions we can identify  $\sigma = (1, 2), (1, 3), (2, 3)$  with 3, 2, 1, respectively, and the process yields the identity

$$T_{r,\omega}f = \sum_{1 \le j \le 3} T_{r,\omega}K_{j,i_j}f - \sum_{1 \le j < k \le 3} T_{r,\omega}K_{j,i_j}K_{k,i_k}f + T_{r,\omega}K_{1,i_1}K_{2,i_2}K_{3,i_3}f$$

plus a remainder term. More generally, we obtain

$$T_{r,\omega}f = \sum_{\emptyset \neq \Gamma \subset \Sigma} (-1)^{|\Gamma|+1} T_{r,\omega} \Big[ \prod_{\sigma \in \Gamma} K_{\sigma,i_{\sigma}} \Big] f + T_{r,\omega} \mathcal{R}_{\mathbf{i}} f.$$

In effect, we have expanded the polynomial  $1 - \prod_{\sigma} (1 - x_{\sigma})$ , and so the remainder  $\mathcal{R}_{\mathbf{i}}$  is given by

$$(\mathcal{R}_{\mathbf{i}}f)^{\wedge}(\xi) = \prod_{\sigma \in \Sigma} (1 - \psi_{\sigma, i_{\sigma}})(\xi) f^{\wedge}(\xi).$$

In contrast with the operators  $K_{\sigma,i}$ , which are essentially two-dimensional, the operators  $\mathcal{R}_i$  are genuinely higher dimensional objects, however once we see that the multiplier associated to  $T_{r,\omega}\mathcal{R}_{\mathbf{i}}$  is identically zero whenever  $\omega \in \Omega_{\mathbf{i}}$  and r > 0,

(3) 
$$m(r(\omega_1\xi_1,\ldots,\omega_n\xi_n))\prod_{\sigma\in\Sigma}(1-\psi_{\sigma,i_\sigma})(\xi)\equiv 0,$$

we obtain

$$T_{r,\omega}f = \sum_{\emptyset \neq \Gamma \subset \Sigma} (-1)^{|\Gamma|+1} T_{r,\omega} \Big[ \prod_{\sigma \in \Gamma} K_{\sigma,i\sigma} \Big] f, \qquad \omega \in \Omega_{\mathbf{i}}$$

which yields (2). Given that the cones are invariant under scaling, by taking r large, (3) is little more than the assertion that the hyperplane is covered by the cones.

After the scaling  $\omega_j \xi_j \to \xi_j$  for  $1 \leq j \leq n$ , it will suffice to prove that the region defined by

(4) 
$$\left|\sum_{j=1}^{n} \xi_{j}\right| \leqslant \frac{1}{r}$$
 and  $\left(\sum_{j=1}^{n} \xi_{j}^{2}\right)^{1/2} \geqslant \frac{2n^{2}}{r}$ 

and

$$-\frac{\xi_j}{\xi_k}\frac{\omega_k}{\omega_j} \leqslant \frac{1}{n}\theta_{\sigma,i+1} \quad \text{or} \quad -\frac{\xi_j}{\xi_k}\frac{\omega_k}{\omega_j} > n\theta_{\sigma,i} \quad \text{for all} \quad \sigma \in \Sigma$$

is empty. As  $\omega \in \Omega_i$ , we see that the complements of the scaled cones are contained in

(5) 
$$-\frac{\xi_j}{\xi_k} < \frac{1}{n} \quad \text{or} \quad -\frac{\xi_j}{\xi_k} > n \quad \text{for all} \quad \sigma \in \Sigma.$$

We suppose for a contradiction that the region defined by (4) and (5) is not empty. It is clear by comparing the inequalities in (4) that the components of a vector  $\xi$  in this region cannot all have the same sign. By symmetric invariance of the conditions, we may suppose that

$$\xi_1, \dots, \xi_{m-1} \ge 0$$
 and  $\xi_m, \dots, \xi_n < 0$ 

for some  $1 < m \leq n$ . We can also suppose without loss of generality that  $|\xi_1| \geq |\xi_j|$  for all j > 1 and  $|\xi_m| \geq |\xi_j|$  for all j > m. Then taking j = 1 and k = m in (5) we see that  $|\xi_1| \geq n |\xi_m|$ . On the other hand, by the first condition of (4),

$$|\xi_1| - (n-1)|\xi_m| \leq \left|\sum_{j=1}^n \xi_j\right| \leq \frac{1}{r}.$$

Combining the two estimates we obtain  $|\xi_1| \leq n/r$ . Since  $|\xi_1| \geq |\xi_j|$  for j > 1, this yields

$$|\xi_1| + \ldots + |\xi_n| \leqslant \frac{n^2}{r}$$

which contradicts the second inequality in (4). Thus,  $T_{r,\omega}\mathcal{R}_{\mathbf{i}} \equiv 0$  whenever r > 0 and  $\omega \in \Omega_{\mathbf{i}}$ , and we are done.

We will also require the following square function estimates which follow easily from the two–dimensional theory.

**Lemma 1.2.** Let  $1 and <math>\Gamma \subset \Sigma$ . Then

$$\left\| \left( \sum_{\mathbf{i} \in \mathbb{Z}^{\Gamma}} \left\| \left[ \prod_{\sigma \in \Gamma} K_{\sigma, i_{\sigma}} \right] f \right\|^{2} \right)^{\frac{1}{2}} \right\|_{p} \leqslant C \| f \|_{p}, \quad where \quad \mathbf{i} = (i_{\sigma})_{\sigma \in \Gamma},$$

and C depends only on  $|\Gamma|$ , p and the lacunary constants  $\lambda_{\sigma}$ .

