NON-COMMUTATIVE KHINTCHINE TYPE INEQUALITIES ASSOCIATED WITH FREE GROUPS

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ABSTRACT. Let $\mathbf{F}_n$ denote the free group with n generators $g_1, g_2, \ldots, g_n$. Let λ stand for the left regular representation of $\mathbf{F}_n$ and let τ be the standard trace associated to λ. Given any positive integer d, we study the operator space structure of the subspace $\mathcal{W}_p(n, d)$ of $L_p(\tau)$ generated by the family of operators $\lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$ with $1 \leq i_k \leq n$. Moreover, our description of this operator space holds up to a constant which does not depend on n or p, so that our result remains valid for infinitely many generators. We also consider the subspace of $L_p(\tau)$ generated by the image under λ of the set of reduced words of length d. Our result extends to any exponent $1 \leq p \leq \infty$ a previous result of Buchholz for the space $\mathcal{W}_\infty(n, d)$. The main application is a certain interpolation theorem, valid for any degree d (extending a result of the second author restricted to $d = 1$). In the simplest case $d = 2$, our theorem can be stated as follows: consider the space $\mathcal{K}_p$ formed of all block matrices $a = (a_{ij})$ with entries in the Schatten class S_p, such that a is in S_p relative to $\ell_2 \otimes \ell_2$ and moreover such that $(\sum_{ij} a_{ij}^* a_{ij})^{1/2}$ and $(\sum_{ij} a_{ij} a_{ij}^*)^{1/2}$ both belong to S_p. We equip $\mathcal{K}_p$ with the maximum of the three corresponding norms. Then, for $2 \leq p \leq \infty$ we have $\mathcal{K}_p \simeq (\mathcal{K}_2, \mathcal{K}_\infty)_\theta$ with $1/p = (1 - \theta)/2$.

INTRODUCTION

Let $\mathcal{R}_p(n)$ be the subspace of $L_p[0, 1]$ generated by the classical Rademacher functions $r_1, r_2, \ldots, r_n$. As is well-known, for any exponent $1 \leq p < \infty$, the classical Khintchine inequalities provide a linear isomorphism between $\mathcal{R}_p(n)$ and $\ell_2(n)$ with constants independent of n. However, to describe the operator space structure of $\mathcal{R}_p(n)$ we need the so-called non-commutative Khintchine inequalities, introduced by F. Lust-Piquard in [7] and extended in [8] to the case $p = 1$, see also [2] for an analysis of the optimal constants.

To describe the non-commutative Khintchine inequalities, let us consider a family $x_1, x_2, \ldots, x_n$ of elements in the Schatten class S_p. Then we have the following equivalences of norms for $1 \leq p \leq 2$

$$\Big\| \sum_{k=1}^n x_k r_k \Big\|_{L_p([0,1];S_p)} \simeq \inf_{x_k = y_k + z_k} \left\{ \Big\| \Big(\sum_{k=1}^n y_k y_k^* \Big)^{1/2} \Big\|_{S_p} + \Big\| \Big(\sum_{k=1}^n z_k^* z_k \Big)^{1/2} \Big\|_{S_p} \right\},$$

while for $2 \leq p < \infty$ we have

$$\Big\| \sum_{k=1}^n x_k r_k \Big\|_{L_p([0,1];S_p)} \simeq \max \left\{ \Big\| \Big(\sum_{k=1}^n x_k x_k^* \Big)^{1/2} \Big\|_{S_p}, \Big\| \Big(\sum_{k=1}^n x_k^* x_k \Big)^{1/2} \Big\|_{S_p} \right\}.$$

Again the constants do not depend on n. According to the purposes of this paper, it will be more convenient to rewrite these inequalities in terms of the spaces R_p^n

and C_p^n. Let us consider the Schatten class S_p^n over the $n \times n$ matrices and let us denote its natural basis by

$$\left\{ e_{ij} \mid 1 \leq i, j \leq n \right\}.$$

We define R_p^n to be the subspace of S_p^n generated by $e_{11}, e_{12}, \ldots, e_{1n}$ while C_p^n will be the subspace generated by $e_{11}, e_{21}, \ldots, e_{n1}$. That is, R_p^n and C_p^n can be regarded as the row and column subspaces of S_p^n respectively. Note that both subspaces have a natural operator space structure inherited from S_p^n. In terms of the spaces R_p^n and C_p^n, the non-commutative Khintchine inequalities can be rephrased by saying that $\mathcal{R}_p(n)$ is completely isomorphic to $R_p^n + C_p^n$ whenever $1 \leq p \leq 2$ and $\mathcal{R}_p(n)$ is completely isomorphic to $R_p^n \cap C_p^n$ for $2 \leq p < \infty$. More concretely, when $1 \leq p \leq 2$ it follows that

$$\Big\| \sum_{k=1}^n x_k \mathrm{r}_k \Big\|_{L_p([0,1];S_p)}$$

$$\simeq \inf_{x_k = y_k + z_k} \left\{ \Big\| \sum_{k=1}^n y_k \otimes e_{1k} \Big\|_{S_p(\ell_2 \otimes \ell_2)} + \Big\| \sum_{k=1}^n z_k \otimes e_{k1} \Big\|_{S_p(\ell_2 \otimes \ell_2)} \right\},$$

while for $2 \leq p \leq \infty$ we have

$$\Big\| \sum_{k=1}^n x_k \mathrm{r}_k \Big\|_{L_p([0,1];S_p)}$$

$$\simeq \max \left\{ \Big\| \sum_{k=1}^n x_k \otimes e_{1k} \Big\|_{S_p(\ell_2 \otimes \ell_2)}, \Big\| \sum_{k=1}^n x_k \otimes e_{k1} \Big\|_{S_p(\ell_2 \otimes \ell_2)} \right\}.$$

Now let $\mathbf{F}_n$ be the free group with n generators $g_1, g_2, \ldots, g_n$. If λ denotes the left regular representation of $\mathbf{F}_n$, the family of operators $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$ appear as the free analog of the sequence $\mathrm{r}_1, \mathrm{r}_2, \ldots, \mathrm{r}_n$ in this framework. Namely, let us consider the standard trace τ on $C_\lambda^*(\mathbf{F}_n)$. Then, by [3] it turns out that the subspace $\mathcal{W}_p(n)$ of $L_p(\tau)$ generated by the operators $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$ is completely isomorphic to $\mathcal{R}_p(n)$ (with constants independent of n) for $1 \leq p < \infty$. Moreover, for $p = \infty$ we have

$$\Big\| \sum_{k=1}^n a_k \otimes \lambda(g_k) \Big\|_{L_\infty(\mathrm{tr} \otimes \tau)}$$

$$\simeq \max \left\{ \Big\| \sum_{k=1}^n a_k \otimes e_{1k} \Big\|_{S_\infty(\ell_2 \otimes \ell_2)}, \Big\| \sum_{k=1}^n a_k \otimes e_{k1} \Big\|_{S_\infty(\ell_2 \otimes \ell_2)} \right\},$$

where R_n and C_n are the usual expressions for R_∞^n and C_∞^n. In other words, we also have $\mathcal{W}_\infty(n) \simeq R_n \cap C_n$ completely isomorphically. In this paper we shall generalize the mentioned complete isomorphism for $\mathcal{W}_p(n)$, where the results above appear as the case of degree one. More concretely, given any positive integer d, we shall consider (operator-valued) homogeneous polynomials of degree d in the variables

$$\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n).$$

Let $\mathcal{W}_p(n, d)$ be the subspace of $L_p(\tau)$ generated by the operators $\lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$, where $1 \leq i_k \leq n$ for $1 \leq k \leq d$. The aim of this paper is to describe the operator space structure of $\mathcal{W}_p(n, d)$ for any value of d. To that aim, we consider an auxiliary non-commutative L_p space $L_p(\varphi)$ equipped with a faithful normal semi-finite trace

φ. Then, describing the operator space structure of $\mathcal{W}_p(n,d)$ becomes equivalent to describing the norm of

$$\sum_{i_1,i_2,\ldots,i_d=1}^{n} a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$$

in $L_p(\varphi \otimes \tau)$ up to constants not depending on n or p. Let $\mathcal{A}$ be the family of operators $\left\{ a_{i_1 i_2 \cdots i_d} \mid 1 \leq i_k \leq n \right\}$. This family can be regarded as an element of a matrix-valued $L_p(\varphi)$-space in several ways. Namely, given $0 \leq k \leq d$, we can construct a matrix $\mathcal{A}_k$ with entries in $\mathcal{A}$ by taking the first k indices $i_1, i_2, \ldots, i_k$ as the row index and the last $d-k$ indices $i_{k+1}, i_{k+2}, \ldots, i_d$ as the column index. In other words, we consider the matrix

$$\mathcal{A}_k = \left(a_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)} \right).$$

In particular, we see $\mathcal{A}_k$ as an element of the Haagerup tensor product $C_p^{n^k} \otimes_h R_p^{n^{d-k}}$ with values in $L_p(\varphi)$. This allows us to define the following family of spaces

$$\mathcal{K}_p(n,d) = \sum_{k=0}^{d} C_p^{n^k} \otimes_h R_p^{n^{d-k}} \qquad \text{for } 1 \leq p \leq 2,$$

$$\mathcal{K}_p(n,d) = \bigcap_{k=0}^{d} C_p^{n^k} \otimes_h R_p^{n^{d-k}} \qquad \text{for } 2 \leq p \leq \infty.$$

In this paper we shall prove that $\mathcal{W}_p(n,d)$ and $\mathcal{K}_p(n,d)$ are completely isomorphic operator spaces with constants depending only on the degree d. More concretely, the following result holds.

