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FATOU-TYPE THEOREMS AND BOUNDARY VALUE PROBLEMS FOR ELLIPTIC SYSTEMS IN THE UPPER HALF-SPACE

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Dedicated with great pleasure and respect to Vladimir Maz'ya on the occasion of his 80th birthday

ABSTRACT. We survey recent progress in a program which to date has produced [18]-[25], aimed at proving general Fatou-type results and establishing the well-posedness of a variety of boundary value problems in the upper half-space \mathbb{R}^n_+ for second-order, homogeneous, constant complex coefficient, elliptic systems L, formulated in a manner that emphasizes pointwise nontangential boundary traces of the null-solutions of L in \mathbb{R}^n_+ .

1. INTRODUCTION

The topic of boundary value problems for elliptic operators in the upper half-space is a venerable subject which has received much attention throughout the years. While there is a wealth of results in which the smoothness of solutions and boundary data are measured on the scales of Sobolev, Besov, and Triebel-Lizorkin spaces (cf., e.g., [1], [2], [10], [12], [14], [15], [16], [17], [26], [27], [29], [31], [32], [33], [37], [38], [39], [40] and the literature cited therein), the scenario in which the boundary traces are taken in a nontangential pointwise sense and the size of the solutions is measured using the nontangential maximal operator is considerably less understood. A notable exception is the case when the differential operator involved is the Laplacian, a situation dealt with at length in a number of monographs (cf., e.g., [3], [11], [34], [35], [36]). However, such undertakings always seem to employ rather specialized properties of harmonic functions, so new ideas are required when dealing with more general second-order elliptic systems in place of

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the Laplacian. In a sequence of recent works (cf. [18], [19], [20], [21], [22], [23], [24], [25]) the authors have systematically studied Fatou-type theorems and boundary value problems in the upper half-space for second-order elliptic homogeneous constant complex coefficient systems, in a formulation which emphasizes the nontangential pointwise behavior of the solutions on the boundary.

The goal of this paper is to present for the first time a coherent, inclusive account of the progress registered so far in [18]-[25]. In §2, much attention is devoted to the topics of Poisson kernel and Fatou-type theorem. Complex problems typically call for a structured approach, and this is the path we follow vis-a-vis to the notion of Poisson kernel. Its original development is typically associated with the names of Agmon, Douglis, Nirenberg, Lopatinskii, Shapiro, Solonnikov, among others (cf. [1]-[2], [17], [31]-[33]), and here we further contribute to the study of Poisson kernels associated with second-order elliptic systems from the point of view of harmonic analysis. As regards the second topic of interest in §2 mentioned earlier, recall that the trademark blueprint of a Fatou-type theorem is that certain size and integrability properties of a null-solution of an elliptic equation in a certain domain (often formulated in terms of the nontangential maximal operator) imply the a.e. existence of the pointwise nontangential boundary trace of the said function. Our Fatou-type theorems follow this design and are also quantitative in nature since the boundary trace does not just simply exist but encodes significant information regarding the size of the original function.

In §3 such results are used as tools for proving that a variety of boundary value problems for elliptic systems in the upper half-space are well-posed. In particular, here we monitor how the format of the problem changes as the space of boundary data morphs from the Lebesgue scale L^p with 1 , to the space of essentially boundedfunctions, to the space of functions of bounded mean oscillations and, further, to thespace of Hölder continuous functions (or, more generally, the space of functions withsublinear growth). A significant number of results are new, and particular care is paidto understanding the extent to which the emerging theory is optimal. Along the way, alarge number of relevant open problems are singled out for further study.

We proceed to describe the class of systems employed in this work. Throughout, fix $n \in \mathbb{N}$ satisfying $n \geq 2$, along with $M \in \mathbb{N}$. Consider a second-order, homogeneous, $M \times M$ system, with constant complex coefficients, written (with the usual convention of summation over repeated indices always in place, unless otherwise mentioned) as

$$Lu := \left(\partial_r (a_{rs}^{\alpha\beta} \partial_s u_\beta)\right)_{1 \le \alpha \le M},\tag{1}$$

when acting on $u = (u_{\beta})_{1 \leq \beta \leq M}$ whose components are distributions in an open subset of \mathbb{R}^n . Assume that L is elliptic in the sense that there exists some $c \in (0, \infty)$ such that

$$\operatorname{Re}\left[a_{rs}^{\alpha\beta}\xi_{r}\xi_{s}\overline{\eta_{\alpha}}\eta_{\beta}\right] \geq c|\xi|^{2}|\eta|^{2} \text{ for every}$$

$$\xi = (\xi_{r})_{1 \leq r \leq n} \in \mathbb{R}^{n} \text{ and } \eta = (\eta_{\alpha})_{1 \leq \alpha \leq M} \in \mathbb{C}^{M}.$$

$$(2)$$

Examples include scalar operators, such as the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ or, more generally, operators of the form div $A\nabla$ with $A = (a_{rs})_{1 \le r,s \le n}$ an $n \times n$ matrix with complex entries

satisfying the ellipticity condition

$$\inf_{\xi \in S^{n-1}} \operatorname{Re}\left[a_{rs}\xi_r\xi_s\right] > 0,\tag{3}$$

