Noncommutative Cotlar identities for groups acting on tree-like structures

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Abstract

Let T_m be a noncommutative Fourier multiplier. In recent work, Mei and Ricard introduced a noncommutative analogue of Cotlar's identity in order to prove that certain multipliers are bounded on the non-commutative L_p-spaces of a free group. Here, we study Cotlar type identities in full generality, giving a closed characterization for them in terms of m:

$$\big(m(gh) - m(g)\big)\,\big(m(g^{-1}) - m(h)\big) = 0,\ \forall g \in \mathrm{G} \setminus \{e\}, h \in \mathrm{G}.$$

We manage to prove, using a geometric argument, that if X is a tree —or more generally an $\mathbb{R}$-tree— on which G acts and m lifts to a function $\widetilde{m} : X \to \mathbb{C}$ that is constant on the connected subsets of $X \setminus \{x_0\}$, then m satisfies Cotlar's identity and thus T_m is bounded in L_p for $1 < p < \infty$. This result establishes a new connection between groups actions on $\mathbb{R}$-trees and Fourier multipliers. We show that m is trivial when the action has global fixed points. This machinery allows us to simultaneously generalize the free group transforms of Mei and Ricard and the theory of Hilbert transforms in left-orderable groups, which follows from Arveson's subdiagonal algebras. Using Bass-Serre theory, we construct new examples of Fourier multipliers in groups. Those include new families like Baumslag-Solitar groups. We also show that a natural Hilbert transform in $\mathrm{PSL}_2(\mathbb{C})$ satisfies Cotlar's identity when restricted to the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$.

Introduction

The Hilbert transform was introduced by Hilbert in 1912 as part of his investigation of the Riemann problem in the realm of complex analysis [31]. Indeed, it may be regarded as the operator describing the boundary behavior of the harmonic conjugate in the upper half plane. Explicitly, it is given by

$$Hf(x) \,=\, \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy. \tag{HT}$$

Equivalently, it is the Fourier multiplier $(Hf)^\wedge(\xi) = i\,\mathrm{sgn}(\xi)\,\widehat{f}(\xi)$ [20]. In 1924, M. Riesz proved its L_p-boundedness for all $p \in 2\mathbb{Z}_+$ using an *ad hoc* complex analysis argument [57, 58]. By duality and Marcinkiewicz's interpolation this yields L_p-boundedness for every $1 < p < \infty$. Afterwards, Kolmogorov and Calderón-Zygmund found proofs giving the weak type $(1,1)$ [40, 63, 9]. Among other consequences, the mapping properties of the Hilbert transform are crucial for the L_p-convergence of partial Fourier series/integrals on Euclidean spaces and their integer lattices. In fact, by elementary manipulations, the frequency restriction of $f \in L_p(\mathbb{R}^n)$ to $N \cdot \mathcal{P}$ converges to f in the L_p-norm as $N \to \infty$, where $N \cdot \mathcal{P} \subset \mathbb{R}^n$ is the dilation of any convex polyhedron $\mathcal{P}$ containing the origin.

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According to Feffermann's construction in his solution of the ball multiplier problem, this result is false if $\mathcal{P}$ is convex set such that $\partial\mathcal{P}$ has nonvanishing curvature. The intermediate case in which $\mathcal{P}$ is an infinitely-faceted polyhedron has been completely solved in dimension 2 but remains open in higher dimensions [1, 2, 4, 51].

In 1955, Cotlar proved the L_p-boundedness of the Hilbert transform through an extremely simple argument [15]. He showed that H is L_p-bounded for $p = 2^k$ recursively —from the trivial case $p = 2$— using his elegant *Cotlar identity*

$$|Hf|^2 \, = \, 2H\big(f\, Hf\big) - H\big(H|f|^2\big). \tag{Classical Cotlar}$$

A key point in our work is to notice that Cotlar's identity and their generalizations have a nicer expression at the frequency side of the Fourier transform. As an illustration, notice that an Euclidean Fourier multiplier $(T_m f)^\wedge(\xi) = m(\xi)\, \widehat{f}(\xi)$ satisfies Cotlar's identity precisely when

$$\big(m(\xi+\eta) - m(\xi)\big)\big(m(-\xi) - m(\eta)\big) = 0, \quad \text{a.e } \xi, \eta \in \mathbb{R}. \tag{Classical $\widehat{\text{Cotlar}}$}$$

The Hilbert transform is just one example among the many multipliers that satisfy the classical Cotlar identity above, which in turn implies the L_p-boundedness of T_m. In this article we will investigate similar identities on more general locally compact groups and their von Neumann algebras. A pioneering work in this direction was due to Mei and Ricard [43], where the authors deployed a noncommutative analogue of the Cotlar identity, that holds in the context of amalgamated free product von Neumann algebras. The main goal of this article is to further Mei and Ricard's technique beyond free groups by illuminating the hidden connection between noncommutative Cotlar identities and group actions on trees and other tree-like structures —like $\mathbb{R}$-trees and uniquely arcwise connected spaces.

Noncommutative Fourier multipliers. We will deal here with operators analogous to the Hilbert transform over group algebras. Let G be a locally compact group and let us define its group von Neumann algebra as

$$\mathcal{L}\mathrm{G} := \overline{\mathrm{span}}^{\mathrm{w}^*} \big\{ \lambda_g : g \in \mathrm{G} \big\} \subset \mathcal{B}(L^2\mathrm{G}).$$

When G is unimodular, the algebra $\mathcal{L}\mathrm{G}$ admits a normal, semifinite and faithful tracial weight called the *Plancherel trace* [53, Chapter 8], with respect to which the noncommutative L_p-spaces $L_p(\mathcal{L}\mathrm{G})$ are defined [64, 55]. When G is abelian, $L_p(\mathcal{L}\mathrm{G})$ is isomorphic to $L_p(\widehat{\mathrm{G}})$, the L_p-space over the Pontryagin dual of G. In the general case, $L_p(\mathcal{L}\mathrm{G})$ can be thought of as a natural generalization of the L_p elements over the dual of G. As such, many classical problems of Fourier analysis over L_p-spaces find an analogue in the noncommutative setting. A prominent example is the study of the L_p-boundedness of Fourier multipliers.

Given $m : \mathrm{G} \to \mathbb{C}$, the *Fourier multiplier of symbol* m will be the, potentially unbounded, linear operator $T_m : D \subset \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}$ given by linear extension of

$$T_m(\lambda_g) = m(g)\, \lambda_g.$$

The boundedness of Fourier multipliers over noncommutative L_p-spaces presents challenges absent in the classical setting. A key difficulty is the extension of singular integral techniques to von Neumann algebras. Although steps towards a noncommutative Calderón-Zygmund theory had been taken [49, 28, 37, 8], a fully noncommutative Calderón-Zygmund theory capable of yielding weak $(1,1)$ has not been found outside of semi-commutative or nilpotent settings.

As hinted before, one possible way of overcoming this difficulty was found in [43], the noncommutative analogue of the Cotlar identity developed in their paper allowed them to prove that functions $m : \mathbb{F}_2 \to \mathbb{C}$ over the free group whose value $m(\omega)$ depends only on the starting letter $\{a, a^{-1}, b, b^{-1}\}$ of the reduced word $\omega \in \mathbb{F}_2$ give rise to bounded Fourier multipliers on L_p, i.e.,

$$\big\| T_m : L_p(\mathcal{L}\mathbb{F}_2) \to L_p(\mathcal{L}\mathbb{F}_2) \big\|_{\mathrm{cb}} \, < \, \infty, \quad \text{for every } 1 < p < \infty. \tag{MR}$$

In this paper, we will study a new geometric way to define L_p-bounded Fourier multipliers on groups admitting actions on tree-like structures, and we will see in Section 4 that this recovers (MR) as a particular example.

Noncommutative Cotlar identities. Let $\mathrm{G}_0 \subset \mathrm{G}$ be an open subgroup of the locally compact and unimodular group G. It is trivial to see that G_0 is also unimodular and that $\mathcal{L}\mathrm{G}_0 \subset \mathcal{L}\mathrm{G}$ is a complemented inclusion, that is, an inclusion admitting a normal conditional expectation $\mathbb{E} : \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}_0$. We will say that a potentially unbounded multiplier T_m satisfies a *noncommutative Cotlar identity* with respect to the von Neumann subalgebra $\mathcal{L}\mathrm{G}_0$ iff

$$\mathbb{E}^{\perp}\big[T_m(f)\,T_m(f)^*\big] \;=\; \mathbb{E}^{\perp}\Big[T_m\big(f\,T_m(f)^*\big) + T_m\big(f\,T_m(f)^*\big)^* - T_m\big(T_m(ff^*)^*\big)\Big], \tag{Cotlar}$$

where $\mathbb{E}^{\perp} = (\mathrm{id} - \mathbb{E})$.

We have distilled an easily verifiable closed formula ($\widehat{\mathrm{Cotlar}}$) for m that is equivalent to (Cotlar) above, see Theorem 1.5. Since with an additional assumption on the symbol, the Cotlar's formula implies L_p-boundedness, we obtain the following theorem.

Theorem A. *Let* G *be a locally compact unimodular group,* $\mathrm{G}_0 \subset \mathrm{G}$ *an open subgroup and* $m : \mathrm{G} \to \mathbb{C}$ *a left* G_0*-invariant bounded function. If* m *satisfies that*

$$\big(m(g^{-1}) - m(h)\big)\,\big(m(gh) - m(g)\big) \;=\; 0, \quad \forall g \in \mathrm{G} \setminus \mathrm{G}_0, h \in \mathrm{G}, \tag{$\widehat{\mathrm{Cotlar}}$}$$

then T_m *is* L_p*-bounded for* $1 < p < \infty$ *and furthermore*

$$\big\|T_m : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathrm{G})\big\| \lesssim \left(\frac{p^2}{p-1}\right)^{\beta}, \qquad \text{with } \beta = \log_2(1+\sqrt{2}). \tag{1}$$

The result above holds true in the non-relative case in which the expectation $\mathbb{E}$ is removed. This case can be thought of as a degenerate case in which G_0 is empty. This is specially useful when dealing with continuous groups and allows us to see the classical Cotlar identity in $\mathbb{R}$ as a particular case of our theory, see Remark 1.6.

The first advantage of the result above is that it makes the verification of the Cotlar identity for previously known cases almost trivial. Indeed, we have that it holds in the following situations.

(1) Classical case. In the classical case of $\mathrm{G} = \mathbb{Z}$ and $\mathrm{G}_0 = \{0\}$ with $m(x) = \mathrm{sgn}(x)$ we only have to verify that either $m(x+y) = m(x)$ or $m(-x) = m(y)$. But that is trivial since either x and y have different signs or $x+y$ and x share the same sign.

(2) Free product case. If $\mathrm{G} = \mathrm{G}_1 * \mathrm{G}_2$ and $\mathrm{G}_0 = \{e\}$ with both G_2 and G_2 discrete, then any function $m(\omega)$ such that its value depends on the first letter of the reduced word of g satisfies ($\widehat{\mathrm{Cotlar}}$). Indeed, we need to prove that either $m(gh) = m(g)$ or that $m(g^{-1}) = m(h)$. Assume the first equality fails, then the first letter of gh and that of g are different, but that can only happen if the reduced word of h begins with the reduced word of g^{-1}. If that is the case g^{-} and h have the same starting letter and thus $m(g^{-1}) = m(h)$. This family of examples was explored by Mei and Ricard [43].

A natural question that we answer affirmatively is whether there are examples of groups that go beyond those two categories. In order to explore that question it seems natural to search for bounded functions $m : \mathrm{G} \to \mathbb{C}$ satisfying ($\widehat{\mathrm{Cotlar}}$) with G having Serre's property (FA) [61]. A group G is said to have Serre's property (FA) iff every action of G on a tree has a global fixed point (a vertex in the tree which is fixed by the action of any $g \in \mathrm{G}$). More deeply, Serre proved that a discrete group G has property (FA) iff it is finitely generated, it does not possess a quotient isomorphic to $\mathbb{Z}$ and it cannot be expressed as a nontrivial amalgamated free product $\mathrm{G} = \mathrm{G}_1 *_A \mathrm{G}_2$. Therefore a group with property (FA) is excluded from examples 1 and 2. Although we manage to show that such examples with property (FA) exist, we also give examples —like Baumslag-Solitar groups $\mathrm{BS}(1,m)$ and the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_{-1})$— which despite failing property (FA) admit bounded functions satisfying Cotlar's identity for reasons unrelated to them having $\mathbb{Z}$-quotients or being free products.

Groups acting on uniquely arcwise connected spaces. The closed formula in ($\widehat{\mathrm{Cotlar}}$) highlights a surprising connection between Cotlar's identity and geometric group theory. A topological space X is arcwise connected iff given two points $x, y \in X$ there exists an injective continuous path $\gamma : [0,1] \to X$ joining x and y. An arcwise connected space will be said to be a *uniquely arcwise connected space* or *UAC space* iff the path joining x and y is unique [6]. Let $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space and fix a root $x_0 \in X$ with G_0 being the stabilizer St_{x_0} of x_0. Observe that $X \setminus \{x_0\}$ is given by

$$X \setminus \{x_0\} = \bigsqcup_{\beta} X_\beta, \tag{2}$$

where each X_β is arcwise connected. In some cases, the disjoint union above can be taken as a topological characterization of X and the subsets X_β as connected components. Nevertheless, there are UAC spaces for which $X \setminus \{x_0\}$ is not a disjoint union of connected components. Observe that the action of G_0 restricted to $X \setminus \{x_0\}$ permutes the X_β. The following theorem gives a machinery to get multipliers satisfying Cotlar's identity from actions on UAC spaces.

Theorem B. *Let* $\mathrm{G} \curvearrowright X$ *be a topological action on a UAC space. Fix* $x_0 \in X$*,* $\mathrm{G}_0 = \mathrm{St}_{x_0}$*, and let* $\widetilde{m} : X \to \mathbb{C}$ *be a bounded function satisfying that*

- **(i)** $\widetilde{m}$ *is constant over each* X_β *of* (2)*.*
- **(ii)** $\widetilde{m}$ *is constant over* G_0 *orbits, ie* $\widetilde{m}|_{X_\beta} = \widetilde{m}|_{X_\alpha}$ *if there is an element* $h \in \mathrm{G}_0$ *such that* $X_\beta = h \cdot X_\alpha$*.*

Then, the function $m : \mathrm{G} \to \mathbb{C}$ *given by* $m(g) = \widetilde{m}(g \cdot x_0)$ *satisfies* $(\widehat{\mathrm{Cotlar}})$ *and therefore*

$$\big\| T_m : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G}) \big\| \lesssim \Big(\frac{p^2}{p-1} \Big)^\beta \qquad \text{with } \beta = \log_2(1+\sqrt{2}). \tag{3}$$

The proof is so neat that can be tightly summarized here. First, notice that the condition (ii) ensures that m is left-G_0-invariant. The condition (i) on the other hand implies that $(\widehat{\mathrm{Cotlar}})$ holds. To see this, assume that $m(gh) \neq m(g)$. Since the two values are different, the arcwise connected subsets of $X \setminus \{x_0\}$ in which $gh \cdot x_0$ and $g \cdot x_0$ lay are different, see Figure 1. Thus, there is a unique arc joining them that passes through x_0. Applying g^{-1} to this arc gives an arc starting at x_0 and which passes by $g^{-1} \cdot x_0$ and $h \cdot x_0$ in that order. But, as such, $g^{-1} \cdot x_0$ and $h \cdot x_0$ must lay in the same arcwise connected subset and thus $m(h) = m(g^{-1})$.

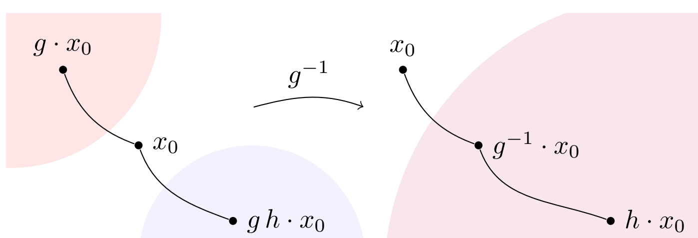

Figure 1: Action over paths.

Left orderable groups. A family of examples that fits right into the model of Theorem A is that of *left orderable groups*. Those are groups admitting a *total* or *linear* order $(\mathrm{G}, \preceq)$ that is invariant under left translations, ie $g \preceq h \iff ag \preceq ah$ for every $a, g, h \in \mathrm{G}$. We will write $g \prec h$ when $g \preceq h$ but $g \neq h$. For left orderable groups the following multiplier boundedness result holds for their sign function.

Theorem C. *Let* G *be a left-orderable group and* $\mathrm{sgn} : \mathrm{G} \to \mathbb{C}$ *be the function*

$$\mathrm{sgn}(g) = \begin{cases} 1 & \text{when } e \prec g \\ 0 & \text{when } g = e \\ -1 & \text{when } g \prec e. \end{cases}$$

Then $H = T_{\mathrm{sgn}} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$ satisfies that

$$\big\| H : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G}) \big\| \lesssim \frac{p^2}{p-1}, \quad \textit{for every } 1 < p < \infty.$$

The boundedness of H in Theorem C can be obtained by showing directly that $(\widehat{\mathrm{Cotlar}})$ holds. Alternatively, it is known that, in the discrete case, a group is left-orderable iff it acts on $\mathbb{R}$ by order-preserving homeomorphisms. The observation that $\mathbb{R}$ is a UAC space allows us to prove the result as a consequence of Theorem B. In principle, proving Cotlar's identity gives just that the L_p-norm grows like $\mathrm{O}(p^\beta)$ with $\beta = \log_2(1+\sqrt{2})$, as $p \to \infty$. Nevertheless, a more careful argument allows us to show that the constant can be lowered down to the optimal order $\mathrm{O}(p)$, as long as $m(g)\overline{m(g^{-1})} = -1$ for $g \in \mathrm{G} \setminus \{e\}$, see Corollary 1.9. It is also worth noticing that the above transforms for left-orderable group algebras can be seen as a particular example of the Hilbert transforms associated with Arveson's subdiagonal algebras [3], for which the weak type $(1,1)$ was proved by Randrianantoanina [56], see also [55, Theorem 8.4]. Therefore our geometric model in Theorem B generalizes simultaneously Mei and Ricard's free Hilbert transforms and subdiagonal Hilbert transforms, recovering the best known constants in both cases.

Left orderable groups include:

- Torsion-free abelian groups;
- Torsion-free nilpotent groups;
- Free groups $\mathbb{F}_r$;
- Braid groups B_n;
- Right-angled Artin groups;
- Baumslag-Solitar groups $\mathrm{BS}(1,n)$ for $n \geq 2$;
- Surface groups;
- The Thompson group F.

We manage to obtain explicit examples of L_p-bounded Fourier multipliers on each of these families of groups. Furthermore, there are known examples of left orderable groups that have Serre's property (FA). For instance, let $\widetilde{\mathrm{PSL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{R})$ be the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and let $D(2,3,7) \subset \mathrm{PSL}_2(\mathbb{R})$ be the $(2,3,7)$-triangular group. Then, the lifting Γ of $D(2,3,7)$ to $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ is an example of a group with Serre's property (FA) that is also left-orderable [5, 14, 39, 61].

Graphs of groups and Bass-Serre theory. A wealth of examples of multipliers satisfying $(\widehat{\mathrm{Cotlar}})$ can be obtained from Bass-Serre theory —which allows to classify groups acting on trees without edge inversions— see [61]. Indeed, given a group acting on a tree $\mathrm{G} \curvearrowright T$, it is possible to build a graph by taking the quotient with respect to the action $X = T/\mathrm{G}$ and associating to each vertex and to each edge its corresponding stabilizer. Observe that, due to the lack of edge inversions, the stabilizer of an edge embeds into the stabilizers of its extremes. This structure —a graph with groups on its edges and vertices and such that the groups at the edges embed into the extremes of said edge— is called a *graph of groups*. In our case, we will denote it by $\mathbb{X}$. Like in the case of graphs, it is possible to define the universal cover of $\mathbb{X}$, $\widetilde{\mathbb{X}} \to \mathbb{X}$, such that its underlying graph is a tree and its fundamental group $\pi_1(\mathbb{X})$ acts as Deck's transformations of $\widetilde{\mathbb{X}} \to \mathbb{X}$. The main point of the theory is that $\pi_1(\mathbb{X}) \cong \mathrm{G}$ and the action of $\pi_1(\mathbb{X}) \curvearrowright \widetilde{\mathbb{X}}$ recovers $\mathrm{G} \curvearrowright T$.

Elementary examples of graphs of groups include Higman-Neumann-Neumann (HNN) extensions and free products. While the case of free products gives examples in the spirit of Mei-Ricard [43], the multipliers associated with actions of HNN extensions on their Bass-Serre trees gives new families of examples. A simple example of a HNN extension is given by the *Baumslag-Solitar groups* $B(n,m)$ with $m, n \in \mathbb{Z}$. First, notice that every subgroup of $\mathbb{Z}$ is of the form $\ell\mathbb{Z}$ for $\ell \in \mathbb{Z}_+$, and as such they are all isomorphic. Pick two of them $n\mathbb{Z}, m\mathbb{Z} \subset \mathbb{Z}$. The Baumlag-Solitar group is the minimal extension of $\mathbb{Z}$ for which the two subgroups are conjugate

$$\mathrm{BS}(n,m) \;=\; \big\langle t, r \,|\, t\, r^m\, t^{-1} \;=\; r^n \big\rangle$$

By Theorem B, the action of $\mathrm{BS}(n,m)$ on its Bass-Serre tree yields a bounded Hilbert transform satisfying (Cotlar). While $\mathrm{BS}(n,m)$ has $\mathbb{Z}$ quotients, and therefore fails Serre's property (FA), the Hilbert transform obtained is not covered by the examples of Mei and Ricard. This is explained in more detail in Section 4

$\mathbf{PSL_2(K)}$, its lattices and open questions. Natural models of Hilbert transforms on a group G often appear via the following straightforward idea. Let $\mathcal{X}$ be a geometric object on which G acts and assume $\mathcal{X}$ contains a *barrier* $\mathcal{F} \subset \mathcal{X}$ such that $\mathcal{X} \setminus \mathcal{F}$ is divided into two separated halves $\mathcal{X} \setminus \mathcal{F} = \mathcal{X}_+ \sqcup \mathcal{X}_-$. Then, given $x_0 \in \mathcal{F}$, a natural Hilbert transform can be defined with symbol

$$m(g) = \mathbf{1}_{\mathcal{X}_+}(g \cdot x_0) - \mathbf{1}_{\mathcal{X}_-}(g \cdot x_0).$$

Important instances of this model include:

(1) Hilbert space model. Let $\mathcal{X} = \mathcal{H}$ be a (real) Hilbert space in which G acts by affine isometries $\pi(g)$. These isometries are given by $\xi \mapsto \alpha(g)\xi + \beta(g)$, where $\alpha(g)$ is an orthogonal transformation. Let $\mathcal{F} = \langle v \rangle^\perp$ be the codimension 1 subspace of vectors perpendicular to $v \in \mathcal{H} \setminus \{0\}$. Choosing $x_0 = 0$ gives the symbol $m(g) = \mathrm{sgn}\left(\langle \beta(g), v \rangle\right)$. These symbols have been studied for finite dimensional $\mathcal{H}$ in [10, Appendix A] and [52].

(2) Manifold model. Choose $\mathcal{X} = M$ as a n-dimensional Riemannian manifold in which G acts by isometries $\alpha : \mathrm{G} \to \mathrm{Iso}(M)$ and let $\mathcal{F} \subset \mathcal{X}$ be a $(n-1)$-dimensional geodesic submanifold such that $\mathcal{X} \setminus \mathcal{F}$ has two connected components.

(3) Tree model. $\mathcal{X} = T$ being a tree on which G acts. Choose $x_0 \in T$ to be a vertex, that we will henceforth call the *root*. Then, $\mathcal{X} \setminus \{x_0\}$ is made up of r connected components, with r being the valence of x_0, that we can arrange into two families $\mathcal{X}_+$ and $\mathcal{X}_-$. This is an instance of the model described in Theorem B above.

