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FRACTIONAL INTEGRALS, POTENTIAL OPERATORS AND TWO-WEIGHT, WEAK TYPE NORM INEQUALITIES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. We prove two-weight, weak type norm inequalities for potential operators and fractional integrals defined on spaces of homogeneous type. We show that the operators in question are bounded from $L^p(v)$ to $L^{q,\infty}(u)$, $1 < p \le q < \infty$, provided the pair of weights (u, v) verifies a Muckenhoupt condition with a "power-bump" on the weight u.

1. Introduction.

A space of homogenous type $(\mathcal{X}, d, \mu)$ is a set $\mathcal{X}$ endowed with a quasimetric d and a non-negative Borel measure μ such that the doubling condition

$$\mu(B(x, 2\,r)) \le C_0\, \mu(B(x, r)) < \infty \tag{1}$$

holds for all $x \in \mathcal{X}$ and $r > 0$, where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$ is the ball with center x and radius r. Since d is a quasimetric, there exists $\kappa \ge 1$ such that

$$d(x, y) \le \kappa \big(d(x, z) + d(z, y)\big), \qquad \text{for all } x, y, z \in \mathcal{X}.$$

Besides, by [1] there exists another quasimetric d', continuous and equivalent to d, for which every ball is open. So, without loss of generality, the quasimetric d is assumed to be continuous and the balls to be open.

We will use the following notation: for any given ball B we write $B = B(x_B, r(B))$ where x_B denotes its center and $r(B)$ its radius. Given $\tau > 0$, we will write $\tau\, B$ for the ball with the same center as B and with radius $r(\tau\, B) = \tau\, r(B)$. In what follows, a weight w will be a non-negative locally integrable function with respect to μ. For any measurable set E we will write $w(E) = \int_E w(x)\, d\mu(x)$.

If C_0 is the smallest constant for which the measure μ satisfies (1), the number $D = \log_2 C_0$ is called the doubling order of μ. Iterating (1) we have

$$\frac{\mu(B_1)}{\mu(B_2)} \le C_\mu \left(\frac{r(B_1)}{r(B_2)}\right)^D, \quad \text{for all balls } B_2 \subset B_1. \tag{2}$$

We additionally assume that all annuli in $\mathcal{X}$ are not empty, that is, for all $x \in \mathcal{X}$ and $0 < r < R$, $B(x, R) \setminus B(x, r) \ne \emptyset$. In this way, μ satisfies the following reverse doubling

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property (see [7]): there exist $\delta > 0$ and $c_\mu > 0$ such that

$$\frac{\mu(B_1)}{\mu(B_2)} \geq c_\mu \left(\frac{r(B_1)}{r(B_2)}\right)^\delta, \quad \text{for all balls } B_2 \subset B_1. \tag{3}$$

Consider $\alpha > 0$. For $f \geq 0$, $f \in L^\infty_c(\mu)$ (f bounded with bounded support), we define the fractional integral of order α as

$$I_\alpha f(x) = \int_{\mathcal{X}} f(y) \frac{d(x,y)^\alpha}{\mu(B(x,d(x,y)))} \, d\mu(y).$$

We devote this paper to prove some two-weight, weak type norm inequalities for these fractional integrals. Precisely, we obtain the following result:

Theorem 1.1. *Let $1 < p \leq q < \infty$ and $\alpha > 0$. Let (u,v) be a pair of weights for which there exists $r > 1$ such that for every ball $B \subset \mathcal{X}$,*

$$r(B)^\alpha \, \mu(B)^{\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{\mu(B)} \int_B u(x)^r \, d\mu(x)\right)^{\frac{1}{r\,q}} \left(\frac{1}{\mu(B)} \int_B v(x)^{1-p'} \, d\mu(x)\right)^{\frac{1}{p'}} \leq C_{u,v} < \infty. \tag{4}$$

Then the fractional operator I_α verifies the following weak type (p,q) inequality

$$\sup_{\lambda>0} \lambda \, u\big\{x \in \mathcal{X} : |I_\alpha f(x)| > \lambda\big\}^{\frac{1}{q}} \leq C \left(\int_{\mathcal{X}} |f(x)|^p \, v(x) \, d\mu(x)\right)^{\frac{1}{p}}. \tag{5}$$

The corresponding strong type analog of (5) was proved in [5]. For a version of this result in the euclidean setting when $p = q$ see [3]. Working in spaces of homogeneous type leads to some difficulties. We will discretize the operator I_α by means of some dyadic sets introduced in [6]. This dyadic structure has a lot of properties in common with the dyadic cubes in the euclidean setting. A very importante difference is that these sets are built "upwards" in the following sense, one starts with a fixed generations and only the ancestors are defined, that is, parents, grandparents, ... Therefore the corresponding dyadic Hardy-Littlewood maximal function will not differentiate since the sets can not be shrunk to a given point $x \in \mathcal{X}$.

The method used to prove Theorem 1.1 can be further applied to derive similar estimates for more general potential operators. Indeed, we are going to see that Theorem 1.1 can be obtained as a consequence of Theorem 1.2 below. We consider potential operators T given by

$$Tf(x) = \int_{\mathcal{X}} K(x,y) \, f(y) \, d\mu(y),$$

where the kernel $K(x,y)$ is a non-negative measurable function defined for $x \neq y$. Associated with T we define a functional φ, given a ball $B \subset \mathcal{X}$,

$$\varphi(B) = \sup_{\substack{x,y \in B \\ d(x,y) \geq c\, r(B)}} K(x,y), \tag{6}$$

where c is some sufficiently small geometric constant (see [6]). We assume that φ satisfies the following hypotheses: there is C_φ such that

(a) The functional φ is doubling, that is,

$$\varphi(2\,B) \leq C_\varphi \, \varphi(B), \qquad \text{for all balls } B \subset \mathcal{X}. \tag{7}$$

(*b*) There exists $\varepsilon > 0$ such that

$$\varphi(B_1)\,\mu(B_1) \leq C_\varphi \left(\frac{r(B_1)}{r(B_2)}\right)^{\varepsilon} \varphi(B_2)\,\mu(B_2), \qquad \text{for all balls } B_1 \subset B_2. \tag{8}$$

We would like to point out that these potential operators are more general than those considered in [5] where two-weight strong type estimates are proved for them, see this reference for more details and some examples.