(where S^{n-1} denotes the unit sphere in \mathbb{R}^n), as well as complex versions of the Lamé system of elasticity

$$L := \mu \Delta + (\lambda + \mu) \nabla \text{div where the Lamé moduli } \lambda, \mu \in \mathbb{C}$$

satisfy $\operatorname{Re} \mu > 0$ and $\operatorname{Re} (2\mu + \lambda) > 0.$ (4)

The last condition above is equivalent to the demand that the Lamé system (4) is Legendre-Hadamard elliptic (in the sense of (2)). While the Lamé system is symmetric, we stress that the results in this paper require no symmetry for the systems involved.

We shall work in the upper half-space

$$\mathbb{R}^{n}_{+} := \left\{ x = (x', x_n) \in \mathbb{R}^{n} = \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0 \right\}$$
(5)

whose topological boundary we shall henceforth identify with the horizontal hyperplane \mathbb{R}^{n-1} via $\partial \mathbb{R}^n_+ \ni (x', 0) \equiv x' \in \mathbb{R}^{n-1}$. The origin in \mathbb{R}^{n-1} is denoted by 0', and we agree to let $B_{n-1}(x', r) := \{y' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$ stand for the (n-1)-dimensional ball centered at $x' \in \mathbb{R}^{n-1}$ and of radius r > 0. We shall also let \mathbb{N}_0 stand for the collection of all non-negative integers. Finally, we will adopt the standard custom of allowing the letter C to denote constants which may vary from one occurrence to another.

2. POISSON KERNELS AND GENERAL FATOU-TYPE RESULTS

Poisson kernels for elliptic operators in a half-space have a long history (see, e.g., [1], [2], [17], [31], [32], [33]). In the theorem below we single out the most essential features which identify these objects uniquely.

Theorem 1. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Then there exists a matrix-valued function

$$P^{L} = \left(P^{L}_{\alpha\beta}\right)_{1 \le \alpha, \beta \le M} : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M \times M}$$

$$\tag{6}$$

(called the Poisson kernel for L in \mathbb{R}^n_+) satisfying the following properties:

(a) There exists $C \in (0, \infty)$ such that

$$|P^{L}(x')| \le \frac{C}{(1+|x'|^{2})^{\frac{n}{2}}} \quad for \ each \ x' \in \mathbb{R}^{n-1}.$$
(7)

(b) The function P^L is Lebesgue measurable and

$$\int_{\mathbb{R}^{n-1}} P^L(x') \, dx' = I_{M \times M},\tag{8}$$

where $I_{M \times M}$ denotes the $M \times M$ identity matrix.

(c) If one sets

$$\begin{aligned}
K^{L}(x',t) &:= P^{L}_{t}(x') = t^{1-n} P^{L}(x'/t) \\
for each \ x' \in \mathbb{R}^{n-1} \quad and \ t > 0,
\end{aligned} \tag{9}$$

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then the $\mathbb{C}^{M \times M}$ -valued function K^L satisfies (with L acting on the columns of K^L in the sense of distributions)

$$LK^{L} = 0 \cdot I_{M \times M} \quad in \quad \left[\mathcal{D}'(\mathbb{R}^{n}_{+})\right]^{M \times M}.$$
(10)

(d) The Poisson kernel P^L is unique in the class of $\mathbb{C}^{M \times M}$ -valued functions defined in \mathbb{R}^{n-1} and satisfying (a)-(c) above.

Concerning Theorem 1, we note that the existence part follows from the classical work of S. Agmon, A. Douglis, and L. Nirenberg in [2] (cf. also [17], [31]-[33]). The uniqueness property has been recently proved in [21].

