Abstract The notion of B-convexity for operator spaces, which a priori depends on a set of parameters indexed by $\Sigma$, is defined. Some of the classical characterizations of this geometric notion for Banach spaces are studied in this new context. For instance, an operator space is $B_\Sigma$-convex if and only if it has $\Sigma$-subtype. The class of uniformly non-$L^1(\Sigma)$ operator spaces, which is also the class of $B_\Sigma$-convex operator spaces, is introduced. Moreover, an operator space having non-trivial $\Sigma$-type is $B_\Sigma$-convex. However, the converse is false. The row and column operator spaces are nice counterexamples of this fact, since both are Hilbertian. In particular, this result shows that a version of the Maurey-Pisier theorem does not hold in our context. Some other examples of Hilbertian operator spaces will be treated. In the last part of this paper, the independence of $B_\Sigma$-convexity with respect to $\Sigma$ is studied. This provides some interesting problems which will be posed.

Keywords: $B_\Sigma$-convex operator space; uniformly non-$L^1(\Sigma)$ operator space; non-trivial $\Sigma$-type; $K_\Sigma$-convex operator space; row and column operator spaces

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1. Introduction

In the last thirty years, the notions of type and cotype of a Banach space with respect to several orthonormal systems have been deeply investigated. It turns out that, in order to study some geometric properties of the Banach space considered, these notions are very useful. In a series of papers with José García-Cuerva and José Manuel Marco, we have initiated a theory of type and cotype for operator spaces. As it might be expected, the question is to what extent these notions are related to the geometry of operator spaces.

Our previous results can be summarized as follows. In [3], we study the validity of the Hausdorff-Young inequality for vector-valued functions defined on a non-commutative compact group. This inequality does not make sense when our functions take values in a Banach space, we need an operator space structure on it. This requirement goes back to Pisier’s work [14], where non-commutative vector-valued Lebesgue spaces are studied.
In particular, this gives rise to the notions of Fourier type and cotype of an operator space with respect to a non-commutative compact group. We investigate in [3] the basic properties of this notion. The paper [2] is devoted to the study of the sharp Fourier type and cotype exponents of Lebesgue spaces and Schatten classes. This is basic in the commutative theory. However, the problem of finding the sharp exponents of a given operator space is highly non-trivial, even for the simplest case of Lebesgue spaces. When dealing with compact semisimple Lie groups, we have solved part of this problem in [2] using the well-developed representation theory on this kind of groups. Finally, the work [4] deals with the general theory of type and cotype for operator spaces. First, we define the notion of quantized orthonormal system, which plays the role of the orthonormal system in the classical setting. Then we introduce the notions of type and cotype with respect to such a system. In particular, this provides non-commutative extensions of the notions of Rademacher, Steinhaus and Gaussian type and cotype. The main result in [4] is an operator space version of the classical result of Kwapień [7], which characterizes Hilbert spaces up to isomorphism by means of vector-valued orthogonal series. We give several approaches to this result in the operator space setting, characterizing in such a way OH operator spaces up to complete isomorphism by a type 2 / cotype 2 condition.

These previous results show that, also in the non-commutative setting, there exists some interaction between the theory of type and cotype and the geometry of operator spaces. Hence, it seems that the next step in the process should be to study the notion of B-convex operator space. Interest on B-convexity for Banach spaces was generated in [1], where Beck studied certain strong law of large numbers for Banach space valued random variables. However, our motivation to study this notion for operator spaces lies in some other characterizations of this geometric condition. For instance, the following are equivalent for any Banach space $B$:

(a) $B$ is B-convex.
(b) $B$ does not contain $l^1_n$ uniformly.
(c) $B$ has Rademacher subtype.
(d) $B$ has non-trivial Rademacher type.
(e) $B$ is K-convex.

Giesy proved in [5] the equivalence between (a) and (b), while Pisier proved the others. The proof of the equivalence between (a), (c) and (d) can be found in [11]. Finally, the equivalence between B-convexity and K-convexity was given in [12] and is much more complicated. For some other characterizations of B-convexity, the reader is referred to [10]. Our aim is to study the validity of these classical characterizations for operator spaces.

We shall follow a notation similar to that employed in [4]. Namely, let $(\Omega, \mathcal{M}, \mu)$ be a probability measure space with no atoms and let $d_\Sigma = \{d_\sigma : \sigma \in \Sigma\}$ be a family of positive integers indexed by $\Sigma$. The quantized Steinhaus system of parameters $(\Sigma, d_\Sigma)$ is a collection $S_\Sigma = \{\zeta^\sigma : \Omega \to U(d_\sigma)\}_{\sigma \in \Sigma}$ of independent random unitary matrices, uniformly distributed on the unitary group $U(d_\sigma)$ equipped with its normalized Haar measure $\lambda_\sigma$. Such a system lead us to the notion of $\Sigma$-type. There is a similar definition for the quantized Rademacher system of parameters $(\Sigma, d_\Sigma)$. However, since we work with
operator spaces (which are defined over the complex field), it will be more convenient to deal with the Steinhaus system. This does not make any difference since, as it was mentioned in [4], Rademacher and Steinhaus Σ-type are equivalent notions.

The organization of the paper is as follows. In Section 2 we define the main notions we shall work with, such as BΣ-convexity, Σ-type, Σ-subtype and uniformly non-L1(Σ) operator spaces. We also prove some basic results which will be applied all throughout the paper. Some of these notions are equivalent. Namely, an operator space has Σ-subtype if and only if it is uniformly non-L1(Σ) and each of these properties are equivalent to the condition of being BΣ-convex. The proof of this is the content of Section 3. In Section 4 we work with certain tensor submultiplicativity which generalizes the classical submultiplicativity of the constants involved in the theory, see [11] for the details. This will be useful in order to see the validity in our context, for the main sets of parameters (Σ, dΣ), of some well-known classical results. In Section 5 we show that the property of having non-trivial Σ-type is stronger than BΣ-convexity. Moreover, we provide examples of BΣ-convex Hilbertian operator spaces failing the non-trivial Σ-type condition. Some other interesting examples will be treated. Finally, Section 6 is devoted to the study of the dependence of BΣ-convexity with respect to (Σ, dΣ). First, we prove that there is no dependence when we work only with those sets of parameters having dΣ unbounded. Then, we introduce the notion of KΣ-convexity, and we show that the independence with respect to (Σ, dΣ) is equivalent to see that an operator space is BΣ-convex if and only if it is KΣ-convex. Also a sufficient condition for the independence, in terms of the way in which S1n embeds in SΩ(l1), is given.

All throughout this paper, some basic notions of operator spaces and vector-valued Schatten classes will be assumed. The main results that we will be using can be found in [14]. Also, the main results in the theory of type for operator spaces will be assumed. The reader is referred to [4] for a brief summary of them. Finally, along this work we make a slight abuse of notation since sometimes, when there is no risk of confusion, we shall write Σ to denote the set of parameters (Σ, dΣ).

2. The main definitions

Given an operator space E, a set of parameters (Σ, dΣ) and 1 ≤ p < ∞, we define the spaces LpΣ(E) as follows

\[ L^p_Σ(E) = \left\{ A \in \prod_{σ \in Σ} M_{d_σ} \otimes E : \|A\|_{L^p_Σ(E)} = \left( \sum_{σ \in Σ} d_σ \|A^σ\|_{S^p_Σ(E)}^p \right)^{1/p} < ∞ \right\} \]

\[ L^∞_Σ(E) = \left\{ A \in \prod_{σ \in Σ} M_{d_σ} \otimes E : \|A\|_{L^∞_Σ(E)} = \sup_{σ \in Σ} \|A^σ\|_{S^∞_Σ(E)} < ∞ \right\} , \]

where we write Mn for the vector space of n × n complex matrices and S^p_Σ(E) stands for the Schatten p-class over Mn with values in E. We shall also use the infinite-dimensional Schatten classes S^p(E) with values in E, the reader is referred to [14] for a precise definition of these spaces. L^p(Σ) will denote the case E = C. We impose on LpΣ(E) its natural operator space structure, see [3] or Chapter 2 of [14] for the details. Now
let $1 \leq p \leq \infty$ and let $\Gamma$ be any subset of $\Sigma$, if $\zeta^\sigma$ are the elements of the quantized Steinhaus system (as defined in the introduction), then we define the mapping $T_p(\Gamma, E)$ by the relation

$$A \in L^p_E(\Gamma) \mapsto \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \in L^2_E(\Omega).$$

We shall denote $T_p(\Sigma, E)$ by $T_p(E)$. For $\Gamma$ finite, we also define the number $\Delta_\Gamma = \sum_{\sigma \in \Gamma} d_\sigma^2$.

