GUNDY'S DECOMPOSITION FOR NON-COMMUTATIVE MARTINGALES AND APPLICATIONS

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Abstract. We provide an analogue of Gundy's decomposition for L_1-bounded non-commutative martingales. An important difference from the classical case is that for any L_1-bounded non-commutative martingale, the decomposition consists of four martingales. This is strongly related with the row/column nature of non-commutative Hardy spaces of martingales. As applications, we obtain simpler proofs of the weak type $(1,1)$ boundedness for non-commutative martingale transforms and the non-commutative analogue of Burkholder's weak type inequality for square functions. A sequence $(x_n)_{n\geq 1}$ in a normed space X is called 2-co-lacunary if there exists a bounded linear map from the closed linear span of $(x_n)_{n\geq 1}$ to l_2 taking each x_n to the n-th vector basis of l_2. We prove (using our decomposition) that any relatively weakly compact martingale difference sequence in $L_1(\mathcal{M},\tau)$ whose sequence of norms is bounded away from zero is 2-co-lacunary, generalizing a result of Aldous and Fremlin to non-commutative L_1-spaces.

Introduction

The main motivation for this paper comes from a fundamental decomposition of martingales due to Gundy [13] which is generally referred to as the *Gundy's decomposition theorem*. Gundy's theorem has been very useful in establishing weak type $(1,1)$ boundedness of certain quasi-linear mappings such as square functions and Doob's maximal functions. In particular, certain classical inequalities such as the weak type $(1,1)$ boundedness of martingale transforms and Burkholder's weak type inequality for square functions can be deduced from Gundy's theorem. We refer to [4, 14, 21] for some variations of Gundy's result and more applications and to Garsia's notes [12] for a complete discussion on classical martingale inequalities.

Gundy's decomposition theorem played a central role in classical martingale theory and it can be regarded as a probabilistic counterpart of the well known Calderón-Zygmund decomposition for integrable functions [5] in harmonic analysis. Due to its relevance in the classical theory, it is natural to consider whether or not such decomposition theorem can be generalized to the non-commutative setting. In this paper, we investigate possible analogues of Gundy's theorem for non-commutative martingales. We first recall this classical result:

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Gundy's decomposition theorem [13]**.** *Let $f = (f_n)_{n\geq 1}$ be a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is bounded in $L_1(\Omega, \mathbb{P})$, and λ be a positive real number. Then there are three martingales a, b, and c relative to the same filtration and satisfying the following properties for some absolute constant* c*:*

(i) $f = a + b + c$;

(ii) the martingale a satisfies

$$\|a\|_1 \leq \mathrm{c}\|f\|_1, \quad \|a\|_2^2 \leq \mathrm{c}\lambda\|f\|_1, \quad \|a\|_\infty \leq \mathrm{c}\lambda;$$

(iii) the martingale b satisfies

$$\sum_{k=1}^{\infty} \|db_k\|_1 \leq \mathrm{c}\|f\|_1;$$

(iv) the martingale c satisfies

$$\lambda\mathbb{P}\Big(\Big\{\sup_{k\geq 1} |dc_k| > 0\Big\}\Big) \leq \mathrm{c}\,\|f\|_1\,.$$

As a prominent subfield of the theory of non-commutative probability, the theory of non-commutative martingale inequalities has achieved considerable progress in recent years. Indeed, many classical inequalities have been reformulated to include non-commutative martingales. This general theme started from the fundamental paper of Pisier and Xu [24] where they introduced non-commutative martingale Hardy spaces and formulated the right analogue of Burkholder-Gundy inequalities. It was their general functional analytic approach that led to the renewed interests in this topic. Shortly after [24], Junge obtained in [15] a non-commutative analogue of Doob's maximal functions. Extensions of Burkholder/Rosenthal inequalities for conditioned square functions were proved by Junge and Xu in [17]. Martingale BMO spaces were studied in [24, 22, 16] and some weak type inequalities can be found in [27, 28]. We also refer the reader to a recent survey by Xu [32] for a rather complete exposition of the subject.

Following this general theme, we analyze analogues of Gundy's decomposition for non-commutative martingales. For this, we note first that since the notion of supremum does not necessarily make sense for a family of operators, we require an appropriate reformulation of condition (iv) above. It is clear that the following equalities of measurable sets hold:

$$\Big\{\sup_{k\geq 1} |dc_k| > 0\Big\} = \bigcup_{k\geq 1} \Big\{|dc_k| > 0\Big\} = \bigcup_{k\geq 1} \mathrm{supp}|dc_k|.$$

That is, condition (iv) is equivalent to:

$$\lambda\mathbb{P}\Big(\bigcup_{k\geq 1} \mathrm{supp}|dc_k|\Big) \leq \mathrm{c}\|f\|_1.$$

A non-commutative analogue of this condition can be formulated using the notion of support projection of a measurable operator. Our main result in this paper reads as follows:

Theorem A. *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite trace τ and let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$. If $x = (x_n)_{n\geq 1}$ is a L_1-bounded non-commutative martingale* *with respect to* $(\mathcal{M}_n)_{n\geq 1}$ *and* λ *is a positive real number, there exist four martingales* α, β, γ *and* υ *relative to the same filtration and satisfying:*

(i) $x = \alpha + \beta + \gamma + \upsilon$;

(ii) the martingale α satisfies

$$\|\alpha\|_1 \leq \mathrm{c}\|x\|_1, \quad \|\alpha\|_2^2 \leq \mathrm{c}\lambda\|x\|_1, \quad \|\alpha\|_\infty \leq \mathrm{c}\lambda;$$

(iii) the martingale β satisfies

$$\sum_{k=1}^{\infty} \|d\beta_k\|_1 \leq \mathrm{c}\|x\|_1;$$

(iv) γ and υ are L_1-martingales with

$$\max\Big\{\lambda\tau\Big(\bigvee_{k\geq 1}\mathrm{supp}|d\gamma_k|\Big), \lambda\tau\Big(\bigvee_{k\geq 1}\mathrm{supp}|d\upsilon_k^*|\Big)\Big\} \leq \mathrm{c}\|x\|_1.$$

An important difference between classical and non-commutative martingales is that the decomposition stated in Theorem A requires four martingales versus the three martingales of Gundy's classical decomposition. This difference is highlighted in Section 2 and is essentially due to the row and column nature of Hardy spaces for non-commutative martingales from [24]. We also remark that Gundy's paper [13] is based mainly on stopping time arguments which at the time of this writing do not appear to have a trackable non-commutative extension. Our approach is based on a non-commutative analogue of Doob's maximal inequality formulated by Cuculescu for positive martingales in [6]. Let us also mention that a weaker version of Gundy's decomposition was obtained by Burkholder in [3]. This alternative decomposition $f = a + b + c$ does not satisfies the L_∞-estimate for the martingale a. However, it only uses one stopping time (Gundy's approach uses two) in the construction and is therefore easier to handle for many applications. We shall also obtain a non-commutative analogue of Burkholder's decomposition which will be used in some of the applications we present in this paper.

As in the classical context, our decomposition is a powerful tool to prove weak type inequalities for non-commutative martingales. We illustrate this by reproving the main results in [27] and [28] respectively. More concretely, the weak type $(1,1)$ boundedness of non-commutative martingale transforms and the non-commutative analogue of Burkholder's weak type inequality for square functions. The latter result was recently proved in [28] and can be regarded as the weak type extension of non-commutative Burkholder-Gundy inequality from [24]. The contribution of our approach lies in the simplicity of the proofs, derived from the new insight provided by Gundy's decomposition.

The last application is a non-commutative extension of a classical result of Aldous and Fremlin [1] on basic sequences on L_1-spaces. Recall that a basic sequence $(x_n)_{n\geq 1}$ in a Banach space X is said to be 2-co-lacunary if there is a constant $\delta > 0$ so that for any finite sequence $(a_n)_{n\geq 1}$ of scalars,

$$\delta\Big(\sum_{n\geq 1}|a_n|^2\Big)^{1/2} \leq \Big\|\sum_{n\geq 1} a_n x_n\Big\|_{\mathrm{X}}.$$

As application of Theorem A, we shall prove that any relatively weakly compact martingale difference sequence in $L_1(\mathcal{M},\tau)$ whose sequence of norms is bounded away from zero is a 2-co-lacunary sequence. Using this, we shall also prove that for any semifinite and hyperfinite von Neumann algebra $\mathcal{M}$, every bounded sequence in $L_1(\mathcal{M},\tau)$ has either a convergent or a 2-co-lacunary subsequence.

The paper is organized as follows. In Section 1, we set some basic preliminary background concerning non-commutative symmetric spaces and martingale theory that will be needed throughout. Section 2 is devoted mainly to the statement and proof of the main result along with some reformulations. In Section 3, we present the three applications mentioned above. Our notation and terminology are standard as may be found in the books [19] and [29]. The letter c will denote an absolute constant which might change from one instance to another.

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1. Preliminary definitions and results

This section is devoted to some preliminary definitions and results that might be well-known to experts in the field and that will be needed in the rest of the paper. Throughout, $\mathcal{M}$ is a semifinite von Neumann algebra with a normal faithful semifinite trace τ. The identity element of $\mathcal{M}$ is denoted by $\mathbf{1}$. For $0 < p \leq \infty$, let $L_p(\mathcal{M},\tau)$ be the associated non-commutative L_p-space, see for instance [23, 25]. Note that if $p=\infty$, $L_\infty(\mathcal{M},\tau)$ is just $\mathcal{M}$ with the usual operator norm; also recall that for $0<p<\infty$, the (quasi)-norm on $L_p(\mathcal{M},\tau)$ is defined by

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p}, \quad \text{where} \quad |x| = (x^*x)^{1/2}.$$

1.1. **Non-commutative symmetric spaces.** Assume that $\mathcal{M}$ is acting on a Hilbert space H. A closed densely defined operator x on H is *affiliated* with $\mathcal{M}$ if x commutes with every unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If a is a densely defined self-adjoint operator on H and $a = \int_{\mathbb{R}} s de_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{\mathbb{R}} \chi_B(s) de_s^a$. An operator x affiliated with $\mathcal{M}$ is called *τ-measurable* if there exists $s>0$ such that

$$\tau\big(\chi_{(s,\infty)}(|x|)\big) < \infty.$$

The *generalized singular-value* $\mu(x) : \mathbb{R}_+ \to \mathbb{R}_+$ of a τ-measurable x is defined by

$$\mu_t(x) = \inf\Big\{ s>0 \ \big| \ \tau\big(\chi_{(s,\infty)}(|x|)\big) \leq t \Big\}.$$