It is very interesting to point out that, in many examples, the same idempotent Fourier multiplier on a group can be obtained from more than one of the three different models above. Here we will illustrate that phenomenon with the continuous groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$, which will explain part of our original motivation. Let $\mathrm{SL}_2(\mathbb{K})$ be the group of 2×2 matrices of determinant 1 with entries over a field $\mathbb{K}$ that in our examples will be $\mathbb{R}$ or $\mathbb{C}$. $\mathrm{PSL}_2(\mathbb{K})$ will denote the quotient of $\mathrm{SL}_2(\mathbb{K})$ by scalar matrices $\{\pm \mathrm{id}\}$. Both groups, $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$, act faithfully and transitively by isometries on the real hyperbolic spaces of dimension 2 and 3, $\mathrm{PSL}_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ and $\mathrm{PSL}_2(\mathbb{C}) \curvearrowright \mathbb{H}^3$ —which we will identify with their upper half plane and upper half space models. Let us denote the coordinates of $\mathbb{H}^2$ by (x, y) and the coordinates of $\mathbb{H}^3$ by (x, y, z). We can take the geodesic $\{x = 0\} \subset \mathbb{H}^2$ as separating space in the first example. A calculation yields that the Hilbert transform in the sense of the manifold model is

$$m\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{g} \;=\; \mathrm{sgn}\big(\mathrm{Re}\{g \cdot i\}\big) \;=\; \mathrm{sgn}\big(a_{11}a_{21} + a_{12}a_{22}\big), \tag{4}$$

where g is the class $\pm[a_{i,j}]_{i,j}$. This multiplier can be related to the other two models. Indeed, for the Hilbert space model, it is possible to construct a metrically proper 1-cocycle $\beta : \mathrm{PSL}_2(\mathbb{K}) \to \mathcal{H}$ into an infinite dimensional Hilbert space $\mathcal{H}$ and choose a unit vector $u \in \mathcal{H}$ such that $m(g) = \mathrm{sgn}\langle \beta(g), u \rangle$, see [21, 11]. While the group $\mathrm{PSL}_2(\mathbb{R})$ is continuous, and thus it is unable to act on trees in an interesting way, the tree model interpretation is indeed available for the restriction of (4) to $\mathrm{PSL}_2(\mathbb{Z})$. The key observation is that

$$\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

This free-product decomposition yields an action of $\mathrm{PSL}_2(\mathbb{Z})$ on its Bass-Serre tree with respect to which the multiplier $m|_{\mathrm{PSL}_2(\mathbb{Z})}$ can be recovered.

For the complex case, the separating subspace is given by the 2-dimensional geodesic submanifold $\{x = 0\} \subset \mathbb{H}^2$, which readily gives that

$$m\underbrace{\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}}_{g} = \mathrm{sgn}\big(\mathrm{Re}\{p(g \cdot x_0)\}\big) = \mathrm{sgn}\big(\mathrm{Re}\{z_{11}\overline{z}_{21} + z_{12}\overline{z}_{22}\}\big), \tag{5}$$

where $x_0 = (0, 1) \in \mathbb{C} \times \mathbb{R}_+$, $p(z, r) = z$ and $g = \pm[z_{i,j}]_{i,j}$. In this case, the relationship with the other two models is more involved. Nevertheless, it is still possible to describe the multiplier (5) above in terms of proper infinite-dimensional 1-cocycles with respect to a natural direction u. For the tree model the situation is a lot more contentious. Indeed, let $\mathcal{O}_{-d}$ be the ring of integers of the algebraic field $\mathbb{Q}(\sqrt{-d})$, where d is a square-free integer. The lattices $\mathrm{PSL}_2(\mathcal{O}_{-d}) \subset \mathrm{PSL}_2(\mathbb{C})$ are the *Bianchi groups*. It is known that all of them except for $d = 3$ admit nontrivial actions on trees, see [23]. Indeed, for $d = 1$ this yield the following, quite involved, isomorphism

$$\mathrm{PSL}_2(\mathcal{O}_{-1}) \cong \big(S_3 *_{\mathbb{Z}_3} A_4\big) *_{\mathrm{PSL}_2(\mathbb{Z})} \big(S_3 *_{\mathbb{Z}_2} V\big),$$

where S_n are the permutation groups, A_n are the alternating groups and V is the Klein 4 group, see [23, Theorem 2.1.(i)]. It is possible that m, when restricted to $\mathrm{PSL}_2(\mathcal{O}_{-d})$, may have an expression in terms of a nontrivial action on a tree. Nevertheless, the complexity of the amalgamated free product decompositions obtained make it a difficult approach to work with. On the other hand, the strength of our characterization in Theorem A allows us to prove the boundedness of $m|_{\mathrm{PSL}_2(\mathcal{O}_{-d})}$ directly. We have also verified that $d = 1$ is the only Bianchi group for which the restriction of (5) satisfies $(\widehat{\text{Cotlar}})$.

Theorem D. *Let* $\mathrm{G} = \mathrm{PSL}_2(\mathcal{O}_{-1}) \subset \mathrm{PSL}_2(\mathbb{C})$ *and* $m : \mathrm{PSL}_2(\mathcal{O}_{-1}) \to \mathbb{C}$ *be the function given by*

$$m\begin{pmatrix} a_{11} + ib_{11} & a_{12} + ib_{12} \\ a_{21} + ib_{21} & a_{22} + ib_{22} \end{pmatrix} = \mathrm{sgn}\big(a_{11}a_{21} + b_{11}b_{21} + a_{12}a_{22} + b_{12}b_{22}\big).$$

Then m satisfies $(\widehat{\text{Cotlar}})$ *and therefore*

$$\big\|T_m : L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathcal{O}_{-1})) \to L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathcal{O}_{-1}))\big\| \lesssim \Big(\frac{p^2}{p-1}\Big)^{\beta} \quad \text{with } \beta = \log_2(1 + \sqrt{2}).$$

This leaves open whether $m|_\Gamma$ is bounded for lattices other than $\mathrm{PSL}_2(\mathcal{O}_{-1})$. In the same spirit, the boundedness of (4) and (5) over the whole group is an natural problem that we leave open.

Problem A.

- **(A.1)** Let m be as (4). Is $T_m : L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{R})) \to L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{R}))$ bounded?
- **(A.2)** Let m be as (5). Is $T_m : L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{C})) \to L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{C}))$ bounded?

In the classical case of $\mathrm{G} = \mathbb{R}$, the boundedness of the Hilbert transform can be obtained from smooth multiplier results, in particular it satisfies the hypothesis of both the Hörmander-Mikhlin and Marcinkiewicz theorems [20]. Therefore, Problem A seems closely related to the question of whether smoothness conditions of a function $\widetilde{m} : \mathbb{H}^2 \to \mathbb{C}$ yield L_p-boundedness of the lifted multiplier $m(g) = \widetilde{m}(g \cdot i)$. Results that point in that direction have already appeared in the literature. For instance, a result for local smooth Fourier multipliers in $\mathrm{SL}_2(\mathbb{R})$ features in [50]. In the case of S_p-bounded Schur multipliers, results for global Hörmander-Mikhlin Schur multipliers have been obtained in [13, 12]. This smooth multiplier approach to Problem A above presents two main obstacles. The first is that —contrary to the results in [12]— the singularity isn't located in a single point, instead it is a codimension 1 subset containing the stabilizer of a point in $\mathbb{H}^2$. The second is that currently available tecniques for the passage from Schur to Fourier multipliers require either to work with compactly supported multipliers or in homogeneous groups, see [50, 12].

1. Cotlar identities and multipliers

Noncommutative integration. Throughout this text we will use liberally *noncommutative integration theory* and the theory of noncommutative L_p-spaces. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a *von neumann algebra* admitting a *normal semifinite and faithful tracial weight* $\tau : \mathcal{M}_+ \to [0, \infty]$ that we will henceforth just refer to as a *n.s.f trace.* It is possible to construct the noncommutative L_p-spaces associated to $(\mathcal{M}, \tau)$ as the subset of τ-measurable operators $L_p(\mathcal{M}, \tau) \subset L_0(\mathcal{M}, \tau)$ satisfying that

$$f \in L_p(\mathcal{M}, \tau) \quad \Longleftrightarrow \quad \|f\|_p := \tau\big(|f|^p\big)^{\frac{1}{p}} < \infty.$$

This theory, which goes back all the way to Dixmier and Segal [18, 60], is already well understood and the interested reader can consult it in [64, 55, 26].

Let G be a locally compact group that we will throughout the text assume to be second countable, and let $L_2(\mathrm{G})$ be its L_2-space with respect to the *left Haar measure* μ [22]. As usual, we will denote by $\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}))$ the *left regular representation*, which is the unitary representation $g \mapsto \lambda_g$ that acts by sending $\xi(h)$ to $\xi(g^{-1}h)$. The *left regular von Neumann algebra* $\mathcal{L}\mathrm{G} \subset \mathcal{B}(L_2(\mathrm{G}))$ of G is given by

$$\overline{\lambda[L_1(\mathrm{G})]^{\mathrm{w}^*}} \; = \; \overline{\Big\{ \lambda(\varphi) := \int_{\mathrm{G}} \varphi(g)\, \lambda_g \, d\mu(g) : \varphi \in L_1(\mathrm{G}) \Big\}^{\mathrm{w}^*}} \; \subset \; \mathcal{B}(L_2(\mathrm{G})).$$

This von Neumann algebra admits a normal, semifinite and faithful weight $\tau : \mathcal{L}\mathrm{G}_+ \to [0, \infty]$ that satisfies the Plancherel identity. This weight is usually referred as the *Plancherel weight* [53, Chapter 7]. The weight τ is a n.s.f trace precisely when G is unimodular. Thus, we will work in the natural setting of unimodular groups and refer to τ as the *Plancherel trace.* In this context, the Plancherel trace is given by

$$\tau\left(\int_{\mathrm{G}} \varphi(g)\, \lambda_g \, d\mu(g) \right) = \varphi(e), \qquad \text{for every } \; \varphi \in C_c(\mathrm{G}) * C_c(\mathrm{G}).$$

We will denote the noncommutative L_p-spaces associated to τ simply by $L_p(\mathcal{L}\mathrm{G})$. By analogy with the classical Fourier transform, we will denote by $\widehat{f}(g)$ the value $\tau(\lambda_g^* f)$, which is well defined whenever $f \in L_1(\mathcal{L}\mathrm{G})$. It is also worth noticing that, by Plancherel's theorem, the map $f \mapsto \widehat{f}$ is defined almost everywhere in $g \in \mathrm{G}$ for any $f \in L_2(\mathcal{L}\mathrm{G})$ and gives an isometry $L_2(\mathcal{L}\mathrm{G}) \to L_2(\mathrm{G})$.

Conditional expectations. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra of $\mathcal{M}$, that is a $*$-subalgebra that is also ultraweakly closed. If τ is a n.s.f. trace over $\mathcal{M}$ and $\tau|_{\mathcal{N}}$ is still semifinite, then it is easy to see that the inclusion $\iota : L_1(\mathcal{N}) \hookrightarrow L_1(\mathcal{M})$ is isometric and its dual is a normal (i.e: ultraweakly continuous) *conditional expectation* $\mathbb{E} : \mathcal{M} \to \mathcal{N} \subset \mathcal{M}$. By conditional expectation we mean a *unital and completely positive* normal map such that $\mathbb{E}|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ and $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$. Recall that by Tomiyama's Theorem, see [7, Theorem 1.5.10], $\mathbb{E}$ is automatically $\mathcal{N}$-bimodular.

Let $\mathrm{G}_0 \subset \mathrm{G}$ be two groups such that G_0 is open inside G. Then G_0 is unimodular if G is. Furthermore, the Plancherel trace τ_{G_0} of $\mathcal{L}\mathrm{G}_0$ coincides with the Plancherel trace of $\mathcal{L}\mathrm{G}$ restricted to $\mathcal{L}\mathrm{G}_0$. Therefore there is a normal and trace-preserving conditional expectation $\mathbb{E} : \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}_0 \subset \mathcal{L}\mathrm{G}$ that is given by the Fourier multiplier associated to $\mathbf{1}_{\mathrm{G}_0}$ ie:

$$\mathbb{E}(f) \; = \; \mathbb{E}\left(\int_{\mathrm{G}} \widehat{f}(g)\, \lambda_g \, d\mu(g) \right) \; = \; \int_{\mathrm{G}} \mathbf{1}_{\mathrm{G}_0}(g)\, \widehat{f}(g)\, \lambda_g \, d\mu(g).$$

The fact that $\mathbb{E}$ is trace preserving allows us to extend $\mathbb{E}$ as a contraction to all the L_p-spaces $1 \leq p \leq \infty$, $\mathbb{E} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G}_0) \subset L_p(\mathcal{L}\mathrm{G})$.

Noncommutative Cotlar identities. Most of this section up until the closed-formula characterization of the Cotlar identity in the proof of Theorem A follows closely the results obtained by Mei and Ricard and it is included here for the sake of completion.

First, let us assume that $\mathcal{A} \subset \mathcal{M}$ is a unital $*$-subalgebra such that $\mathcal{A} \cap L_p(\mathcal{M})$ is norm dense in $L_p(\mathcal{M})$ for every $2 \leq p < \infty$. When dealing with a complemented von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ we will assume that $\mathcal{A} \cap L_p(\mathcal{N})$ is again norm dense in $L_p(\mathcal{N})$ and that $\mathcal{A} \cap \mathcal{N}$ is ultraweakly dense in $\mathcal{N}$. Whenever we say that an operator $H : \mathcal{A} \to \mathcal{A}$ is $\mathcal{N}$-modular, we mean that it is modular with respect to $\mathcal{A} \cap \mathcal{N}$.

Definition 1.1 ([43, from Proposition 3.2(iv)]). *Let $\mathcal{A} \subset \mathcal{M}$ be as above and $\mathbb{E} : \mathcal{M} \to \mathcal{N} \subset \mathcal{M}$ be a conditional expectation. A linear operator $H : \mathcal{A} \to \mathcal{A}$ is said to satisfy Cotlar identity relative to $\mathcal{N}$ iff*

$$\mathbb{E}^{\perp}\big[H(f)\,H(f)^*\big] \;=\; \mathbb{E}^{\perp}\Big[H\big(f\,H(f)^*\big) + H\big(f\,H(f)^*\big)^* - H\big(H(ff^*)^*\big)\Big], \qquad (\mathrm{Cotlar}_{\mathbb{E}^{\perp}})$$

for every $f \in \mathcal{A} \cap L_2(\mathcal{M})$, where $\mathbb{E}^{\perp} = (\mathrm{id} - \mathbb{E})$.

We will use Definition 1.1 mainly in the case in which $H = T_m$ is a Fourier multiplier, $\mathcal{M} = \mathcal{L}\mathrm{G}$ and $\mathcal{N} = \mathcal{L}\mathrm{G}_0$ is given by an open subgroup $\mathrm{G}_0 \subset \mathrm{G}$. For the choice of the algebra $\mathcal{A}$ we need to be a little bit more careful. First, let $L_\infty(\mathrm{G})_{\mathrm{c}}$ be the convolution algebra of essentially bounded and compactly supported functions, then we define

$$\mathcal{A} \;=\; \begin{cases} \lambda[L_\infty(\mathrm{G})_{\mathrm{c}}] & \text{when G is discrete,} \\ \lambda[L_\infty(\mathrm{G})_{\mathrm{c}}] + \mathbb{C}\cdot\mathbf{1} & \text{when G is not discrete.} \end{cases}$$

The reason why we need to artificially add a unit to $\mathcal{A}$ is the case of non-discrete groups will be clear after the proof of Lemma 1.2, where the value $H(\mathbf{1})$ would be used, see also Remark 1.3.

Notice that there are also nonrelative versions of the Cotlar identity in which the subalgebra $\mathcal{N} \subset \mathcal{M}$ is in effect taken to be 0:

$$H(f)\,H(f)^* \;=\; H\big(f\,H(f)^*\big) + H\big(f\,H(f)^*\big)^* - H\big(H(ff^*)\big)^*, \quad \forall f \in \mathcal{A} \cap L_2(\mathrm{G}). \qquad (\mathrm{Cotlar}_{\mathrm{nr}})$$

When working with groups we will, perhaps ambiguously, refer to this identity as the Cotlar identity —without specifying any subgroup— only when the group G is continuous i.e. $\mu(\{e\}) = 0$. In the case in which the group G is discrete we will say T_m satisfies the Cotlar identity if it satisfies the relative Cotlar identity $(\mathrm{Cotlar}_{\mathbb{E}^{\perp}})$ with respect to the subgroup $\{e\}$, which gives $\mathbb{E}^{\perp}[f] = f - \tau(f)\mathbf{1}$.

We will need the following lemma.

Lemma 1.2 ([43, Proposition 3.4]). *Let $\mathbb{E} : \mathcal{M} \to \mathcal{N}$ be a conditional expectation and $H : \mathcal{A} \to \mathcal{A}$ a left $\mathcal{N}$-modular map. It holds that*

$$\mathbb{E}\big[H(f)H(f)^*\big] \leq \|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\|^2\, \mathbb{E}\big[ff^*\big]. \qquad (1.1)$$

Furthermore, if $\mathbb{E}H = H\mathbb{E}$, we have that for every $1 \leq p \leq \infty$

$$\begin{aligned} \big\|\mathbb{E}\big[H\big(f\,H(f)^*\big)\big]\big\|_p &\leq \big\|\mathbb{E}H(\mathbf{1})\big\|_\infty \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|\, \big\|\mathbb{E}\big[ff^*\big]\big\|_p & (1.2) \\ \big\|\mathbb{E}\left[H\big(H(ff^*)^*\big)\right]\big\|_p &\leq \big\|\mathbb{E}H(\mathbf{1})\big\|_\infty^2\, \big\|\mathbb{E}\big[ff^*\big]\big\|_p & (1.3) \end{aligned}$$

Proof. All of the points are elementary. For (1.1) first notice that every state of $\mathcal{N}$ is of the form $f \mapsto \tau(\delta\, f)$, where δ is a positive element of norm 1 in the space $L_1(\mathcal{N})$. Decomposing it as $\delta = \delta^{\frac{1}{2}}\delta^{\frac{1}{2}}$ gives

$$\begin{aligned} \tau\left\{\delta\,\mathbb{E}\big[H(f)H(f)^*\big]\right\} &= \tau\left\{\delta^{\frac{1}{2}}\mathbb{E}\big[H(f)H(f)^*\big]\delta^{\frac{1}{2}}\right\} \\ &= \tau\left\{\mathbb{E}\big[\delta^{\frac{1}{2}}H(f)H(f)^*\delta^{\frac{1}{2}}\big]\right\} \\ &= \tau\left\{\mathbb{E}\big[H(\delta^{\frac{1}{2}}f)H(\delta^{\frac{1}{2}}f)^*\big]\right\} \\ &= \tau\left\{H(\delta^{\frac{1}{2}}f)H(\delta^{\frac{1}{2}}f)^*\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \; \big\|H : L^2(\mathcal{M}) \to L^2(H)\big\|^2 \, \tau \left\{ (\delta^{\frac12} f)(\delta^{\frac12} f)^* \right\} \\
&= \; \big\|H : L^2(\mathcal{M}) \to L^2(H)\big\|^2 \, (\tau \circ \mathbb{E}) \left\{ (\delta^{\frac12} f)(\delta^{\frac12} f)^* \right\} \\
&= \; \big\|H : L^2(\mathcal{M}) \to L^2(H)\big\|^2 \, \tau \left\{ \delta \, \mathbb{E}\big[f f^* \big] \right\}.
\end{aligned}
$$

Since this is true for every state, the operator inequality (1.1) holds.

For (1.2) we use that $\mathbb{E}H = H\mathbb{E}$ to rewrite $\mathbb{E}\left[H\big(f\,H(f)^*\big)\right]$ as $(\mathbb{E}H\mathbb{E}) \circ \mathbb{E}\left[\big(f\,H(f)^*\big)\right]$. The operator norm on $\mathbb{E}H\mathbb{E} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is bounded by that of $\mathbb{E}H\iota : L_p(\mathcal{N}) \to L_p(\mathcal{N})$, where $\iota : L_p(\mathcal{N}) \hookrightarrow L_p(\mathcal{M})$ is the natural isometric inclusion. By the left $\mathcal{N}$-modularity of H we have that

$$
\mathbb{E}H(f) \,=\, \mathbb{E}H(f\,\mathbf{1}) \,=\, f\,\mathbb{E}H(\mathbf{1}) \quad \forall f \in L_p(\mathcal{N}) \cap \mathcal{A}. \tag{1.4}
$$

By the density of $L_p(\mathcal{N}) \cap \mathcal{A}$ in $L_p(\mathcal{N})$ we obtain that $\mathbb{E}H\iota$ is a right multiplication operator on $L_p(\mathcal{N})$ and thus its norm is $\|\mathbb{E}H(\mathbf{1})\|_\infty$. To estimate the term $\mathbb{E}\left[\big(f\,H(f)^*\big)\right]$ we will use the following version of Hölder's inequality [34, Inequality (2.1)]

$$
\big\|\mathbb{E}\big[f\,g^*\big]\big\|_p \leq \big\|\mathbb{E}\big[f\,f^*\big]^{\frac12}\big\|_r \, \big\|\mathbb{E}\big[g\,g^*\big]^{\frac12}\big\|_s \quad \text{when} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s},
$$

with $r = s = 2p$ and $g = H(f)$ to obtain that

$$
\big\|\mathbb{E}\left[f\,H(f)^*\right]\big\|_p \leq \big\|\mathbb{E}\left[f\,f^*\right]\big\|_p^{\frac12} \, \big\|\mathbb{E}\left[H(f)\,H(f)^*\right]\big\|_p^{\frac12}.
$$

Applying the inequality in (1.1) gives the result.

Identity (1.3) follows immediately after using two times the fact that H and $\mathbb{E}$ commutes and that $\mathbb{E}H\mathbb{E}$ has a norm in L_p bounded by $\|\mathbb{E}H(\mathbf{1})\|_\infty$. □

Remark 1.3. The reason why we need to include the unit in the algebra $\mathcal{A}$ is in order to make sense of (1.4). In many natural examples, like in the classical Hilbert transform (HT), the use of a principal value in the integral automatically sends the constant functions to 0, this trivially including them in the domain of definition. Nevertheless, definitions based on functions of the Schwartz class may be undefined over constants. In order to apply the framework of this Section to such operators it is necessary to extend H to $\mathbb{C}\mathbf{1}$ in a way that preserves the left $\mathcal{N}$-modularity. This will be trivial in cases in which $\mathcal{N} = \mathbb{C}\mathbf{1}$ or —when dealing with multipliers— if $H(\mathbf{1})$ is chosen to be $m(e)\mathbf{1}$ where $m(e)$ is the essentially unique value of m over G_0.

We can now prove the following extrapolation result.

Proposition 1.4 ([43, Theorem 3.5])**.** *Let $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{A}$ be as before and let $H : \mathcal{A} \to \mathcal{A}$ be a left $\mathcal{N}$-modular operator commuting with $\mathbb{E} : \mathcal{M} \to \mathcal{N}$. If H satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *then* $\forall\, 2 \leq p < \infty$

$$
\big\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\| \,\lesssim\, p^\beta \, \max\left\{ \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|, \, \big\|\mathbb{E}H(\mathbf{1})\big\|_\infty \right\}
$$

where $\beta = \log_2(1 + \sqrt{2})$.