**Definition 2.1.** Let $1 \leq p \leq 2$, we say that an operator space $E$ has $\Sigma$-type $p$ if the mapping $T_p(E)$ is completely bounded.

**Remark 2.2.** This notion of type depends on the election of the quantized Steinhaus system $S_\Sigma$ we are working with. If we take $\Sigma_0 = \mathbb{N}$ and $d_\sigma = 1$ for all $\sigma \in \Sigma_0$, then we work with the classical Steinhaus system and Definition 2.1 provides a completely bounded version of the classical definition of type. We shall refer to these parameters as the classical set of parameters $\Sigma_0$. One could think that this is the right definition of type in the operator space setting and that there is no reason to introduce all the quantized Steinhaus systems in the theory. But in fact, those quantized Steinhaus systems $S_\Sigma$ with $d_\Sigma$ unbounded are sometimes the right systems to work with. One example of this assertion is given in [4], where we show that for those systems we can obtain an operator space version of Kwapień’s theorem with weaker hypothesis than for the classical Steinhaus system. Another example will be given in Section 6, where we shall prove that the notion of $\Sigma_\Sigma$-convexity is independent of $\Sigma$ whenever $d_\Sigma$ is unbounded.

**Remark 2.3.** Let $S^q_\Sigma(E)$ be the closure in $L^q_E(\Omega)$ of the subspace spanned by the entries $\{\zeta^\sigma_{ij} : \sigma \in \Sigma, 1 \leq i, j \leq d_\sigma\}$ of the functions of $S_\Sigma$ with $E$-valued coefficients. A version of the Khintchine-Kahane inequalities for random matrices given in [8] implies that the norm of $S^q_\Sigma(E)$, regarded as a Banach space, is equivalent to that of $S^2_\Sigma(E)$ whenever $1 \leq q_1, q_2 < \infty$. In particular, given $1 \leq p \leq 2$ and $1 \leq q < \infty$, the validity of the inequality

$$\left\| \sum_{\sigma \in \Sigma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right\|_{L^q_E(\Omega)} \leq c \left( \sum_{\sigma \in \Sigma} d_\sigma \|A^\sigma\|_{S^p_\sigma(E)}^p \right)^{1/p}$$

(2.1)

does not depend on the value of $q$. However, for $1 \leq q_1 \neq q_2 < \infty$, Pisier showed in [14] that in general $S^q_\Sigma(E)$ and $S^2_\Sigma(E)$ are not completely isomorphic as operator spaces with their natural operator space structure. Therefore, in contrast with (2.1), each election of the exponent $1 \leq q < \infty$ in Definition 2.1 gives different notions of $\Sigma$-type.

**Remark 2.4.** The absence of Khintchine inequalities for operator spaces forces us to choose an exponent $q$ in the definition of $\Sigma$-type. Our election differs from that of [4]. The reason is that in [4] (where we took $q = p'$ in the definition of $\Sigma$-type $p$) the aim was to have a unified theory of type and cotype for uniformly bounded quantized orthonormal systems, while here the election $q = 2$ facilitates our work. However, although the notion of $\Sigma$-type given here is not the same as in [4], we shall mainly be concerned with the
notion of non-trivial Σ-type for which the election of q does not matter! Namely, let $1 \leq q_1 < q_2 < \infty$ and $1 < p \leq 2$. Let us consider the mapping $T^p_q(E)$ given by

$$A \in L^p_E(\Sigma) \mapsto \sum_{\sigma \in \Sigma} d_{\sigma} \text{tr}(A^\sigma \zeta^\sigma) \in L^q_E(\Omega).$$

The complete boundedness of $T^p_q(E)$ obviously implies the complete boundedness of $T^p_q(E)$. Conversely, let us assume that $T^p_q(E)$ is completely bounded. Then, since the $cb$-norm of $T^\infty_1(E)$ is 1 for any operator space $E$, by complex interpolation there exists some $1 < r < p$ such that $T^p_q(E)$ is also completely bounded. That is, the notion of non-trivial Σ-type does not depend on the election of $q$ in the definition of Σ-type.

We also generalize to our context some other notions of the commutative theory. For that we shall need the numbers $\Delta_\Gamma$ defined above for each finite $\Gamma \subset \Sigma$. Let $E$ be an operator space and let $(\Sigma, d_\Sigma)$ be any set of parameters. We say that $E$ has Σ-subtype if there exists a finite subset $\Gamma$ of $\Sigma$, such that

$$\|T_2(\Gamma, E)\|_{cb} < \frac{1}{2} \Delta_\Gamma^1.$$

Now we define $B_\Sigma$-convex and uniformly non-$L^1(\Sigma)$ operator spaces. As in the previous definitions, when dealing with the classical set of parameters $\Sigma_0$, we obtain a completely bounded version of the classical notion.

**Definition 2.5.** Let $E$ be an operator space and let us fix a set of parameters $(\Sigma, d_\Sigma)$.

- $E$ is said to be $B_\Sigma$-convex if there exists a finite subset $\Gamma$ of $\Sigma$ and $0 < \delta \leq 1$ such that, for any family $\{A^\sigma \in M_{d_\sigma} \otimes S^2(E)\}_{\sigma \in \Gamma}$, we have

$$\frac{1}{\delta} \inf_{B^\sigma \text{ unitary}} \left\| \sum_{\sigma \in \Gamma} d_{\sigma} \text{tr}(A^\sigma B^\sigma) \right\|_{S^2(E)} \leq (1 - \delta) \max_{\sigma \in \Gamma} \|A^\sigma\|_{S^\infty(E)}.$$

- $E$ is said to contain $L^1(\Gamma)$ $\lambda$-uniformly if, for each finite subset $\Gamma$ of $\Sigma$, there exists a subspace $F_\Gamma$ of $S^2(E)$ and a linear isomorphism $\Lambda_\Gamma : L^1(\Gamma) \to F_\Gamma$ such that

$$\|\Lambda_\Gamma\|_{cb} \|\Lambda_\Gamma^{-1}\| \leq \lambda.$$

$E$ is uniformly non-$L^1(\Sigma)$ if it does not contain $L^1(\Gamma)$ $\lambda$-uniformly for some $\lambda > 1$.

**Remark 2.6.** The reader could expect that, in the given definition of containing $L^1(\Gamma)$ $\lambda$-uniformly, we should require that $\|\Lambda_\Gamma\|_{cb} \|\Lambda_\Gamma^{-1}\|_{cb} \leq \lambda$. However we have given an intermediate notion between that condition and the classical notion, which uses the Banach-Mazur distance. The reason for that election will become clear in Theorem 3.2.

**Remark 2.7.** For the classical set of parameters $\Sigma_0$, the given definition of containing $L^1(\Gamma)$ $\lambda$-uniformly can be rephrased by saying that $S^2(E)$ contains $l_1^\lambda$ $\lambda$-uniformly in the Banach space sense. Namely, the $cb$-norm of $\Lambda_\Gamma$ coincides in this case with its operator norm since it is defined on a max operator space.
suppose that, for all a technical lemma. For that purpose we begin by fixing a set of parameters \((\Sigma, d_{\Sigma})\). The equivalent notions 3. The equivalent notions follow from (a). This completes the proof. Moreover, taking \(\|A\|_{S_2^1(E)} = \sqrt{n} \|A\|_{S_2^1(E)}\), then \(\|A\|_{S_2^1(E)} = n \|A\|_{S_2^1(E)}\).