The reader is referred to [11] for a detailed exposition of the function $\mu(x)$. For a rearrangement invariant quasi-Banach function space E on the interval $(0,\tau(\mathbf{1}))$, we define the *non-commutative symmetric space* $E(\mathcal{M},\tau)$ by setting:

$$E(\mathcal{M},\tau) := \Big\{ x \in L_0(\mathcal{M},\tau) \ \big| \ \mu(x) \in E \Big\}, \quad \text{and} \quad \|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$$

where $L_0(\mathcal{M},\tau)$ stands for the $*$-algebra of τ-measurable operators. It is known that $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a Banach (respectively, quasi-Banach) space whenever E is a Banach (respectively, quasi-Banach) function space. We refer the reader to [7, 31] for more in depth discussion of this construction. For the case where E is a Banach space, the inclusions

$$L_1(\mathcal{M},\tau) \cap \mathcal{M} \subseteq E(\mathcal{M},\tau) \subseteq L_1(\mathcal{M},\tau) + \mathcal{M}$$

hold with the inclusion maps being of norm one (here, the norms in $L_1(\mathcal{M},\tau)\cap\mathcal{M}$ and $L_1(\mathcal{M},\tau)+\mathcal{M}$ are the usual norms of the intersection and sum of Banach spaces). The *Köthe dual* $E(\mathcal{M},\tau)^\times$ of $E(\mathcal{M},\tau)$ is defined to be the set of all $x \in L_0(\mathcal{M},\tau)$ such that $xy \in L_1(\mathcal{M},\tau)$ for all $y \in E(\mathcal{M},\tau)$. With the norm defined by setting:

$$\|x\|_{E(\mathcal{M},\tau)^\times} := \sup\Big\{\tau(|xy|)\ \Big|\ y \in E(\mathcal{M},\tau),\ \|y\|_{E(\mathcal{M},\tau)} \leq 1\Big\},$$

the Köthe dual $E(\mathcal{M},\tau)^\times$ is a Banach space. Basic properties of Köthe duality for the commutative case may be found in [20]. For the non-commutative setting, the reader is referred to [8]. We note from [8] that if E is a rearrangement invariant function space E then $(E(\mathcal{M},\tau)^\times, \|\cdot\|_{E(\mathcal{M},\tau)^\times})$ may be identified with the space $(E^\times(\mathcal{M},\tau), \|\cdot\|_{E^\times(\mathcal{M},\tau)})$. In particular,

$$\begin{aligned}(L_1(\mathcal{M},\tau)+\mathcal{M})^\times &= L_1(\mathcal{M},\tau)\cap\mathcal{M},\\ (L_1(\mathcal{M},\tau)\cap\mathcal{M})^\times &= L_1(\mathcal{M},\tau)+\mathcal{M}.\end{aligned}$$

Relative weak compactness in non-commutative spaces plays a role in this paper. Below, we explicitly state a characterization that we need in the subsequent sections. First, we set $S_0(\mathcal{M},\tau) := \mathcal{M}_0 \cap (L_1(\mathcal{M},\tau)+\mathcal{M})$ with

$$\mathcal{M}_0 := \Big\{x \in L_0(\mathcal{M},\tau)\ \Big|\ \mu_t(x) \to 0 \text{ as } t \to \infty\Big\}.$$

Theorem 1.1. [9, Theorem 5.4]. *Assume that the symmetric space $E(\mathcal{M},\tau)$ is contained in $S_0(\mathcal{M},\tau)$ and that K is a bounded subset of $E(\mathcal{M},\tau)^\times$. Then, the following statements are equivalent:*

(i) *$\mu(K)$ is relatively $\sigma(E^\times, E)$-compact;*
(ii) *K is relatively $\sigma(E(\mathcal{M},\tau)^\times, E(\mathcal{M},\tau))$-compact.*

Our interest in this paper is mainly restricted to non-commutative L_p-spaces and *non-commutative weak L_1-spaces.* Following the construction of symmetric spaces of measurable operators, the non-commutative weak L_1-space $L_{1,\infty}(\mathcal{M},\tau)$, is defined as the set of all x in $L_0(\mathcal{M},\tau)$ for which the quasi-norm

$$\|x\|_{1,\infty} = \sup_{t>0} t\mu_t(x) = \sup_{\lambda>0} \lambda\tau\big(\chi_{(\lambda,\infty)}(|x|)\big) < \infty.$$

As in the commutative case, it can be easily verified that if $x_1, x_2 \in L_{1,\infty}(\mathcal{M},\tau)$ then $\|x_1+x_2\|_{1,\infty} \leq 2\|x_1\|_{1,\infty} + 2\|x_2\|_{1,\infty}$. In fact, the following more general quasi-triangle inequality holds and will be used repeatedly in the sequel. A short proof can be found in [28, Lemma 1.2].

Lemma 1.2. *Given two operators x_1, x_2 in $L_{1,\infty}(\mathcal{M},\tau)$ and $\lambda > 0$, we have*

$$\lambda\,\tau\Big(\chi_{(\lambda,\infty)}\big(|x_1+x_2|\big)\Big) \leq 2\lambda\,\tau\Big(\chi_{(\lambda/2,\infty)}\big(|x_1|\big)\Big) + 2\lambda\,\tau\Big(\chi_{(\lambda/2,\infty)}\big(|x_2|\big)\Big).$$

Let $\mathsf{P} = \{p_i\}_{i=1}^m$ be a finite sequence of mutually orthogonal projections in $\mathcal{M}$. We consider the *triangular truncation* with respect to P as the mapping on $L_0(\mathcal{M},\tau)$ defined by:

$$\mathcal{T}^{(\mathsf{P})}x = \sum_{i=1}^m \sum_{i\leq j} p_i x p_j.$$

The following lemma will be used in the sequel.

Lemma 1.3. [28, Proposition 1.4] *There exists an absolute constant* $\mathrm{c} > 0$ *so that if* $(\mathsf{P}_k)_{k \geq 1}$ *is a family of finite sequences of mutually orthogonal projections and* $(x_k)_{k \geq 1}$ *is a sequence in* $L_1(\mathcal{M}, \tau)$*, then*

$$\Big\| \Big(\sum_{k \geq 1} \big| \mathcal{T}^{(\mathsf{P}_k)} x_k \big|^2 \Big)^{1/2} \Big\|_{1,\infty} \leq \mathrm{c} \sum_{k \geq 1} \|x_k\|_1.$$

1.2. **Non-commutative martingales.** Consider a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ (i.e. a weak* closed $*$-subalgebra of $\mathcal{M}$). A *conditional expectation* $\mathsf{E} : \mathcal{M} \to \mathcal{N}$ from $\mathcal{M}$ onto $\mathcal{N}$ is a positive contractive projection. The conditional expectation E is called *normal* if the adjoint map E^* satisfies $\mathsf{E}^*(\mathcal{M}_*) \subset \mathcal{N}_*$. In this case, there is map $\mathsf{E}_* : \mathcal{M}_* \to \mathcal{N}_*$ whose adjoint is E. Note that such normal conditional expectation exists if and only if the restriction of τ to the von Neumann subalgebra $\mathcal{N}$ remains semifinite (see for instance [29, Theorem 3.4]). Any such conditional expectation is trace preserving (that is, $\tau \circ \mathsf{E} = \tau$) and satisfies the bimodule property:

$$\mathsf{E}(axb) = a\mathsf{E}(x)b \quad \text{for all} \quad a, b \in \mathcal{N} \text{ and } x \in \mathcal{M}.$$

Let $(\mathcal{M}_n)_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is weak* dense in $\mathcal{M}$. Assume that for every $n \geq 1$, there is a normal conditional expectation $\mathsf{E}_n : \mathcal{M} \to \mathcal{M}_n$. Note that for every $1 \leq p < \infty$ and $n \geq 1$, E_n extends to a positive contraction $\mathsf{E}_n : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}_n, \tau|_{\mathcal{M}_n})$. A *non-commutative martingale* with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ is a sequence $x = (x_n)_{n \geq 1}$ in $L_1(\mathcal{M}, \tau)$ such that

$$\mathsf{E}_m(x_n) = x_m \quad \text{for all} \quad 1 \leq m \leq n < \infty.$$

If additionally $x \subset L_p(\mathcal{M}, \tau)$ for some $1 \leq p \leq \infty$, then x is called a L_p-*martingale*. In this case, we set

$$\|x\|_p := \sup_{n \geq 1} \|x_n\|_p \,.$$

If $\|x\|_p < \infty$, x is called a L_p-*bounded martingale*. Given a martingale $x = (x_n)_{n \geq 1}$, we assume the convention that $x_0 = 0$. Then, the martingale difference sequence $dx = (dx_k)_{k \geq 1}$ associated to x is defined by

$$dx_k = x_k - x_{k-1}.$$

We now describe square functions of non-commutative martingales. Following Pisier and Xu [24], we will consider the following row and column versions of square functions. Given a martingale difference sequence $dx = (dx_k)_{k \geq 1}$ and $n \geq 1$, we define the *row* and *column square functions* of x as

$$\mathcal{S}_{C,n}(x) := \Big(\sum_{k=1}^n |dx_k|^2 \Big)^{1/2} \quad \text{and} \quad \mathcal{S}_{R,n}(x) := \Big(\sum_{k=1}^n |dx_k^*|^2 \Big)^{1/2}.$$

Let us consider a rearrangement invariant (quasi)-Banach function space E on the interval $[0, \tau(\mathbf{1}))$. Then, we define the spaces $E(\mathcal{M}, \tau; l_C^2)$ and $E(\mathcal{M}, \tau; l_R^2)$ as the completions of the vector space of finite sequences $a = (a_k)_{k \geq 1}$ in $E(\mathcal{M}, \tau)$ with respect to the following norms

$$\|a\|_{E(\mathcal{M},\tau;l_C^2)} \quad = \quad \Big\| \Big(\sum_{k \geq 1} |a_k|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M},\tau)},$$

$$\|a\|_{E(\mathcal{M},\tau;l^2_R)} = \Big\| \Big(\sum_{k\geq 1} |a_k^*|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M},\tau)}.$$

The martingale difference sequence dx belongs to $E(\mathcal{M},\tau;l^2_C)$ (respectively, $E(\mathcal{M},\tau;l^2_R)$) if and only if the sequence $(\mathcal{S}_{C,n}(x))_{n\geq 1}$ (respectively, $(\mathcal{S}_{R,n}(x))_{n\geq 1}$) is bounded in $E(\mathcal{M},\tau)$. In this case, the limits

$$\mathcal{S}_C(x) := \Big(\sum_{k=1}^{\infty} |dx_k|^2 \Big)^{1/2} \quad \text{and} \quad \mathcal{S}_R(x) := \Big(\sum_{k=1}^{\infty} |dx_k^*|^2 \Big)^{1/2}$$

are elements of $E(\mathcal{M},\tau)$. These two versions of square functions are very crucial in the subsequent sections.

The next result is by now well known for positive martingales [6] and can be viewed as a non-commutative analogue of the classical weak type $(1,1)$ boundedness of Doob's maximal function. The extension to self-adjoint martingales stated below plays a crucial role in the next section. Its proof is a minor adjustment of the original argument of Cuculescu but we will include the details for completeness.

