Topics in Clifford Analysis: Special Volume in Honor of Wolfgang Sprößig, 105-124, Trends Math., Birkhäuser, Cham, 2019.
DOI 10.1007/978-3-030-23854-4_5

A FATOU THEOREM AND POISSON'S INTEGRAL REPRESENTATION FORMULA FOR ELLIPTIC SYSTEMS IN THE UPPER-HALF SPACE

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Dedicated to Wolfgang Sprössig on the occasion of his 70th birthday

Abstract. Let L be a second-order, homogeneous, constant (complex) coefficient elliptic system in $\mathbb{R}^n$. The goal of this article is to prove a Fatou-type result, regarding the a.e. existence of the nontangential boundary limits of any null-solution u of L in the upper-half space, whose nontangential maximal function satisfies an integrability condition with respect to the weighted Lebesgue measure $(1+|x'|^{n-1})^{-1}dx'$ in $\mathbb{R}^{n-1}\equiv\partial\mathbb{R}^n_+$. This is the best result of its kind in the literature. In addition, we establish a naturally accompanying integral representation formula involving the Agmon-Douglis-Nirenberg Poisson kernel for the system L. Finally, we use this machinery to derive well-posedness results for the Dirichlet boundary value problem for L in $\mathbb{R}^n_+$ formulated in a manner which allows for the simultaneous treatment of a variety of function spaces.

1. Introduction

Let $n\in\mathbb{N}$ with $n\geq 2$ denote the dimension of the Euclidean ambient space. Fix an integer $M\in\mathbb{N}$ and consider the second-order, homogeneous, $M\times M$ system, with constant complex coefficients in $\mathbb{R}^n$, written (with the usual convention of summation over repeated indices in place) as

$$Lu := \Big(a^{\alpha\beta}_{rs}\partial_r\partial_s u_\beta\Big)_{1\leq\alpha\leq M}, \tag{1.1}$$

when acting on vector-valued distributions $u=(u_\beta)_{1\leq\beta\leq M}$ in an open subset of $\mathbb{R}^n$. Throughout, we shall assume that L is `elliptic` in the sense that there exists a real number $c>0$ such that the following Legendre-Hadamard condition is satisfied:

$$\begin{gathered}\mathrm{Re}\left[a^{\alpha\beta}_{rs}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\right]\geq c|\xi|^2|\eta|^2 \quad \text{for every}\\ \xi=(\xi_r)_{1\leq r\leq n}\in\mathbb{R}^n \quad\text{and}\quad \eta=(\eta_\alpha)_{1\leq\alpha\leq M}\in\mathbb{C}^M.\end{gathered}\tag{1.2}$$

Examples to keep in mind are the Laplacian and the Lamé system.

As is known from the classical work of S. Agmon, A. Douglis, and L. Nirenberg in [1] and [2], every operator L as in (1.1)-(1.2) has a Poisson kernel, denoted by P^L (an object whose

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Date: May 29, 2018.

2010 *Mathematics Subject Classification.* Primary: 31A20, 35C15, 35J57, 42B37. Secondary: 42B25.

Key words and phrases. Fatou type theorem, Poisson integral representation formula, nontangential maximal function, second-order elliptic system, Dirichlet problem, Hardy-Littlewood maximal operator, Green function.

The first and second authors acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV2015-0554). They also acknowledge support from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC agreement no. 615112 HAPDEGMT. The third author has been supported in part by a Simons Foundation grant # 426669, the fourth author has been supported in part by Simons Foundation grant # 318658, while the fifth author has been supported in part by the Simons Foundation grant # 281566. This work has been possible thanks to the support and hospitality of *Temple University* (USA), *University of Missouri* (USA), and *ICMAT, Consejo Superior de Investigaciones Científicas* (Spain). The authors express their gratitude to these institutions.

properties mirror the most basic characteristics of the classical harmonic Poisson kernel). For details, see Theorem 2.3 below.

The main goal of this paper is to establish a Fatou-type theorem and a naturally accompanying Poisson integral representation formula for null-solutions of an elliptic system L, as above, in the upper-half space defined in the upper-half space

$$\mathbb{R}^n_+ := \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} :\, x_n > 0\big\}. \tag{1.3}$$

Among other things, this is going to yield versatile well-posedness results for the Dirichlet problem in $\mathbb{R}^n_+$ for such systems. Prior to formulating the main result, some comments on the notation used are in order. Given a function u defined in $\mathbb{R}^n_+$, by $\mathcal{N}_\kappa u$ we shall denote the nontangential maximal function of u with aperture κ; see (2.2) for a precise definition. Next, by $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}$ we denote the (κ-)nontangential limit of the given function u on the boundary of the upper half-space (canonically identified with $\mathbb{R}^{n-1}$), as defined in (2.3). Finally, given any $d \in \mathbb{N}$, the Lebesgue measure in $\mathbb{R}^d$ will be denoted by $\mathscr{L}^d$.

Theorem 1.1 (A Fatou-Type Theorem and Poisson's Integral Formula)**.** *Let L be an $M \times M$ system with constant complex coefficients as in* (1.1)-(1.2)*, and fix some aperture parameter $\kappa > 0$. Then*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \\[4pt] \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x')\, \frac{dx'}{1+|x'|^{n-1}} < \infty, \end{cases} \tag{1.4}$$

implies that

$$\begin{cases} u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ exists at } \mathscr{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1}, \\[4pt] u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ belongs to } \ \Big[L^1\Big(\mathbb{R}^{n-1}\,,\, \dfrac{dx'}{1+|x'|^{n-1}}\Big)\Big]^M, \\[4pt] u(x',t) = \Big(P^L_t * \big(u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\big)\Big)(x') \ \text{ for each } \ (x',t) \in \mathbb{R}^n_+, \end{cases} \tag{1.5}$$

where $P^L = \big(P^L_{\beta\alpha}\big)_{1\leq\beta,\alpha\leq M}$ is the Agmon-Douglis-Nirenberg Poisson kernel for the system L in $\mathbb{R}^n_+$ and $P^L_t(x') := t^{1-n}P^L(x'/t)$ for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.

This refines [7, Theorem 6.1, p. 956]. We also wish to remark that even in the classical case when $L := \Delta$, the Laplacian in $\mathbb{R}^n$, Theorem 1.1 is more general (in the sense that it allows for a larger class of functions) than the existing results in the literature. Indeed, the latter typically assume an L^p integrability condition for the harmonic function which, in the range $1 < p < \infty$, implies our weighted L^1 integrability condition for the nontangential maximal function demanded in (1.4). In this vein see, e.g., [4, Theorems 4.8-4.9, pp. 174-175], [13, Corollary, p. 200], [14, Proposition 1, p. 119].