Proof. First, let us denote the operator norm on L_p of H by $c_p := \big\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\|$. We are going to proceed by induction, assuming that $c_p < \infty$ to prove that $c_{2p} < \infty$. Choose $f \in \mathcal{A} \cap L_p(\mathcal{M})$ with $\|f\|_p \leq 1$ and notice that

$$
\begin{aligned}
\|H(f)\|_{2p}^2 \;&=\; \|H(f)H(f)^*\|_p \\
&\leq\; \big\|\mathbb{E}\big[H(f)H(f)^*\big]\big\|_p + \big\|\mathbb{E}^\perp\big[H(f)H(f)^*\big]\big\|_p \\
&\leq\; c_2^2 \,\big\|\mathbb{E}\big[ff^*\big]\big\|_p + \big\|\mathbb{E}^\perp\big[H\big(f\,H(f)^*\big) + H\big(f\,H(f)^*\big)^* - H\big(H(ff^*)^*\big)\big]\big\|_p \qquad (1.5) \\
&\leq\; c_2^2 \,\|f\|_{2p}^2 + \big\|H\big(f\,H(f)^*\big)\big\|_p + \big\|H\big(f\,H(f)^*\big)^*\big\|_p + \big\|H\big(H(ff^*)^*\big)\big\|_p
\end{aligned}
$$

$$+\big\|\mathbb{E}\big[H\big(f\,H(f)^*\big)\big]\big\|_p + \big\|\mathbb{E}\big[H\big(f\,H(f)^*\big)^*\big]\big\|_p + \big\|\mathbb{E}\big[H\big(H(ff^*)^*\big)\big]\big\|_p \tag{1.6}$$

$$\leq \quad \big(c_2^2 + 2\,c_p\,c_{2p} + c_p^2 + 2\kappa\,c_2 + \kappa^2\big)\|f\|_{2p}^2, \tag{1.7}$$

where $\kappa = \|\mathbb{E}H(\mathbf{1})\|_\infty$. We have used (1.1) in the second term of the sum of (1.5) and estimate (1.1) of Lemma 1.2 in the first. To pass from (1.6) to (1.7) we have used the other two identities of Lemma 1.2. Now, taking supremum over $\|f\|_{2p} \leq 1$ and using the norm density of $\mathcal{A} \cap L_{2p}(\mathcal{M})$ in $L_{2p}(\mathcal{M})$ allows to get c_{2p}^2 on the left hand side. Setting $a_p = c_p/\max\{\kappa, c_2\}$ gives the recursive inequality

$$a_{2p}^2 \leq 2\,a_p\,a_{2p} + a_p^2 + 4. \tag{1.8}$$

Adding a_{2p}^2 to both sides in order to complete squares gives

$$2a_{2p}^2 \;\leq\; a_{2p}^2 + 2\,a_p\,a_{2p} + a_p^2 + 4 \;=\; (a_{2p} + a_p)^2 + 4 \;\leq\; (a_{2p} + a_p + 2)^2.$$

After taking square roots and recursively applying the inequality above, the following is obtained

$$a_{2^k} \;\leq\; C\,(1+\sqrt{2})^{k-1}, \quad \text{with} \quad C \leq 3 + \sqrt{2}.$$

This, together with Marcinkiewicz interpolation for intermediate values of p, gives the desired inequality. □

We are now going to prove the equivalence between Cotlar's identity ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) and the closed formula in Theorem A.

Theorem 1.5. *Let* $\mathrm{G}_0 \subset \mathrm{G}$ *be an open subgroup of* G *and* $m : \mathrm{G} \to \mathbb{C}$ *be a bounded function. The following properties are equivalent*

- **(i)** T_m *satisfies* ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$).
- **(ii)** *The function* m *satisfies that*

$$\big(m(g^{-1}) - m(h)\big)\,\big(m(gh) - m(g)\big) \;=\; 0, \quad \forall g \in \mathrm{G} \setminus \mathrm{G}_0, h \in \mathrm{G},$$

Proof. Expanding ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) for T_m gives

$$0 \;=\; \underbrace{\mathbb{E}^\perp\Big[T_m(f)\,T_m(f)^*\Big]}_{\text{(I)}} - \underbrace{\mathbb{E}^\perp\Big[T_m\big(f\,T_m(f)^*\big)\Big]}_{\text{(II)}} - \underbrace{\mathbb{E}^\perp\Big[T_m\big(f\,T_m(f)^*\big)^*\Big]}_{\text{(III)}} + \underbrace{\mathbb{E}^\perp\Big[T_m\big(T_m(ff^*)^*\big)\Big]}_{\text{(IV)}}.$$

Now, elementary computations yield that

$$\begin{aligned}
\text{(I)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(gh)\,\overline{m(h)}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(II)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(g)\,\overline{m(h)}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(III)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(gh)\,\overline{m(g^{-1})}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(IV)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(g)\,\overline{m(g^{-1})}\,\lambda_g\,d\mu(h)\,d\mu(g)
\end{aligned}$$

which in turn imply, using the Plancherel theorem, that

$$0 = \int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,\big(m(gh) - m(g)\big)\,\big(\overline{m(h) - m(g^{-1})}\big)\,d\mu(h) \quad \text{for almost every } g \in \mathrm{G} \setminus \mathrm{G}_0.$$

Obviously, if the factor $G_g(h) = (m(gh) - m(g))\,(\overline{m(h) - m(g^{-1})})$ is equal to 0 so is the above integral and therefore ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) holds. The reciprocal is immediate in the case of discrete groups. Indeed, choose any $h_0 \in \mathrm{G}$ and assume that $g \in \mathrm{G} \setminus \mathrm{G}_0$ is fixed. Pick $\widehat{f} = \delta_{h_0} + \delta_{gh_0}$. In order to evaluate the integral, notice that

$$\widehat{f}(gh)\, \overline{\widehat{f}(h)} = \delta_{\{h=h_0\}} + \delta_{\{g^2=e\}} \cdot \delta_{\{h=gh_0\}}.$$

The term $\delta_{h=h_0}$ in the above sum gives $G_g(h_0)$ in the integral. The term in which $g^2 = e$ and $h = gh_0$ gives $\overline{G_g(h_0)}$. Therefore, $\operatorname{Re} G_g(h) = 0$ for any $h \in \mathrm{G}$. The imaginary part is similarly shown to be 0. In the case of a continuous group G it is necessary to change δ_{h_0} with a modification of the unit. $\square$

With all that at hand we are ready to prove Theorem A.

Proof (of Theorem A). Observe that, if m is left-G_0 invariant, then T_m is left $\mathcal{L}\mathrm{G}_0$-modular. We also have that $\|T_m : L_2(\mathcal{L}\mathrm{G}) \to L_2(\mathcal{L}\mathrm{G})\| = \|m\|_\infty$ and that $\|\mathbb{E}T_m(\mathbf{1})\|_\infty = |m(e)|$. Since the closed formula in (ii) is equivalent to $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ by Theorem 1.5, we can apply Proposition 1.4 to obtain the bound (1) for $p \geq 2$, while for $1 < p \leq 2$ the result follows by standard duality arguments. $\square$

Remark 1.6. Notice Theorem A follows equally in the non-relative case in which we assume $(\mathrm{Cotlar}_{\mathrm{nr}})$ instead of $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$. In this case, the extrapolation theorem works verbatim while the computations to obtain the factorization identity $(\widehat{\mathrm{Cotlar}})$ follow by repeating all the calculations without $\mathbb{E}^\perp$. In fact, the whole reason for which a unified statement has not been given is that the empty set can not be a subgroup since it doesn't contain $e \in \mathrm{G}$.

Remark 1.7. Let $\alpha : \mathcal{N} \to \mathcal{N}$ be a normal and trace preserving $*$-homomorphism. It is immediate that both Proposition 1.4 as well as Lemma 1.2 hold if we change the condition of H being left $\mathcal{N}$-modular by that of being left $\mathcal{N}$-modular relative to α, i.e.,

$$H(f\, g) = \alpha(f)\, H(g), \qquad \text{for } f \in \mathcal{N} \cap \mathcal{A},\ g \in \mathcal{A}.$$

In the case of multipliers this easy observation has deep consequences. For instance, let $\chi : \mathrm{G}_0 \to \mathbb{T}$ be a (multiplicative) character. It is a straightforward consequence of Fell's absorption principle that the map $\lambda_g \mapsto \chi(g)\, \lambda_g$ induces a normal and trace-preserving $*$-homomorphism $\alpha_\chi : \mathcal{L}\mathrm{G}_0 \to \mathcal{L}\mathrm{G}_0$. Let $H = T_m$ be a Fourier multiplier on $\mathcal{L}\mathrm{G}$. We have that it is left $\mathcal{L}\mathrm{G}_0$-modular with respect to α_χ, ie $H(f\, g) = \alpha_\chi(f)\, H(g)$, for every $f \in \mathcal{A} \cap \mathcal{N}$ and $g \in \mathcal{A}$ iff

$$m(k\, g) = \chi(k)\, m(g), \qquad \text{for every } k \in \mathrm{G}_0,\ g \in \mathrm{G}. \tag{1.9}$$

This is specially useful when G_0 is abelian since, in that case, every function in G_0 can be expressed as a convex combination of characters by the Fourier transform. This will be exploited in a forthcoming paper of the third named author [66].

Tightening the constant. It is known that, in the real line $\mathrm{G} = \mathbb{R}$ the operator L_p-norm of the classical Hilbert transform (HT) is given by

$$\big\| H : L_p(\mathbb{R}) \to L_p(\mathbb{R}) \big\| \, = \, \max \Big\{ \tan\Big(\frac{\pi}{2\,p}\Big),\ \cot\Big(\frac{\pi}{2\,p}\Big) \Big\}, \qquad \text{for } 1 < p < \infty$$

see [54] or [29] for a simplified proof. These constants grow asymptotically like p as $p \to \infty$ and like $1/(p-1)$ as $p \to 1^+$, and those are the growth orders that we conjecture optimal in the noncommutative case as well. An interesting observation, originally made by Gokhberg and Krupnik in the classical case [25] is that Cotlar's identity in the real line gives the optimal order of growth for the constant in terms of p. Indeed, in the classical case, the fact that $H^2 = -\mathrm{id}$ yields a recurrence relation of the form

$$c_{2p}^2 \, \leq \, 2 c_p\, c_{2p} + 1 \tag{1.10}$$

instead of (1.8). The lack of a term depending on c_p^2 gives a decisively smaller bound. Solving the quadratic inequality in (1.10), gives

$$c_{2p} \leq c_p + \sqrt{c_p^2 + 1}$$

and that results, after applying duality and interpolation, in the optimal growth order for the constant.

In the noncommutative case, the same type of argument holds for operators $H : \mathcal{A} \to \mathcal{A}$ satisfying that $H\, H_{\mathrm{op}} = -\mathrm{id}$, where $H_{\mathrm{op}}(f) = H(f^*)^*$. We have the following improvement over Proposition 1.4.

Proposition 1.8. *Let $\mathcal{N} \subset \mathcal{M}$, $\mathcal{A}$ and $H : \mathcal{A} \to \mathcal{A}$ be as before and assume H is left $\mathcal{N}$-modular, commutes with $\mathbb{E} : \mathcal{M} \to \mathcal{N}$ and satisfies that $\mathbb{E}^{\perp} H\, H_{\mathrm{op}} = -\mathbb{E}^{\perp}$. If H satisfies* ($\mathrm{Cotlar}_{\mathbb{E}^{\perp}}$)*, then*

$$\big\| H : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \big\| \, \lesssim \, p \, \max \Big\{ \big\| H : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \big\|, \| \mathbb{E} H(\mathbf{1}) \|_\infty \Big\} \quad \text{for every } p \geq 2.$$

Proof. The proof is immediate once it is noticed that the property $\mathbb{E}^{\perp} H\, H_{\mathrm{op}} = -\mathbb{E}^{\perp}$ implies that ($\mathrm{Cotlar}_{\mathbb{E}^{\perp}}$) can be rewritten as

$$\mathbb{E}^{\perp} \big[H(f)\, H(f)^* \big] \, = \, \mathbb{E}^{\perp} \Big[H\big(f\, H(f)^*\big) + H\big(f\, H(f)^*\big)^* + f f^* \Big].$$

Applying the same proof of Proposition 1.4 gives the recurrence

$$c_{2p}^2 \, \leq \, 2\, c_{2p}\, c_p + 2 + 3 \max \big\{ c_2^2, \| \mathbb{E} H(\mathbf{1}) \|_\infty^2 \big\}. \tag{1.11}$$

After solving the quadratic inequality, we obtain

$$c_{2p} \leq c_p + \sqrt{c_p^2 + \kappa},$$

where $\kappa = 2 + 3 \max \big\{ c_2^2, \| \mathbb{E} H(\mathbf{1}) \|_\infty^2 \big\}$. Iterating and applying Marcinkiewicz interpolation gives the bound. □

Observe that if $H = T_m$ is a Fourier multiplier, then $(T_m)_{\mathrm{op}} = T_{\tilde{m}}$, for $\widetilde{m}(g) = \overline{m(g^{-1})}$. Thus, we are asking that $m(g)\, \overline{m(g^{-1})} = -1$ for every $g \in \mathrm{G} \setminus \mathrm{G}_0$. Similarly, since in the case of multipliers $\| \mathbb{E} H(\mathbf{1}) \|_\infty = \| m\, \mathbf{1}_{\mathrm{G}_0} \|_\infty \leq \| m \|_\infty = c_2$ we can simplify the recurrence above assuming $c_2 = 1$. In particular, we obtain the following corollary.

Corollary 1.9. *Let* G *be a group and let* $m : \mathrm{G} \to \mathbb{C}$ *be a function satisfying* ($\widehat{\mathrm{Cotlar}}$) *relative to a subgroup* $\mathrm{G}_0 \subset \mathrm{G}$*, and such that* m *is left* G_0*-invariant and* $m(g)\overline{m(g^{-1})} = -1$*, for every* $g \in \mathrm{G} \setminus \mathrm{G}_0$*. Then*

$$\big\| T_m : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G}) \big\|_{\mathrm{cb}} \, \lesssim \, \Big(\frac{p^2}{p-1} \Big) \| m \|_\infty.$$

Both the Proposition 1.8 and Corollary 1.9 are also true if one changes the condition that $m(g)\, \overline{m(g^{-1})} = -1$ for $g \in \mathrm{G} \setminus \mathrm{G}_0$ by any other constant λ independent of g.

The convex hull of Cotlar-type multipliers. A natural question is which class of multipliers m can be shown to be bounded in $L_p(\mathcal{L}\mathrm{G})$ by being represented as a convex combination of multipliers satisfying ($\mathrm{Cotlar}_{\mathbb{E}^{\perp}}$) or natural modifications of them. To that end notice that if m is an L_p-bounded multiplier, then so is $g \mapsto m(h^{-1} \theta(g) r)$, where $h, r \in \mathrm{G}$, $\theta \in \mathrm{Aut}(\mathrm{G})$ and their norms coincide. Let us denote the group of transformations of G given by $g \mapsto h^{-1} \theta(g) r$ as the *affine transformations* $\mathrm{Aff}(\mathrm{G})$ *of* G. Observe that, if we define $\mathrm{G}^{\Delta} = (\mathrm{G} \oplus \mathrm{G})/\Delta$, where $\Delta = \{(a,a) : a \in \mathcal{Z}(\mathrm{G})\} \subset \mathrm{G} \oplus \mathrm{G}$, then there is a faithful representation that sends (h, r) to $g \mapsto h^{-1}\, g\, r$. A trivial computation gives that

$$\mathrm{Aff}(\mathrm{G}) \cong \mathrm{G}^{\Delta} \rtimes \mathrm{Aut}(\mathrm{G}),$$

with the natural action. It is clear that, if $\mu \in M(\mathrm{Aut}(\mathrm{G}))$ is a finite signed measure and $m : \mathrm{Aff}(\mathrm{G}) \times \mathrm{G} \to \mathbb{C}$ is a bounded map such that $g \mapsto m(\alpha, g)$ satisfies ($\mathrm{Cotlar}_{\mathbb{E}^{\perp}}$) for every α, then

$$m(g) \, = \, \int_{\mathrm{Aff}(\mathrm{G})} m(\alpha, \alpha(g))\, d\mu(\alpha),$$

is clearly bounded in $L_p(\mathcal{L}\mathrm{G})$ for every $1 < p < \infty$. We could add more flexibility to this technique by allowing the map $g \mapsto m(\alpha, g)$ to be the product of k terms satisfying ($\mathrm{Cotlar}_{\mathbb{E}^{\perp}}$). Let us call this class $\mathrm{coCot}^k(\mathrm{G})$. We leave mostly unexplored the following natural problem

Problem 1.10. *Let* $\mathrm{G}_0 \subset \mathrm{G}$ *and* $m : \mathrm{G} \to \mathbb{C}$ *be as above. Are there sufficient conditions, for example in terms of smoothness, implying that* $m \in \mathrm{coCot}^k(\mathrm{G})$*?*

This remains as a underexplored approach to prove the boundedness of Fourier multipliers over groups without recourse to noncommutative analogues of singular integral theory.

Observe that in the classical case of $\mathbb{R}$ any function of bounded variation lays in the convex hull of (translations of) the classical Hilbert transform and therefore $m \in \mathrm{BV}(\mathbb{R}) \implies m \in \mathrm{coCot}^1(\mathbb{R})$, see [20, Corollary 3.8]. In higher dimensions the behavior is even richer. For instance, let $m : \mathbb{R}^2 \to \mathbb{C}$ be a function satisfying that $m(\lambda \xi) = m(\xi)$ for every $\lambda > 0$. Clearly, m depends only on its angular component $m|_{\mathbb{T}}$. We have that

$$m|_{\mathbb{T}} \in \mathrm{BV}(\mathbb{T}) \qquad \implies \qquad m \in \mathrm{coCot}^2(\mathbb{R}^2).$$

To see that, let $v_\theta = (\cos\theta, \sin\theta)$ and let $\Sigma_\theta \subset \mathbb{R}^2$ be the sector of all vectors whose polar angle ω lays in $[0,\theta)$. We can write its characteristic function as

$$\mathbf{1}_{\Sigma_\theta}(\underbrace{\xi_1,\xi_2}_{\xi}) \; = \; \left(\frac{\mathrm{sgn}\langle \xi, v_{\frac{\pi}{2}} \rangle + 1}{2} \right) \left(\frac{\mathrm{sgn}\langle \xi, v_{\theta - \frac{\pi}{2}} \rangle + 1}{2} \right),$$

when $\theta < \pi$ and a similar expression otherwise. But, any function $\widetilde{m} : \mathbb{T} \to \mathbb{C}$ of bounded variation can be identified with a function on $[0,2\pi]$ such that $\widetilde{m}(2\pi) - \widetilde{m}(0)$ equals the jump discontinuity of the original function around 0. Elementary manipulations show that $\widetilde{m}$ lays in the convex hull of the functions $\mathbf{1}_{[0,\theta]}$. Radially extending the argument gives that m is in the convex hull of $\mathbf{1}_{\Sigma_\theta}$.

Generalizations. It worth noticing that the identity ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) can be generalized in a natural way by changing the equality by an operator inequality

$$\mathbb{E}^\perp \big[H(f)\, H(f)^* \big] \; \le \; \mathbb{E}^\perp \Big[H\big(f\, H(f)^*\big) + H\big(f\, H(f)^*\big)^* - H\big(H(ff^*)^*\big) \Big], \qquad (\mathrm{Cotlar}^{\le}_{\mathbb{E}^\perp})$$

for every $f \in \mathcal{A} \cap L_2(\mathcal{M})$. It is clear that this inequality implies the same bound $\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| \lesssim p^\beta \, \|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\|$ of Proposition 1.4 for left $\mathcal{N}$-modular operators. A more interesting question is whether there exists a closed-formula characterization of Fourier multipliers T_m with left G_0 invariant symbol m satisfying ($\mathrm{Cotlar}^{\le}_{\mathbb{E}^\perp}$). To formulate such characterization let us define

$$\Omega_m(g,h) = \big(m(g) - m(gh^{-1})\big)\, \overline{\big(m(h) - m(hg^{-1})\big)}\, \mathbf{1}_{\mathrm{G} \setminus \mathrm{G}_0}(gh^{-1})$$

and notice that ($\mathrm{Cotlar}^{\le}_{\mathbb{E}^\perp}$) is actually equivalent to

$$0 \; \le \; \int_{\mathrm{G}\setminus\mathrm{G}_0} \int_{\mathrm{G}} \widehat{f}(gh)\, \overline{\widehat{f}(h)}\, \Omega_m(gh,h)\, \lambda_g \, d\mu(h)\, d\mu(g).$$

This is the key to the following

Theorem 1.11. *Let* G *be a unimodular group and let* $\mathrm{G}_0 \subset \mathrm{G}$ *be an open subgroup and* m *be a left* G_0*-invariant function. The following are equivalent.*

- **(i)** *The operator* $H = T_m$ *satisfies* ($\mathrm{Cotlar}^{\le}_{\mathbb{E}^\perp}$).
- **(ii)** *There is a Hilbert space* $\mathcal{H}$ *and a bounded measurable function* $\xi : \mathrm{G} \to \mathcal{H}$ *such that*

$$\Omega_m(g,h) = \big\langle \xi(h), \xi(g) \big\rangle.$$

If any of the two conditions hold, then bound (1) *is satisfied.*

Proof. It is clear that if there exists a map $\xi : \mathrm{G} \to \mathcal{H}$ as above, then

$$\begin{aligned} &\int_{\mathrm{G}\setminus\mathrm{G}_0} \int_{\mathrm{G}} \widehat{f}(gh)\, \overline{\widehat{f}(h)}\, \Omega_m(gh,h)\, \lambda_g \, d\mu(h)\, d\mu(g) \\ &\qquad\qquad = \Big\langle \int_{\mathrm{G}} \widehat{f}(g)\, \xi(g) \otimes \lambda_g \, d\mu(g), \, \int_{\mathrm{G}} \widehat{f}(h)\, \xi(h) \otimes \lambda_h \, d\mu(h) \Big\rangle_{\mathcal{X}}, \end{aligned}$$

where $\mathcal{X}$ is the left Hilbert $\mathcal{L}\mathrm{G}$-module given by completing $\mathcal{H} \otimes \mathcal{L}\mathrm{G}$ with the inner product $\langle \xi \otimes f, \eta \otimes g \rangle_{\mathcal{X}} = \langle \eta, \xi \rangle f g^*$. The order of the entries ξ and η is switched to maintain the convention that every scalar Hilbert product is antilinear in the first component. The same applies to the statement in point (ii).

For the reciprocal, first notice that the identity ($\mathrm{Cotlar}^{\leq}_{\mathbb{E}^\perp}$) can be understood as a positivity condition for a quadratic form. Thus, applying polarization gives the sesquilinear form $B_H : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ given by

$$B_H(f,g) \, = \, \mathbb{E}^\perp \Big[- H(g) \, H(f)^* + H\big(g \, H(f)^*\big) + H\big(f \, H(g)^*\big)^* - H\big(H(f \, g^*)^*\big) \Big],$$

where $\mathcal{D} := \mathcal{A} \cap L_2(\mathcal{M}) = \lambda[L_\infty(\mathrm{G})_\mathrm{c}]$. Observe that the unbounded map $\mathcal{E} : \mathcal{D} \subset \mathcal{L}\mathrm{G} \to \mathbb{C}$ given by

$$\mathcal{E}\left(\int_\mathrm{G} \widehat{f}(g) \, \lambda_g \, d\mu(g) \right) \, = \, \int_\mathrm{G} \widehat{f}(g) \, d\mu(g)$$

is positive and therefore $\langle f, g \rangle_H = \mathcal{E}(B_H(f,g))$ is an inner product. Now, we can construct a Hilbert space $\mathcal{H}_0$ by quotienting out the nulspace and taking closures of $\mathcal{D}$ as usual. Notice that, in the case of discrete G, it holds that $\lambda_g \in \mathcal{D}$ and that $\langle \lambda_g, \lambda_h \rangle_H = \Omega_m(g,h)$. Therefore, defining $\xi(g)$ as the class in $\mathcal{H}_0$ of λ_g gives the desired result. In the continuous case we can substitute λ_g for $\lambda(\delta_g * \psi_\alpha)$, where $(\psi_\alpha)_\alpha \subset \mathcal{D}$ is an approximation of the unit and apply standard ultraproduct arguments. □

This more general Cotlar identity ($\mathrm{Cotlar}^{\leq}_{\mathbb{E}^\perp}$) leads to the following problem, which we leave unexplored.

Problem 1.12. Let $\mathrm{G}_0 \subset \mathrm{G}$ and $m : \mathrm{G} \to \mathbb{C}$ be as above. Is there a geometric model of $\mathrm{G} \curvearrowright \mathcal{X}$, possibly generalizing that of actions on UAC spaces in Theorem B, such that if $m(g)$ lifts to $\mathcal{X}$ via a function $m(g) = \widetilde{m}(g \cdot x_0)$, then it satisfies the condition in Theorem 1.11.(ii).