The Poisson kernel introduced above is the basic tool used to construct solutions for the Dirichlet problem for the system L in the upper half-space. This is most apparent from Theorem 2 stated a little further below. For now, we proceed to define the nontangential maximal operator and the nontangential boundary trace. Specifically, having fixed some aperture parameter $\kappa > 0$, at each point $x' \in \partial \mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ we define the conical nontangential approach region with vertex at x' as

$$\Gamma_{\kappa}(x') := \{ y = (y', t) \in \mathbb{R}^{n}_{+} : |x' - y'| < \kappa t \}.$$
(11)

Given a continuous vector-valued function $u : \mathbb{R}^n_+ \to \mathbb{C}^M$, we then define the nontangential maximal operator acting on u by setting

$$\left(\mathcal{N}_{\kappa}u\right)(x') := \sup\left\{|u(y)|: y \in \Gamma_{\kappa}(x')\right\}, \qquad x' \in \mathbb{R}^{n-1}.$$
 (12)

We shall also need a version of the nontangential maximal operator in which the supremum is now taken over cones truncated near the vertex. Specifically, given a continuous vector-valued function $u: \mathbb{R}^n_+ \to \mathbb{C}^M$, for each $\varepsilon > 0$ define

$$\left(\mathcal{N}_{\kappa}^{(\varepsilon)}u\right)(x') := \sup\left\{|u(y)|: y = (y',t) \in \Gamma_{\kappa}(x') \text{ with } t > \varepsilon\right\}$$
(13)

at each $x' \in \mathbb{R}^{n-1}$. Whenever meaningful, the κ -nontangential pointwise boundary trace of a continuous vector-valued function $u : \mathbb{R}^n_+ \to \mathbb{C}^M$ is given by

$$\left(u\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}\right)(x') := \lim_{\Gamma_\kappa(x')\ni y\to(x',0)} u(y) \quad \text{for} \quad x'\in\partial\mathbb{R}^n_+\equiv\mathbb{R}^{n-1}.$$
(14)

It is then clear from definitions that for any continuous vector-valued function $u : \mathbb{R}^n_+ \to \mathbb{C}^M$ and any $\varepsilon, \kappa > 0$ we have

$$\mathcal{N}_{\kappa}^{(\varepsilon)}u, \ \mathcal{N}_{\kappa}u \text{ are lower semicontinuous,}$$
$$0 \leq \mathcal{N}_{\kappa}^{(\varepsilon)}u \leq \mathcal{N}_{\kappa}u \text{ on } \partial \mathbb{R}_{+}^{n} \equiv \mathbb{R}^{n-1}.$$
(15)

In addition, for each such function u we have

$$\|u\|_{[L^{\infty}(\mathbb{R}^{n}_{+})]^{M}} = \|\mathcal{N}_{\kappa}u\|_{L^{\infty}(\mathbb{R}^{n-1})}.$$
(16)

Finally, whenever the nontangential boundary trace exists, we have

$$\begin{aligned} u\Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.} & \text{ is a Lebesgue measurable function} \\ \text{ and } \left|u\Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}\right| \leq \mathcal{N}_{\kappa} u \quad \text{ on } \quad \partial \mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}. \end{aligned}$$
(17)

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Prior to stating our next result, we make some comments further clarifying notation and terminology. Throughout, we agree to denote by \mathcal{M} the Hardy-Littlewood maximal operator on \mathbb{R}^{n-1} . This acts on each vector-valued function f with components in $L^1_{\text{loc}}(\mathbb{R}^{n-1})$ according to

$$\left(\mathcal{M}f\right)(x') := \sup_{Q \ni x'} \frac{1}{\mathcal{L}^{n-1}(Q)} \int_{Q} |f| \, d\mathcal{L}^{n-1}, \qquad \forall x' \in \mathbb{R}^{n-1}, \tag{18}$$

where the supremum runs over all cubes Q in \mathbb{R}^{n-1} containing x', and where \mathcal{L}^{n-1} denotes the (n-1)-dimensional Lebesgue measure in \mathbb{R}^{n-1} .

Next, pick some integrability exponent $q \in (1, \infty)$ (whose actual choice is ultimately immaterial), and fix an arbitrary $p \in \left(\frac{n-1}{n}, 1\right]$. Recall that a Lebesgue measurable function $a : \mathbb{R}^{n-1} \to \mathbb{C}$ is said to be an (p, q)-atom if for some cube $Q \subset \mathbb{R}^{n-1}$ one has

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^q(\mathbb{R}^{n-1})} \le \mathcal{L}^{n-1}(Q)^{1/q-1/p}, \quad \int_{\mathbb{R}^{n-1}} a \, d\mathcal{L}^{n-1} = 0. \tag{19}$$

One may then define the Hardy space $H^p(\mathbb{R}^{n-1})$ as the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^{n-1})$ which may be written as

$$f = \sum_{j \in \mathbb{N}} \lambda_j \, a_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^{n-1}) \tag{20}$$

for some sequence $\{a_j\}_{j\in\mathbb{N}}$ of (p,q)-atoms and a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p$. For each $f \in H^p(\mathbb{R}^{n-1})$ we then set $||f||_{H^p(\mathbb{R}^{n-1})} := \inf\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p}$ with the infimum taken over all atomic decompositions of f as $\sum_{j\in\mathbb{N}} \lambda_j a_j$.

