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Weighted norm inequalities for singular integral operators

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Abstract

We study weighted norm inequalities for singular integral operators with different smoothness conditions assumed on the kernels. The weakest one is the so-called classical Hörmander condition, which is an L^1 regularity, and the strongest is given by a Hölder or Lipschitz smoothness. Between them we have some kind of L^r-regularity, $1 < r \leq \infty$. We will present some results that are known for singular integrals with these kernels. We will be focused on studying Coifman's inequality:

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x)dx \leq C \int_{\mathbb{R}^n} Mf(x)^p \, w(x)dx,$$

for any $0 < p < \infty$ and $w \in A_\infty$, where T is a singular integral operator with kernel satisfying a Hölder regularity condition and M is the Hardy-Littlewood maximal function. We will see that such an inequality is no longer true when the hypotheses on the kernel are relaxed. This is the case for kernels satisfying the Hörmander condition. For the intermediate regularity conditions some positive and negative results of this kind are shown. In these cases the operator on the right hand side is changed in such a way that it can measure the singularity of T. Some of the results we will present are in a collaboration paper with Carlos Pérez and Rodrigo Trujillo-González.

1 Introduction.

Some of the most significant and studied operators in Harmonic Analysis are the Hardy-Littlewood maximal function, the Hilbert transform and the Riesz transforms. The first one is defined as the supremum of the averages of the function over all the cubes $Q \subset \mathbb{R}^n$ (with sides parallel to the coordinate axes in the sequel), that is,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

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The Hilbert transform is defined in $\mathbb{R}$ and the Riesz transforms are the analogs in $\mathbb{R}^n$, $n \geq 2$, and they are given in the following way

$$Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy, \qquad\qquad R_j f(x) = p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}\,f(y)\,dy.$$

These integrals have to be defined in such a way they make sense. Note that the kernels $1/x$ in $\mathbb{R}$ or $x_j/|x|^{n+1}$ in $\mathbb{R}^n$, $n \geq 2$, are singular and they are not locally integrable at the origin and this is the reason why the integrals are understood in a principal value sense. The Hardy-Littlewood maximal function is very related to Hilbert or Riesz transforms since it controls them as we will see later. Studying maximal operators turns out to be easier and this control might be crucial to understand the singular integral operators.

A generalization of the Hilbert or Riesz transforms is given by the following convolution type operators

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)\,f(y)\,dy$$

with kernel K having bounded Fourier transform $\widehat{K} \in L^\infty(\mathbb{R}^n)$. Thus, T is a linear and bounded operator on $L^2(\mathbb{R}^n)$. Further generalizations can be considered with two-variable kernels that do not give a convolution type operator and some of them play an important role in Analysis. Nevertheless, we are going to concentrate in the simplest case on which the operators are of convolution type, the reader is referred, for instance, to [Duo] for the general case.

Coming back to the singular integral operators defined above, so far we only know that they are continuous in $L^2(\mathbb{R}^n)$. To get better properties on T some conditions can be imposed about the size or the smoothness of K. The size condition of the kernel that generalizes the case of the Hilbert or Riesz transforms is $|K(x)| \leq A\,|x|^{-n}$. Note that this decay has a problem of integrability both at the origin and at infinity. For the operators that we want to consider this condition will not be assumed, we will be focused on different smoothness conditions on K and the results that can be achieved by assuming them. The regularity conditions will be scaled in the Lebesgue spaces and we will use the notation H_r, $1 \leq r \leq \infty$. The weakest one is the so-called Hörmander condition

$$(H_1) \qquad\qquad \sup_{y \in \mathbb{R}^n} \int_{|x|>c\,|y|} |K(x-y) - K(x)|\,dx < \infty,$$

which is understood as an L^1-regularity. A singular integral operator with kernel satisfying (H_1) is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. This a classical result obtained by Calderón and Zygmund in the 50's, see [CZ]. The main tool for this proof is the Calderón-Zygmund decomposition of the function into a good and a bad part. This decomposition is performed by means of the Hardy-Littlewood maximal operator, fact that reflects the connection between this maximal function and the singular integral operators.

