MULTI-INDEXED p-ORTHOGONAL SUMS IN NON-COMMUTATIVE LEBESGUE SPACES

JAVIER PARCET

ABSTRACT. In this paper we extend a recent Pisier's inequality for p-orthogonal sums in non-commutative Lebesgue spaces. To that purpose, we generalize the notion of p-orthogonality to the class of multi-indexed families of operators. This kind of families appear naturally in certain non-commutative Khintchine type inequalities associated with free groups. Other p-orthogonal families are given by the homogeneous operator-valued polynomials in the Rademacher variables or the multi-indexed martingale difference sequences. As in Pisier's result, our tools are mainly combinatorial.

INTRODUCTION

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful, normal trace τ satisfying $\tau(1) = 1$ and let us consider the associated non-commutative Lebesgue space $L_p(\tau)$ for an even integer p. Let Γ be the product set $\{1, 2, \ldots, n\}^d$ and let $f = (f_\gamma)_{\gamma \in \Gamma}$ be a family of operators in $L_p(\tau)$ indexed by Γ. We shall say that f is *p-orthogonal with d indices* if

$$\tau\big(f_{h(1)}^* f_{h(2)} f_{h(3)}^* f_{h(4)} \cdots f_{h(p-1)}^* f_{h(p)}\big) = 0$$

whenever the function $h : \{1, 2, \ldots, p\} \to \Gamma$ has an injective projection. In other words, whenever the coordinate function $\pi_k \circ h : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, n\}$ is an injective function for some $1 \le k \le d$. Of course, as it is to be expected, the product above can be replaced by

$$f_{h(1)} f_{h(2)}^* \cdots f_{h(p-1)} f_{h(p)}^*,$$

with no consequences in the forthcoming results. The case of one index $d = 1$ was already considered by Pisier in [6]. The main result in [6] is the following inequality, which holds for any p-orthogonal family $f_1, f_2, \ldots, f_n$ with one index

$$\Big\| \sum_{k=1}^n f_k \Big\|_{L_p(\tau)} \le \frac{3\pi}{2} p \max \left\{ \Big\| \Big(\sum_{k=1}^n f_k^* f_k \Big)^{1/2} \Big\|_{L_p(\tau)}, \Big\| \Big(\sum_{k=1}^n f_k f_k^* \Big)^{1/2} \Big\|_{L_p(\tau)} \right\}.$$

Some natural examples of 1-indexed p-orthogonal sequences of operators are the (non-commutative) *martingale difference sequences*, the operators associated to a *p-dissociate subset* of any discrete group (via the left regular representation) or a *free circular family* in Voiculescu's sense [10]. In particular, several relevant inequalities in Harmonic Analysis such as the Littlewood-Paley inequalities, the (non-commutative) Burkholder-Gundy inequalities [8], or the (non-commutative) Khintchine inequalities [3, 4] appear as particular cases. Moreover, it turns out that the combinatorial techniques applied in [6] led to the sharp order of growth

Partially supported by the Project BFM 2001/0189, Spain.
2000 Mathematics Subject Classification: Primary 46L52. Secondary 05A18.
Key words and phrases: Khintchine inequality, Möbius inversion, p-orthogonal sums.

of the constant appearing in the non-commutative Burkholder-Gundy inequalities. For the more general case of d indices, we are interested in upper bounds for the norm in $L_p(\tau)$ of the sum

$$\sum_{\gamma \in \Gamma} f_\gamma.$$

To explain the main result of this paper, let us introduce some notation. Let $[m]$ be an abbreviation for the set $\{1, 2, \ldots, m\}$. Then, if $\mathbb{P}_d(2)$ denotes the set of partitions (α, β) of $[d]$ into two disjoint subsets (where we allow α and β to be the empty set), we denote by

$$\pi_\alpha : \Gamma \to [n]^{|\alpha|}$$

the canonical projection given by $\pi_\alpha(\gamma) = (i_k)_{k \in \alpha}$ for any $\gamma = (i_1, \ldots, i_d) \in [n]^d$. Then, if e_{ij} denotes the natural basis of the Schatten class S_p, the sum

$$\sum_{\gamma \in \Gamma} f_\gamma \otimes e_{\pi_\alpha(\gamma), \pi_\beta(\gamma)}$$

can be understood as an $L_p(\tau)$-valued matrix with $n^{|\alpha|}$ rows and $n^{|\beta|}$ columns. In particular, we always obtain an element of the vector-valued space $L_p(\tau; S_p)$. Our main result can be stated as follows. Let p be an even integer and let $f = (f_\gamma)_{\gamma \in \Gamma}$ be a p-orthogonal family in $L_p(\tau)$ with d indices, then

$$\Big\| \sum_{\gamma \in \Gamma} f_\gamma \Big\|_{L_p(\tau)} \le \mathrm{k}_d \, p^{\frac{d(d+1)}{2}} \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma \in \Gamma} f_\gamma \otimes e_{\pi_\alpha(\gamma), \pi_\beta(\gamma)} \Big\|_{L_p(\tau; S_p)} \right\}.$$

Here, k_d denotes an absolute constant depending only on d. Recall that Pisier's inequality follows from our result for 1-indexed p-orthogonal sums since α is either $\{1\}$ or the empty set while β is the complement of α. The general picture of our proof follows similar ideas to those in [6]. Indeed, let $\mathbf{F}_n$ be the free group with n generators $g_1, g_2, \ldots, g_n$ and let λ stand for the left regular representation of $\mathbf{F}_n$. Then it is easy to check that the family of operators

$$f_\gamma = \lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}) \qquad \text{with} \qquad \gamma = (i_1, i_2, \ldots, i_d),$$

is p-orthogonal with d indices for any even integer p. Using the non-commutative Khintchine inequality for free generators, we show that this family satisfies the inequality appearing in our main result. After that, the basic idea is to show that the norm of any p-orthogonal sum with d indices is controlled by the behaviour of this family. To that aim, we use the same combinatorial techniques employed in [6] to obtain a factorization result which allows us to use Hölder inequality. Then, the result follows easily.

In Section 1, we describe the inequalities which arise when applying several times the non-commutative Khintchine inequality for free generators to the family $\lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d})$. These inequalities will be used in the proof of our result. In Section 2, we give a brief summary of results about the theory of partitions that we shall need in the proof. Section 3 is devoted to the proof of the stated inequality for multi-indexed p-orthogonal sums. Section 4 contains two particularly interesting examples of multi-indexed p-orthogonal sums. The first one generalizes the notion of p-dissociate set in a discrete group. The second one is related to a Burkholder-Gundy type inequality for multi-indexed martingale difference sequences.