We prove two-weight, weak type norm inequalities for these potential operators:

Theorem 1.2. *Let $1 < p \leq q < \infty$. Assume that T is given as above and that φ satisfies* (7) *and* (8)*. Let (u, v) be a pair of weights for which there exists $r > 1$ such that for every ball $B \subset \mathcal{X}$,*

$$\varphi(B)\,\mu(B)^{\frac{1}{q}+\frac{1}{p'}} \left(\frac{1}{\mu(B)}\int_B u(x)^r\,d\mu(x)\right)^{\frac{1}{r\,q}} \left(\frac{1}{\mu(B)}\int_B v(x)^{1-p'}\,d\mu(x)\right)^{\frac{1}{p'}} \leq C_{u,v} < \infty. \tag{9}$$

Then the potential operator T verifies the following weak type (p, q) inequality

$$\sup_{\lambda>0} \lambda\, u\big\{x \in \mathcal{X} : |Tf(x)| > \lambda\big\}^{\frac{1}{q}} \leq C \left(\int_{\mathcal{X}} |f(x)|^p\, v(x)\, d\mu(x)\right)^{\frac{1}{p}}. \tag{10}$$

Remark 1.3. When $T = I_\alpha$, the kernel is $K(x, y) = d(x, y)^\alpha/\mu(B(x, d(x, y)))$ and therefore we have $\varphi(B) \approx r(B)^\alpha/\mu(B)$. Note that φ satisfies (7), and (8) with $\varepsilon = \alpha$. Observe that (9) coincides with (4) and therefore Theorem 1.1 is a particular case of Theorem 1.2.

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2. Dyadic sets and the Hardy-Littlewood maximal function.

We are going to consider certain dyadic sets introduced in [6]. Let us fix $\rho = 8\,\kappa^5$. For every (large negative) integer m, there exist a collection of points $\{x_j^k\}$ and a family of sets $\mathcal{D}_m = \{E_j^k\}$ with $k = m, m+1, \dots$ and $j = 1, 2, \dots$ such that

- $B(x_j^k, \rho^k) \subset E_j^k \subset B(x_j^k, \rho^{k+1})$.
- For every $k \geq m$, the sets $\{E_j^k\}_j$ are pairwise disjoint in j, and $\mathcal{X} = \bigcup_j E_j^k$.
- If $m \leq k < l$, then either $E_j^k \bigcap E_i^l = \emptyset$ or $E_j^k \subset E_i^l$.

Thus, we call $\mathcal{D} = \bigcup_m \mathcal{D}_m$ a dyadic cube decomposition of $\mathcal{X}$ and we refer to the sets in $\mathcal{D}$ as dyadic cubes. A dyadic cube will be written as Q, and Q^* will denote the ball that contains Q in such a way that $\frac{1}{\rho}\,Q^* \subset Q \subset Q^*$, that is, if $Q = E_j^k$, then $Q^* = B(x_j^k, \rho^{k+1})$. We will call $\ell(Q) = \frac{r(Q^*)}{\rho}$ $(= \rho^k)$ the "sidelength" of Q and so $Q^* = B(x_Q, \rho\,\ell(Q))$. Note, that the cubes of each $\mathcal{D}_m$ satisfy the dyadic properties above, but, in general, for different values of m these nestedness properties might fail.

We set $\mathcal{D}_m^k = \{E_j^k\}_j = \{Q \in \mathcal{D}_m : \ell(Q) = \rho^k\}$. We will refer to these cubes as the cubes of the generation ρ^k. For $M \geq m$, we also define $\widetilde{\mathcal{D}}_m^M$ which consists of the cubes between the generations ρ^m and ρ^M. Then,

$$\widetilde{\mathcal{D}}_m^m \subset \widetilde{\mathcal{D}}_m^{m+1} \subset \widetilde{\mathcal{D}}_m^{m+2} \subset \cdots \subset \mathcal{D}_m \quad \text{and thus} \quad \mathcal{D}_m = \bigcup_{M=m}^{\infty} \widetilde{\mathcal{D}}_m^M.$$

Associated with the cubes of $\mathcal{D}_m$, the dyadic Hardy-Littlewood maximal function can be defined:

$$\mathcal{M}_m^d f(x) = \sup_{x \in Q \in \mathcal{D}_m} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y).$$

Observe that the lengths of the sides of the cubes in $\mathcal{D}_m$ are at least ρ^m, and so the averages in this maximal operator are taken over sets that are not arbitrarily small.

We will use the following standard notation: f_Q stands for the μ-average of f over Q. For this maximal operator, a Calderón-Zygmund decomposition can be performed which yields the weak type $(1,1)$ for $\mathcal{M}_m^d$. We leave the proofs, which follow the ideas of the classical case, to the reader.

Lemma 2.1 (Calderón-Zygmund decomposition)**.** *Let* $0 \le f \in L^1_{\mathrm{loc}}(\mu)$ *be such that* $f_Q \longrightarrow 0$ *as* $\mu(Q) \to \infty$. *For every* $\lambda > 0$, *we set* $\Omega_\lambda = \{x \in \mathcal{X} : \mathcal{M}_m^d f(x) > \lambda\}$. *Then, there exists a collection of pairwise disjoint dyadic cubes* $\{Q_j^\lambda\}_j \subset \mathcal{D}_m$ *in such a way that*

$$\Omega_\lambda = \bigcup_j Q_j^\lambda \qquad \text{and} \qquad \frac{1}{\mu(Q_j^\lambda)} \int_{Q_j^\lambda} f(y)\, d\mu(y) > \lambda.$$

Furthermore, these cubes are maximal: if $Q \in \mathcal{D}_m$ *and* $f_Q > \lambda$ *then* $Q \subset Q_j^\lambda$ *for some* j. *Besides, for* $Q \supsetneq Q_j^\lambda$ *we have* $f_Q \le \lambda$.

Next, we consider a functional introduced in [3]. For a further generalization see [2].