*Proof.* In order to bound directional maximal operators in  $L^2$ , the required square function estimate, with p = 2, follows directly from Plancherel's theorem and the finite overlapping of the supports of  $\{\psi_{\sigma,i}\}_{i\in\mathbb{Z}}$ . This is where we use the lacunarity of the sequences  $\{\theta_{\sigma,i}\}_{i\in\mathbb{Z}}$ . When  $p \neq 2$ , by a standard randomisation argument, using Khintchine's inequality, the square function estimates follow from the uniform  $L^p$ -boundedness, independent of the choice of the signs, of the Fourier multiplier operators

$$f \mapsto \Big(\sum_{\mathbf{i} \in \mathbb{Z}^{\Gamma}} \pm \prod_{\sigma \in \Gamma} \psi_{\sigma, i_{\sigma}} f^{\wedge} \Big)^{\vee}.$$

This in turn is a consequence of the Marcinkiewicz multiplier theorem (see for example [24, pp. 109]), for which it suffices to check a number of conditions involving integrals of derivatives of the multipliers. After applying the product rule, the calculation reduces to the case  $|\Gamma| = 1$ . Applying Fubini's theorem so as to ignore the trivial variables, this was originally checked by A. Córdoba and R. Fefferman [9, Section 4] in their proof of a two-dimensional angular Littlewood–Paley inequality. Again, this boils down to checking that the supports of  $\{\psi_{\sigma,i}\}_{i\in\mathbb{Z}}$  are finite overlapping, which follows from the lacunarity.

Armed with these lemmas, the proof is completed easily as follows. In order to establish the idea, we treat the easiest case first.

**Case** n = 3 and p = 2. We can identify  $\sigma = (1, 2), (1, 3), (2, 3)$  with 3, 2, 1, respectively, and suppose for simplicity that the supremum in Lemma 1.1 is attained when  $\Gamma = \{(1, 2), (1, 3)\}$ , say, which we have identified with  $\{2, 3\}$ . Then

$$\|M_{\Omega}\|_{2\to 2} \leqslant C \left\| \sup_{\mathbf{i}\in\mathbb{Z}^3} M_{\Omega_{\mathbf{i}}} K_{2,i_2} K_{3,i_3} \right\|_{2\to 2}.$$

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Using the inclusion  $\ell^2(\mathbb{Z}^2) \hookrightarrow \ell^\infty(\mathbb{Z}^2)$  and interchanging the order of the sum and the integral,

$$\begin{split} \left\| \sup_{\mathbf{i} \in \mathbb{Z}^{3}} M_{\Omega_{\mathbf{i}}} K_{2,i_{2}} K_{3,i_{3}} f \right\|_{2} &\leq \left( \sum_{i_{2},i_{3} \in \mathbb{Z}} \left\| \sup_{i_{1} \in \mathbb{Z}} M_{\Omega_{\mathbf{i}}} K_{2,i_{2}} K_{3,i_{3}} f \right\|_{2}^{2} \right)^{\frac{1}{2}} \\ &\leq \sup_{i_{2},i_{3} \in \mathbb{Z}} \left\| \sup_{i_{1} \in \mathbb{Z}} M_{\Omega_{\mathbf{i}}} \right\|_{2 \to 2} \left( \sum_{i_{2},i_{3} \in \mathbb{Z}} \left\| K_{2,i_{2}} K_{3,i_{3}} f \right\|_{2}^{2} \right)^{\frac{1}{2}} \\ &= \sup_{i_{2},i_{3} \in \mathbb{Z}} \left\| M_{\Omega_{2,i_{2}} \cap \Omega_{3,i_{3}}} \right\|_{2 \to 2} \left( \sum_{i_{2},i_{3} \in \mathbb{Z}} \left\| K_{2,i_{2}} K_{3,i_{3}} f \right\|_{2}^{2} \right)^{\frac{1}{2}} \\ &\leq C \sup_{i_{3} \in \mathbb{Z}} \left\| M_{\Omega_{3,i_{3}}} \right\|_{2 \to 2} \left( \sum_{i_{3} \in \mathbb{Z}} \left\| K_{3,i_{3}} f \right\|_{2}^{2} \right)^{\frac{1}{2}} \\ &\leq C \sup_{i_{3} \in \mathbb{Z}} \left\| M_{\Omega_{3,i_{3}}} \right\|_{2 \to 2} \left\| f \right\|_{2}, \end{split}$$

and so we are done. In the final two inequalities we used nothing more that the finite overlapping of the two–dimensional conic frequency supports.

More generally, we consider  $\mathbb{Z}^{\Sigma} = \mathbb{Z}^{\Gamma} \times \mathbb{Z}^{\Sigma \setminus \Gamma}$ , and given  $\mathbf{i} = (i_{\sigma})_{\sigma \in \Sigma}$ , we write  $\mathbf{i} = \mathbf{j} \times \mathbf{k}$  where  $\mathbf{j} = (i_{\sigma})_{\sigma \in \Gamma}$  and  $\mathbf{k} = (i_{\sigma})_{\sigma \in \Sigma \setminus \Gamma}$ . Using the inclusion  $\ell^{p}(\mathbb{Z}^{\Gamma}) \hookrightarrow \ell^{\infty}(\mathbb{Z}^{\Gamma})$  and interchanging the order of the sum and the integral,

(6) 
$$\left\| \sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_{p} \leq \left( \sum_{\mathbf{j}\in\mathbb{Z}^{\Gamma}} \left\| \sup_{\mathbf{k}\in\mathbb{Z}^{\Sigma\setminus\Gamma}} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_{p}^{p} \right)^{\frac{1}{p}} \\ \leq \sup_{\mathbf{j}\in\mathbb{Z}^{\Gamma}} \left\| \sup_{\mathbf{k}\in\mathbb{Z}^{\Sigma\setminus\Gamma}} M_{\Omega_{\mathbf{i}}} \right\|_{p\to p} \left( \sum_{\mathbf{j}\in\mathbb{Z}^{\Gamma}} \left\| f_{\mathbf{j}} \right\|_{p}^{p} \right)^{\frac{1}{p}} \\ \leq \sup_{\sigma\in\Sigma} \sup_{i\in\mathbb{Z}} \left\| M_{\Omega_{\sigma,i}} \right\|_{p\to p} \left\| \left( \sum_{\mathbf{j}\in\mathbb{Z}^{\Gamma}} \left| f_{\mathbf{j}} \right|^{p} \right)^{\frac{1}{p}} \right\|_{p}.$$