Theorem. *$\mathcal{K}_p(n,d)$ and $\mathcal{W}_p(n,d)$ are completely isomorphic for $1 \leq p \leq \infty$ and there exists an absolute constant c_d depending only on d such that the following inequalities hold*

$$\frac{1}{\mathrm{c}_d} \left\| \mathcal{A} \right\|_{L_p(\varphi; \mathcal{K}_p(n,d))} \leq \left\| \sum_{i_1 \cdots i_d=1}^{n} a_{i_1 \cdots i_d} \otimes \lambda(g_{i_1} \cdots g_{i_d}) \right\|_{L_p(\varphi \otimes \tau)} \leq \mathrm{c}_d \left\| \mathcal{A} \right\|_{L_p(\varphi; \mathcal{K}_p(n,d))}.$$

Moreover, the natural projection $\mathrm{P} : L_p(\tau) \to \mathcal{W}_p(n,d)$ *is c.b. with* $\|\mathrm{P}\|_{cb} \leq \mathrm{c}_d$.

The case $p = \infty$ of this result is the content of Buchholz's paper [1]. However, a slightly different definition of $\mathcal{W}_\infty(n,d)$ was used there. Indeed, Buchholz takes the subspace of $L_\infty(\tau)$ generated by the image under λ of the set of reduced words of length d. That is, inverses of generators are also allowed to appear. As we shall see, our arguments apply in this context for any $1 \leq p \leq \infty$ and we will also provide the proof. In Section 1 we set some preliminary results while Section 2 is devoted to the proof of our main result and its analog for reduced words of length d.

1. Preliminary results

One of the key points in the proof of our main result lies in the iteration of the non-commutative Khintchine inequality applied to the operator space $\mathcal{W}_p(n)^{\otimes d}$. This process was already pointed in [12, Section 9.8], but we provide here an explicit description of the resulting inequalities. After that, we state an L_p-valued version of Fell's absorption principle that will be also needed in the proof.

1.1. **Iterations of the Khintchine inequality.** Let $\mathbf{F}_n$ be the free group with n generators $g_1, g_2, \ldots, g_n$. If $(\delta_t)_{t \in \mathbf{F}_n}$ denotes the natural basis of $\ell_2(\mathbf{F}_n)$, the left regular representation λ of $\mathbf{F}_n$ is defined by the relation $\lambda(t_1)\delta_{t_2} = \delta_{t_1 t_2}$. The reduced C^*-algebra $C^*_\lambda(\mathbf{F}_n)$ is defined as the C^*-algebra generated in $\mathcal{B}(\ell_2(\mathbf{F}_n))$ by the operators $\lambda(t)$ when t runs over $\mathbf{F}_n$. Let us denote by τ the standard trace on $C^*_\lambda(\mathbf{F}_n)$ defined by $\tau(x) = \langle x\delta_e, \delta_e \rangle$, where e denotes the identity element of $\mathbf{F}_n$. Then, we construct the non-commutative L_p space $L_p(\tau)$ in the usual way and consider the subspace $\mathcal{W}_p(n)$ of $L_p(\tau)$ generated by the operators $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$. The aim of this section is to describe the operator space structure of $\mathcal{W}_p(n)^{\otimes d}$ as a subspace of $L_p(\tau^{\otimes d})$ for the exponents $2 \leq p \leq \infty$. As it was pointed out in [12], the case $1 \leq p \leq 2$ follows easily by duality. However, we shall not write the explicit inequalities in that case since we are not using them and the notation is considerably more complicated.

The following result can be regarded as a particular case of our main result for homogeneous polynomials of degree 1. It was proved in [3] for $p = \infty$ while the proof for $2 \leq p < \infty$ can be found in Corollary 9.7.2 of [12]. We notice that its proof uses the fact that $R_p^n \cap C_p^n$ is an interpolation family for $2 \leq p \leq \infty$.

Lemma 1.1. *The following equivalence of norms holds for $2 \leq p \leq \infty$,*

$$\Big\| \sum_{k=1}^n a_k \otimes \lambda(g_k) \Big\|_{L_p(\varphi \otimes \tau)} \simeq \max \left\{ \Big\| \sum_{k=1}^n a_k \otimes e_{1k} \Big\|_{L_p(\varphi; R_p^n)}, \Big\| \sum_{k=1}^n a_k \otimes e_{k1} \Big\|_{L_p(\varphi; C_p^n)} \right\}.$$

In fact, the linear map $u : R_p^n \cap C_p^n \to \mathcal{W}_p(n)$ defined by

$$u(e_{1k} \oplus e_{k1}) = \lambda(g_k),$$

is a complete isomorphism with $\|u\|_{cb} \leq 2$ and completely contractive inverse. On the other hand, the canonical projection $\mathrm{P} : L_p(\tau) \to \mathcal{W}_p(n)$ satisfies $\|\mathrm{P}\|_{cb} \leq 2$.

Let us consider the group product $\mathrm{G}_d = \mathbf{F}_n \times \mathbf{F}_n \times \cdots \times \mathbf{F}_n$ with d factors. The left regular representation λ_d of G_d has the form

$$\lambda_d(t_1, t_2, \ldots, t_d) = \lambda(t_1) \otimes \lambda(t_2) \otimes \cdots \otimes \lambda(t_d),$$

where λ still denotes the left regular representation of $\mathbf{F}_n$. In particular, the reduced C^*-algebra $C^*_{\lambda_d}(\mathrm{G}_d)$ is endowed with the trace $\tau_d = \tau \otimes \tau \otimes \cdots \otimes \tau$ with d factors. This allows us to consider the non-commutative space $L_p(\tau_d)$ for any $1 \leq p \leq \infty$. Then we define the space $\mathcal{W}_p(n)^{\otimes d}$ to be the subspace of $L_p(\tau_d)$ generated by the family of operators

$$\lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}).$$

If we apply repeatedly Lemma 1.1 to the sum

$$\mathcal{S}_d(a) = \sum_{i_1, \ldots, i_d = 1}^n a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}) \in L_p(\varphi \otimes \tau_d),$$

then we easily get

$$\|\mathcal{S}_d(a)\|_{L_p(\varphi \otimes \tau_d)} \leq 2^d \max \left\{ \Big\| \sum_{i_1, \ldots, i_d = 1}^n a_{i_1 \cdots i_d} \otimes \xi_1(i_1) \otimes \cdots \otimes \xi_d(i_d) \Big\|_{S_p^{n^d}(L_p(\varphi))} \right\},$$

where the maximum runs over all possible ways to choose the functions $\xi_1, \xi_2, \ldots, \xi_d$ among $\xi_k(\cdot) = e_{\cdot 1}$ and $\xi_k(\cdot) = e_{1 \cdot}$. That is, each function ξ_k can take values either

in the space R_p^n or in the space C_p^n. For a given selection of $\xi_1, \xi_2, \ldots, \xi_d$ we split up these functions into two sets, one made up of the functions taking values in R_p^n and the other taking values in C_p^n. More concretely, let us consider the sets

$$\begin{aligned} \mathrm{R}_\xi &= \big\{ k \,|\, \xi_k(i) = e_{1i} \big\}, \\ \mathrm{C}_\xi &= \big\{ k \,|\, \xi_k(i) = e_{i1} \big\}. \end{aligned}$$

Then, if C_ξ has s elements, the sum

$$\sum_{i_1, \ldots, i_d = 1}^n a_{i_1 \cdots i_d} \otimes \xi_1(i_1) \otimes \cdots \otimes \xi_d(i_d)$$

can be regarded as a $n^s \times n^{d-s}$ matrix with entries in $L_p(\varphi)$. Now we introduce a simpler notation already employed in [3]. Let $[m]$ be an abbreviation for the set $\{1, 2, \ldots, m\}$. Then, if $\mathbb{P}_d(2)$ denotes the set of partitions (α, β) of $[d]$ into two disjoint subsets α and β, we denote by

$$\pi_\alpha : [n]^d \to [n]^{|\alpha|}$$

the canonical projection given by $\pi_\alpha(\mathrm{I}) = (i_k)_{k \in \alpha}$ for any $\mathrm{I} = (i_1, \ldots, i_d) \in [n]^d$. This notation allows us express the inequality above in a much more understandable way. Namely, we have

$$(1) \quad \|\mathcal{S}_d(a)\|_{L_p(\varphi \otimes \tau_d)} \leq 2^d \max_{(\alpha, \beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{\mathrm{I} \in [n]^d} a_\mathrm{I} \otimes e_{\pi_\alpha(\mathrm{I}), \pi_\beta(\mathrm{I})} \Big\|_{L_p(\varphi; S_p^{n^d})} \right\}.$$

Remark 1.2. By the same arguments, the converse of (1) holds with constant 1.

1.2. **Fell's absorption principle in L_p.** In the following, we shall use repeatedly the following L_p-valued version of the so-called Fell's absorption principle. This result might be known as folklore in the theory. However, we include the proof since we were not able to provide a reference.

Absorption Principle in L_p. *Given a discrete group* G, *let us denote by* λ_G *the left regular representation of* G *and by* τ_G *the associated trace on the reduced* C^*-*algebra of* G. *Then, given any other unitary representation* $\pi : \mathrm{G} \to \pi(\mathrm{G})''$, *the following representations are unitarily equivalent*

$$\lambda_\mathrm{G} \otimes \pi \simeq \lambda_\mathrm{G} \otimes 1,$$

where 1 *stands for the trivial representation of* G *in* $\pi(\mathrm{G})''$. *Moreover, let us take any faithful normalized trace* ψ *on* $\pi(\mathrm{G})''$. *Then, given any function* $a : \mathrm{G} \to L_p(\varphi)$ *finitely supported on* G, *the following equality holds for* $1 \leq p \leq \infty$

$$(2) \quad \Big\| \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_\mathrm{G}(t) \otimes \pi(t) \Big\|_{L_p(\varphi \otimes \tau_\mathrm{G} \otimes \psi)} = \Big\| \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_\mathrm{G}(t) \Big\|_{L_p(\varphi \otimes \tau_\mathrm{G})}.$$

Proof. We refer the reader to Proposition 8.1 of [12] for a proof of the claimed unitary equivalence. For the second assertion, it is easy to reduce to the case when φ is a tracial state. Then, we fix a pair of operators

$$\begin{aligned} \mathsf{S} &= \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_\mathrm{G}(t), \\ \mathsf{T} &= \sum_{t \in \mathrm{G}} a(t) \otimes \lambda_\mathrm{G}(t) \otimes \pi(t), \end{aligned}$$

with $a : \mathrm{G} \to L_p(\varphi) \cap L_\infty(\varphi)$ finitely supported. Since ψ is normalized, it is clear that

$$(\varphi \otimes \tau_\mathrm{G})(\mathsf{S}) = \varphi\big(a(e)\big) = \varphi\big(a(e)\big)\psi\big(\pi(e)\big) = (\varphi \otimes \tau_\mathrm{G} \otimes \psi)(\mathsf{T}).$$

