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WEIGHTED NORM INEQUALITIES AND EXTRAPOLATION

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Abstract. We presente a very general extrapolation principle for weights in the classes of Muckenhoupt which provides a method to obtain weighted norm inequalities in Lebesgue and more general function spaces, and also weighted modular inequalities. Vector-valued estimates are derived almost automatically. We will exploit this technique paying special attention to operators that are controlled in weighted Lebesgue spaces by the Hardy-Littlewood maximal function or, more in general, by its iterations. This is the case for regular Calderón-Zygmund operators and their commutators with bounded mean oscillation functions. We will show that these operators behave as the corresponding maximal operator that controls them. Some of the results we will present are in collaboration papers with David Cruz-Uribe and Carlos Pérez, also with Guillermo Curbera, José García-Cuerva and Carlos Pérez.

1. Introduction.

We start by introducing some of the needed background. Consider the Hardy-Littlewood maximal function in $\mathbb{R}^n$ defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the cubes $Q \subset \mathbb{R}^n$ are always considered with their sides parallel to the coordinate axes. This operator is bounded on L^p for every $1 < p \le \infty$ and it maps L^1 into $L^{1,\infty}$. One can change the underlying measure in the Lebesgue spaces by introducing a weight w, which is a non-negative measurable locally integrable function. The estimates of M on weighted Lebesgue spaces $L^p(w) = L^p(w(x)\, dx)$ are governed by the Muckenhoupt conditions, which are defined as follows: we say that $w \in A_p$, $1 \le p < \infty$, if there exists a constant C such that for every cube $Q \subset \mathbb{R}^n$ we have

$$\Big(\frac{1}{|Q|} \int_Q w(x)\, dx\Big) \Big(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\, dx\Big)^{p-1} \le C,$$

when $1 < p < \infty$, and, for $p = 1$,

$$\frac{1}{|Q|} \int_Q w(x)\, dx \le C\, w(x), \qquad \text{for a.e. } x \in Q.$$

This latter condition can be rewritten in terms of the Hardy-Littlewood maximal function: $w \in A_1$ if and only if $Mw(x) \le C\, w(x)$ for a.e. $x \in \mathbb{R}^n$. The class A_∞ is defined as $A_\infty = \bigcup_{p \ge 1} A_p$.

Muckenhoupt in [Muc] proved that the weighted norm inequalities of the Hardy-Littlewood maximal function are characterized by the classes A_p, namely, M maps

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$L^1(w)$ into $L^{1,\infty}(w)$ if and only if $w \in A_1$ and M is bounded on $L^p(w)$, $1 < p < \infty$, if and only if $w \in A_p$.

Let T be an operator which is defined on some class of nice functions $\mathcal{D}_T$. Let us point out that nothing else is assumed on T, in particular, T does not have to be linear or quasilinear. We assume that there exists $0 < p_0 < \infty$ such that M controls T on $L^{p_0}(w)$ for all $w \in A_\infty$, that is, for all $w \in A_\infty$

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} Mf(x)^{p_0}\, w(x)\, dx, \qquad f \in \mathcal{D}_T, \tag{1.1}$$

whenever the left-hand side is finite. The aim of this paper is to show that from this assumption one can prove that T satisfies weighted norm inequalities on Lebesgue spaces and function spaces, and weighted modular inequalities as M does. Besides, all these estimates admit vector-valued extensions. In other words we are able to show that most of the weighted estimates that M satisfies can be proved for T. We also see that similar results are obtained when the operator T is controlled by a given iteration of the Hardy-Littlewood maximal function. We will apply the results obtained to Calderón-Zygmund operators with standard kernel which are controlled by M (see Coifman's estimate (3.5)). We will also consider the commutators of these operators with bounded mean oscillation functions. In this case, the appropriate operators to be written in the right-hand side are the iterations of the Hardy-Littlewood maximal functions.

To work with this kind of estimates we collect the extrapolation results obtained in [CMP] and [CGMP]. Before that, we introduce some notation: as mentioned, there is no assumption on the operator T and in (1.1) one can replace M by any other given operator. In fact, the operators do not need to appear explicitly and one can work with pairs of functions. In what follows, $\mathcal{F}$ is a family of ordered pairs of non-negative measurable functions (f, g). If we say that for some p_0, $0 < p_0 < \infty$, and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f, g) \in \mathcal{F}, \tag{1.2}$$

we always mean that (1.2) holds for any $(f, g) \in \mathcal{F}$ such that the left hand side is finite, and that the constant C depends only upon p and the A_∞ constant of w. We will make similar abbreviated statements involving other function norms or quasi-norms, or even modular type estimates; they will be always interpreted in the same way. Note that using this notation, (1.1) is (1.2) with $\mathcal{F}$ consisting of the pairs $(|Tf|, Mf)$ for $f \in \mathcal{D}_T$.

In [CMP] it is shown that starting from (1.2) one can extrapolate and the same estimate holds for the full range of exponents $0 < p < \infty$ and for all $w \in A_\infty$. In that paper it is also proved that the spaces $L^p(w)$ can be replaced by the Lorentz spaces $L^{p,q}(w)$ for all $0 < p < \infty$ and $0 < q \le \infty$. This was generalized in [CGMP] obtaining that (1.2) implies estimates on very general rearrangement invariant quasi-Banach function spaces (RIQBFS in the sequel) and also very general weighted modular inequalities. Furthermore, the fact that one can work with general families $\mathcal{F}$ allows one to prove, in an almost automatic way, that all these estimates extend to sequence-valued functions. The next result collects all these extrapolation results. The needed background is collected in Section 2.