1. Iterations of the Khintchine inequality

Let $\mathbf{F}_n$ be the free group with n generators $g_1, g_2, \ldots, g_n$. If δ_t denotes the generic element of the natural basis of $\ell_2(\mathbf{F}_n)$, the left regular representation λ of $\mathbf{F}_n$ is defined by the relation

$$\lambda(t_1)\delta_{t_2} = \delta_{t_1 t_2}.$$

The reduced C^*-algebra $C^*_\lambda(\mathbf{F}_n)$ is defined as the C^*-algebra generated in $\mathcal{B}(\ell_2(\mathbf{F}_n))$ by the operators $\lambda(t)$ when t runs over $\mathbf{F}_n$. Let us denote by τ the standard trace on $C^*_\lambda(\mathbf{F}_n)$ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$, where e denotes the identity element of $\mathbf{F}_n$. Then, we construct the non-commutative Lebesgue space $L_p(\tau)$ in the usual way and consider the subspace $\mathcal{W}_p(n)$ of $L_p(\tau)$ generated by the operators $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$. The next result was proved by Haagerup and Pisier in [2] when $p = \infty$ and extended to any exponent $2 \le p \le \infty$ in [7].

Lemma 1.1. *Let $a_1, a_2, \ldots, a_n$ be a family of operators in some non-commutative Lebesgue space $L_p(\varphi)$. The following equivalence of norms holds for $2 \le p \le \infty$,*

$$\Big\| \sum_{k=1}^n a_k \otimes \lambda(g_k) \Big\|_{L_p(\varphi\otimes\tau)} \simeq \max\left\{ \Big\| \sum_{k=1}^n a_k \otimes e_{1k} \Big\|_{L_p(\varphi; R_p^n)}, \Big\| \sum_{k=1}^n a_k \otimes e_{k1} \Big\|_{L_p(\varphi; C_p^n)} \right\}.$$

In fact, the linear map $u : R_p^n \cap C_p^n \to \mathcal{W}_p(n)$ defined by

$$u(e_{1k} \oplus e_{k1}) = \lambda(g_k),$$

is a complete isomorphism with $\|u\|_{cb} \le 2$ and completely contractive inverse.

The row and column Hilbert spaces R_p^n and C_p^n are defined as the operator spaces generated by $\{e_{1j} \mid 1 \le j \le n\}$ and $\{e_{i1} \mid 1 \le i \le n\}$ respectively in S_p. Now, let us consider the group product $\mathrm{G}_d = \mathbf{F}_n \times \mathbf{F}_n \times \cdots \times \mathbf{F}_n$ with d factors. The left regular representation λ_d of G_d has the form

$$\lambda_d(t_1, t_2, \ldots, t_d) = \lambda(t_1) \otimes \lambda(t_2) \otimes \cdots \otimes \lambda(t_d).$$

Hence, the reduced C^*-algebra $C^*_{\lambda_d}(\mathrm{G}_d)$ is endowed with the trace $\tau_d = \tau \otimes \tau \otimes \cdots \otimes \tau$ with d factors. This allows us to consider the non-commutative space $L_p(\tau_d)$ for any $1 \le p \le \infty$. Then we define the space $\mathcal{W}_p(n)^{\otimes d}$ to be the subspace of $L_p(\tau_d)$ generated by the family of operators

$$\lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}).$$

The aim of this section is to describe the operator space structure of $\mathcal{W}_p(n)^{\otimes d}$ as a subspace of $L_p(\tau_d)$ for the exponents $2 \le p \le \infty$. This operator space structure has been already described in [7, Section 9.8], but here we shall give a more detailed exposition. As it was pointed out in [7], the case $1 \le p \le 2$ follows easily by duality. However, we shall not write the explicit inequalities in that case since we are not using them and the notation is considerably more complicated. If we apply repeatedly Lemma 1.1 to the sum

$$\mathcal{S}_d(a) = \sum_{i_1,\ldots,i_d=1}^n a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}) \in L_p(\varphi \otimes \tau_d),$$

then we easily get

$$\|\mathcal{S}_d(a)\|_{L_p(\varphi\otimes\tau_d)} \le 2^d \max\left\{ \Big\| \sum_{i_1,\ldots,i_d=1}^n a_{i_1\cdots i_d} \otimes \xi_1(i_1) \otimes \cdots \otimes \xi_d(i_d) \Big\|_{L_p(\varphi; S_p)} \right\},$$

where the maximum runs over all possible ways to choose the functions $\xi_1, \xi_2, \ldots, \xi_d$ among $\xi_k(\cdot) = e_{\cdot 1}$ and $\xi_k(\cdot) = e_{1\cdot}$. That is, each function ξ_k can take values either in the space R_p^n or in the space C_p^n. For a given selection of $\xi_1, \xi_2, \ldots, \xi_d$ we split up these functions into two sets, one made up of the functions taking values in R_p^n and the other taking values in C_p^n. More concretely, let us consider the sets

$$\begin{aligned} \mathrm{R}_\xi &= \big\{ k \mid \xi_k(i) = e_{1i} \big\}, \\ \mathrm{C}_\xi &= \big\{ k \mid \xi_k(i) = e_{i1} \big\}. \end{aligned}$$

Then, if C_ξ has s elements, the sum

$$\sum_{i_1,\ldots,i_d=1}^{n} a_{i_1\cdots i_d} \otimes \xi_1(i_1) \otimes \cdots \otimes \xi_d(i_d)$$

can be regarded as a $n^s \times n^{d-s}$ matrix with entries in $L_p(\varphi)$. Now, using the notation already presented in the Introduction, we express the inequality above in a much more understandable way. Namely, we have

$$(1) \qquad \|\mathcal{S}_d(a)\|_{L_p(\varphi\otimes\tau_d)} \le 2^d \max_{(\alpha,\beta)\in\mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma\in\Gamma} a_\gamma \otimes e_{\pi_\alpha(\gamma),\pi_\beta(\gamma)} \Big\|_{L_p(\varphi;S_p)} \right\}.$$

Remark 1.2. By the same arguments, the converse of (1) holds with constant 1.

2. Möbius inversion

Given a positive integer m, we denote by $\mathbb{P}_m$ the lattice of partitions of the set $[m] = \{1, 2, \ldots, m\}$. If ρ and σ are elements of $\mathbb{P}_m$, we shall write $\rho \le \sigma$ when every block of ρ is contained in some block of σ. The minimal and maximal elements of $\mathbb{P}_m$ with respect to this partial order are denoted by $\dot{0}$ and $\dot{1}$ respectively. That is, $\dot{0}$ stands for the partition into m singletons and $\dot{1}$ coincides with $\{[m]\}$. The Möbius function μ is a complex-valued function defined on the set of pairs of partitions (ρ, σ) in $\mathbb{P}_m \times \mathbb{P}_m$ satisfying $\rho \le \sigma$. The following Lemma summarizes the main properties of this function that we shall use below.