**Proof.** If \(p = 1\) and \(q = \infty\), then (a) follows easily from Corollary 9.8 of [13]. The general case follows by complex interpolation. Let us prove (b). By homogeneity, we assume without lost of generality that \(\|A\|_{S_2^1(E)} = 1\). On the other hand, by Theorem 1.5 of [14], we know that  
\[
\|A\|_{S_2^1(E)} = \inf_{A = aB_0B} \|a\|_{S_2^1} \|B\|_{S_2^1(E)} \|B_0\|_{S_2^1} = 1
\]

where \(a, B_0, B \in M_n\) and \(B_0 \in M_n \otimes E\). Moreover, if \(F = \text{span}\{A_{ij} : 1 \leq i, j \leq n\}\), we can take \(B \in M_n \otimes F\). In particular, for all \(k \geq 1\) there exist \(\alpha_k, \beta_k \in M_n\) and \(B_k \in M_n \otimes F\) such that \(A = \alpha_k B_k \beta_k\), \(1 \leq \|\alpha_k\|_{S_2^1} < 1 + 1/k\) and \(\|B_k\|_{S_2^1(E)} = \|\beta_k\|_{S_2^1} = 1\). By the finite-dimensionality of \(F\), we know that the sequence \((\alpha_k, \beta_k, B_k)\) belongs to a compact subset of \(S_2^1_n \times S_2^1(E) \times S_2^1_n\). Thus, there exist \(\alpha_0, \beta_0 \in M_n\) and \(B_0 \in M_n \otimes F\) such that \(A = \alpha_0 B_0 \beta_0\) and \(\|\alpha_0\|_{S_2^1} = \|B_0\|_{S_2^1(E)} = \|\beta_0\|_{S_2^1} = 1\). But then, again by Theorem 1.5 of [14],
\[
\|\alpha_0\|_{S_2^1} \|\beta_0\|_{S_2^1} \geq \|A\|_{S_2^1(E)} = \sqrt{n}.
\]
Moreover, taking \(p = 2\) and \(q = 4\), (a) gives \(\|\alpha_0\|_{S_2^1} \|\beta_0\|_{S_2^1} \leq n^{1/4}\). In summary, we get \(\|\alpha_0\|_{S_2^1} = \|\beta_0\|_{S_2^1} = n^{1/4}\) and \(\|\alpha_0\|_{S_2^1} = \|\beta_0\|_{S_2^1} = 1\). Then, it is well-known that there exists \(U, V \in U(n)\) such that \(\alpha_0 = n^{-1/4} U\) and \(\beta_0 = n^{-1/4} V\). Therefore, \(\|A\|_{S_2^1(E)} \leq \|\alpha_0\|_{S_2^1} \|B_0\|_{S_2^1(E)} \|\beta_0\|_{S_2^1} = n^{-1/2}\). This gives \(\|A\|_{S_2^1(E)} \geq n \|A\|_{S_2^1(E)}\), the reverse inequality follows from (a). This completes the proof. \(\square\)

3. The equivalent notions

In this section we show the equivalence between some of the notions previously defined. For that purpose we begin by fixing a set of parameters \((\Sigma, d_{\Sigma})\). First, we need to prove a technical lemma.

**Lemma 3.1.** Let \(F\) be an operator space and let \(\Gamma\) be a finite subset of \(\Sigma\). Let us suppose that, for all \(\varepsilon > 0\), there exist a family of matrices \(X^\sigma \in M_{d_{\Sigma}} \otimes F\) with \(\sigma \in \Gamma\) and such that
\[
1. \left( \int_{\Omega} \left\| \sum_{\sigma \in \Gamma} d_{\sigma} \text{tr}(X^\sigma_\omega \zeta^\sigma(\omega)) \right\|_F^2 \, d\mu(\omega) \right)^{1/2} \geq \Delta_\Gamma - \varepsilon.
\]
\[
2. \left( \sum_{\sigma \in \Gamma} d_{\sigma} \|X^\sigma_\omega\|_{S_2^1(F)}^2 \right)^{1/2} = \Delta_\Gamma^{1/2}.
\]
Then \( \max_{\sigma \in \Gamma} \| X^\sigma \|_{S^\infty_{\mathcal{U}}(F)} \leq 1 + \xi(\varepsilon) \), with \( \xi(\varepsilon) \to 0^+ \) as \( \varepsilon \to 0^+ \).

**Proof.** Let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{R}_+ \) containing all the intervals \((0, \varepsilon)\) with \( \varepsilon > 0 \) and let \( F_{\mathcal{U}} \) be the corresponding ultraproduct operator space. Then we define \( X^\sigma = (X^\sigma)_{\mathcal{U}} \) for \( \sigma \in \Gamma \). That is, \( X^\sigma \in M_{\mathcal{U}} \otimes F_{\mathcal{U}} \). We obviously have

\[
\left( \int_{\Omega} \left( \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X^\sigma \zeta^\sigma(\omega)) \right)^2 d\mu(\omega) \right)^{1/2} \geq \Delta_{\Gamma} \quad \text{and} \quad \left( \sum_{\sigma \in \Gamma} d_\sigma \| X^\sigma \|^2_{S^\infty_{\mathcal{U}}(F_{\mathcal{U}})} \right)^{1/2} = \Delta_{\Gamma}^{1/2}.
\]

By Hölder inequality and Lemma 2.9, we can write

\[
\Delta_{\Gamma}^2 \leq \int_{\Omega} \left( \sum_{\sigma \in \Gamma} d_\sigma \| X^\sigma \|_{S^1_{\mathcal{U}}(F_{\mathcal{U}})} \| \zeta^\sigma(\omega) \|_{S_{\mathcal{U}}(\omega)} \right)^2 d\mu(\omega) = \left( \sum_{\sigma \in \Gamma} d_\sigma \sqrt{d_\sigma} \| X^\sigma \|_{S^2_{\mathcal{U}}(F_{\mathcal{U}})} \right)^2 \leq \Delta_{\Gamma} \sum_{\sigma \in \Gamma} d_\sigma \| X^\sigma \|^2_{S^2_{\mathcal{U}}(F_{\mathcal{U}})} = \Delta_{\Gamma}^2.
\]

In particular, in this case, Lemma 2.9 and Hölder inequality are equalities. Therefore, we obtain

\[
\| X^\sigma \|_{S^1_{\mathcal{U}}(F_{\mathcal{U}})} = \sqrt{d_\sigma} \| X^\sigma \|_{S^2_{\mathcal{U}}(F_{\mathcal{U}})} \quad \text{and} \quad d_\sigma = c_0 \sqrt{d_\sigma} \| X^\sigma \|_{S^2_{\mathcal{U}}(F_{\mathcal{U}})}
\]

for some positive constant \( c_0 \) and any \( \sigma \in \Gamma \). Now, Lemma 2.9 gives \( \| X^\sigma \|_{S^\infty_{\mathcal{U}}(F_{\mathcal{U}})} = 1/c_0 \). But \( c_0 = 1 \) since

\[
\frac{1}{c_0^2} \Delta_{\Gamma} = \sum_{\sigma \in \Gamma} d_\sigma \| X^\sigma \|^2_{S^2_{\mathcal{U}}(F_{\mathcal{U}})} = \Delta_{\Gamma}.
\]

So, we have \( \max_{\sigma \in \Gamma} \| X^\sigma \|_{S^\infty_{\mathcal{U}}(F_{\mathcal{U}})} = 1 \). Finally, by the isometry \( S^\infty_{\mathcal{U}}(F_{\mathcal{U}}) = S^\infty_{\mathcal{U}}(F)_{\mathcal{U}} \) (see Chapter 5 of [14]) and the definition of ultraproduct operator space, the result follows. \( \square \)

**Theorem 3.2.** Let \( E \) be an operator space and let us fix a set of parameters \((\Sigma, d_\Sigma)\). Then, the following are equivalent:

(a) \( E \) has \( \Sigma \)-subtype.

(b) \( E \) is \( B_2 \)-convex.

(c) \( E \) is uniformly non-\( L^1(\Sigma) \).