Theorem 1.1 ([CMP], [CGMP]). *Let $\mathcal{F}$ be a family of ordered pairs of non-negative, measurable functions (f, g). Assume that there exists $0 < p_0 < \infty$ such that*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f, g) \in \mathcal{F}, \tag{1.3}$$

for all $w \in A_\infty$ and whenever the left-hand side is finite. Then, for all $(f,g) \in \mathcal{F}$ and all $\{(f_j, g_j)\}_j \subset \mathcal{F}$ we have the following estimates:

(a) **Lebesgue spaces,** [CMP]**:** *For all $0 < p, q < \infty$ and $w \in A_\infty$,*

$$\|f\|_{L^p(w)} \le C\,\|g\|_{L^p(w)}, \qquad \Big\|\Big(\sum_j (f_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \le C\,\Big\|\Big(\sum_j (g_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)}.$$

(b) **Rearrangement invariant quasi-Banach function spaces,** [CGMP]**:** *Let $\mathbb{X}$ be a RIQBFS such that $\mathbb{X}$ is p-convex for some $0 < p \le 1$ —equivalently $\mathbb{X}^r$ is Banach for some $r \ge 1$— and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for all $0 < q < \infty$ and $w \in A_\infty$ we have*

$$\|f\|_{\mathbb{X}(w)} \le C\,\|g\|_{\mathbb{X}(w)}, \qquad \Big\|\Big(\sum_j (f_j)^q\Big)^{\frac{1}{q}}\Big\|_{\mathbb{X}(w)} \le C\,\Big\|\Big(\sum_j (g_j)^q\Big)^{\frac{1}{q}}\Big\|_{\mathbb{X}(w)}.$$

(c) **Modular inequalities,** [CGMP]**:** *Let $\phi \in \Phi$ with $\phi \in \Delta_2$ and suppose that there exist some exponents $0 < r, s < \infty$ such that $\phi(t^r)^s$ is quasi-convex. Then for all $0 < q < \infty$ and all $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} \phi\big(f(x)\big)\, w(x)\, dx \;\le\; C \int_{\mathbb{R}^n} \phi\big(g(x)\big)\, w(x)\, dx,$$

$$\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j f_j(x)^q\Big)^{\frac{1}{q}}\bigg)\, w(x)\, dx \;\le\; C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j g_j(x)^q\Big)^{\frac{1}{q}}\bigg)\, w(x)\, dx,$$

Furthermore, for $\mathbb{X}$ as before one can also get that $\phi(f)$ is controlled by $\phi(g)$ on $\mathbb{X}(w)$. In particular, taking $\mathbb{X} = L^{1,\infty}$ we have the following weak-type modular inequalities

$$\sup_\lambda \phi(\lambda)\, w\{x : f(x) > \lambda\} \;\le\; C \sup_\lambda \phi(\lambda)\, w\{x : g(x) > \lambda\},$$

$$\sup_\lambda \phi(\lambda)\, w\bigg\{x : \Big(\sum_j f_j(x)^q\Big)^{\frac{1}{q}} > \lambda\bigg\} \;\le\; C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j g_j(x)^q\Big)^{\frac{1}{q}}\bigg)\, w(x)\, dx,$$

for all $w \in A_\infty$.

We will use this result starting with (1.1) which will allow us to obtain inequalities for T using those that are known for M. The advantage of this method is that once (1.1) is known, no property of T is used and everything reduces to prove estimates for M.

The plan of the paper is as follows. The next section is devoted to introduce the needed background. In Section 3 we study those operators that satisfy (1.1): we will present a collection of weighted estimates for the Hardy-Littlewood maximal function to show that T behaves in the same way. Finally, in Section 4 we consider operators with a higher degree of singularity in the sense that the operator appearing in the right hand side of (1.1) is an iteration of the Hardy-Littlewood maximal function. We will establish weighted estimates for T as a consequence of the extrapolation results. We will pay special attention to those estimates near L^1.

2. Preliminaries

In this section we present the needed background.

2.1. **Basics on Function Spaces.** We collect several basic facts about rearrangement invariant quasi-Banach function spaces (RIQBFS). We start with the Banach case. For a complete account the reader is referred to [BS]. Let (Ω, Σ, μ) be a σ-finite non-atomic measure space. We write $\mathcal{M}$ for the set of measurable functions and $\mathcal{M}^+$ for the non-negative ones. Given a Banach function norm ρ we the Banach function space $\mathbb{X} = \mathbb{X}(\rho)$ as

$$\mathbb{X} = \big\{ f \in \mathcal{M} : \|f\|_{\mathbb{X}} = \rho(|f|) < \infty \big\}.$$

The associate space of $\mathbb{X}$ is the space $\mathbb{X}'$ defined by the Banach function norm ρ':

$$\rho'(f) = \sup \Big\{ \int_\Omega f\, g\, d\mu : g \in \mathcal{M}^+, \ \rho(g) \le 1 \Big\}.$$

Note that, by definition, it follows that for all $f \in \mathbb{X}$, $g \in \mathbb{X}'$ the following generalized Hölder's inequality holds:

$$\int_\Omega |f\, g|\, d\mu \le \|f\|_{\mathbb{X}}\, \|g\|_{\mathbb{X}'}.$$

The distribution function μ_f of a measurable function f is

$$\mu_f(\lambda) = \mu\big\{ x \in \Omega : |f(x)| > \lambda \big\}, \qquad \lambda \ge 0.$$