A special case of Theorem 1.1 worth singling out is as follows. Recall the Agmon-Douglis-Nirenberg kernel function

$$\begin{gathered} K^L \in \textstyle\bigcap_{\varepsilon>0} \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon)\big)\big]^{M\times M}, \\ K^L(x) := P^L_t(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+, \end{gathered} \tag{1.6}$$

associated with the elliptic system L as in Theorem 2.3. Fix some $t_o > 0$ and define

$$u(x) := K^L(x', t+t_o) = P^L_{t+t_o}(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+. \tag{1.7}$$

Then

$$u \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+}\big)\big]^{M\times M}, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad u\big|_{\partial\mathbb{R}^n_+} = P^L_{t_o} \ \text{ on } \ \mathbb{R}^{n-1}. \tag{1.8}$$

In addition, (2.12) ensures that there exists a finite constant $C_{t_o} > 0$ with the property that $|u(x)| \leq C_{t_o}(1+|x|)^{1-n}$ for each $x \in \mathbb{R}^n_+$. For each fixed $\kappa > 0$ this readily entails

$$\big(\mathcal{N}_\kappa u\big)(x') \leq \frac{C}{1+|x'|^{n-1}}, \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{1.9}$$

This, in turn, guarantees that the finiteness condition demanded in (1.9) is presently satisfied. Having verified all hypotheses of Theorem 1.1, from the Poisson integral representation formula in the last line of (1.5) and (1.7)-(1.8) we conclude that

$$P^L_{t+t_o}(x') = u(x',t) = \big(P^L_t * P^L_{t_o}\big)(x') \ \text{ for all } \ (x',t) \in \mathbb{R}^n_+, \tag{1.10}$$

where the convolution between the two matrix-valued functions in (1.10) is understood in a natural fashion, taking into account the algebraic multiplication of matrices. Ultimately, this provides an elegant proof of the following result (first established in [7, Theorem 5.1] via a conceptually different argument):

the Agmon-Douglis-Nirenberg Poisson kernel P^L associated with any given elliptic system L as in Theorem 2.3 satisfies the semi-group property $P^L_{t_0+t_1} = P^L_{t_0} * P^L_{t_1}$ for all $t_0, t_1 > 0$. (1.11)

Here is another important corollary of Theorem 1.1, which refines [7, Theorem 3.2, p. 935].

Corollary 1.2 (A General Uniqueness Result)**.** *Let L be an $M \times M$ system with constant complex coefficients as in* (1.1)-(1.2)*, and fix an aperture parameter $\kappa > 0$. Then*

$$\left.\begin{array}{l} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \\[6pt] \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x')\,\frac{dx'}{1+|x'|^{n-1}} < +\infty, \\[6pt] u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathscr{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}, \end{array}\right\} \Longrightarrow u = 0 \ \text{ in } \ \mathbb{R}^n_+. \tag{1.12}$$

Theorem 1.1 also interfaces tightly with the topic of boundary value problems. To elaborate on this aspect, we need more notation. Denote by $\mathbb{M}$ the collection of all (equivalence classes of) Lebesgue measurable functions $f : \mathbb{R}^{n-1} \to [-\infty, \infty]$ such that $|f| < \infty$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Also, call a subset $\mathbb{Y}$ of $\mathbb{M}$ a `function lattice` if the following properties hold:

(i) whenever $f, g \in \mathbb{M}$ satisfy $0 \leq f \leq g$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$ and $g \in \mathbb{Y}$ then necessarily $f \in \mathbb{Y}$;
(ii) $0 \leq f \in \mathbb{Y}$ implies $\lambda f \in \mathbb{Y}$ for every $\lambda \in (0,\infty)$;
(iii) $0 \leq f, g \in \mathbb{Y}$ implies $\max\{f, g\} \in \mathbb{Y}$.

In passing, note that, granted (i), one may replace (ii)-(iii) above by the condition: $0 \leq f, g \in \mathbb{Y}$ implies $f + g \in \mathbb{Y}$. As usual, we set $\log_+ t := \max\big\{0\,,\,\ln t\big\}$ for each $t \in (0,\infty)$. Also, the symbol $\mathcal{M}$ is reserved for the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$; see (2.6).

We are now in a position to discuss the following refinement of [7, Theorem 1.1, p. 915].

Corollary 1.3 (A Template for the Dirichlet Problem)**.** *Let L be an $M \times M$ system with constant complex coefficients as in* (1.1)-(1.2)*, and fix an aperture parameter $\kappa > 0$. Also, assume that*

$$\mathbb{Y} \subseteq L^1\Big(\mathbb{R}^{n-1}\,,\,\frac{dx'}{1+|x'|^{n-1}}\Big), \quad \mathbb{Y} \textit{ is a function lattice}, \tag{1.13}$$

and that

$$\begin{array}{l}\mathbb{X} \textit{ is a collection of } \mathbb{C}^M\textit{-valued measurable} \\ \textit{functions on } \mathbb{R}^{n-1} \textit{ satisfying } \mathcal{M}\mathbb{X} \subseteq \mathbb{Y}.\end{array} \tag{1.14}$$

Then the $(\mathbb{X}, \mathbb{Y})$*-Dirichlet boundary value problem for the system* L *in the upper-half space, formulated as*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \\ Lu = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+, \\ \mathcal{N}_\kappa u \in \mathbb{Y}, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathbb{X}, \end{cases} \tag{1.15}$$

has a unique solution. Moreover, the solution u *of* (1.15) *is given by*

$$u(x) = (P^L_t * f)(x') \quad \textit{for all} \quad x = (x', t) \in \mathbb{R}^{n-1} \times (0, \infty) = \mathbb{R}^n_+, \tag{1.16}$$

where P^L *is the Poisson kernel for* L *in* $\mathbb{R}^n_+$*, and satisfies*

$$\big(\mathcal{N}_\kappa u\big)(x') \leq C\,\mathcal{M}f(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{1.17}$$

for some constant $C \in (0, \infty)$ *that depends only on* L*,* n*, and* κ*.*

Corollary 1.3 contains as particular cases a multitude of well-posedness results for elliptic systems in the upper-half space. For example, one may take Muckenhoupt weighted Lebesgue spaces $\mathbb{X} := \big[L^p(\mathbb{R}^{n-1}, w\mathscr{L}^{n-1})\big]^M$ and $\mathbb{Y} := L^p(\mathbb{R}^{n-1}, w\mathscr{L}^{n-1})$ with $p \in (1, \infty)$ and $w \in A_p$, or Morrey spaces in $\mathbb{R}^{n-1}$; for more on this, as well as other examples, see [7].