Let us consider the operators $x = \mathsf{S}^*\mathsf{S}$ and $z = \mathsf{T}^*\mathsf{T}$. Recalling that the equality above holds for any pair of operators of the same kind, we deduce that

$$(\varphi \otimes \tau_\mathrm{G})(x^n) = (\varphi \otimes \tau_\mathrm{G} \otimes \psi)(z^n)$$

for any integer $n \geq 0$. Now we let A_x (resp. A_z) be the (commutative) algebra generated by x (resp. z) in $L_\infty(\varphi \otimes \tau_\mathrm{G})$ (resp. $L_\infty(\varphi \otimes \tau_\mathrm{G} \otimes \psi)$). If μ_x (resp. μ_z) denotes the inherited probability measure on A_x (resp. A_z), we have

$$\int \mathrm{Q}(x)\, d\mu_x = \int \mathrm{Q}(z)\, d\mu_z,$$

for any polynomial Q. By the Stone-Weierstrass theorem, we conclude that the distribution of x with respect to μ_x coincides with the distribution of z with respect to μ_z. Therefore, $\|\mathsf{S}\|_{L_p(\varphi \otimes \tau_\mathrm{G})} = \|x\|_{L_{p/2}(\mu_x)} = \|z\|_{L_{p/2}(\mu_z)} = \|\mathsf{T}\|_{L_p(\varphi \otimes \tau_\mathrm{G} \otimes \psi)}$. $\square$

2. KHINTCHINE TYPE INEQUALITIES FOR $\mathcal{W}_p(n,d)$

We first prove the case of degree 2, since the notation is simpler and it contains almost all the ingredients employed in the proof of the general case. This will simplify the reading of the paper. After the proof of the general case, we study the same problem when redefining the spaces $\mathcal{K}_p(n,d)$ and $\mathcal{W}_p(n,d)$ so that we consider all the reduced words of length d.

2.1. The case of degree 2. As we did in the Introduction, we define $\mathcal{W}_p(n,2)$ to be the subspace of $L_p(\tau)$ generated by the operators $\lambda(g_i g_j)$ for $1 \leq i,j \leq n$. We shall also consider the spaces

$$\mathcal{K}_p(n,2) = \sum_{k=0}^{2} C_p^{n^k} \otimes_h R_p^{n^{d-k}} \qquad \text{for } 1 \leq p \leq 2,$$

$$\mathcal{K}_p(n,2) = \bigcap_{k=0}^{2} C_p^{n^k} \otimes_h R_p^{n^{d-k}} \qquad \text{for } 2 \leq p \leq \infty.$$

That is, if $\mathcal{A} = \big\{a_{ij} \mid 1 \leq i,j \leq n\big\} \subset L_p(\varphi)$, we consider the norms

$$\Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{1,ij} \Big\|_{L_p(\varphi; R_p^{n^2})}, \quad \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \Big\|_{L_p(\varphi; S_p^n)}, \quad \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij,1} \Big\|_{L_p(\varphi; C_p^{n^2})}.$$

We label them by $\|\mathcal{A}\|_0, \|\mathcal{A}\|_1$ and $\|\mathcal{A}\|_2$ respectively. Then, we have

$$\|\mathcal{A}\|_{L_p(\varphi; \mathcal{K}_p(n,2))} = \max\Big\{ \|\mathcal{A}\|_k \,\Big|\, 0 \leq k \leq 2 \Big\} \qquad \text{for } 2 \leq p \leq \infty.$$

This identity describes the operator space structure of $\mathcal{K}_p(n,2)$ for $2 \leq p \leq \infty$. On the other hand, the obvious modifications lead to a description of the operator space structure of $\mathcal{K}_p(n,2)$ for $1 \leq p \leq 2$. We shall prove the following result

Theorem 2.1. $\mathcal{K}_p(n,2)$ and $\mathcal{W}_p(n,2)$ are completely isomorphic for $1 \leq p \leq \infty$. More concretely, there exists an absolute constant c independent of n and p such that the following inequalities hold

$$\frac{1}{\mathrm{c}} \|\mathcal{A}\|_{L_p(\varphi; \mathcal{K}_p(n,2))} \leq \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes \lambda(g_i g_j) \Big\|_{L_p(\varphi \otimes \tau)} \leq \mathrm{c}\, \|\mathcal{A}\|_{L_p(\varphi; \mathcal{K}_p(n,2))}.$$

Moreover, the natural projection $\mathrm{P} : L_p(\tau) \rightarrow \mathcal{W}_p(n, 2)$ *is c.b. with* $\|\mathrm{P}\|_{cb} \leq \mathrm{c}$.

A similar statement to this result was proved by Haagerup (unpublished) and Buchholz [1] for $p = \infty$, see also Theorem 9.7.4 in [12]. Namely, the only difference is that Buchholz considered the whole set of words of length 2 instead of those words composed only of generators. Therefore, it is clear that Theorem 2.1 holds for $p = \infty$. By transposition, P also defines a completely bounded projection from $L_1(\tau)$ onto $\mathcal{W}_1(n, 2)$. Hence, the last assertion of Theorem 2.1 follows by complex interpolation. In particular, if p' stands for the conjugate exponent of p, it turns out that the dual $\mathcal{W}_p(n, 2)^*$ is completely isomorphic to $\mathcal{W}_{p'}(n, 2)$ for $1 \leq p \leq \infty$. On the other hand, it is obvious that $\mathcal{K}_p(n, 2)^*$ is completely isometric to $\mathcal{K}_{p'}(n, 2)$. In summary, it suffices to prove Theorem 2.1 for $2 \leq p \leq \infty$ since the case $1 \leq p \leq 2$ follows by duality.

Remark 2.2. From the previous considerations, it is clear that the space $\mathcal{W}_p(n, 2)$ is completely isomorphic to $(\mathcal{W}_\infty(n, 2), \mathcal{W}_2(n, 2))_\theta$ for $\theta = 2/p$. Hence, if we knew a priori that $\mathcal{K}_p(n, 2)$ is an interpolation family for $2 \leq p \leq \infty$, Theorem 2.1 would follow by complex interpolation between the obvious case for $p = 2$ and Buchholz's result. Conversely, Theorem 2.1 implies that $\mathcal{K}_p(n, 2)$ is an interpolation family for $2 \leq p \leq \infty$. The fact that $\mathcal{K}_p(n, 2)$ is an interpolation family was known to M. Junge [4] at the time of the preparation of this paper. He communicated to us the following sketch of his argument. Let τ_n stand for the normalized trace on the matrix algebra M_n and let us write $\mathcal{N}$ for the free product of algebras

$$\mathcal{N} = \overset{n}{\underset{k=1}{*}} \mathcal{A}_k \qquad \text{with} \qquad \mathcal{A}_k = (M_n \oplus_\infty M_n, \varphi_n) \qquad \text{and} \qquad \varphi_n = \frac{1}{2}(\tau_n \oplus \tau_n)$$

for $1 \leq k \leq n$. Then, if $\pi_k : \mathcal{A}_k \rightarrow \mathcal{N}$ denotes the natural inclusion, the map

$$x \in R_p^{n^2} \cap S_p^n \cap C_p^{n^2} \longmapsto \sum_{k=1}^n \pi_k((x, -x)) \in L_p(\mathcal{N})$$

is a complete isomorphism onto its image and the image is completely complemented in $L_p(\mathcal{N})$. Moreover, the constants appearing in the complete isomorphism and the projection considered above do not depend on n. This is based on the L_p version of the operator-valued Voiculescu's inequality given in [5]. Here we shall give a different proof that will be useful in the proof of the general case of degree d.

Now we focus on the proof for the case $2 \leq p \leq \infty$. The lower estimate is much simpler and it even holds with $\mathrm{c} = 1$. Namely, it suffices to check that

$$\|\mathcal{A}\|_k \leq \Big\| \sum_{i,j=1}^n a_{ij} \otimes \lambda(g_i g_j) \Big\|_{L_p(\varphi \otimes \tau)},$$

for any $0 \leq k \leq 2$. But we know that it holds trivially for $p = 2$ and also, by Buchholz's result, for $p = \infty$. Therefore, the lower estimate follows by complex interpolation since $R_p^{n^2}$, S_p^n and $C_p^{n^2}$ are interpolation families. Hence, we just need to prove the upper estimate. To that aim, we go back to Section 1, where we considered the group $\mathrm{G}_2 = \mathbf{F}_n \times \mathbf{F}_n$ and the subspace $\mathcal{W}_p(n) \otimes \mathcal{W}_p(n)$ of $L_p(\tau_2)$ generated by the family of operators

$$\lambda(g_i) \otimes \lambda(g_j).$$

We consider the subspace $\mathcal{V}_p(n,2)$ of $L_p(\tau_3)$ defined by

$$\mathcal{V}_p(n,2) = \Big\{ \sum_{i,j=1}^n \alpha_{ij}\,\lambda(g_i) \otimes \lambda(g_j) \otimes \lambda(g_i g_j) \in L_p(\tau_3) \,\big|\, \alpha_{ij} \in \mathbb{C} \Big\}.$$

Lemma 2.3. $\mathcal{V}_p(n,2)$ *is a completely complemented subspace of* $L_p(\tau_3)$.