A Banach function space $\mathbb{X}$ is rearrangement invariant if $\rho(f) = \rho(g)$ for every pair of functions f, g which are equimeasurable, that is, $\mu_f = \mu_g$. In this case, we say that the Banach function space $\mathbb{X} = \mathbb{X}(\rho)$ is rearrangement invariant. It follows that $\mathbb{X}'$ is also rearrangement invariant. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf \big\{ \lambda \ge 0 : \mu_f(\lambda) \le t \big\}, \qquad t \ge 0.$$

The main property of f^* is that it is equimeasurable with f, that is,

$$\mu\big\{ x \in \Omega : |f(x)| > \lambda \big\} = \big| \big\{ t \in \mathbb{R}^+ : f^*(t) > \lambda \big\} \big|.$$

This allows one to obtain a representation of $\mathbb{X}$ on the measure space $(\mathbb{R}^+, dt)$. That is, there exists a RIBFS $\overline{\mathbb{X}}$ over $(\mathbb{R}^+, dt)$ such that $f \in \mathbb{X}$ if and only if $f^* \in \overline{\mathbb{X}}$, and in this case $\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}}$ (Luxemburg's representation theorem, see [BS, p. 62]). Furthermore, the associate space $\mathbb{X}'$ of $\mathbb{X}$ is represented in the same way by the associate space $\overline{\mathbb{X}}'$ of $\overline{\mathbb{X}}$, and so $\|f\|_{\mathbb{X}'} = \|f^*\|_{\overline{\mathbb{X}}'}$.

From now on let $\mathbb{X}$ be rearrangement invariant Banach function spaces (RIBFS) in $(\mathbb{R}^n, dx)$ and let $\overline{\mathbb{X}}$ be its corresponding RIBFS in $(\mathbb{R}^+, t)$.

Next, we define the Boyd indices of $\mathbb{X}$, which are closely related to some interpolation properties, see [BS, Ch. 3] for more details. First we introduce the dilation operator

$$D_t f(s) = f(s/t), \qquad 0 < t < \infty, \quad f \in \overline{\mathbb{X}},$$

and its norm $h_{\mathbb{X}}(t) = \|D_t\|_{\mathcal{B}(\overline{\mathbb{X}})}$ where $\mathcal{B}(\overline{\mathbb{X}})$ denotes the space of bounded linear operators on $\overline{\mathbb{X}}$. Then, the lower and upper Boyd indices are defined respectively by

$$p_{\mathbb{X}} = \lim_{t \to \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \sup_{1 < t < \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)}, \qquad q_{\mathbb{X}} = \lim_{t \to 0^+} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \inf_{0 < t < 1} \frac{\log t}{\log h_{\mathbb{X}}(t)}.$$

We have that $1 \le p_{\mathbb{X}} \le q_{\mathbb{X}} \le \infty$. The relationship between the Boyd indices of $\mathbb{X}$ and $\mathbb{X}'$ is the following: $p_{\mathbb{X}'} = (q_{\mathbb{X}})'$ and $q_{\mathbb{X}'} = (p_{\mathbb{X}})'$, where, as usual, p and p' are conjugate exponents.

Take w an A_∞-weight on $\mathbb{R}^n$. We use the standard notation $w(E) = \int_E w(x)\,dx$. The distribution function and the decreasing rearrangement with respect to w are given by

$$w_f(\lambda) = w\big\{x \in \mathbb{R}^n : |f(x)| > \lambda\big\}; \qquad f_w^*(t) = \inf\big\{\lambda \geq 0 : w_f(\lambda) \leq t\big\}.$$

We define the weighted version of the space $\mathbb{X}$:

$$\mathbb{X}(w) = \big\{f \in \mathcal{M} : \|f_w^*\|_{\mathbb{X}} < \infty\big\},$$

and the norm associated to it $\|f\|_{\mathbb{X}(w)} = \|f_w^*\|_{\mathbb{X}}$. By construction $\mathbb{X}(w)$ is a Banach function space built over $\mathcal{M}(\mathbb{R}^n, w(x)\,dx)$. By doing the same procedure with the associate spaces we can see that the associate space $\mathbb{X}(w)'$ coincides with the weighted space $\mathbb{X}'(w)$.

Given a Banach function space $\mathbb{X}$, for each $0 < r < \infty$, as in [JS], we define

$$\mathbb{X}^r = \big\{f \in \mathcal{M} : |f|^r \in \mathbb{X}\big\} = \big\{f \in \mathcal{M} : \|f\|_{\mathbb{X}^r} = \big\||f|^r\big\|_{\mathbb{X}}^{\frac{1}{r}}\big\}.$$

Note that this notation is natural for the Lebesgue spaces since L^r coincides with $(L^1)^r$. If $\mathbb{X}$ is a RIBFS and $r \geq 1$ then, $\mathbb{X}^r$ still is a RIBFS but, in general, for $0 < r < 1$, the space $\mathbb{X}^r$ is not necessarily Banach. Note that in the same way we can also define powers of weighted spaces and we have $(\mathbb{X}(w))^r = \mathbb{X}^r(w)$.

In this paper we work with spaces $\mathbb{X}$ so that $\mathbb{X} = \mathbb{Y}^s$ for some RIBFS $\mathbb{Y}$ and some $0 < s < \infty$. The space $\mathbb{X}$ is in particular a rearrangement quasi-Banach space (RIQBFS in the sequel), see [GK] or [Mon] for more details. Let us observe that another equivalent approach consists in introducing first the quasi-Banach case and then one restricts the attention to those RIQBFS for which a large power is a Banach space. This latter property turns out to be equivalent to the fact that the RIQBFS $\mathbb{X}$ is p-convex for some $0 < p \leq 1$, that is, there exists C such that for all $N \geq 1$ and $f_1, \cdots, f_N \in \mathbb{X}$, all

$$\Big\|\Big(\sum_{j=1}^N |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{\mathbb{X}} \leq C\,\Big(\sum_{j=1}^N \|f_j\|_{\mathbb{X}}^p\Big)^{\frac{1}{p}}.$$

In this case, after renorming if necessary, one has that $\mathbb{X}^{\frac{1}{p}}$ is a RIBFS.