2. Groups acting on $\mathbb{R}$-trees

Let X be a Hausdorff topological space. An arc γ on X is a subset of X that is the image of an injective continuous function mapping $[0,1]$ onto γ. The space X is said to be *uniquely arcwise connected*, or UAC, if any two points in X are joined by a unique arc. We say that a group G acts on an UAC space X if G acts by homeomorphisms on X.

If, in addition, the UAC space X is metrisable and there is metric $d : X \times X \to \mathbb{C}$ such that the unique arc joining two points is isometric to a closed interval of the real line, then (X,d) is called an $\mathbb{R}$-tree. We will say that a group G acts on an $\mathbb{R}$-tree X if it acts on it by isometries.

Observe that this definition is topological in nature since the underlying space is required to be arcwise connected. An alternative route to $\mathbb{R}$-trees can be taken by defining them as hyperbolic spaces with $\delta = 0$, i.e., every triangle is a *tripod*. These two definitions, although equivalent in spirit, are slightly different. Namely, a tree seen as a discrete set with the edge metric is a 0-hyperbolic metric space but not an $\mathbb{R}$-tree in our definition. This is not a problem since trees can still be seen as a subclass of $\mathbb{R}$-trees by treating them as *simplicial trees*, i.e. the one-dimensional simplicial complexes obtained from the incidence information of the tree. We also have that, given an $\mathbb{R}$-tree X, if the set of points whose complement has three or more connected components is discrete in X, then X is a simplicial tree. The first definition of $\mathbb{R}$-trees was given by Tits [65], then Morgan and Shalen [46], following earlier results of Alperin and Moss, drew attention to the theory of $\mathbb{R}$-trees by showing how to compactify a generalization of Teichmuller space for a finitely generated group using $\mathbb{R}$-trees. We refer the reader to [6] for more on $\mathbb{R}$-trees.

We define the following two models for a group acting on a UAC space. Let $\mathrm{G} \curvearrowright X$ be a topological action and $x_0 \in X$ a selected point. We will say that a bounded measurable function $\varphi : X \setminus \{x_0\} \to \mathbb{C}$ is *constant along arcs* iff $\varphi(x) = \varphi(y)$ if there is an arc connecting x and y inside $X \setminus \{x_0\}$. This is equivalent to decomposing $X \setminus \{x_0\}$ as a union of arcwise connected subsets and imposing the function to be constant over those subsets.

Model 1. Let $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space, $x_0 \in X$ a point and $\widetilde{m} : X \to \mathbb{C}$ a bounded measurable function such that

(i) $\widetilde{m}$, restricted to $X \setminus \{x_0\}$, is constant along arcs.

(ii) $\widetilde{m}$, is invariant under the action $\mathrm{St}_{x_0} \curvearrowright X \setminus \{x_0\}$.

Then, we define the multiplier $m : \mathrm{G} \to \mathbb{C}$ as

$$m(g) = \widetilde{m}(g \cdot x_0)$$

and fix $(\mathrm{G}_0, \mathrm{G})$ as $(\mathrm{St}_{x_0}, \mathrm{G})$.

This definition has the drawback that the invariance under St_{x_0} of m can make the symbol constant outside St_{x_0} in some cases. We introduce the following, more involved, model.

Model 2. Let us fix two distinct constants $C_1, C_2 \in \mathbb{C}$. Let similarly $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space and $x_0 \in X$ a point. Choose $X_0 \subset X \setminus \{x_0\}$ an arcwise connected subset. We define $m : \mathrm{G} \to \mathbb{C}$ to be

$$m(g) = \begin{cases} 0 & \text{when } g \in \mathrm{St}_{x_0} \text{ and } g \cdot X_0 = X_0 \\ C_1 & \text{when } g \in \mathrm{St}_{x_0} \text{ and } g \cdot X_0 \neq X_0 \\ C_1 & \text{when } g \notin \mathrm{St}_{x_0} \text{ and } g \cdot x_0 \notin X_0 \\ C_2 & \text{when } g \cdot x_0 \in X_0. \end{cases}$$

We will also fix G_0 to be $\mathrm{St}_{x_0} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\}$.

Observe that Model 2 is a natural modification of Model 1 for the function $\widetilde{m} : X \to \mathbb{C}$ given by $\widetilde{m} = C_1 \mathbf{1}_{X \setminus (\{x_0\} \cup X_0)} + C_2 \mathbf{1}_{X_0}$. The main difference is that we extend the value of C_1 to a portion of the stabilizer.

Proposition 2.1. *Let* $\mathrm{G} \curvearrowright X$ *be an action as above.*

(i) *Let* $m : \mathrm{G} \to \mathbb{C}$ *and* G_0 *be like in Model 1. Then,* T_m *satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *relative to* G_0 *and is left* $\mathcal{L}\mathrm{G}_0$*-modular.*

(ii) *Let* $m : \mathrm{G} \to \mathbb{C}$ *and* G_0 *be like in Model 2. Then,* T_m *satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *relative to* G_0 *and is left* $\mathcal{L}\mathrm{G}_0$*-modular.*

The points above imply that T_m *in both model 1 and 2 are bounded in* $L_p(\mathcal{L}\mathrm{G})$ *for* $1 < p < \infty$.

Proof. The statement in point (i) has already been proved in the introduction. Thus, we concentrate on point (ii). The fact that m is left invariant under the action of $\mathrm{G}_0 = \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\}$ is immediate. Now, all we have to do is to verify $(\widehat{\mathrm{Cotlar}})$. To that end, let us divide the group as a disjoint union $\mathrm{G} = \mathrm{G}_0 \cup \mathrm{G}_1 \cup \mathrm{G}_2 \cup \mathrm{G}_3$, where

$$\begin{aligned} \mathrm{G}_0 &= \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\} \\ \mathrm{G}_1 &= \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 \neq X_0\} \\ \mathrm{G}_2 &= \{g \in \mathrm{G} : g \cdot x_0 \neq x_0\} \cap \{g \in \mathrm{G} : g \cdot x_0 \notin X_0\} \\ \mathrm{G}_3 &= \{g \in \mathrm{G} : g \cdot x_0 \neq x_0\} \cap \{g \in \mathrm{G} : g \cdot x_0 \in X_0\}. \end{aligned}$$

Observe also that, since m is both left and right G_0-invariant, it is enough to verify $(\widehat{\mathrm{Cotlar}})$ for $g, h \in \mathrm{G} \setminus \mathrm{G}_0$. Assume that $m(g^{-1}) \neq m(h)$, otherwise we are done, our aim is to show that $m(gh) = m(g)$. We will proceed by cases. First, assume that $g \in \mathrm{G}_1$. This is equivalent to $g^{-1} \in \mathrm{G}_1$ and therefore $h \in \mathrm{G}_3$. But then, $g\,h \cdot x_0 \notin X_0$ and $g\,h \cdot x_0 \neq x_0$. Therefore $g\,h \in \mathrm{G}_2$ and we get $m(gh) = m(h)$. In the case of $g \in \mathrm{G}_2$ we have the whole range of possibilities and h can belong to G_1, G_2 or G_3. In the case of $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_1$ we have that $gh \in \mathrm{G}_2$. Indeed, $gh \cdot x_0 = g \cdot x_0 \notin X_0$ and it is immediate that gh is not in the stabilizer of x_0. For the second case of $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_2$ the condition $m(g^{-1}) \neq m(h)$ implies that $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in distinct arcwise connected subsets of $X \setminus \{x_0\}$. But then there is a unique path connecting both points that passes through the root. Applying g to the whole arc gives that $g \cdot x_0$ and $gh \cdot x_0$ lay in the same arcwise connected subset and and do not stabilize x_0, see Figure 1. Therefore $m(gh) = m(g)$. The third case is given by $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_3$. Observe that if $g \in \mathrm{G}_2$, then g^{-1} can only be inside G_2 or G_3. But the case $g^{-1} \in \mathrm{G}_3$ can be easily discarded since it will contradict the assumption $m(g^{-1}) \neq m(h)$. But if $g^{-1} \in \mathrm{G}_2$ and $h \in \mathrm{G}_3$, then $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in distinct arcwise connected subsets and we can proceed like in the previous case. It remains to check the case of $g \in \mathrm{G}_3$. We have that h can be in either G_1, G_2 or G_3. In the first case we deduce that $gh \in \mathrm{G}_3$. For the second one we have that if $g \in \mathrm{G}_3$ and $h \in \mathrm{G}_2$, then we can assume that $g^{-1} \in \mathrm{G}_3$, the only other choice being $g^{-1} \in \mathrm{G}_2$ which will contradict the assumption $m(g^{-1}) \neq m(h)$. But this implies that $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in different arcwise connected subsets of $X \setminus \{x_0\}$ and we can apply the argument in Figure 1. Lastly, if $g \in \mathrm{G}_3$ and $h \in \mathrm{G}_3$ we obtain similarly that g^{-1} can only lay in G_2. The same path argument applies. Applying Theorem B, we get the L_p-boundedness of T_m. □

Observe that in the case of $\mathrm{G} = \mathbb{R}$, the Hilbert transform is also of weak type $(1,1)$, ie $H : L_1(\mathbb{R}) \to L_{1,\infty}(\mathbb{R})$ and bounded between $L_\infty(\mathbb{R})$ and the space of *bounded mean oscillation functions* $\mathrm{BMO}(\mathbb{R})$. Both endpoint spaces give —by either complex or real interpolation with L_2— the optimal order for the operator L_p norm of H. The following problem remains open

Problem 2.2. Let G be a group and m a multiplier like in Model 1 or 2. Is it possible to construct spaces X_1 and X_∞, in place of $L_{1,\infty}$ and BMO, such that

- **(i)** $\|T_m : L_1(\mathcal{L}\mathrm{G}) \to \mathrm{X}_1\| < \infty$ and $\|T_m : L_\infty(\mathrm{G}) \to \mathrm{X}_\infty\| < \infty$.
- **(ii)** Interpolation of X_1 or X_∞ with L_2 yields growth of $p^2/(p-1)$ for the operator L_p norm of T_m.

This problem presents at least three challenges. The first difficulty comes from the fact that weak type $(1,1)$ bounds are difficult to obtain for noncommutative singular integral type operators. There are known in some semicommutative examples [49, 8] but open in the case of Quantum Euclidean spaces [28] and in most group settings beside left-orderable groups [56]. The second challenge is that the specific endpoint space X_∞ to be used for m in Model 1 or Model 2 has to be defined in terms of the geometry of $\mathrm{G} \curvearrowright X$ and it cannot just be the usual noncommutative BMO space, see [35, 41, 42]. Indeed, there is a natural *unital and completely positive* semigroup $S_t : \mathcal{L}\mathbb{F}_2 \to \mathcal{L}\mathbb{F}_2$ in the free group algebra given by $S_t(\lambda_g) = e^{-t|g|}$, see [30]. This semigroup allows to construct a natural and interpolating semigroup BMO space $\mathrm{BMO}(\mathcal{L}\mathbb{F}_2)$, see [36]. But it is known that the multipliers (MR) are unbounded from $L_\infty(\mathcal{L}\mathbb{F}_2)$ to $\mathrm{BMO}(\mathcal{L}\mathbb{F}_2)$, see [44, Appendix A]. The third difficulty comes from the fact that the classical technique, employed by Kolmogorov [40], of comparing a singular integral operator with a maximal one is delicate in this context since some of the operators obtained are not positivity preserving, which makes interpolating maximal functions an open problem [38]. The technique of Kolmogorov has been used in the noncommutative case in [33]. We will also mention that some tentative progress in the direction of Problem 2.2 has been made. For instance, Gałązka and Osękowski [24] have lowered the operator L_p-norm of the Fourier multiplier $m : \mathbb{F}_2 \to \mathbb{C}$ that depend on the starting letter from $\mathrm{O}(p^\beta)$ to $\mathrm{O}(p \log p)$ as $p \to \infty$.

Now, we are going to study Models 1 and 2 in the context of $\mathbb{R}$-trees. Observe that, since any $\mathbb{R}$-tree action is an action of the underlying UAC topological space, Proposition 2.1 above works for actions on $\mathbb{R}$-trees. Furthermore, in the case of group actions on $\mathbb{R}$-trees we have the following result connecting the existence of global fixed points with the form of the Fourier multiplier T_m. We are going to say that the multipliers m coming from Models 1 and 2 are *trivial* iff they are constant for any $g \in \mathrm{G} \setminus \mathrm{G}_0$.

Proposition 2.3. *Let X be an $\mathbb{R}$-tree and $\mathrm{G} \curvearrowright X$ an (isometric) action of discrete group. The following holds*

- **(i)** *If the action $\mathrm{G} \curvearrowright X$ has a global fixed point, then for any choice of a root $x_0 \in X$ the multipliers in Model 1 and Model 2 are trivial.*
- **(ii)** *If G is finitely generated, for any action on an $\mathbb{R}$-tree X, there is either a global fixed point or there exists $x_0 \in X$ such that the corresponding symbol given in Model 2 is nontrivial.*

Proof. We will prove first (i) for m as in Model 1. Assume that the actions has a global fixed point $x_1 \in X$. If x_1 coincides with the root x_0 then $m = 0$ and there is nothing to prove. Therefore we assume $x_0 \neq x_1$. Similarly, we can assume without loss of generality that $\mathrm{St}_{x_0} \neq \mathrm{G}$, since otherwise $m = 0$. Pick $g \in \mathrm{G}$ and assume that $g \cdot x_0$ lays in an arcwise connected subset of $X \setminus \{x_0\}$ different from that of x_1. Then, there is a unique path joining x_1 and $g \cdot x_0$ that passes through the root x_0, see Figure 2. But since x_1 is fixed by any $g \in \mathrm{G}$, after applying g to the path we obtain a larger path, which contradict the fact that the action is isometric. This implies that $g^2 \cdot x_0 = g \cdot x_0$ and so

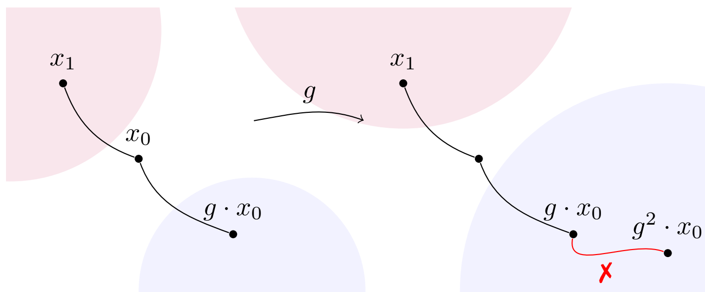

Figure 2: The action of g over the path connecting the global fixed point and $g \cdot x_0$.

$g \cdot x_0 = x_0$, which contradicts the assumptions. Therefore, $g \cdot x_0$ belongs to the same connected subset of x_1 for every g that do not stabilize the root x_0, that is, $m(g)$ is constant for any $g \in \mathrm{G} \setminus \mathrm{St}_{x_0}$.

For the case of m as in Model 2, let $x_1 \in X$ be a global fixed point. We can again consider without loss of generality that $x_0 \neq x_1$ and that $\mathrm{St}_{x_0} \neq \mathrm{G}$ G. By those assumptions, there exists $g_0 \in \mathrm{G}$ with $g_0 \cdot x_0 \in X \setminus \{x_0\}$. We have two possibilities for g_0. If $g_0 \cdot x_0 \neq x_0$ and $g_0 \cdot x_0 \notin X_0$, then $m(g_0) = C_1$, while if $g_0 \cdot x_0 \in X_0$ then $m(g_0) = C_2$. If we are in the first case $g_0 \cdot x_0 \notin X_0$, then the multiplier m will be trivial unless there exists a $g_1 \in \mathrm{G}$ such that $g_1 \cdot x_0 \in X_0$. Let us obtain a contradiction. First, we claim that the global fixed point $x_1 \notin X_0$. Assume $x_1 \in X_0$. Then, there is a path joining x_1 and $g_0 \cdot x_0$ that passes through the root x_0, see Figure 3 But applying g_0 to the path gives that

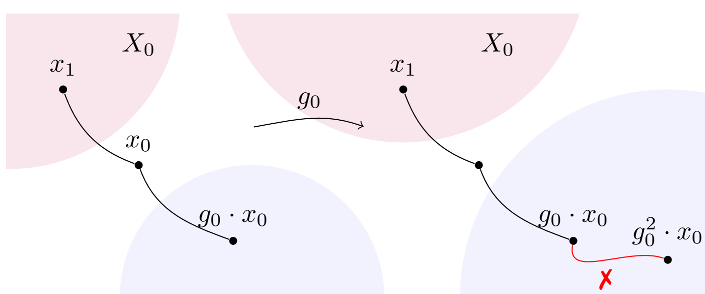

Figure 3: The action of g_0 over the path connecting x_1 and $g_0 \cdot x_0 \notin X_0$.

$g_0 \cdot x_0 = x_0$ which is a contradiction. Therefore $x_1 \notin X_0$, But since $g_1 \cdot x_0 \in X_0$, we can build a path from x_1 to $g_1 \cdot x_0$ that passes through the root x_0 and, repeating the same argument as before, obtain that $g_1 \cdot x_0 = x_0$, which is a contradiction. For the second case $g_0 \cdot x_0 \in X_0$. First, we notice that x_1 must live in X_0, if not, proceeding as before we will get that $g_0 \cdot x_0 = x_0$, which is a contradiction. In order for m to be nontrivial there should be a $g_1 \in \mathrm{G}$ such that either $g_1 \cdot x_0 = x_0$ and $g_1 \cdot X_0 \neq X_0$ or $g_1 \cdot x_0 \neq x_0$ and $g_1 \cdot x_0 \notin X_0$. For the first case, let us choose a point $x_2 \in X_0$. Then, $g_1 \cdot x_2 \notin X_0$ and similarly $g_1 \cdot x_2 \neq x_0$. Now, construct a path joining x_1 with $g_1 \cdot x_2$. Since $x_1 \in X_0$ but $g_1 \cdot x_2$ belongs to a different connected subset, the arc joining them passes through the root x_0. Let us apply g_1^{-1} to the whole arc. Since we have that $g_1^{-1} \cdot x_0 = x_0$, we obtain an arc that joins x_1 and x_2 and passes through the root x_0. But this is a contradiction with the fact that x_1 and x_2 both belong to X_0, see Figure 4, since X is a UAC space the arc joining two elements in the same connected subset of $X \setminus \{x_0\}$ cannot pass trough the root x_0. The remaining case is when $g_1 \cdot x_0 \neq x_0$ and $g_1 \cdot x_0 \notin X_0$.

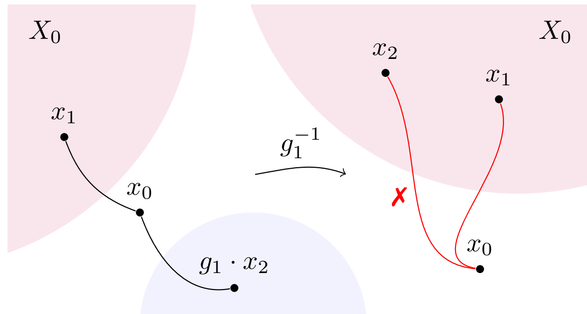

Figure 4: The action of g_1^{-1} over the path connecting x_1 and $g_1 \cdot x_2$.

Then, the arc joining $x_1 \in X_0$ and $g_1 \cdot x_1$ passes through x_0. Applying g_1 gives a contradiction with the fact that the action is isometric and that $g_1 \cdot x_0 \neq x_0$. $\square$

Many groups we are familiar with admit actions on $\mathbb{R}$-trees. For example, every finitely generated hyperbolic group. Indeed, a finitely generated group is hyperbolic if and only if every asymptotic cone of the group is an $\mathbb{R}$-tree [27]. Then, the action of a hyperbolic group on its Cayley graph induces an action on the asymptotic cone of the Cayley graph. Moreover, any surface group having Euler characteristic less than -1 acts freely on an $\mathbb{R}$-tree [47].

On the other hand, there are many examples of groups for which any action on an $\mathbb{R}$-tree has a global fixed point. When this happens the group G is said to have property (F$\mathbb{R}$), see [6, 62]. The following corollary of Proposition 2.3 characterizes groups with property (F$\mathbb{R}$).

Corollary 2.4. G *has property* (F$\mathbb{R}$) *if and only if the symbol m in Model 2 is trivial for any $x_0 \in X$ in any $\mathbb{R}$-tree X on which* G *acts (isometrically).*

One natural question, that we leave open, is whether there are functions $m : \mathrm{G} \to \mathbb{C}$ satisfying (Cotlar) for $\mathrm{G}_0 = \{e\}$ and such that they do not lift to a function $\widetilde{m}$ on a UAC space like in Model 1 or 2 on which G acts. One way to prove the non-existence of such lift $\widetilde{m}$ would be a procedure to assemble from the group G and the function m a G-space X on which G acts naturally and a lift $\widetilde{m} : X \to \mathbb{C}$ satisfying hypothesis like those of Models 1 and 2. So far, this reverse construction has escaped us.

3. Left orderable groups

Recall that a *total* or *linear* order $\preceq$ is an order relation such that, given any two points x, y then it holds that $x \preceq y$ or $y \preceq x$, with both of them happening simultaneously precisely when $x = y$. A *left-orderable group* is a group admitting a left-invariant total order, ie $(\mathrm{G}, \preceq)$ satisfies that $g \preceq h$ if and only if $kg \preceq kh$. Recall that, as we defined in the introduction, every left orderable group has a *sign function* $\mathrm{sgn} : \mathrm{G} \to \mathbb{C}$ that assigns $+1$ or -1 depending on whether $e \prec g$ or $g \prec e$. We will prove that $H = T_{\mathrm{sgn}}$ is bounded.

Proof (of Theorem C). First, assume that G is discrete. All that is required to do is to prove the identity ($\widehat{\text{Cotlar}}$) relative to $\mathrm{G}_0 = \{e\}$. Assume that $\mathrm{sgn}(g) \neq \mathrm{sgn}(g\,h)$, where both g and h are different from e. If both g and h had the same sign, so would gh, therefore the sign of h has to be different from that of g. But that implies that the signs of g and h^{-1} coincide. To see that the norm grows as $\mathrm{O}(p)$ as $p \to \infty$ just notice that $\mathrm{sgn}(g^{-1}) = -\mathrm{sgn}(g)$ and therefore $m(g)\overline{m(g^{-1})} = -1$, which allows to apply the same technique in Proposition 1.8 and Corollary 1.9. The case of continuous groups follows similarly using the nonrelative Cotlar identity in ($\text{Cotlar}_{\mathrm{nr}}$). $\square$

Apart from the direct proof above, it is interesting to notice that these Hilbert transform type multipliers on left orderable groups can be put into the framework of Model 1 and Theorem B. To do that we need to use the well-known characterization of countable left-orderable groups as order preserving groups of homeomorphisms of the real line $\mathbb{R}$, which is a UAC space. The first reference we found on this is [32], we include a proof below for the reader's convenience. Before proceeding to the proof recall that a total ordering is *dense* iff for any x, y such that $x \prec y$, there exists z such that $x \prec z \prec y$.

Proposition 3.1. *Every countable left-orderable group acts on the real line $\mathbb{R}$ by orientation preserving homeomorphisms and without global fixed point.*

Proof. Consider the set $X = \mathrm{G} \times \mathbb{Q}$. We put a total order relation on X declaring $(a, x) \preceq (b, y)$ if either $a \preceq b$ or $a = b$ and $x \leq y$ in the canonical order of $\mathbb{Q}$. Clearly, $(X, \preceq)$ is an unbounded, dense ordered set. Since, up to isomorphisms of ordered sets, $(\mathbb{Q}, \leq)$ is the only countable totally ordered set with these properties, there is an isomorphism of ordered sets $\phi : X \to \mathbb{Q}$.