Proof. We know that both $\mathcal{W}_p(n)$ and $\mathcal{W}_p(n,2)$ are completely complemented in $L_p(\tau)$. In particular, the projection which maps

$$\sum_{u,v,w \in \mathbf{F}_n} \alpha_{uvw}\,\lambda(u) \otimes \lambda(v) \otimes \lambda(w) \in L_p(\tau_3)$$

to the sum

$$\Sigma_2(\alpha) = \sum_{i,j,r,s=1}^n \alpha_{ijrs}\,\lambda(g_i) \otimes \lambda(g_j) \otimes \lambda(g_r g_s) \in \mathcal{W}_p(n) \otimes \mathcal{W}_p(n) \otimes \mathcal{W}_p(n,2),$$

is completely bounded with cb norm uniformly bounded in n and p. This shows that $\mathcal{W}_p(n) \otimes \mathcal{W}_p(n) \otimes \mathcal{W}_p(n,2)$ is completely complemented in $L_p(\tau_3)$. After that, we project onto $\mathcal{V}_p(n,2)$ by using the standard diagonal projection

$$\mathrm{P}\big(\Sigma_2(\alpha)\big) = \sum_{i,j,r,s=1}^n \int\int \varepsilon_i \delta_j \Big[\alpha_{ijrs}\,\lambda(g_i) \otimes \lambda(g_j) \otimes \lambda(g_r g_s) \Big] \varepsilon_r \delta_s\, d\mu(\varepsilon) d\mu(\delta),$$

where μ is the normalized counting measure on $\{-1,1\}^n$. Now, in the case $p = \infty$ it is not difficult to see that the norm of

$$\Sigma_2^{\pm}(\alpha) = \sum_{i,j,r,s=1}^n \varepsilon_i \delta_j \Big[\alpha_{ijrs}\,\lambda(g_i) \otimes \lambda(g_j) \otimes \lambda(g_r g_s) \Big] \varepsilon_r \delta_s$$

in $L_\infty(\tau_3)$ is equivalent (in the category of operator spaces) to that of $\Sigma_2(\alpha)$ in $L_\infty(\tau_3)$ for any choice of signs $\varepsilon_i, \delta_j, \varepsilon_r, \delta_s$. Moreover, the constants do not depend on the signs taken. Indeed, by Buchholz's result the norm of $\Sigma_2^{\pm}(\alpha)$ is equivalent to the norm of an element in $L_\infty(\tau_2; \mathcal{K}_\infty(n,d))$. In that case, the signs $\varepsilon_r \delta_s$ can be regarded as a Schur multiplier. Hence, ε_r and δ_s can be dropped by means of [12, Exercise 1.5]. On the other hand, the signs ε_i and δ_j disappear by applying Fell's absorption principle. Moreover, the norms of $\Sigma_2(\alpha)$ and $\Sigma_2^{\pm}(\alpha)$ clearly coincide for $p = 2$. Therefore, since $\mathcal{W}_p(n) \otimes \mathcal{W}_p(n) \otimes \mathcal{W}_p(n,2)$ is an interpolation family, both norms are equivalent for any $2 \leq p \leq \infty$. In particular, by Jensen's inequality we have

$$\big\| \mathrm{P}\big(\Sigma_2(\alpha)\big) \big\|_{L_p(\tau_3)} \leq \mathrm{c}\, \big\| \Sigma_2(\alpha) \big\|_{L_p(\tau_3)},$$

for some absolute constant c. Since the same holds taking values in S_p, it turns out that P is a completely bounded projection with constants independent of n and p. In summary, putting all together the result follows. This completes the proof. $\qquad\square$

The next step in the proof of Theorem 2.1 is to show that $\mathcal{W}_p(n,2)$ and $\mathcal{V}_p(n,2)$ are completely isomorphic operator spaces for any exponent $2 \leq p \leq \infty$. Namely, given a family $\mathcal{A} = \big\{ a_{ij} \,|\, 1 \leq i,j \leq n \big\}$ in $L_p(\varphi)$, we consider the sum

$$\Sigma_2(a) = \sum_{i,j=1}^n \lambda(g_i) \otimes \lambda(g_j) \otimes a_{ij} \otimes \lambda(g_i g_j) \in L_p(\tau_2 \otimes \varphi \otimes \tau).$$

Applying Buchholz's result to $\Sigma_2(a)$, we get the equivalence of norms

$$\big\|\Sigma_2(a)\big\|_{L_\infty(\tau_2\otimes\varphi\otimes\tau)} \simeq \max\Big\{\|\mathcal{A}'\|_k \;\Big|\; 0 \le k \le 2\Big\},$$

where $\mathcal{A}' = \big\{a'_{ij} \mid 1 \le i,j \le n\big\}$ with

$$a'_{ij} = \lambda(g_i) \otimes \lambda(g_j) \otimes a_{ij}.$$

Remark 2.4. Note that

$$\sum_{i,j=1}^n a'_{ij} \otimes e_{ij} = \Phi_1 \cdot \Big[\sum_{i,j=1}^n 1 \otimes 1 \otimes a_{ij} \otimes e_{ij}\Big] \cdot \Phi_2,$$

where

$$\begin{aligned}
\Phi_1 &= \sum_{i=1}^n \lambda(g_i) \otimes 1 \otimes 1 \otimes e_{ii}, \\
\Phi_2 &= \sum_{j=1}^n 1 \otimes \lambda(g_j) \otimes 1 \otimes e_{jj}.
\end{aligned}$$

Therefore, according to Remark 2.4 and since Φ_1 and Φ_2 are unitary, we obtain $\|\mathcal{A}\|_1 = \|\mathcal{A}'\|_1$. The obvious modifications lead to $\|\mathcal{A}\|_k = \|\mathcal{A}'\|_k$ for $k = 0$ and $k = 2$. In summary, if

$$\mathcal{S}_2(a) = \sum_{i,j=1}^n a_{ij} \otimes \lambda(g_i g_j),$$

we conclude that the norm of $\mathcal{S}_2(a)$ in $L_\infty(\varphi \otimes \tau)$ is equivalent (in the category of operator spaces) to the norm of $\Sigma_2(a)$ in $L_\infty(\tau_2\otimes\varphi\otimes\tau)$. Now recall that by Lemma 2.3, $\mathcal{V}_p(n,2)$ is an interpolation family for $2 \le p \le \infty$. Then, since the norm of these sums obviously coincide when $p = 2$, we get by complex interpolation

$$(3) \qquad \Big\|\sum_{i,j=1}^n a_{ij} \otimes \lambda(g_i g_j)\Big\|_p \le \mathrm{c}\Big\|\sum_{i,j=1}^n \lambda(g_i) \otimes \lambda(g_j) \otimes a_{ij} \otimes \lambda(g_i g_j)\Big\|_p$$

for any $2 \le p \le \infty$. Here c denotes an absolute constant independent of n and p. In what follows, the value of c might change from one instance to another. Now we apply the iteration of the Khintchine inequality (1) to inequality (3) to obtain

$$(4) \quad \Big\|\sum_{i,j=1}^n a_{ij} \otimes \lambda(g_i g_j)\Big\|_p \le \mathrm{c} \max_{(\alpha,\beta)\in\mathbb{P}_2(2)} \left\{\Big\|\sum_{\mathrm{I}\in[n]^2} \tilde{a}_{\mathrm{I}} \otimes e_{\pi_\alpha(\mathrm{I}),\pi_\beta(\mathrm{I})}\Big\|_{L_p(\varphi\otimes\tau;S_p^2)}\right\},$$

with $\tilde{a}_{ij} = a_{ij} \otimes \lambda(g_i g_j)$. Hence, we have four terms on the right

- $\Big\|\sum_{i,j=1}^n \tilde{a}_{ij} \otimes e_{1,ij}\Big\|_{L_p(\varphi\otimes\tau;R_p^{n^2})}$ $\bullet\ \Big\|\sum_{i,j=1}^n \tilde{a}_{ij} \otimes e_{ij}\Big\|_{L_p(\varphi\otimes\tau;S_p^n)}$

- $\Big\|\sum_{i,j=1}^n \tilde{a}_{ij} \otimes e_{ij,1}\Big\|_{L_p(\varphi\otimes\tau;C_p^{n^2})}$ $\bullet\ \Big\|\sum_{i,j=1}^n \tilde{a}_{ij} \otimes e_{ji}\Big\|_{L_p(\varphi\otimes\tau;S_p^n)}$

If $\widetilde{\mathcal{A}} = \{\tilde{a}_{ij} \mid 1 \le i,j \le n\}$, the first three terms are nothing but $\|\widetilde{\mathcal{A}}\|_0, \|\widetilde{\mathcal{A}}\|_1, \|\widetilde{\mathcal{A}}\|_2$. Arguing as above we have

$$\max\Big\{\|\mathcal{A}\|_k \;\Big|\; 0 \le k \le 2\Big\} = \max\Big\{\|\widetilde{\mathcal{A}}\|_k \;\Big|\; 0 \le k \le 2\Big\}.$$

In particular, the proof will be completed if we see that

$$\Big\| \sum_{i,j=1}^{n} \tilde{a}_{ij} \otimes e_{ji} \Big\|_{L_p(\varphi \otimes \tau; S_p^n)} \le c \max \Big\{ \|\mathcal{A}\|_k \,\Big|\, 0 \le k \le 2 \Big\}.$$

This is the content of the following Lemma. The proof is not complicated but, as we shall see in the next paragraph, it constitutes one of the key points in the proof of the general case.