Lemma 2.1. *Let us consider a pair of functions $\Phi : \mathbb{P}_m \to V$ and $\Psi : \mathbb{P}_m \to V$ taking values in some vector space V. Then the following implication holds*

$$\Psi(\rho) = \sum_{\sigma\ge\rho} \Phi(\sigma) \ \Rightarrow\ \Phi(\rho) = \sum_{\sigma\ge\rho} \mu(\rho,\sigma)\Psi(\sigma).$$

Besides, the Möbius function satisfies the following identities

- $\displaystyle\sum_{\sigma\in\mathbb{P}_m} |\mu(\dot{0},\sigma)| = m!.$
- *For any $\sigma > \dot{0}$, we have* $\displaystyle\sum_{\dot{0}\le\rho\le\sigma} \mu(\rho,\sigma) = 0.$

For a more detailed exposition of these topics we refer the reader to [1]. Now, let p be an even integer and let $\varphi : E_1 \times \cdots \times E_p \to V$ be a multilinear map defined on certain vector spaces $E_1, E_2, \ldots, E_p$ and taking values in the vector space V. For each $1 \le s \le p$ we consider elements $f_\gamma(s) \in E_s$ indexed by Γ. Then, we define the sums

$$\mathrm{F}_s = \sum_{\gamma\in\Gamma} f_\gamma(s) \in E_s.$$

Clearly we have

$$\varphi(\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_p) = \sum_h \varphi\big(f_{h(1)}(1), f_{h(2)}(2), \ldots, f_{h(p)}(p)\big),$$

where the sum runs over the set of functions $h : \{1, 2, \ldots, p\} \to \Gamma$. Now, for any such function h and for each $1 \leq k \leq d$, we consider the partition $\sigma_k(h) \in \mathbb{P}_p$ associated to the coordinate function $\pi_k \circ h$. In other words, given $1 \leq r, s \leq p$ we have the following characterization

$$r \sim s \,(\mathrm{mod}\, \sigma_k(h)) \Leftrightarrow \pi_k(h(r)) = \pi_k(h(s)),$$

where $\sim$ $(\mathrm{mod}\, \sigma)$ means belonging to the same block of σ. Let us also consider the d-tuple $\delta(h) = (\sigma_1(h), \sigma_2(h), \ldots, \sigma_d(h))$ in the product $\mathbf{P}(p, d) = \mathbb{P}_p \times \cdots \times \mathbb{P}_p$ with d factors. Then we can write

$$\varphi(\mathrm{F}_1, \ldots, \mathrm{F}_p) = \sum_{\eta \in \mathbf{P}(p,d)} \Phi(\eta),$$

where $\Phi : \mathbf{P}(p, d) \to V$ has the form

$$\Phi(\eta) = \sum_{h:\, \delta(h)=\eta} \varphi\big(f_{h(1)}(1), \ldots, f_{h(p)}(p)\big).$$

Now, if $\eta = (\rho_1, \ldots, \rho_d)$ we shall write $\eta \sim \dot{0}$ whenever $\rho_k = \dot{0}$ for some $1 \leq k \leq d$. Then, we obtain the following decomposition

$$\varphi(\mathrm{F}_1, \ldots, \mathrm{F}_p) = \sum_{\eta \sim \dot{0}} \Phi(\eta) + \sum_{\rho_1 > \dot{0}} \cdots \sum_{\rho_d > \dot{0}} \Phi(\eta). \tag{2}$$

Similarly, the expression $h \sim \dot{0}$ will denote the existence of some $1 \leq k \leq d$ such that $\sigma_k(h) = \dot{0}$. In other words, $h \sim \dot{0}$ whenever h has an injective projection. Then, since $\delta(h) = (\sigma_1(h), \ldots, \sigma_d(h))$, we have

$$\sum_{\eta \sim \dot{0}} \Phi(\eta) = \sum_{h \sim \dot{0}} \varphi\big(f_{h(1)}(1), \ldots, f_{h(p)}(p)\big). \tag{3}$$

For the second sum in (2), we define

$$\Psi_d(\rho_1, \rho_2, \ldots, \rho_d) = \sum_{\sigma_d \geq \rho_d} \Phi(\rho_1, \rho_2, \ldots, \rho_{d-1} | \sigma_d).$$

Then, if we fix $\rho_1, \rho_2, \ldots, \rho_{d-1}$, we can apply Lemma 2.1 to obtain

$$\begin{aligned}
\sum_{\rho_d > \dot{0}} \Phi(\rho_1, \rho_2, \ldots, \rho_d) &= \sum_{\rho_d > \dot{0}} \Big(\sum_{\sigma_d \geq \rho_d} \mu(\rho_d, \sigma_d) \Psi_d(\rho_1, \ldots, \rho_{d-1} | \sigma_d) \Big) \\
&= \sum_{\sigma_d > \dot{0}} \Psi_d(\rho_1, \ldots, \rho_{d-1} | \sigma_d) \sum_{\dot{0} < \rho_d \leq \sigma_d} \mu(\rho_d, \sigma_d) \\
&= \sum_{\sigma_d > \dot{0}} (-\mu(\dot{0}, \sigma_d)) \Psi_d(\rho_1, \ldots, \rho_{d-1} | \sigma_d)
\end{aligned}$$

Similarly, we define

$$\begin{aligned}
\Psi_{d-1}(\rho_1, \rho_2, \ldots, \rho_{d-1} | \sigma_d) &= \sum_{\sigma_{d-1} \geq \rho_{d-1}} \Psi_d(\rho_1, \rho_2, \ldots, \rho_{d-2} | \sigma_{d-1}, \sigma_d), \\
\Psi_{d-2}(\rho_1, \rho_2, \ldots | \sigma_{d-1}, \sigma_d) &= \sum_{\sigma_{d-2} \geq \rho_{d-2}} \Psi_{d-1}(\rho_1, \rho_2, \ldots | \sigma_{d-2}, \sigma_{d-1}, \sigma_d),
\end{aligned}$$

and so on until

$$\Psi_1(\rho_1|\sigma_2,\dots,\sigma_d) = \sum_{\sigma_1 \geq \rho_1} \Psi_2(\sigma_1,\sigma_2,\dots,\sigma_d).$$

Then, applying Lemma 2.1 as above, we have for $1 \leq k \leq d-1$

$$\sum_{\rho_k > \dot{0}} \Psi_{k+1}(\rho_1,\dots|\sigma_{k+1},\dots,\sigma_d) = -\sum_{\sigma_k > \dot{0}} \mu(\dot{0},\sigma_k)\Psi_k(\rho_1,\dots|\sigma_k,\dots,\sigma_d).$$

Putting all together, we get

$$\sum_{\rho_1 > \dot{0}} \cdots \sum_{\rho_d > \dot{0}} \Phi(\eta) \quad = \quad (-1)^d \sum_{\sigma_1 > \dot{0}} \cdots \sum_{\sigma_d > \dot{0}} \Big[\prod_{k=1}^d \mu(\dot{0},\sigma_k) \Big] \Psi_1(\sigma_1,\dots,\sigma_d), \tag{4}$$

where the function Ψ_1 can be easily rewritten as

$$\Psi_1(\sigma_1,\dots,\sigma_d) = \sum_{\substack{h \,:\, \sigma_k(h) \geq \sigma_k \\ 1 \leq k \leq d}} \varphi\big(f_{h(1)}(1), f_{h(2)}(2), \dots, f_{h(p)}(p)\big). \tag{5}$$

In summary, looking at (2), (3), (4) and (5) we have the following result.