Definition 2.2. *Given* $r > 1$ *and a weight* u, *define de the set function* A_u^r *on measurable sets* $E \subset \mathcal{X}$ *by*

$$A_u^r(E) = \mu(E)^{\frac{1}{r'}} \Big(\int_E u(x)^r\, d\mu(x) \Big)^{\frac{1}{r}} = \mu(E) \Big(\frac{1}{\mu(E)} \int_E u(x)^r\, d\mu(x) \Big)^{\frac{1}{r}},$$

where the second equality holds provided $\mu(E) > 0$.

Lemma 2.3 ([3, Lemma 3.2])**.** *For any* $r > 1$ *and weight* u, *the set function* A_u^r *has the following properties:*

(*i*) *If* $E \subset F$ *then* $A_u^r(E) \le \left(\dfrac{\mu(E)}{\mu(F)} \right)^{\frac{1}{r'}} A_u^r(F)$.

(*ii*) $u(E) \le A_u^r(E)$.

(*iii*) *If* $\{E_j\}_j$ *is a sequence of disjoint sets and* $\bigcup_j E_j = E$, *then* $\sum_j A_u^r(E_j) \le A_u^r(E)$.

We conclude this section with some auxiliary result to be used later.

Proposition 2.4. *Given* $0 \le f \in L_c^\infty(\mu)$, $0 < q < \infty$, $r > 1$ *and a weight* u, *there exist* $\varepsilon, C > 0$ (*which only depend on the space,* q *and* r) *such that for every* $\lambda > 0$ *there exists a subcollection* $\{R_j^\lambda\}_j$ *of dyadic cubes from the Calderón-Zygmund decomposition of* f *at height* λ (*see Lemma* 2.1), *in such a way that*

$$\frac{1}{\mu(R_j^\lambda)} \int_{R_j^\lambda} |f(y) - f_{R_j^\lambda}|\, d\mu(y) > \varepsilon\, \lambda$$

and

$$\sup_{\lambda>0} \lambda^q\, u\{x \in \mathcal{X} : \mathcal{M}_m^d f(x) > \lambda\} \le C \sup_{\lambda>0} \lambda^q \sum_j A_u^r(R_j^\lambda). \tag{11}$$

Proof. Set $r_0 = \min\{r, \frac{1}{1-q}\}$ for $0 < q < 1$ and $r_0 = r$ for $q \ge 1$. Note that $r_0 > 1$. Since $r_0 \le r$, then $A_u^{r_0}(E) \le A_u^r(E)$ for any measurable set E. Thus, it will be enough to prove (11) for r_0. We can assume that the right-hand side in (11) is finite, otherwise there is nothing to prove. On the other hand, we can suppose that u is bounded and has compact support. To prove the general case, take $u_k = \min\{u, k\}\ \chi_{B(x_0,k)}$ which is bounded and has compact support. Then (11) holds with u_k. Since $u = \lim_k u_k = \sup_k u_k$, by the monotone convergence theorem we get the desired inequality for u.

Given $0 \le f \in L_c^\infty(\mu)$, we apply Lemma 2.1 to f and $\Omega_\lambda = \bigcup_i Q_i^\lambda$ for every $\lambda > 0$. Set $N = 1 + C_\mu\, \rho^{2D} > 1$. Then, $\Omega_{N\lambda} = \bigcup_j Q_j^{N\lambda} \subset \Omega_\lambda$ and by maximality, $Q_j^{N\lambda} \subset Q_i^\lambda$ for some i. Thus, by Lemma 2.3 parts (ii), (iii):

$$\lambda^q\, u(\Omega_{N\lambda}) \le \lambda^q \sum_j A_u^{r_0}(Q_j^{N\lambda}) \le \lambda^q \sum_i A_u^{r_0}\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big). \tag{12}$$

Take $0 < \varepsilon < N^{-p\, r_0'}$. We split the indices i in two sets:

$$\begin{aligned} i \in F \qquad &\text{if} \qquad \frac{1}{\mu(Q_i^\lambda)} \int_{Q_i^\lambda} |f(y) - f_{Q_i^\lambda}|\, d\mu(y) \le \varepsilon\, \lambda, \\ i \in G \qquad &\text{if} \qquad \frac{1}{\mu(Q_i^\lambda)} \int_{Q_i^\lambda} |f(y) - f_{Q_i^\lambda}|\, d\mu(y) > \varepsilon\, \lambda. \end{aligned}$$

Observe that $\{Q_i^\lambda : i \in G\}$ are the desired cubes and so we relabel them as $\{R_j^\lambda\}_j$. On the other hand, we take $x \in \Omega_{N\lambda} \bigcap Q_i^\lambda$. So, $\mathcal{M}_m^d f(x) > N\lambda > \lambda$ and since $f_Q \le \lambda$ for $Q_i^\lambda \subsetneq Q$ we have that $\mathcal{M}_m^d(f\ \chi_{Q_i^\lambda})(x) = \mathcal{M}_m^d f(x)$. Moreover,

$$N\lambda < \mathcal{M}_m^d(f\ \chi_{Q_i^\lambda})(x) \le \mathcal{M}_m^d\big(|f - f_{Q_i^\lambda}|\ \chi_{Q_i^\lambda}\big)(x) + f_{Q_i^\lambda} \le \mathcal{M}_m^d\big(|f - f_{Q_i^\lambda}|\ \chi_{Q_i^\lambda}\big)(x) + C_\mu\, \rho^{2D}\, \lambda$$

where the latter estimate is obtained passing to the parent cube of Q_i^λ. Hence, we have that $\mathcal{M}_m^d\big(|f - f_{Q_i^\lambda}|\ \chi_{Q_i^\lambda}\big)(x) > \lambda$. For $i \in F$, by the weak type $(1,1)$ of $\mathcal{M}_m^d$ we observe

$$\mu\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big) \le \mu\{x \in Q_i^\lambda : \mathcal{M}_m^d\big(|f - f_{Q_i^\lambda}|\ \chi_{Q_i^\lambda}\big)(x) > \lambda\} \le \varepsilon\, \mu(Q_i^\lambda).$$

Since $\Omega_{N\lambda} \bigcap Q_i^\lambda \subset Q_i^\lambda$, by Lemma 2.3 part (i),

$$A_u^{r_0}\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big) \le \left(\frac{\mu\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big)}{\mu(Q_i^\lambda)}\right)^{\frac{1}{r_0'}} A_u^{r_0}(Q_i^\lambda) \le \varepsilon^{\frac{1}{r_0'}}\, A_u^{r_0}(Q_i^\lambda), \qquad \text{for all } i \in F.$$