In relation to this we wish to make three comments. First, the very definition of the quasi-norm $\|\cdot\|_{H^p(\mathbb{R}^{n-1})}$ implies that whenever $f \in H^p(\mathbb{R}^{n-1})$ is written as in (20) then the series actually converges in $H^p(\mathbb{R}^{n-1})$. Second, from the definition of $\|\cdot\|_{H^p(\mathbb{R}^{n-1})}$ we also see that each $f \in H^p(\mathbb{R}^{n-1})$ has a quasi-optimal atomic decomposition, i.e., f may be written as in (20) with

$$\frac{1}{2} \left(\sum_{j \in \mathbb{N}} |\lambda_j|^p\right)^{1/p} \le \|f\|_{H^p(\mathbb{R}^{n-1})} \le \left(\sum_{j \in \mathbb{N}} |\lambda_j|^p\right)^{1/p}.$$
(21)

Third, consider the vector case, i.e., the space $[H^p(\mathbb{R}^{n-1})]^M$. In such a setting we find it convenient to work with \mathbb{C}^M -valued (p,q)-atoms. Specifically, these are functions

$$a \in \left[L^{q}(\mathbb{R}^{n-1})\right]^{M} \text{ such that for some cube } Q \subset \mathbb{R}^{n-1} \text{ one has}$$

$$\sup p a \subset Q, \quad \|a\|_{[L^{q}(\mathbb{R}^{n-1})]^{M}} \leq \mathcal{L}^{n-1}(Q)^{1/q-1/p},$$

and
$$\int_{\mathbb{R}^{n-1}} a \, d\mathcal{L}^{n-1} = 0 \in \mathbb{C}^{M}.$$
(22)

Suppose now that some $f = (f_{\beta})_{1 \leq \beta \leq M} \in [H^p(\mathbb{R}^{n-1})]^M$ has been given. Then each f_{β} has an atomic decomposition $f_{\beta} = \sum_{j=1}^{\infty} \lambda_{\beta j} a_{\beta j}$ (no summation on β here) where each

 $a_{\beta j}$ is a (p,q)-atom and $\{\lambda_{\beta j}\}_{j\in\mathbb{N}}\in\ell^p$, which is quasi-optimal, hence

$$\|f_{\beta}\|_{H^p(\mathbb{R}^{n-1})} \approx \left(\sum_{j=1}^{\infty} |\lambda_{\beta j}|^p\right)^{1/p} \text{ for each } \beta \in \{1, \dots, M\}.$$
(23)

Using the Kronecker symbol formalism, introduce $\mathbf{e}_{\beta} := (\delta_{\gamma\beta})_{1 \leq \gamma \leq M} \in \mathbb{C}^M$ for each index $\beta \in \{1, \ldots, M\}$, then write

$$f = \sum_{\beta=1}^{M} f_{\beta} \mathbf{e}_{\beta} = \sum_{\beta=1}^{M} \sum_{j=1}^{\infty} \lambda_{\beta j} a_{\beta j} \mathbf{e}_{\beta} = \sum_{\beta=1}^{M} \sum_{j=1}^{\infty} \lambda_{\beta j} A_{\beta j}$$
(24)

with convergence in $\left[H^p(\mathbb{R}^{n-1})\right]^M$, where

$$A_{\beta j} := a_{\beta j} \mathbf{e}_{\beta} \text{ for each } \beta \in \{1, \dots, M\} \text{ and } j \in \mathbb{N}$$

$$(25)$$

are \mathbb{C}^M -valued functions as in (22), hence \mathbb{C}^M -valued (p, q)-atoms. If we then relabel the sequences $\{A_{\beta j}\}_{\substack{1 \leq \beta \leq M \\ j \in \mathbb{N}}}$ and $\{\lambda_{\beta j}\}_{\substack{1 \leq \beta \leq M \\ j \in \mathbb{N}}}$ simply as $\{a_j\}_{j \in \mathbb{N}}$ and $\{\lambda_j\}_{j \in \mathbb{N}}$, respectively, we may re-cast (23)-(25) as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ with convergence in } \left[H^p(\mathbb{R}^{n-1}) \right]^M,$$

where each a_j is a \mathbb{C}^M -valued (p,q) -atom (cf. (22)),
and $\|f\|_{[H^p(\mathbb{R}^{n-1})]^M} \approx \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}.$ (26)

Since the Poisson kernel P^L and the kernel function K^L from Theorem 1 are of fundamental importance to the work described in this paper, a more in-depth analysis of their main properties is in order. Before stating our theorem addressing this analysis, for each real number m we agree to denote by $L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^m})$ the space of \mathbb{C} -valued Lebesgue measurable functions which are absolutely integrable in \mathbb{R}^{n-1} with respect to the weighted Lebesgue measure $\frac{dx'}{1+|x'|^m}$.