Lemma 2.2. *The following identity holds*

$$\begin{aligned} \varphi(\mathrm{F}_1,\dots,\mathrm{F}_p) \quad &= \quad \sum_{h \sim \dot{0}} \varphi\big(f_{h(1)}(1),\dots,f_{h(p)}(p)\big) \\ &+ \quad (-1)^d \sum_{\sigma_1 > \dot{0}} \cdots \sum_{\sigma_d > \dot{0}} \Big[\prod_{k=1}^d \mu(\dot{0},\sigma_k) \Big] \Psi(\sigma_1,\dots,\sigma_d), \end{aligned}$$

where Ψ has the following form

$$\Psi(\sigma_1,\dots,\sigma_d) = \sum_{\substack{h \,:\, \sigma_k(h) \geq \sigma_k \\ 1 \leq k \leq d}} \varphi\big(f_{h(1)}(1), f_{h(2)}(2), \dots, f_{h(p)}(p)\big).$$

3. Proof of the main result

In this section we shall prove the result stated below. We start by factorizing the sum which defines the function Ψ above. This will allow us to show that the behaviour of any p-orthogonal sum with d indices is majorized by the estimates obtained in Section 1, with the aid of non-commutative Khintchine inequalities.

Theorem 3.1. *If $f = (f_\gamma)_{\gamma \in \Gamma}$ is p-orthogonal in $L_p(\tau)$ with d indices, then*

$$\Big\| \sum_{\gamma \in \Gamma} f_\gamma \Big\|_{L_p(\tau)} \leq \mathrm{k}_d \, p^{\frac{d(d+1)}{2}} \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma \in \Gamma} f_\gamma \otimes e_{\pi_\alpha(\gamma),\pi_\beta(\gamma)} \Big\|_{L_p(\tau;S_p)} \right\}.$$

3.1. **Factorization of Ψ.** Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal trace τ satisfying $\tau(1) = 1$ and let p be an even integer. Following the notation above, we shall take in what follows $E_s = L_p(\tau)$ for all $1 \le s \le p$ and the multilinear map φ will be replaced by the trace τ acting on a product of p operators in $L_p(\tau)$. That is, $f = (f_\gamma)_{\gamma \in \Gamma}$ is assumed to be p-orthogonal in $L_p(\tau)$ with d indices and we have

$$\varphi\big(f_{h(1)}(1), f_{h(2)}(2), \ldots, f_{h(p)}(p)\big) = \tau\big(f_{h(1)}(1) f_{h(2)}(2) \cdots f_{h(p)}(p)\big),$$

where

$$f_{h(s)}(s) = \begin{cases} f_{h(s)}^* & \text{if } s \text{ is odd,} \\ f_{h(s)} & \text{if } s \text{ is even.} \end{cases}$$

The aim now is to factorize the sum

$$\Psi(\sigma_1, \ldots, \sigma_d) = \sum_{\substack{h \,:\, \sigma_k(h) \ge \sigma_k \\ 1 \le k \le d}} \tau\big(f_{h(1)}(1) f_{h(2)}(2) \cdots f_{h(p)}(p)\big).$$

We shall need below the following version of Fell's absorption principle.

Absorption Principle in L_p. *Given a discrete group* G*, let us denote by* λ_{G} *the left regular representation of* G *and by* τ_{G} *the associated trace on the reduced* C^**-algebra of* G*. Then, given any other unitary representation* $\pi : \mathrm{G} \to \pi(\mathrm{G})''$*, the following representations are unitarily equivalent*

$$\lambda_{\mathrm{G}} \otimes \pi \simeq \lambda_{\mathrm{G}} \otimes 1,$$

where 1 *stands for the trivial representation of* G *in* $\pi(\mathrm{G})''$*. Let us consider any faithful normalized trace* ψ *on* $\pi(\mathrm{G})''$*. Then, given any finitely supported function* $a : \mathrm{G} \to L_p(\varphi)$*, the following equality holds for* $1 \le p \le \infty$

$$\Big\| \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_{\mathrm{G}}(t) \otimes \pi(t) \Big\|_{L_p(\varphi \otimes \tau_{\mathrm{G}} \otimes \psi)} = \Big\| \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_{\mathrm{G}}(t) \Big\|_{L_p(\varphi \otimes \tau_{\mathrm{G}})}.$$

Proof. See Proposition 8.1 of [7] for the first part and [5] for the second. □

Lemma 3.2. *Let* $\sigma_1, \sigma_2, \ldots, \sigma_d$ *be a family of partitions in* $\mathbb{P}_p$ *different from* $\dot{0}$*. If we are given* $0 \le q \le d$*, let* B_q *be the set of elements* s *in* $\{1, 2, \ldots, p\}$ *being a singleton exactly in* q *partitions among* $\sigma_1, \sigma_2, \ldots, \sigma_d$*. Then, there exists a discrete group* G *and a family* $\mathsf{F}_1, \mathsf{F}_2, \ldots, \mathsf{F}_p$ *in* $L_p(\tau_{\mathrm{G}} \otimes \tau)$ *satisfying*

$$\|\mathsf{F}_s\|_p \le \mathrm{k}_d \, p^{\frac{q(q+1)}{2}} \Big\| \sum_{i_1, \ldots, i_d = 1}^{n} \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes f_{i_1 \cdots i_d} \Big\|_{L_p(\tau_d \otimes \tau)}$$

for each $s \in \mathrm{B}_q$ *whenever* $0 \le q < d$ *and also*

$$\|\mathsf{F}_s\|_p = \Big\| \sum_{\gamma \in \Gamma} f_\gamma \Big\|_{L_p(\tau)}$$

for each $s \in \mathrm{B}_d$*. Moreover, we have*

$$\sum_{\substack{h \,:\, \sigma_k(h) \ge \sigma_k \\ 1 \le k \le d}} \tau\big(f_{h(1)}(1) f_{h(2)}(2) \cdots f_{h(p)}(p)\big) = (\tau_{\mathrm{G}} \otimes \tau)(\mathsf{F}_1 \mathsf{F}_2 \cdots \mathsf{F}_p). \tag{6}$$

Remark 3.3. As we have pointed out, Theorem 3.1 was already proved in [6] for 1-indexed families. In particular, we can assume that Theorem 3.1 holds for any k-indexed family whenever $1 \le k \le d-1$ and prove Theorem 3.1 by induction. In the proof of Lemma 3.2, we shall need to use this induction hypothesis.

Remark 3.4. From now on, k_d might change from one instance to another.