Proposition 1.4. *If $x=(x_n)_{n\geq 1}$ is a self-adjoint L_1-bounded martingale and λ is a positive real number, there exists a sequence of decreasing projections in the von Neumann algebra $\mathcal{M}$*

$$q_0^{(\lambda)} \geq q_1^{(\lambda)} \geq q_2^{(\lambda)} \geq \dots$$

satisfying the following properties:

(i) *for every $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$;*
(ii) *for every $n\geq 1$, $q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}$;*
(iii) *for every $n \geq 1$, $|q_n^{(\lambda)} x_n q_n^{(\lambda)}| \leq \lambda q_n^{(\lambda)}$;*
(iv) *if we set $q^{(\lambda)} = \bigwedge_{n=1}^{\infty} q_n^{(\lambda)}$, then*

$$\tau\Big(\mathbf{1} - q^{(\lambda)}\Big) \leq \frac{1}{\lambda}\|x\|_1.$$

Proof. Let $q_0^{(\lambda)} = \mathbf{1}$ and inductively on $n\geq 1$, define

$$q_n^{(\lambda)} := q_{n-1}^{(\lambda)} \chi_{[-\lambda,\lambda]}\Big(q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}\Big) = \chi_{[-\lambda,\lambda]}\Big(q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}\Big) q_{n-1}^{(\lambda)}.$$

The identity above follows since $q_{n-1}^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}$ because $q_{n-1}^{(\lambda)}$ is a projection. This clearly gives a decreasing sequence of projections. By induction, condition (i) holds. Moreover, condition (ii) follows directly from the definition above. For (iii), note that for every $n\geq 1$,

$$\begin{aligned} q_n^{(\lambda)} x_n q_n^{(\lambda)} &= q_n^{(\lambda)} (q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}) q_n^{(\lambda)} \\ &= q_{n-1}^{(\lambda)} \chi_{[-\lambda,\lambda]}(q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}) q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)} \chi_{[-\lambda,\lambda]}(q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}) q_{n-1}^{(\lambda)}. \end{aligned}$$

Therefore $-\lambda q_n^{(\lambda)} \leq q_n^{(\lambda)} x_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$ and (iii) follows. To prove (iv), we use the non-commutative analogue of Krickeberg's decomposition [6], so that we may write $x_n = w_n - z_n$ where $w=(w_n)_{n\geq 1}$ and $z=(z_n)_{n\geq 1}$ are positive martingales with

$$\|x\|_1 = \tau(w_1 + z_1).$$

For every $n \geq 1$,

$$\begin{aligned} \|x\|_1 &= \tau\big((w_n+z_n)q_n^{(\lambda)}\big) + \sum_{k=1}^n \tau\Big((w_n+z_n)(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big) \\ &= \tau\big(q_n^{(\lambda)}(w_n+z_n)q_n^{(\lambda)}\big) + \sum_{k=1}^n \tau\Big(\mathsf{E}_k(w_n+z_n)(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big). \end{aligned}$$

Since $\tau\big(q_n^{(\lambda)}(w_n+z_n)q_n^{(\lambda)}\big) \geq 0$, we have

$$\begin{aligned} \|x\|_1 &\geq \sum_{k=1}^n \tau\Big((q_{k-1}^{(\lambda)} - q_k^{(\lambda)})(w_k+z_k)(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big) \\ &\geq \tau\Big(\sum_{k=1}^n \Big|(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})(w_k-z_k)(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big|\Big) \\ &= \tau\Big(\sum_{k=1}^n \Big|(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big|\Big). \end{aligned}$$

From the definition of $q_k^{(\lambda)}$, it is clear that

$$\begin{aligned} q_{k-1}^{(\lambda)} - q_k^{(\lambda)} &= q_{k-1}^{(\lambda)}\Big(\chi_{(-\infty,-\lambda)}(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)}) + \chi_{(\lambda,\infty)}(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})\Big) \\ &= \Big(\chi_{(-\infty,-\lambda)}(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)}) + \chi_{(\lambda,\infty)}(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})\Big)q_{k-1}^{(\lambda)}. \end{aligned}$$

Therefore, if $q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)} = \int_{\mathbb{R}} t de_t^{(k)}$ is the spectral decomposition of $q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)}$, we find that

$$\begin{aligned} \lambda(q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) &\leq \int_{-\infty}^{-\lambda} |t|\, de_t^{(k)} + \int_{\lambda}^{\infty} |t|\, de_t^{(k)} \\ &= \Big|(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})\Big|. \end{aligned}$$

We can now conclude that

$$\tau\Big(\mathbf{1} - q_n^{(\lambda)}\Big) \leq \frac{1}{\lambda}\|x\|_1.$$

Taking the limit as $n \to \infty$, we obtain (iv). This completes the proof. □

In the following, we will refer to the sequence of projections of Proposition 1.4 as the sequence of Cuculescu's projections associated to the (self-adjoint) martingale x and the (positive) parameter λ. In the next result, we collect some basic properties of this sequence that are very useful for the presentation in the next section.

Proposition 1.5. *Let $x = (x_n)_{n\geq 1}$ be a self-adjoint L_1-bounded martingale and λ a positive real number. Then, the sequence of Cuculescu's projections associated to x and λ satisfies the following estimates for every $n \geq 1$:*

$$\sum_{k=1}^n \big\|q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)} - q_k^{(\lambda)}x_k q_k^{(\lambda)}\big\|_1 \leq \|x\|_1, \tag{1}$$

$$\sum_{k=1}^n \big\|q_{k-1}^{(\lambda)}x_{k-1} q_{k-1}^{(\lambda)} - q_k^{(\lambda)}x_{k-1} q_k^{(\lambda)}\big\|_1 \leq 2\|x\|_1, \tag{2}$$

$$\sum_{k=1}^{n} \big\| q_{k-1}^{(\lambda)} dx_k q_{k-1}^{(\lambda)} - q_k^{(\lambda)} dx_k q_k^{(\lambda)} \big\|_1 \quad \leq \quad 3\|x\|_1. \tag{3}$$

Moreover, the following identity holds

$$\sum_{k=1}^{n} q_{k-1}^{(\lambda)} dx_k q_{k-1}^{(\lambda)} = q_n^{(\lambda)} x_n q_n^{(\lambda)} + \sum_{k=1}^{n} (q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) x_k (q_{k-1}^{(\lambda)} - q_k^{(\lambda)}). \tag{4}$$

In particular, we obtain

$$\Big\| \sum_{k=1}^{n} q_{k-1}^{(\lambda)} dx_k q_{k-1}^{(\lambda)} \Big\|_1 \leq 2\|x\|_1. \tag{5}$$

Proof. We will write $(q_n)_{n\geq 0}$ for $(q_n^{(\lambda)})_{n\geq 0}$ and q for $q^{(\lambda)}$ (see Proposition 1.4). Let $v_k = q_{k-1}x_k q_{k-1} - q_k x_k q_k$. Since q_k commutes with $q_{k-1}x_k q_{k-1}$, we have $v_k = (q_{k-1} - q_k)x_k(q_{k-1} - q_k)$ and therefore for any given $n \geq 1$,

$$\begin{aligned} \sum_{k=1}^{n} \|v_k\|_1 &= \sum_{k=1}^{n} \big\| (q_{k-1} - q_k)x_k(q_{k-1} - q_k) \big\|_1 \\ &= \sum_{k=1}^{n} \big\| \mathsf{E}_k\big((q_{k-1} - q_k)x_n(q_{k-1} - q_k)\big) \big\|_1 \\ &\leq \sum_{k=1}^{n} \big\| (q_{k-1} - q_k)x_n(q_{k-1} - q_k) \big\|_1 \leq \|x_n\|_1. \end{aligned}$$

Hence, inequality (1) is satisfied. For inequality (2), set

$$\sigma_k = q_k x_{k-1} q_k - q_{k-1} x_{k-1} q_{k-1}.$$

Then we have,

$$\begin{aligned} \sigma_k &= q_k x_{k-1}(q_k - q_{k-1}) + (q_k - q_{k-1})x_{k-1}q_{k-1} \\ &= q_k q_{k-1} x_{k-1} q_{k-1}(q_k - q_{k-1}) + (q_k - q_{k-1}) q_{k-1} x_{k-1} q_{k-1}. \end{aligned}$$

By Hölder's inequality and Proposition 1.4, we deduce that

$$\sum_{k=1}^{\infty} \|\sigma_k\|_1 \leq 2 \sum_{k=1}^{\infty} \tau(q_{k-1} - q_k) \big\| q_{k-1} x_{k-1} q_{k-1} \big\|_\infty \leq 2\lambda\tau(\mathbf{1} - q) \leq 2\|x\|_1,$$

which proves inequality (2). The estimate in (3) follows directly from (1, 2) and triangle inequality. The identity (4) follows immediately from summing by parts. Indeed, for $n \geq 1$ we have

$$\begin{aligned} \sum_{k=1}^{n} q_{k-1} dx_k q_{k-1} &= \sum_{k=1}^{n} \big(q_{k-1} x_k q_{k-1} - q_{k-1} x_{k-1} q_{k-1} \big) \\ &= \sum_{k=1}^{n-1} \big(q_{k-1} x_k q_{k-1} - q_k x_k q_k \big) + q_{n-1} x_n q_{n-1} \\ &= \sum_{k=1}^{n} \big(q_{k-1} x_k q_{k-1} - q_k x_k q_k \big) + q_n x_n q_n \end{aligned}$$

Finally, (5) follows from (1) and (4). The proof is complete. □

2. Non-commutative Gundy's decomposition

In this section we present the non-commutative analogue of Gundy's theorem, which is the main result of this paper. All adapted sequences and martingales are understood to be with respect to a fixed filtration $(\mathcal{M}_n)_{n\geq 1}$ of von Neumann subalgebras of $\mathcal{M}$. For convenience, we assume that $\mathsf{E}_0 = \mathsf{E}_1$.

Theorem 2.1. *If $x = (x_n)_{n\geq 1}$ is a L_1-bounded non-commutative martingale and λ is a positive real number, there exist four martingales α, β, γ, and υ satisfying the following properties for some absolute constant* c*:*

(i) $x = \alpha + \beta + \gamma + \upsilon$*;*

(ii) *the martingale α satisfies*

$$\|\alpha\|_1 \leq \mathrm{c}\|x\|_1, \quad \|\alpha\|_2^2 \leq \mathrm{c}\lambda\|x\|_1, \quad \|\alpha\|_\infty \leq \mathrm{c}\lambda;$$

(iii) *the martingale β satisfies*

$$\sum_{k=1}^{\infty} \|d\beta_k\|_1 \leq \mathrm{c}\|x\|_1;$$

(iv) *γ and υ are L_1-martingales with*

$$\max\Big\{\lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}|d\gamma_k|\Big),\, \lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|d\upsilon_k^*|\Big)\Big\} \leq \mathrm{c}\|x\|_1.$$

Proof. Without loss of generality, we can and do assume that the martingale x is positive. Denote by $(q_n)_{n\geq 0}$ the sequence of Cuculescu's projections associated with the martingale x and a fixed $\lambda > 0$. The construction is done in two steps.

Step 1. We consider the following martingale difference sequence

$$dy_k := q_k dx_k q_k - \mathsf{E}_{k-1}(q_k dx_k q_k) \quad \text{for} \quad k \geq 1.$$

It is clear that $(dy_k)_{k\geq 1}$ is a martingale difference sequence and the corresponding martingale $y = (y_n)_{n\geq 1}$ is a self-adjoint L_1-martingale. The following intermediate lemma is essential for our construction.