Here we wish to identify the most inclusive setting in which Corollary 1.3 yields a well-posedness result. Specifically, in view of the assumptions made in (1.13)-(1.14) it is natural to consider the linear space

$$\begin{aligned} \mathscr{Z} &:= \Big\{ f \in \big[L^1\big(\mathbb{R}^{n-1}\,,\, \tfrac{dx'}{1+|x'|^{n-1}}\big)\big]^M : \mathcal{M}f \in L^1\big(\mathbb{R}^{n-1}\,,\, \tfrac{dx'}{1+|x'|^{n-1}}\big) \Big\} \\ &= \Big\{ f : \mathbb{R}^{n-1} \to \mathbb{C}^M : \ \text{measurable and} \ \ \mathcal{M}f \in L^1\big(\mathbb{R}^{n-1}\,,\, \tfrac{dx'}{1+|x'|^{n-1}}\big) \Big\} \end{aligned} \tag{1.18}$$

(recall that $\mathcal{M}$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$) equipped with the norm

$$\begin{aligned} \|f\|_{\mathscr{Z}} &:= \|f\|_{[L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}})]^M} + \|\mathcal{M}f\|_{L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\big)} \\ &\approx \|\mathcal{M}f\|_{L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\big)}, \qquad \forall\, f \in \mathscr{Z}. \end{aligned} \tag{1.19}$$

Then, Corollary 1.3 applied with $\mathbb{X} := \mathscr{Z}$ and $\mathbb{Y} := L^1\big(\mathbb{R}^{n-1}\,,\, \frac{dx'|}{1+|x'|^{n-1}}\big)$ yields the following result.

Corollary 1.4 (The Most Inclusive Well-Posedness Result)**.** *Let* L *be an* $M \times M$ *system with constant complex coefficients as in* (1.1)*-*(1.2)*, and fix an aperture parameter* $\kappa > 0$*. Then the following boundary-value problem is well-posed:*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x')\, \frac{dx'}{1 + |x'|^{n-1}} < \infty, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{Z}. \end{cases} \tag{1.20}$$

The relevance of the fact that (1.4) implies (1.5) in the context of all the aforementioned boundary value problems (cf. (1.15), (1.20)) is that the nontangential boundary trace $u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}$ is guaranteed to exist by the other conditions imposed on the function u in the formulation of the said problems, and that the solution may be recovered from the boundary datum via convolution with the Poisson kernel canonically associated with the system L.

The type of boundary value problems treated here, in which the size of the solution is measured in terms of its nontangential maximal function and its trace is taken in a nontangential pointwise sense, has been dealt with in the particular case when $L=\Delta$, the Laplacian in $\mathbb{R}^n$, in a number of monographs, including [3], [4], [13], [14], and [15]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle, or the Schwarz reflection principle for harmonic functions. Neither of the latter techniques may be adapted successfully to prove uniqueness in the case of general systems treated here, and our approach is more in line with the work in [7] (which involves Green function estimates and a sharp version of the Divergence Theorem), with some significant refinements. A remarkable aspect is that our approach works for the entire class of elliptic systems L as in (1.1)-(1.2).

2. Preliminary Matters

Throughout, $\mathbb{N}$ stands for the collection of all strictly positive integers, and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. As such, for each $k\in\mathbb{N}$, we denote by $\mathbb{N}_0^k$ the collection of all multi-indices $\alpha=(\alpha_1,\dots,\alpha_k)$ with $\alpha_j\in\mathbb{N}_0$ for $1\leq j\leq k$. Also, fix $n\in\mathbb{N}$ with $n\geq 2$. We shall work in the upper-half space $\mathbb{R}^n_+$, whose topological boundary $\partial\mathbb{R}^n_+=\mathbb{R}^{n-1}\times\{0\}$ will be frequently identified with the horizontal hyperplane $\mathbb{R}^{n-1}$ via $(x',0)\equiv x'$. The origin in $\mathbb{R}^{n-1}$ is denoted by $0'$ and we let $B_{n-1}(x',r)$ stand for the $(n-1)$-dimensional Euclidean ball of radius r centered at $x'\in\mathbb{R}^{n-1}$. Having fixed $\kappa>0$, for each boundary point $x'\in\partial\mathbb{R}^n_+$ introduce the conical nontangential approach region with vertex at x' as

$$\Gamma_\kappa(x'):=\big\{y=(y',t)\in\mathbb{R}^n_+:\,|x'-y'|<\kappa\,t\big\}. \tag{2.1}$$

Given a vector-valued function $u:\mathbb{R}^n_+\to\mathbb{C}^M$, the nontangential maximal function of u is defined by

$$\big(\mathcal{N}_\kappa u\big)(x'):=\sup\big\{|u(y)|:\,y\in\Gamma_\kappa(x')\big\},\qquad x'\in\partial\mathbb{R}^n_+\equiv\mathbb{R}^{n-1}. \tag{2.2}$$

Whenever meaningful, we also define the nontangential trace of u as

$$u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}(x'):=\lim_{\Gamma_\kappa(x')\ni y\to(x',0)}u(y)\quad\text{for }\,x'\in\partial\mathbb{R}^n_+\equiv\mathbb{R}^{n-1}. \tag{2.3}$$

In the sequel, we shall need to consider a localized version of the nontangential maximal operator. Specifically, given any $E\subset\mathbb{R}^n_+$, for each $u:E\to\mathbb{C}^M$ we set

$$\big(\mathcal{N}_\kappa^E u\big)(x'):=\sup\big\{|u(y)|:\,y\in\Gamma_\kappa(x')\cap E\big\},\qquad x'\in\partial\mathbb{R}^n_+\equiv\mathbb{R}^{n-1}. \tag{2.4}$$

Hence, $\mathcal{N}_\kappa^E u=\mathcal{N}_\kappa\widetilde{u}$ where $\widetilde{u}$ is the extension of u to $\mathbb{R}^n_+$ by zero outside E. In the scenario when u is originally defined in the entire upper-half space $\mathbb{R}^n_+$ we may therefore write

$$\mathcal{N}_\kappa^E u=\mathcal{N}_\kappa(\mathbf{1}_E u), \tag{2.5}$$

where $\mathbf{1}_E$ denotes the characteristic function of E.

The action of the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$ on any Lebesgue measurable function f defined in $\mathbb{R}^{n-1}$ is given by

$$\big(\mathcal{M}f\big)(x'):=\sup_{r>0}\fint_{B_{n-1}(x',r)}|f|\,d\mathscr{L}^{n-1},\qquad\forall\,x'\in\mathbb{R}^{n-1}, \tag{2.6}$$

where the barred integral denotes mean average (for functions which are $\mathbb{C}^M$-valued the average is taken componentwise).

We next recall a useful weak compactness result from [7, Lemma 6.2, p. 956]. To state it, denote by $\mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$ the space of continuous functions in $\mathbb{R}^{n-1}$ vanishing at infinity.