Proof. Let us consider an integer $2 \le m \le p$. As it is customary, we write τ_{m-1} for the standard trace associated to the reduced C^*-algebra of the group product $\mathbf{F}_n \times \mathbf{F}_n \times \cdots \times \mathbf{F}_n$ with $m-1$ factors. Then, for each $1 \le i \le n$, we consider the following family $\xi_1(i), \xi_2(i), \ldots, \xi_m(i)$ of operators in $L_p(\tau_{m-1})$

$$\begin{array}{rcl} \xi_1(i) & = & \lambda(g_i)^* \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1, \\ \xi_2(i) & = & \lambda(g_i) \otimes \lambda(g_i)^* \otimes 1 \otimes \cdots \otimes 1, \\ \xi_3(i) & = & 1 \otimes \lambda(g_i) \otimes \lambda(g_i)^* \otimes \cdots \otimes 1, \\ & \cdots & \\ \xi_{m-1}(i) & = & 1 \otimes \cdots \otimes 1 \otimes \lambda(g_i) \otimes \lambda(g_i)^*, \\ \xi_m(i) & = & 1 \otimes \cdots \otimes 1 \otimes \lambda(g_i). \end{array}$$

Given $g : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$, this family has the following property

$$\tau_{m-1}\big(\xi_1(g(1)) \cdots \xi_m(g(m))\big) = \begin{cases} 1 & \text{if } g \text{ is constant,} \\ 0 & \text{if } g \text{ is non-constant.} \end{cases} \tag{7}$$

Let us make explicit the blocks of the partitions $\sigma_1, \sigma_2, \ldots, \sigma_d$ by

$$\sigma_k = \Big\{ \mathrm{A}_{kj_k} \ \Big| \ 1 \le j_k \le \mathrm{m}_k \Big\}.$$

Now we fix σ_k and, for each A_{kj_k} with cardinality $m_{j_k} > 1$, we construct the family $\Pi(i, j_k) = \{\xi_1(i, j_k), \xi_2(i, j_k), \ldots, \xi_{m_{j_k}}(i, j_k)\}$ in $L_p(\tau_{m_{j_k}-1})$ as above. Notice that $1 \le i \le n$ and $1 \le j_k \le \mathrm{m}_k$. If the set A_{kj_k} has only one element, we take $\Pi(i, j_k) = \{\xi_1(i, j_k)\}$ with $\xi_1(i, j_k) = 1$. Then we consider the following families of m_k-fold tensor products

$$\begin{array}{rcl} \Sigma(i, 1) & = & \Pi(i, 1) \otimes 1 \otimes 1 \otimes \cdots \otimes 1, \\ \Sigma(i, 2) & = & 1 \otimes \Pi(i, 2) \otimes 1 \otimes \cdots \otimes 1, \\ & \cdots & \\ \Sigma(i, \mathrm{m}_k) & = & 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \Pi(i, \mathrm{m}_k). \end{array}$$

Here, the r-th '1' in $\Sigma(i, j_k)$ denotes the identity operator in $L_p(\tau_{m_r-1})$. Recall that, fixed $1 \le i \le n$, each $\Sigma(i, j_k)$ is an *ordered family* with m_{j_k} elements. On the other hand, for each $1 \le s \le p$, there exist a unique set of indices $j_1(s), j_2(s), \ldots, j_d(s)$ such that s belongs to the corresponding blocks of $\sigma_1, \sigma_2, \ldots, \sigma_d$. In other words, we pick up the indices $j_k(s)$ satisfying

$$s \in \bigcap_{k=1}^{d} \mathrm{A}_{kj_k(s)}.$$

This allows us to consider the family of operators

$$\Lambda(\gamma, s) = \bigotimes_{k=1}^{d} \Sigma(i_k, j_k(s)),$$

where $i_k = \pi_k(\gamma)$ and $\gamma \in \Gamma$. Now we select an element of $\Lambda(\gamma, s)$ as follows. If s is the r_1-th element in the block $\mathrm{A}_{1j_1(s)}$, then we pick up the r_1-th operator in the family $\Sigma(i_1, j_1(s))$. Let us denote it by $x_{1s}(i_1)$. Similarly, if s is the r_2-th element in $\mathrm{A}_{2j_2(s)}$, we pick up the r_2-th operator in $\Sigma(i_2, j_2(s))$, say $x_{2s}(i_2)$. We iterate this process to get an element

$$x_{1s}(i_1) \otimes x_{2s}(i_2) \otimes \cdots \otimes x_{ds}(i_d) \in \Lambda(\gamma, s).$$

Then we define,

$$\mathsf{F}_s = \sum_{\gamma \in \Gamma} \Big(\bigotimes_{k=1}^d x_{ks}(\pi_k(\gamma)) \Big) \otimes f_\gamma(s).$$

Clearly, there exists a collection of discrete groups $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_d$ (all of them being direct products of $\mathbf{F}_n$) such that $\mathsf{F}_s \in L_p(\tau_\mathrm{G} \otimes \tau)$ with $\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G}_d$. Let us check that identity (6) holds. Notice that

$$(\tau_\mathrm{G} \otimes \tau)(\mathsf{F}_1 \mathsf{F}_2 \cdots \mathsf{F}_p) = \sum_{\gamma_1, \ldots, \gamma_p \in \Gamma} \prod_{k=1}^d \tau_{\mathrm{G}_k} \Big(\prod_{s=1}^p x_{ks}(\pi_k(\gamma_s)) \Big) \tau(f_{\gamma_1}(1) \cdots f_{\gamma_p}(p)).$$

Recalling the definition of x_{ks} and property (7), it can be checked that

$$\prod_{k=1}^d \tau_{\mathrm{G}_k} \Big(\prod_{s=1}^p x_{ks}(\pi_k(\gamma_s)) \Big)$$

is 1 when the condition $j_k(s) = j_k(s') \Rightarrow \pi_k(\gamma_s) = \pi_k(\gamma_{s'})$ holds for $k = 1, 2, \ldots, d$ and is 0 otherwise. In particular, identity (6) follows. Now we look at the norm of F_s in $L_p(\tau_\mathrm{G} \otimes \tau)$. First assume that $s \in \mathrm{B}_d$. That is, s is a singleton of σ_k for every $k = 1, 2, \ldots, d$. Then

$$\mathsf{F}_s = \sum_{\gamma \in \Gamma} 1 \otimes f_\gamma(s) = \Big(\sum_{\gamma \in \Gamma} 1 \otimes f_\gamma \Big)^{(*)}$$

where $(*)$ is $*$ when s is odd and 1 otherwise. Therefore the stated assertion follows. Finally, assume that $s \in \mathrm{B}_q$ with $q < d$. If $q = 0$ our estimation for the norm of F_s is easy. Namely, a quick inspection of the definition of F_s allows us to write

$$\mathsf{F}_s \simeq \sum_{i_1, \ldots, i_d = 1}^n \Big(\bigotimes_{k=1}^d \chi(i_k) \Big) \otimes f_{i_1 \cdots i_d}(s),$$

where $\chi(i_k)$ can be either $\lambda(g_{i_k})^*$ or $\lambda(g_{i_k}) \otimes \lambda(g_{i_k})^*$ or $\lambda(g_{i_k})$. However, by Fell's absorption principle these terms are unitarily equivalent. In other words, in this particular case we obtain an equality

$$\|\mathsf{F}_s\|_p = \Big\| \sum_{i_1, \ldots, i_d = 1}^n \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes f_{i_1 \cdots i_d}(s) \Big\|_{L_p(\tau_d \otimes \tau)}.$$