**Case**  $p \ge 2$ . Using the inclusion  $\ell^2(\mathbb{Z}^{\Gamma}) \hookrightarrow \ell^p(\mathbb{Z}^{\Gamma})$ , from (6) we obtain

$$\left\|\sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}}M_{\Omega_{\mathbf{i}}}f_{\mathbf{j}}\right\|_{p} \leqslant \sup_{\sigma\in\Sigma}\sup_{i\in\mathbb{Z}}\left\|M_{\Omega_{\sigma,i}}\right\|_{p\to p}\left\|\left(\sum_{\mathbf{j}\in\mathbb{Z}^{\Gamma}}\left|f_{\mathbf{j}}\right|^{2}\right)^{\frac{1}{2}}\right\|_{p}.$$

Taking  $f_{\mathbf{j}} = \left[\prod_{\sigma \in \Gamma} K_{\sigma, i_{\sigma}}\right] f$ , where  $\mathbf{j} = (i_{\sigma})_{\sigma \in \Gamma}$ , and applying Lemmas 1.1 and 1.2, we obtain the desired estimate.

**Case**  $1 . This is based on an argument of M. Christ used in [6, 1] which refined the argument of Nagel–Stein–Wainger [20]. We suppose initially that <math>\Omega$  is finite, so that by the triangle inequality and the

Hardy–Littlewood maximal theorem,  $M_{\Omega}$  is bounded. Then by interpolating between

$$\left\|\sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}}M_{\Omega_{\mathbf{i}}}f_{\mathbf{j}}\right\|_{p}\leqslant\|M_{\Omega}\|_{p\rightarrow p}\left\|\sup_{\mathbf{j}\in\mathbb{Z}^{\Gamma}}|f_{\mathbf{j}}|\right\|_{p}$$

and (6), we see that  $\|\sup_{\mathbf{i}\in\mathbb{Z}^{\Sigma}}M_{\Omega_{\mathbf{i}}}f_{\mathbf{j}}\|_{p}$  is bounded above by

$$\|M_{\Omega}\|_{p\to p}^{1-\frac{p}{2}} \Big(\sup_{\sigma\in\Sigma}\sup_{i\in\mathbb{Z}}\|M_{\Omega_{\sigma,i}}\|_{p\to p}\Big)^{\frac{p}{2}} \left\|\Big(\sum_{\mathbf{j}\in\mathbb{Z}^{\Gamma}}|f_{\mathbf{j}}|^{2}\Big)^{\frac{1}{2}}\right\|_{p}$$

Taking  $f_{\mathbf{j}} = \left[\prod_{\sigma \in \Gamma} K_{\sigma, i_{\sigma}}\right] f$ , where  $\mathbf{j} = (i_{\sigma})_{\sigma \in \Gamma}$ , and applying Lemmas 1.1 and 1.2 as before, we see that

$$\|M_{\Omega}\|_{p \to p} \leqslant C \|M_{\Omega}\|_{p \to p}^{1-\frac{p}{2}} \Big( \sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}} \|M_{\Omega_{\sigma,i}}\|_{p \to p} \Big)^{\frac{p}{2}}.$$

Rearranging, we obtain the desired estimate with C independent of  $\Omega$ , so we can drop the restriction that  $\Omega$  is finite. This completes the proof.  $\Box$ 

In both [20] and [6], a single conic Fourier multiplier was introduced for each direction. This multiplier had to cover (the bulk of) the hyperplane perpendicular to the direction, and so was necessarily multidimensional in nature. Rather restrictive conditions on the directions were then required to ensure finite overlapping of the supports of the multipliers, yielding a bound via orthogonality as above. In order to achieve greater flexibility, we introduced a number of essentially two-dimensional multipliers instead. This is only possible via a covering rather than a partition, however after adding and subtracting a number of products of these multipliers we obtain a *signed partition of unity*. This came at essentially no cost and in fact simplifies matters because the orthogonality in two dimensions, summing over one index at a time, is trivial to check. On the other hand, our multipliers are naturally associated to partitions of the directions allowing us to introduce a multiplier for each segment instead of one for each direction.

# 2. Proof of Theorem C

First we prove the inclusion  $\operatorname{Max}_{p}(n) \subset \operatorname{Lsh}(n)$  which is restated in the following lemma. We appeal to a quantitative version of Bateman's theorem [4], allowing us to treat the shadows simultaneously and thus uniformly. We also use that the cross product of a two-dimensional Kakeya set with a cube is a Kakeya set.

**Lemma 2.1.** Let  $n \ge 2$  and  $1 , and suppose that <math>M_{\Omega}$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$ . Then  $\Omega \in \mathsf{Lsh}(n)$ .

Proof. As  $M_{\Omega}$  is bounded if and only if  $M_{\overline{\Omega}}$  is bounded, we can suppose that  $\Omega$  is closed. We appeal to Bateman's terminology [4]. In particular we will consider the binary tree  $\mathcal{T}_{\Pi}$  associated to the shadow of  $\Omega$  on  $\Pi$ , for any two-dimensional subspace  $\Pi$ , and their splitting numbers split( $\mathcal{T}_{\Pi}$ ). We say that  $\Omega$  admits Kakeya shadows if there exists a constant C such that for any  $N \ge 1$  there exists a two-dimensional subspace  $\Pi(N)$  and a finite collection of rectangles  $\mathcal{R}_{\Pi(N)}$  contained in  $\Pi(N)$ , with longest side pointing in a direction of the shadow of  $\Omega$  on  $\Pi(N)$ , that satisfy

(7) 
$$\left| \bigcup_{R \in \mathcal{R}_{\Pi(N)}} R \right| \leq \frac{C}{N} \left| \bigcup_{R \in \mathcal{R}_{\Pi(N)}} 3R \right|$$

Here, 3R has the same center and width as R, but three times the length.