Lemma 2.1. *Let $v:\mathbb{R}^{n-1}\to(0,\infty)$ be a Lebesgue measurable function and consider a sequence $\{f_j\}_{j\in\mathbb{N}}$ in the weighted Lebesgue space $L^1(\mathbb{R}^{n-1},v\mathscr{L}^{n-1})$ such that*

$$F:=\sup_{j\in\mathbb{N}}|f_j|\in L^1(\mathbb{R}^{n-1},v\mathscr{L}^{n-1}). \tag{2.7}$$

Then there exists a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ and a function $f\in L^1(\mathbb{R}^{n-1},v\mathscr{L}^{n-1})$ with the property that

$$\int_{\mathbb{R}^{n-1}} f_{j_k}(x')\varphi(x')v(x')\,dx'\longrightarrow\int_{\mathbb{R}^{n-1}} f(x')\varphi(x')v(x')\,dx'\quad\text{as}\quad k\to\infty, \tag{2.8}$$

for every $\varphi\in\mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$.

We next discuss the notion of Poisson kernel in $\mathbb{R}^n_+$ for an operator L as in (1.1)-(1.2).

Definition 2.2. *Let L be an $M\times M$ system with constant complex coefficients as in* (1.1)-(1.2)*. A* `Poisson kernel` *for L in $\mathbb{R}^n_+$ is a matrix-valued function*

$$P^L=\big(P^L_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}:\mathbb{R}^{n-1}\longrightarrow\mathbb{C}^{M\times M} \tag{2.9}$$

such that the following conditions hold:

(a) there exists $C\in(0,\infty)$ such that $|P^L(x')|\leq\dfrac{C}{(1+|x'|^2)^{\frac{n}{2}}}$ for each $x'\in\mathbb{R}^{n-1}$;

(b) the function P^L is Lebesgue measurable and $\displaystyle\int_{\mathbb{R}^{n-1}}P^L(x')\,dx'=I_{M\times M}$, the $M\times M$ identity matrix;

(c) if $K^L(x',t):=P^L_t(x'):=t^{1-n}P^L(x'/t)$, for each $x'\in\mathbb{R}^{n-1}$ and $t\in(0,\infty)$, then the function $K^L=\big(K^L_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ satisfies (*in the sense of distributions*)

$$LK^L_{\cdot\beta}=0\quad\text{in}\quad\mathbb{R}^n_+\quad\text{for each}\quad\beta\in\{1,\dots,M\}, \tag{2.10}$$

where $K^L_{\cdot\beta}:=\big(K^L_{\alpha\beta}\big)_{1\leq\alpha\leq M}$.

Poisson kernels for elliptic boundary value problems in a half-space have been studied extensively in [1], [2], [5, §10.3], [10], [11], [12]. Here we record a corollary of more general work done by S. Agmon, A. Douglis, and L. Nirenberg in [2].

Theorem 2.3. *Any $M\times M$ system L with constant complex coefficients as in* (1.1)-(1.2) *has a Poisson kernel P^L in the sense of Definition 2.2, which has the additional property that the function*

$$K^L(x',t):=P^L_t(x')\quad\text{for all}\quad(x',t)\in\mathbb{R}^n_+, \tag{2.11}$$

satisfies $K^L\in\big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+}\setminus B(0,\varepsilon)\big)\big]^{M\times M}$ for every $\varepsilon>0$, and has the property that for each multi-index $\alpha\in\mathbb{N}_0^n$ there exists $C_\alpha\in(0,\infty)$ such that

$$\big|(\partial^\alpha K^L)(x)\big|\leq C_\alpha\,|x|^{1-n-|\alpha|},\quad\text{for every}\quad x\in\overline{\mathbb{R}^n_+}\setminus\{0\}. \tag{2.12}$$

Here and elsewhere, the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

The next result we recall has been proved in [7, Theorem 3.1, p. 934].

Proposition 2.4. *Let L be an $M \times M$ system with constant complex coefficients as in* (1.1)-(1.2)*, and recall the Poisson kernel P^L for L in $\mathbb{R}^n_+$ from Theorem 2.3. Also, fix some arbitrary aperture parameter $\kappa > 0$. Given a function*

$$f \in \Big[L^1\Big(\mathbb{R}^{n-1}\,,\,\frac{dx'}{1+|x'|^n}\Big)\Big]^M, \tag{2.13}$$

set

$$u(x',t) := (P^L_t * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+. \tag{2.14}$$

Then u is meaningfully defined via an absolutely convergent integral,

$$u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \ \text{ at } \mathscr{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1} \tag{2.15}$$

(*with the last identity valid in the set of Lebesgue points of f*)*, and there exists a constant $C = C(n, L, \kappa) \in (0,\infty)$ with the property that*

$$\big(\mathcal{N}_\kappa u\big)(x') \le C\big(\mathcal{M}f\big)(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{2.16}$$

A key ingredient in the proof of our main result is understanding the nature of the Green function associated with a given elliptic system. While we elaborate on this topic in Theorem 2.6 below, we begin by providing a suitable definition for the said Green function (which, in particular, is going to ensure its uniqueness). To set the stage, denote by $\mathcal{D}'(\mathbb{R}^n_+)$ the space of distributions in $\mathbb{R}^n_+$.

Definition 2.5. *Let L be an $M \times M$ system with constant complex coefficients as in* (1.1)-(1.2)*. Call $G^L(\cdot,\cdot) : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag} \to \mathbb{C}^{M\times M}$ a* `Green function` *for L in $\mathbb{R}^n_+$ provided for each $y = (y', y_n) \in \mathbb{R}^n_+$ the following properties hold* (*for some aperture parameter $\kappa > 0$*)*:*

$$G^L(\cdot\,,y) \in \big[L^1_{\mathrm{loc}}(\mathbb{R}^n_+)\big]^{M\times M}, \tag{2.17}$$

$$G^L(\cdot\,,y)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathscr{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+, \tag{2.18}$$

$$\int_{\mathbb{R}^{n-1}} \Big(\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus\overline{B(y,y_n/2)}}\, G^L(\cdot\,,y)\Big)(x')\frac{dx'}{1+|x'|^{n-1}} < \infty, \tag{2.19}$$

$$L\big[G^L(\cdot\,,y)\big] = -\delta_y\, I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\mathbb{R}^n_+)\big]^{M\times M}, \tag{2.20}$$

where the $M \times M$ system L acts in the "dot" variable on the columns of G.