If (H_1) is the weakest regularity assumption, the strongest one will be of Hölder or Lipschitz type, namely,

$$(H_\infty^*) \qquad\qquad |K(x-y) - K(x)| \leq C\,\frac{|y|^\alpha}{|x|^{\alpha+n}}, \qquad \text{whenever } |x| > c\,|y|,$$

for some $c > 1$ and $0 < \alpha \leq 1$. The reason why we have used (H_∞^*) rather than (H_∞) will be clear later —we keep this latter notation for an L^∞ condition that is weaker—. Note that this condition implies (H_1) and also that the kernels of the Hilbert or Riesz transforms satisfy (H_∞^*) with $c = 2$ and $\alpha = 1$. Indeed, they verify an estimate that is better: $|\nabla K(x)| \le A\,|x|^{-(n+1)}$. We will see after a while that (H_∞^*) is key when weighted norm inequalities are studied.

Between (H_1) and (H_∞^*) the following variant of the Hörmander condition can be considered: let $1 \le r \le \infty$, we say that the kernel K verifies the L^r-Hörmander condition, if there are $c, C_r > 0$ such that for any $y \in \mathbb{R}^n$ and $R > c\,|y|$

$$(H_r) \qquad \sum_{m=1}^{\infty} (2^m\,R)^{\frac{n}{r'}} \Big(\int_{2^m\,R<|x|\le 2^{m+1}\,R} |K(x-y) - K(x)|^r\,dx \Big)^{\frac{1}{r}} \le C_r,$$

in the case $r < \infty$, and

$$(H_\infty) \qquad \sum_{m=1}^{\infty} (2^m\,R)^n \sup_{2^m\,R<|x|\le 2^{m+1}\,R} |K(x-y) - K(x)| \le C_\infty,$$

when $r = \infty$. We will use the notation (H_r) for the previous conditions and H_r for the classes of kernels satisfying them, the same is applied to (H_∞^*).

This definition is implicit in the work of D. Kurtz and R. Wheeden [KW], where it is shown that the classical Dini condition for K implies that $K \in H_r$ (see [KW, p. 359]). Later on, these classes H_r were considered in [RRT] and [Wat]. In fact, in this last paper the L^r-Hörmander condition plays an essential role when studying rough singular integral operators. Namely, for such an operator T, one can write $T = \sum T_j$ where the kernel of T_j satisfies the L^r-Hörmander condition with constant growing linearly in j.

Our aim is twofold. Firstly, we will review the weighted norm estimates that are known for the singular integral operators with the kernels in the previous classes. We will study how sharp they are. Secondly, we present some lack of weighted norm inequalities when the kernels are less regular. In particular, for $K \in H_1$ we are going to provide some counterexamples on which the expected weighted norm inequalities do not hold. To prove these negative results we will use some extrapolation results taken from [CMP].

The source of this presentation is the paper [MPT] written in collaboration with C. Pérez and R. Trujillo-González to whom the author wants to express his gratitude.

2 Weighted norm inequalities and Coifman's type estimates

In what follows a weight w is a non-negative locally integrable function. As usual $L^p(w)$ will denote the L^p space with the underlying measure $w(x)\,dx$.

Muckenhoupt in [Muc] found some classes of weights when he characterized the boundedness of the Hardy-Littlewood maximal function in weighted Lebesgue spaces. The classes A_p, $1 \le p < \infty$, are defined as

$$(A_p) \qquad \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\Big)^{p-1} \le C < \infty, \qquad \text{for } p > 1,$$

$$(A_1) \qquad \frac{1}{|Q|}\int_Q w(x)\,dx \le C\,w(x), \qquad \text{for a.e. } x \in Q.$$

The class A_1 can be equivalently defined as $Mw(x) \le C\, w(x)$ a.e. We also remind that $A_\infty = \bigcup_{p\ge 1} A_p$. The result proved in [Muc] establishes that M maps $L^1(w)$ into $L^{1,\infty}(w)$ if and only if $w \in A_1$ and that w is bounded on $L^p(w)$, $1 < p < \infty$, if and only if $w \in A_p$.