Notice that the dependence on s on the right can be ignored since the two possible expressions that come out (for s odd and s even) turn out to be equal. It remains to check the cases $0 < q < d$. For simplicity of notation, we assume that s is a singleton in $\sigma_1, \sigma_2, \ldots, \sigma_q$. As we shall see, the general case can be proved in a similar way. Then, again by Fell's absorption principle, we have

$$\|\mathsf{F}_s\|_p = \Big\| \sum_{i_1, \ldots, i_d = 1}^n \lambda(g_{i_{q+1}}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes f_{i_1 \cdots i_d}(s) \Big\|_{L_p(\tau_{d-q} \otimes \tau)}.$$

Applying the iteration of Khintchine inequality described in (1), we have

$$\|\mathsf{F}_s\|_p \le \mathrm{k}_d \max_{(\alpha,\beta)\in\mathbb{P}_{d-q}(2)} \left\{ \Big\| \sum_{\nu\in[n]^{d-q}} \Big[\sum_{\zeta\in[n]^q} f_{\zeta,\nu}(s) \Big] \otimes e_{\pi_\alpha(\nu),\pi_\beta(\nu)} \Big\|_{L_p(\tau;S_p)} \right\}.$$

The sum on the right can be rewritten as follows

$$\begin{aligned} \sum_{\nu\in[n]^{d-q}} \Big[\sum_{\zeta\in[n]^q} f_{\zeta,\nu}(s) \Big] \otimes e_{\pi_\alpha(\nu),\pi_\beta(\nu)} &= \sum_{\zeta\in[n]^q} \Big[\sum_{\nu\in[n]^{d-q}} f_{\zeta,\nu}(s) \otimes e_{\pi_\alpha(\nu),\pi_\beta(\nu)} \Big] \\ &= \sum_{\zeta\in[n]^q} f_\zeta^{\alpha\beta}(s). \end{aligned}$$

Now we observe that the family $f_\zeta^{\alpha\beta}(s)$ is p-orthogonal with q indices for any (α,β,s) as a simple consequence of the p-orthogonality of f. Since $q<d$, we can apply the induction hypothesis recalled in Remark 3.3 to obtain

$$\Big\| \sum_{\zeta\in[n]^q} f_\zeta^{\alpha\beta}(s) \Big\|_{L_p(\tau;S_p)} \le \mathrm{k}_d\, p^{\frac{q(q+1)}{2}} \max_{(\varepsilon,\delta)\in\mathbb{P}_q(2)} \left\{ \Big\| \sum_{\zeta\in[n]^q} f_\zeta^{\alpha\beta}(s) \otimes e_{\pi_\varepsilon(\zeta),\pi_\delta(\zeta)} \Big\|_p \right\}.$$

Putting it all together, the assertion follows by Remark 1.2. □

3.2. **Concluding estimates.** Now we are ready to prove Theorem 3.1. First we recall that the p-orthogonality of f can be combined with Lemma 2.2 to drop those terms for which the indices admit an injective projection. In other words,

$$\begin{aligned} \Big\| \sum_{\gamma\in\Gamma} f_\gamma \Big\|_{L_p(\tau)}^p &= \sum_{\gamma_1,\ldots,\gamma_p\in\Gamma} \tau\big(f_{\gamma_1}^* f_{\gamma_2} f_{\gamma_3}^* f_{\gamma_4} \cdots f_{\gamma_{p-1}}^* f_{\gamma_p}\big) \\ &= (-1)^d \sum_{\sigma_1>\dot{0}} \cdots \sum_{\sigma_d>\dot{0}} \Big[\prod_{k=1}^d \mu(\dot{0},\sigma_k) \Big] \Psi(\sigma_1,\ldots,\sigma_d). \end{aligned}$$

On the other hand, let us write

$$\begin{aligned} \mathsf{A} &= \Big\| \sum_{\gamma\in\Gamma} f_\gamma \Big\|_{L_p(\tau)}, \\ \mathsf{B} &= \Big\| \sum_{i_1,\ldots,i_d=1}^n \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes f_{i_1\cdots i_d} \Big\|_{L_p(\tau_d\otimes\tau)}, \\ \mathsf{C} &= \max_{(\alpha,\beta)\in\mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma\in\Gamma} f_\gamma \otimes e_{\pi_\alpha(\gamma),\pi_\beta(\gamma)} \Big\|_{L_p(\tau;S_p)} \right\}. \end{aligned}$$

Let us write $\delta=(\sigma_1,\sigma_2,\ldots,\sigma_d)$ and let $r(\delta)$ be the number of common singletons. That is, $r(\delta)$ coincides with the cardinality of B_d. Then, Lemma 3.2 and Hölder's inequality provide the following estimate

$$\begin{aligned} \big|\Psi(\sigma_1,\ldots\sigma_d)\big| &\le \prod_{s=1}^p \|\mathsf{F}_s\|_p \\ &\le \mathsf{A}^{r(\delta)} \prod_{q=0}^{d-1} \big(\mathrm{k}_d p^{\frac{q(q+1)}{2}} \mathsf{B}\big)^{|\mathrm{B}_q|} \end{aligned}$$

$$\leq \Big[\prod_{q=0}^{d-1} p^{\frac{q(q+1)}{2}|\mathrm{B}_q|}\Big] \mathsf{A}^{r(\delta)} \big(\mathrm{k}_d \mathsf{C}\big)^{p-r(\delta)}.$$

Notice that $\mathsf{B} \leq \mathrm{k}_d\, \mathsf{C}$ by inequality (1). Now, recalling that

$$\sum_{q=0}^{d-1} \frac{q(q+1)}{2} |\mathrm{B}_q| \leq \frac{d(d-1)}{2} \sum_{q=0}^{d-1} |\mathrm{B}_q| = \frac{d(d-1)}{2} (p - r(\delta)),$$

we obtain the following estimate for Ψ

$$\big|\Psi(\sigma_1, \ldots \sigma_d)\big| \leq \mathsf{A}^{r(\delta)} \Big(\mathrm{k}_d\, p^{\frac{d(d-1)}{2}} \mathsf{C}\Big)^{p-r(\delta)}.$$

Putting it all together, we get

$$\mathsf{A}^p \leq \sum_{\sigma_1 > \dot{0}} \cdots \sum_{\sigma_d > \dot{0}} \Big[\prod_{k=1}^{d} |\mu(\dot{0}, \sigma_k)|\Big] \mathsf{A}^{r(\delta)} \Big(\mathrm{k}_d\, p^{\frac{d(d-1)}{2}} \mathsf{C}\Big)^{p-r(\delta)}.$$

Since $\sigma_k > \dot{0}$ for all k, we have $0 \leq r(\delta) \leq p - 1$. Therefore, we can write

$$\mathsf{A}^p \leq \sum_{r=0}^{p-1} \varphi_r\, \mathsf{A}^r \Big(\mathrm{k}_d\, p^{\frac{d(d-1)}{2}} \mathsf{C}\Big)^{p-r},$$

with φ_r given by

$$\varphi_r = \sum_{\delta_0 \colon r(\delta_0) = r} \prod_{k=1}^{d} |\mu(\dot{0}, \sigma_k)|.$$