We plug this estimate into (12):

$$\begin{aligned} \lambda^q \sum_j A_u^{r_0}(Q_j^{N\lambda}) \ &\le\ \lambda^q \sum_{i \in F} A_u^{r_0}\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big) + \lambda^q \sum_{i \in G} A_u^{r_0}\big(\Omega_{N\lambda} \bigcap Q_i^\lambda\big) \\ &\le\ \varepsilon^{\frac{1}{r_0'}}\, \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) + \lambda^q \sum_j A_u^{r_0}(R_j^\lambda). \end{aligned} \tag{13}$$

If $q \geq 1$, then $r_0 = r > 1$ and $q - \frac{1}{r_0'} > 0$. Otherwise, $0 < q < 1$, we have $r_0 \leq \frac{1}{1-q}$ and $q - \frac{1}{r_0'} \geq 0$. In both cases, for every $\Lambda > 0$, by (iii) of Lemma 2.3, we observe

$$\sup_{0<\lambda<\Lambda} \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) \leq \sup_{0<\lambda<\Lambda} \lambda^q A_u^{r_0}(\Omega_\lambda) \leq \sup_{0<\lambda<\Lambda} \lambda^{q-\frac{1}{r_0'}} \|f\|_{L^1(\mu)}^{\frac{1}{r_0'}} \|u\|_{L^{r_0}(\mu)} < \infty,$$

because u and f belong to $L_c^\infty(\mu)$. We take the supremum in (13):

$$\sup_{0<\lambda<\Lambda/N} \lambda^q \sum_i A_u^{r_0}(Q_j^{N\lambda}) \leq \varepsilon^{\frac{1}{r_0'}} \sup_{0<\lambda<\Lambda} \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) + \sup_{0<\lambda<\Lambda} \lambda^q \sum_j A_u^{r_0}(R_j^\lambda),$$

and we get

$$\sup_{0<\lambda<\Lambda} \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) \leq N^q \varepsilon^{\frac{1}{r_0'}} \sup_{0<\lambda<\Lambda} \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) + N^q \sup_{\lambda>0} \lambda^q \sum_j A_u^{r_0}(R_j^\lambda).$$

Note that $0 < \varepsilon < N^{-q\, r_0'}$ and that the first term in the right hand side is finite. Thus we move it to the other side and, as in (12), we obtain

$$\sup_{0<\lambda<\Lambda} \lambda^q u(\Omega_\lambda) \leq \sup_{0<\lambda<\Lambda} \lambda^q \sum_i A_u^{r_0}(Q_i^\lambda) \leq C \sup_{\lambda>0} \lambda^q \sum_j A_u^{r_0}(R_j^\lambda)$$

for every $\Lambda > 0$. This leads to (11) with r_0 instead of r as desired. □

Later on, we will need to estimate the number of cubes (or dilated cubes) of a fixed generation which meet a ball. The doubling condition of the measure provides a bound for this number.

Remark 2.5. Let $\tau \geq 1$ and B be a ball. If $\{Q_j\}_{j=1}^{M_0} \subset \mathcal{D}_m^k$ is a collection of cubes verifying $\tau\, Q_j^* \bigcap B \neq \emptyset$, for $1 \leq j \leq M_0$, then

$$M_0 \leq C_\mu\, \kappa^D \left(2\,\rho\,\tau\,\kappa + \frac{r(B)}{\rho^k}\right)^D.$$

To see this, we set $r = \rho^k$. Since $\tau\, Q_j^* \bigcap B \neq \emptyset$, for $1 \leq j \leq M_0$, then $\tau\, Q_j^* \subset \widetilde{\tau}\, B$ with $\widetilde{\tau} = \kappa\,(2\,\rho\,\tau\,\kappa\, \frac{r}{r(B)} + 1)$. By using (2) we obtain

$$\mu(\widetilde{\tau}\, B) \geq \sum_{j=1}^{M_0} \mu(Q_j) \geq \frac{1}{C_\mu} \left(\frac{r}{\widetilde{\tau}\, r(B)}\right)^D \sum_{j=1}^{M_0} \mu(\widetilde{\tau}\, B) = \frac{1}{C_\mu} \left(\frac{r}{\widetilde{\tau}\, r(B)}\right)^D \mu(\widetilde{\tau}\, B)\, M_0.$$

3. Discretizing the potential operators.

By using the dyadic cubes we introduced before, we are going to get a discrete version of the potential operator T as in [5]. We assume throughout that φ defined in (6) satisfies (7) and (8). We fix $\rho = 8\,\kappa^5$ and a large negative integer m. In the sequel, we will always consider bounded functions $f \geq 0$ with compact support. We set

$$T^m f(x) = \int_{d(x,y)>\rho^m} K(x,y)\, f(y)\, d\mu(y).$$

Note that $T^m f(x) \nearrow T f(x)$ as $m \to -\infty$. For x, y with $d(x,y) > \rho^m$, there exists $k \geq m$ such that $\rho^k < d(x,y) \leq \rho^{k+1}$. Besides, since $\mathcal{X}$ can be written as the pairwise disjoint union of the cubes of $\mathcal{D}_m^k$, there exists an unique cube $Q \in \mathcal{D}_m^k$ with $Q \ni x$ and so $y \in 2\,\kappa\, Q^*$. Thus,

$x, y \in 2\kappa Q^*$ and $d(x,y) > c\, r(2\kappa Q^*)$ for c sufficiently small, namely, $0 < c < (2\kappa\rho)^{-1}$. In this way, by (7) we have

$$K(x,y) \le \varphi(2\kappa Q^*) \le C_1\,\varphi(Q^*) \le C_1 \sum_{Q\in\mathcal{D}_m} \varphi(Q^*)\ \chi_Q(x)\ \chi_{2\kappa Q^*}(y),$$