On the other hand, Coifman's inequality, see [Coi], states a precise control of Calderón-Zygmund operators T with kernel $K \in H^*_\infty$ in terms of M:

$$(C) \qquad \int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx \le C \int_{\mathbb{R}^n} Mf(x)^p\, w(x)dx,$$

for any $0 < p < \infty$ and $w \in A_\infty$. Thus, we can get, for instance, that T is bounded on $L^p(w)$ for $w \in A_p$, $p > 1$. Similar estimates hold replacing the $L^p(w)$ norms in both sides by the weak norms in $L^{p,\infty}(w)$ which, for $p = 1$, yields that $T : L^1(w) \longrightarrow L^{1,\infty}(w)$ for $w \in A_1$. Coifman proved (C) by establishing a good-λ inequality relating T and M. There is another approach using the sharp maximal function (see [AP] for details of this technique). Recall that

$$M^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx,$$

where f_Q stands for the average of f over Q, and that $M^\#_\delta f(x) = M^\#\big(|f|^\delta\big)(x)^{1/\delta}$. Then, for T with $K \in H^*_\infty$ we have the pointwise estimate $M^\#_\delta(Tf)(x) \le C_\delta\, Mf(x)$, $0 < \delta < 1$. This fact plus Fefferman-Stein inequality for M and $M^\#$ (proved as well by means of a good-λ inequality) also yield Coifman's inequality (C). There is still another approach with no use at all of the good-λ technique, this way combines ideas from [Ler] and [CMP], we will give more details later.

When K is less regular, say $K \in H_r$ for $1 < r \le \infty$, some substitutes of (C) arise. Now the operator is worse and it is expectable to get a bigger maximal function on the right hand side. Let us set $M_q f(x) = M\big(|f|^q\big)(x)^{1/q}$ and note that $Mf(x) \le M_q f(x)$ for $1 \le q < \infty$. In [RRT], [Wat] we can find the pointwise estimate $M^\#(Tf)(x) \le c_r\, M_{r'} f(x)$ whenever $K \in H_r$, $1 < r < \infty$. Then, the following Coifman's type inequality holds

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx \le C \int_{\mathbb{R}^n} M_{r'} f(x)^p\, w(x)dx, \tag{1}$$

for any $0 < p < \infty$ and $w \in A_\infty$. As a direct consequence, we have that T is bounded on $L^p(w)$, if $w \in A_{p/r'}$ for $r' < p < \infty$, or if $w^{1-p'} \in A_{p'/r'}$ for $1 < p < r$, or if $w^{r'} \in A_p$ for $1 < p < \infty$. The case $p = r'$ follows by interpolation with change of measure and by the reverse Hölder property (see [RRT] for more details).

When T is a singular integral operator with kernel in the class H_∞, then we get (C), or what is the same, (1) with M in place of $M_{r'}$. As a consequence, T is bounded on $L^p(w)$ for $w \in A_p$, $1 < p < \infty$. In this case the proof of (C) is also obtained by proving the pointwise estimate $M^\#_\delta(Tf)(x) \le C_\delta\, Mf(x)$, $0 < \delta < 1$. For more examples of this kind the reader is referred to [AP]. We remark that this gives an improvement of (C) since, as we noted, $H^*_\infty \subsetneq H_\infty$. An explicit example can be easily adapted from the proof of Theorem 4.6 by taking $K = \chi_{B_1(0)} \in H_\infty$ but it is not in H^*_∞.

These positive results drive us to the following questions:

- Is it possible to get similar estimates for $r = 1$, in other words, what kind of weighted estimates can be proved when the kernel is in H_1?

- For $1 < r < \infty$, can we replace $M_{r'}$ in (1) by the pointwise smaller operator M_t with $1 \le t < r'$?
- Is the operator T bounded on $L^p(w)$ for every $1 < p < \infty$ and for every $w \in A_p$ or, even more, for $w \in A_1$?

We are going to show that the answer to each of the above questions is negative.