Setting $g \cdot (a, x) = (ga, x)$ we obtain a faithful action of G on X. By conjugating this action with the isomorphism $\phi : X \to \mathbb{Q}$, we get an action of G on $\mathbb{Q}$ such that $g \cdot x = \phi(g \cdot \phi^{-1}(x))$. So for each $e \neq g \in \mathrm{G}$ such that $e \preceq g$, we get an strictly increasing bijection $\varphi_g : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto g \cdot x$. By density, φ_g can be uniquely extended to an orientation preserving homeomorphism of the real line $\widetilde{\varphi}_g \in \mathrm{Homeo}_+(\mathbb{R})$ setting for each $x \in \mathbb{R}$,

$$\widetilde{\varphi}_g(x) = \sup\{\varphi_g(q) : q \leq x, q \in \mathbb{Q}\}.$$

Now we claim that for any $x \in \mathbb{R}$, there exists $g \in \mathrm{G}$ such that $\widetilde{\varphi}_g(x) > x$. If the claim is true, then the action of G on $\mathbb{R}$ defined above admits no global fixed point. To prove the claim, it is enough to show that for any $q \in \mathbb{Q}$, there is a $g \in \mathrm{G}$ satisfying $q + 1 \leq \varphi_g(q)$. Suppose $\phi(x) = q$ and $\phi(y) = q + 1$ with $x, y \in X$. By construction we have $\phi(g \cdot x) = \varphi_g(q)$. By the definition of the order on X, it is clear that for any $x, y \in X$, there exists $g \in \mathrm{G}$ such that $y \preceq g \cdot x$. Then since ϕ preserves the order on X, we get $q + 1 = \phi(y) \leq \phi(g \cdot x) = \varphi_g(q)$. The claim is proved. □

Observe as well that, given a homeomorphism of the real line $f : \mathbb{R} \to \mathbb{R}$ it is either orientation preserving or orientation reversing. In fact, we will say that f is orientation reversing if there is a couple of points $x, y \in \mathbb{R}$ such that $x < y$ but $f(y) < f(x)$. The composition of two such maps is again orientation preserving. Therefore, there is a multiplicative map $\mathrm{Homeo}(\mathbb{R}) \to \mathbb{Z}_2 \cong \{1, -1\}$ that associate to each homeomorphism with $+1$ or -1 depending on whether the homeomorphism is orientation preserving or reversing. As a consequence each discrete group G acting on $\mathbb{R}$ has an index 2 subgroup that is left-orderable. So, when the UAC space X in Model 1 is equal to $\mathbb{R}$, the group G is left-orderable and have an index 2 subgroup in which the multiplier m essentially coincides with the sign function.

Observe that the left invariant order of a left orderable group is by no means unique. Thus, the sign funtions associated to different orders may give different L_p-bounded Fourier multiplier. Similarly, left orderable groups can have actions on UAC spaces other than $\mathbb{R}$ which will yield different multipliers still within our Model 1. An example of this will be that of Baumslag-Solitar groups $\mathrm{BS}(1, n)$, which both admit nontrivial actions on their Bass-Serre trees and are left-orderable.

Let us mention that the Hilbert transform $H = T_{\mathrm{sgn}} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$ of a left-orderable group can be considered as a particular case of the Hilbert transforms associated with *subdiagonal algebras* which have been studied in [56], see also [55, Section 8]. Such algebras were introduced by Arveson [3]. Indeed, let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a n.s.f trace and let $\mathbb{E} : \mathcal{M} \to \mathcal{D} \subset \mathcal{M}$ be a τ-preserving conditional expectation onto a von Neumann subalgebra $\mathcal{D} \subset \mathcal{M}$. A *finite subdiagonal algebra* $\mathcal{H}^\infty(\mathcal{M}) \subset \mathcal{M}$ with respect to $\mathbb{E}$ is a weak-$*$ closed, non self-adjoin algebra such that

- **(i)** $\mathbb{E}(f\, g) = \mathbb{E}(f)\, \mathbb{E}(g)$,
- **(ii)** $\big\{f + g^* : f, g \in \mathcal{H}^\infty(\mathcal{M})\big\} = \mathcal{M}$ and
- **(iii)** $\mathcal{H}^\infty(\mathcal{M}) \cap (\mathcal{H}^\infty(\mathcal{M}))^* = \mathcal{D}$.

The archetypal example of such algebra is the $n \times n$ upper triangular matrices, which are a finite subdiagonal algebra with respect to the diagonal subalgebra $\ell_\infty^n \subset M_n(\mathbb{C})$. The notation comes from the fact that for the algebra $L_\infty(\mathbb{T})$ and the integral as expectation, the Hardy space $\mathcal{H}^\infty(\mathbb{T})$ gives a finite subdiagonal algebra. Another family of examples comes from the so called Nest algebras [16] of totally ordered families of projections. In this setting any $f \in \mathcal{M}$ admits a unique decomposition as

$$f = g + \delta + h^*, \quad \text{with } g, h \in \mathcal{H}^\infty_0(\mathcal{M}),\ \delta \in \mathcal{D},$$

where $\mathcal{H}^\infty_0(\mathcal{M}) = \{f \in \mathcal{H}^\infty(\mathcal{M}) : \mathbb{E}(f) = 0\}$. The Hilbert transform associated to the finite subdiagonal algebra $\mathcal{H}^\infty(\mathcal{M})$ is thus $H(f) = -ig + ih^*$. For these family of operators Randrianantoanina [56] proved their weak type $(1,1)$ bound, which after interpolation gives the optimal constant in terms of p. In the case of a discrete left-orderable group $\mathcal{L}\mathrm{G}$ with the subalgebra $\mathbb{C}\mathbf{1} \subset \mathcal{L}\mathrm{G}$ and the conditional expectation given by $f \mapsto \tau(f)\mathbf{1}$, we have that

$$\mathcal{H}^\infty_0(\mathcal{L}\mathrm{G}) \, = \, \left\{ f = \sum_{e \prec g} \widehat{f}(g)\, \lambda_g : f \in \mathcal{L}\mathrm{G} \right\}$$

is a finite subdiagonal algebra whose Hilbert transform coincides with the multiplier $-iT_{\mathrm{sgn}}$ in Theorem C. Thus, by Corollary 1.9, Cotlar identities can be used to recover previously known results with optimal constant.

Examples: Free groups and the Magnus embedding. It is known that the free group $\mathbb{F}_2 = \langle a, b \rangle$ can be totally ordered in a way that is both left and right invariant. This was first proved in [48] but we will use the more explicit order from [19] using Magnus expansions. Let Λ be the non-abelian $\mathbb{Z}$ algebra

$$\Lambda \, = \, \mathbb{Z}\langle\!\langle a, b \rangle\!\rangle,$$

of infinite power series in two non-commuting variables a, b with coefficients in $\mathbb{Z}$. The magnus map $\mu : \mathbb{F}_2 \to \Lambda$ is the multiplicative map given by extension of

$$\begin{aligned} \mu(a) &= 1 + a \\ \mu(a^{-1}) &= 1 - a + a^2 - a^3 + a^4 - \cdots, \end{aligned}$$

and the same changing a by b. Observe that the Magnus map is a well defined and an injective group homomorphism from $\mathbb{F}_2$ into the group of invertible elements of the form $1 + R$, with R a series without independent term. This group can be bi-invariantly ordered in the following way. First, note that every element in Λ can be seen as an infinite sum spanned by (unital) free monoid A^* generated by $A = \{a, b\}$. We can order A^* by using its *shortlex* order, by which to words satisfying that $\omega_1 \preceq_{\mathrm{slex}} \omega_2$ if the usual word length satisfies $|\omega_1| \leq |\omega_2|$ or if $|\omega_1| = |\omega_2|$ and $\omega_1 \preceq \omega_2$ in the lexicographic order. Then we can use the usual lexicographic ordering on Λ seeing it as the product space $(A^*)^{\mathbb{Z}}$ The positive and negative cones associated to this order, restricted to the series that start with 1, can be easily described. Let $A = (\omega_k)_{k \geq 0}$ be the ordered enumeration and let

$$s = 1 + \sum_{k=1}^{\infty} a_k\, \omega_k$$

then

$$1 \preceq s \iff 0 \leq a_{k_0} \ \text{ for the first non-zero term.}$$

This cone is trivially closed under products and it is similarly easy to see that every element is either positive or its symbolic inverse is. The order induced by the Magnus embedding is thus left-invariant (it can actually be proved that it is bi-invariant). Let $\omega \in \mathbb{F}_2$ be a reduced word in the free group and $v \in A^*$, then we have the coefficients $\mu(\omega, v) \in \mathbb{Z}$ are the only integers satisfying

$$\mu(\omega) = 1 + \sum_{v \in A^* \setminus \{1\}} \mu(\omega, v)\, v.$$

The coefficients $\mu(\omega, v)$ can be computed in polynomial time giving a way of computing its sign function. The associated Hilbert transform $H = T_{\mathrm{sgn}}$ is bounded in $L_p(\mathcal{L}\mathbb{F}_2)$ with optimal constants by Theorem C.

Examples: Thompson group *F*. The Thompson group F consists of orientation preserving, piecewise linear homeomorphisms f of the interval $[0,1]$ such that the non-differentiable points of f are dyadic rationals in $[0,1]$ and that the slopes of f on dyadic open intervals are integer powers of 2. The Thompson group F has been studied for its unusual properties that make it a limit case within amenable groups. In particular it is finitely presented, of exponential growth, torsion free and doesn't contain free subgroups. Its amenability is still open, but if it were true, it will provide a new counterexample to the disproved von Neumann conjecture (also known as Day's Problem). F is also one of the simplest examples of a left-orderable group that is not residually nilpotent. There are many ways to define a left-invariant order on F, see for instance [17], one of them is the following. For any $f \in F$, denote by x_f the left-most point y for which $f'_+(y) \neq 1$

$$x_f \, = \, \inf \big\{ y : f'_+(y) \neq 1 \big\}$$

where f'_+ stands for the right derivative of f. A left-invariant order on F is given by setting that f is positive if and only if $f'_+(x_f) > 1$. Therefore the sign symbol given by

$$m(f) \, = \, \begin{cases} 1 & \text{when } f'_+(x_f) > 1 \\ 0 & \text{when } f = \mathrm{id} \\ -1 & \text{when } f'_+(x_f) < 1 \end{cases}$$

satisfies Cotlar's identity and by Theorem C we have that the operator $H = T_m$ is L_p-bounded

$$\big\| H : L_p(\mathcal{L}F) \to L_p(\mathcal{L}F) \big\| \, \lesssim \, \frac{p^2}{p-1} \qquad \text{for every } 1 < p < \infty.$$

4. Multipliers from Bass-Serre theory

The theory of discrete groups acting on trees (without edge inversions) is very well understood due to the work of Serre. The interested reader can find more on this topic in [61]. Here we are going to briefly explain how it is possible to use this theory to build examples of multipliers satisfying Cotlar's identity and to understand previously known examples like Free group multipliers from [43][44] under a geometric lens.

Recall that a *graph of groups* is an object $\mathbb{X}$ composed of a connected graph X together with a group G_x for each vertex $x \in \mathrm{Vert}(X)$ and another group H_y for every edge $y \in \mathrm{Edge}(X)$ satisfying that the group in the edge embeds into the groups of both of its extremes. More formally, we will consider that our graph is oriented, that both directions of the edge occur and that there can be multiple edges between two given vertices. As it is customary in this context, we will denote by $\bar{y}$ the *reverse edge* and by $o(y) \in \mathrm{Vert}(X)$ and $t(y) \in \mathrm{Vert}(X)$ the *origin* and *target* vertices of the edge. We will say that $\mathrm{Edge}_+ \subset \mathrm{Edge}$ is an orientation of the graph if Edge_+ contains either y or $\bar{y}$. By definition, we have that $\mathrm{H}_y = \mathrm{H}_{\bar{y}}$ and that there are injective homomorphisms $\alpha_y : \mathrm{H}_y \to \mathrm{G}_{t(y)}$ and $\alpha_{\bar{y}} : \mathrm{H}_y \to \mathrm{G}_{o(y)}$. We will denote the image group $\alpha_y[\mathrm{H}_y] \subset \mathrm{G}_{t(y)}$ by H_y^y. Similarly $\mathrm{H}_{\bar{y}}^{\bar{y}}$ will denote the image of $\alpha_{\bar{y}}$.

Given a graph of groups $\mathbb{X}$ it is possible to define its *fundamental group* $\pi_1(\mathbb{X})$. In order to do that, let us introduce the group $F(\mathbb{X})$ generated by all the G_x, for $x \in \mathrm{Vert}(X)$ and all the edges $y \in \mathrm{Edge}(X)$ subject to the relationships

$$\bar{y} = y^{-1}, \quad y \, \alpha_y(h) \, y^{-1} = \alpha_{\bar{y}}(h), \ \forall \, h \in \mathrm{H}_y.$$

There are two possible definitions of $\pi_1(\mathbb{X})$.

(D.i) For the first construction choose a base point $x_0 \in \mathrm{Vert}(X)$ and a closed path c that starts and ends at x_0. We will denote the path by its ordered sequence of edges $c = y_1 \, y_2 \cdots y_m$, with $t(y_i) = o(y_{i+1})$ and $o(y_1) = x_0 = t(y_m)$. The group $\pi_1(\mathbb{X}, x_0)$ is given by elements of the form:

$$|c, r| \, = \, r_0 \, y_1 \, r_1 \, y_2 \, r_2 \cdots y_m \, r_m, \tag{4.1}$$

where c is a closed loop as above, $r_j \in \mathrm{G}_{o(y_{j+1})}$ for $j \leq m-1$ and $r_m \in \mathrm{G}_{x_0}$. Here r is just notation for the tuple $r = (r_0, r_1, \cdots r_m)$. We will call the pair (c, r) a *word of type* c. The group $\pi_1(\mathbb{X}, x_0)$ is given by such elements $\big|c, r\big|$ with the concatenation as multiplication.

(D.ii) For the second construction fix a *spanning tree* $T \subset X$, that is a tree containing all vertices of X. The fundamental group $\pi_1(\mathbb{X}; T)$ can be constructed as

$$\pi_1(\mathbb{X}; T) \;=\; F(\mathbb{X}) / \langle\!\langle y : y \in \mathrm{Edge}(T) \rangle\!\rangle,$$

where $\langle\!\langle y : y \in \mathrm{Edge}(T) \rangle\!\rangle$ is the normal subgroup generated by all the edges y in the spanning tree.

The first definition is independent on the choice of x_0, while the second is independent of the choice of spanning tree T. Both are isomorphic and we will usually denote the resulting group just by $\pi_1(\mathbb{X})$. We are going to sketch why they are equal. All we have to see is that the canonical projection

$$p : F(\mathbb{X}) \longrightarrow \pi_1(\mathbb{X}; T) \tag{4.2}$$

restricts to an isomorphism of $\pi_1(\mathbb{X}, x_0) \subset F(\mathbb{X})$ onto $\pi_1(\mathbb{X}; T)$. The injectivity of the restricted map is clear, since no nontrivial closed loop can be fully contained in the spanning tree. To see the surjectivity let us construct a partial inverse $\phi : \pi_1(\mathbb{X}; T) \to \pi_1(\mathbb{X}, x_0)$. Recall that for every point $x \in \mathrm{Vert}(X)$ there is a unique path starting from x_0 and ending in x contained within the spanning tree T. Let us denote by $\gamma_x \in F(\mathbb{X})$ the element associated to the path $\gamma_x = y_1\, y_2 \cdots y_r$ with $o(y_1) = t(y_r) = x_0$. If $y \in \mathrm{Edge}(X)$ we define ϕ as

$$\phi(y) = \begin{cases} e & \text{when } y \in \mathrm{Edge}(T) \\ \gamma_{o(y)}\, y\, \gamma_{t(y)}^{-1} & \text{when } y \notin \mathrm{Edge}(T). \end{cases}$$

Observe that in both cases we obtain words associated to a closed path starting with x_0 like in (4.1). Similarly, if $g \in \mathrm{G}_x$ for a vertex $x \in \mathrm{Vert}(X)$, then we define ϕ as

$$\phi(g) \;=\; \gamma_x\, g\, \gamma_x^{-1}.$$

Straightforward computations give the desired properties of the partial inverse.

Using the definition of $\pi_1(\mathbb{X})$ in Definition (D.i), a notion of *normal form* generalizing the intuitive notion of reduced word can be easily defined. It is said that a word $g = |c, r|$ of type c is in normal form if either $m = 0$ and $r_0 \neq e$ or, in the case in which $m \geq 1$, it holds that for every $1 \leq j \leq m-1$, $r_j \notin \mathrm{H}_{y_j}^{y_j}$ when $y_{j+1} = \overline{y}_j$. Let $|c, r|$ and $|c, \mu|$ be two words of type c with $r = (r_0, r_1, \ldots, r_m)$ and $\mu = (\mu_0, \mu_1, \ldots, \mu_m)$. They are said to be *equivalent* if $\mu_0 = r_0 a_1^{\bar{y}_1}$ and $\mu_j = (a_j^{y_j})^{-1} r_j a_{j+1}^{\bar{y}_{j+1}}$. Here $a_j \in \mathrm{H}_{y_j}$ and $a_j^{y_j}$ and $a_j^{\bar{y}_j}$ are the corresponding images in $\mathrm{H}_{y_j}^{y_j}$ and $\mathrm{H}_{\bar{y}_j}^{\bar{y}_j}$. We have that (c, r) and (c, μ) are equivalent if and only if $|c, r| = |c, \mu|$. This implies that any $g \in \pi_1(\mathbb{X})$ admits a unique normal form up to equivalence.

The fundamental group of a graph of groups unifies several constructions.

- **Homotopy groups of 1-dim. simplicial complexes.** Let X be a connected graph as above. We can trivially turn it into a graph of groups by putting the trivial group in each vertex and edge. In that case $\pi_1(\mathbb{X})$ is naturally isomorphic to the usual fundamental group $\pi_1(X)$ —also called the first homotopy group— of the associated 1-dimensional simplicial complex. Note that to build the simplicial complex we just use a single edge for each pair y and $\bar{y}$. The reason why we obtain the homotopy group is more apparent with the first definition. To see the equivalence with the second, notice that every spanning tree is contractible and, as such, we can retract it to a singe point and have a loop for each cycle of the graph, see [59, Chapter 11].

- **Amalgamated free products.** Let X be a connected graph with two vertices x_1 and x_2 associated to groups G_1 and G_2 and connected by a single edge y. When the group associated to the edge is trivial $\{e\}$, the fundamental group of this graph of groups is the free product

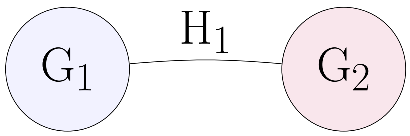

Figure 5: Amalgamated free product.

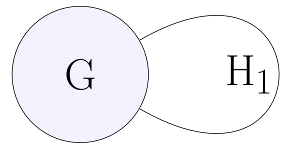

Figure 6: HNN extension.

$\pi_1(\mathbb{X}) = \mathrm{G}_1 * \mathrm{G}_2$. This construction can easily be tweaked by putting a larger subgroup H on the edge y that embeds into both into G_1 and G_2, see Figure 5. Then the corresponding fundamental group is the amalgamated free product $\mathrm{G}_1 *_H \mathrm{G}_2$.

- **HNN extensions.** Let $\mathrm{G} = \langle S \,|\, R \rangle$ be a finitely presented group and H_1 and H_2 two isomorphic subgroups H_1, $\mathrm{H}_2 \subset \mathrm{G}$. Fix an isomorphism $\theta : \mathrm{H}_1 \to \mathrm{H}_2$. The *HNN* extension of G relative to α, see [59], is the smallest extension of G such that θ is implemented by an inner automorphism, i.e. $\theta(h) = t\, h\, t^{-1}$, for some t and every $h \in \mathrm{H}_1$. In the case of a finitely presented group, its HNN extension is given by
$$\big\langle S,\, t \,\big|\, R,\, tht^{-1} = \theta(h) \,\big\rangle.$$
This construction can be obtained as the fundamental group of a graph of groups with a single vertex with group G and an edge y whose associated group is H_1 and the two inclusions are given by $\alpha_y(h) = h$ and $\alpha_{\bar{y}}(h) = \theta(h)$, see Figure 6.

Let $\mathbb{X}$ be a graph of groups whose underlying graph we will denote by X and let $\pi = \pi_1(\mathbb{X})$ be its fundamental group. We will recall the construction of the Bass-Serre tree $\widetilde{X}$ of $\mathbb{X}$ whose vertices are

$$\mathrm{Vert}(\widetilde{X}) = \bigsqcup_{x \in \mathrm{Vert}(X)} \pi / \mathrm{G}_x,$$

while its edges are given by

$$\mathrm{Edge}_+(\widetilde{X}) = \bigsqcup_{y \in \mathrm{Edge}_+(X)} \pi / \mathrm{H}^{\bar{y}}_{\bar{y}} \quad \text{and} \quad \mathrm{Edge}_-(\widetilde{X}) = \bigsqcup_{y \in \mathrm{Edge}_-(X)} \pi / \mathrm{H}^{\bar{y}}_{\bar{y}},$$

where $\mathrm{Edge}_+(X)$ is a fixed orientation of the edges of X. In the definition above we are including only the edges of a fixed orientation of $\widetilde{X}$ induced by the orientation of X. Observe that there is a canonical projection $\widetilde{X} \twoheadrightarrow X$ that maps vertices onto vertices and edges onto edges. We can define the endpoint of each edge as follows

$$\begin{aligned} &o(g\,\mathrm{H}^{\bar{y}}_{\bar{y}}) = g\mathrm{G}_{o(y)}, && t(g\mathrm{H}^{\bar{y}}_{\bar{y}}) = gg_y\mathrm{G}_{t(y)} && \text{for } \ y \in \mathrm{Edge}_+(X),\ g \in \pi, \\ &o(g\mathrm{H}^{\bar{y}}_{\bar{y}}) = gg_y^{-1}\mathrm{G}_{o(y)}, && t(g\mathrm{H}^{\bar{y}}_{\bar{y}}) = g\mathrm{G}_{t(y)} && \text{for } y \in \mathrm{Edge}_-(X),\ g \in \pi, \end{aligned} \tag{4.3}$$

where $g_y = p(y)$ is the image of y under the canonical projection p in (4.2). The group π acts on $\widetilde{X}$ by left multiplication.

Let $\mathrm{G} \curvearrowright T$ be a group acting on a tree without edge inversions. Then, it is possible to build a graph of groups by taking X as the space of orbits $X = \mathrm{G}\backslash T$. Furthermore, to each of the vertices of X, we can associate it with stabilizer group and do the same for edges. The fact that there are no edge inversions gives that the groups on the edges embed into the groups on the vertices and thus we have a graph of groups $\mathbb{X}$ constructed from the action $\mathrm{G} \curvearrowright T$. Serre's fundamental theorem attests that the group G, as well as the tree action $\mathrm{G} \curvearrowright T$ can be recovered from $\mathbb{X}$. Indeed, it holds that $\pi_1(\mathbb{X}) \cong \mathrm{G}$ and that the action of $\pi_1(\mathbb{X})$ on its Bass-Serre tree $\widetilde{X}$ is isomorphic to $\mathrm{G} \curvearrowright T$. It is interesting to point out that in the case in which $\mathbb{X}$, as a graph of groups, have only trivial group on its vertices and edges, its bass-Serre tree its just the *universal covering space* $\widetilde{X}$ of the underlying simplicial graph X

and the action of $\pi_1(X)$ on $\widetilde{X}$ is the usual action by Deck transformations. We will further describe the Bass-Serre trees of the examples in Figures 5 and 6 in the subsections below.

The Bass-Serre tree construction also allows us to produce actions on trees for the fundamental group of any graph of groups. We will exploit this to obtain examples of Fourier multipliers satisfying Cotlar's identity.