Regarding the statement of Theorem 1.1 we have to make several remarks.

Remark 2.1. Note that in (b) of Theorem 1.1 we have restricted ourselves to the case of $\mathbb{X}$ p-convex with $q_{\mathbb{X}} < \infty$. As we have just mentioned, this means that $\mathbb{X}^r$ is a Banach space (with $r = 1/p$). Thus, by Lorentz-Shimogaki's theorem (see [Lor], [Shi] and [BS, p. 54]) $q_{\mathbb{X}} < \infty$ is equivalent to the boundedness of the Hardy-Littlewood maximal function on $(\mathbb{X}^r)'$.

Remark 2.2. Theorem 1.1 part (b) can be equivalently formulated in terms of RIBFS rather than quasi-Banach spaces. The conclusion would be as follows:

> *Then, for all RIBFS $\mathbb{X}$ such that $q_{\mathbb{X}} < \infty$ —or equivalently, that the Hardy-Littlewood maximal function is bounded on $\mathbb{X}'$—, all p such that $0 < p < \infty$, and all $w \in A_\infty$, we have*
>
> $$\|f\|_{\mathbb{X}^p(w)} \leq C\,\|g\|_{\mathbb{X}^p(w)}, \qquad (f,g) \in \mathcal{F},$$
>
> *and the corresponding vector-valued inequalities also hold.*

The equivalence is based on the fact that if $\mathbb{Y} = \mathbb{X}^r$ then $q_{\mathbb{Y}} = r \cdot q_{\mathbb{X}}$.

Remark 2.3. The formulation given in (b) of Theorem 1.1 and the equivalent one presented in the previous remark reflect that there are two different points of view: suppose that one wants to get estimates in $L^{\frac{1}{2}}$. The first formulation consists in looking at the RIQBFS $\mathbb{X} = L^{\frac{1}{2}}$ which has the property that $\mathbb{X}^2 = L^1$ is a Banach space. This convexity allows us to apply Theorem 1.1 to $\mathbb{X}$. Alternatively one can start from $\mathbb{X} = L^1$ which is a RIBFS and by the second formulation get estimates in $\mathbb{X}^p$ for all $0 < p < \infty$, and in particular in $\mathbb{X}^{\frac{1}{2}} = L^{\frac{1}{2}}$.

Some examples of RIQBFS are Lebesgue spaces, classical Lorentz spaces, Lorentz Λ-spaces, Orlicz spaces, Marcinkiewicz spaces, etc, see [CGMP] for more details. In some of these examples, the Boyd indices can be computed very easily, for instance if $\mathbb{X}$ is L^p, $L^{p,q}$, $L^p(\log L)^\alpha$ or $L^{p,q}(\log L)^\alpha$ (where $0 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$) then $p_{\mathbb{X}} = q_{\mathbb{X}} = p$. In this cases, it is easy to compute the powers of $\mathbb{X}$ and one obtains

$$(L^{p,q})^r = L^{p\,r,q\,r}, \qquad\qquad (L^{p,q}(\log L)^\alpha)^r = L^{p\,r,q\,r}(\log L)^\alpha,$$

note the same applies to $L^p = L^{p,p}$ and $L^p(\log L)^\alpha = L^{p,p}(\log L)^\alpha$.

2.2. **Basics on modular inequalities.** We introduce some notation, the terminology used is taken from [KK] and [RR]. Let Φ be the set of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ which are nonnegative, increasing and such that $\phi(0^+) = 0$ and $\phi(\infty) = \infty$. If $\phi \in \Phi$ is convex we say that ϕ is a Young function. An N-function (from *nice Young function*) ϕ is a Young function such that

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

The function $\phi \in \Phi$ is said to be quasi-convex if there exists a convex function $\widetilde{\phi}$ and $a \geq 1$ such that

$$\widetilde{\phi}(t) \leq \phi(t) \leq a\,\widetilde{\phi}(a\,t), \qquad t \geq 0. \tag{2.1}$$

We say that $\phi \in \Phi$ satisfies the Δ_2 condition, we will write $\phi \in \Delta_2$, if ϕ is doubling, that is, if

$$\phi(2\,t) \leq C\,\phi(t), \qquad t \geq 0.$$

Given $\phi \in \Phi$ we define the complementary function $\overline{\phi}$ by

$$\overline{\phi}(s) = \sup_{t>0}\{s\,t - \phi(t)\}, \qquad s \geq 0.$$

By definition we have Young's inequality

$$s\,t \leq \phi(s) + \overline{\phi}(t), \qquad s,t \geq 0. \tag{2.2}$$

When ϕ is an N-function, then $\overline{\phi}$ is an N-function too, and we have the following

$$t \leq \phi^{-1}(t)\,\overline{\phi}^{-1}(t) \leq 2\,t, \qquad t \geq 0. \tag{2.3}$$

The lower and upper dilation indices of $\phi \in \Phi$ are defined respectively by

$$i_\phi = \lim_{t\to 0^+} \frac{\log h_\phi(t)}{\log t} = \sup_{0<t<1} \frac{\log h_\phi(t)}{\log t}, \qquad I_\phi = \lim_{t\to\infty} \frac{\log h_\phi(t)}{\log t} = \inf_{1<t<\infty} \frac{\log h_\phi(t)}{\log t},$$

where

$$h_\phi(t) = \sup_{s>0} \frac{\phi(s\,t)}{\phi(s)}, \qquad t > 0,$$

see [KPS] and [KK]. Observe that $0 \leq i_\phi \leq I_\phi \leq \infty$. It is easy to see that if ϕ is quasi-convex, then $i_\phi \geq 1$. If ϕ is an N-function, then we have that the indices for ϕ and $\overline{\phi}$ satisfy the following: $i_{\overline{\phi}} = (I_\phi)'$ and $I_{\overline{\phi}} = (i_\phi)'$.