We prove the contrapositive. If  $\Omega \notin \mathsf{Lsh}(n)$ , then by Theorem 3 (combined with Remark 2) in [4], for any  $N \ge 1$ , there is a shadow of  $\Omega$  on  $\Pi(N)$  for which split( $\mathcal{T}_{\Pi(N)}$ )  $\ge 2^N$ . Bateman proved (see pages 61–62 and Claim 7 of [4]) that split( $\mathcal{T}_{\Pi(N)}$ )  $\ge 2^N$  implies the existence of a finite family  $\mathcal{R}_{\Pi(N)}$  of rectangles R satisfying (7). Now for each  $N \in \mathbb{N}$ , we pick an orthonormal basis  $(e_1, \ldots, e_n)$  so that span $(e_1, e_2) = \Pi(N)$ . For each rectangle R in the subcollection  $\mathcal{R}_{\Pi(N)}$ , we set

$$\beta \equiv \beta(R) = \operatorname{diam}(R)(\omega_1^2 + \omega_2^2)^{-1/2},$$

where  $\omega$  is a direction of  $\Omega$  whose shadow points in the direction of R, and let  $\alpha \equiv \alpha(N)$  to be ten times the maximum  $\beta(R)$  with  $R \in \mathcal{R}_{\Pi(N)}$ . Taking

$$E_N = \bigcup_{R \in \mathcal{R}_{\Pi(N)}} R \times [0, \alpha]^{n-2},$$

defined with respect to the basis  $(e_1, \ldots, e_n)$ , we then have

$$M_{\Omega}[\chi_{E_N}](x) \ge 1/8$$
 for all  $x \in \bigcup_{R \in \mathcal{R}_{\Pi(N)}} 3R \times [3\beta, \alpha - 3\beta]^{n-2}.$ 

Using (7), we see that for all  $N \ge 1$ ,

$$\left\|M_{\Omega}[\chi_{E_N}]\right\|_p \geqslant cN^{\frac{1}{p}} \|\chi_{E_N}\|_p,$$

so that  $M_{\Omega}$  is not bounded from  $L^{p}(\mathbb{R}^{n})$  to  $L^{p}(\mathbb{R}^{n})$  when p is finite.  $\Box$ 

Using Theorem A in order to bound the maximal operators associated to the sets of Lac(n) and  $Lac_p(n, \varepsilon)$ , at this stage we have proven that

$$\_\mathsf{ac}(n) \subset \mathsf{Lac}_{\mathsf{p}}(n) \subset \mathsf{Max}_{\mathsf{p}}(n) \subset \mathsf{Lsh}(n).$$

It therefore remains to prove that  $Max_p(n) \subset Lac_p(n)$ . It is tempting to suppose that the job is already done – that Theorem A can be applied repeatedly in order to reduce a shadow to a single direction, thus reducing the dimension of the problem. That is to say  $Lsh(n) \subset Lac(n)$  yielding a full chain of equivalences. However the lacunary orders of the shadows are unstable in the sense that shadows on two-dimensional subspaces which are close can have dramatically different lacunary orders, and so it is not clear that it helps to apply Theorem A and then change the basis in order to apply it again. One may be faced each time with lacunary orders which are as bad as before, and the process may never end. This would not be a problem if a slightly more flexible version of Theorem A were true, however the obvious candidates for such a theorem are false (see the following section).

We get round the problem via a topological argument – we prove that the process must stop as otherwise there would be a direction of arbitrarily large accumulation order which is not possible by the following lemma. The reason why we are able to carry out this argument for p-lacunarity and not for lacunarity, is that we are assured that if a set of directions is not placunary (but gives rise to a bounded maximal operator) then a dissection must always contain a segment which is not p-lacunary. This is not the case with lacunarity – the segments may all be lacunary but without a uniform bound on their lacunary orders. We do not know if this is a merely technical problem or if it could be reflected in the geometry of the directions.

Given an *m*-dimensional subspace  $\Pi \subset \mathbb{R}^n$  and a set of directions  $\Omega$ , we define the *m*-shadow  $\Xi$  of  $\Omega$  on  $\Pi$  by

$$\Xi = \left\{ \frac{\mathbf{P}_{\Pi}(\omega)}{|\mathbf{P}_{\Pi}(\omega)|} : \omega \in \Omega \setminus \Pi^{\perp} \right\} \subset \Pi \cap \mathbb{S}^{n-1},$$

where  $P_{\Pi}$  denotes the orthogonal projection onto  $\Pi$  (see Figure 2 for an illustration with n = 3 and m = 2). Note that a 2-shadow is the same thing as a shadow. For an *m*-shadow  $\Xi$ , we consider  $A_{\ell} = \operatorname{Ac}(A_{\ell-1})$ , where  $A_1 = \operatorname{Ac}(\Xi)$ , the accumulation points of  $\Xi$ . We say that  $\Xi$  has accumulation order *L* if  $A_L$  is a finite set.

**Lemma 2.2.** Let  $2 \leq m \leq n$  and  $1 , and suppose that <math>M_{\Omega}$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$ . Then the *m*-shadows of  $\Omega$  have uniformly bounded accumulation order.