Lemma 2.2. *The martingale y is L_1-bounded with $\|y\|_1 \leq 9\|x\|_1$.*

Proof. This follows essentially from Proposition 1.5. Indeed, for every $n \geq 1$,

$$\begin{aligned}
\|y_n\|_1 &= \Big\|\sum_{k=1}^{n} q_k dx_k q_k - \mathsf{E}_{k-1}(q_k dx_k q_k)\Big\|_1 \\
&\leq \Big\|\sum_{k=1}^{n} q_{k-1} dx_k q_{k-1}\Big\|_1 \\
&+ \sum_{k=1}^{n} \big\|q_{k-1} dx_k q_{k-1} - q_k dx_k q_k\big\|_1 \\
&+ \sum_{k=2}^{n} \big\|\mathsf{E}_{k-1}(q_k dx_k q_k)\big\|_1 + \big\|q_1 x_1 q_1\big\|_1.
\end{aligned}$$

Since for $2 \leq k \leq n$,

$$\mathsf{E}_{k-1}\big(q_k dx_k q_k\big) = \mathsf{E}_{k-1}\big(q_k dx_k q_k - q_{k-1} dx_k q_{k-1}\big),$$

the assertion follows from estimates (3) and (5). Thus the lemma is proved. □

Step 2. Let $(\pi_n)_{n\geq 0}$ stand for the sequence of Cuculescu's projections relative to the (self-adjoint) martingale y and the parameter λ fixed above. We define the martingales α, β, γ, and υ as follows:

$$(\mathbf{G}_\lambda) \qquad \begin{cases} d\alpha_k & := \pi_{k-1}\big[q_k dx_k q_k - \mathsf{E}_{k-1}(q_k dx_k q_k)\big]\pi_{k-1}, \\ d\beta_k & := \pi_{k-1}\big[q_{k-1} dx_k q_{k-1} - q_k dx_k q_k + \mathsf{E}_{k-1}(q_k dx_k q_k)\big]\pi_{k-1}, \\ d\gamma_k & := dx_k - dx_k q_{k-1}\pi_{k-1}, \\ d\upsilon_k & := dx_k q_{k-1}\pi_{k-1} - \pi_{k-1} q_{k-1} dx_k q_{k-1}\pi_{k-1}. \end{cases}$$

Clearly, $d\alpha$, $d\beta$, $d\gamma$ and $d\upsilon$ are martingale difference sequences and $x = \alpha+\beta+\gamma+\upsilon$.

Lemma 2.3. *The martingale α satisfies*

$$\|\alpha\|_1 \leq 18\|x\|_1, \quad \|\alpha\|_2^2 \leq 72\lambda\|x\|_1, \quad \|\alpha\|_\infty \leq 4\lambda.$$

Proof. Note that for $k \geq 1$, $d\alpha_k = \pi_{k-1} dy_k \pi_{k-1}$. Thus, the L_1-estimate follows directly from (5) and Lemma 2.2. For the L_∞-estimate, we recall from (4) that for every $n \geq 1$,

$$\alpha_n = \sum_{k=1}^n \pi_{k-1} dy_k \pi_{k-1} = \pi_n y_n \pi_n + \sum_{k=1}^n (\pi_{k-1} - \pi_k) y_k (\pi_{k-1} - \pi_k).$$

The key observation is that $\sup_{k\geq 1} \|dy_k\|_\infty \leq 2\lambda$. We have the following estimates:

$$\begin{aligned} \|\alpha_n\|_\infty \quad \leq \quad & \|\pi_n y_n \pi_n\|_\infty \\ + \quad & \Big\| \sum_{k=1}^n (\pi_{k-1} - \pi_k) dy_k (\pi_{k-1} - \pi_k) \Big\|_\infty \\ + \quad & \Big\| \sum_{k=1}^n (\pi_{k-1} - \pi_k) y_{k-1} (\pi_{k-1} - \pi_k) \Big\|_\infty. \end{aligned}$$

We deduce from the definition of $(\pi_n)_{n\geq 0}$ that,

$$\|\alpha_n\|_\infty \leq \lambda + \sup_{k\leq n} \|dy_k\|_\infty + \sup_{k\leq n} \|(\pi_{k-1} - \pi_k) y_{k-1} (\pi_{k-1} - \pi_k)\|_\infty \leq 4\lambda.$$

The L_2-estimate follows from those of L_1 and L_∞ using Hölder's inequality. □

Lemma 2.4. *The martingale β satisfies*

$$\sum_{k=1}^\infty \|d\beta_k\|_1 \leq 7\|x\|_1.$$

Proof. From the definition of $(d\beta_k)_{k\geq 1}$, we have

$$\sum_{k=1}^\infty \|d\beta_k\|_1 \leq \|d\beta_1\|_1 + \sum_{k=2}^\infty \big\|q_{k-1} dx_k q_{k-1} - q_k dx_k q_k\big\|_1 + \sum_{k=2}^\infty \big\|\mathsf{E}_{k-1}\big(q_k dx_k q_k\big)\big\|_1.$$

Since for every $k \geq 2$,

$$\mathsf{E}_{k-1}\big(q_k dx_k q_k\big) = \mathsf{E}_{k-1}\big(q_k dx_k q_k - q_{k-1} dx_k q_{k-1}\big),$$

we conclude from (3) that

$$\sum_{k=1}^\infty \|d\beta_k\|_1 \leq \|x_1\|_1 + 2\sum_{k=2}^\infty \big\|q_{k-1} dx_k q_{k-1} - q_k dx_k q_k\big\|_1 \leq 7\|x\|_1.$$

Thus we have the estimate as stated. □

Lemma 2.5. *For every* $k \geq 1$,

(a) $\mathrm{supp}|d\gamma_k| \leq \mathbf{1} - \pi_{k-1} \wedge q_{k-1}$;
(b) $\mathrm{supp}|dv_k^*| \leq \mathbf{1} - \pi_{k-1} \wedge q_{k-1}$.

Consequently, the martingales γ *and* v *satisfy*

$$\max\Big\{\lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|d\gamma_k|\Big),\, \lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|dv_k^*|\Big)\Big\} \leq 10\,\|x\|_1.$$

Proof. It is immediate from $(\mathbf{G}_\lambda)$ that $d\gamma_k = d\gamma_k(\mathbf{1} - \pi_{k-1} \wedge q_{k-1})$ and using polar decomposition, we obtain $|d\gamma_k| = |d\gamma_k|(\mathbf{1} - \pi_{k-1} \wedge q_{k-1})$ which shows that $\mathrm{supp}\,|d\gamma_k| \leq \mathbf{1} - \pi_{k-1} \wedge q_{k-1}$. As a consequence,

$$\bigvee_{k\geq 1} \mathrm{supp}\,|d\gamma_k| \leq \mathbf{1} - \pi \wedge q.$$

Therefore, we deduce

$$\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}\,|d\gamma_k|\Big) \leq \tau(\mathbf{1}-\pi) + \tau(\mathbf{1}-q) \leq \frac{1}{\lambda}\big(\|y\|_1 + \|x\|_1\big) \leq \frac{10}{\lambda}\,\|x\|_1.$$

The same argument applies to the martingale difference sequence $(dv_k^*)_{k\geq 1}$. □

It is now clear that by combining Lemma 2.3, Lemma 2.4, and Lemma 2.5, all the estimates from items (ii), (iii), and (iv) of Theorem 2.1 are verified. Moreover, the fact that γ and v are L_1-martingales follows directly from $(\mathbf{G}_\lambda)$. □

Remark 2.6. It is important to note that in strong contrast with the commutative case, the consideration of the fourth martingale v is necessary in the decomposition stated in Theorem 2.1. Indeed, assume that the decomposition in Theorem 2.1 can be done with only three martingales. That is, if for every L_1-bounded martingale x and $\lambda > 0$, there is a decomposition $x = \alpha + \beta + \gamma$ satisfying (ii), (iii) and (iv) of Theorem 2.1. Then a straightforward adjustment of the argument from the classical case used in [13] would prove that there is an absolute constant c such that

$$\max\Big\{\|\mathcal{S}_R(x)\|_{1,\infty},\, \|\mathcal{S}_C(x)\|_{1,\infty}\Big\} \leq \mathrm{c}\,\|x\|_1.$$

In particular, a standard use of the real interpolation method shows that there is a constant c_p depending only on p such that, if x is a L_p-bounded martingale with $1 < p < 2$, then we have

$$\max\Big\{\|\mathcal{S}_R(x)\|_p,\, \|\mathcal{S}_C(x)\|_p\Big\} \leq \mathrm{c}_p\,\|x\|_p.$$

This is in a direct conflict with the non-commutative analogue of Burkholder-Gundy inequality proved in [24]. Hence in general, decomposition into three martingales is not possible in Theorem 2.1. This observation confirms the relation between our decomposition and the row/column nature of Hardy spaces for non-commutative martingales. Moreover, a detailed inspection of the arguments sketched above shows that in fact, the martingales γ and v can be regarded as the 'column' and 'row' part of its commutative counterpart in [13].

Remark 2.7. In the construction $(\mathbf{G}_\lambda)$ above, we have $\alpha_1 = 0$, $\beta_1 = x_1$, $\gamma_1 = 0$, and $v_1 = 0$. Any other choice of these first terms could have been taken without any difference on the properties stated in Theorem 2.1. Our choice is motivated in part by our second application below, where we need to have $\gamma_1 = v_1 = 0$.

In the next formulation, we observe that if one wants to use three martingales in the decomposition of Theorem 2.1, then we have to consider a weaker notion of support projections.

Definition 2.8. For a non necessarily self-adjoint operator $x \in \mathcal{M}$, we define the *two-sided null projection* of x to be the greatest projection q satisfying $qxq = 0$. In this case, we set $\operatorname{supp}^* x := \mathbf{1} - q$.

Clearly, $\operatorname{supp} x = \operatorname{supp}^* x$ if $\mathcal{M}$ is abelian. In general, supp^* is weaker than the usual support in the sense that $\operatorname{supp}^* x \leq \operatorname{supp} x$ for any self-adjoint $x \in \mathcal{M}$ and for a non-self adjoint $x \in \mathcal{M}$, $\operatorname{supp}^* x$ is a subprojection of both the right and left supports of x. Using this weaker notion of support projections, we can state:

Corollary 2.9. *If $x = (x_n)_{n\geq 1}$ is a L_1-bounded non-commutative martingale and λ is a positive real number, there exist three martingales a, b, and c satisfying the following properties for some absolute constant* c*:*

(i) $x = a + b + c$;

(ii) *the martingale a satisfies*

$$\|a\|_1 \leq \mathrm{c}\|x\|_1, \quad \|a\|_2^2 \leq \mathrm{c}\lambda\|x\|_1, \quad \|a\|_\infty \leq \mathrm{c}\lambda;$$

(iii) *the martingale b satisfies*

$$\sum_{k=1}^{\infty} \|db_k\|_1 \leq \mathrm{c}\|x\|_1;$$

(iv) *the martingale c satisfies*

$$\lambda\tau\Big(\bigvee_{k\geq 1} \operatorname{supp}^* dc_k\Big) \leq \mathrm{c}\|x\|_1.$$

Proof. According to $(\mathbf{G}_\lambda)$, it is enough to set $a := \alpha$, $b := \beta$ and $c := \gamma + \upsilon$. Then, (ii) and (iii) follow directly from Theorem 2.1. For (iv), we note from $(\mathbf{G}_\lambda)$ that for every $k \geq 1$, $dc_k = dx_k - \pi_{k-1}q_{k-1}dx_k q_{k-1}\pi_{k-1}$. Thus we deduce that for $k \geq 1$, $(\pi_{k-1} \wedge q_{k-1})dc_k(\pi_{k-1} \wedge q_{k-1}) = 0$ and therefore, $\operatorname{supp}^* dc_k \leq \mathbf{1} - (\pi_{k-1} \wedge q_{k-1})$. In particular, we obtain

$$\tau\Big(\bigvee_{k\geq 1} \operatorname{supp}^* dc_k\Big) \leq \tau(\mathbf{1} - \pi) + \tau(\mathbf{1} - q).$$

At this point, (iv) follows as in Lemma 2.5. The proof is complete. □

Let us note that Gundy's original proof in [13] uses two stopping times. This essentially explains why we need two steps in our construction of $(\mathbf{G}_\lambda)$. In [3], Burkholder provided a weaker version of Gundy's decomposition where only the L_2-estimate is required for the first martingale in his decomposition. His approach uses only one stoping time. In the next result, we provide a non-commutative analogue of Burkholder's approach. This provide us with a simpler decomposition which is more useful for some applications.