Lemma 2.5. *The following inequality holds*

$$\Big\| \sum_{i,j=1}^{n} a_{ij} \otimes \lambda(g_i g_j) \otimes e_{ji} \Big\|_{L_p(\varphi \otimes \tau; S_p^n)}$$

$$\le \quad c \max \Big\{ \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{1,ij} \Big\|_{L_p(\varphi; R_p^{n^2})}, \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij,1} \Big\|_{L_p(\varphi; C_p^{n^2})} \Big\}.$$

Proof. When $p = \infty$ we can apply Buchholz's result to obtain

$$\Big\| \sum_{i,j=1}^{n} a_{ij} \otimes \lambda(g_i g_j) \otimes e_{ji} \Big\|_{L_\infty(\varphi \otimes \tau; S_\infty^n)} \le c \max \Big\{ A, B, C \Big\},$$

where the terms A and C are given by

$$A \quad = \quad \Big\| \Big(\sum_{i,j=1}^{n} (a_{ij} \otimes e_{ji})(a_{ij} \otimes e_{ji})^* \Big)^{1/2} \Big\|_\infty = \sup_{1 \le j \le n} \Big\| \Big(\sum_{i=1}^{n} a_{ij} a_{ij}^* \Big)^{1/2} \Big\|_\infty$$

$$C \quad = \quad \Big\| \Big(\sum_{i,j=1}^{n} (a_{ij} \otimes e_{ji})^* (a_{ij} \otimes e_{ji}) \Big)^{1/2} \Big\|_\infty = \sup_{1 \le i \le n} \Big\| \Big(\sum_{j=1}^{n} a_{ij}^* a_{ij} \Big)^{1/2} \Big\|_\infty.$$

In particular, we have the following estimates

$$A \quad \le \quad \Big\| \Big(\sum_{i,j=1}^{n} a_{ij} a_{ij}^* \Big)^{1/2} \Big\|_\infty = \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{1,ij} \Big\|_{L_\infty(\varphi; R_\infty^{n^2})}$$

$$C \quad \le \quad \Big\| \Big(\sum_{i,j=1}^{n} a_{ij}^* a_{ij} \Big)^{1/2} \Big\|_\infty = \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij,1} \Big\|_{L_\infty(\varphi; C_\infty^{n^2})}.$$

It remains to estimate the middle term B. We have

$$B \quad = \quad \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ji} \otimes e_{ij} \Big\|_{L_\infty(\varphi; S_\infty^n \otimes_{\min} S_\infty^n)}$$

$$= \quad \Big\| \Big(\sum_{i,j=1}^{n} a_{ij}^* a_{ij} \otimes e_{ii} \otimes e_{jj} \Big)^{1/2} \Big\|_{L_\infty(\varphi; S_\infty^n \otimes_{\min} S_\infty^n)}$$

$$= \quad \sup_{1 \le i,j \le n} \|a_{ij}\|_{L_\infty(\varphi)}.$$

Therefore, the term B is even smaller that A, C. This completes the proof for the case $p = \infty$. On the other hand, for the case $p = 2$ we clearly have an equality. Finally, we recall that

$$L_p \big(\varphi; S_p^n (\mathcal{W}_p(n,2))^{op} \big) \qquad \text{and} \qquad L_p \big(\varphi; R_p^{n^2} \cap C_p^{n^2} \big)$$

are interpolation families, see Section 9.5 in [12] for the details. Therefore, the result follows for $2 \leq p \leq \infty$ by complex interpolation. This completes the proof. $\square$

2.2. **The general case.** Now we prove the analog of Theorem 2.1 for any positive integer d. As we shall see, there exist a lot of similarities with the proof for degree 2. Therefore, we shall not repeat in detail those arguments which already appeared above. The statement of this result is the following.

Theorem 2.6. $\mathcal{K}_p(n,d)$ and $\mathcal{W}_p(n,d)$ are completely isomorphic for $1 \leq p \leq \infty$. More concretely, there exists an absolute constant c_d depending only on d such that the following inequalities hold

$$\frac{1}{\mathrm{c}_d} \left\| \mathcal{A} \right\|_{L_p(\varphi;\mathcal{K}_p(n,d))} \leq \left\| \sum_{i_1 \ldots i_d=1}^{n} a_{i_1 \cdots i_d} \otimes \lambda(g_{i_1} \cdots g_{i_d}) \right\|_{L_p(\varphi \otimes \tau)} \leq \mathrm{c}_d \left\| \mathcal{A} \right\|_{L_p(\varphi;\mathcal{K}_p(n,d))}.$$

Moreover, the natural projection $\mathrm{P} : L_p(\tau) \to \mathcal{W}_p(n,d)$ is c.b. with $\|\mathrm{P}\|_{cb} \leq \mathrm{c}_d$.

Before starting the proof of Theorem 2.6, we point out some remarks analogous to those given for Theorem 2.1. The arguments needed to prove the assertions given below are the same as the ones we used for the case of degree 2.

- Again, a similar statement to this result was proved by Buchholz [1] for $p = \infty$. Buchholz's considered the whole set of words of length d. Therefore, Theorem 2.6 holds for $p = \infty$ by Buchholz's more general statement.
- By transposition and complex interpolation, the last assertion of Theorem 2.6 follows. Hence, $\mathcal{W}_p(n,d)$ interpolates well up to complete isomorphism for $1 \leq p \leq \infty$. Moreover, $\mathcal{W}_{p'}(n,d)$ is completely isomorphic to $\mathcal{W}_p(n,d)^*$. In particular, since $\mathcal{K}_p(n,d)$ behaves well with respect to duality, it suffices to prove Theorem 2.6 for $2 \leq p \leq \infty$.
- Given a family of operators $\mathcal{A} = \left\{ a_{i_1 \cdots i_d} \mid 1 \leq i_k \leq n \right\}$ in $L_p(\varphi)$, we define

$$\|\mathcal{A}\|_k = \left\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1 \cdots i_d} \otimes e_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)} \right\|_{L_p\left(\varphi; C_p^{n^k} \otimes_h R_p^{n^{d-k}}\right)}.$$

- If $2 \leq p \leq \infty$, the lower estimate holds with $\mathrm{c}_d = 1$. Namely, it follows by Buchholz's result and complex interpolation since we are allowed to look separately at the inequalities

$$\|\mathcal{A}\|_k \leq \left\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1 \cdots i_d} \otimes \lambda(g_{i_1} \cdots g_{i_d}) \right\|_{L_p(\varphi \otimes \tau)}, \qquad \text{for } 0 \leq k \leq d.$$

Recall that $C_p^{n^k} \otimes_h R_p^{n^{d-k}}$ is a rectangular Schatten p-class of size $n^k \times n^{d-k}$. In particular, it follows that it is an interpolation family.

In what follows, c_d will denote a constant depending only on d and whose value might change from one instance to another. Now we start the proof of Theorem 2.6. By the considerations above, it suffices to prove the upper estimate for $2 \leq p \leq \infty$. Let G stand for the free group $\mathbf{F}_{nd}$ with nd generators and let ψ_{nd} be the natural trace on the reduced C^*-algebra of G. We label the generators by $g_{1k}, g_{2k}, \ldots, g_{dk}$ with $1 \leq k \leq n$. If λ_{G} denotes the left regular representation of G, we consider the family of operators

$$\mathcal{A}^\dagger = \left\{ a_{i_1 i_2 \cdots i_d} \otimes \lambda_{\mathrm{G}}(g_{1i_1} g_{2i_2} \cdots g_{di_d}) \,\Big|\, 1 \leq i_k \leq n \right\}.$$

That is, we take the image under λ_{G} of the set of reduced words of length d where the first letter is one of the first n generators, the second letter is one of the second n generators and so on. Let us write $\mathcal{W}_p(nd, n, d)$ to denote the subspace of $L_p(\psi_{nd})$ generated by the operators $\lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d})$. Then, we have

$$(5) \quad \Big\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1\cdots i_d} \otimes \lambda(g_{i_1} \cdots g_{i_d}) \Big\|_p \leq \mathrm{c}_d \Big\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1\cdots i_d} \otimes \lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d}) \Big\|_p.$$

Namely, for $p = \infty$ this follows by Buchholz's result. Then, since both $\mathcal{W}_p(n, d)$ and $\mathcal{W}_p(nd, n, d)$ are interpolation families, inequality (5) holds for any $2 \leq p \leq \infty$ by complex interpolation. Now, proceeding as in the previous paragraph, we consider the group $\mathrm{G}_d = \mathbf{F}_n \times \mathbf{F}_n \times \cdots \times \mathbf{F}_n$ and the subspace $\mathcal{W}_p(n)^{\otimes d}$ of $L_p(\tau_d)$ generated by the family of operators

$$\lambda(g_{i_1}) \otimes \lambda(g_{i_2}) \otimes \cdots \otimes \lambda(g_{i_d}).$$

Then we define $\mathcal{V}_p(n, d)$ as the subspace of $L_p(\tau_d \otimes \psi_{nd})$ defined by

$$\mathcal{V}_p(n, d) = \Big\{ \sum_{i_1,\ldots,i_d=1}^{n} \alpha_{i_1\cdots i_d} \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes \lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d}) \,\big|\, \alpha_{i_1\cdots i_d} \in \mathbb{C} \Big\}.$$

Recalling that both $\mathcal{W}_p(n)$ and $\mathcal{W}_p(nd, n, d)$ are completely complemented in their respective L_p spaces, it can be showed just like in Lemma 2.3 that $\mathcal{V}_p(n, d)$ is completely complemented in $L_p(\tau_d \otimes \psi_{nd})$ with constants depending only on the degree d. The next step in the proof is to obtain the analog of inequality (3). Namely, the inequality

$$(6) \quad \Big\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1\cdots i_d} \otimes \lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d}) \Big\|_p$$

$$\leq \quad \mathrm{c}_d \Big\| \sum_{i_1,\ldots,i_d=1}^{n} \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes a_{i_1\cdots i_d} \otimes \lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d}) \Big\|_p,$$

for $2 \leq p \leq \infty$. The proof of this inequality is identical to the one given for inequality (3). Indeed, we have just showed that both $\mathcal{W}_p(nd, n, d)$ and $\mathcal{V}_p(n, d)$ are interpolation families. Therefore, the proof of (6) works by complex interpolation between the obvious case $p = 2$ and the case $p = \infty$. When $p = \infty$, the idea consists in applying to both terms in (6) Buchholz's result for degree d. Then, inequality (6) becomes equivalent to

$$\max \Big\{ \|\mathcal{A}\|_k \,\big|\, 0 \leq k \leq d \Big\} \leq \mathrm{c}_d \max \Big\{ \|\mathcal{A}'\|_k \,\big|\, 0 \leq k \leq d \Big\},$$

where $\mathcal{A}'$ is given by

$$\mathcal{A}' = \Big\{ a_{i_1 i_2 \cdots i_d} \otimes \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \,\big|\, 1 \leq i_k \leq n \Big\}.$$

Remark 2.7. Again it is clear that $\|\mathcal{A}\|_k = \|\mathcal{A}'\|_k$ for any $0 \leq k \leq d$. Namely, as we pointed out in Remark 2.4, the sum

$$\sum_{i_1,\ldots,i_d=1}^{n} \lambda(g_{i_1}) \otimes \cdots \otimes \lambda(g_{i_d}) \otimes a_{i_1 i_2 \cdots i_d} \otimes e_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)}$$

factorizes as

$$\Phi_1 \cdot \Big[\sum_{i_1,\ldots,i_d=1}^{n} a_{i_1 i_2 \cdots i_d} \otimes e_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)} \Big] \cdot \Phi_2,$$

where Φ_1 (resp. Φ_2) is a $n^k \times n^k$ (resp. $n^{d-k} \times n^{d-k}$) unitary mapping.