Proof. As  $M_{\Omega}$  is bounded, the 2-shadows are uniformly lacunary of finite order by Lemma 2.1, so that in particular the 2-shadows of  $\Omega$  have uniformly bounded accumulation order. Thus, it will suffice to prove that if the accumulation order of an m-shadow  $\Xi$  of  $\Omega$  on  $\Pi$  is greater than L, then there exists a 2-shadow of  $\Xi$ , and hence also of  $\Omega$ , whose accumulation order is greater than L. We take  $\xi \in A_{L+1}$  and consider a sequence  $\{\xi_j\}_{j\geq 1}$  in  $A_L$  which accumulates at  $\xi$ . Then for all but (at most) one (m-1)-dimensional subspace of  $\Pi$ , the (m-1)-shadows of  $\{\xi_j\}_{j\geq 1}$  on the (m-1)-dimensional subspaces accumulate at the (m-1)-shadows of  $\xi$ . Then we consider sequences in  $A_{L-1}$  which accumulate at  $\xi_j$ . Again for all but one (m-1)-dimensional subspaces accumulate at the (m-1)-shadows on the (m-1)-dimensional subspaces accumulate at the (m-1)-shadows of  $\xi_j$ . Continuing the process, we see that for all but a countable number of (m-1)-dimensional subspaces of  $\Pi$ , the (m-1)-shadow of  $\xi$  is of accumulation order  $\geq L+1$ . We take one such shadow and repeat the process. This yields an (m-2)-dimensional shadow of the (m-1)-dimensional shadow of  $\xi$ , which is an (m-2)-dimensional shadow of  $\xi$ , that is of accumulation order  $\geq L+1$ . Repeating the process, we obtain the desired result.  $\Box$ 

We will require an auxiliary definition, similar to the definition of placunarity. For a fixed  $\varepsilon > 0$ , we say that an m-shadow  $\Xi$  of  $\Omega$  is

- (n, p)-lacunary of order 0 if it consists of a single direction
- (n, p)-lacunary of order L if there are members  $\{\Xi_{\sigma,i\sigma}\}_{\sigma\in\Sigma(d)}$  of a dissection of  $\Xi$  which are (n, p)-lacunary of order  $\leqslant L 1$  and for which the sets  $\{\Omega_{\sigma,i\sigma}\}_{\sigma\in\Sigma(d)}$  that shade them are dominating;

$$\|M_{\Omega_{\sigma,i}}\|_{p\to p} \leqslant \|M_{\Omega_{\sigma,i_{\sigma}}}\|_{p\to p} + \varepsilon \quad \text{for all} \quad i \in \mathbb{Z}^*.$$

We say that an *m*-shadow  $\Xi$  of  $\Omega$  is (n, p)-lacunary if it is (n, p)-lacunary of finite order. Note that in this context the dominating sets (which from now on we refer to as dominating segments) need only dominate the rest of their partition, not the whole dissection.

The final inclusion,  $Max_p(n) \subset Lac_p(n)$ , is a consequence of the fact that

(8) 
$$\operatorname{Max}_{p}(n) \subset \operatorname{Lac}_{p}(n,\varepsilon)$$
 for all  $\varepsilon > 0$ .

When  $\Omega \in \mathsf{Max}_p(n)$ , the existence of dominating segments is always given, and so the 2-shadows of  $\Omega$  are (n, p)-lacunary, as by Lemma 2.1 we have that  $\Omega \in \mathsf{Lsh}(n)$ . Thus (8) can be obtained by n-2 applications of the following lemma, observing that (n, p)-lacunary *n*-shadows are *p*-lacunary.

**Lemma 2.3.** Let  $2 \leq m \leq n-1$  and  $1 , and suppose that <math>M_{\Omega}$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$ . Then, if the m-shadows of  $\Omega$  are (n,p)lacunary, then the (m+1)-shadows of  $\Omega$  are (n,p)-lacunary.

Proof. As  $M_{\Omega}$  is bounded, the (m+1)-shadows of  $\Omega$  have finite accumulation order by Lemma 2.2. We suppose for a contradiction that the *m*-shadows of  $\Omega$  are (n, p)-lacunary, but that there is an (m+1)-shadow of  $\Omega$  which is not. Thus, we will have our desired contradiction if we can show that this (m+1)-shadow of  $\Omega$ , which from now on we call  $\Xi$ , is of arbitrarily large accumulation order.

We take a basis  $(e_1, \ldots, e_{m+1})$  with  $e_{m+1}$  being an accumulation point of  $\Xi$ (which exists by compactness as finite sets are (n, p)-lacunary when  $M_{\Omega}$  is bounded) and write  $\Pi = \operatorname{span}(e_1, \ldots, e_m)$ . Note that the *m*-shadow of  $\Xi$  on  $\Pi$  is the same as the *m*-shadow of  $\Omega$  on  $\Pi$ . In fact we choose the basis vectors



**Figure 2.** If  $\sigma_1 \in \Sigma(m)$  we reduce the order of the shadow on  $\Pi$ .

 $(e_1, \ldots, e_m)$  more carefully: We dissect  $\Xi$  (simultaneously dissecting the m-shadow on  $\Pi$  and partially dissecting  $\Omega$ ) with  $(e_1, \ldots, e_m)$  and  $\{\theta_{\sigma,i}\}_{i\in\mathbb{Z}}$  for each  $\sigma \in \Sigma(m)$  chosen in order to reduce the (n, p)-lacunary order of the m-shadow on  $\Pi$  (see Figure 2 for a three dimensional illustration). We are free to choose any lacunary  $\{\theta_{\sigma,i}\}_{i\in\mathbb{Z}}$  for  $\sigma = (j, m + 1)$  with  $1 \leq j \leq m$ ). Now as the (m + 1)-shadow  $\Xi$  is not (n, p)-lacunary, there must be a  $\sigma_1 \in \Sigma(m + 1)$  for which the (m + 1)-shadow on span $(e_1, \ldots, e_{m+1})$  of the dominating segments in each partition as  $M_\Omega$  is bounded. If  $\sigma_1 = (j, m+1) \in \Sigma(m+1) \setminus \Sigma(m)$  we have found such a shadow which is separated from  $e_{m+1}$ , and hence we have found a new accumulation point by compactness (see Figure 3). If  $\sigma_1 \in \Sigma(m)$ , we choose one of the dominating segments whose m-shadow on  $\Pi$  has reduced (n, p)-lacunary order (see Figure 2).