**Proof.** Let us suppose that \( E \) has \( \Sigma \)-subtype, we shall see that \( E \) is \( B_2 \)-convex. We know that \( \| T_2(\Gamma, E) \|_{cb} = (1 - \delta) \Delta_{\Gamma}^{1/2} \) for some \( \Gamma \subset \Sigma \) finite and some \( 0 < \delta \leq 1 \). Hence, we can write

\[
\frac{1}{\Delta_{\Gamma}} \inf_{B^s(\mathcal{U})) \text{unitary}} \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma B^\sigma) \right\|_{S^2(\mathcal{E})} \leq \frac{1}{\Delta_{\Gamma}} \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma \zeta^\sigma) \right\|_{S^2(L^2_{\mathcal{U}}(\Omega))} \leq \frac{1 - \delta}{\Delta_{\Gamma}^{1/2}} \| A \|_{S^2(\mathcal{L}_k(\Gamma))}.
\]
Then the result follows since, by Lemma 2.9, we have
\[ \|A\|_{L^2(S^2(E), \Gamma)} \leq \Delta_1^{1/2} \max_{\sigma \in \Gamma} \|A_\sigma\|_{S^2_S(S^2(E))}. \]

Now, to see that $B_\Sigma$-convex operator spaces are uniformly non-$\mathcal{L}_1(\Sigma)$, we assume that $E$ contains $\mathcal{L}_1(\Gamma)$ $\lambda$-uniformly for all $\lambda > 1$ and we have to see that $E$ is not $B_\Sigma$-convex. We know that, for all $\lambda > 1$ and all $\Gamma \subset \Sigma$ finite, there exists a subspace $F_\Gamma$ of $S^2(E)$ and some isomorphism $\Lambda_\Gamma : L^1(\Gamma) \to F_\Gamma$

\[ A \in L^1(\Gamma) \mapsto \sum_{\sigma \in \Gamma} \sum_{i,j=1}^{d_\sigma} a_{ij}^\sigma x_{ij}^\sigma \in F_\Gamma \quad \text{where} \quad A_\sigma = (a_{ij}^\sigma), \]

such that $\|\Lambda_\Gamma\|_{cb} = 1$ and $\|\Lambda_\Gamma^{-1}\| \leq \lambda$. On the other hand, if $\sigma \in \Gamma$ and we define the matrix $X_\sigma = d_{\sigma}^{-1}(x_{ij}^\sigma)$, we have
\[ \|X_\sigma\|_{S^2_S(S^2(E))} = \|\text{tr}(X_\sigma \cdot )\|_{cb(S^2_S, S^2(E))} \leq \|\Lambda_\Gamma\|_{cb} = 1. \quad (3.1) \]

Hence, by the estimate for the norm of $\Lambda_\sigma^{-1}$ and (3.1), we obtain
\[ \inf_{\|B\|_{L^1(\Gamma)} = \Delta} \frac{1}{\lambda} \inf_{B^\sigma \text{ unital}} \|\sum_{\sigma \in \Gamma} \sum_{i,j=1}^{d_\sigma} a_{ij}^\sigma x_{ij}^\sigma \|_{S^2(E)} \geq \Delta_\Gamma \]
since $\|B\|_{L^1(\Gamma)} = \Delta$ whenever $B^\sigma \in U(d_\sigma)$ for all $\sigma \in \Gamma$. In particular, taking $\lambda \to 1^+$, we conclude that the operator space $E$ is not $B_\Sigma$-convex.

Finally, let us assume that $\|T_\Sigma(\Gamma, E)\|_{cb} = \Delta_1^{1/2}$ for all $\Gamma \subset \Sigma$ finite. We have to see that $E$ contains $\mathcal{L}_1(\Gamma)$ $\lambda$-uniformly for all $\lambda > 1$. By Lemma 1.7 of [14], we know that $\|T_\Sigma(\Gamma, E)\|_{cb} = \|T_\Sigma(\Gamma, S^2(E))\|$. In particular, for all $\epsilon > 0$ there exists a family of matrices $X_\epsilon^\sigma \in M_{d_\sigma} \otimes S^2(E)$ such that
\[ \int_\Omega \left( \sum_{\sigma \in \Gamma} \|d_{\sigma} \text{tr}(X_\epsilon^\sigma \sigma(\omega))\|_{S^2(E)}^2 \right) d\mu(\omega) \geq \Delta_\Gamma - \epsilon \quad \text{and} \quad \sum_{\sigma \in \Gamma} d_{\sigma} \|X_\epsilon^\sigma\|_{S^2_S(S^2(E))}^2 = \Delta_\Gamma. \]

Moreover, by Hölder inequality we obtain
\[ \left\| \sum_{\sigma \in \Gamma} \|d_{\sigma} \text{tr}(X_\epsilon^\sigma \sigma(\omega))\|_{S^2(E)} \right\|_{S^2(S^2(E))} \leq \sum_{\sigma \in \Gamma} d_{\sigma} \|X_\epsilon^\sigma\|_{S^2_S(S^2(E))} \|\sigma(\omega)\|_{S^2_S} \leq \Delta_\Gamma. \]

That is, if we set $f_\epsilon(\omega) = \left\| \sum_{\sigma \in \Gamma} d_{\sigma} \text{tr}(X_\epsilon^\sigma \sigma(\omega)) \right\|_{S^2(S^2(E))}^2$ for $\omega \in \Omega$, then we have
\[ 0 \leq f_\epsilon(\omega) \leq \Delta_\Gamma^2 \quad \text{and} \quad \Delta_\Gamma^2 - \epsilon \leq \int_\Omega f_\epsilon(\omega) d\mu(\omega) \leq \Delta_\Gamma^2. \]

In particular, $\mu(\{\omega \in \Omega : f_\epsilon(\omega) < \Delta_\Gamma^2 - k\epsilon\} \leq 1/k$ for all $k \geq 1$. On the other hand, if we fix $U_0^\sigma \in U(d_\sigma)$ for any $\sigma \in \Gamma$, we define $U_0(\sigma, \delta) = \{U^\sigma = U(d_\sigma) : \|U^\sigma - U_0^\sigma\|_{S^2_S} < \delta\}$.
That is, there exists some \( \varepsilon \) and where \( \varepsilon \) Taking we obtain the following inequality

\[
\mu_\Gamma \{ \omega \in \Omega : \mathcal{C}(\omega) \in \mathcal{U}_0(\sigma, \delta), \; \sigma \in \Gamma \} = \prod_{\sigma \in \Gamma} \lambda_\sigma(\mathcal{U}_0(\sigma, \delta)) > 0.
\]

Therefore, by choosing \( k_0(\delta) \) such that \( k_0(\delta)^{-1} \leq \mu_\Gamma \{ \omega \in \Omega : \mathcal{C}(\omega) \in \mathcal{U}_0(\sigma, \delta), \; \sigma \in \Gamma \} \), we obtain the following inequality

\[
\mu_\Gamma \{ \omega \in \Omega : f_\varepsilon(\omega) < \Delta_0^2 - k_0(\delta)\varepsilon \} \leq \mu_\Gamma \{ \omega \in \Omega : \mathcal{C}(\omega) \in \mathcal{U}_0(\sigma, \delta), \; \sigma \in \Gamma \}.
\]

That is, there exists some \( \omega_0 \in \Omega \) such that \( \mathcal{C}(\omega_0) \in \mathcal{U}_0(\sigma, \delta) \) for all \( \sigma \in \Gamma \) and such that \( f_\varepsilon(\omega_0) \geq \Delta_0^2 - k_0(\delta)\varepsilon \). These two properties give us the following sequence of inequalities

\[
\sqrt{\Delta_0^2 - k_0(\delta)\varepsilon} \leq \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma(\omega_0) - U_0^\sigma) \right\|_{S^2(\mathcal{E})} + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma U_0^\sigma) \right\|_{S^2(\mathcal{E})}
\]

\[
\leq \sum_{\sigma \in \Gamma} d_\sigma \left\| X_\sigma \right\|_{S^2(\mathcal{E})} \left\| \mathcal{C}(\omega_0) - U_0^\sigma \right\|_{S^2(\mathcal{E})} + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma U_0^\sigma) \right\|_{S^2(\mathcal{E})}
\]

\[
< \delta \Delta_0 + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma U_0^\sigma) \right\|_{S^2(\mathcal{E})}
\]