Thus we define the discrete version of T^m as

$$\mathcal{T}^m f(x) = \sum_{Q\in\mathcal{D}_m} \varphi(Q^*) \int_{2\kappa Q^*} f(y)\,d\mu(y)\ \chi_Q(x) = \sum_{Q\in\mathcal{D}_m} a(Q)\ \chi_Q(x),$$

and we have that $T^m f(x) \le C_1\,\mathcal{T}^m f(x)$. We truncate the later sum in the following way:

$$\mathcal{T}^m f(x) = \sup_{M\ge m} \sum_{k=m}^{M} \sum_{Q\in\mathcal{D}_m^k} a(Q)\ \chi_Q(x) = \sup_{M\ge m} \sum_{Q\in\widetilde{\mathcal{D}}_m^M} a(Q)\ \chi_Q(x) = \sup_{M\ge m} \mathcal{T}^{m,M} f(x).$$

Hence,

$$T^m f(x) \le C_1\,\mathcal{T}^m f(x) = C_1 \sup_{M\ge m} \mathcal{T}^{m,M} f(x) = C_1 \lim_{M\to\infty} \mathcal{T}^{m,M} f(x). \tag{14}$$

Proposition 3.1. *For every $M \ge m$ and for every $0 \le f \in L_c^\infty(\mu)$, we have that $0 \le \mathcal{T}^{m,M} f \in L_c^\infty(\mu)$ and thus $\mathcal{T}^{m,M} f \in L^q(\mu)$ for all $1 \le q \le \infty$.*

Proof. Let B be a ball such that $\operatorname{supp} f \subset B$ and $Q \in \mathcal{D}_m^k$ such that $2\kappa Q^*$ meets B (otherwise $a(Q) = 0$). By Remark 2.5 with $\tau = 2\kappa \ge 1$ we have

$$\#\{Q \in \mathcal{D}_m^k : 2\kappa Q^* \cap B \ne \emptyset\} \le C_\mu\,\kappa^D \left(4\rho\kappa^2 + \frac{r(B)}{\rho^k}\right)^D \le C_\mu\,\kappa^D \left(4\rho\kappa^2 + \frac{r(B)}{\rho^m}\right)^D = \widetilde{M}.$$

Besides, $2\kappa Q^* \subset \widetilde{\tau}_k B$ with $\widetilde{\tau}_k = \kappa\left(4\rho\kappa^2 \frac{\rho^k}{r(B)} + 1\right)$. Since $\widetilde{\tau}_k \le \widetilde{\tau}_M$, it follows that $Q \subset 2\kappa Q^* \subset \widetilde{\tau}_k B \subset \widetilde{\tau}_M B$ and $\chi_Q(x) \le \chi_{\widetilde{\tau}_M B}(x)$. On the other hand, by (8),

$$\begin{aligned} a(Q) &= \varphi(Q^*) \int_{2\kappa Q^*} f(y)\,d\mu(y) \le C_\mu\,(2\kappa)^D\,\|f\|_{L^\infty(\mu)}\,\varphi(Q^*)\,\mu(Q^*) \\ &\le C_\mu\,(2\kappa)^D\,\|f\|_{L^\infty(\mu)}\,C_\varphi \left(\frac{r(Q^*)}{r(\widetilde{\tau}_M B)}\right)^\varepsilon \varphi(\widetilde{\tau}_M B)\,\mu(\widetilde{\tau}_M B) \\ &\le C_\mu\,(2\kappa)^D\,\|f\|_{L^\infty(\mu)}\,C_\varphi \left(\frac{\rho^{M+1}}{r(\widetilde{\tau}_M B)}\right)^\varepsilon \varphi(\widetilde{\tau}_M B)\,\mu(\widetilde{\tau}_M B) = C\,\|f\|_{L^\infty(\mu)} \end{aligned}$$

Putting these estimates together, we conclude as desired

$$\mathcal{T}^{m,M} f(x) = \sum_{k=m}^{M} \sum_{\substack{Q\,\in\,\mathcal{D}_m^k \\ 2\kappa Q^*\,\cap\, B \ne \emptyset}} a(Q)\ \chi_Q(x) \le C\,\|f\|_{L^\infty(\mu)}\,\widetilde{M}\,(M - m + 1)\ \chi_{\widetilde{\tau}_M B}(x).$$

□

4. Auxiliary results.

This section is devoted to get some lemmas which will be used to prove Theorem 1.2. The following result was originally obtained in [6] in the euclidean case ($\mathbb{R}^d$ with the Lebesgue measure), and for the classical fractional integrals. In our case of spaces of homogeneous type, it was essentially obtained in [5]. Although the hypotheses assumed in [5] are stronger, it is not difficult to realize that the same arguments work for our the potential operators T. We sketch the proof for completeness.

Lemma 4.1 ([5])**.** *Let $0 \le f \in L^1_{\mathrm{loc}}(\mu)$. There exists C (only depending on the space and φ) such that for every $Q_0 \in \mathcal{D}_m$,*

$$\sum_{\substack{Q \in \mathcal{D}_m \\ Q \subset Q_0}} \varphi(Q^*)\,\mu(Q^*) \int_{2\,\kappa\,Q^*} f(x)\,d\mu(x) \le C\,\varphi(Q_0^*)\,\mu(Q_0^*) \int_{\kappa\,(2\,\kappa+1)\,Q_0^*} f(x)\,d\mu(x).$$

Proof. We write

$$\mathcal{D}_m(Q_0) = \{Q \in \mathcal{D}_m : Q \subset Q_0\}; \qquad \mathcal{D}_m^k(Q_0) = \{Q \in \mathcal{D}_m(Q_0) : \ell(Q) = \rho^{-k}\,\ell(Q_0)\}, \quad k \ge 0.$$