Multipliers from normal forms. Let $\mathbb{X}$ be a graph of group whose underlying connected graph is X and fix $x_0 \in \mathrm{Vert}X$ and $\pi = \pi_1(\mathbb{X}, x_0)$. Let c be a closed path starting and ending in x_0 and let $c = y_1\, y_2\, \cdots\, y_n$ be its sequence of edges. We have that any element $e \neq g = |c, r| \in \pi$ admits a normal form as

$$g = r_0\, y_1\, r_1\, y_2\, r_2\, \cdots\, y_n\, r_n.$$

We would like to define a multiplier depending only on the starting segment $r_0\, y_1$. But recall that two words in normal form $|c, r|$ and $|c, \mu|$ being equivalent implies that $r_0 = \mu_0\, a_1^{\bar{y}_1}$ with $a_1 \in \mathrm{H}_{y_1}$ and $a_1^{\bar{y}_1}$ is its image under $\alpha_{\bar{y}_1}$. Therefore, we will say that a symbol $m : \pi_1(\mathbb{X}, x_0) \to \mathbb{C}$ *depends on the starting segment* iff there exists a function

$$\widetilde{m} : \underbrace{\bigsqcup_{y} \big(\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}\big)}_{W} \longrightarrow \mathbb{C},$$

where y runs over every edge that starts at x_0, such that

$$m(g) \; = \; \begin{cases} \widetilde{m}\big(r_0 \cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}\big), & \text{when } g \notin \mathrm{G}_{x_0} \\ 0 & \text{when } g \in \mathrm{G}_{x_0} \end{cases} \tag{4.4}$$

with $g = |c, r|$ being a word of type c in normal form. Notice that, when we say that y runs over every edge such that $o(y) = x_0$, we are logically counting edges connecting x_0 with other vertices once but we counting loops starting and ending at x_0 twice since both $o(y)$ and $o(\bar{y})$ are equal to x_0. It is easy to see that it is always possible to find nontrivial multipliers depending on the starting segment if $|W| > 1$. We will first show directly that any symbol depending on the starting segment satisfies a Cotlar identity relative to G_{x_0}. The corresponding Fourier multiplier T_m will thus be L_p-bounded provided that the symbol is left G_{x_0}-invariant. We are also going to see that any symbol depending on the starting segment lifts naturally to a function on the Bass-Serre tree of π and therefore it falls inside Model 1.

Proposition 4.1. *Let $\pi = \pi_1(\mathbb{X}, x_0)$ be as above and let $m : \pi \to \mathbb{C}$ be a function depending only on the initial segment. It holds that*

- **(i)** *m satisfies Cotlar's identity of Theorem 1.5.*(ii) *relative to G_{x_0}, i.e.,*

$$\big(m(g^{-1}) - m(h)\big)\,\big(m(gh) - m(g)\big) \; = \; 0, \qquad \textit{for every } g \in \pi \setminus \mathrm{G}_{x_0}, h \in \pi.$$

- **(ii)** *Furthermore, if $\widetilde{m}$ in* (4.4) *is constant in every element of the disjoint union W, i.e., its value depends only on the outgoing edge y, then m is left G_{x_0}-invariant.*

Proof. When $h \in \mathrm{G}_{x_0}$, then it is immediate that $m(gh) = m(g)$. Therefore, it is enough to prove the identity for $g, h \notin \mathrm{G}_{x_0}$. Assume that $m(gh) \neq m(g)$, since otherwise the identity holds. Let us write g in its normal form $g = r_0\, y_1\, r_1\, y_2\, r_2 \cdots y_n\, r_n$ with $r_0 \in g_0 \cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Since $m(gh) \neq m(g)$, h must admit a normal form $b = r_n^{-1} y_n^{-1} \cdots y_1^{-1} r$ with $r = r_0^{-1} \cdot r'$ for some $r' \in g_0' \cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Here g_0, $g_0' \in \mathrm{G}_{x_0}$ and $g_0^{-1} g_0' \notin \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Moreover, since $r_n^{-1} y_n^{-1} \cdots y_1^{-1} r_0^{-1}$ is a normal form of g^{-1}, by the definition of m, it is clear that $m(g^{-1}) = m(h)$. For the second point, notice that the left action $\mathrm{G}_{x_0} \curvearrowright \mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$ is transitive for each y, therefore a left G_{x_0}-invariant function has to be constant in each of the elements of the disjoint union W. $\square$

Observe that, by the definition of the Bass-Serre tree of $\mathbb{X}$ in equation (4.3), the set of edges in $\widetilde{X}$ connected to the vertex $\widetilde{x}_0 = e \cdot \mathrm{G}_{x_0} \in \mathrm{Vert}(\widetilde{X})$ is given, up to orientation, by the disjoint union

$$W = \bigsqcup_y \left(\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}\right),$$

where, again, y runs over every edge in X starting with x_0. This already established that the functions depending only on the initial segment and the functions that are constant on the connected components of $\widetilde{X} \setminus \{\widetilde{x}_0\}$ are in natural and bijective correspondence. The following proposition, whose proof is immediate, asserts that m falls into Model 1.

Proposition 4.2. *Let $\pi = \pi_1(\mathbb{X}, x_0)$ be the fundamental group of a connected graph of groups $\mathbb{X}$ and let $\widetilde{X}$ be its Bass-Serre tree.*

(i) *Every symbol $m : \pi \to \mathbb{C}$ depending on the first segment satisfies that*

$$m(g) = \widetilde{m}(g \cdot \widetilde{x}_0),$$

where $\widetilde{m} : \widetilde{X} \to \mathbb{C}$ is the function constant over the connected components of $\widetilde{X} \setminus \{\widetilde{x}_0\}$ that is naturally associated to (4.4) *and $\widetilde{x}_0$ is a vertex of X labelled by G_{x_0}.*

(ii) *If furthermore m depends only on the edge of the starting segment, then $\widetilde{m} : \widetilde{X} \setminus \{\widetilde{x}_0\} \to \mathbb{C}$ has the same value over each two connected components $\widetilde{X}_\alpha$ and $\widetilde{X}_\beta$ such that $r \cdot \widetilde{X}_\alpha = \widetilde{X}_\beta$, for some $r \in \mathrm{G}_{x_0}$.*

Remark 4.3. The above proposition together with Proposition 4.1 and Theorem B imply that every symbol m on $\pi = \pi_1(\mathbb{X}, x_0)$ depending only on the edge of the starting segment gives a Fourier multiplier T_m bounded on $L_p(\mathcal{L}\pi)$ for any $1 < p < \infty$.

Amalgamated free products. Let $\{\mathrm{G}_i : i \geq 1\}$ be a family of groups and A a common subgroup. Let us denote the injective homomorphisms by $\alpha_i : A \to \mathrm{G}_i$, for $i \geq 1$. Now, consider the graph of groups $\mathbb{X}$ in figure 7, which is based on a tree X connecting the points x_0, whose group is A, with the point x_i, $i \geq 1$, whose groups are G_i. Meanwhile, the groups on the edges y_i are all isomorphic to A with the embedding into $\mathrm{G}_{x_0} = A$ being the identity map and the embedding into G_i being α_i.

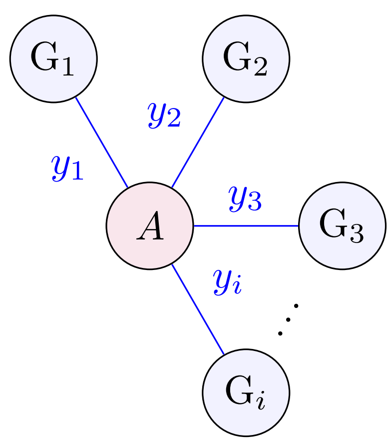

Figure 7: Graph of groups for the amalgamated free product.

In this case, we have that $\pi = \pi_1(\mathbb{X}, x_0)$ is the free product of $\{\mathrm{G}_i : i \geq 1\}$ with A as amalgam

$$\pi = \mathop{\ast}_{i,\,A} \mathrm{G}_i,$$

see [61, Theorem 9]. As a free product, any element of π can be represented by a unique normal word

$$g = a\, s_{i_1} \, \cdots \, s_{i_n}, \tag{4.5}$$

where each s_j is a right coset representative of G_j/A modulo A. Hilbert transforms on amalgamated free products have been studied in the pioneering work of Mei and Ricard [43]. In particular, given a collection of signs $\epsilon = (\epsilon_i)_i$, they studied the operator $H_\epsilon : \mathbb{C}[\pi] \to \mathbb{C}[\pi]$

$$H_\epsilon(f) \, = \, \sum_{i \geq 1} \epsilon_i \, L_i(f),$$

where $\epsilon_i = \pm 1$ and the operator L_i is the Fourier multiplier whose symbol m_i is the characteristic function of the words starting by $a\, s_{i_0}$, where $s_{i_0} \in \mathrm{G}_i$. Note that this operator falls into Model 1 for the action of π on its Bass-Serre tree associated to $\mathbb{X}$ in Figure 7. Moreover, the symbol of H_ϵ is a function depending only on the outgoing edges y_i for $i \geq 1$. By remark 4.3, H_ϵ is bounded on $L_p(\mathcal{L}(\ast_{i,\, A}\mathrm{G}_i))$ for any $1 < p < \infty$. In particular, in the case where $A = \{e\}$ and $\mathrm{G}_1 = \mathrm{G}_2 = \mathbb{Z}$, we have that $\pi = \mathbb{F}_2 = \mathbb{Z} \ast \mathbb{Z}$. Denote the generators by a and b. In this case, the multipliers from Model 1 coming from the action of $\mathbb{F}_2$ on the Bass-Serre tree associated to the graph of groups in Figure 7 distinguish the staring letter of the reduced words of group elements. Therefore, we recover (MR).

Remark 4.4. More interestingly, if we represent $\mathbb{F}_2$ as the fundamental group of the graph of groups of the graph $\mathbb{X}$ given by two vertices $\{x_0, x_1\}$ whose associated groups are isomorphic to $\mathbb{Z}$ and joining by a single edge with trivial group in it. Now, let us consider the Bass-Serre tree $\widetilde{X}$ of this graph of groups (see Figure 8), it is easy to see group elements in $\mathbb{F}_2$ can only map pink vertices to pink vertices and blue to blue. Denote the set containing the blue vertices in the first layer and the pink vertex labelled by $\mathbb{Z}$ in the middle by B_1. It is straightforward to realize that any function on $\widetilde{X}$ that is constant along the connected components of $\widetilde{X} \setminus B_1$ induces a symbol of the form

$$m\big(s_1^{n_1}\, s_2^{n_2} \, \cdots \, s_r^{n_r}\big) \; = \; \begin{cases} \widetilde{m}_1(n_1), & \text{if } \; s_1 = a \\ \widetilde{m}_2(n_1), & \text{if } \; s_1 = b \end{cases}$$

for any normal word $g = s_1^{n_1}\, s_2^{n_2} \, \cdots \, s_r^{n_r}$, with $s_j \in \{a, b\}$ and $n_j \in \mathbb{Z}$. Now the symbol m depends on the first building block of g. Observe that in this case, the function m is not left $\mathbb{Z} = \langle a \rangle$-invariant unless $\widetilde{m}_1 = \widetilde{m}_2 \equiv c$ for some $c \in \mathbb{C}$. Therefore, Theorem B does not directly give L_p-boundedness. Nevertheless, it is possible to obtain L_p-boundedness when $\widetilde{m}_1$ and $\widetilde{m}_2$ are characters, i.e., $\widetilde{m}_1(k) = z_1^k$ and $\widetilde{m}_2(k) = z_2^k$ for some $|z_1| = |z_2| = 1$. In this case, the extrapolation result in Proposition 1.4 also works by using that m is left $\mathbb{Z}$-invariant in the sense that

$$m(a^k\, \omega) = z_1^k\, m(\omega) \quad \text{and} \quad m(b^k\, \omega) = z_2^k\, m(\omega) \qquad \text{for every } k \in \mathbb{Z},\, \omega \in \mathbb{F}_2,$$

see Remark 1.7. This allows to reprove some of the key results in [44] in a geometric fashion. In particular, this makes the map

$$\begin{array}{rcl} \mathbb{C}\big[\mathbb{F}_\infty\big] & \longrightarrow & \mathbb{C}\big[\mathbb{F}_\infty\big] \otimes L_\infty\big(\mathbb{T}\big) \\ \lambda(s_1^{n_1}\, s_2^{n_2} \, \cdots \, s_r^{n_r}) & \longmapsto & \lambda(s_1^{n_1}\, s_2^{n_2} \, \cdots \, s_r^{n_r}) \otimes z_j^{n_1}, \end{array}$$

where $z_j \in \mathbb{T}$ with $j \in \mathbb{N}$, bounded on $L_p(\mathcal{L}\mathbb{F}_\infty)$ for every $1 < p < \infty$. This geometric interpretation will be the subject of a forthcoming paper by the third-named author [66]. This approach also opens the possibility of extending the results from [44] to more general groups acting on trees as long as the stabilizers are all abelian, for instance a natural candidate will be Baumslag-Solitar groups.

HNN extensions and Baumslag-Solitar groups. Suppose H_1 and $\mathrm{H}_2 \subset \mathrm{G}$ are two subgroups isomorphic under $\theta : \mathrm{H}_1 \to \mathrm{H}_2$ and let π be the HNN-extension of G relative to θ

$$\pi = \big\langle S, t \,\big|\, R,\, trt^{-1} = \theta(r), r \in \mathrm{H}_1 \big\rangle,$$

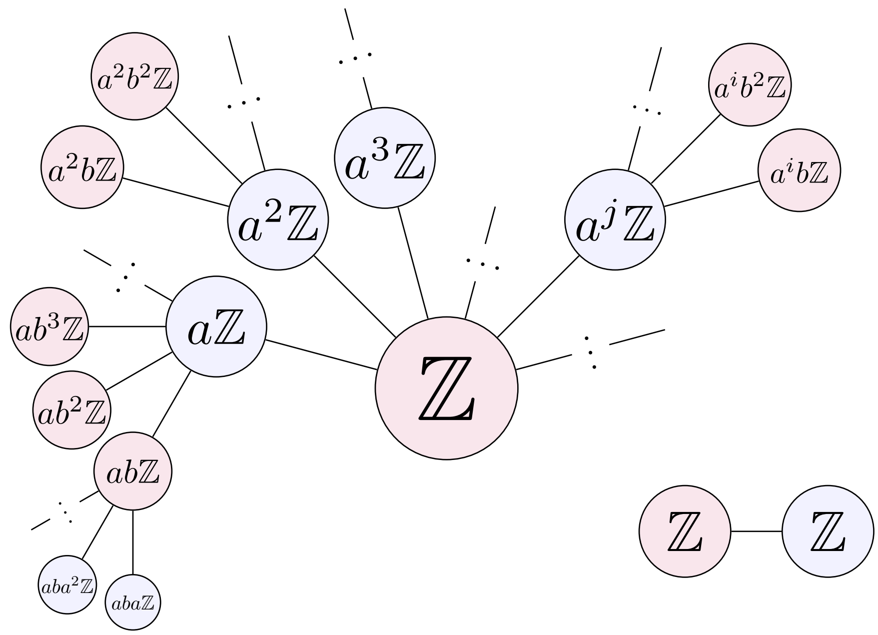

Figure 8: Graph of groups whose fundamental group is $\mathbb{F}_2$ (in the corner) and its Bass-Serre tree. For readability, we have omitted branches going from the root towards vertices labeled $a^k\mathbb{Z}$ with $k \leq 0$.

where $\mathrm{G} = \langle S|R\rangle$ is a presentation for G. The HNN-extension $\pi = H \rtimes T$ is the semi-direct product of the infinite cyclic group T generated by t and the normal subgroup H generated by $t^n \mathrm{G} t^{-n}$, $n \in \mathbb{Z}$. In particular, π has a quotient is isomorphic to $\mathbb{Z}$.

We have already seen, see Figure 6, that $\pi = \pi_1(\mathbb{X}, x_0)$ is the fundamental group of a graph of groups based on a single loop. By the definition of fundamental group, every element of π is represented by word $r_0, t^{e_1} r_1 \cdots t^{e_k}, r_k$ with $k \geq 0$, $e_i = \pm 1$ and $r_i \in \mathrm{G}$. A word in this form will be reduced if it contains neither a substring of the form tat^{-1} with $a \in \mathrm{H}_1$ nor one of the form $t^{-1}bt$ with $b \in \mathrm{H}_2$. Fix coset representatives of G/H_1 and G/H_2, for any $g \in \pi$, there exists a unique reduced word such that

$$g \;=\; r_0\, t^{e_1}\, r_1 \,\cdots\, t^{e_k}\, r_k \tag{4.6}$$

with $r_0 \in \mathrm{G}$, $r_i \in \mathrm{G}/\mathrm{H}_1$ if $e_i = 1$, and $r_i \in \mathrm{G}/\mathrm{H}_2$ if $e_i = -1$.

Now, we are going to give two algebraic forms for Fourier multipliers satisfying ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$), one falling within Model 1 and another within Model 2 by considering the action of the HNN-extension π on its Bass-Serre tree $\widetilde{X}$ by left multiplication. For the first one, we choose the root $\widetilde{x}_0$ to be the vertex labelled by G. Note that if we set the orientation $\mathrm{Edge}_+(X) = \{t\}$, then in the induced orientation of $\widetilde{X}$ there are $[\mathrm{G} : \mathrm{H}_2]$ many edges starting with $\widetilde{x}_0$ and $[\mathrm{G} : \mathrm{H}_1]$ many edges ending in $\widetilde{x}_0$. It is immediate that a function $m : \pi \to \mathbb{C}$ depends on the starting segment iff there is a function $\widetilde{m} : (\mathrm{G}/\mathrm{H}_1) \sqcup (\mathrm{G}/\mathrm{H}_2) \to \mathbb{C}$ such that

$$m(g) \;=\; \begin{cases} \widetilde{m}_1\big(r_0 \cdot \mathrm{H}_1\big) & \text{when } \; e_1 = +1 \\ 0 & \text{when } g \in \mathrm{G} \\ \widetilde{m}_2\big(r_0 \cdot \mathrm{H}_2\big) & \text{when } \; e_1 = -1, \end{cases}$$

where $\widetilde{m}_i$ is the restriction of $\widetilde{m}$ to G/H_i for $i \in \{1, 2\}$ and g is equal to its normal form like in (4.6). Furthermore, by Proposition 4.2, we have that m is left G-invariant if and only if it depends only on the first edge in the normal form, which can only be t or t^{-1} in our setting. Therefore, we introduce the following function

$$m(g) \;=\; \begin{cases} C_1 & \text{when } \; e_1 = +1 \\ 0 & \text{when } g \in \mathrm{G} \\ C_2 & \text{when } \; e_1 = -1. \end{cases} \tag{4.7}$$

Now, we can use the definition in Model 2 to the action $\pi \curvearrowright \widetilde{X}$. Choose $\widetilde{x}_0$ as the root and let $\widetilde{X}_0 \subset \widetilde{X}$ be the connected component separated from the root by the edge H_1. It is easy to check the vertices in the connected component of H_2 always take the form $g \cdot \mathrm{G}$ with the expression (4.6) of g starting with t. Hence we get the following symbol $\varphi : \pi \to \mathbb{C}$:

$$\varphi(g) \;=\; \begin{cases} 0 & \text{if } g \in \mathrm{H}_2 \subset \mathrm{G} \\ D_1 & \text{if } r_0 = e \text{ and } e_1 = 1 \\ D_2 & \text{otherwise}, \end{cases} \tag{4.8}$$

where D_1 and D_2 are two different constants.

Proposition 4.5. *Let π be the HNN extension of* G *with respect to θ as before. We have that*

- **(i)** *Let $m : \pi \to \mathbb{C}$ be like in* (4.7)*, then T_m satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *relative to* G *and is left $\mathcal{L}$G-modular. As a consequence*

$$\big\| T_m : L_p(\mathcal{L}\pi) \to L_p(\mathcal{L}\pi) \big\| \;\lesssim\; \Big(\frac{p^2}{p-1} \Big)^{\beta} \qquad \textit{where } \beta = \log_2(1+\sqrt{2}).$$

- **(ii)** *Let $\varphi : \pi \to \mathbb{C}$ be like in* (4.8)*, then T_φ satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *relative to* $\mathrm{H}_2 \subset \mathrm{G}$ *and is left $\mathcal{L}\mathrm{H}_2$-modular. As a consequence*

$$\big\| T_\varphi : L_p(\mathcal{L}\pi) \to L_p(\mathcal{L}\pi) \big\| \;\lesssim\; \Big(\frac{p^2}{p-1} \Big)^{\beta} \qquad \textit{where } \beta = \log_2(1+\sqrt{2}).$$

Both multipliers fall within the scope of Model 1 and Model 2 respectively.

The result above can be illustrated in the particular case of the Baumslag-Solitar group

$$\mathrm{BS}(n,m) \;=\; \big\langle r, t \,\big|\, t\, r^m\, t^{-1} = r^n \big\rangle,$$

which can be seen as a HNN extension of $\mathbb{Z} = \langle r \rangle$ with respect to the map θ that sends $r^{mk} \mapsto r^{nk}$ that establishes an isomorphism between the subgroups $m\mathbb{Z}$ and $n\mathbb{Z} \subset \mathbb{Z}$. It holds that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ and $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_m$. We can take representatives $\{0,\, 1,\, \ldots\, n-1\}$ and $\{0,\, 1,\, \ldots\, m-1\}$ of the quotients. Let $\mathbb{X}$ be the single loop graph of groups and $\widetilde{X}$ be its Bass-Serre tree depicted in Figure 9. The two directions of the loop give us the two subgroups $m\mathbb{Z}$, $n\mathbb{Z}$, and by definition of the Bass-Serre tree you get that $\widetilde{X} \setminus \{\widetilde{x}_0\}$ has $n+m$ connected components, see Figure 9. In the first model of (4.7) we get a function $\widetilde{m} : \widetilde{X} \to \mathbb{C}$ that takes the value 0 at the root labelled by G and two different values for the vertices above and below the root.

In the case of the Model 2 in equation (4.8) we select the connected component of the vertex tG.

Now consider the unique normal form of g:

$$g = r^{k_0} t^{e_1} r^{k_1} \cdots t^{e_m} r^{k_m} \qquad \text{with } k_0 \in \mathbb{Z}, e_1, \ldots e_j \in \{\pm 1\}$$

if $e_j = 1$, then $k_j \in \mathbb{Z}/m\mathbb{Z}$ for $j \geq 1$, if $e_j = -1$, then $k_j \in \mathbb{Z}/n\mathbb{Z}$ for $j \geq 1$. We can apply both definitions (4.7) and (4.8) to obtain bounded multipliers satisfying Cotlar's identity.

Observe that, even though $\mathrm{BS}(n,m)$ has an abelian quotient $q : \mathrm{BS}(n,m) \to \mathbb{Z}$ given by moding out the generator t. More concretely, q sends g to $k_0 + k_1 + \cdots + k_m$. This will allow to define what

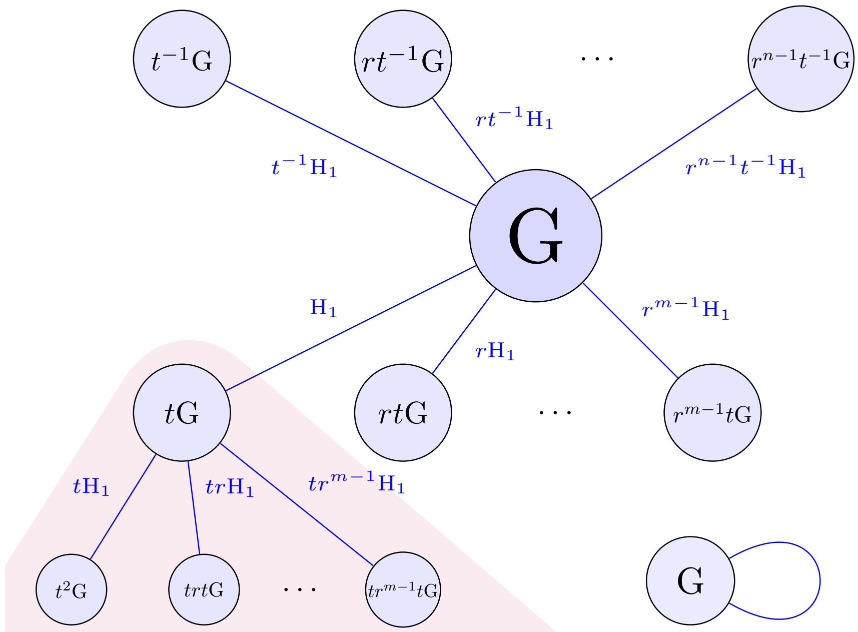

Figure 9: Graph of groups whose fundamental group is the Baumslag-Solitar group (in the corner) and its Bass-Serre tree. Here $\mathrm{H}_1 = n\mathbb{Z}$.

is basically a classical Hilbert transform by taking the sign of $q(g)$. Nevertheless, the multipliers φ and m do not come from the signs of abelian quotient. Similarly, even though $\mathrm{BS}(n,m)$ clearly fails Serre's property (FA), to the best of our knowledge, there is no easy way to write the Baumslag-Solitar group $\mathrm{BS}(m,n)$ as an amalgamated free product in a way that will allow us to interpret the multiplier formulas in (4.7) and (4.8) as examples of the Mei and Ricard's free Hilbert transforms.