3 Extrapolation for A_∞ weights

One of the main ingredients to negatively answer the latter questions will be some extrapolation results taken from [CMP]. We will see that to disprove (C) or (1), or their weak type–weak type analogs, it suffices to show that they fail for just one exponent p_0.

In what follows G and S are two operators defined on some class of smooth functions $\mathcal{S}$ such that $Gf \ge 0$, $Sf \ge 0$ for $f \in \mathcal{S}$. When we write an estimate like

$$\|Gf\|_{L^p(w)} \le C\,\|Sf\|_{L^p(w)}, \tag{2}$$

we always understand that it holds for any $f \in \mathcal{S}$ such that the left hand side is finite and that C depends only upon the A_∞ constant of w and p. We are not assuming any linearity or sublinearity on the operators, the only thing we need is that they are reasonably defined: Gf and Sf are measurable functions for any $f \in \mathcal{S}$. Indeed, one can formulate the result in terms of pairs of functions since the operators play no role. This is the approach used in [CMP] and its generality is extensively used there to deal with several implications, among them we remark those vector-valued that arise almost automatically.

Theorem 3.1 ([CMP]) *Let G, S be as above. Consider the following estimates*:

(*a*) $\|Gf\|_{L^{p_0}(w)} \le C\,\|Sf\|_{L^{p_0}(w)}$, *for some* $0 < p_0 < \infty$ *and all* $w \in A_\infty$.

(*b*) $\|Gf\|_{L^p(w)} \le C\,\|Sf\|_{L^p(w)}$, *for all* $0 < p < \infty$ *and all* $w \in A_\infty$.

(*c*) $\|Gf\|_{L^p(w)} \le C\,\|Sf\|_{L^p(w)}$, *for all* $0 < p < p_0$, *for some* p_0, *and all* $w \in A_1$.

(*d*) $\|Gf\|_{L^{p_0,\infty}(w)} \le C\,\|Sf\|_{L^{p_0,\infty}(w)}$, *for some* $0 < p_0 < \infty$ *and all* $w \in A_\infty$.

(*e*) $\|Gf\|_{L^{p,\infty}(w)} \le C\,\|Sf\|_{L^{p,\infty}(w)}$, *for all* $0 < p < \infty$ *and all* $w \in A_\infty$.

Then,

$$(a) \iff (b) \iff (c) \implies (e) \qquad \text{and} \qquad (d) \iff (e).$$

The reader is referred to the original source [CMP] for a complete account of this technique and also for a great deal of examples that can be used to exploit the latter result.

4 Negative results

Now we have the ingredients needed to answer the questions posed above.

Theorem 4.1 ([MPT]) *Let $1 \le r < \infty$. There exists a singular integral operator T with kernel in H_r for which the following estimates do* ***not*** *hold*:

(*i*) $\displaystyle\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_t f(x)^p\, w(x)\, dx$, *for* $0 < p < \infty$, $w \in A_\infty$ *and* $1 \le t < r'$.

(*ii*) $\|Tf\|_{L^{p,\infty}(w)} \leq C\,\|M_t f\|_{L^{p,\infty}(w)}$, *for* $0 < p < \infty$, $w \in A_\infty$ *and* $1 \leq t < r'$.

(*iii*) $\displaystyle\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} M_t f(x)^p\, Mw(x)\, dx$, *for* $0 < p \leq 1$, w *an arbitrary weight* (*that is, a non-negative locally integrable function*) *and* $1 \leq t < r'$.

(*iv*) $\displaystyle\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx$, *where, either* $1 < p < r'$, $w \in A_1$; *or* $1 < p < \infty$, $w \in A_p$.

Remark 4.2 Note that for kernels satisfying just the classical Hörmander condition (H_1), none of the maximal operators M_t can be written in the right hand side of (*i*), (*ii*) or (*iii*). Observe that no weighted estimate as (*iv*) holds even for the best class of weights A_1. In short, no weighted norm estimate is satisfied in general for operators with kernels satisfying the classical Hörmander condition (H_1). Some other results in this direction are given in [Hof].

Remark 4.3 As we have just mentioned (H_1) is not sufficient for showing weighted norm inequalities for T. However, it has recently obtained that (H_1) yields the boundedness of the supremum of the truncated integrals, see [Gra].