Note that $\mathcal{D}_m^k(Q_0) = \emptyset$ for $\rho^{-k}\,\ell(Q_0) < \rho^m$. In any case, for $Q \subset Q_0$ we have $Q^* \subset 2\,\kappa\,Q_0^*$ and $2\,\kappa\,Q^* \subset \kappa\,(2\,\kappa+1)\,Q_0^*$. Thus by (2), (7) and (8) we get

$$\begin{aligned}
\sum_{\substack{Q \in \mathcal{D}_m \\ Q \subset Q_0}} \varphi(Q^*)\,\mu(Q^*) \int_{2\,\kappa\,Q^*} f(x)\,d\mu(x) &= \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{D}_m^k(Q_0)} \varphi(Q^*)\,\mu(Q^*) \int_{2\,\kappa\,Q^*} f(x)\,d\mu(x) \\
&\le C_\varphi\,(2\,\kappa)^{-\varepsilon}\,\varphi(2\,\kappa\,Q_0^*)\,\mu(2\,\kappa\,Q_0^*) \sum_{k=0}^{\infty} \rho^{-k\,\varepsilon} \sum_{Q \in \mathcal{D}_m^k(Q_0)} \int_{2\,\kappa\,Q^*} f(x)\,\chi_{2\,\kappa\,Q^*}(x)\,d\mu(x) \\
&\le C\,\varphi(Q_0^*)\,\mu(Q_0^*) \sum_{k=0}^{\infty} \rho^{-k\,\varepsilon} \int_{\kappa\,(2\,\kappa+1)\,Q_0^*} f(x) \Big(\sum_{Q \in \mathcal{D}_m^k(Q_0)} \chi_{2\,\kappa\,Q^*}(x) \Big)\,d\mu(x).
\end{aligned}$$

Set $\rho^{k_0} = \ell(Q_0)$. Then $\mathcal{D}_m^k(Q_0) \subset \mathcal{D}_m^{k_0-k}$ and setting $B = B(x, \rho^{-k}\,\ell(Q_0)) = B(x, \rho^{k_0-k})$ we have

$$\sum_{Q \in \mathcal{D}_m^k(Q_0)} \chi_{2\,\kappa\,Q^*}(x) \le \#\{Q \in \mathcal{D}_m^{k_0-k} : 2\,\kappa\,Q^* \cap B \neq \emptyset\} \le C_\mu\,\kappa^D\,(4\,\rho\,\kappa^2+1)^D,$$

where the latter estimate holds by Remark 2.5 when $k_0 - k \ge m$, and it is trivial when $k_0 - k < m$ since $\mathcal{D}_m^{k_0-k} = \emptyset$. To complete the estimate we only have to used that $\varepsilon > 0$ and $\rho > 1$. $\square$

Lemma 4.2. *Let $M \ge m$, $0 \le f \in L^\infty_c(\mu)$ and $Q_0 \in \mathcal{D}_m$. Then*

$$\frac{1}{\mu(Q_0)} \int_{Q_0} |\mathcal{T}^{m,M} f(x) - (\mathcal{T}^{m,M} f)_{Q_0}|\,d\mu(x) \le C\,\varphi(Q_0^*) \int_{\kappa\,(2\,\kappa+1)\,Q_0^*} f(x)\,d\mu(x),$$

where C depends on the space and φ.

Proof. We split $\mathcal{T}^{m,M} f$ as

$$\mathcal{T}^{m,M} f(x)\;\chi_{Q_0}(x) = \sum_{\substack{Q \in \widetilde{\mathcal{D}}_m^M \\ Q \subset Q_0}} a(Q)\;\chi_Q(x) + \Big(\sum_{\substack{Q \in \widetilde{\mathcal{D}}_m^M \\ Q_0 \subsetneq Q}} a(Q) \Big)\;\chi_{Q_0}(x) = I(x) + II\;\chi_{Q_0}(x),$$

where we can observe that the second term is constant over Q_0. If $Q_0 \notin \widetilde{\mathcal{D}}_m^M$, then the second term does not appear since there is no cube in $\widetilde{\mathcal{D}}_m^M$ containing Q_0. In any case, applying Lemma 4.1 we conclude

$$\frac{1}{\mu(Q_0)} \int_{Q_0} |\mathcal{T}^{m,M} f(x) - (\mathcal{T}^{m,M} f)_{Q_0}|\,d\mu(x) \le \frac{2}{\mu(Q_0)} \int_{Q_0} I(x)\,d\mu(x)$$

$$\leq \frac{C}{\mu(Q_0^*)} \sum_{\substack{Q\in\mathcal{D}_m \\ Q\subset Q_0}} \varphi(Q^*)\,\mu(Q^*) \int_{2\,\kappa\,Q^*} f(y)\,d\mu(y) \leq C\,\varphi(Q_0^*) \int_{\kappa\,(2\,\kappa+1)\,Q_0^*} f(y)\,d\mu(y).$$

□

Lemma 4.3. *Let $f \geq 0$, $f \in L^1_{\rm loc}(\mu)$. Let $Q_0 \in \mathcal{D}_m$ and $s > 0$, such that*

$$\varphi(Q_0^*) \int_{\kappa\,(2\,\kappa+1)\,Q_0^*} f(y)\,d\mu(y) > s,$$

Then, there exists $P \in \mathcal{D}_m$ with $\ell(P) = \ell(Q_0)$ such that $P \bigcap \kappa\,(2\,\kappa+1)\,Q_0^* \neq \emptyset$;

$$\kappa\,(2\,\kappa+1)\,Q_0^* \subset \big(2\,\kappa^3\,(2\,\kappa+1)+\kappa\big)\,P^* = \tau_1\,P^*, \qquad P^* \subset \big(\kappa+\kappa^2\,(1+\kappa\,(2\,\kappa+1))\big)\,Q_0^* = \tau_2\,Q_0^*$$

and, for some C, which depends on $\mathcal{X}$ and φ,

$$\varphi(P^*) \int_P f(y)\,d\mu(y) > C\,s.$$

Proof. Put $\tau = \kappa\,(2\,\kappa+1)$ and $\rho^{k_0} = \ell(Q_0)$. Let $Q \subset \mathcal{D}_m$ with $\ell(Q) = \ell(Q_0)$ and $Q \bigcap \tau\,Q_0^* \neq \emptyset$. Then, $\kappa\,(2\,\kappa+1)\,Q_0^* \subset \tau_1\,Q^*$ and $Q^* \subset \tau_2\,Q_0^*$, where τ_1, τ_2 are the constants defined above. By Remark 2.5 with $B = \tau\,Q_0^*$,

$$\#\{Q \in \mathcal{D}_m^{k_0} : Q \cap \tau\,Q_0^* \neq \emptyset\} \leq \#\{Q \in \mathcal{D}_m^{k_0} : Q^* \cap \tau\,Q_0^* \neq \emptyset\} \leq C_\mu\,\kappa^D\,(2\,\rho\,k + \tau\,\rho) = M_0.$$