The existence and basic properties of the Green function just defined are discussed in our next theorem (a proof of which may be found in [6]). Before stating it, we make two conventions regarding notation. First, we agree to abbreviate $\mathrm{diag} := \{(x,x) : x \in \mathbb{R}^n_+\}$ for the diagonal in the Cartesian product $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. Second, given a function $G(\cdot,\cdot)$ of two vector variables, $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag}$, for each $k \in \{1,\dots,n\}$ we agree to write $\partial_{X_k}G$ and $\partial_{Y_k}G$, respectively, for the partial derivative of G with respect to x_k, and y_k. This convention may be iterated, lending a natural meaning to $\partial^\alpha_X\partial^\beta_Y G$, for each pair of multi-indices $\alpha, \beta \in \mathbb{N}^n_0$. We are now ready to present the result alluded to above.

Theorem 2.6. *Assume that L is an $M \times M$ system with constant complex coefficient as in* (1.1)-(1.2)*. Then there exists a unique Green function $G^L(\cdot,\cdot)$ for L in $\mathbb{R}^n_+$, in the sense of Definition 2.5. Moreover, this Green function also satisfies the following additional properties:*

(1) Given $\kappa > 0$, for each $y \in \mathbb{R}^n_+$ and each compact neighborhood K of y in $\mathbb{R}^n_+$ there exists a finite constant $C_y = C(n, L, \kappa, K, y) > 0$ such that for every $x' \in \mathbb{R}^{n-1}$ there holds

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K}\big(G^L(\cdot, y)\big)(x') \leq C_y \, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}}. \tag{2.21}$$

Moreover, for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha| + |\beta| > 0$, there exists some constant $C_y = C(n, L, \kappa, \alpha, \beta, K, y) \in (0, \infty)$ such that

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K}\big((\partial_X^\alpha \partial_Y^\beta G^L)(\cdot, y)\big)(x') \leq \frac{C_y}{1 + |x'|^{n-2+|\alpha|+|\beta|}}. \tag{2.22}$$

(2) For each fixed $y \in \mathbb{R}^n_+$, there holds

$$G^L(\cdot\,, y) \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(y, \varepsilon)\big)\big]^{M \times M} \quad \textit{for every} \;\; \varepsilon > 0. \tag{2.23}$$

As a consequence of (2.23) and (2.18), for each fixed $y \in \mathbb{R}^n_+$ one has

$$G^L(\cdot, y)\Big|_{\partial \mathbb{R}^n_+} = 0 \quad \textit{everywhere on} \;\; \mathbb{R}^{n-1}. \tag{2.24}$$

(3) For each $\alpha, \beta \in \mathbb{N}_0^n$ the function $\partial_X^\alpha \partial_Y^\beta G^L$ is translation invariant in the tangential variables, in the sense that

$$\begin{gathered} \big(\partial_X^\alpha \partial_Y^\beta G^L\big)\big(x - (z', 0), y - (z', 0)\big) = \big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x, y) \\ \textit{for each} \;\; (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag} \;\; \textit{and} \;\; z' \in \mathbb{R}^{n-1}, \end{gathered} \tag{2.25}$$

and is positive homogeneous, in the sense that

$$\begin{gathered} \big(\partial_X^\alpha \partial_Y^\beta G^L\big)(\lambda x, \lambda y) = \lambda^{2-n-|\alpha|-|\beta|}\big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x, y) \\ \textit{for each} \;\; x, y \in \mathbb{R}^n_+ \;\; \textit{with} \;\; x \neq y \;\; \textit{and} \;\; \lambda \in (0, \infty), \\ \textit{provided either } n \geq 3, \textit{ or } |\alpha| + |\beta| > 0. \end{gathered} \tag{2.26}$$

(4) If $G^{L^\top}(\cdot, \cdot)$ denotes the (unique, by the first part of the statement) Green function for $L^\top$ (the transposed of L) in $\mathbb{R}^n_+$, then

$$G^L(x, y) = \Big[G^{L^\top}(y, x)\Big]^\top, \qquad \forall\, (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag}. \tag{2.27}$$

Hence, as a consequence of (2.27), (2.18), and (2.23), for each fixed $x \in \mathbb{R}^n_+$ and $\varepsilon > 0$,

$$G^L(x, \cdot) \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(x, \varepsilon)\big)\big]^{M \times M} \quad \textit{and} \quad G^L(x, \cdot)\Big|_{\partial \mathbb{R}^n_+} = 0 \;\; \textit{on} \;\; \mathbb{R}^{n-1}. \tag{2.28}$$

(5) For any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a finite constant $C_{\alpha\beta} > 0$ such that

$$\begin{gathered} \big|\big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x, y)\big| \leq C_{\alpha\beta} |x - y|^{2-n-|\alpha|-|\beta|}, \\ \forall\, (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag}, \textit{ if either } n \geq 3, \textit{ or } |\alpha| + |\beta| > 0, \end{gathered} \tag{2.29}$$

and, corresponding to $|\alpha| = |\beta| = 0$ and $n = 2$, there exists $C \in (0, \infty)$ such that

$$\big|G^L(x, y)\big| \leq C + C\big|\ln|x - \overline{y}|\big|, \quad \forall\, (x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \setminus \mathrm{diag}, \tag{2.30}$$

where $\overline{y} := (y', -y_n) \in \mathbb{R}^n$ is the reflexion of $y = (y', y_n) \in \mathbb{R}^n_+$ across the boundary of the upper-half space.

(6) *The Agmon-Douglis-Nirenberg Poisson kernel* $P^L = \big(P^L_{\gamma\alpha}\big)_{1\leq\gamma,\alpha\leq M}$ *for* L *in* $\mathbb{R}^n_+$ *from Theorem 2.3 is related to the Green function* G^L *for* L *in* $\mathbb{R}^n_+$ *according to the formula*

$$\begin{gathered} P^L_{\gamma\alpha}(z') = a^{\beta\alpha}_{nn}\big(\partial_{Y_n} G^L_{\gamma\beta}\big)\big((z',1),0\big), \quad \forall\, z' \in \mathbb{R}^{n-1}, \\ \textit{for each} \ \ \alpha,\gamma \in \{1,\dots,M\}. \end{gathered} \tag{2.31}$$

We shall now record the following versatile version of interior estimates for second-order elliptic systems. A proof may be found in [8, Theorem 11.9, p. 364].