Theorem 2. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Then the Agmon-Douglis-Nirenberg Poisson kernel P^L and the kernel function K^L from Theorem 1 satisfy the following properties:

(a) The function P^L belongs to $\left[\mathcal{C}^{\infty}(\mathbb{R}^{n-1})\right]^{M \times M}$ and satisfies the following non-degeneracy property:

for each
$$a \in \mathbb{C}^M \setminus \{0\}$$
 one
may find some $\lambda > 0$ such that
 $\int_{S^{n-2}} |P^L(\lambda\omega)a| d\omega > 0.$ (27)

One may extend K^L to a function belonging to $\left[\mathcal{C}^{\infty}(\overline{\mathbb{R}^n_+} \setminus \{0\})\right]^{M \times M}$. Consequently, formula (10) also holds in a pointwise sense in \mathbb{R}^n_+ . Moreover, there exists some constant $C \in (0, \infty)$ such that

$$|K^{L}(x',t)| \le Ct/(t^{2} + |x'|^{2})^{n/2} \quad for \ each \ (x',t) \in \mathbb{R}^{n}_{+},$$
(28)

an estimate which further implies $K^L(x',0) = 0$ for each $x' \in \mathbb{R}^{n-1} \setminus \{0'\}$. In addition, one has $\int_{\mathbb{R}^{n-1}} K^L(x'-y',t) \, dy' = I_{M \times M}$ for all $(x',t) \in \mathbb{R}^n_+$, as well as

 $K^{L}(\lambda x) = \lambda^{1-n} K^{L}(x)$ for all $x \in \mathbb{R}^{n}_{+}$ and $\lambda > 0$. In particular, for each multi-index $\alpha \in \mathbb{N}^{n}_{0}$ there exists $C_{\alpha} \in (0, \infty)$ with the property that

$$\left| (\partial^{\alpha} K^{L})(x) \right| \leq C_{\alpha} |x|^{1-n-|\alpha|}, \qquad \forall x \in \overline{\mathbb{R}^{n}_{+}} \setminus \{0\}.$$
(29)

(b) The following semi-group property holds:

$$P_{t_0+t_1}^L = P_{t_0}^L * P_{t_1}^L \quad for \ all \ t_0, t_1 > 0.$$
(30)

(c) Given a Lebesgue measurable function $f = (f_{\beta})_{1 \leq \beta \leq M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M}$ satisfying

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^n} \, dx' < \infty,\tag{31}$$

at each point $(x',t) \in \mathbb{R}^n_+$ set

$$u(x',t) := (P_t^L * f)(x')$$

$$:= \left(\int_{\mathbb{R}^{n-1}} t^{1-n} P_{\alpha\beta}^L((x'-y')/t) f_\beta(y') \, dy' \right)_{1 \le \alpha \le M}.$$
 (32)

Then $u : \mathbb{R}^n_+ \to \mathbb{C}^M$ is meaningfully defined via an absolutely convergent integral, satisfies (for each given aperture parameter $\kappa > 0$)

$$u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+},$$

and $\left.u\right|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ each \ Lebesgue \ point \ of \ f$
(hence, in particular, at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1}),
(33)

and there exists a constant $C = C(L, \kappa) \in (0, \infty)$ with the property that

$$(\mathcal{N}_{\kappa}u)(x') \leq C \mathcal{M}f(x') \text{ for each } x' \in \mathbb{R}^{n-1}.$$
 (34)

Also, if $L = \Delta$, the Laplacian in \mathbb{R}^n , then the opposite inequality in (34) is true as well. Furthermore, the following unrestricted convergence result holds

$$\lim_{\substack{\mathbb{R}^n_+ \ni x \to (x'_0, 0) \\ \text{if } x'_0 \in \mathbb{R}^{n-1} \text{ is a continuity point for } f.}} (35)$$

In other words, u given by (32) extends by continuity to $\mathbb{R}^n_+ \cup \{(x'_0, 0)\}$ whenever $x'_0 \in \mathbb{R}^{n-1}$ is a continuity point for f. In particular,

whenever
$$f \in \left[L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right) \cap \mathcal{C}^0(\mathbb{R}^{n-1})\right]^M$$
 and u
is as in (32), then u extends uniquely to a function in $\left[\mathcal{C}^0(\overline{\mathbb{R}^n_+})\right]^M$. (36)