Corollary 2.10. *Let $x = (x_n)_{n\geq 1}$ be a L_1-bounded positive martingale and λ be a positive real number. Let us consider the decomposition of x as a sum of four* *martingales* $x = \alpha' + \beta' + \gamma' + \upsilon'$ *with martingale differences given by*

$$(\mathbf{G}'_\lambda) \qquad \begin{cases} d\alpha'_k & := q_k^{(\lambda)} dx_k q_k^{(\lambda)} - \mathsf{E}_{k-1}\big(q_k^{(\lambda)} dx_k q_k^{(\lambda)}\big), \\ d\beta'_k & := q_{k-1}^{(\lambda)} dx_k q_{k-1}^{(\lambda)} - q_k^{(\lambda)} dx_k q_k^{(\lambda)} + \mathsf{E}_{k-1}\big(q_k^{(\lambda)} dx_k q_k^{(\lambda)}\big), \\ d\gamma'_k & := dx_k - dx_k q_{k-1}^{(\lambda)}, \\ d\upsilon'_k & := dx_k q_{k-1}^{(\lambda)} - q_{k-1}^{(\lambda)} dx_k q_{k-1}^{(\lambda)}. \end{cases}$$

Then, the following properties hold:

(i) *the martingale* α' *satisfies*

$$\|\alpha'\|_1 \leq \mathrm{c}\|x\|_1, \quad \|\alpha'\|_2^2 \leq \mathrm{c}\lambda\|x\|_1;$$

(ii) *the martingales* β', γ' *and* υ' *behave as* β, γ *and* υ *in Theorem* 2.1.

Proof. The estimates for β', γ' and υ' can be verified verbatim as in the proof of Theorem 2.1. For the L_1-estimate of α', we use Lemma 2.2. To estimate the L_2-norm of α', we note by orthogonality that for $n \geq 1$,

$$\|\alpha'_n\|_2^2 = \sum_{k=1}^n \|d\alpha'_k\|_2^2 \leq 4 \sum_{k=1}^n \|q_k dx_k q_k\|_2^2 .$$

On the other hand, since for every $k \geq 1$, $q_k dx_k q_k = q_k(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) q_k$, we get that

$$\|\alpha'_n\|_2^2 \leq 4 \sum_{k=1}^n \|q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}\|_2^2 .$$

Finally, according to [27, Lemma 3.4], this gives $\|\alpha'_n\|_2^2 \leq 24\lambda \|x\|_1$. □

Remark 2.11. Corollary 2.10 trivially extends to non-positive martingales.

3. Applications

3.1. **Non-commutative martingale transforms.** As a first application of our decomposition, we provide a very simple proof of the weak type $(1,1)$ boundedness of non-commutative martingale transforms obtained in [27]. This has implications in non-commutative martingale theory as well as for estimating UMD constants of certain non-commutative function spaces. The reader is referred to [27] and to Xu's survey [32] for a detailed exposition of these implications.

Theorem 3.1. *There exists an absolute constant* c *such that for every martingale* $x = (x_k)_{k\geq 1}$ *bounded in* $L_1(\mathcal{M}, \tau)$ *and every sequence* $(\xi_k)_{k\geq 0}$ *in* $\mathcal{M}$ *satisfying the following properties:*

(i) $\xi_0 = \mathbf{1}$;

(ii) $\sup_{k\geq 1} \|\xi_k\|_\infty \leq 1$;

(iii) $\xi_{k-1} \in \mathcal{M}_{k-1} \cap \mathcal{M}'_k$ *for* $k \geq 1$;

the following estimate holds for all $n \geq 1$,

$$\Big\| \sum_{k=1}^n \xi_{k-1} dx_k \Big\|_{1,\infty} \leq \mathrm{c}\|x\|_1.$$

Proof. We have to show that

$$\lambda\, \tau\Big(\chi_{(\lambda,\infty)}\Big(\Big|\sum_{k=1}^n \xi_{k-1} dx_k\Big|\Big)\Big) \leq \mathrm{c}\|x\|_1, \tag{6}$$

for every $0 < \lambda < \infty$. For this, we fix $\lambda > 0$ and consider the decomposition $x = \alpha + \beta + \gamma + \upsilon$ of x associated to λ from Theorem 2.1. Using the elementary inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ for operators, we have

$$\begin{aligned}\Big|\sum_{k=1}^n \xi_{k-1} dx_k\Big|^2 &\leq 4\Big|\sum_{k=1}^n \xi_{k-1} d\alpha_k\Big|^2 + 4\Big|\sum_{k=1}^n \xi_{k-1} d\beta_k\Big|^2\\ &+ 4\Big|\sum_{k=1}^n \xi_{k-1} d\gamma_k\Big|^2 + 4\Big|\sum_{k=1}^n \xi_{k-1} d\upsilon_k\Big|^2.\end{aligned}$$

Taking the trace, we obtain from Lemma 1.2

$$\begin{aligned}&\lambda\tau\Big(\chi_{(\lambda,\infty)}\Big(\Big|\sum_{k=1}^n \xi_{k-1} dx_k\Big|\Big)\Big) = \lambda\tau\Big(\chi_{(\lambda^2,\infty)}\Big(\Big|\sum_{k=1}^n \xi_{k-1} dx_k\Big|^2\Big)\Big)\\ \leq\;& 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n \xi_{k-1} d\alpha_k\Big|^2\Big)\Big) + 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n \xi_{k-1} d\beta_k\Big|^2\Big)\Big)\\ +\;& 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n \xi_{k-1} d\gamma_k\Big|^2\Big)\Big) + 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n \xi_{k-1} d\upsilon_k\Big|^2\Big)\Big)\\ =\;& I + II + III + IV.\end{aligned}$$

For the first term I, we use Chebychev's inequality to deduce:

$$I \leq \frac{64}{\lambda}\Big\|\sum_{k=1}^n \xi_{k-1} d\alpha_k\Big\|_2^2 = \frac{64}{\lambda}\sum_{k=1}^n \|\xi_{k-1} d\alpha_k\|_2^2 \leq \frac{64}{\lambda}\sum_{k=1}^n \|d\alpha_k\|_2^2 \leq \mathrm{c}\|x\|_1. \tag{7}$$

For the second term II, we proceed similarly,

$$\begin{aligned} II &= 4\lambda\,\tau\Big(\chi_{(\lambda/2,\infty)}\Big(2\Big|\sum_{k=1}^n \xi_{k-1} d\beta_k\Big|\Big)\Big)\\ &\leq 16\Big\|\sum_{k=1}^n \xi_{k-1} d\beta_k\Big\|_1 \leq 16\sum_{k=1}^n \|d\beta_k\|_1 \leq \mathrm{c}\|x\|_1.\end{aligned} \tag{8}$$

For III, we note that $|\sum_{k=1}^n \xi_{k-1} d\gamma_k|^2$ is supported by the projection

$$\bigvee_{k\geq 1} \mathrm{supp}|d\gamma_k|.$$

With this observation, it follows that

$$III \leq 4\lambda\tau\Big(\bigvee_{k\geq 1} \mathrm{supp}|d\gamma_k|\Big) \leq \mathrm{c}\|x\|_1. \tag{9}$$

For the last term IV, we remark first that since ξ_{k-1} commutes with $d\upsilon_k$, we have

$$d\upsilon_k^* \xi_{k-1}^* = \xi_{k-1}^* d\upsilon_k^*.$$

Therefore

$$IV = 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n d\upsilon_k^* \xi_{k-1}^*\Big|^2\Big)\Big) = 4\lambda\tau\Big(\chi_{(\lambda^2/4,\infty)}\Big(4\Big|\sum_{k=1}^n \xi_{k-1}^* d\upsilon_k^*\Big|^2\Big)\Big).$$

Using the same argument as in III, we can conclude that

$$IV \leq \mathrm{c}\|x\|_1. \tag{10}$$

Inequality (6) follows immediately from (7, 8, 9, 10). The proof is complete. □

3.2. **Non-commutative Burkholder inequality on square functions.** This subsection is devoted to the non-commutative extension of the weak type $(1,1)$ boundedness of square functions of classical martingales [2]. In [28], the following non-commutative extension of Burkholder's result was obtained:

Theorem 3.2. *There exists an absolute constant* $\mathrm{c} > 0$ *such that for any given martingale* $x = (x_n)_{n\geq 1}$ *that is bounded in* $L_1(\mathcal{M},\tau) \cap L_2(\mathcal{M},\tau)$*, there exist two martingales* y *and* z *with* $x = y + z$ *and:*

$$\Big\| \Big(\sum_{n=1}^{\infty} |dy_n|^2 \Big)^{1/2} \Big\|_{1,\infty} + \Big\| \Big(\sum_{n=1}^{\infty} |dz_n^*|^2 \Big)^{1/2} \Big\|_{1,\infty} \leq \mathrm{c} \|x\|_1 .$$

We also refer to [28] for some applications of Theorem 3.2. Our purpose is to highlight that most of the complicated estimates from the proof of Theorem 3.2 in [28] can be explained through the decomposition in Section 2. Namely, we will use the decomposition $(\mathbf{G}'_\lambda)$.

Proof. First, we recall from [28] that the general case can be deduced easily from the special case where x is a positive martingale. Therefore, without loss of generality, we shall assume in what follows that x is a positive L_1-bounded martingale and with norm $\|x\|_1 = 1$. Moreover, we shall only present the special case where $\mathcal{M}$ is a finite von Neumann algebra with the trace τ being normalized. The semifinite case only requires slight changes, see [28] for further details.