This completes the proof of inequality (6). Another possible approach to (6) is given by iterating Fell's absorption principle d times, with the suitable choice for π each time. We leave the details to the reader. Notice that Fell's absorption principle shows that (6) is in fact an equality with $c_d = 1$. Then we apply the iteration of Khintchine inequality (1) to inequality (6). This gives

$$(7) \quad \Big\| \sum_{i_1,\ldots,i_d=1}^{n} a_{i_1 \cdots i_d}^{\dagger} \Big\|_p \leq c_d \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \Big\| \sum_{I \in [n]^d} a_I^{\dagger} \otimes e_{\pi_\alpha(I),\pi_\beta(I)} \Big\|_{L_p(\varphi \otimes \psi_{nd}; S_p^{n^d})} \right\},$$

with $a_{i_1 \cdots i_d}^{\dagger} = a_{i_1 \cdots i_d} \otimes \lambda_{\mathrm{G}}(g_{1i_1} \cdots g_{di_d})$. Now we have 2^d terms on the right. Before going on, let us look for a moment at the norm of the space $\mathcal{K}_p(n,d)$. Concretely, if we rewrite the definition of $\mathcal{K}_p(n,d)$ for $2 \leq p \leq \infty$ with the notation employed at the end of Section 1, we obtain

$$(8) \quad \|\mathcal{A}\|_{L_p(\varphi; \mathcal{K}_p(n,d))} = \max_{0 \leq k \leq d} \left\{ \Big\| \sum_{I \in [n]^k} \sum_{J \in [n]^{d-k}} a_{IJ} \otimes e_{I,J} \Big\|_{L_p(\varphi; C_p^{n^k} \otimes_h R_p^{n^{d-k}})} \right\}.$$

To complete the proof of Theorem 2.6 it remains to see that the right side of (7) is controlled by the right side of (8). Here is where the proof of the general case differs from that of degree 2. Let us sketch briefly how we shall conclude the proof. If $\mathrm{RHS}_{(7)}$ stands for the right hand side of (7), we shall prove that

$$(9) \quad \mathrm{RHS}_{(7)} \leq c_d \max_{0 \leq k \leq d} \left\{ \Big\| \sum_{I \in [n]^k} \sum_{J \in [n]^{d-k}} a_{IJ}^{\dagger} \otimes e_{I,J} \Big\|_{L_p(\varphi \otimes \psi_{nd}; C_p^{n^k} \otimes_h R_p^{n^{d-k}})} \right\}.$$

Assuming we have (9), the proof is completed since the right side of (9) coincides with the right side of (8). Namely, it clearly follows by the same factorization argument as above. That is,

$$\sum_{I \in [n]^k} \sum_{J \in [n]^{d-k}} a_{IJ}^{\dagger} \otimes e_{I,J} = \Phi_1 \cdot \Big[\sum_{I \in [n]^k} \sum_{J \in [n]^{d-k}} a_{IJ} \otimes e_{I,J} \Big] \cdot \Phi_2,$$

with Φ_1 and Φ_2 suitably chosen unitary mappings. In summary, it remains to prove inequality (9). Recall that the $d+1$ terms which appear on the right hand side of (9) also appear on its left hand side. They correspond to $\alpha = \emptyset, [1], [2], \ldots, [d]$. The remaining terms correspond to certain transpositions just like the term we bounded with the aid of Lemma 2.5. Thus, we just need to show that the transposed terms are controlled by the non-transposed ones. To that aim, we need to introduce some notation. Given $(\alpha,\beta) \in \mathbb{P}_d(2)$, we define

$$\mathrm{a} = \max \Big\{ k \mid k \in \alpha \Big\},$$
$$\mathrm{b} = \min \Big\{ k \mid k \in \beta \Big\}.$$

We define $\mathrm{a} = 0$ for $\alpha = \emptyset$ and $\mathrm{b} = d+1$ for $\beta = \emptyset$. We shall say that (α,β) is *non-transposed* whenever $\mathrm{a} < \mathrm{b}$ and (α,β) will be called *transposed* otherwise. Let

us introduce the number $T(\alpha,\beta) = a - b$, so that (α,β) is transposed whenever $T(\alpha,\beta) > 0$. The proof of the remaining inequality lies on the following claim.

Claim 2.8. *Let (α,β) be a transposed element of $\mathbb{P}_d(2)$. Then, we have*

$$\Big\| \sum_{I \in [n]^d} a_I^\dagger \otimes e_{\pi_\alpha(I),\pi_\beta(I)} \Big\|_p$$

$$\leq \quad c_d \max\left\{ \Big\| \sum_{I \in [n]^d} a_I^\dagger \otimes e_{\pi_{\alpha_1}(I),\pi_{\beta_1}(I)} \Big\|_p , \Big\| \sum_{I \in [n]^d} a_I^\dagger \otimes e_{\pi_{\alpha_2}(I),\pi_{\beta_2}(I)} \Big\|_p \right\},$$

for some (α_1,β_1) and (α_2,β_2) in $\mathbb{P}_d(2)$ satisfying

$$\begin{aligned} T(\alpha_1,\beta_1) &< T(\alpha,\beta), \\ T(\alpha_2,\beta_2) &< T(\alpha,\beta). \end{aligned}$$

Remark 2.9. Clearly, iteration of Claim 2.8 concludes the proof of Theorem 2.6.

Proof. If $G = \mathbf{F}_{nd}$, we define $\pi : G \to \mathcal{B}(\ell_2(G))$ by

$$\pi(g_{rs}) = \begin{cases} \lambda_G(g_{rs}) & \text{if } r = a, b \\ 1 & \text{otherwise} \end{cases},$$

where λ_G denotes the left regular representation of G and $1 \leq s \leq n$. Clearly, the mapping π extends to a unitary representation of G. Therefore, by Fell's absorption principle we have

$$\Big\| \sum_{I \in [n]^d} a_I^\dagger \otimes e_{\pi_\alpha(I),\pi_\beta(I)} \Big\|_p = \Big\| \sum_{I \in [n]^d} a_I^\dagger \otimes \lambda_G(g_{bi_b} g_{ai_a}) \otimes e_{\pi_\alpha(I),\pi_\beta(I)} \Big\|_p.$$

Recall that $\alpha, \beta \neq \emptyset$ since otherwise (α,β) would be non-transposed. Hence we can assume that $1 \leq b < a \leq d$. Then we define

$$\begin{array}{rclcrcl} \alpha_1 &=& \alpha \setminus \{a\} & \quad & \alpha_2 &=& \alpha \cup \{b\} \\ \beta_1 &=& \beta \cup \{a\} & & \beta_2 &=& \beta \setminus \{b\}. \end{array}$$

In particular, we can write

$$(10) \quad \sum_{I \in [n]^d} a_I^\dagger \otimes \lambda_G(g_{bi_b} g_{ai_a}) \otimes e_{\pi_\alpha(I),\pi_\beta(I)} = \sum_{i_a,i_b=1}^{n} x_{i_b i_a} \otimes \lambda_G(g_{bi_b} g_{ai_a}) \otimes e_{i_a,i_b},$$

where $x_{i_b i_a}$ has the following form

$$x_{i_b i_a} = \sum_{i_1,\ldots,i_{b-1}=1}^{n} \sum_{i_{b+1},\ldots,i_{a-1}=1}^{n} \sum_{i_{a+1},\ldots,i_d=1}^{n} a_{i_1 \cdots i_d}^\dagger \otimes e_{\pi_{\alpha_1}(i_1 \cdots i_d),\pi_{\beta_2}(i_1 \cdots i_d)},$$

with the obvious modifications on the sum indices if $a = d$ or $b = 1$ or $a = b + 1$. On the other hand, since $x_{i_b i_a}$ lives in some non-commutative L_p space $L_p(\varphi)$, we can apply Lemma 2.5 to the right hand side of (10) to obtain

$$\Big\| \sum_{i_a,i_b=1}^{n} x_{i_b i_a} \otimes \lambda_G(g_{bi_b} g_{ai_a}) \otimes e_{i_a,i_b} \Big\|_p$$

$$\leq \quad c_d \max\left\{ \Big\| \sum_{i_a,i_b=1}^{n} x_{i_b i_a} \otimes e_{1,i_b i_a} \Big\|_{L_p(\varphi; R_p^{n^2})} , \Big\| \sum_{i_a,i_b=1}^{n} x_{i_b i_a} \otimes e_{i_b i_a,1} \Big\|_{L_p(\varphi; C_p^{n^2})} \right\}.$$

Finally, we observe that

$$\sum_{i_{\mathrm a},i_{\mathrm b}=1}^{n} x_{i_{\mathrm b} i_{\mathrm a}} \otimes e_{1,i_{\mathrm b} i_{\mathrm a}} \;=\; \sum_{\mathrm I \in [n]^d} a_{\mathrm I}^{\dagger} \otimes e_{\pi_{\alpha_1}(\mathrm I),\pi_{\beta_1}(\mathrm I)},$$

$$\sum_{i_{\mathrm a},i_{\mathrm b}=1}^{n} x_{i_{\mathrm b} i_{\mathrm a}} \otimes e_{i_{\mathrm b} i_{\mathrm a},1} \;=\; \sum_{\mathrm I \in [n]^d} a_{\mathrm I}^{\dagger} \otimes e_{\pi_{\alpha_2}(\mathrm I),\pi_{\beta_2}(\mathrm I)}.$$