(d) For each $p \in \left(\frac{n-1}{n}, 1\right]$ and each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$, the kernel function K^L satisfies

$$(\partial^{\alpha} K^{L})(x' - \cdot, t) \in \left[H^{p}(\mathbb{R}^{n-1})\right]^{M \times M} \text{ for all } (x', t) \in \mathbb{R}^{n}_{+}, \text{ and}$$
$$\sup_{(x',t)\in\mathbb{R}^{n}_{+}} \left\|t^{|\alpha|-(n-1)(\frac{1}{p}-1)}(\partial^{\alpha} K^{L})(x' - \cdot, t)\right\|_{[H^{p}(\mathbb{R}^{n-1})]^{M \times M}} < \infty.$$
(37)

In fact, for each $p \in \left(\frac{n-1}{n}, 1\right]$, each $q \in [1, \infty]$ with q > p, each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$, and each $(x', t) \in \mathbb{R}_+^n$, the function

$$m_{x',t}^{\alpha} := t^{|\alpha| - (n-1)(\frac{1}{p} - 1)} (\partial^{\alpha} K^{L})(x' - \cdot, t)$$
(38)

is, up to multiplication by some fixed constant $C \in (0, \infty)$ (which depends exclusively on L, p, q, n, α), a $\mathbb{C}^{M \times M}$ -valued L^q -normalized molecule relative to the ball $B_{n-1}(x', t)$ for the Hardy space $[H^p(\mathbb{R}^{n-1})]^{M \times M}$. More precisely,

$$\int_{\mathbb{R}^{n-1}} m_{x',t}^{\alpha} \, d\mathcal{L}^{n-1} = 0 \cdot I_{M \times M},\tag{39}$$

and there exists $C \in (0, \infty)$ such that one has

$$\left\|m_{x',t}^{\alpha}\right\|_{\left[L^{q}(B_{n-1}(x',t))\right]^{M\times M}} \leq C\mathcal{L}^{n-1}\left(B_{n-1}(x',t)\right)^{\frac{1}{q}-\frac{1}{p}},\tag{40}$$

and, using the abbreviation $\varepsilon := |\alpha|/(n-1)$, for each $k \in \mathbb{N}$ one also has

$$\| m_{x',t}^{\alpha} \|_{[L^{q}(B_{n-1}(x',2^{k}t)\setminus B_{n-1}(x',2^{k-1}t))]^{M\times M}} \le C 2^{k(n-1)\left(\frac{1}{q}-1-\varepsilon\right)} \mathcal{L}^{n-1} (B_{n-1}(x',t))^{\frac{1}{q}-\frac{1}{p}}.$$

$$(41)$$

(e) Given any $\alpha \in \mathbb{N}_0^n$, for each fixed t > 0 the function $(\partial^{\alpha} K^L)(\cdot, t)$ belongs to the Hölder space $\left[\mathcal{C}^{\theta}(\mathbb{R}^{n-1})\right]^{M \times M}$ for each exponent $\theta \in (0, 1)$. As a consequence of this, given an arbitrary $f = (f_{\beta})_{1 \leq \beta \leq M} \in \left[H^p(\mathbb{R}^{n-1})\right]^M$ with $p \in \left(\frac{n-1}{n}, 1\right]$, one may meaningfully define

$$u(x',t) := \left(P_t^L * f\right)(x') \tag{42}$$

$$:= \left\{ \left\langle f_{\beta}, \left[K_{\alpha\beta}^{L}(x' - \cdot, t) \right] \right\rangle \right\}_{1 \le \alpha \le M} \text{ for } (x', t) \in \mathbb{R}_{+}^{n},$$

where $\langle \cdot, \cdot \rangle$ is the pairing between distributions belonging to the Hardy space $H^p(\mathbb{R}^{n-1})$ and equivalence classes (modulo constants) of functions belonging to the homogeneous Hölder space $\dot{\mathcal{C}}^{(n-1)(1/p-1)}(\mathbb{R}^{n-1})$ if p < 1, and to BMO(\mathbb{R}^{n-1}) if p = 1 (cf., e.g., [11, Theorem 5.30, p.307]). Then

$$u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+},$$

and $\left.u\right|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} exists \mathcal{L}^{n-1}-a.e. \ in \ \mathbb{R}^{n-1}.$ (43)

Moreover, there exists a constant $C = C(L, \kappa, p) \in (0, \infty)$ with the property that

$$\begin{aligned} \left\| \mathcal{N}_{\kappa} u \right\|_{L^{p}(\mathbb{R}^{n-1})} &\leq C \| f \|_{[H^{p}(\mathbb{R}^{n-1})]^{M}} \quad whenever \\ f \in \left[H^{p}(\mathbb{R}^{n-1}) \right]^{M} \quad and \ u \ is \ as \ in \ (42). \end{aligned}$$

$$\tag{44}$$

We wish to note that, in sharp contrast with (33), in the context of (43) we no longer expect the nontangential pointwise trace $u\Big|_{\partial \mathbb{R}^n_+}^{\kappa^{-n.t.}}$ to be directly related to the (generally speaking tempered distribution) $f \in [H^p(\mathbb{R}^{n-1})]^M$. For example, in (127)-(132) we present an example in which the said trace vanishes at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} even though $f \neq 0$.