These indices give, among other things, information about the growth of ϕ in terms of power functions. Indeed, if $0 < i_\phi \le I_\phi < \infty$, given ε small enough, we have for all $t \ge 0$

$$\begin{aligned} \phi(\lambda\, t) &\le C_\varepsilon\, \lambda^{I_\phi+\varepsilon}\, \phi(t), \qquad \text{for} \quad \lambda \ge 1, \\ \phi(\lambda\, t) &\le C_\varepsilon\, \lambda^{i_\phi-\varepsilon}\, \phi(t), \qquad \text{for} \quad \lambda \le 1. \end{aligned}$$

It is clear then, that $\phi \in \Delta_2$ if and only if $I_\phi < \infty$.

Remark 2.4. We would like to stress the analogy between the hypotheses of Theorem 1.1 parts (b) and (c). The facts that $\mathbb{X}^r$ is Banach for some $r \ge 1$ and $\phi(t^r)^s$ is quasi-convex for some $0 < r, s < \infty$ play the same role. Indeed in the proofs these properties are used to ensure the existence of a dual space and a complementary function which allow one to perform a duality argument in both cases. On the other hand, in (b) one assumes that $q_{\mathbb{X}} < \infty$ and in (c) it is supposed that $\phi \in \Delta_2$ which, as mentioned, means $I_\phi < \infty$. So, in both cases, we are assuming the finiteness of the upper indices. In the proofs, these conditions are used to assure that the Hardy-Littlewood maximal function is bounded on the dual of $\mathbb{X}^r$ and also it satisfies a modular inequality with respect to the complementary function of $\phi(t^r)^s$.

Remark 2.5. As in Remark 2.2, one can reformulate part (c) in Theorem 1.1 in the following way: one can start with an N-function ϕ such that $I_\phi < \infty$, or equivalently, M satisfies a modular inequality with respect to $\overline{\phi}$, and then get weighted modular inequalities with respect to the functions $\phi(t^r)^s$ for all $0 < r, s < \infty$.

Some examples to whom these results can be applied are $\phi(t) = t^p$, $\phi(t) = t^p\,(1 + \log^+ t)^\alpha$, $\phi(t) = t^p\,(1 + \log^+ t)^\alpha\,(1 + \log^+ \log^+ t)^\beta$ with $0 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. In all these cases one can see that $i_\phi = I_\phi = p$ and also that $\phi(t^r)$ is quasi-convex for r large enough.

3. Operators controlled by the Hardy-Littlewood maximal function

We are going to apply Theorem 1.1 to (1.1) in order to get all those inequalities for the pairs $(|Tf|, Mf)$. Then next goal consists in proving weighted norm inequalities for T as a consequence of the ones known for M.

We already know that M maps $L^p(w)$ into $L^p(w)$ for all $w \in A_p$, $1 < p < \infty$, and $L^{1,\infty}(w)$ into $L^1(w)$ for all $w \in A_1$. Regarding the vector-valued inequalities it is also known that M satisfies the corresponding ℓ^q-valued weighted estimates for $1 < q < \infty$. In order to show that M satisfies vector-valued estimates on RIQBFS or of modular type we will use the following inequality, see [CGMP]: if $1 < q < \infty$, we have for all $0 < p < \infty$, and all $w \in A_\infty$

$$\Big\| \Big(\sum_j (Mf_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C\, \Big\| M\Big(\Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big) \Big\|_{L^p(w)}. \tag{3.1}$$

This allows us to use Theorem 1.1 with the pairs given by this estimate and therefore the vector-valued inequalities for M follows from its scalar estimates. Next, we collect the weighted vector-valued inequalities obtained for M by this method:

Theorem 3.1. *Let $\mathbb{X}$ be a RIQBFS which is p-convex for some $p > 0$ and let $\phi \in \Phi$ be a quasi-convex function.*

(*i*) *If* $1 < p_{\mathbb{X}} \le \infty$, *for all* $w \in A_{p_{\mathbb{X}}}$ *we have*

$$\|Mf\|_{\mathbb{X}(w)} \le C\,\|f\|_{\mathbb{X}(w)}. \tag{3.2}$$

(*ii*) *If* $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$ *we have that for all* $1 < q < \infty$ *and for all* $w \in A_{p_{\mathbb{X}}}$, M *satisfies the following weighted vector-valued inequality*

$$\Big\|\Big(\sum_j (Mf_j)^q\Big)^{\frac{1}{q}}\Big\|_{\mathbb{X}(w)} \le C\,\Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{\mathbb{X}(w)}. \tag{3.3}$$

(*iii*) *For all* $w \in A_{i_\phi}$,

$$\begin{aligned}\int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\,w(x)\,dx \quad &\le \quad C\int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\,w(x)\,dx, \qquad \text{if } 1 < i_\phi \le \infty,\\ \sup_\lambda \phi(\lambda)\,w\big\{x : Mf(x) > \lambda\big\} \quad &\le \quad C\int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\,w(x)\,dx, \qquad \text{if } i_\phi = 1.\end{aligned}$$