This completes the proof since it is clear that $\mathrm T(\alpha_1,\beta_1), \mathrm T(\alpha_2,\beta_2) < \mathrm T(\alpha,\beta)$. $\qquad\square$

Remark 2.10. A consequence of our proof is the following equivalence of norms

$$\max_{0 \le k \le d} \left\{ \Big\| \sum_{\mathrm I \in [n]^k} \sum_{\mathrm J \in [n]^{d-k}} a_{\mathrm{IJ}}^{\dagger} \otimes e_{\mathrm I,\mathrm J} \Big\|_p \right\} \simeq \max_{(\alpha,\beta) \in \mathbb P_d(2)} \left\{ \Big\| \sum_{\mathrm I \in [n]^d} a_{\mathrm I}^{\dagger} \otimes e_{\pi_\alpha(\mathrm I),\pi_\beta(\mathrm I)} \Big\|_p \right\}.$$

This means that, for operators of the form $a_{i_1 \cdots i_d} \otimes \lambda(g_{1i_1} \cdots g_{di_d})$, the transposed terms are controlled by the non-transposed ones. The presence of $\lambda(g_{1i_1} \cdots g_{di_d})$ is essential to apply Fell's absorption principle. Moreover, this equivalence is no longer true for arbitrary families of operators. A simple counterexample is given by the 2-indexed family $a_{ij} = e_{ji} \in S_p^n$. Namely, it is easy to check that

$$\Big\| \sum_{i,j=1}^{n} e_{ji} \otimes e_{1,ij} \Big\|_{S_p^n(R_p^{n^2})} = n^{1/2+1/p}, \qquad\qquad \Big\| \sum_{i,j=1}^{n} e_{ji} \otimes e_{ij} \Big\|_{S_p^n(S_p^n)} = n^{2/p},$$

$$\Big\| \sum_{i,j=1}^{n} e_{ji} \otimes e_{ij,1} \Big\|_{S_p^n(C_p^{n^2})} = n^{1/2+1/p}, \qquad\qquad \Big\| \sum_{i,j=1}^{n} e_{ji} \otimes e_{ji} \Big\|_{S_p^n(S_p^n)} = n.$$

In other words, the non-transposed term is not controlled by the transposed ones. In particular, we conclude that the estimation given in Section 1 for the iteration of Khintchine inequality is not equivalent to that provided by Theorem 2.6.

2.3. **The main result for words of length** d**.** As we have pointed out several times in this paper, Buchholz's result also holds for the whole set of reduced words of length d. Therefore, it is natural to seek for the analog of Theorem 2.6 in this case. We shall need the following modified version of Lemma 2.5.

Lemma 2.11. *The following inequality holds for any exponent* $2 \le p \le \infty$

$$\Big\| \sum_{i,j=1}^{n} a_{ij} \otimes \lambda(g_i g_j^{-1}) \otimes e_{ji} \Big\|_{L_p(\varphi \otimes \tau; S_p^n)}$$

$$\le \quad \mathrm c \max \left\{ \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{1,ij} \Big\|_{L_p(\varphi; R_p^{n^2})}, \Big\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij,1} \Big\|_{L_p(\varphi; C_p^{n^2})} \right\}.$$

Proof. We can split the sum on the left hand side as follows

$$\sum_{i,j=1}^{n} a_{ij} \otimes \lambda(g_i g_j^{-1}) \otimes e_{ji} = \sum_{k=1}^{n} a_{kk} \otimes 1 \otimes e_{kk} + \sum_{1 \le i \ne j \le n} a_{ij} \otimes \lambda(g_i g_j^{-1}) \otimes e_{ji}.$$

Since $g_i g_j^{-1}$ is a reduced word of length 2 whenever $i \ne j$, the arguments employed in the proof of Lemma 2.5 apply to estimate the norm of the second sum on the right in $L_p(\varphi \otimes \tau; S_p^n)$. For the first sum, the estimation is obvious since it follows by complex interpolation when we replace max by min above. $\qquad\square$

Before stating the announced result, we redefine the analogs of the operator spaces $\mathcal{K}_p(n,d)$ and $\mathcal{W}_p(n,d)$ in this new framework. To that aim, let us define the elements $h_1, h_2, \ldots, h_{2n}$ of $\mathbf{F}_n$ as follows

$$
h_k = \begin{cases} g_k & \text{if } 1 \le k \le n \\ g_{k-n}^{-1} & \text{otherwise} \end{cases},
$$

where $g_1, g_2, \ldots, g_n$ are the generators of $\mathbf{F}_n$. In this paragraph, $\mathsf{W}_p(n,d)$ will denote the subspace of $L_p(\tau)$ generated by the image under λ of the set of reduced words of length d. In other words, an element of $\mathsf{W}_p(n,d)$ has the form

$$
\sum_{|t|=d} \alpha_t \lambda(t) = \sum_{i_1,\ldots,i_d=1}^{2n} \alpha_{i_1 i_2 \cdots i_d} \lambda(h_{i_1} h_{i_2} \cdots h_{i_d}) \in L_p(\tau),
$$

where the family of scalars

$$
\mathsf{A} = \left\{ \alpha_{i_1 i_2 \cdots i_d} \,\middle|\, 1 \le i_k \le 2n \right\},
$$

satisfies the following *cancellation property*

(11) $$ \alpha_{i_1 i_2 \cdots i_d} = 0 \quad \text{if} \quad i_s \equiv n + i_{s+1} \,(\mathrm{mod}\, 2n), $$

for some $1 \le s < d$. Note that the cancellation property (11) is taken so that we only consider reduced words of length d. This notation will allow us to handle the space $\mathsf{W}_p(n,d)$ just like $\mathcal{W}_p(n,d)$ in the previous paragraph. On the other hand, as pointed out in the Introduction, we can regard the family A as a $(2n)^k \times (2n)^{d-k}$ matrix as follows

$$
\mathsf{A}_k = \left(\alpha_{(i_1 \cdots i_k),(i_{k+1} \cdots i_d)} \right) \in C_p^{(2n)^k} \otimes_h R_p^{(2n)^{d-k}}.
$$

Then, we define $\mathsf{K}_p(n,d)$ as the subspace of

$$
\mathcal{J}_p(n,d) = \sum_{k=0}^{d} C_p^{(2n)^k} \otimes_h R_p^{(2n)^{d-k}} \qquad \text{if } 1 \le p \le 2,
$$
$$
\mathcal{J}_p(n,d) = \bigcap_{k=0}^{d} C_p^{(2n)^k} \otimes_h R_p^{(2n)^{d-k}} \qquad \text{if } 2 \le p \le \infty,
$$

where certain entries are zero according to (11). We shall prove the following result.

Corollary 2.12. $\mathsf{K}_p(n,d)$ and $\mathsf{W}_p(n,d)$ *are completely isomorphic for* $1 \le p \le \infty$. *More concretely, there exists an absolute constant* c_d *depending only on* d *such that the following inequalities hold for any family* $\mathcal{A}$ *of operators in* $L_p(\varphi)$ *satisfying the cancellation property* (11)

$$
\frac{1}{\mathrm{c}_d} \|\mathcal{A}\|_{L_p(\varphi;\mathsf{K}_p(n,d))} \le \left\| \sum_{i_1 \ldots i_d=1}^{2n} a_{i_1 \cdots i_d} \otimes \lambda(h_{i_1} \cdots h_{i_d}) \right\|_{L_p(\varphi \otimes \tau)} \le \mathrm{c}_d \|\mathcal{A}\|_{L_p(\varphi;\mathsf{K}_p(n,d))}.
$$

Moreover, the natural projection $\mathrm{P} : L_p(\tau) \to \mathsf{W}_p(n,d)$ *is c.b. with* $\|\mathrm{P}\|_{cb} \le \mathrm{c}_d$.

The proof we are giving is quite similar to that of Theorem 2.6. In particular, we shall skip those arguments which already appeared above. The first remark is that the case $p = \infty$ is exactly the content of Buchholz's result in [1]. Therefore, arguing as we did after the statement of Theorem 2.6, we have:

- The last assertion of Corollary 2.12 holds.
- The spaces $\mathsf{W}_p(n,d)$ are an interpolation family for $1 \le p \le \infty$.

- The dual of the space $\mathsf{W}_p(n,d)$ is completely isomorphic to $\mathsf{W}_{p'}(n,d)$.
- The case $1 \le p \le 2$ in Corollary 2.12 follows from the case $2 \le p \le \infty$.
- The lower estimate for the case $2 \le p \le \infty$ holds with some constant c_d.

The last two points use that $\mathsf{K}_p(n,d)$ is completely complemented in $\mathcal{J}_p(n,d)$ with constants independent on n and p. The proof of this fact is simple. Indeed, by transposition and complex interpolation it suffices to prove it for $p = \infty$. Now, since $\mathcal{J}_\infty(n,d)$ is an intersection space, we just need to see it for each space appearing in the intersection. Let $\mathcal{I}$ be the set of indices in $[2n]^d$ satisfying the cancellation property (11) and let $\mathsf{H}_\infty(n,d)$ be the subspace of elements of $\mathcal{J}_\infty(n,d)$ supported in $\mathcal{I}$. Then, it is clear that the projection Q onto $\mathsf{H}_\infty(n,d)$ is completely bounded since it decomposes as a sum of $d-1$ *diagonal* projections. In particular, the projection P onto $\mathsf{K}_p(n,d)$ is also completely bounded.