Remark 4.4 The estimates in (*i*) say that both (1) and the pointwise estimate $M^{\#}(Tf)(x) \leq c_r\, M_{r'} f(x)$ are sharp. Note also, that in (*iv*) the range of exponents $1 < p < r'$ and $w \in A_1$ is optimal, since for $r' \leq p < \infty$ and $w \in A_1 \subset A_{p/r'}$, T is bounded on $L^p(w)$ as mentioned before.

Remark 4.5 The importance of (*iii*) is given by the following argument. A. Lerner has recently obtained the following estimate

$$\int_{\mathbb{R}^n} |Tf(x)|\, w(x)\, dx \leq C \int_{\mathbb{R}^n} Mf(x)\, Mw(x)\, dx$$

for a singular integral operator T with kernel satisfying (H_∞^*) and for any arbitrary weight w. His proof is not based on the good-λ technique but uses the so called local sharp maximal function of F. John. Pushing Lerner techniques one can get the same estimate with exponents $0 < p \leq 1$. Taking in particular $w \in A_1$ which means $Mw(x) \leq C\, w(x)$ we get

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p\, w(x)\, dx$$

for any $0 < p \leq 1$ and for any $w \in A_1$. Applying Theorem 3.1 to the latter estimate, which corresponds to (*c*), we eventually get Coifman's inequality. We would like to emphasize that this combination of [Ler] and [CMP] has not used the good-λ technique and provides a new proof of (C).

The proof of Theorem 4.1 will be a consequence of the extrapolation technique in [CMP], Theorem 3.1 above, plus the following negative result for power weights.

Theorem 4.6 ([MPT]) *Let* $1 \leq r < \infty$, $1 \leq p < r'$, $-n < \alpha < -np/r'$ *and* $w_\alpha(x) = |x|^\alpha$. *There exists a singular integral operator* T *with kernel in* H_r *for which the following estimate does **not** hold*:

$$\|Tf\|_{L^{p,\infty}(w_\alpha)} \leq C\,\|f\|_{L^p(w_\alpha)}. \tag{3}$$

This negative result should be compared with the following positive result: let r, p be as in the theorem and let $-np/r' < \alpha \le 0$, then the following estimate holds

$$\|Tf\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}, \tag{4}$$

where $w(x) = |x|^\alpha$. This arises essentially from the results by Watson [Wat] using interpolation with change of measures.

Next, we are going to sketch the proof of Theorem 4.1 and the counterexample for Theorem 4.6 will be given afterwards.

Proof of Theorem 4.1. The estimate in (iii) with $w \in A_1$, that is, with $Mw(x) \le C\,w(x)$ a.e., implies (i), since, in Theorem 3.1, (a) and (c) are equivalent. On the other hand, by Theorem 3.1, (i) yields (ii). So, if we show that (ii) leads to a contradiction then (i) and (iii) have to failed. Furthermore, by the extrapolation result Theorem 3.1, it suffices to get some fixed exponent p_0 for which the weak type–weak type (ii) does not hold. Fix $1 \le t < r'$ and $w \in A_1 \subset A_\infty$. Then we take any p_0 such that $t < p_0 < r'$. Assume that (ii) holds, then

$$\|Tf\|_{L^{p_0,\infty}(w)} \le C\,\|M_t f\|_{L^{p_0,\infty}(w)} \le C\,\big\|M\big(|f|^t\big)\big\|_{L^{\frac{p_0}{t}}(w)}^{\frac{1}{t}} \le C\,\|f\|_{L^{p_0}(w)},$$

where in the latter estimate we have used that $p_0/t > 1$ and that $w \in A_1$, so that M is bounded on $L^{\frac{p_0}{t}}(w)$. Note that this estimate says that T is bounded from $L^{p_0}(w)$ to $L^{p_0,\infty}(w)$ for any $w \in A_1$ where $1 < p_0 < r'$. In particular, this estimate holds for the A_1-weight $w(x) = |x|^\alpha$ with $-n < \alpha < -n\,p_0/r'$, contradicting Theorem 4.6.