(*iv*) *If* $\phi \in \Delta_2$ (*or, what is the same,* $I_\phi < \infty$), *for all* $1 < q < \infty$, M *satisfies the following vector-valued weighted modular inequalities: for all* $w \in A_{i_\phi}$,

$$\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j Mf_j(x)^q\Big)^{\frac{1}{q}}\bigg)\,w(x)\,dx \le C\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j |f_j(x)|^q\Big)^{\frac{1}{q}}\bigg)\,w(x)\,dx,$$

if $1 < i_\phi < \infty$, *and if* $i_\phi = 1$ *we have the weak-type modular inequality*

$$\sup_\lambda \phi(\lambda)\,w\bigg\{x : \Big(\sum_j Mf_j(x)^q\Big)^{\frac{1}{q}} > \lambda\bigg\} \le C\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j |f_j(x)|^q\Big)^{\frac{1}{q}}\bigg)\,w(x)\,dx.$$

Remark 3.2. This result can be seen as an extension of the classical Theorem of Lorentz-Shimogaki (see [Lor], [Shi] and [BS, p. 54]) which states that the Hardy-Littlewood maximal function is bounded on a RIBFS $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$. Note that Theorem 3.1 contains weighted, vector-valued and modular extensions of this result.

Remark 3.3. As in Remark 2.4 one can see the analogy between the hypotheses of parts (*i*), (*ii*) and respectively (*iii*) and (*iv*). Note, for instance, that we have obtained weighted vector-valued inequalities for M on $\mathbb{X}$ provided $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$ and $w \in A_{p_{\mathbb{X}}}$. Analogously, M satisfies strong weighted modular inequalities with respecto to ϕ whenever $1 < i_\phi \le I_\phi < \infty$. Note that this same comment applies to Corollaries 3.4 and 4.1 below.

The proof of Theorem 3.1 can be found in [CGMP]. The first part is obtained directly, while (*ii*) follows by (*i*) and by extrapolation applying (*b*) in Theorem 1.1 to (3.1). Part (*iii*) can be proved directly using the convexity properties of ϕ. This inequality was first consider in [KT] under slightly hypotheses, see also [KK]. Part (*iv*) can be shown as before from (*iii*) and by applying (*c*) in Theorem 1.1 to (3.1). Similar results are proved by different methods in [KK].

The following result shows that if T satisfies (1.1), then T behaves as the Hardy-Littlewood maximal function in terms of the weighted estimates.

Corollary 3.4. *Let* T *be an operator defined in some class of nice functions* $\mathcal{D}_T$. *Assume that there is* $0 < p_0 < \infty$ *such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\,w(x)\,dx \le C\int_{\mathbb{R}^n} Mf(x)^{p_0}\,w(x)\,dx, \qquad f \in \mathcal{D}_T \tag{3.4}$$

for all $w \in A_\infty$ and whenever the left-hand side is finite. Then the pairs $(|Tf|, Mf)$, for $f \in \mathcal{D}_T$, satisfy all the estimates contained in Theorem 1.1. Hence, for all $1 < p, q < \infty$ and all $w \in A_p$

$$\|Tf\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}, \qquad \Big\|\Big(\sum_j |Tf_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \le C\,\Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)}.$$

If $w \in A_1$ and $1 < q < \infty$ we have

$$\|Tf\|_{L^{1,\infty}(w)} \le C\,\|f\|_{L^1(w)}, \qquad \Big\|\Big(\sum_j |Tf_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{1,\infty}(w)} \le C\,\Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^1(w)}.$$

Furthermore, let $\mathbb{X}$ be a RIQBFS such that $\mathbb{X}$ is p-convex for some $0 < p < 1$ and such that $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$. Then, T satisfies (3.2) and (3.3). On the other hand, let $\phi \in \Phi$ be a quasi-convex function such that $\phi \in \Delta_2$, (or, what is the same, $I_\phi < \infty$). Then, T satisfies the weighted modular inequalities contained in (iii) and (iv) of Theorem 3.1.

Remark 3.5. This result extends the classical Theorem of Boyd (see [Bo] and [BS, p. 154]) on which it is obtained that the Hilbert transform is bounded on a RIBFS $\mathbb{X}$ if and only if $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$. As we see below, Coifman's inequality (3.5) implies that the Hilbert transform satisfies (3.4) and so all the weighted estimates in Corollary 3.4 hold. Furthermore, any Calderón-Zygmund operator can be controlled by the Hardy-Littlewood maximal function (see (3.5) below) and therefore we obtain this family of weighted estimates for this class of operators. Thus, we are extending Boyd's result in the way that the class of operators is wider, we get weighted estimates, modular inequalities and also all of them admit vector-valued versions.

Remark 3.6. In addition to the previous remark, notice that Corollary 3.4 can be applied to operators which are not necessarily linear or quasilinear, this means that the general interpolation results can not be used. Thus, it is not clear how to get estimates on RIQBFS following the classical ways (see [BS]). The idea behind this latter comment is that estimates for T are proved through M, for which classical interpolation results can be employed. On the other hand, it should be pointed out that it is not clear how to interpolate between estimates like (3.4) —even if the operator T is linear— since M appears in the right-hand side.