We begin by recalling the construction of the martingales y and z from [28]. We consider collections of sequences of pairwise disjoint projections as follows: for $n \geq 1$, set

$$\begin{cases} p_{0,n} & := \bigwedge_{k=0}^{\infty} q_n^{(2^k)}, \quad \text{and} \\ p_{i,n} & := \bigwedge_{k=i}^{\infty} q_n^{(2^k)} - \bigwedge_{k=i-1}^{\infty} q_n^{(2^k)} \quad \text{for } i \geq 1, \end{cases} \tag{11}$$

where $(q_n^{(s)})_{n\geq 0}$ denotes the sequence of Cuculescu's projections relative to x and $s > 0$. The martingales y and z are defined from their respective martingale difference sequences as follows:

$$\begin{cases} dy_1 & := \sum_{j=0}^{\infty} \sum_{i\leq j} p_{i,1} dx_1 p_{j,1}; \\ dy_k & := \sum_{j=0}^{\infty} \sum_{i\leq j} p_{i,k-1} dx_k p_{j,k-1} \quad \text{for } k \geq 2; \\ dz_1 & := \sum_{j=0}^{\infty} \sum_{i> j} p_{i,1} dx_1 p_{j,1}; \\ dz_k & := \sum_{j=0}^{\infty} \sum_{i> j} p_{i,k-1} dx_k p_{j,k-1} \quad \text{for } k \geq 2. \end{cases} \tag{12}$$

We refer to [28] for the fact that indeed $(dy_k)_{k\geq 1}$ and $(dz_k)_{k\geq 1}$ are martingale difference sequences. Moreover, as already explained in [28], it suffices to see that there exists an absolute constant c such that for every non-negative integer m,

$$2^m \, \tau\Big(\chi_{(2^m,\infty)}\big(\mathcal{S}_C(y)\big)\Big) \leq \mathrm{c}. \tag{13}$$

This will be done in two steps:

Step 1. In what follows, we take $\lambda = 2^m$ for some $m \geq 0$. We begin as in [28] by truncating the series which defines dy_k. More concretely, applying Lemma 1.2 and Proposition 1.4 we make the following reduction, see [28] for further details.

Proposition 3.3. [28, Proposition A]. We have

$$\lambda\tau\Big(\chi_{(\lambda,\infty)}(\mathcal{S}_C(y))\Big) \leq 2\lambda\tau\Big(\chi_{(\lambda/2,\infty)}(\mathcal{S}_C^{(m)}(y))\Big) + 4,$$

where $\mathcal{S}_C^{(m)}(y)$ denotes the following square function

$$\mathcal{S}_C^{(m)}(y) = \left(\Big|\sum_{j=0}^{m}\sum_{i\leq j} p_{i,1}dx_1p_{j,1}\Big|^2 + \sum_{k=2}^{\infty}\Big|\sum_{j=0}^{m}\sum_{i\leq j} p_{i,k-1}dx_kp_{j,k-1}\Big|^2\right)^{1/2}.$$

Step 2. According to (13) and Proposition 3.3, we need to estimate

$$2\lambda\tau\Big(\chi_{(\lambda/2,\infty)}(\mathcal{S}_C^{(m)}(y))\Big).$$

For this, we consider the decomposition $(\mathbf{G}'_\lambda)$ of x associated to the parameter $\lambda = 2^m$. This gives $x = \alpha' + \beta' + \gamma' + \upsilon'$ as in Corollary 2.10. For $k \geq 1$, set

$$\mathsf{P}_k^{(m)} := (p_{i,k})_{i=0}^m.$$

Then with this notation,

$$\mathcal{S}_C^{(m)}(y) = \Big(\big|\mathcal{T}^{\mathsf{P}_1^{(m)}}(dx_1)\big|^2 + \sum_{k=2}^{\infty}\big|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(dx_k)\big|^2\Big)^{1/2}.$$

We make the following crucial observation:

Lemma 3.4. *The martingales γ' and υ' do not contribute to the quantity $\mathcal{S}_C^{(m)}(y)$. In particular,*

$$\mathcal{S}_C^{(m)}(y) = \left(\big|\mathcal{T}^{\mathsf{P}_1^{(m)}}(d\alpha'_1 + d\beta'_1)\big|^2 + \sum_{k=2}^{\infty}\big|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\alpha'_k + d\beta'_k))\big|^2\right)^{1/2}.$$

Proof. Recall from $(\mathbf{G}'_\lambda)$ that $d\gamma'_1 + d\upsilon'_1 = 0$ and for every $k \geq 2$,

$$d\gamma'_k + d\upsilon'_k = dx_k - q_{k-1}^{(2^m)}dx_kq_{k-1}^{(2^m)}.$$

The key point here is that

$$p_{i,k-1}(d\gamma'_k + d\upsilon'_k)p_{j,k-1} = 0 \quad \text{for any} \quad 0 \leq i,j \leq m \quad \text{whenever} \quad k \geq 2.$$

This gives $\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\gamma'_k + d\upsilon'_k) = 0$ for every $k \geq 2$, which proves the lemma. □

Now, using the elementary inequality for operators $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ and Lemma 1.2 above, we deduce that

$$\begin{aligned}
&2\lambda\tau\Big(\chi_{(\lambda/2,\infty)}(\mathcal{S}_C^{(m)}(y))\Big) \\
\leq\ & 4\lambda\tau\left(\chi_{(\lambda^2/8,\infty)}\Big(2\big|\mathcal{T}^{\mathsf{P}_1^{(m)}}(d\alpha'_1)\big|^2 + 2\sum_{k=2}^{\infty}\big|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\alpha'_k)\big|^2\Big)\right) \\
+\ & 4\lambda\tau\left(\chi_{(\lambda^2/8,\infty)}\Big(2\big|\mathcal{T}^{\mathsf{P}_1^{(m)}}(d\beta'_1)\big|^2 + 2\sum_{k=2}^{\infty}\big|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\beta'_k)\big|^2\Big)\right)
\end{aligned}$$

$$= \quad I + II.$$

Chebychev's inequality gives

$$I \quad \leq \quad \frac{64}{\lambda}\Big(\big\|\mathcal{T}^{\mathsf{P}_1^{(m)}}(d\alpha_1')\big\|_2^2 + \sum_{k=2}^\infty \big\|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\alpha_k')\big\|_2^2\Big).$$

Since triangular truncations are orthogonal projections in $L_2(\mathcal{M},\tau)$, we deduce

$$I \leq \frac{64}{\lambda}\sum_{k=1}^\infty \|d\alpha_k'\|_2^2 \leq \mathrm{c}.$$

where the last inequality follows from Corollary 2.10 (i). For II, we have

$$II \leq 16\Big\|\Big(\big|\mathcal{T}^{\mathsf{P}_1^{(m)}}(d\beta_1')\big|^2 + \sum_{k=2}^\infty \big|\mathcal{T}^{\mathsf{P}_{k-1}^{(m)}}(d\beta_k')\big|^2\Big)^{1/2}\Big\|_{1,\infty}.$$

Therefore we can apply Lemma 1.3 to obtain that

$$II \leq \mathrm{c}\sum_{k=1}^\infty \|d\beta_k'\|_1 \leq \mathrm{c}.$$

The last estimate follows once more from Corollary 2.10 (ii). Thus, combining the estimates for I and II with Proposition 3.3, the desired inequality (13) follows. □

Remark 3.5. Let us consider a positive non-commutative martingale x and let $x = \alpha' + \beta' + \gamma' + \upsilon'$ stand for the decomposition given in Corollary 2.10 associated to λ. Then, using the properties stated in Corollary 2.10, it is not difficult to check that

$$\begin{aligned}
\max\Big\{\lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_R(\alpha'))\big),\, \lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_C(\alpha'))\big)\Big\} \quad &\leq \quad \mathrm{c}\|x\|_1,\\
\max\Big\{\lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_R(\beta'))\big),\, \lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_C(\beta'))\big)\Big\} \quad &\leq \quad \mathrm{c}\|x\|_1,\\
\max\Big\{\lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_R(\upsilon'))\big),\, \lambda\tau\big(\chi_{(\lambda,\infty)}(\mathcal{S}_C(\gamma'))\big)\Big\} \quad &\leq \quad \mathrm{c}\|x\|_1.
\end{aligned}$$

Unfortunately, in contrast with the commutative case, Theorem 3.2 does not follow automatically from these estimates. Namely, in the commutative case for any given martingale x and any $\lambda > 0$, we can take Gundy's decomposition associated to λ and then the commutative analogue of the estimates given above provides the desired weak type inequality. However, in the non-commutative setting we are forced to decompose the martingale x into two other martingales $x = y + z$ *before* being able to apply Gundy's decomposition. This is justified by the fact that the non-commutative weak Hardy space $\mathcal{H}_{1,\infty}(\mathcal{M},\tau)$ is the sum of two quasi-Banach spaces, see [24, 28] for further details. Thus, the classical proof of Theorem 3.2 for commutative martingales does not work here since we would have to consider the Gundy's decomposition of y and z separately. However, even in the case where both y and z were L_1-bounded, this only allows us to control our terms by the norms of y and z in $L_1(\mathcal{M})$, not by the norm of x in $L_1(\mathcal{M})$.

3.3. **Co-lacunary sequences in non-commutative L_1-spaces.** Let X be a Banach space. A sequence $(x_n)_{n\geq 1}$ in X is said to be 2-*co-lacunary* if there is $\delta > 0$ such that for any finite sequence $(a_n)_{n\geq 1}$ of scalars,

$$\delta\Big(\sum_{n\geq 1}|a_n|^2\Big)^{1/2} \leq \Big\|\sum_{n\geq 1} a_n x_n\Big\|_{\mathrm{X}}.$$

This property can also be described by saying that $(x_n)_{n\geq 1}$ dominates the unit vector basis of l_2, but we will follow the term 2-co-lacunary from [1] which was motivated by the terminology *lacunary sequences* used in [18] for a dual property. In [1], Aldous and Fremlin proved the remarkable result that if $(\Omega, \mathcal{F}, \mu)$ is a probability space, then every uniformly integrable martingale difference sequence which is bounded away from zero is 2-co-lacunary in $L_1(\Omega, \mu)$. Using such result, they deduced the following subsequence principle in L_1-spaces: every bounded sequence in $L_1(\Omega, \mu)$ has either a convergent or a 2-co-lacunary subsequence in $L_1(\Omega, \mu)$.

The principal result of this section is Theorem 3.6 below which extends the main result of Aldous and Fremlin in [1] on classical martingale difference sequences to the non-commutative setting. The proof in [1] is very involved and based on several use of stopping times. Another proof was also given by Dor in [10]. Our proof below uses the decomposition ($\mathbf{G}'_\lambda$). This approach seems to be overlooked for the commutative case.

Theorem 3.6. *Let $(d_k)_{k\geq 1}$ be a martingale difference sequence in $L_1(\mathcal{M}, \tau)$ with:*
 (i) $\gamma = \inf\big\{\|d_k\|_1 \bigm| k \geq 1\big\} > 0$*;*
 (ii) $\big\{d_k \bigm| k \geq 1\big\}$ *is relatively weakly compact in $L_1(\mathcal{M}, \tau)$.*
Then, the sequence $(d_k)_{k\geq 1}$ is a 2-co-lacunary sequence in $L_1(\mathcal{M}, \tau)$.

From the closed graph theorem, Theorem 3.6 can be reformulated as follows:

Theorem 3.6'. *Let $(d_k)_{k\geq 1}$ be the sequence described in Theorem 3.6. Suppose that $(a_k)_{k\geq 1}$ is a sequence of scalars such that the series $\sum_{k\geq 1} a_k d_k$ is convergent in $L_1(\mathcal{M}, \tau)$. Then we have*

$$\sum_{k\geq 1} |a_k|^2 < \infty.$$

Proof. The proof will be divided into several cases.