Sketch of the proof. We only prove the upper estimate for $2 \le p \le \infty$. As above, let us write G for the free group $\mathbf{F}_{nd}$ and ψ_{nd} for the standard trace on its reduced C^*-algebra. Now, following the notation just introduced, we label the set of generators and its inverses by $h_{1k}, h_{2k}, \ldots, h_{dk}$ with $1 \le k \le 2n$. If λ_{G} denotes the left regular representation of G, we consider the family of operators

$$\mathcal{A}^\dagger = \left\{ a_{i_1 i_2 \cdots i_d} \otimes \lambda_{\mathsf{G}}(h_{1i_1} h_{2i_2} \cdots h_{di_d}) \,\middle|\, 1 \le i_k \le 2n \right\}.$$

The following chain of inequalities can be proved applying the same arguments as for the proof of Theorem 2.6. Namely, essentially we use Buchholz's result, complex interpolation and the iteration of Khintchine inequality described in Section 1.

$$(12) \quad \left\| \sum_{i_1,\ldots,i_d=1}^{2n} a_{i_1 \cdots i_d} \otimes \lambda(h_{i_1} \cdots h_{i_d}) \right\|_p$$

$$\le \; \mathsf{c}_d \left\| \sum_{i_1,\ldots,i_d=1}^{2n} a_{i_1 \cdots i_d} \otimes \lambda_{\mathsf{G}}(h_{1i_1} \cdots h_{di_d}) \right\|_p$$

$$\le \; \mathsf{c}_d \left\| \sum_{i_1,\ldots,i_d=1}^{2n} \lambda(h_{i_1}) \otimes \cdots \otimes \lambda(h_{i_d}) \otimes a_{i_1 \cdots i_d} \otimes \lambda_{\mathsf{G}}(h_{1i_1} \cdots h_{di_d}) \right\|_p$$

$$\le \; \mathsf{c}_d \max_{(\alpha,\beta) \in \mathbb{P}_d(2)} \left\{ \left\| \sum_{\mathrm{I} \in [2n]^d} a_{\mathrm{I}} \otimes \lambda_{\mathsf{G}}(h_{1i_1} \cdots h_{di_d}) \otimes e_{\pi_\alpha(\mathrm{I}),\pi_\beta(\mathrm{I})} \right\|_p \right\}.$$

Then, the proof reduces again to the proof of

$$(13) \quad \mathrm{RHS}_{(12)} \le \mathsf{c}_d \max_{0 \le k \le d} \left\{ \left\| \sum_{\mathrm{I} \in [2n]^k} \sum_{\mathrm{J} \in [2n]^{d-k}} a_{\mathrm{IJ}} \otimes \lambda_{\mathsf{G}}(h_{1i_1} \cdots h_{di_d}) \otimes e_{\mathrm{I},\mathrm{J}} \right\|_p \right\}.$$

By Fell's absorption principle, we have

$$\left\| \sum_{\mathrm{I} \in [2n]^d} a_{\mathrm{I}}^\dagger \otimes e_{\pi_\alpha(\mathrm{I}),\pi_\beta(\mathrm{I})} \right\|_p = \left\| \sum_{\mathrm{I} \in [2n]^d} a_{\mathrm{I}}^\dagger \otimes \lambda_{\mathsf{G}}(h_{\mathrm{b}i_\mathrm{b}} h_{\mathrm{a}i_\mathrm{a}}) \otimes e_{\pi_\alpha(\mathrm{I}),\pi_\beta(\mathrm{I})} \right\|_p,$$

with $a_{i_1 \cdots i_d}^\dagger = a_{i_1 \cdots i_d} \otimes \lambda_{\mathsf{G}}(h_{1i_1} \cdots h_{di_d})$. Moreover, we can write

$$(14) \quad \sum_{\mathrm{I} \in [2n]^d} a_{\mathrm{I}}^\dagger \otimes \lambda_{\mathsf{G}}(h_{\mathrm{b}i_\mathrm{b}} h_{\mathrm{a}i_\mathrm{a}}) \otimes e_{\pi_\alpha(\mathrm{I}),\pi_\beta(\mathrm{I})} = \sum_{i_\mathrm{a},i_\mathrm{b}=1}^{2n} x_{i_\mathrm{b}i_\mathrm{a}} \otimes \lambda_{\mathsf{G}}(h_{\mathrm{b}i_\mathrm{b}} h_{\mathrm{a}i_\mathrm{a}}) \otimes e_{i_\mathrm{a},i_\mathrm{b}},$$

where $x_{i_b i_a}$ has the form

$$x_{i_b i_a} = \sum_{i_1,\dots,i_{b-1}=1}^{2n} \sum_{i_{b+1},\dots,i_{a-1}=1}^{2n} \sum_{i_{a+1},\dots,i_d=1}^{2n} a^\dagger_{i_1 \cdots i_d} \otimes e_{\pi_{\alpha_1}(i_1 \cdots i_d), \pi_{\beta_2}(i_1 \cdots i_d)},$$

with the obvious modifications if a $= d$ or b $= 1$ or a $=$ b $+ 1$. Recall that the operators x_{i_b, i_a} do not necessarily satisfy the cancellation property (11). However, we can decompose the sum in (14) as follows

$$\sum_{i_a, i_b = 1}^{2n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b} = \sum_{i_a, i_b = 1}^{n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b}$$

$$+ \sum_{i_a = 1}^{n} \sum_{i_b = n+1}^{2n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b}$$

$$+ \sum_{i_b = 1}^{n} \sum_{i_a = n+1}^{2n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b}$$

$$+ \sum_{i_a, i_b = n+1}^{2n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b}.$$

Then it is clear than Lemma 2.5 applies to the first and the fourth sums while Lemma 2.11 applies to the second and third sums. In summary, we have

$$\left\| \sum_{i_a, i_b = 1}^{2n} x_{i_b i_a} \otimes \lambda_G(h_{b i_b} h_{a i_a}) \otimes e_{i_a, i_b} \right\|_p$$

$$\leq c_d \max \left\{ \left\| \sum_{i_a, i_b = 1}^{2n} x_{i_b i_a} \otimes e_{1, i_b i_a} \right\|_{L_p(\varphi; R_p^{n^2})}, \left\| \sum_{i_a, i_b = 1}^{2n} x_{i_b i_a} \otimes e_{i_b i_a, 1} \right\|_{L_p(\varphi; C_p^{n^2})} \right\}.$$

Finally, we conclude as in Claim 2.8. This completes the proof of (13).   □

Remark 2.13. In Voiculescu's free probability theory, stochastic independence of random variables is replaced by freeness of non-commutative random variables. In this setting, the Wigner's probability distribution

$$d\mu_W(t) = 1_{[-2,2]} \frac{\sqrt{4 - t^2}}{2\pi} \, dt$$

plays a crucial role. Namely, given a free family $x_1, x_2, \dots, x_n$ of self-adjoint random variables in a non-commutative probability space $(\mathcal{M}, \tau)$, we say that $x_1, x_2, \dots, x_n$ is a *free semi-circular system* if each x_k is equipped with Wigner's distribution. This family is the free analog of a system of n independent standard real-valued gaussian random variables. Explicit constructions of free semi-circular systems are available by means of the creation and annihilation operators on the full Fock space, see [12, 15] for more on this. The free analog of n independent complex-valued gaussians is now given by taking

$$z_k = \frac{1}{\sqrt{2}}(x_k' + ix_k''),$$

with $x_1', x_1'', x_2', x_2'', \dots, x_n', x_n''$ being a free semi-circular system. This new system is called a *free circular system*. At this point, it is natural to guess that the analog of

Theorem 2.6 should hold when we replace free generators by free circular random variables. Indeed, as it was pointed out in the Introduction, the family of operators $\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)$ is the free analog of the sequence of Rademacher functions $r_1, r_2, \ldots, r_n$. Therefore, a free version of the central limit theorem is exactly what is needed here. A precise statement of this result can be found in [15] and supports the previous identification between real-valued gaussians and semi-circular random variables. Although we are not giving the details, it can be checked that the central limit theorem for free random variables provides the analog of Theorem 2.6 for free circular variables. In other words, if we replace the operators $\lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$ by the products $z_{i_1} z_{i_2} \cdots z_{i_d}$ in Theorem 2.6, then the same conclusions hold. In passing, we also refer the interested reader to Nou's paper [9], which contains the analog of Buchholz's result for q-gaussian randon variables.

Remark 2.14. The paper [11] deals with the notion of p-orthogonal sums in non commutative L_p spaces. Applying some combinatorial techniques, it is shown that the Khintchine type inequality that applies for $\mathcal{W}_p(n)$ majorizes the behaviour of a much larger class of operators, the so-called p-orthogonal sums, for any even integer p. On the other hand, the bounds given in Section 1 for $\mathcal{W}(n)^{\otimes d}$ constitute an upper bound of a more general family of operators. Namely, let $(\mathcal{M}, \tau)$ be a von Neumann algebra endowed with a standard trace satisfying $\tau(1) = 1$ and let $L_p(\tau)$ be the associated non-commutative L_p space. Let Γ stand for the product set $[n] \times \cdots \times [n]$ with d factors. Then, given an even integer p and a family $f = (f_\gamma)_{\gamma \in \Gamma}$ of operators in $L_p(\tau)$ indexed by Γ, we shall say that f is p-orthogonal with d indices if

$$\tau\big(f_{h(1)}^* f_{h(2)} f_{h(3)}^* f_{h(4)} \cdots f_{h(p-1)}^* f_{h(p)}\big) = 0$$

whenever the function $h : \{1, 2, \ldots, p\} \to \Gamma$ has an injective projection. In other words, whenever the coordinate function $\pi_k \circ h : \{1, 2, \ldots, p\} \to [n]$ is an injective function for some index $1 \le k \le d$. The paper [10] extends the results in [11] to this more general setting by studying the norm in $L_p(\tau)$ of the sum

$$\sum_{\gamma \in \Gamma} f_\gamma.$$

More concretely, the norm of this sum in $L_p(\tau)$ is bounded above by the expressions given in Section 1, see [10] for a precise statement. Moreover, we should point out that, in contrast with the image under λ of the words of length d, the family $\lambda(g_{i_1} g_{i_2} \cdots g_{i_d})$ is also a p-orthogonal family with d indices.

Note added in proof. Recently, Ricard and Xu [13] have extended Buchholz's result [1] to arbitrary free product C^*-algebras. As they point out in their paper, the same construction holds for amalgamated free products of von Neumann algebras. Moreover, after Ricard/Xu's work, Junge and the first-named author have generalized in [6] the main result in [13] to arbitrary indices $2 \le p \le \infty$ as a consequence of the free analogue of Rosenthal's inequality [14], also proved in [6].

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