Proof of Theorem 2. For items (a)-(c) see [21], [22], [24], [25]. To deal with the claims in item (d), fix p, q, α, x', t as in the statement. Then

$$\int_{\mathbb{R}^{n-1}} m_{x',t}^{\alpha}(y') \, dy' = t^{|\alpha| - (n-1)(\frac{1}{p} - 1)} \, \partial^{\alpha} \int_{\mathbb{R}^{n-1}} K^L(x' - y', t) \, dy' = 0 \tag{45}$$

since $\int_{\mathbb{R}^{n-1}} K^L(x'-y',t) \, dy' = I_{M \times M}$ and $|\alpha| > 0$. This proves (39). Also, based on (29) we may estimate

$$\int_{B_{n-1}(x',t)} \left| m_{x',t}^{\alpha}(y') \right|^{q} dy' \leq C \int_{B_{n-1}(x',t)} \frac{t^{q|\alpha|-q(n-1)(\frac{1}{p}-1)}}{(t+|x'-y'|)^{q(n-1+|\alpha|)}} dy'$$
$$\leq C \int_{B_{n-1}(x',t)} \frac{t^{q|\alpha|-q(n-1)(\frac{1}{p}-1)}}{t^{q(n-1+|\alpha|)}} dy'$$
$$= C \left[\mathcal{L}^{n-1} \left(B_{n-1}(x',t) \right)^{\frac{1}{q}-\frac{1}{p}} \right]^{q}, \tag{46}$$

and, if $\varepsilon := |\alpha|/(n-1)$, for every $k \in \mathbb{N}$ we may write

$$\int_{B_{n-1}(x',2^{k}t)\setminus B_{n-1}(x',2^{k-1}t)} \left| m_{x',t}^{\alpha}(y') \right|^{q} dy' \\
\leq C \int_{2^{k-1}t < |x'-y'| < 2^{k}t} \frac{t^{q|\alpha|-q(n-1)(\frac{1}{p}-1)}}{(t+|x'-y'|)^{q(n-1+|\alpha|)}} dy' \\
\leq C \int_{B_{n-1}(x',2^{k}t)} \frac{t^{q|\alpha|-q(n-1)(\frac{1}{p}-1)}}{(2^{k}t)^{q(n-1+|\alpha|)}} dy' \\
= C \Big[2^{k(n-1)\left(\frac{1}{q}-1-\varepsilon\right)} \mathcal{L}^{n-1} \Big(B_{n-1}(x',t) \Big)^{\frac{1}{q}-\frac{1}{p}} \Big]^{q},$$
(47)

for some constant $C \in (0, \infty)$ independent of k, x', and t. From these, the estimates claimed in (40), (41) readily follow. Going further, the first claim in item (e) is a consequence of the fact that, as seen from (29), for each $\alpha \in \mathbb{N}_0^n$ there exists $C_\alpha \in$ $(0, \infty)$ such that $\|(\partial^{\alpha} K^L)(\cdot, t)\||_{[L^{\infty}(\mathbb{R}^{n-1})]^{M \times M}} \leq C_{\alpha} t^{1-n-|\alpha|}$ for every t > 0, together with an elementary observation to the effect that any bounded Lipschitz function in \mathbb{R}^{n-1} belongs to the Hölder space $\mathcal{C}^{\theta}(\mathbb{R}^{n-1})$ for each exponent $\theta \in (0, 1)$. In concert with the identification of the duals of Hardy spaces (cf., e.g., [11]) this shows that the pairings in (42) are meaningful.

To prove (44), fix some $q \in (1, \infty)$ and assume first that the scalar components of f are (p, q)-atoms. Hence, we need to consider

$$u(x',t) := (P_t^L * a)(x'), \qquad \forall (x',t) \in \mathbb{R}^n_+,$$
(48)

where $a : \mathbb{R}^{n-1} \to \mathbb{C}^M$ is a (p,q)-atom (cf. (22)). Then, on account of (34), Hölder's inequality, the L^q -boundedness of the Hardy-Littlewood maximal operator, and the normalization of the atom we may write