Case A: Assume first that $(d_k)_{k\geq 1}$ is a sequence of self-adjoint operators and $(a_k)_{k\geq 1}$ is a sequence in $\mathbb{R}$. We start by noting that since $(d_k)_{k\geq 1}$ is relatively weakly compact (and thus, equiintegrable in the sense of [26]), condition (i) of Theorem 3.6 is equivalent to:

$$\sigma := \inf\Big\{\|d_k\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \ \Big|\ k \geq 1\Big\} > 0. \tag{14}$$

Indeed, if $\liminf_{k\to\infty} \|d_k\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} = 0$, then there is a subsequence $(d_{k_j})_{j\geq 1}$ that converges to zero in $L_1(\mathcal{M}, \tau) + \mathcal{M}$, and a fortiori, it converges to zero in the measure topology. By [26, Proposition 2.11], we have $\lim_{j\to\infty} \|d_{k_j}\|_1 = 0$ which violates condition (i).

Next, we observe that $(a_k d_k)_{k\geq 1}$ is a (self-adjoint) martingale difference sequence and denote by $y = (y_n)_{n\geq 1}$ the corresponding martingale. By assumption, y is a

L_1-bounded self-adjoint martingale. For every $\lambda > 0$, we can consider the sequence of Cuculescu's projections associated to the martingale y and $\lambda > 0$. We claim that

$$\lim_{\lambda\to\infty} \sup \Big\{ \big\| d_k \big(\mathbf{1} - q_k^{(\lambda)} \big) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \ \big| \ k \in \mathbb{N} \Big\} = 0.$$

Indeed, for every $k \in \mathbb{N}$ and $\lambda > 0$, we have (see e.g. [20, Proposition 2.a.2]):

$$\big\| d_k \big(\mathbf{1} - q_k^{(\lambda)} \big) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} = \int_0^1 \mu_t \big(d_k \big(\mathbf{1} - q_k^{(\lambda)} \big) \big) \ dt.$$

Using properties of singular value functions from [11],

$$\begin{aligned} \big\| d_k \big(\mathbf{1} - q_k^{(\lambda)} \big) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} &\leq \int_0^1 \mu_t(d_k)\, \mu_t \big(\mathbf{1} - q_k^{(\lambda)} \big) \ dt \\ &\leq \int_0^1 \mu_t(d_k)\, \chi_{[0,\tau(\mathbf{1}-q_k^{(\lambda)})]}(t) \ dt \\ &\leq \int_0^{1\wedge\lambda^{-1}\|y\|_1} \mu_t(d_k) \ dt. \end{aligned}$$

Since $\{d_k \mid k \geq 1\}$ is relatively weakly compact in $L_1(\mathcal{M},\tau)$, it is a fortiori relatively $\sigma(L_1(\mathcal{M},\tau)+\mathcal{M}, L_1(\mathcal{M},\tau)\cap\mathcal{M})$-compact in $L_1(\mathcal{M},\tau)+\mathcal{M}$. According to the terminology used in Section 1, we note that $L_1(\mathcal{M},\tau)\cap\mathcal{M} \subset S_0(\mathcal{M},\tau)$. Then, by Theorem 1.1 $\{\mu(d_k) \mid k \geq 1\}$ is relatively $\sigma(L_1+L_\infty, L_1\cap L_\infty)$-compact in $L_1[0,\tau(\mathbf{1})) + L_\infty[0,\tau(\mathbf{1}))$ and therefore $\{\mu(d_k)\chi_{[0,1]} \mid k \geq 1\}$ is relatively weakly compact in $L_1[0,1]$. In other words, it is uniformly integrable in $L_1[0,1]$. Thus we conclude that

$$\lim_{\lambda\to\infty} \sup \left\{ \int_0^{1\wedge\lambda^{-1}\|y\|_1} \mu_t(d_k) \ dt \ \Big| \ k \geq 1 \right\} = 0,$$

which proves the claim. For the remainder of the proof, we fix $\lambda > 0$ so that,

$$\sup \Big\{ \big\| d_k (\mathbf{1} - q_k^{(\lambda)}) \big\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} \ \big| \ k \geq 1 \Big\} \leq \sigma/5 \tag{15}$$

where σ is from (14). We will simply write $(q_n)_{n\geq 0}$ for $(q_n^{(\lambda)})_{n\geq 0}$. We note from (15) that for every $k \geq 1$,

$$\begin{aligned} \sigma &\leq \|d_k\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &\leq \big\| q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &+ \big\| q_k d_k (\mathbf{1}-q_k) - \mathsf{E}_{k-1}(q_k d_k (\mathbf{1}-q_k)) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &+ \big\| (\mathbf{1}-q_k) d_k - \mathsf{E}_{k-1}((\mathbf{1}-q_k) d_k) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &\leq \big\| q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k) \big\|_2 \\ &+ 2\big\| q_k d_k (\mathbf{1}-q_k) \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &+ 2\big\| (\mathbf{1}-q_k) d_k \big\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \\ &\leq \big\| q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k) \big\|_2 + 4\sigma/5. \end{aligned}$$

Therefore,

$$\inf \Big\{ \big\| q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k) \big\|_2 \ \big| \ k \geq 1 \Big\} \geq \sigma/5. \tag{16}$$

Let us consider the decomposition $y = \alpha' + \beta' + \gamma' + \upsilon'$ of the martingale y according to Corollary 2.10 and Remark 2.11 and relative to the parameter $\lambda > 0$ fixed above. Then we have $\|\alpha'\|_2^2 \leq c\lambda\|y\|_1$. Recall that for every $k \geq 1$,

$$d\alpha'_k = q_k dy_k q_k - \mathsf{E}_{k-1}(q_k dy_k q_k) = a_k \left(q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k)\right).$$

This gives,

$$\sum_{k\geq 1} |a_k|^2 \left\|q_k d_k q_k - \mathsf{E}_{k-1}(q_k d_k q_k)\right\|_2^2 \leq c\lambda \left\|y\right\|_1,$$

and therefore by (16) we conclude,

$$\sigma^2 \sum_{k\geq 1} |a_k|^2 \leq 25c\lambda \left\|y\right\|_1 < \infty.$$

The proof for this case is complete.

Case B: Assume now that $(d_k)_{k\geq 1}$ is not necessarily a sequence of self-adjoint operators and $(a_k)_{k\geq 1}$ is a sequence in $\mathbb{R}$. Consider the (semifinite) von Neumann algebra $\mathcal{M} \oplus_\infty \mathcal{M}$ with the trace $\widetilde{\tau} = \tau \oplus_\infty \tau$ and the filtration $(\mathcal{M}_n \oplus_\infty \mathcal{M}_n)_{n\geq 1}$. For $k \geq 1$, let

$$\widetilde{d}_k := \Big((d_k + d_k^*)/2, (d_k - d_k^*)/2i\Big) \in \mathcal{M}_k \oplus_\infty \mathcal{M}_k.$$

Then $(\widetilde{d}_k)_{k\geq 1}$ is a self-adjoint martingale difference sequence in $L_1(\mathcal{M} \oplus_\infty \mathcal{M}, \widetilde{\tau})$ that clearly verifies the conditions (i) and (ii) of Theorem 3.6. If $\sum_{k\geq 1} a_k d_k$ is convergent, then so is the series $\sum_{k\geq 1} a_k d_k^*$. Consequently, the series

$$\sum_{k\geq 1} a_k \widetilde{d}_k$$

converges in $L_1(\mathcal{M} \oplus_\infty \mathcal{M}, \widetilde{\tau})$. Hence, from Case A, we get $\sum_{k\geq 1} |a_k|^2 < \infty$.

Case C: For the general case where $a_k \in \mathbb{C}$, we set

$$\gamma_k := \begin{cases} a_k/|a_k| & \text{if } a_k \neq 0 \\ 1 & \text{if } a_k = 0. \end{cases}$$

Then it is clear that $(\hat{d}_k)_{k\geq 1} = (\gamma_k d_k)_{k\geq 1}$ is a martingale difference sequence that satisfies conditions (i) and (ii) of Theorem 3.6. Moreover, since we assume by hypothesis that the series

$$\sum_{k\geq 1} a_k d_k = \sum_{k\geq 1} |a_k| \hat{d}_k$$

is convergent, Case B insures that $\sum_{k\geq 1} |a_k|^2 < \infty$. The proof is complete. □

As a consequence of Theorem 3.6, we have the following result that generalizes a result from [1] (see also [10] for a quantitative version) to non-commutative spaces.

Corollary 3.7. *Assume that $\mathcal{M}$ is semifinite and hyperfinite. Let $(x_n)_{n\geq 1}$ be a bounded sequence in $L_1(\mathcal{M}, \tau)$. Then either $(x_n)_{n\geq 1}$ has a convergent subsequence or it has a 2-co-lacunary subsequence.*

For the proof, we will use the following perturbation lemma from [1].

Lemma 3.8. *Let X be a normed space and $(x_n)_{n\geq 1}$ be a bounded sequence in X. Then the following properties hold:*

(a) *If* $(x_n)_{n\geq 1}$ *is* 2*-co-lacunary and* $x \in \mathrm{X}$*, then there exists* $m \in \mathbb{N}$ *such that* $(x_n - x)_{n\geq m}$ *is* 2*-co-lacunary;*
(b) *If* $(x_n)_{n\geq 1}$ *is* 2*-co-lacunary and* $(y_n)_{n\geq 1} \subset \mathrm{X}$ *with* $\sum_{n\geq 1} \|x_n - y_n\|_{\mathrm{X}}$ *being convergent, then there exists* $m \in \mathbb{N}$ *such that* $(y_n)_{n\geq m}$ *is* 2*-co-lacunary.*

Proof of Corollary 3.7. Assume that $(x_n)_{n\geq 1}$ has no convergent subsequence. By Rosenthal's l_1-theorem, either $(x_n)_{n\geq 1}$ has a subsequence equivalent to the unit vector basis of l_1, and therefore 2-co-lacunary (in fact, 1-co-lacunary), or $(x_n)_{n\geq 1}$ has a weakly convergent subsequence. Assume w.l.o.g. that $(x_n)_{n\geq 1}$ converges to x weakly. Then, according to Lemma 3.8 (a), it suffices to show that $(x_n - x)_{n\geq 1}$ has a 2-co-lacunary subsequence.