Taking \( \varepsilon(\delta) = \delta/k_0(\delta) \), it is easy to check that there exists \( \gamma_1(\delta) > 0 \) such that

\[
\left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma(\omega_0) - U_0^\sigma) \right\|_{S^2(\mathcal{E})} \geq \Delta_0 - \gamma_1(\delta) \tag{3.2}
\]

and where \( \varepsilon(\delta), \gamma_1(\delta) \to 0^+ \) as \( \delta \to 0^+ \). In particular, since for some other election of the unitary matrices \( U_0^\sigma \) (\( \sigma \in \Gamma \)) we have the same value for \( \lambda_\sigma(\mathcal{U}_0(\sigma, \delta)) \) (by the translation invariance of the Haar measure \( \lambda_\sigma \)), we obtain that \( k_0(\delta) \) does not depend on the chosen matrices \( U_0^\sigma \) (\( \sigma \in \Gamma \)) and (3.2) holds for any family of unitary matrices \( U^\sigma \in U(d_\sigma) \) with \( \sigma \in \Gamma \). Now, given \( A \in L^1(\Sigma) \) of norm 1, we use polar decomposition to write \( A^\sigma = U_0^\sigma A^\sigma \) with \( U_0^\sigma \in U(d_\sigma) \). Then, we have

\[
\Delta_0 - \gamma_1(\delta) \leq \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma(\omega_0) - U_0^\sigma I - A^\sigma) \right\|_{S^2(\mathcal{E})} + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma A^\sigma) \right\|_{S^2(\mathcal{E})}
\]

\[
\leq \sum_{\sigma \in \Gamma} d_\sigma \left\| X_\sigma \right\|_{S^2(\mathcal{E})} \left\| U_0^\sigma I - A^\sigma \right\|_{S^2(\mathcal{E})} + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma A^\sigma) \right\|_{S^2(\mathcal{E})}
\]

\[
\leq (1 + \xi(\delta))\Delta_0 - 1 + \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_\sigma A^\sigma) \right\|_{S^2(\mathcal{E})}
\]

where the last inequality follows by Lemma 3.1. Let us consider the subspace \( \mathcal{F}_\Gamma \) of \( S^2(\mathcal{E}) \) spanned by the entries of \( X_{\varepsilon(\delta)}^\sigma \) where \( \sigma \) runs over \( \Gamma \). Then, the last inequality gives that the linear isomorphism \( \Lambda_\Gamma \), given by

\[
A \in L^1(\Gamma) \mapsto \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_{\varepsilon(\delta)}^\sigma A^\sigma) \in \mathcal{F}_\Gamma,
\]
satisfies \( \|A_\Gamma^{-1}\| \leq 1 + \gamma_2(\delta) \) for some \( \gamma_2(\delta) > 0 \) satisfying \( \gamma_2(\delta) \to 0^+ \) as \( \delta \to 0^+ \). Moreover, we have that \( \|A_\Gamma\|_{cb} \leq 1 + \xi(\delta) \). Namely, given \( A \in S^1(\mathcal{L}^1(\Gamma)) \), we have

\[
\left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(X_{\varepsilon(\delta)} A^\sigma) \right\|_{S^1(E)} \leq \left\| \sum_{\sigma \in \Gamma} d_\sigma \|X_{\varepsilon(\delta)}\| S_{\sigma}^{\infty}(S^2(E)) \|A^\sigma\| S_{d_\sigma}^{1}(S^1) \leq (1 + \xi(\delta)) \|A\|_{S^1(\mathcal{L}^1(\Gamma))}.
\]

The first inequality follows by an inequality of Holder type, see e.g. Lemma 3.3 of [3]. That is, we have seen that \( \|A_\Gamma\|_{cb} = \|A_\Gamma \otimes I_{S^1}\| \leq 1 + \xi(\delta) \). Therefore \( \|A_\Gamma\|_{cb} \|A_\Gamma^{-1}\| \leq 1 + \gamma_3(\delta) \) with \( \gamma_3(\delta) \to 0^+ \) as \( \delta \to 0^+ \). Therefore, taking \( \delta \to 0^+ \) we obtain that \( E \) contains \( \mathcal{L}^1(\Gamma) \) \( \lambda \)-uniformly for all \( \lambda > 1 \) as we wanted. This completes the proof. \( \boxdot \)

**Remark 3.3.** As in Lemma 3.1, we could have used an argument with ultraproducts to show that uniformly non-\( \mathcal{L}^1(\Sigma) \) operator spaces have \( \Sigma \)-subtype. This alternative proof is a bit shorter. However, for the sake of clarity, we have preferred to give the more explicit argument used in the proof of Theorem 3.2.

As it was pointed out in Remark 2.8, it is very natural to wonder whether Definition 2.5 if affected if we change \( S^2(E) \) by \( S^p(E) \) with \( 1 < p < \infty \). The notion of \( B_2 \)-convexity should not depend on the election of the exponent \( p \) and, fortunately, this is the case.

**Corollary 3.4.** An operator space \( E \) is \( B_\Sigma \)-convex if and only if there exists a finite subset \( \Gamma \) of \( \Sigma \) such that

\[
\frac{1}{\Delta_{\Gamma}} \inf_{B^\sigma \text{unitary}} \left\| \sum_{\sigma \in \Gamma} d_\sigma \text{tr}(A^\sigma B^\sigma) \right\|_{S^p(E)} \leq (1 - \delta) \max_{\sigma \in \Gamma} \|A^\sigma\| S_{d_\sigma}^{\infty}(S^p(E))
\]

for some \( 1 < p < \infty \) and any family \( \{A^\sigma \in M_{d_\sigma} \otimes S^p(E)\}_{\sigma \in \Gamma} \).

**Proof.** By Lemma 1.7 of [14] and Theorem 3.2, we know that \( E \) is \( B_\Sigma \)-convex if and only if there exists a finite subset \( \Gamma \) of \( \Sigma \) such that \( \|T_2(\Gamma, S^2(E))\| < \Delta_{\Gamma}^{1/2} \). On the other hand, given \( 1 < p < \infty \), we claim that

\[
\|T_2(\Gamma, S^p(E))\| < \Delta_{\Gamma}^{1/2} \iff \|T_2(\Gamma, S^3(E))\| < \Delta_{\Gamma}^{1/2}.
\]

By Holder inequality we have \( \|T_2(\Gamma, F)\| \leq \Delta_{\Gamma}^{1/2} \) for any operator space \( F \). Then, (3.3) follows by complex interpolation with \( S^1(E) \) and \( S^{\infty}(E) \). In particular, given \( 1 < p < \infty \), we have that \( E \) is \( B_2 \)-convex if and only if there exists \( \Gamma \subset \Sigma \) finite such that \( \|T_2(\Gamma, S^p(E))\| < \Delta_{\Gamma}^{1/2} \). But, proceeding as in the proof of Theorem 3.2, we can see that the desired inequality is equivalent to this last condition. This completes the proof. \( \boxdot \)

**Remark 3.5.** By similar arguments, if \( 1 < p < \infty \), we can also replace \( S^2(E) \) by \( S^p(E) \) in the definition of uniformly non-\( \mathcal{L}^1(\Sigma) \) operator spaces.
4. Tensor submultiplicativity

As we pointed out in Remark 2.2, it seems that the classical set of parameters and those \( \Sigma \) having \( d_\Sigma \) unbounded are the most relevant ones. In this Section we shall see that, for these sets of parameters \( \Sigma \), the notion of \( B_\Sigma \)-convexity is stable under complete isomorphy and the notion of containing \( \ell^1(\Gamma) \) \( \lambda \)-uniformly does not depend on \( \lambda > 1 \).

The commutative analogs of these results are very well-known, see e.g. [5] or [11]. In order to prove these results, we need to fix some notation. Let \((\Sigma, d_\Sigma)\) be a set of parameters. Given two subsets \( \Gamma^1 \) and \( \Gamma^2 \) of \( \Sigma \), we define their tensor product as

\[
\Gamma^1 \otimes \Gamma^2 = \{ \sigma_1 \otimes \sigma_2 : \ \sigma_j \in \Gamma^j, \ j = 1, 2 \ \text{where} \ d_{\sigma_1 \otimes \sigma_2} = d_{\sigma_1} d_{\sigma_2} \}.
\]

We say that \((\Sigma, d_\Sigma)\) is \( \otimes \)-closed if, for any pair of subsets \( \Gamma^1 \) and \( \Gamma^2 \) of \( \Sigma \), there exists an injective mapping \( j : \Gamma^1 \otimes \Gamma^2 \to \Sigma \) such that \( d_{j(\sigma_1 \otimes \sigma_2)} = d_{\sigma_1 \otimes \sigma_2} \) for all \( \sigma_1 \otimes \sigma_2 \in \Gamma^1 \otimes \Gamma^2 \).

Also, given \( \Gamma \subset \Sigma \) finite, we define

\[
N_\Gamma(E) = \frac{1}{\sqrt{D_\Gamma}} \| T_2(\Gamma, E) \|_{cb}.
\]

**Lemma 4.1.** If \((\Sigma, d_\Sigma)\) is \( \otimes \)-closed, then \( N_{\Gamma^1 \otimes \Gamma^2}(E) \leq N_{\Gamma^1}(E) N_{\Gamma^2}(E) \) for any pair of finite subsets \( \Gamma^1 \) and \( \Gamma^2 \) of \( \Sigma \).