Note that M_0 only depends on the space. Since $\mathcal{X}$ can be written as the pairwise disjoint union of the cubes in $\mathcal{D}_m^{k_0}$, there exist $\{Q_j\}_{j=1}^J \subset \mathcal{D}_m^{k_0}$ with $Q_j \cap \tau\,Q_0^* \neq \emptyset$ and $\tau\,Q_0^* \subset \bigcup_{j=1}^J Q_j$. Moreover, we know that $J \leq M_0$. If for all $1 \leq j \leq J$

$$\int_{Q_j} f(x)\,d\mu(x) \leq \frac{s}{\varphi(Q_0^*)\,M_0}, \tag{15}$$

then we get into a contradiction

$$\int_{\tau\,Q_0^*} f(x)\,d\mu(x) \leq \sum_{j=1}^J \int_{Q_j} f(x)\,d\mu(x) \leq \frac{s}{\varphi(Q_0^*)\,M_0}\,J \leq \frac{s}{\varphi(Q_0^*)}.$$

Therefore, at least one of these cubes, say P, does not verify (15), and so

$$\varphi(P^*) \int_P f(y)\,d\mu(y) > \frac{\varphi(P^*)}{\varphi(Q_0^*)\,M_0}\,s \geq C\,s.$$

The last estimate follows observing that $Q_0^* \subset \tau\,Q_0^* \subset \tau_1\,P^*$ and, by (8),

$$\varphi(Q_0^*)\,\mu(Q_0^*) \leq C_\varphi \left(\frac{r(Q_0^*)}{r(\tau_1\,P^*)}\right)^\varepsilon \varphi(\tau_1\,P^*)\,\mu(\tau_1\,P^*) \leq C\,\varphi(P^*)\,\mu(Q_0^*),$$

where we have used (7) and that P^* and Q^* have comparable measures. □

5. Proof of Theorem 1.2.

We make some reductions. It is clear, that it is enough to obtain (10) for $0 \le f \in L^\infty_c(\mu)$. Furthermore, $T^m f(x) \nearrow T f(x)$ as $m \to -\infty$. Thus by (14) and by the monotone convergence theorem it is enough to get

$$\sup_{\lambda>0} \lambda^q\, u\big\{x \in \mathcal{X} : \mathcal{T}^{m,M} f(x) > \lambda\big\} \le C \Big(\int_{\mathcal{X}} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}}$$

with C independent of m and $M \ge m$. We fix $0 \le f \in L^\infty_c(\mu)$, a large negative integer m and $M \ge m$. For $Q_0 \in \mathcal{D}^m_m$, we have a sequence of cubes $Q_0 \subset Q_1 \subset Q_2 \subset \dots$, with $Q_k \in \mathcal{D}^{m+k}_m$. In this way,

$$\mathcal{T}^{m,M} f(y)\; \chi_{Q_0}(y) = \sum_{k=0}^{M-m} \sum_{Q\in\mathcal{D}^{k+m}_m} a(Q)\; \chi_Q(y)\; \chi_{Q_0}(y) = \Big(\sum_{k=0}^{M-m} a(Q_k)\Big)\; \chi_{Q_0}(y)$$

and $\mathcal{T}^{m,M} f$ is constant on Q_0. Then, $\mathcal{T}^{m,M} f(x) \le \mathcal{M}^d_m(\mathcal{T}^{m,M} f)(x)$ for $x \in Q_0$. Since this is done for any $Q_0 \in \mathcal{D}^m_m$ and these cubes cover $\mathcal{X}$, we conclude that $\mathcal{T}^{m,M} f(x) \le \mathcal{M}^d_m(\mathcal{T}^{m,M} f)(x)$ for all $x \in \mathcal{X}$, and so

$$\lambda^q\, u\{x \in \mathcal{X} : \mathcal{T}^{m,M} f(x) > \lambda\} \le \lambda^q\, u\{x \in \mathcal{X} : \mathcal{M}^d_m(\mathcal{T}^{m,M} f)(x) > \lambda\}. \tag{16}$$

By Proposition 3.1, we know that $0 \le \mathcal{T}^{m,M} f \in L^\infty_c(\mu)$. Thus, we use Proposition 2.4 and there exist $\varepsilon, C > 0$ such that for every $\lambda > 0$, there is collection of pairwise disjoint dyadic cubes $\{R^\lambda_j\}_j$ in such a way that the following conditions hold:

$$\frac{1}{\mu(R^\lambda_j)} \int_{R^\lambda_j} |\mathcal{T}^{m,M} f(y) - (\mathcal{T}^{m,M} f)_{R^\lambda_j}|\, d\mu(y) > \varepsilon\, \lambda$$

and

$$\sup_{\lambda>0} \lambda^q\, u\{x \in \mathcal{X} : \mathcal{M}^d_m(\mathcal{T}^{m,M} f)(x) > \lambda\} \le C \sup_{\lambda>0} \lambda^q \sum_j A^r_u(R^\lambda_j). \tag{17}$$

By Lemma 4.2 we get

$$\varepsilon\, \lambda < \frac{1}{\mu(R^\lambda_j)} \int_{R^\lambda_j} |\mathcal{T}^{m,M} f(x) - (\mathcal{T}^{m,M} f)_{R^\lambda_j}|\, d\mu(x) \le C_1\, \varphi((R^\lambda_j)^*) \int_{\kappa\,(2\,\kappa+1)\,(R^\lambda_j)^*} f(x)\, d\mu(x).$$

Lemma 4.3 with $s = \varepsilon\, \lambda\, C_1^{-1}$ assures the existence of $P^\lambda_j \in \mathcal{D}_m$ with $\ell(P^\lambda_j) = \ell(R^\lambda_j)$;

$$P^\lambda_j \bigcap \kappa\,(2\,\kappa+1)\,(R^\lambda_j)^* \ne \emptyset, \qquad \kappa\,(2\,\kappa+1)\,(R^\lambda_j)^* \subset \tau_1\,(P^\lambda_j)^*, \qquad (P^\lambda_j)^* \subset \tau_2\,(R^\lambda_j)^*$$

where τ_1 and τ_2 are the constants given in that result; and

$$\varphi((P^\lambda_j)^*) \int_{P^\lambda_j} f(y)\, d\mu(y) > C_2\, \lambda. \tag{18}$$