Theorem 2.7. *Consider a homogeneous, constant coefficient, second-order, system* L *satisfying the weak ellipticity condition* $\det[L(\xi)] \neq 0$ *for each* $\xi \in \mathbb{R}^n \setminus \{0\}$*. Then for each null-solution* u *of* L *in a ball* $B(x,R)$ *(where* $x \in \mathbb{R}^n$ *and* $R > 0$*),* $0 < p < \infty$*,* $\lambda \in (0,1)$*,* $\ell \in \mathbb{N}_0$*, and* $0 < r < R$*, one has*

$$\sup_{z\in B(x,\lambda r)} |\nabla^\ell u(z)| \leq \frac{C}{r^\ell}\left(⨍_{B(x,r)} |u|^p \, d\mathscr{L}^n\right)^{1/p}, \tag{2.32}$$

where $C = C(L,p,\ell,\lambda,n) > 0$ *is a finite constant.*

We conclude by recording a suitable version of the divergence theorem recently obtained in [9]. To state it requires a few preliminaries which we dispense with first. We shall write $\mathcal{E}'(\mathbb{R}^n_+)$ for the subspace of $\mathcal{D}'(\mathbb{R}^n_+)$ consisting of those distributions which are compactly supported. Hence,

$$\mathcal{E}'(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+) \quad \text{and} \quad L^1_{\rm loc}(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+). \tag{2.33}$$

For each compact set $K \subset \mathbb{R}^n_+$, define $\mathcal{E}'_K(\mathbb{R}^n_+) := \big\{u \in \mathcal{E}'(\mathbb{R}^n_+) : \operatorname{supp} u \subset K\big\}$ and consider

$$\begin{aligned} \mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) := \big\{u \in \mathcal{D}'(\mathbb{R}^n_+) :\, &\exists\, v_1 \in \mathcal{E}'_K(\mathbb{R}^n_+) \ \text{and} \ \exists\, v_2 \in L^1(\mathbb{R}^n_+) \\ &\text{such that} \ \ u = v_1 + v_2 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n_+)\big\}. \end{aligned} \tag{2.34}$$

Also, introduce $\mathscr{C}^\infty_b(\mathbb{R}^n_+) := \mathscr{C}^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ and let $\big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*$ denote its algebraic dual. Moreover, we let ${}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\cdot\,,\,\cdot\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)}$ denote the natural duality pairing between these spaces. It is useful to observe that for every compact set $K \subset \mathbb{R}^n_+$ one has

$$\mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) \subset \big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*. \tag{2.35}$$

Theorem 2.8 ([9])**.** *Assume that* $K \subset \mathbb{R}^n_+$ *is a compact set and that* $\vec{F} \in \big[L^1_{\rm loc}(\mathbb{R}^n_+)\big]^n$ *is a vector field satisfying the following conditions (for some aperture parameter* $\kappa > 0$*):*

(a) $\operatorname{div}\vec{F} \in \mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+)$*, where the divergence is taken in the sense of distributions;*

(b) *the nontangential maximal function* $\mathcal{N}^{\mathbb{R}^n_+\setminus K}_\kappa \vec{F}$ *belongs to* $L^1(\mathbb{R}^{n-1})$*;*

(c) *the nontangential boundary trace* $\vec{F}\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^n_+}$ *exists (in* $\mathbb{C}^n$*) at* $\mathscr{L}^{n-1}$*-a.e. point in* $\mathbb{R}^{n-1}$*.*

Then, with $e_n := (0,\dots,0,1) \in \mathbb{R}^n$ *and "dot" denoting the standard inner product in* $\mathbb{R}^n$*,*

$${}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\operatorname{div}\vec{F},1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\,\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^n_+}\big)\, d\mathscr{L}^{n-1}. \tag{2.36}$$

3. Proofs of Main Results

We take on the task of presenting the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix an arbitrary point $x^\star \in \mathbb{R}^n_+$ and bring in $G^{L^\top}(\cdot, x^\star)$, the Green function with pole at $x^\star$ for $L^\top$, the transposed of the operator L (cf. Definition 2.5 and Theorem 2.6 for details on this matter). For ease of notation, abbreviate

$$G(\cdot) := G^{L^\top}(\cdot, x^\star) \text{ in } \mathbb{R}^n_+ \setminus \{x^\star\}. \tag{3.1}$$

By design, this is a matrix-valued function, say $G = (G_{\alpha\gamma})_{1\leq\alpha,\gamma\leq M}$. We shall apply Theorem 2.8 to a suitably chosen vector field. To set the stage, consider the compact set

$$K_\star := \overline{B(x^\star, r)} \subset \mathbb{R}^n_+, \qquad \text{where } r := \operatorname{dist}(x^\star, \partial\mathbb{R}^n_+) \cdot \tfrac{\kappa}{2\sqrt{4+\kappa^2}}. \tag{3.2}$$

For each $\varepsilon > 0$ consider the function $u^\varepsilon : \overline{\mathbb{R}^n_+} \to \mathbb{C}^M$ given by

$$u^\varepsilon(x) := u(x', x_n + \varepsilon) \text{ for all } x = (x', x_n) \in \overline{\mathbb{R}^n_+}. \tag{3.3}$$

Then

$$u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M, \quad Lu^\varepsilon = 0 \text{ in } \mathbb{R}^n_+, \text{ and } \mathcal{N}_\kappa u^\varepsilon \leq \mathcal{N}_\kappa u \text{ on } \mathbb{R}^{n-1}. \tag{3.4}$$

Fix $\varepsilon > 0$ along with some $\beta \in \{1, \ldots, M\}$ and, using the summation convention over repeated indices, define the vector field

$$\vec{F} := \Big(u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\, \partial_k G_{\gamma\beta} - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\, \partial_k u^\varepsilon_\gamma\Big)_{1\leq j\leq n} \quad \text{at } \mathscr{L}^n\text{-a.e. point in } \mathbb{R}^n_+. \tag{3.5}$$

From (3.5), Theorem 2.6, and the fact that $u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M$ it follows that

$$\vec{F} \in \big[L^1_{\rm loc}(\mathbb{R}^n_+)\big]^n \cap \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus K_\star)\big]^n \tag{3.6}$$

and, on account of (2.24) (used for $L^\top$ in place of L), we have

$$\vec{F}\Big|_{\partial\mathbb{R}^n_+} = \Big(\big(u^\varepsilon_\alpha\big|_{\partial\mathbb{R}^n_+}\big) a^{\gamma\alpha}_{kj} \big(\partial_k G_{\gamma\beta}\big)\big|_{\partial\mathbb{R}^n_+}\Big)_{1\leq j\leq n}. \tag{3.7}$$

Next, in the sense of distributions in $\mathbb{R}^n_+$, we may compute

$$\begin{aligned}
\operatorname{div}\vec{F} &= (\partial_j u^\varepsilon_\alpha)\, a^{\gamma\alpha}_{kj}\,(\partial_k G_{\gamma\beta}) + u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\,(\partial_j\partial_k G_{\gamma\beta}) \\
&\quad - (\partial_j G_{\alpha\beta})\, a^{\alpha\gamma}_{jk}\,(\partial_k u^\varepsilon_\gamma) - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\,(\partial_j\partial_k u^\varepsilon_\gamma) \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.8}$$

where the last equality defines the I_i's. Changing variables $j' = k$, $k' = j$, $\alpha' = \gamma$, and $\gamma' = \alpha$ in I_3 yields

$$I_3 = -(\partial_{k'} G_{\gamma'\beta})\, a^{\gamma'\alpha'}_{k'j'}\,(\partial_{j'} u^\varepsilon_{\alpha'}) = -I_1. \tag{3.9}$$