**Proof.** Let us consider a family \( A = \{ A^{\sigma_1 \otimes \sigma_2} \in M_{d_{\sigma_1} d_{\sigma_2}} \otimes S^2(E) : \ \sigma_1 \in \Gamma^1, \sigma_2 \in \Gamma^2 \} \). Let \( A^{\sigma_1 \otimes \sigma_2}(\omega) = \zeta^{j(\sigma_1 \otimes \sigma_2)}(\omega) A^{\sigma_1 \otimes \sigma_2} \) for \( \omega \in \Omega \). Since \( \zeta^{j(\sigma_1 \otimes \sigma_2)} \) is uniformly distributed on the unitary group \( U(d_{\sigma_1} d_{\sigma_2}) \) and \( \zeta^{\sigma_1}(\omega_1) \otimes \zeta^{\sigma_2}(\omega_2) \) is unitary, we have

\[
\int_\Omega \left\| \sum_{\sigma_j \in \Gamma^j} d_{\sigma_1} d_{\sigma_2} tr(A^{\sigma_1 \otimes \sigma_2}(\omega)) \right\|^2_{S^2(E)} \ d\mu(\omega)
\]

\[
= \int_\Omega \left\| \sum_{\sigma_j \in \Gamma^j} d_{\sigma_1} d_{\sigma_2} tr(A^{\sigma_1 \otimes \sigma_2}(\omega)(\zeta^{\sigma_1}(\omega_1) \otimes \zeta^{\sigma_2}(\omega_2))) \right\|^2_{S^2(E)} \ d\mu(\omega)
\]

for all \( \omega_1, \omega_2 \in \Omega \). Therefore, if we write

\[
X^{\sigma_1}(\omega, \omega_2) = \sum_{\sigma_2 \in \Gamma^2} d_{\sigma_2} tr(A^{\sigma_1 \otimes \sigma_2}(\omega) \zeta^{\sigma_2}(\omega_2)) \in M_{d_{\sigma_1}} \otimes S^2(E),
\]

we obtain the following estimate

\[
\int_\Omega \left\| \sum_{\sigma_j \in \Gamma^j} d_{\sigma_1} d_{\sigma_2} tr(A^{\sigma_1 \otimes \sigma_2}(\omega)) \right\|^2_{S^2(E)} \ d\mu(\omega)
\]

\[
= \int_{\Omega \times \Omega \times \Omega} \left\| \sum_{\sigma_j \in \Gamma^j} d_{\sigma_1} d_{\sigma_2} tr(A^{\sigma_1 \otimes \sigma_2}(\omega)(\zeta^{\sigma_1}(\omega_1) \otimes \zeta^{\sigma_2}(\omega_2))) \right\|^2_{S^2(E)} \ d\mu(\omega_1) d\mu(\omega_2) d\mu(\omega)
\]

\[
= \int_{\Omega \times \Omega \times \Omega} \left\| \sum_{\sigma_1 \in \Gamma^1} d_{\sigma_1} tr(X^{\sigma_1}(\omega, \omega_2) \zeta^{\sigma_1}(\omega_1)) \right\|^2_{S^2(E)} \ d\mu(\omega_1) d\mu(\omega_2) d\mu(\omega)
\]
for all $\Gamma \subset \Sigma$ finite and certain family of matrices $X^\sigma \in M_{d_\sigma} \otimes S^2(\mathcal{E})$, with $\sigma \in \Gamma$. On the other hand, from this inequality it is not difficult to see using Lemma 2.9 that

\[ \inf_{\|X^\sigma\|_{S^2(\mathcal{E})} = 1} \sum_{\sigma \in \Gamma} d_{\sigma} \text{tr}(X^\sigma B^\sigma) \]
\[ N_\Gamma(E) \geq 1/\lambda \] for all \( \Gamma \subseteq \Sigma \) finite. Now, if \( E \) is \( B_\Sigma \)-convex, we know that there exist some finite subset \( \Gamma_0 \) of \( \Sigma \) such that \( N_{\Gamma_0}(E) < 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma_0}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Sigma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
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\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
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\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
This gives that \( E \) is not \( B_\Sigma \)-convex. But, by Theorem 3.2, this is equivalent to say that \( E \) contains \( L_1(\Gamma) \) \( \tau \)-uniformly for all \( \tau > 1 \). Therefore, by Lemma 4.1, we would have
\[
\frac{1}{\lambda} \leq N_{\Gamma \otimes n}(E) \leq N_{\Gamma_0}(E)^n \to 0^+ \quad \text{as} \quad n \to \infty.
\]
The following result has its origins in an unpublished result of Magdalena Musat which asserts that $S^2(R)$ and $S^2(C)$ are superreflexive Banach spaces. After some conversations, initiated by Marius Junge and Gilles Pisier, Timur Oikhberg found a surprisingly simple proof of this fact. The next Theorem is based on the techniques employed there. First, we fix some notation. As usual, given $0 < \theta < 1$, we shall denote by $R(\theta)$ the complex interpolation operator space $(R,C)_\theta$. Analogously, $C(\theta)$ stands for $(C,R)_\theta = R(1-\theta)$. By convention, we also set that $R(0) = C(1) = R$ and $C(0) = R(1) = C$.

**Theorem 5.2.** Let $1 \leq p \leq 2$, then $R(1/p)$ and $C(1/p)$ are $B_{S_0}$-convex Hilbertian operator spaces having $\Sigma_0$-type $p$. Moreover, if $1 \leq p < 2$, then $R(1/p)$ and $C(1/p)$ do not have $\Sigma_0$-type $q$ for any $p < q \leq 2$.

**Proof.** Let us suppose that $R(1/p)$ and $C(1/p)$ are not $B_{S_0}$-convex. Then, by Theorem 3.2 and Remark 2.7, the spaces $S^2(R(1/p))$ and $S^2(C(1/p))$ should not have type $> 1$ in the Banach space sense. However, we claim that both spaces have type $4/3$. Therefore, $R(1/p)$ and $C(1/p)$ are $B_{S_0}$-convex. To prove our claim we recall that, by Theorem 1.1 of [14], we have $S^2(R) = R(1/2) \otimes_h R \otimes_h R(1/2)$ and $S^2(C) = C(1/2) \otimes_h C \otimes_h C(1/2)$ completely isometrically. Now, since the Haagerup tensor product commutes with the complex interpolation functor, we can write

\[
S^2(R) = (R \otimes_h R \otimes_h R, C \otimes_h R \otimes_h C)_{1/2},
\]

\[
S^2(C) = (C \otimes_h C \otimes_h C, R \otimes_h C \otimes_h R)_{1/2},
\]

completely isometrically. But, as Banach spaces, $R \otimes_h R \otimes_h R$ and $C \otimes_h C \otimes_h C$ are isometrically isomorphic to a Hilbert space. In particular, (5.1) gives that $S^2(R)$ and $S^2(C)$ have type $4/3$ in the Banach space sense. Hence, by complex interpolation, the same happens with $S^2(R(1/p))$ and $S^2(C(1/p))$.