If $\mathcal{M}$ is hyperfinite then $\mathcal{M} = \overline{\cup_\alpha \mathcal{M}_\alpha}$ (weak* closure) where $(\mathcal{M}_\alpha)_{\alpha\in I}$ is a net of finite dimensional *-subalgebras directed by inclusion. There exist contractive projections $E_\alpha : \mathcal{M} \to \mathcal{M}_\alpha$ which are simultaneously contractions from $\mathcal{M} \to \mathcal{M}_\alpha$ and $L_1(\mathcal{M}, \tau) \to L_1(\mathcal{M}_\alpha, \tau_\alpha)$, where τ_α denotes the restriction of τ on $\mathcal{M}_\alpha$. The projections E_α's satisfy $E_\alpha = E_\alpha E_\beta$ for $\alpha \leq \beta$. Moreover, for every $f \in L_1(\mathcal{M}, \tau)$, $\lim_\alpha \|E_\alpha(f) - f\|_1 = 0$. For $n \geq 1$, let $f_n = x_n - x$. Then by assumption $(f_n)_{n\geq 1}$ is a weakly null sequence in $L_1(\mathcal{M}, \tau)$. Moreover, since $(f_n)_{n\geq 1}$ does not converges in the norm of $L_1(\mathcal{M}, \tau)$, we may assume w.l.o.g. that the sequence $(f_n)_{n\geq 1}$ itself satisfies

$$\inf \Big\{ \|f_n\|_1 \ \big| \ n \geq 1 \Big\} > 0. \tag{17}$$

Set $n_1 = 1$, choose $\alpha_1 \in I$ such that

$$\big\| f_1 - E_{\alpha_1}(f_1) \big\|_1 < 2^{-2}. \tag{18}$$

Since $(f_n)_{n\geq 1}$ converges weakly to zero and $\mathcal{M}_{\alpha_1}$ is finite dimensional, we have

$$\lim_{n\to\infty} \|E_{\alpha_1}(f_n)\|_1 = 0.$$

Choose $n_2 > n_1 = 1$ such that,

$$\|E_{\alpha_1}(f_{n_2})\|_1 < 2^{-3}$$

and $\alpha_2 > \alpha_1$ so that,

$$\big\| f_{n_2} - E_{\alpha_2}(f_{n_2}) \big\|_1 < 2^{-3}.$$

Inductively, one gets a sequence $(n_k)_{k\geq 1} \subseteq \mathbb{N}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \cdots$ in I such that for every $k \geq 2$,

$$\max \Big\{ \big\| E_{\alpha_{k-1}}(f_{n_k}) \big\|_1, \ \big\| f_{n_k} - E_{\alpha_k}(f_{n_k}) \big\|_1 \Big\} < 2^{-(k+1)}. \tag{19}$$

For $k \geq 2$, set

$$\begin{cases} v_1 & := E_{\alpha_1}(f_1), \\ v_k & := E_{\alpha_k}(f_{n_k}) - E_{\alpha_{k-1}}(f_{n_k}). \end{cases}$$

Lemma 3.9. *The sequence* $(v_k)_{k\geq 1}$ *satisfies:*

(a) $\big\| f_{n_k} - v_k \big\|_1 \leq 2^{-k}$ *for every* $k \geq 1$*;*
(b) $\liminf_{k\to\infty} \|v_k\|_1 > 0$*;*
(c) $\big\{ v_k \mid k \geq 1 \big\}$ *is relatively weakly compact in* $L_1(\mathcal{M}, \tau)$*.*

Proof. Property (a) follows from (18, 19) and the triangle inequality. Property (b) follows from (a) and (17). Finally, (c) follows directly from (a) and the fact that $(f_{n_k})_{k\geq 1}$ is weakly null. □

Note that if $\mathcal{M}$ is finite, then the projections E_α's defined above are conditional expectations so the sequence $(v_k)_{k\geq 1}$ is clearly a martingale difference sequence with respect to the filtration $(\mathcal{M}_{\alpha_k})_{k\geq 1}$. Assume that $\mathcal{M}$ is infinite. For every $k \geq 1$, let p_k be the self-adjoint projection which is the unit element of $\mathcal{M}_{\alpha_k}$. When τ is infinite, $p_k \neq \mathbf{1}$. The contractive projection E_{α_k} can be written as $E_{\alpha_k} = \mathcal{E}_k \circ Q_{\alpha_k}$ where $Q_{\alpha_k}(x) = p_k x p_k$ and $\mathcal{E}_k$ is the unit preserving conditional expectation from the finite von Neumann algebra $p_k \mathcal{M} p_k$ onto the von Neumann subalgebra $\mathcal{M}_{\alpha_k}$. Since $\{v_k \mid k \geq 1\}$ is relatively weakly compact in $L_1(\mathcal{M}, \tau)$ and $(p_k - p_{k-1})_{k\geq 2}$ is a disjoint sequence of projections, we have (see [26])

$$\lim_{k\to\infty} \big\|(p_k - p_{k-1})v_k(p_k - p_{k-1})\big\|_1 = 0.$$

By taking subsequence if necessary, we can assume that for every $k \geq 2$,

$$\big\|(p_k - p_{k-1})v_k(p_k - p_{k-1})\big\|_1 \leq 2^{-k}. \tag{20}$$

For $k \geq 2$, set

$$\begin{cases} w_1 & := v_1, \\ w_k & := v_k - (p_k - p_{k-1})v_k(p_k - p_{k-1}). \end{cases}$$

Lemma 3.10. *The sequence $(w_k)_{k\geq 1}$ is a martingale difference sequence with:*

(a) $\big\|f_{n_k} - w_k\big\|_1 \leq 2^{-(k-1)}$ *for every* $k \geq 1$*;*
(b) $\liminf_{k\to\infty} \|w_k\|_1 > 0$*;*
(c) $\{w_k \mid k \geq 1\}$ *is relatively weakly compact in* $L_1(\mathcal{M}, \tau)$*.*

Proof. Properties (a), (b), and (c) follow directly from Lemma 3.9 and (20). Hence it remains to show that $(w_k)_{k\geq 1}$ is a martingale difference sequence. For $k \geq 1$, we set

$$\mathcal{S}_k := \mathcal{M}_{\alpha_k} + \sum_{s\geq k} (p_{s+1} - p_s)\mathcal{M}_{\alpha_{s+1}}(p_{s+1} - p_s).$$

Clearly, $(\mathcal{S}_k)_{k\geq 1}$ is an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ and, by the definition of the $\mathcal{M}_\alpha$'s, we may assume that $\mathcal{M} = \overline{\cup_{k\geq 1}\mathcal{S}_k}$ (weak* closure). Define $\mathbb{E}_k : \mathcal{M} \to \mathcal{S}_k$ by setting

$$\mathbb{E}_k(x) = E_{\alpha_k}(x) + \sum_{s\geq k} (p_{s+1} - p_s)E_{\alpha_{s+1}}(x)(p_{s+1} - p_s),$$

where the sum is taken with respect to the weak* topology. The right hand side is well defined since the series $\sum_{s\geq k}(p_{s+1} - p_s)E_{\alpha_{s+1}}(x)(p_{s+1} - p_s)$ is weakly unconditionally Cauchy in $\mathcal{M}$. In fact, for any finite set $S \subset [k, \infty)$,

$$\Big\|\sum_{s\in S}(p_{s+1} - p_s)E_{\alpha_{s+1}}(x)(p_{s+1} - p_s)\Big\|_\infty \leq \|x\|_\infty.$$

The operator $\mathbb{E}_k$ is clearly a (contractive) positive projection and $\mathbb{E}_k(\mathbf{1}) = \mathbf{1}$. Thus, according to [30], the mapping $\mathbb{E}_k$ is a conditional expectation. Moreover, for every $x \in \mathcal{M}$, we have

$$\tau(\mathbb{E}_k(x)) \quad = \quad \tau(E_{\alpha_k}(x)) + \sum_{s\geq k} \tau\big((p_{s+1} - p_s)E_{\alpha_{s+1}}(x)(p_{s+1} - p_s)\big)$$

$$= \tau\big(\mathcal{E}_k(p_k x p_k)\big) + \sum_{s\geq k} \tau\big(\mathcal{E}_{s+1}((p_{s+1}-p_s)x(p_{s+1}-p_s))\big)$$
$$= \tau(p_k x p_k) + \sum_{s\geq k} \tau\big((p_{s+1}-p_s)x(p_{s+1}-p_s)\big) = \tau(x).$$

It is clear that for every $k \geq 2$, $E_{\alpha_{k-1}}(v_k) = 0$ and $E_{\alpha_{k-1}}((p_k - p_{k-1})v_k(p_k - p_{k-1})) = 0$. We can deduce from the definition of $\mathbb{E}_{k-1}$ that $\mathbb{E}_{k-1}(w_k) = 0$. Therefore, $(w_k)_{k\geq 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{S}_k)_{k\geq 1}$. The lemma is proved. □

From Theorem 3.6 and Lemma 3.10, we know that $(w_k)_{k\geq m_1}$ is 2-co-lacunary for some $m_1 \geq 1$. Moreover, since $\sum_{k\geq 1} \|f_{n_k} - w_k\|_1 < \infty$, it follows from Lemma 3.8 that there exists $m_2 \geq m_1$ such that $(f_{n_k})_{k\geq m_2} = (x_{n_k} - x)_{k\geq m_2}$ is 2-co-lacunary. The proof of Corollary 3.7 is complete. □

Remark 3.11. We do not know if Corollary 3.7 is valid without the hyperfinite assumption. We leave this as a problem for the interested reader.

Note added in proof. After this paper was accepted for publication, Gilles Pisier communicated to us the following surprisingly simple proof of the Aldous/Fremlin result and its non-commutative generalization (stated in Theorem 3.6 above) for the case where $\mathcal{M}$ is finite and τ is a tracial state. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ be a sequence of independent Bernoulli random variables equidistributed in ± 1. According to the weak type boundedness of martingale transforms from Theorem 3.1, we know that every martingale x in $L_1(\mathcal{M}, \tau)$ satisfies the following estimate for every $0 < p < 1$

$$\int_0^1 \Big\| \sum_{k=1}^n \varepsilon_k(t) dx_k \Big\|_p dt \leq \mathrm{c}_p^1 \int_0^1 \Big\| \sum_{k=1}^n \varepsilon_k(t) dx_k \Big\|_{1,\infty} dt \leq \mathrm{c}_p^2 \Big\| \sum_{k=1}^n dx_k \Big\|_1.$$

Therefore, since $L_p(\mathcal{M}, \tau)$ is of cotype 2 (see e.g. [25, Corollary 5.8]), we deduce

$$\Big(\sum_{k=1}^\infty \|dx_k\|_p^2 \Big)^{1/2} \leq \mathrm{k}_p \Big\| \sum_{k=1}^\infty dx_k \Big\|_1,$$

for every $0 < p < 1$. Now, assume that $(dx_k)_{k\geq 1}$ satisfies the assumptions of Theorem 3.6. It is clear from Proposition 2.11 in [26] that the sequence of norms $\|dx_1\|_p, \|dx_2\|_p, \ldots$ is bounded away from 0 when $0 < p < 1$, so that

$$\delta_p = \inf \Big\{ \|dx_k\|_p \ \big| \ k \geq 1 \Big\} > 0.$$

In particular, given any square-summable sequence $(\alpha_k)_{k\geq 1}$ of scalars and taking the martingale differences $dz_k = \alpha_k dx_k$, we obtain a martingale $z = \sum_k dz_k$ in $L_1(\mathcal{M}, \tau)$ and the considerations above give rise to

$$\delta_p \Big(\sum_{k=1}^\infty |\alpha_k|^2 \Big)^{1/2} \leq \Big(\sum_{k=1}^\infty \|dz_k\|_p^2 \Big)^{1/2} \leq \mathrm{k}_p \Big\| \sum_{k=1}^\infty dz_k \Big\|_1 = \mathrm{k}_p \Big\| \sum_{k=1}^\infty \alpha_k dx_k \Big\|_1.$$

In conclusion, we obtain

$$\Big(\sum_{k=1}^\infty |\alpha_k|^2 \Big)^{1/2} \leq \big(\lim_{p\to 1} \delta_p^{-1} \mathrm{k}_p \big) \Big\| \sum_{k=1}^\infty \alpha_k dx_k \Big\|_1$$

and the martingale difference sequence $(dx_k)_{k\geq 1}$ is 2-co-lacunary. At the time of this writing, we do not know whether or not this elementary argument extends to the infinite case.

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