It remains to show that (iv) does not hold. When $1 < p < r'$ and $w \in A_1$, Theorem 4.6 is contradicted since the weights w_α are in A_1. In the other case, $1 < p < \infty$ and $w \in A_p$. If the estimate holds for some p_0 and any $w \in A_{p_0}$ then, by the Rubio de Francia extrapolation theorem (see [Duo, p. 141]), the estimate will be valid for all $1 < p < \infty$ and $w \in A_p$ which will contradict again Theorem 4.6. □

Proof of Theorem 4.6. We briefly present the counterexample leaving the details to the reader, (see [MPT]). Let $\beta > 0$ and consider the kernel $K(x) = k(|x|)$ where

$$k(t) = t^{-\frac{n}{r}} \left(\log \frac{e}{t}\right)^{-\frac{1+\beta}{r}} \chi_{(0,1)}(t).$$

Note that $K \in L^r(\mathbb{R}^n)$. Take $0 \ne \eta \in \mathbb{R}^n$ far enough from the origin, for instance $|\eta| = 4$. We define the kernel $\widetilde{K}(x) = K(x-\eta)$ and the operator T as

$$Tf(x) = \widetilde{K} * f(x) = \int_{\mathbb{R}^n} K(x-\eta-y)\, f(y)\, dy.$$

Observe that $\widetilde{K} \in L^r(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and hence the operator T is bounded on $L^q(\mathbb{R}^n)$ for every $1 \le q \le \infty$. Just by using that $K \in L^r(\mathbb{R}^n)$ and that it is supported in the unit ball, we can show that $\widetilde{K} \in H_r$ (see [MPT]). Note that when $r = 1$, since $\widetilde{K} \in L^1(\mathbb{R}^n)$, we automatically have $\widetilde{K} \in H_1$. Assume that T maps $L^p(w_\alpha)$ into $L^{p,\infty}(w_\alpha)$ and take

$$0 < \varepsilon < -\alpha - \frac{n}{r'}\,p \qquad \text{and} \qquad f(x) = |x+\eta|^{\frac{-n+\varepsilon}{p}} \chi_{B_1(-\eta)}(x) \in L^p(\mathbb{R}^n).$$

If $x \in B_1(-\eta)$ then $3 < |x| < 5$ and therefore

$$\sup_{\lambda>0} \lambda\, w\big\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\big\}^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p\,|x|^\alpha\,dx\right)^{\frac{1}{p}} \le C\,3^{\frac{\alpha}{p}}\left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

The contradiction arises here because one can show that the left hand side of this estimate is infinity. □

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References

[AP] J. Álvarez and C. Pérez, *Estimates with A_∞ weights for various singular integral operators*, Boll. Un. Mat. Ital. A (7) **8** (1994), no. 1, 123–133.

[CZ] A.P. Calderón, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[Coi] R. Coifman, *Distribution function inequalities for singular integrals*, Proc. Acad. Sci. U.S.A. **69** (1972), 2838–2839.

[CMP] D. Cruz-Uribe, J.M. Martell, C. Pérez, *Extrapolation results for A_∞ weights and applications,* Preprint 2002.

[Duo] J. Duoandikoetxea, *Fourier Analysis*, American Math. Soc., Grad. Stud. Math. **29**, Providence, RI, 2000.

[Hof] S. Hofmann, *Singular integrals with power weights*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 343–353.

[Gra] L. Grafakos, *Estimates for maximal singular integrals*, to appear in Colloq. Math.

[KW] D.S. Kurtz and R.L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[Ler] A.K. Lerner, *Weighted norm inequalities for the local sharp maximal function*, Preprint 2002.

[MPT] J.M. Martell, C. Pérez and R. Trujillo-González, *Lack of weighted estimates for some singular integral operators*, Prerpint 2002.

[Muc] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[RRT] J.L. Rubio de Francia, F.J. Ruiz and J.L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. Math. **62** (1986), 7–48.

[Wat] D.K. Watson, *Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. **60** (1990), no. 2, 389–399.

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