On the other hand, by the reiteration theorem for the complex interpolation method, we have $R(1/p) = (R(1/2),C)_{\frac{p}{2}-1}$ and $C(1/p) = (C(1/2),R)_{\frac{p}{2}-1}$. But $R(1/2) = C(1/2)$ is an $OH$ operator space, see [13]. Therefore, it is easy to check that $R(1/2)$ has $\Sigma_0$-type $2$, see [4] for more on this topic. In particular, by complex interpolation, we get that $R(1/p)$ and $C(1/p)$ have at least $\Sigma_0$-type $q$ where

\[
\frac{1}{q} = \frac{1 - (2/p - 1)}{2} + \frac{2/p - 1}{1} = \frac{1}{p'}
\]

That is, $R(1/p)$ and $C(1/p)$ have $\Sigma_0$-type $p$. Now let $p < q \leq 2$, we want to see that $R(1/p)$ and $C(1/p)$ do not have $\Sigma_0$-type $q$. Following the notation introduced in Remark 2.4, we obviously have that

\[
\|T_q(\Sigma_0,E)\|_{cb} \geq \|T_q^2(\Sigma_0,E)\|_{cb} = \|T_q^2(\Sigma_0,S^q(E))\|
\]

for any operator space $E$. Hence, we just need to check that $T_q^2(\Sigma_0,S^q(E))$ is not bounded if $E = R(1/p)$ or $E = C(1/p)$. But, by the Khintchine-Kahane inequalities, this is equivalent to saying that $S^q(R(1/p))$ and $S^q(C(1/p))$ do not have type $q$ in the Banach
space sense for any $p < q \leq 2$. Let us recall that

\[
S^q(R(1/p)) = (S^\infty(R(1/p)), S^1(R(1/p)))_{1/q} = (C \otimes_h R(1/p) \otimes_h R, R \otimes_h R(1/p) \otimes_h C)_{1/q} = C(1/q) \otimes_h R(1/p) \otimes_h R(1/q).
\]

Analogously, we obtain $S^q(C(1/p)) = C(1/q) \otimes_h C(1/p) \otimes_h R(1/q)$. Let us consider the subspace of $S^q(R(1/p))$ corresponding to $C(1/q) \otimes_h R(1/p)$. Then we can write

\[
C(1/q) \otimes_h R(1/p) = (C \otimes_h R(1/p), R \otimes_h R(1/p))_{1/q} = ((C \otimes_h R, C \otimes_h C)_{1/p}, (R \otimes_h R, R \otimes_h C)_{1/p})_{1/q}
\]

completely isometrically. But, as Banach spaces, $R \otimes_h R$ and $C \otimes_h C$ are isometrically isomorphic to $S^2$. Therefore, we have

\[
C(1/q) \otimes_h R(1/p) = (S^\infty, S^1)_{1/p}, (S^2, S^1)_{1/p})_{1/q} = (S^{2p}, S^{2p/p+1})_{1/q} = S^{2pq/p+q}
\]

isometrically. Finally, since $2pq < q(p+q)$ whenever $p < q$, we have that $C(1/q) \otimes_h R(1/p)$ can not have type $q$ in the Banach space sense. Consequently, the same happens for $S^q(R(1/p))$. A similar argument gives that $S^q(C(1/p))$ can not have type $q$ in the Banach space sense. This completes the proof. \(\square\)

**Remark 5.3.** In particular, by Theorem 5.2, the row and column operator spaces are examples of $B_{\Sigma_0}$-convex Hilbertian operator spaces failing to have non-trivial $\Sigma_0$-type.

**Remark 5.4.** Although we give more details in Section 6, it is a simple consequence of Theorem 3.2 that $B_{\Sigma_0}$-convexity is the strongest condition among the possible sets of parameters we are working with. That is, a $B_{\Sigma_0}$-convex operator space is automatically $B_{\Sigma}$-convex for any other set of parameters $\Sigma$. In particular, the examples treated in Theorem 5.2 are $B_{\Sigma_2}$-convex. Moreover, the given argument to see that $R(1/p)$ and $C(1/p)$ have $\Sigma_0$-type $p$ for any $1 \leq p \leq 2$ remains valid for any other set of parameters $\Sigma$.

Once we have found examples of $B_{\Sigma_0}$-convex Hilbertian operator spaces having sharp $\Sigma_0$-type $p$ for any $1 \leq p \leq 2$, we now show that $\min l^2$ and $\max l^2$ are Hilbertian operator spaces failing to be $B_{\Sigma_0}$-convex. This is based on Example 4.2 of [14], where Pisier makes the following construction. Let $M_2$ be the algebra of $2 \times 2$ complex-valued matrices equipped with its normalized trace $t$ and let us set $(A_k, t_k) = (M_2, t)$ for any $k \geq 1$. Then we consider the so-called hyperfinite $II_1$ factor

\[
(\mathcal{M}, \tau) = \bigotimes_{k=1}^{\infty} (A_k, t_k).
\]

Let $\mathcal{M}_n$ stand for the subalgebra of $\mathcal{M}$ corresponding to $A_1 \otimes \cdots \otimes A_n$. Let us consider the element of $\mathcal{M} \otimes \min l^2$ given by $d_n = V_n \otimes e_n$ where we write $(e_i)$ for the canonical basis of $l^2$ and $(V_n)$ is a sequence in $\mathcal{M}$ satisfying $V_n \in \mathcal{M}_n$ for all $n \geq 1$, $E^{\mathcal{M}_n}(V_{n+1}) = 0$ and the canonical anticommutation relations

\[
V_i V_j^* + V_j V_i^* = \delta_{ij} I \quad \text{and} \quad V_i V_j + V_j V_i = 0.
\]
Then, from these properties, Pisier shows that for all finite sequence of scalars \((\alpha_k)\) with \(1 \leq k \leq n\) and for all \(1 \leq p \leq \infty\) we have

\[
\frac{1}{2} \sup_{1 \leq k \leq n} |\alpha_k| \leq \left\| \sum_{k=1}^{n} \alpha_k d_k \right\|_{L^p(\mathcal{M}; \min I^2)} \leq \sup_{1 \leq k \leq n} |\alpha_k| \tag{5.2}
\]

where \(L^p(\mathcal{M}; \min I^2)\) denotes the non-commutative \(L^p\) space defined in \((\mathcal{M}, \tau)\) and with values in \(\min I^2\). In particular, inequalities \((5.2)\) tell us that the Banach-Mazur distance between some subspace of \(L^p(\mathcal{M}_n; \min I^2)\) and \(l_1^\infty\) is bounded above by \(2\) for all \(n \geq 1\) and all \(1 \leq p \leq \infty\). Then, recalling the natural embedding of \(l_1^n\) into \(l_2^n\) and taking \(p = 2\) in \((5.2)\), it is easy to see that \(S^2(\min I^2)\) contains \(l_1^n\) uniformly in the Banach space sense. Therefore, by Remark 2.7 and Theorem 3.2, we have that \(\min I^2\) is not \(B_{\Sigma_n}\)-convex. By a duality argument, the same happens for \(\max I^2\).

6. On the independence with respect to \(\Sigma\)

In this last Section we study the dependence of the notion of \(B_{\Sigma}\)-convexity with respect to the set of parameters \(\Sigma\). We begin by showing the independence with respect to \(\Sigma\) when we work with sets of parameters satisfying that \(d_\Sigma\) is an unbounded family. After that, we shall give two interesting equivalent formulations of the possible independence of \(B_{\Sigma}\)-convexity with respect to any set of parameters \(\Sigma\).

**Proposition 6.1.** Let us consider two sets of parameters \((\Sigma_1, d_{\Sigma_1})\) and \((\Sigma_2, d_{\Sigma_2})\) with \(d_{\Sigma_1}\) and \(d_{\Sigma_2}\) unbounded. Let \(E\) be an operator space, then \(E\) is \(B_{\Sigma_1}\)-convex if and only if \(E\) is \(B_{\Sigma_2}\)-convex.

**Proof.** By Theorem 3.2, we just need to check that \(E\) contains \(L^1(\Gamma)\) \(\lambda\)-uniformly \((\Gamma \subset \Sigma_1)\) for all \(\lambda > 1\) if and only if \(E\) contains \(L^1(\Gamma)\) \(\lambda\)-uniformly \((\Gamma \subset \Sigma_2)\) for all \(\lambda > 1\). In particular, it suffices to see that \(L^1(\Sigma_1)\) contains \(L^1(\Gamma)\) \(\lambda\)-uniformly \((\Gamma \subset \Sigma_2)\) for all \(\lambda > 1\) and that \(L^1(\Sigma_2)\) contains \(L^1(\Gamma)\) \(\lambda\)-uniformly \((\Gamma \subset \Sigma_1)\) for all \(\lambda > 1\). But this follows from the unboundedness of \(d_{\Sigma_1}\) and \(d_{\Sigma_2}\). Namely, given \(\Gamma \subset \Sigma_2\) finite, we know that there exists \(\Lambda \subset \Sigma_1\) and a bijection \(\tau : \Gamma \rightarrow \Lambda\) such that \(d_\sigma \leq d_{\tau(\sigma)}\) for all \(\sigma \in \Gamma\). In particular we can consider the linear mapping \(S_\Gamma : L^1(\Gamma) \rightarrow L^1(\Lambda)\) given by

\[
S_\Gamma(A)_{\tau(\sigma)}^{\tau(\sigma)} = \frac{d_\sigma}{d_{\tau(\sigma)}} A_{ij}^{\tau(\sigma)} \quad \text{if} \quad 1 \leq i,j \leq d_\sigma \\
0 \quad \text{otherwise}.
\]

Given the fact that \(S_\Gamma\) is a complete isometry, \(L^1(\Sigma_1)\) contains \(L^1(\Gamma)\) \(1\)-uniformly (where \(\Gamma \subset \Sigma_2\)). Similarly, using the unboundedness of \(d_{\Sigma_2}\), we can see that \(L^1(\Sigma_2)\) contains \(L^1(\Gamma)\) \(1\)-uniformly (\(\Gamma \subset \Sigma_1\)). This completes the proof.