Supposing that the span of this *m*-shadow is  $d_2$ -dimensional, where  $d_2 \leq d_1 \equiv m$ , we take  $e_{d_2+1}$  to be the original accumulation point and dissect the (m+1)-shadow with  $e_1, \ldots, e_{d_2} \in \Pi$  and  $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$  for  $\sigma \in \Sigma(d_2)$  chosen in order to reduce the (n, p)-lacunary order of the *m*-shadow on  $\Pi$  of this segment. Again there is a  $\sigma_2 \in \Sigma(d_2+1)$  for which the (m+1)-shadows of the dominating segments are not (n, p)-lacunary. If  $\sigma_2 = (j, d_2 + 1)$  we have separated from the accumulation point. If not, we choose one of the dominating segments whose *m*-shadow on  $\Pi$  has reduced (n, p)-lacunary order and continue. This division into ever smaller sets of directions, whose (m+1)-shadow is not (n, p)-lacunary, cannot stop, as otherwise  $\Xi$  would be (n, p)-lacunary. Also, the chosen sets which are not (n, p)-lacunary cannot be sliced using hyperplanes which pass through the accumulation point indefinitely as the (n, p)-lacunary order of the *m*-shadow on  $\Pi$  is reduced at each slice so eventually we would reduce to the case where the *m*-shadow



**Figure 3.** If  $\sigma_1 \in \Sigma(m+1) \setminus \Sigma(m)$  we separate from accumulation point.

would be a single direction. In that case the (m + 1)-shadow would be contained in a two-dimensional subspace, and so would have to be (n, p)lacunary by Lemma 2.1. We are choosing the sets to be of infinite (n, p)lacunary order, and so the only way we can keep doing this is if one is eventually chosen which is disconnected from the accumulation point. As this set has an infinite number of directions, by compactness we find a new accumulation point.

For a finite number of accumulation points, we can always make a judicious choice of basis and lacunary sequences so that they are separated in the dissection. Either this yields a segment whose (m + 1)-shadow is not (n, p)-lacunary which contains a new accumulation point, or there is a segment whose (m + 1)-shadow is not (n, p)-lacunary which contains one of the old accumulation points. In this case, we take  $e_{m+1}$  to be this contained accumulation point and repeat the process. At some stage the repetition of the process yields a set which is not (n, p)-lacunary and which is disconnected from the contained accumulation point (and the others). This follows again by hypothesis because the sets produced in the dissection which are not (n, p)-lacunary can only contain the contained accumulation point a finite number of times. Continuing the process, we obtain a sequence of accumulation points, which by compactness have an accumulation point. We take this to be  $e_{m+1}$  and consider again all the directions of  $\Xi$ .

Repeating the process there is a segment whose (m + 1)-shadow is not (n, p)-lacunary separated from the accumulation point of accumulation points. This gives rise to another sequence of accumulation points, and an accumulation point of them, separated from the original accumulation point of

accumulation points. Dividing them either yields a segment whose (m + 1)-shadow is not (n, p)-lacunary and is separated from them, and we continue the process with this, or there is a segment whose (m + 1)-shadow is not (n, p)-lacunary and which contains one of the accumulation points of accumulation points, which we take to be  $e_{m+1}$ , and continue the process with this. Eventually this yields a full sequence of accumulation points of accumulation points, which have an accumulation point by compactness. Continuing the process, we see that  $\Xi$  contains accumulation points of arbitrarily large order which contradicts Lemma 2.2, and so we are done.

#### 3. Our notion of lacunarity and the sharpness of Theorem A

As in the previous section, to construct unbounded directional maximal operators, the directions need only be badly spaced after projecting onto a two-dimensional subspace. Thus, in contrast with the two-dimensional case, it is not enough to constrain the angles between the directions if they are to give rise to a bounded maximal operator in higher dimensions. To see this, we enumerate  $\mathbb{Q} \cap [\frac{1}{2}, \frac{2}{3}] = \{q_\ell\}_{\ell \ge 1}$  and consider

$$\Omega = \left\{ \omega \in \mathbb{S}^{n-1} \cap \mathbb{R}^n_+ : \frac{\omega_2}{\omega_1} = q_\ell, \ \omega_j = 2^{-j\ell}, 1 < j < n; \text{ for some } \ell \ge 1 \right\}.$$

Then the angles between the directions form a lacunary sequence converging to zero with lacunary constant 1/2; see Figure 4. Taking  $\theta_{\sigma,i} = 2^{-i}$ , the segments  $\Omega_{\sigma,i}$  consist of at most one direction for all  $i \in \mathbb{Z}^*$  and  $\sigma \in \Sigma \setminus \{(1,2)\}$ . In spite of this,  $M_{\Omega}$  is unbounded. Indeed, consider the set of rectangles  $\mathcal{R}$  in  $\Pi = \operatorname{span}(e_1, e_2)$  with longest side parallel to the shadow on  $\Pi$  of some  $\omega \in \Omega$ . Then the construction of Besicovitch (see for example [13]) provides finite subsets  $\mathcal{R}_N \subset \mathcal{R}$ , for all  $N \ge 1$ , that satisfy (7). Considering  $\chi_{E_N}$ , defined as in the proof of Lemma 2.1, we find  $M_{\Omega}$  unbounded as before.