Corollary 3.4 follows directly from Theorem 1.1 applied to (3.4) and then by using the weighted estimates for the Hardy-Littlewood maximal function contained in Theorem 3.1. For the estimates in $L^{1,\infty}$ one can apply (b) in Theorem 1.1 with $\mathbb{X} = L^{1,\infty}$ and then employ the well known weak type vector-valued inequalities for M. Another possible way consists in taking $\phi(\lambda) = \lambda$ for which $i_\phi = 1$ and then one can use (iii) and (iv) in Theorem 3.1 with T in place of M.

The main example of operators satisfying (3.4) is given by Calderón-Zygmund operators T which are bounded linear operators on L^2 such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy, \qquad \text{for a.e. } x \notin \operatorname{supp} f$$

where the kernel K satisfies the standard estimates

$$|K(x,y)| \le \frac{A}{|x-y|^n}$$

and

$$|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|\le A\,\frac{|y-y'|^{\tau}}{|x-y|^{n+\tau}},\qquad |x-y|>2\,|y-y'|,$$

for some A, $\tau>0$. These operators satisfy Coifman's inequality, see [Coi]:

$$\int_{\mathbb{R}^n}|Tf(x)|^p\,w(x)\,dx\le C\int_{\mathbb{R}^n}Mf(x)^p\,w(x)\,dx \tag{3.5}$$

for all $0<p<\infty$ and all $w\in A_\infty$ and all $f\in C_0^\infty$ such that the left hand-side is finite. This means that we can apply Corollary 3.4 obtaining all the weighted estimates contained there.

4. Operators controlled by iterations of the Hardy-Littlewood maximal function

In this section we consider operators that are controlled by iterations of the Hardy-Littlewood maximal function. Suppose that we have as before some operator T defined in $\mathcal{D}_T$ such that there exists $0<p_0<\infty$ and for all $w\in A_\infty$,

$$\int_{\mathbb{R}^n}|Tf(x)|^{p_0}\,w(x)\,dx\le C\int_{\mathbb{R}^n}M^{m+1}f(x)^{p_0}\,w(x)\,dx,\qquad f\in\mathcal{D}_T, \tag{4.1}$$

whenever the left-hand side is finite and where M^{m+1} is the Hardy-Littlewood maximal operator iterated $m+1$-times with $m\ge 1$ (note that the case $m=0$ was considered in the previous section). As done before, this implies that T is controlled by M^{m+1} in all the senses of Theorem 1.1. Note that in terms of weighted estimates, M^{m+1} and M behave in the same way provided the space is not "close" to L^1, that is, M^{m+1} satisfies all the estimates in Theorem 3.1 but the weak-type modular estimates in (iii) and (iv). This implies some of the inequalities in Corollary 3.4 but one has to be careful at the end-point $p=1$. Let us first state the result that one can get as a direct consequence of the extrapolation technique and we will study later the issues with the end-point estimates.

Corollary 4.1. *Let T be an operator defined in some class of nice functions $\mathcal{D}_T$. Assume that there are an integer $m\ge 1$ and $0<p_0<\infty$ such that for all $w\in A_\infty$*

$$\int_{\mathbb{R}^n}|Tf(x)|^{p_0}\,w(x)\,dx\le C\int_{\mathbb{R}^n}M^{m+1}f(x)^{p_0}\,w(x)\,dx,\qquad f\in\mathcal{D}_T \tag{4.2}$$

whenever the left-hand side is finite. Then the pairs $(|Tf|,M^{m+1}f)$, for $f\in\mathcal{D}_T$, satisfy all the estimates contained in Theorem 1.1. Hence, for all $1<p,q<\infty$ and all $w\in A_p$

$$\|Tf\|_{L^p(w)}\le C\,\|f\|_{L^p(w)},\qquad \Big\|\Big(\sum_j|Tf_j|^q\Big)^{\frac1q}\Big\|_{L^p(w)}\le C\,\Big\|\Big(\sum_j|f_j|^q\Big)^{\frac1q}\Big\|_{L^p(w)}.$$

Furthermore, let $\mathbb{X}$ be a RIQBFS such that $\mathbb{X}$ is p-convex for some $0<p<1$ and such that $1<p_{\mathbb{X}}\le q_{\mathbb{X}}<\infty$. Then, T satisfies (3.2) *and* (3.3)*. On the other hand, let $\phi\in\Phi$ be a quasi-convex function such that $\phi\in\Delta_2$, (or, what is the same, $I_\phi<\infty$). Then, if $1<i_\phi<\infty$, T satisfies the first estimate in (iii) and the first estimate in (iv) of Theorem 3.1.*

To prove this result we first observe that M^{m+1} satisfy all these estimates since M does. Then, using Theorem 1.1 as in the previous section the proof is completed.

We now study the behavior of M^{m+1} near L^1 to eventually show that T satisfies the same estimates. In terms of RIQBFS the natural end-point estimate for the Hardy-Littlewood maximal inequality is the boundedness of M from L^1 to $L^{1,\infty}$ which turns out to be also a weak-type modular inequality. To find the natural spaces and modular inequalities for M^{m+1}, we first consider the function

$$\varphi_m(t) = \frac{t}{(1+\log^+ t)^m}, \qquad t > 0,$$

which is increasing, quasi-concave (that is, $\varphi_m(t)/t$ is decreasing) and satisfies that $\varphi_m(0^+) = 0$. We can define the Marcinkiewicz type space $\widetilde{\mathbb{M}}_{\varphi_m}$ by the quasi-norm

$$\|f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} = \sup_t \varphi_m(t)\, f^*(t).$$

Thus $\mathbb{X} = \widetilde{\mathbb{M}}_{\varphi_m}$ is a RIQBFS such that $\mathbb{X}^r$ is a Banach space for any $r > 1$ and $p_{\mathbb{X}} = q_{\mathbb{X}} = 1$, see [CGMP]. We note that this allows us to use Theorem 1.1 with $\mathbb{X}$. This space plays the role of $L^{1,\infty}$ as we see below.