**Remark 6.2.** Given two sets of parameters \((\Sigma_1, d_{\Sigma_1})\) and \((\Sigma_2, d_{\Sigma_2})\), we shall say that \(\Sigma_1 \leq \Sigma_2\) if there exists an injective mapping \(j : \Sigma_1 \rightarrow \Sigma_2\) such that \(d_\sigma \leq d_{j(\sigma)}\) for all \(\sigma \in \Sigma_1\). Then, by similar arguments to those used in Proposition 6.1, it is easy to see that \(B_{\Sigma_1}\)-convexity is stronger than \(B_{\Sigma_2}\)-convexity whenever \(\Sigma_1 \leq \Sigma_2\). In particular, if \(\Sigma_0\)
stands for the classical set of parameters, a $B_{\Sigma_0}$-convex operator space is automatically $B_\Sigma$-convex for any other set of parameters $\Sigma$.

Proposition 6.1 is only a little step in order to see the independence of the notion of $B_\Sigma$-convexity with respect to $\Sigma$. However, the general case seems to be more complicated. Now, we give two different conditions which could be useful to decide whether or not the independence with respect to $\Sigma$ holds.

(A) **On the notion of $K_\Sigma$-convexity.** In order to introduce the notion of $K_\Sigma$-convexity, we need to define the quantized version of the Gauss system. It was already defined by Marcus and Pisier in [8]. More in connection with the present paper, this system was also treated in the operator space version of Kwapień’s theorem, see [4].

Given a set of parameters $(\Sigma, d_\Sigma)$, we consider a family of independent standard complex-valued gaussian random variables $\{\gamma^\sigma_{ij} : \Omega \rightarrow \mathbb{C} : \sigma \in \Sigma, \ 1 \leq i, j \leq d_\sigma\}$ indexed by $\Sigma$ and $d_\Sigma$. Then, if we construct the random matrices

$$\gamma^\sigma = \frac{1}{\sqrt{d_\sigma}} \left( \begin{array}{cccc} \gamma^\sigma_{ij} \end{array} \right),$$

we obtain the **quantized Gaussian system of parameters** $(\Sigma, d_\Sigma)$. On the other hand, given an operator space $E$ and $f \in L^2_E(\Omega)$, we can consider the Fourier coefficients of $f$ with respect to this system

$$\hat{f}(\sigma) = \int_{\Omega} f(\omega) \gamma^\sigma(\omega)^* d\mu(\omega) \in M_{d_\sigma} \otimes E.$$ We shall say that an operator space $E$ is $K_\Sigma$-convex if the gaussian projection defined as follows

$$f \in L^2_E(\Omega) \mapsto \sum_{\sigma \in \Sigma} d_\sigma \text{tr}(\hat{f}(\sigma) \gamma^\sigma) \in L^2_E(\Omega)$$

is a completely bounded mapping. However, it is obvious that

$$\sum_{\sigma \in \Sigma} d_\sigma \text{tr}(\hat{f}(\sigma) \gamma^\sigma) = \sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} \int_{\Omega} f(\omega) \overline{\gamma^\sigma_{ij}(\omega)} d\mu(\omega) \cdot \gamma^\sigma_{ij}$$

where $\gamma^\sigma_{ij}$ are independent standard complex-valued gaussian random variables. Hence, it turns out that the notion of $K_\Sigma$-convexity does not depend on the set of parameters $\Sigma$. Moreover, if $\Sigma_0$ stands for the classical set of parameters, then any operator space $E$ satisfies

$$E \ K_\Sigma\text{-convex} \iff E \ K_{\Sigma_0}\text{-convex}$$

$$\iff S^2(E) \ K\text{-convex as a Banach space}$$

$$\iff S^2(E) \ B\text{-convex as a Banach space}$$

$$\iff E \ B_{\Sigma_0}\text{-convex}.$$
Therefore, it follows that the notion of $B_\Sigma$-convexity does not depend on the set of parameters $\Sigma$ if and only if $B_\Sigma$-convexity and $K_\Sigma$-convexity are equivalent notions. In particular, it provides a possible approach to check this independence. That is, the problem is to generalize to the operator space setting Pisier’s theorem which shows that $K$-convex and $B$-convex Banach spaces are the same, see [12] for more details on this topic.

**(B) On how $S^1_n$ embeds in $S^2(l^1)$.** Let us suppose that we are given a subspace $F_n$ of $S^2(l^1_n)$ and a linear isomorphism $\Phi_n : S^1_n \rightarrow F_n$ for each $n \geq 1$. Let us denote by $F^\text{min}_n$ the subspace $F_n$ equipped with the operator space structure inherited as a subspace of $S^2(\min l^1_n)$. Then, if we write $\Psi_n : F^\text{min}_n \rightarrow S^1_n$ for the inverse of $\Phi_n$ with the modified operator space structure on $F_n$, we claim that the condition

$$\|\Phi_n\|_{cb} \|\Psi_n\| \leq k$$

for all $n \geq 1$ and some constant $k > 1$ (6.1) implies the $\Sigma$-independence of $B_\Sigma$-convexity. Namely, let $E$ be an operator space which contains $l^1_n$ uniformly in the sense of Definition 2.5. That is, for all $n \geq 1$ there exists a subspace $K_n$ of $S^2(E)$ and a linear isomorphism $\Lambda_n : l^1_n \rightarrow K_n$ such that $\|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \leq \lambda$. Now, if we consider the linear isomorphism

$$T_n = (I_{S^2} \otimes \Lambda_n) \circ \Phi_n : S^1_n \rightarrow H_n \subset S^2(E),$$

the inverse operator factors as follows

$$H_n \rightarrow F^\text{min}_n \rightarrow S^1_n$$

via de composition $T_n^{-1} = \Psi_n \circ (I_{S^2} \otimes \Lambda_n^{-1})$. In summary, using the well-known properties of the minimal operator space structure, the following estimate follows from condition (6.1)

$$\|T_n\|_{cb} \|T_n^{-1}\| \leq \|\Lambda_n\|_{cb} \|\Lambda_n^{-1}\| \leq k\lambda.$$  

This gives that $E$ contains $L^1(\Gamma)$ uniformly for any set of parameters $(\Sigma, d\Sigma)$. Therefore, we have proved that any $B_\Sigma$-convex operator space is $B_{\Sigma_0}$-convex. In particular, the $\Sigma$-independence of $B_\Sigma$-convexity follows from Remark 6.2.

**Remark 6.3.** As in the Banach space case, the given definition of $K_\Sigma$-convexity should not change if we consider the gaussian projection on $L_p^\text{e} E(\Omega)$ instead of $L_2^\text{e} E(\Omega)$ for any $1 < p < \infty$. Fortunately, this is the case. However, this time the proof can not be supported by the Khintchine-Kahane inequalities as it was explained in Remark 2.3. Nevertheless, the argument is simple. Namely, if we refer to this a priori new notion as $K^p_\Sigma$-convexity, then an operator space $E$ is $K^p_\Sigma$-convex if and only if it is $K^p_{\Sigma_0}$-convex. But this last condition means that $S^p(E)$ is $K$-convex as a Banach space. Now, since $K$-convex and $B$-convex Banach spaces are the same, we conclude that $E$ is $K^p_\Sigma$-convex if and only if $S^p(E)$ is $B$-convex as a Banach space. But, by Corollary 3.4, we know that this is equivalent to say that $E$ is $B_{\Sigma_0}$-convex. Finally, since $K_\Sigma$-convexity is equivalent to $B_{\Sigma_0}$-convexity, we are done.
Remark 6.4. Condition (6.1) holds and so Pisier’s theorem on the equivalence between B-convex and K-convex spaces remains valid in the non-commutative setting. The proof of this fact, which will appear as part of a joint work with Marius Junge [6], became clear after this paper was submitted for publication.

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