If the angles between directions restricted to a great circle are lacunary, or if the angles between directions restricted to the Nagel–Stein–Wainger curves are lacunary, then the associated maximal operators are bounded. It is tempting to suppose that if the angles between directions restricted to any smooth curve (which does not spiral around the sphere infinitely many times) are lacunary then the directions give rise to a bounded maximal operator (the authors thank Antonio Córdoba for asking this question). To see that this is not the case we consider the curve  $\gamma : [0, 1/4] \rightarrow \mathbb{S}^{n-1}$ defined to be the normalisation of  $\tilde{\gamma}(t) = (t, t/\log_2(1/t), t, \dots, t, 1)$ . This is little more than a smooth perturbation of a great circle. We consider the directions  $\Omega = \{\omega_\ell\}_{\ell \ge 1}$  where  $\omega_\ell = \gamma(2^{-\ell})$ ; see Figure 5. As long as  $\ell$  is taken sufficiently large we can safely ignore the normalisation. Then it is easy to see that the angles between the directions are lacunary with lacunary



**Figure 4.** A Kakeya shadow on  $\Pi$  with  $\frac{1}{2}n(n-1) - 1$  lacunary shadows.

constant 1/2:

$$\frac{|\tilde{\gamma}(2^{-(\ell+1)}) - e_n|}{|\tilde{\gamma}(2^{-\ell}) - e_n|} \leqslant \frac{|(2^{-(\ell+1)}, 2^{-(\ell+1)}/(\ell+1), \dots, 2^{-(\ell+1)}, 0)|}{|(2^{-\ell}, 2^{-\ell}/\ell, \dots, 2^{-\ell}, 0)|} \\ \leqslant \frac{1}{2} \frac{|(1, 1/(\ell+1), 1, \dots, 1, 0)|}{|(1, 1/\ell, 1, \dots, 1, 0)|} < \frac{1}{2}.$$

In spite of this,  $M_{\Omega}$  is unbounded. Indeed, consider the set of rectangles  $\mathcal{R}$  in  $\Pi = \operatorname{span}(e_1, e_2)$  with longest side parallel to the shadow on  $\Pi$  of some  $\omega \in \Omega$ . Then the construction of Besicovitch provides finite subsets  $\mathcal{R}_N \subset \mathcal{R}$ , for all  $N \ge 1$ , that satisfy (7). To see this it is enough to show that there are approximately uniformly spaced angles between the shadows of the directions at all scales. We have that

$$\frac{\omega_{\ell} \cdot e_2}{\omega_{\ell} \cdot e_1} = \frac{1}{\ell} \quad \text{and} \quad \frac{\omega_{\ell} \cdot e_2}{\omega_{\ell} \cdot e_1} - \frac{\omega_{\ell+1} \cdot e_2}{\omega_{\ell+1} \cdot e_1} = \frac{1}{\ell(\ell+1)}$$

so that the shadow contains  $\ell$  approximately equally spaced points between the shadow of  $\omega_{\ell}$  and  $e_1$  for all  $\ell$  sufficiently large. Considering  $\chi_{E_N}$ , defined as in the proof of Lemma 2.1, we find  $M_{\Omega}$  unbounded as before.

If Theorem A were more flexible, in the sense that the partitions were allowed to 'accumulate' away from the hyperplanes orthogonal to the basis vectors, then we would obtain  $\operatorname{Max}_{p}(n) \equiv \operatorname{Lsh}(n)$  as we would be able to bound the operators associated to the sets of  $\operatorname{Lsh}(n)$ . However, Theorem A is remarkably sharp in the sense that the supremum in  $\sigma$  must be taken over the whole of  $\Sigma$ , and the partitions must accumulate at the hyperplanes perpendicular to the basis vectors. To see this, we let  $e'_{2}$  and  $e'_{n}$  be orthogonal unit vectors in span $(e_{2}, e_{n})$ , close to  $e_{2}$  and  $e_{n}$ , with  $e'_{n}$  in the first quadrant determined by  $e_{2}$  and  $e_{n}$ . We construct a set of directions, accumulating



Figure 5. A Kakeya shadow of well-spaced directions in a smooth curve.

rapidly at  $e'_n$ , for which the angles between the orthogonal projections onto span $(e_1, e'_2)$  are badly spaced. Indeed, we take  $\Omega = \{\omega_\ell\}_{\ell \ge 1}$  so that  $\omega_\ell \cdot e'_2 = q_\ell \omega_\ell \cdot e_1$ . This does not yet completely determine  $\omega_\ell$ . Supposing that we have chosen  $\omega_{\ell-1}$  we can choose the direction  $\omega_\ell$  sufficiently close to  $e'_n$  so that the angle between  $\omega_{\ell-1}$  and  $e'_n$  is at least double that between  $\omega_\ell$  and  $e'_n$ . We can also choose the directions so that

$$\frac{\omega_{\ell-1} \cdot e'_n}{\omega_{\ell-1} \cdot e'_2} \leqslant \frac{1}{2} \frac{\omega_{\ell} \cdot e'_n}{\omega_{\ell} \cdot e'_2}, \quad \text{and} \quad \frac{\omega_{\ell-1} \cdot e_k}{\omega_{\ell-1} \cdot e_j} \leqslant \frac{1}{2} \frac{\omega_{\ell} \cdot e_k}{\omega_{\ell} \cdot e_j}$$

for all  $(j,k) \in \Sigma(n) \setminus \{(2,n)\}$ . Taking  $\theta_{\sigma,i} = 2^{-i}$ , the segments  $\Omega_{\sigma,i}$ , defined with respect to the orthonormal basis  $(e_1, \ldots, e_n)$ , consist of at most one direction for all  $i \in \mathbb{Z}^*$  and  $\sigma \in \Sigma(n) \setminus \{(2,n)\}$ . On the other hand, if we define the final segments by

$$\Omega_{(2,n),i} = \Big\{ \omega \in \Omega : 2^{-(i+1)} < \Big| \frac{\omega \cdot e'_n}{\omega \cdot e'_2} \Big| \leqslant 2^{-i} \Big\}, \quad i \in \mathbb{Z},$$

accumulating at  $\{e'_2\}^{\perp} \cup \{e'_n\}^{\perp}$ , then they also consist of at most one direction for all  $i \in \mathbb{Z}$ . In spite of this,  $M_{\Omega}$  is unbounded as before. Indeed, consider the set of rectangles  $\mathcal{R}$  in  $\Pi = \operatorname{span}(e_1, e'_2)$  with longest side parallel to the shadow on  $\Pi$  of some  $\omega_{\ell}$ . Then there are finite subsets  $\mathcal{R}_N \subset \mathcal{R}$ , for all  $N \geq 1$ , that satisfy (7). Considering  $\chi_{E_N}$ , defined as in the proof of Lemma 2.1, but with respect to the basis  $(e_1, e'_2, e_3, \ldots, e_{n-1}, e'_n)$ , we again find  $M_{\Omega}$  unbounded on  $L^p(\mathbb{R}^n)$  for finite p.