Examples with Serre's Property (FA). We will present here the examples of groups having multipliers satisfying Cotlar's identity and fitting into our Model 1 while having Serre's property (FA). Recall that a group has Serre's property (FA), the A stemming from the French word for tree, *arbre*, if and only if, every isometric action on a simplicial tree has global fixed point. This property admits a closed characterization. Indeed, a countable group G has property (FA) if and only if the following conditions are satisfied

- G is not an amalgamated free product;
- G has no quotient isomorphic to $\mathbb{Z}$;
- G is finitely generated.

The interest of this property in our context is that any multiplier $m : \mathrm{G} \to \mathbb{C}$ on a group having (FA) and satisfying Cotlar's identity falls strictly outside of the previously known classes of examples, including Mei and Ricard's free products and the classical example of $\mathbb{Z}$.

We will give an example of a left orderable group with property (FA). This gives an example for which Cotlar's identity was previously unknown. On the other hand, it's associated Hilbert transform can be proven to be of weak type $(1,1)$ due to the theory of Hilbert transform on finite subdiagonal algebras [56].

We will denote by $D(2,3,7)$ the $(2,3,7)$-triangle group, which is of particular interest in hyperbolic geometry. It is the group of orientation-preserving isometries of the tiling by the $(2,3,7)$ Schwarz triangle. $D(2,3,7)$ admits a presentation

$$D(2,3,7) \,=\, \big\langle x, y \,\big|\, x^2 = y^3 = (xy)^7 = 1 \big\rangle.$$

From the above presentation, it is easy to see $D(2,3,7)$ is a quotient of the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}_1 * \mathbb{Z}_3$ and it is isomorphic to a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Now let us consider the lifting of $(2,3,7)$-triangle group to the universal cover of $\mathrm{PSL}(2,\mathbb{R})$ and denote it by Γ. We know from [5] that Γ has a presentation

$$\Gamma \,=\, \langle x, y, z \,|\, x^2 = y^3 = z^7 = xyz \rangle.$$

The fact that Γ has Serre's property (FA) and simultaneously acts on the real line by homeomorphisms is already known. We gather the different pieces in the proposition below.

Proposition 4.6. *The group Γ has property* (FA) *and is left-orderable. Therefore, its sign Hilbert transform $H = T_{\mathrm{sgn}}$ satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *and thus*

$$\big\| H : L_p(\mathcal{L}\Gamma) \to L_p(\mathcal{L}\Gamma) \big\| \,<\, \infty \qquad \textit{for } 1 < p < \infty.$$

Proof. It follows from the presentations of $D(2,3,7)$ and its covering group Γ that the kernel of the covering homomorphism is isomorphic to $\mathbb{Z}$. That is, we have a short exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to D(2,3,7) \to 1.$$

Note that Γ is perfect [5], i.e. it does not contain any nontrivial abelian quotient, then [14, Proposition 3.2] tells us that Γ has property (FA) if and only if $D(2,3,7)$ has property (FA). Since $D(2,3,7)$ has property (FA), see [61, p. 61], we deduce Γ also has property (FA). Moreover, the action of $\mathrm{PSL}(2,\mathbb{R})$ denoted by $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ on the circle lifts to an action of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ on $\mathbb{R}$ by orientation-preserving homeomorphism, so in this way, the $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ embeds into $\mathrm{Homeo}_+(\mathbb{R})$, see [39] for more details, it naturally admits a left-invariant total order. Therefore, Γ is also left-orderable. Applying Theorem C, we conclude that Γ admits a nontrivial Hilbert transform. $\square$

Recall that this gives a new example of group multiplier satisfying Cotlar's identity but not a new example of a group multiplier being L_p bounded since it falls within the theory described in [56]. In order to obtain new examples outside both the Classical theory, the theory of free Hilbert transforms [43] and the theory of subdiagonal algebras we will need an edxample of a group with property (FA) acting without global fixed points on an $\mathbb{R}$-tree other than $\mathbb{R}$ itself. There are a few examples on that direction in the literature, see [45], but the complexity of the constructions makes finding explicit formulas for the multipliers more involved.

5. Hilbert transforms over lattices of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Over $\mathrm{PSL}_2(\mathbb{R})$ there is a very natural multiplier symbol, playing a role analogous to that of the Hilbert transform in $\mathbb{R}$. This is given by equation (4). In the statement of following theorem we will abuse our notation and denote by the elements in $\mathrm{PSL}_2(\mathbb{R})$, which are classes of matrices up to sign $\pm\mathrm{id}$ by simply matrices. We will also denote by S and T the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It holds that $S^2 = -\mathrm{id}$ and $(ST)^3 = -\mathrm{id}$, which in the quotient group gives that $S^2 = \mathrm{id}$ and $(ST)^3 = \mathrm{id}$. We have the following result.

Proposition 5.1. *Let $m|_{\mathrm{PSL}_2(\mathbb{Z})} : \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{C}$ be the restriction of* (4) *to the modular group. Then m satisfies* $(\mathrm{Cotlar}_{\mathbb{E}^\perp})$ *with respect to the subgroup $\mathrm{G}_0 = \{\mathrm{id}, S\}$ and as a consequence*

$$\big\| T_m : L_p(\mathcal{L}\mathrm{PSL}_2(\mathbb{Z})) \to L_p(\mathcal{L}\mathrm{PSL}_2(\mathbb{Z})) \big\| \,\lesssim\, \Big(\frac{p^2}{p-1} \Big)^{\beta} \qquad \textit{where } \beta = \log_2(1+\sqrt{2}).$$

Proof. First notice that $m : \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ is right K-invariant, with $K = \mathrm{PSO}(2)$. Similarly, when G_0 acts on the left of m we have that

$$m(S^k\, g) \,=\, (-1)^k m(g), \qquad \text{for } g \in \mathrm{PSL}_2(\mathbb{Z}),\, k \in \mathbb{Z}$$

which means that m is left G_0-invariant with respect to a character $\mathrm{G}_0 \to \mathbb{T}$ and Remark 1.7 applies. We need to prove that either $m(gh) = m(g)$ or $m(g^{-1}) = m(h)$ for $g \in \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{G}_0$ and $h \in \mathrm{PSL}_2(\mathbb{Z}) \backslash \{\mathrm{id}\}$. Without loss of generality we can assume that h lives in the larger group $\mathrm{PSL}_2(\mathbb{R})$ and use the right K-invariance of m to assume that

$$h \,=\, \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \qquad \text{and} \qquad g \,=\, \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a,\, b,\, c,\, d \in \mathbb{Z}$. We are going to use the following elementary identities

$$\begin{aligned} m(g) &= \mathrm{sgn}(ac + bd), \\ m(g^{-1}) &= \mathrm{sgn}(-dc - ab), \\ m(h) &= \mathrm{sgn}(x), \\ m(gh) &= \mathrm{sgn}\big(ac(x^2+y^2) + (ad+bc)x + bd\big). \end{aligned}$$

If $m(g^{-1}) = m(h)$, then the Cotlars's identity is already fulfilled. Thus, we consider the case when $m(g^{-1}) \neq m(h)$, which is equal to saying that $x\,(-ab - dc) < 0$. On the other hand, since $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$, we get $abcd = bc + (bc)^2 \geq 0$. We also get that $x(ab(d^2+c^2) + dc(a^2+b^2)) \geq 0$. This observation gives us the following result

$$\begin{aligned} &\big(ac(x^2{+}y^2) + (ad+bc)x + bd\big)(ac+bd) \\ &\qquad = (a^2c^2 + abcd)(x^2+y^2) + \big(ab(d^2+c^2) + dc(a^2+b^2)\big)x + abcd + b^2d^2 \geq 0. \end{aligned}$$

This translates to $m(gh) = m(g)$, which gives the result. □

The proposition above also admits a geometric proof. As we discussed in the introduction, there is a faithful and transitive isometric action of $\mathrm{PSL}_2(\mathbb{R})$ on the Poincaré plane $\mathbb{H}$ through Möbious transformations. As a lattice of $\mathrm{PSL}_2(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{Z})$ naturally acts on $\mathbb{H}$. Now consider a fundamental domain D of $\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$, it is well known that such a domain D can be taken to be a geodesic triangle with finite area but infinite diameter. Let us consider the hyperbolic tessellation:

$$\mathbb{H} \,=\, \bigsqcup_{g \in \mathrm{PSL}_2(\mathbb{Z})} g \cdot \mathrm{D}.$$

We can build a graph X such that their vertices can be either tiles of the above tessellation or each of the three sides of each tile, let us denote those by X_{tiles} and X_{sides} respectively. Two vertices of $x_0, x_1 \in X$ are connected by an edge in $\mathrm{Edge}_+(X)$ if x_0 can be labelled by a tile and x_1 can be labelled by one of its boundary segments. They will be similarly connected by an edge in $\mathrm{Edge}_-(X)$ if the roles of x_0 and x_1 are reversed. It is easy to see that X is a tree, see figure 10. Obviously the action of the modular group induces an action on X, and for every $x \in X_{\mathrm{tiles}}$, its orbit is equal to X_{tiles}; for each $x \in X_{\mathrm{sides}}$, its orbit is equal to X_{sides}. It is also trivial to see that the stabilizer of each element in X_{sides} is conjugate to $\{\mathrm{id}, S\} \cong \mathbb{Z}_2$, while the stabilizer of each element in X_{tiles} is conjugate to $\{\mathrm{id},\, (ST),\, (ST)^{-1}\} \cong \mathbb{Z}_3$. By the Bass-Serre theory reviewed in Section 4 it is immediate that $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Furthermore, we can take the fundamental domain D such that one of its boundary segments $x_0 \in X_{\mathrm{sides}}$ lays within the geodesic $\{z : \mathrm{Re}\{z\} = 0\}$. Then, it is clear that the function $z \mapsto \mathrm{sgn}(\mathrm{Re}\{z\})$ takes two values, it is constant on each connected component of $X \setminus \{x_0\}$. Now we can apply Model 1 to obtain that Cotlar's identity holds. Observe that Proposition 4.1 and the discussion before the theorem implies that $g \mapsto \mathrm{Re}\{g \cdot i\}$ depends only on the initial segment. Therefore (4) restricted to the modular group $\mathrm{PSL}_2(\mathbb{Z})$ coincides with a free Hilbert transform for $\mathbb{Z}_2 * \mathbb{Z}_3$.

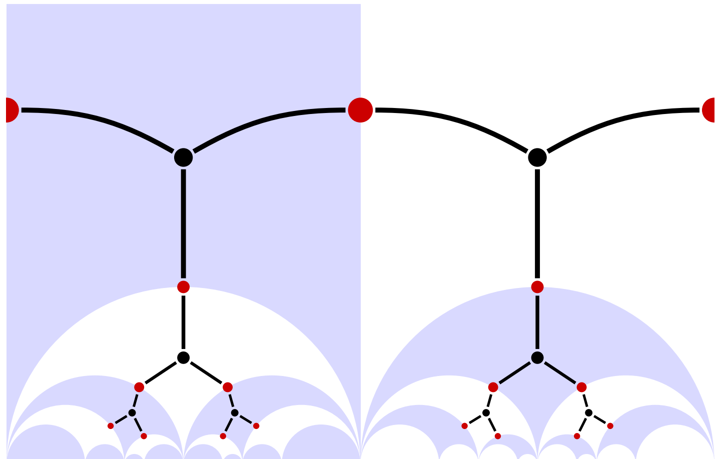

Figure 10: Tesselation associated to the modular group $\mathrm{PSL}_2(\mathbb{Z})$ and its tree X.

The case of $\mathrm{PSL}_2(\mathcal{O}_{-1})$. The purpose of this subsection is to prove Theorem C. Recall that $\mathrm{PSL}_2(\mathcal{O}_{-1}) \subset \mathrm{PSL}_2(\mathbb{C})$ is a lattice, we will denote such group by Γ_1. Recall as well that $\mathrm{PSL}_2(\mathbb{C})$ is semisimple and that any element admits an essentially unique KAN-decomposition with $K = \mathrm{PSU}(2)$, A the abelian group of real diagonal matrices and N the Nilpotent, in fact abelian, group of upper triangular complex matrices with ones on the diagonal. More concretely we have that

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} = \underbrace{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}}_{\in K} \underbrace{\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}}_{\in A} \underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}_{\in N}, \qquad \text{where } s = \sqrt{|a|^2 + |c|^2} \text{ and } t = \frac{\bar{a}b + \bar{c}d}{|a|^2 + |c|^2}.$$

Let $g \in \Gamma_1$ be a matrix of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 + i\, a_2 & b_1 + i\, b_2 \\ c_1 + i\, c_2 & d_1 + i\, d_2 \end{pmatrix}. \tag{5.1}$$

The condition that the determinant equals 1 implies that

$$a_1 d_1 + b_2 c_2 - (b_1 c_1 + a_2 d_2) = 1, \tag{5.2}$$
$$a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1 = 0. \tag{5.3}$$

Using the two identities above, we can elementary prove the following key inequality.

Lemma 5.2. *Let $g \in \Gamma_1$ be as in (5.1). It holds that*

$$0 \le \big(a_1 d_1 + b_2 c_2\big)\big(b_1 c_1 + a_2 d_2\big) \le \mathrm{Re}(a\bar{c})\,\mathrm{Re}(b\bar{d}) = \big(a_1 c_1 + a_2 c_2\big)\big(b_1 d_1 + b_2 d_2\big)$$

Proof. For the first inequality, let $A = a_1 d_1 + b_2 c_2$ and $B = b_1 c_1 + a_2 d_2$. By equation (5.2) we have $A = 1 + B$. But this implies that $AB = B^2 + B$. Since B is an integer $AB \ge 0$. For the second inequality, it is enough to show that $0 \le (\mathrm{I}) - (\mathrm{II})$, where $(\mathrm{I}) = (a_1 c_1 + a_2 c_2)(b_1 d_1 + b_2 d_2)$ and $(\mathrm{II}) = (a_1 d_1 + b_2 c_2)(b_1 c_1 + a_2 d_2)$. We apply (5.3) to obtain

$$\begin{aligned} (\mathrm{I}) &= a_1 b_1 c_1 d_1 + a_1 b_2 c_1 d_2 + a_2 b_1 c_2 d_1 + a_2 b_2 c_2 d_2 \\ &= a_1 b_1 c_1 d_1 + b_1 b_2 c_1 c_2 + \big(b_2 c_1 - a_2 d_1\big) b_2 c_1 + a_2 b_1 c_2 d_1 + a_2 b_2 c_2 d_2, \\ (\mathrm{II}) &= a_1 b_1 c_1 d_1 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + a_2 b_2 c_2 d_2 \\ &= a_1 b_1 c_1 d_1 + a_2 b_1 c_2 d_1 + \big(b_2 c_1 - a_2 d_1\big) a_2 d_2 + b_1 b_2 c_1 c_2 + a_2 b_2 c_2 d_2. \end{aligned}$$

Thus $(\mathrm{I}) - (\mathrm{II}) = \big(b_2 c_1 - a_2 d_1\big) b_2 c_1 - \big(b_2 c_1 - a_2 d_1\big) a_2 d_2 = (X - Y)X - (X - Y)Y = (X - Y)^2 \ge 0,$. where $X = b_2 c_1$ and $Y = a_2 d_1$. $\square$

The next lemma follows by elementary calculations.

Lemma 5.3. *Let* $g \in \Gamma_1$ *as in* (5.1).

(i) *If we assume that* $\mathrm{Re}\{a\bar{c}\} \neq 0$ *and* $\mathrm{Re}\{b\bar{d}\} \neq 0$, *then we get* $\mathrm{sgn}\big(\mathrm{Re}\{a\bar{c}\}\big) = \mathrm{sgn}\big(\mathrm{Re}\{b\bar{d}\}\big)$.

(ii) *On the other hand,* $\mathrm{Re}\,\{a\bar{c}\}\,\mathrm{Re}\,\{b\bar{d}\} = 0$ *if and only if* $(a_1d_1 + b_2c_2)(b_1c_1 + a_2d_2) = 0$ *and* $b_2c_1 = a_2d_1$.

(iii) *If* $\mathrm{Re}\{\bar{a}b\} \neq 0$ *and* $\mathrm{Re}\{\bar{c}d\} \neq 0$, *then* $\mathrm{sgn}(\mathrm{Re}\{\bar{a}b\}) = \mathrm{sgn}(\mathrm{Re}\{\bar{c}d\})$.

(iv) $\mathrm{Re}\,\{\bar{a}b\}\,\mathrm{Re}\,\{\bar{c}d\} = 0$ *if and only if* $\big(a_1d_1 + b_2c_2\big)\big(b_1c_1 + a_2d_2\big) = 0$ *and* $b_1c_2 = a_2d_1$.

(v) *It holds that* $\mathrm{Im}\,\big\{b\bar{c} - a\bar{d}\big\}^2 - 4\,\mathrm{Re}\{a\bar{c}\} \cdot \mathrm{Re}\{b\bar{d}\} \leq 0$.

Proof. We will just given an sketch. Observe that, by Lemma 5.2, we have that $\mathrm{Re}(a\bar{c})\,\mathrm{Re}(b\bar{d}) \geq 0$. Therefore, if both terms are different from zero, they have the same sign. The point (ii) follows similarly by Lemma 5.2 and its proof. Points (iii) and (iv) are a reiteration of the previous two points but changing rows by columns. Lastly, for (v), we start by noticing that

$$\mathrm{Im}\{b\bar{c}\} - \mathrm{Im}\{a\bar{d}\} = b_2c_1 - b_1c_2 - a_2d_1 + a_1d_2 = 2\big(a_1d_2 - b_2c_1\big),$$

where we have used (5.3). Now, we have that

$$\begin{aligned}
& \big(b_2c_1 - a_2d_1\big)^2 - \big(a_1c_1 + a_2c_2\big)\,\big(b_1d_1 + b_2d_2\big) \\
= \;& b_2^2c_1^2 + a_2^2d_1^2 - 2a_2b_2c_1d_1 - a_1b_1c_1d_1 - a_1b_2c_1d_2 - a_2b_1c_2d_1 + a_2b_2c_2d_2 && (5.4)\\
= \;& a_2d_1\big(a_2d_1 - b_2c_1 - b_1c_2\big) - a_1b_1c_1d_1 - b_1b_2c_1c_2 - a_2b_2c_2d_2 && (5.5)\\
= \;& -a_1a_2d_1d_2 - a_1b_1c_1d_1 - b_1b_2c_1c_2 - a_2b_2c_2d_2 \\
= \;& -\big(a_2d_2 + b_1c_1\big)\big(a_1d_1 + b_2c_2\big) && (5.6)\\
= \;& -X(1 + X) \;\leq\; 0,
\end{aligned}$$

where $X = a_2d_2 + b_1c_1$. We have used identity (5.3) in (5.4), and (5.5). For (5.6) we use (5.2). The last term is negative since X is an integer and $X^2 + X$ is always positive when $X \in \mathbb{Z}$. □

Now let us consider the symbol given in (5). When it is restricted to Γ_1, is 0 over a large subset. We are going to show that this subset is in fact a subgroup $\Gamma_0 \subset \Gamma_1$. The following proposition follows from the previous lemma.

Lemma 5.4. *Let* $g \in \Gamma_1$ *be a matrix with coefficients like those in* (5.1) *and such that* $\mathrm{Re}\{a\bar{c}+b\bar{d}\} = 0$. *Then* $g \in \Gamma_0^+ \cup \Gamma_0^-$, *where*

$$\begin{aligned}
\Gamma_0^+ &= \left\{\begin{pmatrix} a_1 & b_2 i \\ c_2 i & d_1 \end{pmatrix} : a_1, b_2, c_2, d_1 \in \mathbb{Z}, a_1d_1 + b_2c_2 = 1\right\}\\
\Gamma_0^- &= \left\{\begin{pmatrix} a_2 i & b_1 \\ c_1 & d_2 i \end{pmatrix} : a_2, b_1, c_1, d_2 \in \mathbb{Z}, -a_2d_2 - b_1c_1 = 1\right\}
\end{aligned}$$

It is clear that Γ_0^+ *and* $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ *are subgroups of* Γ_1.

Proof. Observe that, by Lemma 5.3.(i), $\mathrm{Re}\{a\bar{c}\}$ and $\mathrm{Re}\{b\bar{d}\}$ have the same sign, therefore if $\mathrm{Re}\{a\bar{c} + b\bar{d}\} = 0$ that is because $\mathrm{Re}\{a\bar{c}\} = \mathrm{Re}\{b\bar{d}\} = 0$. Assume that a_1 and a_2 are coprimes and that b_1 and b_2 are also coprimes. Since $\mathrm{Re}\{a\bar{c}\} = a_1c_1 + a_2c_2 = 0$, we have that (a_1, a_2) and (c_1, c_2) are perpendicular vectors with integer coordinates. But, since a_1 and a_2 are coprimes $(c_1, c_2) = \ell(a_2, -a_1)$ for some $\ell \in \mathbb{Z}$. Similarly $(d_1, d_2) = m\,(b_2, -b_1)$ for some $m \in \mathbb{Z}$. Computing the determinant gives

$$\det\begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ \ell a_2 - i\ell a_1 & mb_2 - imb_1 \end{pmatrix} \;=\; \big(m - \ell\big)\big(a_2b_1 + a_1b_2\big) + i\big(m - \ell\big)\big(a_2b_2 - a_1b_1\big) \;=\; 1.$$

But this implies that $m-\ell=\pm1$ and $a_2b_1+a_1b_2=\pm1$. Since the imaginary part has to be 0 and $m-\ell=\pm1$, it follows that $a_2b_2=a_1b_1$. But since a_1 and a_2, like b_1 and b_2, are coprimes, we have that $a_2=b_1$ and $a_1=b_2$. Therefore $a_1^2+a_2^2=1$ and $b_1^2+b_2^2=1$. Since they are integers, one of a_1, a_2 and of b_1 and b_2 has to be 0. So we have either $a_2=b_1=0$ or $a_1=b_2=0$. The case of non-coprime a_1 and a_2 can be proved similarly, by noticing that $a_1+ia_2=k\alpha_1+ik\alpha_2$ with $k=\gcd(a_1,a_2)$ and α_1 and α_2 coprimes. In that case every perpendicular vector to (a_1,a_2) with integer coordinates is of the form $(\ell\alpha_2,-\ell\alpha_1)$. □

Observe that Γ_0^+ is isomorphic to the subgroup $\mathrm{PSL}_2(\mathbb{Z})$ in Γ_1. Indeed, if we take the natural embedding $\mathrm{PSL}_2(\mathbb{Z})\subset\Gamma_1$, we have that

$$\Gamma_0^+ \;=\; J\,\mathrm{PSL}_2(\mathbb{Z})\,J^{-1}, \qquad \text{where } J=\begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{-\frac{i\pi}{4}}\end{pmatrix}.$$

The group $\Gamma_0=\Gamma_0^+\cup\Gamma_0^-$ is generated by Γ_0^+ and the diagonal matrix with eigenvalues i and $-i$. It is trivial to check that Γ_0 it is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})\rtimes\mathbb{Z}_2$. With the order two automorphism given by conjugation.

Now, we introduce the following modification from the symbol in (5).

Definition 5.5. Let us define the symbol $m_2:\Gamma_1\to\mathbb{C}$ by

$$m_2(g)=\begin{cases}\mathrm{sgn}\big(\,\mathrm{Re}\{a\,\bar{c}+b\,\bar{d}\}\big) & \text{when } g\in\Gamma_0 \\ 1 & \text{when } g\in\Gamma_0^- \\ 0 & \text{when } g\in\Gamma_0^+.\end{cases}$$

We call the group elements of Γ the singular points of m_2.