To deal with the modular inequalities we introduce the function

$$\psi_m(t) = t\,(1+\log^+ t)^m, \qquad t > 0.$$

Note that ψ is an increasing convex function with $\psi(0^+) = 0$ and $\psi \in \Delta_2$.

For M^{k+1} we have the following end-point estimates:

Proposition 4.2. *Let $m \geq 1$. Then,*

$$M^{m+1} : L(\log L)^m \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}$$

and

$$\big|\{x \in \mathbb{R}^n : M^{m+1}f(x) > \lambda\}\big| \leq C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Furthermore, for any $w \in A_1$ we have the weighted estimates

$$M^{m+1} : L(\log L)^m(w) \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}(w)$$

and

$$w\{x \in \mathbb{R}^n : M^{m+1}f(x) > \lambda\} \leq C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx.$$

These estimates are the analogs in terms of RIQBFS and modular inequalities of the weak type $(1,1)$ of M. As before, we can show that the operator T satisfies the same estimates.

Corollary 4.3. *Let T be an operator as in Corollary 4.1 satisfying* (4.2). *Then, for all $w \in A_1$*

$$T : L(\log L)^m(w) \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}(w)$$

and

$$w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx.$$

To prove the first estimate we only need to apply Theorem 1.1, part (b), with the pairs $(|Tf|, M^{m+1}f)$ for $f \in \mathcal{D}_T$ and $\mathbb{X} = \widetilde{\mathbb{M}}_{\varphi_m}$, and then Proposition 4.2. Note that as mentioned $\mathbb{X}$ is a RIQBFS with the property that $\mathbb{X}^r$ is Banach for every $r > 1$ and also $p_{\mathbb{X}} = q_{\mathbb{X}} = 1$. Observe that the class of weights A_1 is natural since $p_{\mathbb{X}} = 1$.

The modular inequality is not so automatic. Define the function $\phi_m(t) = \frac{1}{\psi_m(1/t)}$ and observe that $\phi_m \in \Phi$ is such that $\phi_m \in \Delta_2$ (indeed, $i_{\phi_m} = I_{\phi_m} = 1$) and $\phi(t^r)$ is quasi-convex for some large r. Then, we can apply (c) in Theorem 1.1 with ϕ_m and Proposition 4.2 to obtain

$$\begin{aligned}
w\big\{x \in \mathbb{R}^n : |Tf(x)| > 1\big\} &\leq \sup_t \phi_m(t)\, w\big\{x \in \mathbb{R}^n : |Tf(x)| > t\big\} \\
&\leq C \sup_t \phi_m(t)\, w\big\{x \in \mathbb{R}^n : M^{m+1} f(x) > t\big\} \\
&\leq C \sup_t \phi_m(t) \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{t}\right) w(x)\, dx \\
&\leq C \sup_t \phi_m(t)\, \psi_m\left(\frac{1}{t}\right) \int_{\mathbb{R}^n} \psi_m\big(|f(x)|\big)\, w(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} \psi_m\big(|f(x)|\big)\, w(x)\, dx, \qquad (4.3)
\end{aligned}$$

where we have used that ψ_m is submultiplicative. If the operator T is linear, (4.3) implies the desired estimate by homogeneity. Otherwise, we observe that we have proved this estimate starting from (4.2) which for any $\lambda > 0$ implies

$$\int_{\mathbb{R}^n} \left(\frac{|Tf(x)|}{\lambda}\right)^{p_0} w(x)\, dx \leq C \int_{\mathbb{R}^n} \left(\frac{M^{m+1} f(x)}{\lambda}\right)^{p_0} w(x)\, dx, \qquad f \in \mathcal{D}_T,$$

with C independent of λ. This induces a new family of pairs of functions given by $(|Tf|/\lambda, M^{m+1} f/\lambda)$ to whom we can apply (4.3) to conclude as desired

$$w\big\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\big\} \leq C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx,$$

where C does not depend on $\lambda > 0$.

The main example of operators satisfying (4.2) is given by the commutators of Calderón-Zygmund operators with bounded mean oscillation functions. Let T be a Calderón-Zygmund operator with standard kernel as before. Let b be a function of bounded mean oscillation, that is,

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx < \infty$$

where b_Q stands for the average of b on Q. Then we define the first order commutator

$$C_b^1 f(x) = [b, T] f(x) = b(x)\, Tf(x) - T(b\, f)(x),$$

and for $m \geq 2$, the m-order commutator $C_b^m f(x) = [b, C_b^{m-1}] f(x)$. In this way we have

$$C_b^m f(x) = \int_{\mathbb{R}^n} \big(b(x) - b(y)\big)^m K(x, y)\, f(y), \qquad \text{for a.e. } x \notin \operatorname{supp} f.$$

Note that this definition makes sense for $m \geq 0$ and the commutator of order 0 is nothing but T. The maximal operator that controls the commutator C_b^m is M^{m+1} which is the Hardy-Littlewood maximal function iterated $m+1$-times, namely, in [Per] it is shown that

$$\int_{\mathbb{R}^n} |C_b^m f(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} M^{m+1} f(x)^p\, w(x)\, dx \qquad (4.4)$$

for every $0 < p < \infty$ and $w \in A_\infty$ and all $f \in C_0^\infty$ such that the left hand-side is finite. Thus, Corollaries 4.1 and 4.3 can be applied and we obtain all those weighted estimates for C_b^m.

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