As regards I_4, we have

$$I_4 = -G_{\alpha\beta}\,(Lu^\varepsilon)_\alpha = 0, \tag{3.10}$$

by (3.4). Finally,

$$\begin{aligned}
I_2 &= u^\varepsilon_\alpha (L_{A^\top} G_{\cdot\beta})_\alpha = u^\varepsilon_\alpha (L^\top G_{\cdot\beta})_\alpha \\
&= -u^\varepsilon_\alpha \delta_{\alpha\beta}\delta_{x^\star} = -u^\varepsilon_\beta\, \delta_{x^\star} = -u^\varepsilon_\beta(x^\star)\, \delta_{x^\star}.
\end{aligned} \tag{3.11}$$

Collectively, these equalities permit us to conclude that, in the sense of distributions in $\mathbb{R}^n_+$,

$$\operatorname{div}\vec{F} = -u^\varepsilon_\beta(x^\star)\, \delta_{x^\star} \in \mathcal{E}'(\mathbb{R}^n_+). \tag{3.12}$$

In particular,

$$\operatorname{div}\vec{F}\in\mathcal{D}'(\mathbb{R}^n_+)\ \text{ induces a continuous functional in }\ \big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*. \tag{3.13}$$

Moving on, fix $x'\in\mathbb{R}^{n-1}\equiv\partial\mathbb{R}^n_+$ and pick an arbitrary point

$$y=(y',y_n)\in\Gamma_{\kappa/2}(x')\setminus K_\star. \tag{3.14}$$

Choose a rectifiable path $\gamma:[0,1]\to\overline{\mathbb{R}^n_+}$ joining $(x',0)$ with y in $\Gamma_{\kappa/2}(x')\setminus K_\star$ and whose length is $\leq Cy_n$. Then, for some constant $C\in(0,\infty)$ independent of x' and y, we may estimate

$$\begin{aligned}|G(y)|&=|G(y)-G(x',0)|=\Big|\int_0^1\frac{d}{dt}[G(\gamma(t))]\,dt\Big|\\&=\Big|\int_0^1(\nabla G)(\gamma(t))\cdot\gamma'(t)\,dt\Big|\leq\Big(\sup_{\xi\in\gamma((0,1))}|(\nabla G)(\xi)|\Big)\int_0^1|\gamma'(t)|\,dt\\&\leq Cy_n\cdot\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}(\nabla G)(x'),\end{aligned}\tag{3.15}$$

using the fact that G vanishes on $\partial\mathbb{R}^n_+$, the Fundamental Theorem of Calculus, Chain Rule, and (2.4). Next, define

$$a:=\frac{\kappa}{2(\kappa+1)}\in\big(0,\tfrac12\big) \tag{3.16}$$

and write, using interior estimates (cf. Theorem 2.7) for the function u^ε,

$$\begin{aligned}|(\nabla u^\varepsilon)(y)|&\leq\frac{C}{y_n}⨍_{B(y,a\cdot y_n)}|u^\varepsilon(z)|\,dz\\&\leq Cy_n^{-1}\cdot\sup_{z\in\Gamma_\kappa(x')}|u^\varepsilon(z)|\leq Cy_n^{-1}\cdot\big(\mathcal{N}_\kappa u^\varepsilon\big)(x'),\end{aligned}\tag{3.17}$$

since having $z=(z',z_n)\in B(y,a\cdot y_n)$ entails

$$y_n\leq z_n+|z-y|<z_n+a\cdot y_n\Longrightarrow y_n<(1-a)^{-1}z_n, \tag{3.18}$$

which, bearing in mind that y is as in (3.14), permits us to conclude that

$$\begin{aligned}|z'-x'|&\leq|z'-y'|+|y'-x'|\leq|z-y|+(\kappa/2)y_n<a\cdot y_n+(\kappa/2)y_n\\&=(\kappa/2+a)y_n<\frac{\kappa/2+a}{1-a}z_n=\kappa z_n,\quad\text{hence}\quad z\in\Gamma_\kappa(x').\end{aligned}\tag{3.19}$$

Then combining (3.15) with (3.17) gives, on account of (2.22),

$$\begin{aligned}\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}\big(|G||\nabla u^\varepsilon|\big)(x')&\leq C\big(\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}(\nabla G)\big)(x')\big(\mathcal{N}_\kappa u^\varepsilon\big)(x')\\&\leq C\big(\mathcal{N}_\kappa u\big)(x')\frac{1}{1+|x'|^{n-1}}\quad\text{at each point}\quad x'\in\mathbb{R}^{n-1}.\end{aligned}\tag{3.20}$$

Since we also have

$$\begin{aligned}\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}\big(|\nabla G||u^\varepsilon|\big)(x')&\leq\big(\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}(\nabla G)\big)(x')\big(\mathcal{N}_\kappa u^\varepsilon\big)(x')\\&\leq C\big(\mathcal{N}_\kappa u\big)(x')\frac{1}{1+|x'|^{n-1}}\quad\text{at each point}\quad x'\in\mathbb{R}^{n-1},\end{aligned}\tag{3.21}$$

we conclude from (3.5), (3.20), (3.21), and the second line in (1.4) that

$$\mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}\vec{F}\in L^1(\mathbb{R}^{n-1}). \tag{3.22}$$

Having established (3.6), (3.7), (3.13), and (3.22), Theorem 2.8 applies. To write the Divergence Formula (2.36) in this case, express $x^\star$ as $(x',t) \in \mathbb{R}^{n-1} \times (0,\infty)$. Then, in view of (3.12) and (3.7) we may write

$$
\begin{aligned}
u_\beta(x^\star + \varepsilon e_n) = u^\varepsilon_\beta(x^\star) &= -{}_{\left(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\right)^*}\big(\operatorname{div}\vec{F}\,,\,1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} \\
&= \int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\,\big|_{\partial\mathbb{R}^n_+}\big)\, d\mathscr{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{kn}\big(\partial_k G_{\gamma\beta}\big)(y',0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn}\big(\partial_n G_{\gamma\beta}\big)(y',0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn}(\partial_{X_n} G^{L^\top}_{\gamma\beta})\big((y',0),x^\star\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn}(\partial_{Y_n} G^{L}_{\beta\gamma})\big(x^\star,(y',0)\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn}(\partial_{Y_n} G^{L}_{\beta\gamma})\big((x'-y',t),0\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) t^{1-n} a^{\gamma\,\alpha}_{nn}(\partial_{Y_n} G^{L}_{\beta\gamma})\big(((x'-y')/t,1),0\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) (P^L_{\beta\alpha})_t(x'-y')\, dy', \qquad (3.23)
\end{aligned}
$$

where the fifth equality uses the observation that $(\partial_k G)(y',0) = 0$ whenever $k < n$ since G vanishes (in a smooth fashion) on $\mathbb{R}^{n-1} \times \{0\}$, the sixth equality is a consequence of (3.1), the seventh equality is implied by (2.27), the eighth equality makes use of (2.25) (bearing in mind that $x^\star = (x',t)$), the ninth equality is seen from (2.26), and the last equality comes from (2.31).