Proposition 5.6. *The function m_2 given in Definition 5.5 is both left and right Γ_0^+-invariant.*

Proof. Let $g\in\Gamma_1$ be like in (5.1). We first prove m_2 is left Γ_0^+-invariant. If $h\in\Gamma_0^+$, since Γ_0^+ is a subgroup of Γ_1, it is obvious that $m_2(gh)=m_2(h)=0$. If $h\in\Gamma_0^-$, it is easy to check $gh\in\Gamma_0^-$, thus $m_2(gh)=m_2(h)=1$. If $h\notin\Gamma_0$, then m_2 is right K-invariant and we have $m_2(h)=\mathrm{Re}(t)$, where

$$h \;=\; \begin{pmatrix} s & st \\ 0 & s^{-1}\end{pmatrix}k \quad \text{for some } k\in K.$$

Moreover, we have $gh\notin\Gamma_0$ and

$$m_2(gh)=m_2\bigg(\begin{pmatrix} a_1 & b_2 i \\ c_2 i & d_1\end{pmatrix}\begin{pmatrix} s & st \\ 0 & s^{-1}\end{pmatrix}\bigg) \;=\; \big(a_1d_1+b_2c_2\big)\ \mathrm{Re}(t) \;=\; \mathrm{Re}(t)=m_2(h).$$

Now we show m_2 is right Γ_0^+-invariant. Similarly as the above, if $h\in\Gamma_0^+$, $m_2(hg)=m_2(h)=0$; if $h\in\Gamma_0^-$, we have $hg\in\Gamma_0^-$ and $m_2(hg)=m_2(h)=1$. If $h\notin\Gamma_0$ and let

$$h \;=\; \begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Then $hg\notin\Gamma_0$. Therefore we have

$$\begin{aligned} m_2(hg) &= m_2\bigg(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} u & uv \\ 0 & u^{-1}\end{pmatrix}\bigg) \\ &= \mathrm{sgn}\Big[\mathrm{Re}\,(a\bar{c})\,u^2(1+\,\mathrm{Im}\ v^2)+\mathrm{Im}\,(b\bar{c}-a\bar{d})\ \mathrm{Im}\ v+\mathrm{Re}\,(b\bar{d})\,u^{-2}\Big], \end{aligned}$$

where

$$u=\sqrt{d_1^2+c_2^2}\neq0 \quad \text{and} \quad v=\frac{(b_2d_1-a_1c_2)i}{\sqrt{d_1^2+c_2^2}}$$

Recall that $m_2(h) = \operatorname{sgn}(\operatorname{Re}(a\bar{c} + b\bar{d}))$. If $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$, then by Lemmas 5.3 and (v), it is easy to see $m_2(hg) = m_2(h)$. If $\operatorname{Re}(a\bar{c}) = 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$, we have $m_2(hg) = \operatorname{sgn}[\operatorname{Re}(b\bar{d})\, u^{-2}]$ and $m_2(h) = \operatorname{sgn}(\operatorname{Re}(b\bar{d}))$. Thus $m_2(hg) = m_2(h)$. If $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) = 0$, we have $m_2(hg) = \operatorname{sgn}[\operatorname{Re}(a\bar{c})\, u^2(1 + \operatorname{Im} v^2)]$ and $m_2(h) = \operatorname{sgn}(\operatorname{Re}(a\bar{c}))$, which implies $m_2(hg) = m_2(h)$. □

Remark 5.7. Following a similar argument as in the proof of last proposition, we can show that m_2 given in Definition 5.5 is also Γ_0-right invariant (but not Γ_0-left invariant).

Proposition 5.8. *The function m_2 given in Definition 5.5 satisfies the Cotlar identity* ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) *relative to Γ_0^+, i.e., for any $g \in \Gamma_1 \setminus \Gamma_0^+$ and $h \in \Gamma_1$, it holds that*

$$\big(m_2(g^{-1}) - m_2(h)\big)\big(m_2(gh) - m_2(g)\big) \;=\; 0.$$

Proof. Since m_2 is right Γ_0-invariant by the Proposition 5.6 and Remark 5.7, it suffices to prove the Cotlar identity for $g \in \Gamma_1 - \Gamma_0^+$ and $h \in \Gamma_1 \setminus \Gamma_0$. First let us consider the case when $g \in \Gamma_0^-$. In this case we have $m_2(g) = m_2(g^{-1}) = 1$. If $m_2(g^{-1}) = m_2(h)$, then the identity is satisfied. If $m(g^{-1}) \neq m_2(h)$, this implies $m_2(h) = -1$, and then $h, gh \notin \Gamma_0$. By the right K-invariance of m_2 on $\Gamma_1 \setminus \Gamma_0$, we get

$$m_2(gh) \;=\; m_2\left(\begin{pmatrix} a_2 i & b_1 \\ c_1 & d_2 i \end{pmatrix}\begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix}\right) = (a_2 d_2 + b_1 c_1)\operatorname{Re} t = -\operatorname{Re} t = -m_2(h) = 1.$$

Therefore, we get $m_2(gh) = m_2(g)$, the Cotlar identity is satisfied.

Now let us focus on the case when $g \notin \Gamma_0$. Let $g \notin \Gamma_0$, i.e. $\operatorname{Re}(a\bar{c} + b\bar{d}) \neq 0$ and

$$h = \begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix} k \notin \Gamma_0$$

with $k \in \mathrm{PSU}(2)$. We get the following expressions for the four terms appearing in the Cotlar identity:

$$\begin{aligned}
m_2(g) &= \operatorname{sgn}(\operatorname{Re}(a\bar{c} + b\bar{d})), \\
m_2(g^{-1}) &= -\operatorname{sgn}(\operatorname{Re}(\bar{a}b + \bar{c}d)), \\
m_2(h) &= \operatorname{sgn}(\operatorname{Re} t), \\
m_2(gh) &= m_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix}\right) \\
&= \operatorname{sgn}\Big[\operatorname{Re}(a\bar{c})\, s^2(1 + |t|^2) + \operatorname{Re}(a\bar{d} + b\bar{c})\, \operatorname{Re} t + \operatorname{Im}(b\bar{c} - a\bar{d})\, \operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\Big].
\end{aligned}$$

In the identities for $m_2(h)$ and $m_2(gh)$, we have used the property that m_2 is right K-invariant on $\Gamma_1 \setminus \Gamma_0$. Now we prove the Cotlar identity. If $m_2(g^{-1}) = m_2(h)$, then the identity is satisfied. If not, that means

$$\operatorname{Re} t \cdot \operatorname{Re}(\bar{a}b + \bar{c}d) > 0. \tag{5.7}$$

Then we need to show $m(g) = m(gh)$, or in other words,

$$\begin{aligned}
&\Big[\operatorname{Re}(a\bar{c})\, s^2(1 + |t|^2) + \operatorname{Re}(a\bar{d} + b\bar{c})\, \operatorname{Re} t + \operatorname{Im}(b\bar{c} - a\bar{d})\, \operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\Big] \cdot \operatorname{Re}(a\bar{c} + b\bar{d}) \\
= \;& \operatorname{Re}(a\bar{c} + b\bar{d})\, \operatorname{Re}(a\bar{c})\, s^2(1 + (\operatorname{Re} t)^2) + (\operatorname{Re}(a\bar{c} + b\bar{d})\, \operatorname{Re}(a\bar{d} + b\bar{c})\, \operatorname{Re} t \\
&+ \operatorname{Re}(a\bar{c} + b\bar{d}) \cdot \big[\operatorname{Re}(a\bar{c})\, s^2(\operatorname{Im} t)^2 + \operatorname{Im}(b\bar{c} - a\bar{d})\, \operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\big] > 0.
\end{aligned}$$

We first assume that $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$. By Lemma 5.3, we have

$$\operatorname{Re}(a\bar{c} + b\bar{d})\, \operatorname{Re}(a\bar{c}) > 0.$$

Moreover, Lemma 5.3.(v) implies that the determinant of the quadratic function of $\operatorname{Im} t$ is negative, together with the inequality above, we see that

$$\operatorname{Re}(a\bar{c} + b\bar{d}) \cdot \big[\operatorname{Re}(a\bar{c})\, s^2(\operatorname{Im} t)^2 + \operatorname{Im}(b\bar{c} - a\bar{d})\, \operatorname{Im} t + \operatorname{Re}\, b\bar{d}\, s^{-2}\big] \geq 0$$

Moreover, we claim that

$$\mathrm{Re}\,(a\bar{c}+b\bar{d})\,\mathrm{Re}\,(a\bar{d}+b\bar{c})\ \mathrm{Re}\ t\geq 0.$$

If the claim is true then we will obtain $m(g)=m(gh)$. So now it remains to prove the claim. Notice that by (5.7) and Lemma 5.3, it is enough to show

$$\mathrm{Re}\,(\bar{a}b)\ \mathrm{Re}\,(a\bar{c})\ \mathrm{Re}\,(a\bar{d}+b\bar{c})\geq 0.$$

According to (5.2), $\mathrm{Re}\,(a\bar{d}+b\bar{c})=a_1d_1+a_2d_2+b_1c_1+b_2c_2=2(b_1c_1+a_2d_2)+1$. On the other hand, (5.2) and (5.3) imply that

$$\begin{aligned}\mathrm{Re}\,(\bar{a}b)\ \mathrm{Re}\,(a\bar{c}) &= (a_1b_1+a_2b_2)(a_1c_1+a_2c_2)\\ &= a_1a_2(b_1c_2+b_2c_1)+a_1^2b_1c_1+a_2^2b_2c_2\\ &= a_1a_2(a_1d_2+a_2d_1)+a_1^2b_1c_1+a_2^2b_2c_2\\ &= a_1^2(b_1c_1+a_2d_2)+a_2^2(a_1d_1+b_2c_2)\\ &= (b_1c_1+a_2d_2)(a_1^2+a_2^2)+a_2^2.\end{aligned}$$

Let $X=b_1c_1+a_2d_2$, $A=a_1^2+a_2^2$ and $B=a_2^2$. Then $\mathrm{Re}\,(\bar{a}b)\ \mathrm{Re}\,(a\bar{c})\ \mathrm{Re}\,(a\bar{d}+b\bar{c})=(AX+B)(2X+1)$. Since $X\in\mathbb{Z}$ and $|\frac{B}{A}|\leq 1$, we get $(AX+B)(2X+1)\geq 0$, which proves the claim.

Now we deal with the case when $\mathrm{Re}\,(a\bar{c})\ \mathrm{Re}\,(b\bar{d})=0$. By Lemma 5.3, it is equivalent to saying $(a_1d_1+b_2c_2)(b_1c_1+a_2d_2)=0$ and $b_2c_1=a_2d_1$. Without loss of generality, we assume that $b_1c_1+a_2d_2=0$. Then by (5.2), $a_1d_1+b_2c_2=1$. Since $b_2c_1=a_2d_1$, (5.3) tells us that $b_1c_2=a_1d_2$ and $\mathrm{Im}\,(b\bar{c}-a\bar{d})=2(b_2c_1-a_2d_1)=0$. This implies that

$$a_1b_1=a_1b_1(a_1d_1+b_2c_2)=a_1^2(b_1d_1+b_2d_2)=a_1^2\,\mathrm{Re}\,(b\bar{d}). \tag{5.8}$$

Similarly, we have

$$c_2d_2=c_2^2\ \mathrm{Re}\,(b\bar{d}),\quad a_2b_2=b_2^2\ \mathrm{Re}\,(a\bar{c})\quad\text{and}\quad c_1d_1=d_1^2\ \mathrm{Re}\,(a\bar{c}). \tag{5.9}$$

Since $\mathrm{Re}\,(a\bar{c}+b\bar{d})\neq 0$, $\mathrm{Re}\,(a\bar{c})$ and $\mathrm{Re}\,(b\bar{d})$ can not be zero at the same time. Suppose $\mathrm{Re}\,(a\bar{c})=0$ and $\mathrm{Re}\,(b\bar{d})\neq 0$. Then $m(g)=\mathrm{sgn}\,\mathrm{Re}\,(b\bar{d})$, $m(gh)=\mathrm{sgn}\big[\,\mathrm{Re}\,(a\bar{d}+b\bar{c})\ \mathrm{Re}\ t+\mathrm{Re}\,(b\bar{d})\,s^{-2}\big]=\mathrm{sgn}\big[\,\mathrm{Re}\ t+\mathrm{Re}\,(b\bar{d})\,s^{-2}\big]$. Applying (5.8) and (5.9), we get $\mathrm{Re}\,(\bar{a}b+\bar{c}d)=(a_1^2+c_2^2)\,\mathrm{Re}\,(b\bar{d})$. Recall that we have $\mathrm{Re}\ t\cdot\mathrm{Re}\,(\bar{a}b+\bar{c}d)>0$ by the assumption $m(g^{-1})\neq m(h)$. Therefore, we have $\mathrm{Re}\ t\cdot\mathrm{Re}\,(b\bar{d})>0$, which implies $m(g)=m(gh)$. For the case $\mathrm{Re}\,(a\bar{c})\neq 0$ and $\mathrm{Re}\,(b\bar{d})=0$, we omit the proof since it is similar to the previous case. □

Now, we can prove Theorem D by just reducing the L_p-boundedness of m to that of m_2.

Proof (of Theorem D). Notice that $m:\Gamma_1\to\mathbb{C}$ satisfies that

$$m(g)=m_2(g)-\mathbf{1}_{\Gamma_0}+\mathbf{1}_{\Gamma_0^+}.$$

But Fourier multipliers with symbols being characteristic functions of open subgroups are bounded in L_p for every $1\leq p\leq\infty$. Therefore, up to a finite constant smaller that 2, the operator L_p-norm of T_m is bounded by that of T_{m_2}. □

References

[1] Alfonseca, A. Strong type inequalities and an almost-orthogonality principle for families of maximal operators along directions in $\mathbb{R}^2$. *J. Lond. Math. Soc. 67*, 1 (2003), 208–218.

[2] Alfonseca, A., Soria, F., and Vargas, A. An almost-orthogonality principle in L^2 for directional maximal functions. In *Harmonic Analysis at Mount Holyoke: Proceedings of an AMS-IMS-SIAM Joint Summer Research Conference on Harmonic Analysis, June 25-July 5, 2001, Mount Holyoke College, South Hadley, MA* (2003), vol. 320, American Mathematical Soc., pp. 1–10.

[3] Arveson, W. B. Analyticity in operator algebras. *Amer. J. Math. 89*, 3 (1967), 578–642. Publisher: JSTOR.

[4] Bateman, M. Kakeya sets and directional maximal operators in the plane. *Duke Math. J. 147*, 1 (Mar. 2009), 55–77. Publisher: Duke University Press.

[5] Bergman, G. M. Right orderable groups that are not locally indicable. *Pacific J. Math. 147*, 2 (1991), 243–248.

[6] Bestvina, M. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, R. Daverman and R. Sher, Eds. Elsevier, North-holland, 2002, ch. 2, pp. 55–91.

[7] Brown, N. P., and Ozawa, N. *C^*-algebras and finite-dimensional approximations*, vol. 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[8] Cadilhac, L., Conde-Alonso, J. M., and Parcet, J. Spectral multipliers in group algebras and noncommutative Calderón-Zygmund theory. *J. Math. Pures Appl. 163* (July 2022), 450–472.

[9] Calderón, A. P. On Theorems of M. Riesz and Zygmund. *Proc. Am. Math. Soc. 1*, 4 (1950), 533–535.

[10] Caspers, M., Parcet, J., Perrin, M., and Ricard, É. Noncommutative de leeuw theorems. In *Forum of Mathematics, Sigma* (2015), vol. 3, Cambridge Univ Press, p. e21.

[11] Cherix, P., Cowling, M., Jolissaint, P., Julg, P., and Valette, A. *Groups with the Haagerup property*, vol. 197. Birkhäuser Verlag, Basel, 2001.

[12] Conde-Alonso, J., González-Pérez, A., Parcet, J., and Tablate, E. Hörmander-Mikhlin criteria on Lie group von Neumann algebras. *Preprint* (Jan. 2022). arXiv:2201.08740.

[13] Conde-Alonso, J., González-Pérez, A., Parcet, J., and Tablate, E. Schur multipliers in Schatten-von Neumann classes. *Preprint* (Jan. 2022). arXiv:2201.05511.

[14] Cornulier, Y., and Kar, A. On property (FA) for wreath products. *J. Group Theory 14* (2011), 165–174. Publisher: Walter de Gruyter GmbH & Co. KG.

[15] Cotlar, M. A unified theory of Hilbert transforms and ergodic theorems. *Rev. Mat. Cuyana 1*, 2 (1955), 105–167.

[16] Davidson, K. R. *Nest Algebras*. Longman Scientific & Technical, 1988.

[17] Deroin, B., Navas, A., and Rivas, C. Groups, Orders, and Dynamics. *Preprint* (Oct. 2016). arXiv:1408.5805.

[18] Dixmier, J. Formes linéaires sur un anneau d'opérateurs. *Bulletin de la Société Mathématique de France 81* (1953), 9–39.

[19] Duchamp, G., and Thibon, J.-Y. Simple orderings for free partially commutative groups. *International Journal of Algebra and Computation 02*, 03 (Sept. 1992), 351–355. Publisher: World Scientific Publishing Co.

[20] Duoandikoetxea, J. *Fourier analysis*, vol. 29 of *Graduate Studies in Mathematics*. American Matheatical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[21] Erven, J., and Falkowski, B.-J. *Low order cohomology and applications*, vol. 877 of *Lecture Notes in Mathematics*. Springer, 1981.

[22] Folland, G. B. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[23] Frohman, C., and Fine, B. Some amalgam structures for Bianchi groups. *Proc. Amer. Math. Soc. 102*, 2 (1988), 221–229.

[24] Gałązka, T., and Osękowski, A. A probabilistic approach to hilbert transforms on free group von neumann algebras. *Proc. Amer. Math. Soc. 150*, 07 (2022), 2861–2877.

[25] Gokhberg, I. T., and Krupnik, N. Y. Norm of the Hilbert transformation in the L^p space. *Funct. Anal. Appl. 2*, 2 (1968), 180–181.

[26] Goldstein, S., and Labuschagne, L. Notes on noncommutative L^p and Orlicz spaces, 2020.

[27] Gomov, M. Asymptotic invariants of infinite groups. *Geometric Group Theory 2* (1993), 1–295.

[28] González-Pérez, A. M., Junge, M., and Parcet, J. Singular integrals in quantum Euclidean spaces. *Mem. of the Am. Math. Soc 272*, 1334 (2021).

[29] Grafakos, L. Best bounds for the Hilbert transform on $L^p(\mathbb{R}^1)$. *Mathematical Research Letters 4*, 4 (1997), 469–471.

[30] Haagerup, U. An example of a non nuclear C^*-algebra, which has the metric approximation property. *Invent. Math. 50*, 3 (1978), 279–293.

[31] Hilbert, D. Grundzüge einer allgemeinen theorie der linearen integralgleichungen [1912]. In *Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten*. Springer, 1989, pp. 8–171.

[32] Holland, C. The lattice-ordered groups of automorphisms of an ordered set. *Michigan Mathematical Journal 10*, 4 (1963), 399–408. Publisher: University of Michigan, Department of Mathematics.

[33] Hong, G., Lai, X., and Xu, B. Maximal singular integral operators acting on noncommutative L_p-spaces. *Math. Ann.* (Apr. 2022).

[34] Junge, M. Doob's inequality for non-commutative martingales. *J. Reine Angew. Math. 549* (2002), 149–190.

[35] Junge, M., and Mei, T. BMO spaces associated with semigroups of operators. *Math. Ann. 352*, 3 (2012), 691–743.

[36] Junge, M., Mei, T., and Parcet, J. Noncommutative Riesz transforms—dimension free bounds and Fourier multipliers. *J. Eur. Math. Soc. 20*, 3 (2018).

[37] Junge, M., Mei, T., Parcet, J., and Xia, R. Algebraic Calderón-Zygmund theory. *Adv. Math. 376*, 107443 (2021), 1–72.

[38] Junge, M., and Xu, Q. Noncommutative Burkholder/Rosenthal inequalities. *Ann. Probab. 31*, 2 (2003), 948–995.

[39] Khoi, V. T. A cut-and-paste method for computing the Seifert volumes. *Math. Annalen 326* (2003), 759–801.

[40] Kolmogorov, A. N. Sur les fonctions harmoniques conjuguées et les séries de Fourier. *Fund. Math 7* (1925), 23–28.

[41] Mei, T. Tent spaces associated with semigroups of operators. *J. Funct. Anal. 255*, 12 (2008), 3356–3406.

[42] Mei, T. An H_1-BMO duality theory for semigroups of operators. *preprint* (Apr. 2012). arXiv:1204.5082.

[43] Mei, T., and Ricard, É. Free Hilbert transforms. *Duke Math. J. 166*, 11 (2017), 2153–2182.

[44] Mei, T., Ricard, É., and Xu, Q. A Mikhlin multiplier theory for free groups and amalgamated free products of von Neumann algebras. *Adv. Math. 403* (July 2022), 108394.

[45] Minasyan, A. New examples of groups acting on real trees. *J. Topol. 9*, 1 (2016), 192–214. Publisher: Oxford University Press.

[46] Morgan, J. W., and Shalen, P. B. Valuations, Trees, and Degenerations of Hyperbolic Structures, I. *Ann. Math. 120*, 3 (Nov. 1984), 401.

[47] Morgan, J. W., and Shalen, P. B. Free actions of surface groups on $\mathbb{R}$-trees. *Topology 30*, 2 (1991), 143–154.

[48] Neumann, B. H. On ordered division rings. *Trans. Amer. Math. Soc. 66*, 1 (1949), 202–252. Publisher: JSTOR.

[49] Parcet, J. Pseudo-localization of singular integrals and noncommutative Calderón–Zygmund theory. *J. Funct. Anal. 256*, 2 (2009), 509–593.

[50] Parcet, J., Ricard, É., and Salle, M. Fourier multipliers in $\mathrm{SL}_n(\mathbf{R})$. *Duke Math. J. 171*, 6 (Apr. 2022), 1235–1297. Publisher: Duke University Press.

[51] Parcet, J., and Rogers, K. M. Directional maximal operators and lacunarity in higher dimensions. *Amer. J. Math. 137*, 6 (2015), 1535–1557.

[52] Parcet, J., and Rogers, K. M. Twisted Hilbert transforms vs kakeya sets of directions. *Reine Angew. Math. 2016*, 710 (2016), 137–172.

[53] Pedersen, G. K. *C^*-algebras and their automorphism groups*, vol. 14 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.

[54] Pichorides, S. On the best values of the constants in the theorem of M. Riesz, Zygmund and Kolmogorov. *Studia Math. 44* (1972), 165–179.

[55] Pisier, G., and Xu, Q. Non-commutative L^p-spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*. North-Holland, Amsterdam, 2003, pp. 1459–1517.

[56] Randrianantoanina, N. Hilbert transform associated with finite maximal subdiagonal algebras. *J. Aust. Math. Soc. 65*, 3 (1998), 388–404. Publisher: Cambridge University Press.

[57] Riesz, M. Les fonctions conjuguées et les séries de Fourier. *C.R. Acad. Sci. Paris 178* (1924), 1464–1467.

[58] Riesz, M. Sur les fonctions conjuguées. *Mathe. Zeit. 27*, 1 (1928), 218–244. Publisher: Springer.

[59] Rotman, J. *An Introduction to the Theory of Groups*. Springer Science & Business Media, Aug. 1999.

[60] Segal, I. E. A non-commutative extension of abstract integration. *Ann. of Math.* (1953), 401–457. Publisher: JSTOR.

[61] Serre, J.-P. *Trees*. Springer-Verlag, New York, 1980.

[62] Shalen, P. B. Dendrology of groups: an introduction. In *Essays in group theory*. Springer, 1987, pp. 265–319.

[63] Stein, P. On a theorem of M. Riesz. *J. Lond. Math. Soc. s1-8*, 4 (1933), 242–247.

[64] Terp, M. L^p spaces associated with von Neumann algebras. *Notes, Math. Institute, Copenhagen Univ 3* (1981).

[65] Tits, J. A theorem of Lie-Kolchin for trees. In *Contributions to algebra*, H. Bass, C. P.J., and K. J., Eds. 1977, pp. 377–388. Publisher: Elsevier.

[66] Xia, R. Fourier multipliers coming from group actions on trees, 2022. In preparation.

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