The zero subindex in δ is chosen to denote that the sum is taken over the set of $\delta_0 = (\sigma_1, \sigma_2, \ldots, \sigma_d)$ such that $\sigma_k > \dot{0}$ for all k. Ignoring that restriction and applying Lemma 2.1, we easily get

$$\varphi_r \leq \sum_{\delta \colon r(\delta) \geq r} \prod_{k=1}^{d} |\mu(\dot{0}, \sigma_k)| = \binom{p}{r} \prod_{k=1}^{d} \sum_{\sigma_k \in \mathbb{P}_{p-r}} |\mu(\dot{0}, \sigma_k)| = \binom{p}{r} (p-r)!^d.$$

In particular, we obtain

$$\begin{aligned} \mathsf{A}^p \;&\leq\; \sum_{r=0}^{p-1} \binom{p}{r} (p-r)!^d \mathsf{A}^r \Big(\mathrm{k}_d\, p^{\frac{d(d-1)}{2}} \mathsf{C}\Big)^{p-r} \\ &\leq\; \sum_{r=0}^{p-1} \binom{p}{r} (p-r)! \mathsf{A}^r \mathsf{D}^{p-r}, \end{aligned} \tag{8}$$

where D has the form

$$\mathsf{D} = \mathrm{k}_d \Big[\sup_{0 \leq r \leq p-2} (p-r)!^{\frac{d-1}{p-r}}\Big] p^{\frac{d(d-1)}{2}} \mathsf{C} \leq \mathrm{k}_d p^{\frac{(d+2)(d-1)}{2}} \mathsf{C}.$$

The last inequality follows easily from Stirling's formula. Now, we conclude by applying the same arguments as in [6]. More concretely, proceeding as in Sublemma 2.3 of [6], we obtain

$$\mathsf{A} \leq 2p\, \mathsf{D} \leq \mathrm{k}_d\, p^{\frac{d(d+1)}{2}} \mathsf{C}. \tag{9}$$

This estimation completes the proof of Theorem 3.1. Although the proof of (9) follows from (8) and Sublemma 2.3 of [6], we include the proof for completeness. If

$\mathsf{A} \le p\,\mathsf{D}$ there is nothing to prove. Hence, assume that $\mathsf{A} > p\,\mathsf{D}$. Let us divide at both sides of (8) by A^p and let us write $z = \mathsf{D}/\mathsf{A}$, so that $pz < 1$ and

$$1 \le \sum_{r=0}^{p-1} \binom{p}{r} (p-r)! z^{p-r}.$$

Then, we have

$$\begin{aligned} 2 &\le \sum_{r=0}^{p-1} \binom{p}{r} \int_0^\infty (zx)^{p-r} e^{-x}\, dx + 1 \\ &= \int_0^\infty \big[(1+zx)^p - 1\big] e^{-x}\, dx + 1 \\ &= \int_0^\infty (1+zx)^p e^{-x}\, dx \\ &\le \int_0^\infty \exp(pzx - x)\, dx. \end{aligned}$$

Since $pz < 1$, we conclude $2 \le (1-pz)^{-1}$ and $z^{-1} \le 2p$ as desired.

Remark 3.5. Let us look for a moment what happens with Theorem 3.1 when the von Neumann algebra $\mathcal{M}$ is commutative, so that we can think of $L_p(\tau)$ as $L_p(\mu)$ for some probability measure μ. As was recalled in [6], if we are given a p-orthogonal family $f_1, f_2, \ldots, f_n$ with one index in $L_p(\mu)$, then we obtain the natural analog of Burkholder-Gundy inequality for a martingale difference sequence. Namely, we have

$$\Big(\int_\Omega \Big| \sum_{k=1}^n f_k(\omega) \Big|^p d\mu(\omega) \Big)^{1/p} \le \frac{3\pi}{2} p \Big(\int_\Omega \Big[\sum_{k=1}^n |f_k(\omega)|^2 \Big]^{p/2} d\mu(\omega) \Big)^{1/p}.$$

In the general case, Theorem 3.1 provides the following inequality

$$\Big(\int_\Omega \Big| \sum_{\gamma \in \Gamma} f_\gamma(\omega) \Big|^p d\mu(\omega) \Big)^{1/p} \le \mathrm{k}_d\, p^{\frac{d(d+1)}{2}} \Big(\int_\Omega \Big[\sum_{\gamma \in \Gamma} |f_\gamma(\omega)|^2 \Big]^{p/2} d\mu(\omega) \Big)^{1/p}.$$

4. Two examples

We conclude this paper with two examples of multi-indexed p-orthogonal sums. The first one came out during the preparation of [5] and was the motivation of this work. It provides a generalization of the notion of p-dissociate subset of a discrete group. The second provides an analog of the non-commutative Burkholder-Gundy inequalities for multi-indexed martingale difference sequences.

4.1. **Multi-indexed p-dissociate sets.** Let G be a discrete group with identity element e. A subset $\Lambda = \big\{ t_1, t_2, \ldots, t_n \big\}$ of G is called a *p-dissociate set* if for any injective function $h : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, n\}$, the following non-cancellation property holds

$$t_{h(1)}^{-1} t_{h(2)} t_{h(3)}^{-1} t_{h(4)} \cdots t_{h(p-1)}^{-1} t_{h(p)} \neq e.$$

In a similar way, let Γ be as above and let $\Lambda = \big\{ t_\gamma : \gamma \in \Gamma \big\}$ be a subset of G indexed by Γ. Then, we shall say that Λ is a *p-dissociate set with d indices* if the same non-cancellation property is satisfied whenever the function $h : \{1, 2, \ldots, p\} \to \Gamma$ has an injective projection. Let λ_{G} be the left regular representation of G and let us denote by τ_{G} the natural trace on $\lambda_{\mathrm{G}}(\mathrm{G})''$. Then, since $\tau_{\mathrm{G}}(\lambda(t))$ vanishes unless the element t is the identity e, it is clear that

$$f = \Big\{ a_\gamma \otimes \lambda_{\mathrm{G}}(t_\gamma) \,\Big|\, \gamma \in \Gamma \Big\}$$

is p-orthogonal in $L_p(\tau \otimes \tau_{\mathrm{G}})$ with d indices for any given function $a : \Gamma \to L_p(\tau)$. As was pointed out in [6], these kind of sets can be used to obtain the classical Littlewood-Paley inequalities from the case of 1-indexed families in Theorem 3.1. A remarkable p-dissociate set with d indices is provided by the free group $\mathbf{F}_n$ with n generators $g_1, g_2, \ldots, g_n$. Indeed, let us consider the set

$$\Lambda = \Big\{ g_{i_1} g_{i_2} \cdots g_{i_d} \,\Big|\, 1 \le i_k \le n \Big\}.$$