Since $\beta \in \{1,\dots,M\}$ and $x^\star = (x',t) \in \mathbb{R}^n_+$ have been arbitrarily chosen, the argument so far shows that

$$
u(x',t+\varepsilon) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y') f_\varepsilon(y')\, dy' \quad \text{for each} \quad x = (x',t) \in \mathbb{R}^n_+, \tag{3.24}
$$

where we have abbreviated

$$
f_\varepsilon := u(\cdot,\varepsilon) : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^M \quad \text{for each} \quad \varepsilon > 0. \tag{3.25}
$$

If we also consider the weight $\upsilon : \mathbb{R}^{n-1} \to (0,\infty)$ defined as $\upsilon(x') := (1+|x'|^{n-1})^{-1}$ for each $x' \in \mathbb{R}^{n-1}$, then the last condition in (1.4) entails

$$
\sup_{\varepsilon>0} |f_\varepsilon| \le \mathcal{N}_\kappa u \in L^1\big(\mathbb{R}^{n-1}\,,\,\upsilon\,\mathscr{L}^{n-1}\big). \tag{3.26}
$$

Granted this, the weak-$*$ convergence result from Lemma 2.1 may be used for the sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset L^1\big(\mathbb{R}^{n-1}\,,\,\upsilon\,\mathscr{L}^{n-1}\big)$ to conclude that there exists some $f \in L^1\big(\mathbb{R}^{n-1}\,,\,\upsilon\,\mathscr{L}^{n-1}\big)$ and some sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ which converges to zero with the property that

$$
\lim_{j\to\infty} \int_{\mathbb{R}^{n-1}} \varphi(y') f_{\varepsilon_j}(y') \frac{dy'}{1+|y'|^{n-1}} = \int_{\mathbb{R}^{n-1}} \varphi(y') f(y') \frac{dy'}{1+|y'|^{n-1}} \tag{3.27}
$$

for every continuous function $\varphi \in \mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$. The fact that there exists a constant $C \in (0,\infty)$ for which

$$|P^L(z')| \leq \frac{C}{(1+|z'|^2)^{n/2}} \quad \text{for each} \quad z' \in \mathbb{R}^{n-1} \tag{3.28}$$

(see item *(a)* of Definition 2.2) ensures for each fixed point $(x',t) \in \mathbb{R}^n_+$ the assignment

$$\begin{gathered} \mathbb{R}^{n-1} \ni y' \mapsto \varphi(y') := (1+|y'|^{n-1}) P^L_t(x'-y') \in \mathbb{C}^{M\times M} \\ \text{is a continuous function which vanishes at infinity.} \end{gathered} \tag{3.29}$$

At this stage, from (3.24) and (3.27) used for the function φ defined in (3.29) we obtain (bearing in mind that u is continuous in $\mathbb{R}^n_+$) that

$$u(x',t) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y') f(y')\,dy' \quad \text{for each} \quad x=(x',t) \in \mathbb{R}^n_+. \tag{3.30}$$

With this in hand, and since $L^1\big(\mathbb{R}^{n-1}\,,\,v\,\mathscr{L}^{n-1}\big) \subseteq L^1\Big(\mathbb{R}^{n-1}\,,\,\dfrac{dx'}{1+|x'|^n}\Big)$, we may invoke Proposition 2.4 to conclude that

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists and equals } f \text{ at } \mathscr{L}^{n-1}\text{-a.e. point in } \mathbb{R}^{n-1}. \tag{3.31}$$

Once this has been established, all conclusions in (1.5) are implied by (3.30)-(3.31). □

We close by presenting the proof of Corollary 1.3.

Proof of Corollary 1.3. As a preamble, let us first show that

$$\mathbb{X} \subseteq \Big[L^1\Big(\mathbb{R}^{n-1}\,,\,\frac{dx'}{1+|x'|^n}\Big)\Big]^M. \tag{3.32}$$

To justify this, pick some arbitrary $f \in \mathbb{X}$. Then the inclusion in (1.14) gives that $\mathcal{M}f \in \mathbb{Y}$, hence $\mathcal{M}f$ is not identically $+\infty$. This implies that $f \in [L^1_{\rm loc}(\mathbb{R}^{n-1})]^M$ which, in concert with Lebesgue's Differentiation Theorem, implies that $|f| \leq \mathcal{M}f$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since $\mathbb{Y}$ is a function lattice, it follows that $|f| \in \mathbb{Y}$. Thus, ultimately, (3.32) holds by virtue of the inclusion in (1.13).

To prove the existence of a solution for (1.15), given any $f \in \mathbb{X}$ define u as in (1.16). Note that (3.32) ensures that Proposition 2.4 is applicable. In turn, this guarantees that u is a well-defined null-solution of L belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$, satisfying the boundary condition $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$, as well as the pointwise estimate in (1.17). The latter property, together with the last conditions imposed in (1.14) and (1.13), guarantees $\mathcal{N}_\kappa u \in \mathbb{Y}$. Thus, u is indeed a solution for (1.15).

At this stage, there remains to establish that the boundary value problem (1.15) can have at most one solution. To this end, assume that both u_1 and u_2 solve (1.15) for the same datum $f \in \mathbb{X}$ and set $u := u_1 - u_2 \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$. Then $Lu = 0$ in $\mathbb{R}^n_+$ and $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since we also have $\mathcal{N}_\kappa u_1, \mathcal{N}_\kappa u_2 \in \mathbb{Y}$, the pointwise estimate

$$0 \leq \mathcal{N}_\kappa u \leq \mathcal{N}_\kappa u_1 + \mathcal{N}_\kappa u_2 \leq 2\max\big\{\mathcal{N}_\kappa u_1\,,\,\mathcal{N}_\kappa u_2\big\} \quad \text{on} \quad \mathbb{R}^{n-1} \tag{3.33}$$

forces $\mathcal{N}_\kappa u \in \mathbb{Y}$ by the properties of the function lattice $\mathbb{Y}$. Granted this, Corollary 1.2 applies (thanks to the first condition in (1.13)) and gives that $u \equiv 0$ in $\mathbb{R}^n_+$. Hence $u_1 = u_2$, as wanted. □

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