Let us fix $J \in \mathbb{N}$. From the family of cubes $\{P^\lambda_j\}_{j=1}^J$ we take a maximal subcollection $\{S_i\}_{i=1}^I$ with $1 \le I \le J$. In this way, every S_i is actually P^λ_j for some j and hence (18) holds with S_i. Moreover, if $1 \le j \le J$, there exists $1 \le i \le I$ such that $P^\lambda_j \subset S_i$. Then $\tau_1\,(P^\lambda_j)^* \subset \kappa\,(\tau_1+1)\,S^*_i$ and it follows that $R^\lambda_j \subset \kappa\,(\tau_1+1)\,S^*_i = \widetilde{\tau}_1\,S^*_i$. Notice that the cubes

$\{R_j^\lambda\}_{j=1}^J$ are pairwise disjoint. So by Lemma 2.3 part (iii) and (18), we observe

$$\begin{aligned}
\lambda^q \sum_{j=1}^J A_u^r(R_j^\lambda) &\le \lambda^q \sum_{i=1}^I A_u^r\Big(\bigcup_{\substack{1\le j\le J\\ R_j^\lambda \subset \widetilde{\tau}_1 S_i^*}} R_j^\lambda\Big) \le \lambda^q \sum_{i=1}^I A_u^r(\widetilde{\tau}_1 S_i^*)\\
&\le \frac{1}{C_2^q} \sum_{i=1}^I \mu(\widetilde{\tau}_1 S_i^*)\Big(\frac{1}{\mu(\widetilde{\tau}_1 S_i^*)}\int_{\widetilde{\tau}_1 S_i^*} u^r\,d\mu\Big)^{\frac{1}{r}} \Big(\varphi(S_i^*)\int_{S_i} f\,d\mu\Big)^q\\
&\le C\sum_{i=1}^I \mu(\widetilde{\tau}_1 S_i^*)\,\varphi(S_i^*)^q\Big(\frac{1}{\mu(\widetilde{\tau}_1 S_i^*)}\int_{\widetilde{\tau}_1 S_i^*} u^r\,d\mu\Big)^{\frac{1}{r}}\Big(\int_{S_i} v^{1-p'}\,d\mu\Big)^{\frac{q}{p'}}\Big(\int_{S_i} f^p\,v\,d\mu\Big)^{\frac{q}{p}},
\end{aligned}$$

where in the later estimate we have used Hölder's inequality. Observe that (8) and (2) imply that $\varphi(S_i^*) \le C\,\varphi(\widetilde{\tau}_1 S_i^*)$ since

$$\varphi(S_i^*)\,\mu(S_i^*) \le C_\varphi\,\widetilde{\tau}_1^{-\varepsilon}\,\varphi(\widetilde{\tau}_1 S_i^*)\,\mu(\widetilde{\tau}_1 S_i^*) \le C_\varphi\,\widetilde{\tau}_1^{-\varepsilon}\,C_\mu\,\widetilde{\tau}_1^{D}\,\varphi(\widetilde{\tau}_1 S_i^*)\,\mu(S_i^*).$$

Besides, by (9), we observe

$$\begin{aligned}
&\mu(\widetilde{\tau}_1 S_i^*)\,\varphi(S_i^*)^q\Big(\frac{1}{\mu(\widetilde{\tau}_1 S_i^*)}\int_{\widetilde{\tau}_1 S_i^*} u^r\,d\mu\Big)^{\frac{1}{r}}\Big(\int_{S_i} v^{1-p'}\,d\mu\Big)^{\frac{q}{p'}}\\
&\le C\Big[\mu(\widetilde{\tau}_1 S_i^*)^{\frac{1}{q}+\frac{1}{p'}}\,\varphi(\widetilde{\tau}_1 S_i^*)\Big(\frac{1}{\mu(\widetilde{\tau}_1 S_i^*)}\int_{\widetilde{\tau}_1 S_i^*} u^r\,d\mu\Big)^{\frac{1}{r\,q}}\Big(\frac{1}{\mu(\widetilde{\tau}_1 S_i^*)}\int_{\widetilde{\tau}_1 S_i^*} v^{1-p'}\,d\mu\Big)^{\frac{1}{p'}}\Big]^q\\
&\le C\,(C_{u,v})^q.
\end{aligned}$$

Thus since $q/p \ge 1$ and using that the cubes $\{S_i\}_{i=1}^I$ are pairwise disjoint, we have

$$\lambda^q\sum_{j=1}^J A_u^r(R_j^\lambda) \le C\sum_{i=1}^I\Big(\int_{S_i} f^p\,v\,d\mu\Big)^{\frac{q}{p}} \le C\Big(\int_{\bigcup_{i=1}^I S_i} f^p\,v\,d\mu\Big)^{\frac{q}{p}} \le C\Big(\int_{\mathcal{X}} f^p\,v\,d\mu\Big)^{\frac{q}{p}}.$$

This estimate, after taking limit as $J\to\infty$, (17) and (16) allow us to complete the proof. □

References

[1] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. 33 (1979), 257–270.

[2] D. Cruz-Uribe and A. Fiorenza, *The A_∞ property for Young functions and weighted norm inequalities*, Houston J. Math. **28** (2002), no. 1, 169–182.

[3] D. Cruz-Uribe and C. Pérez, *Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators*, Indiana Univ. Math. J. **49** (2000), no. 2, 697–721.

[4] J.M. Martell, *Desigualdades con pesos en el Análisis de Fourier: de los espacios de tipo homogéneo a las medidas no doblantes*, Ph. D. Thesis, Universidad Autónoma de Madrid, 2001.

[5] C. Pérez and R.L. Wheeden, *Uncertainty principle estimates for vector fields*, J. Functional Analysis **181** (2001), 146–188.

[6] E.T. Sawyer and R.L. Wheeden, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

[7] R.L. Wheeden, *A characterization of some weighted norm inequalities for the fractional maximal function*, Studia Math. **107** (1993), 251–272.