Finally we remark that 'cross products' of lacunary sets, like the directions (1), do not give rise to bounded maximal operators in general. To see this we consider the largest set of the form

$$\Omega = \Big\{ \omega \in \mathbb{S}^{n-1} \cap \mathbb{R}^n_+ : \frac{\omega_k}{\omega_j} = 2^{-i}, \ \frac{\omega_n}{\omega_1} = 3^{-\ell}, \ 1 < j < k; \text{ for some } i, \ell \in \mathbb{Z} \Big\},$$

and  $\Theta = \{2^i 3^{-\ell}\}_{i,\ell \in \mathbb{Z}}$  which is the set of the tangents of the angles between the shadows of the directions on  $\Pi = \operatorname{span}(e_1, e_2)$ . To see that this is dense in  $\mathbb{R}_+$ , which is presumably well-known, we note that

$$|2^i 3^{-\ell} - 1| < \epsilon \iff \left| \frac{i}{\ell} - \log_2 3 \right| < \frac{\log_2(1+\epsilon)}{\ell},$$

when  $2^{i}3^{-\ell} > 1$ , so that by Dirichlet's approximation theorem, 1 is an accumulation point of  $\Theta$ . Then if  $\Theta$  were not dense we could find an interval (a, b), with a, b in the closure of  $\Theta$ , which does not contain an element of  $\Theta$ . However, noting that  $\Theta$  is closed under multiplication, by taking a sequence of  $\Theta$  which accumulates to 1 from above and multiplying by elements of  $\Theta$  sufficiently close to a, we come to a contradiction. Considering  $\chi_{E_N}$ , defined as in the proof of Lemma 2.1, we find  $M_{\Omega}$  unbounded as before.

#### 4. The maximal directional Hilbert transform

It is well known that there is a close relationship between the behaviour of the directional maximal operator and the maximal directional Hilbert transform  $H_{\Omega}$ , defined by  $f \mapsto \sup_{\omega \in \Omega} |H_{\omega}f|$ , where

$$H_{\omega}f(x) = p.v. \int_{\mathbb{R}} f(x - \omega t) \frac{dt}{t}.$$

However, the constant in the following corollary must depend on the cardinality of  $\Omega$  due to a result of Karagulyan which showed that the maximal directional Hilbert transform in the plane is unbounded as soon as the number of directions is infinite [14]. On the other hand, we do not recover the sharp estimates for  $H_{\Omega}$  in terms of the power of the logarithm when n = 2; see [12], and so it would be interesting to see if the following inequality could be improved in that regard. The estimate also holds for more general operators, where the kernel 1/t is replaced by the inverse Fourier transform of a Hörmander–Mikhlin multiplier.

**Corollary 4.1.** Let  $n \ge 2$  and p > 1. Then

$$\|H_{\Omega}f\|_{p\to p} \leqslant C \log |\Omega| \sup_{\sigma\in\Sigma} \sup_{i\in\mathbb{Z}^*} \|M_{\Omega_{\sigma,i}}\|_{p\to p},$$

where C depends only on n, p and the lacunary constants  $\lambda_{\sigma}$  for  $\sigma \in \Sigma$ .

*Proof.* First we note that  $\{x : H_{\Omega}f(x) > \log |\Omega|\gamma\}$  is a subset of

$$\left\{x : H_{\Omega}f(x) > \log |\Omega|\gamma, \ M_{\Omega}f(x) \leqslant \gamma\right\} \cup \left\{x : M_{\Omega}f(x) > \gamma\right\},\$$

so we can use Theorem A to deal with the part of the integral coming from the second level set. Thus, it will suffice to prove

(9) 
$$\int_0^\infty \left| \left\{ x : H_\Omega f(x) > \log |\Omega| \gamma, \, M_\Omega f(x) \leqslant \gamma \right\} \right| p \gamma^{p-1} d\gamma \leqslant C ||f||_p^p.$$

To see this we first note that

$$\left| \left\{ x : H_{\Omega}f(x) > \log |\Omega|\gamma, \ M_{\Omega}f(x) \leqslant \gamma \right\} \right| \\ \leqslant \sum_{\omega \in \Omega} \left| \left\{ x : H_{\omega}f(x) > \log |\Omega|\gamma, \ M_{\omega}f(x) \leqslant \gamma \right\} \right|.$$

Then we use a reformulation of a one-dimensional inequality due to Hunt,

$$\left|\left\{x : H_{\omega}f(x) > N\gamma, \ M_{\omega}f(x) \leqslant \gamma\right\}\right| \leqslant e^{-N} \left|\left\{x : H_{\omega}^{\star}f(x) > \gamma\right\}\right|$$

(see [12, Proposition 2.2] for more details), where

$$H^{\star}_{\omega}f(x) = \sup_{\epsilon > 0} \Big| \int_{|t| > \epsilon} f(x - \omega t) \frac{dt}{t} \Big|.$$

Altogether we see that the left-hand side of (9) is bounded by a constant multiple of

$$\sum_{\omega \in \Omega} \frac{1}{|\Omega|} \int_0^\infty \left| \left\{ x \, : \, H_\omega^\star f(x) > \gamma \right\} \right| p \gamma^{p-1} d\gamma$$
  
$$\leqslant \sum_{\omega \in \Omega} \frac{1}{|\Omega|} \| H_\omega^\star f \|_p^p \leqslant C \sum_{\omega \in \Omega} \frac{1}{|\Omega|} \| f \|_p^p \leqslant C \| f \|_p^p,$$

and so we are done.

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