By the freeness of the generators, it is not difficult to check that Λ is p-dissociate with d indices for any even exponent p. In particular, given any collection of operators $\mathcal{A} = (a_\gamma)_{\gamma \in \Gamma}$ indexed by Γ, the family

$$\boldsymbol{f}_{i_1 i_2 \cdots i_d} = a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$$

is p-orthogonal in $L_p(\tau \otimes \tau_{\mathrm{G}})$ with d indices and Theorem 3.1 gives

$$\Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \Big\|_{L_p(\tau \otimes \tau_{\mathrm{G}})} \le \mathrm{k}_d \, p^{\frac{d(d+1)}{2}} \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \otimes e_{\pi_\alpha(\gamma), \pi_\beta(\gamma)} \Big\|_{L_p(\tau \otimes \tau_{\mathrm{G}}; S_p)} \right\}.$$

However, the following equivalence of norms

$$\max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \otimes e_{\pi_\alpha(\gamma), \pi_\beta(\gamma)} \Big\|_p \right\} \simeq \max_{0 \le k \le d} \left\{ \Big\| \sum_{\mathrm{I} \in [n]^k} \sum_{\mathrm{J} \in [n]^{d-k}} a_{\mathrm{IJ}} \otimes e_{\mathrm{I},\mathrm{J}} \Big\|_p \right\},$$

holds for any exponent $2 \le p \le \infty$ and with constants depending only on d. The reader is referred to [5] for the proof of this fact. Moreover, the main result in [5] claims that the inequality above holds with constants independent on p and the same happens for the reverse inequality. In summary, we have

$$\Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \Big\|_{L_p(\tau \otimes \tau_{\mathrm{G}})} \simeq \max_{0 \le k \le d} \left\{ \Big\| \sum_{\mathrm{I} \in [n]^k} \sum_{\mathrm{J} \in [n]^{d-k}} a_{\mathrm{IJ}} \otimes e_{\mathrm{I},\mathrm{J}} \Big\|_{L_p(\tau; S_p)} \right\},$$

with constants depending only on d. This is the second example we meet in this paper for which the inequality in Theorem 3.1 for multi-indexed p-orthogonal sums turns out to be an equivalence of norms, with constants depending only on d. The first was given in Section 1, see (1) and Remark 1.2. In the next paragraph, we analyze one more example of this kind.

4.2. **Multi-indexed martingale difference sequences.** Let us consider a von Neumann algebra $\mathcal{M}$ with a faithful, normal trace τ satisfying $\tau(1) = 1$. For each $1 \le k \le d$, let us consider a filtration $\mathcal{M}_1(k), \mathcal{M}_2(k), \ldots, \mathcal{M}_n(k)$ of $\mathcal{M}$. A family $\boldsymbol{f} = (\boldsymbol{f}_\gamma)_{\gamma \in \Gamma}$ of random variables in $L_p(\tau)$ will be called a *martingale difference sequence with d indices* if the following condition holds for all $k = 1, 2, \ldots, d$

$$\boldsymbol{f}_\gamma = \mathbb{E}_{\mathcal{M}_{i_k}(k)} \big(h^k_{\gamma_k} \big) - \mathbb{E}_{\mathcal{M}_{i_k - 1}(k)} \big(h^k_{\gamma_k} \big).$$

Here, $\gamma_k = (i_1, \ldots, \widehat{i_k}, \ldots, i_d)$ for $\gamma = (i_1, \ldots, i_d)$ with $\widehat{i_k}$ meaning deletion of i_k and each h^k is a $(d-1)$-indexed family in $L_p(\tau)$. In other words, we require $\boldsymbol{f}$ to be a martingale difference sequence when looking at each component $\pi_k(\gamma) = i_k$

of the index set Γ. Notice that we allow different filtrations for each component. Such a construction again leads to a p-orthogonal family with d indices. Namely, let us assume that the k-th projection of $h : \{1, 2, \ldots, p\} \to \Gamma$ is injective. Then we consider the largest value m_k of $\pi_k \circ h$ and we take conditional expectation of index $m_k - 1$ with respect to the filtration $\mathcal{M}_1(k), \mathcal{M}_2(k), \ldots, \mathcal{M}_n(k)$ so that

$$\tau\Big(\boldsymbol{f}_{h(1)}^* \boldsymbol{f}_{h(2)} \cdots \boldsymbol{f}_{h(p-1)}^* \boldsymbol{f}_{h(p)}\Big) = \tau\Big(\mathbb{E}_{\mathcal{M}_{m_k-1}(k)}\Big[\boldsymbol{f}_{h(1)}^* \boldsymbol{f}_{h(2)} \cdots \boldsymbol{f}_{h(p-1)}^* \boldsymbol{f}_{h(p)}\Big]\Big) = 0.$$

Here Theorem 3.1 also admits a converse so that we get an equivalence

$$\Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \Big\|_{L_p(\tau)} \simeq \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\gamma \in \Gamma} \boldsymbol{f}_\gamma \otimes e_{\pi_\alpha(\gamma), \pi_\beta(\gamma)} \Big\|_{L_p(\tau; S_p)} \right\}.$$

However, in contrast with the previous paragraph, in this case the constants depend on d and p. This equivalence can be regarded as the version of Burkholder-Gundy inequalities for multi-indexed martingale difference sequences. In fact, it can be proved without the aid of Theorem 3.1. Namely, it follows easily by iterating the (non-commutative) Khintchine inequality (with Rademacher functions instead of free generators) and applying repeatedly the UMD property of $L_p(\tau)$, which follows itself by the (non-commutative) Burkholder-Gundy inequalities. The reader is referred to the papers [6, 8, 9] for more on this.

Acknowledgment. I wish to thank G. Pisier for introducing me to this subject. I also want to thank the anonymous referee for a very careful reading of this paper.

References

1. G. Andrews, *The theory of partitions*, Cambridge Univ. Press, 1984.
2. U. Haagerup and G. Pisier, *Bounded linear operators between C^*-algebras*, Duke Math. J. **71** (1993), 889 − 925.
3. F. Lust-Piquard, *Inégalités de Khintchine dans C_p* $(1 < p < \infty)$, C.R. Acad. Sci. Paris **303** (1986), 289-292.
4. F. Lust-Piquard and G. Pisier, *Non-commutative Khintchine and Paley inequalities*, *Ark. Mat.* **29** (1991), 241-260.
5. J. Parcet and G. Pisier, *Non-commutative Khintchine type inequalities associated with free groups*. To appear in Indiana Univ. Math. J.
6. G. Pisier, *An inequality for p-orthogonal sums in non-commutative L_p*, Illinois J. Math. **44** (2000), 901 − 923.
7. G. Pisier, *Introduction to Operator Space Theory*, Cambridge Univ. Press, 2003.
8. G. Pisier and Q.Xu, *Non-commutative martingale inequalities*, Comm. Math. Physics **189** (1997), 667 − 698.
9. N. Randrianantoanina, *Non-commutative martingale transforms*, J. Func. Anal. **194** (2002), 181 − 212.
10. D. Voiculescu, K. Dykema and A. Nica, *Free random variables*, CRM Monograph Series **1**, Amer. Math. Soc., 1992.

Universidad Autónoma de Madrid and Texas A&M University
E-mail address: `javier.parcet@uam.es`