$$\int_{\sqrt{n}Q} \left(\mathcal{N}_{\kappa}u\right)^{p} d\mathcal{L}^{n-1} \leq C \int_{\sqrt{n}Q} \left(\mathcal{M}a\right)^{p} d\mathcal{L}^{n-1}
\leq C\mathcal{L}^{n-1}(Q)^{1-p/q} \left(\int_{\sqrt{n}Q} \left(\mathcal{M}a\right)^{q} d\mathcal{L}^{n-1}\right)^{p/q}
\leq C\mathcal{L}^{n-1}(Q)^{1-p/q} \left(\int_{\mathbb{R}^{n-1}} \left(\mathcal{M}a\right)^{q} d\mathcal{L}^{n-1}\right)^{p/q}
\leq C\mathcal{L}^{n-1}(Q)^{1-p/q} \|a\|_{[L^{q}(\mathbb{R}^{n-1})]^{M}}^{p} \leq C,$$
(49)

for some constant $C \in (0, \infty)$ depending only on n, L, κ, p, q . To proceed, fix an arbitrary point $x' \in \mathbb{R}^{n-1} \setminus \sqrt{nQ}$. If $\ell(Q)$ and x'_Q are, respectively, the side-length and center of the cube Q, this choice entails

$$|z' - x'_Q| \le \max\{\kappa, 2\} (t + |z' - \xi'|), \quad \forall (z', t) \in \Gamma_{\kappa}(x'), \ \forall \xi' \in Q.$$

$$(50)$$

Indeed, if $(z',t) \in \Gamma_{\kappa}(x')$ and $\xi' \in Q$ then, first, $|z' - x'_Q| \leq |z' - \xi'| + |\xi' - x'_Q|$ and, second, $|\xi' - x'_Q| \leq \frac{\sqrt{n}}{2}\ell(Q) \leq \frac{1}{2}|x' - x'_Q| \leq \frac{1}{2}(|x' - z'| + |z' - x'_Q|) \leq \frac{1}{2}(\kappa t + |z' - x'_Q|)$, from which (50) follows. Next, using (9), the vanishing moment condition for the atom, the Mean Value Theorem together with (29) and (50), Hölder's inequality and, finally, the support and normalization of the atom, for each $(z',t) \in \Gamma_{\kappa}(x')$ we may estimate

$$\begin{aligned} |(P_t^L * a)(z')| &= \left| \int_{\mathbb{R}^{n-1}} \left[K^L(z' - y', t) - K^L(z' - x'_Q, t) \right] a(y') \, dy' \right| \\ &\leq \int_Q \left| K^L(z' - y', t) - K^L(z' - x'_Q, t) \right| |a(y')| \, dy' \\ &\leq C \frac{\ell(Q)}{\left(t + |z' - x'_Q|\right)^n} \int_Q |a(y')| \, dy' \\ &\leq C \frac{\ell(Q)}{\left(t + |z' - x'_Q|\right)^n} \, \mathcal{L}^{n-1}(Q)^{1-1/q} ||a||_{[L^q(\mathbb{R}^{n-1})]^M} \\ &\leq \frac{C \mathcal{L}^{n-1}(Q)^{1-1/p} \ell(Q)}{\left(t + |z' - x'_Q|\right)^n}. \end{aligned}$$
(51)

In turn, (51) implies that for each $x' \in \mathbb{R}^{n-1} \setminus \sqrt{nQ}$ we have

$$(\mathcal{N}_{\kappa}u)(x') = \sup_{(z',t)\in\Gamma_{\kappa}(x')} |(P_t^L * a)(z')|$$

$$\leq \sup_{(z',t)\in\Gamma_{\kappa}(x')} \frac{C\mathcal{L}^{n-1}(Q)^{1-1/p}\ell(Q)}{\left(t+|z'-x'_Q|\right)^n} = \frac{C\mathcal{L}^{n-1}(Q)^{1-1/p}\ell(Q)}{|x'-x'_Q|^n},$$
(52)

hence

$$\int_{\mathbb{R}^{n-1}\setminus\sqrt{n}Q} \left(\mathcal{N}_{\kappa}u\right)^p d\mathcal{L}^{n-1} \le C \int_{\mathbb{R}^{n-1}\setminus\sqrt{n}Q} \frac{\mathcal{L}^{n-1}(Q)^{p-1}\ell(Q)^p}{|x'-x'_Q|^{np}} \, dx' = C,\tag{53}$$

for some constant $C \in (0, \infty)$ depending only on n, L, p, q. From (49) and (53) we deduce that whenever u is as in (48) then, for some constant $C \in (0, \infty)$ independent of the atom,

$$\int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa} u \right)^p d\mathcal{L}^{n-1} \le C.$$
(54)

Next, consider the general case when the function u is defined as in (42) for some arbitrary $f \in [H^p(\mathbb{R}^{n-1})]^M$. Writing f as in (26) then permits us to express (in view of the specific manner in which the duality pairing in (42) manifests itself), for each fixed $(x',t) \in \mathbb{R}^n_+$,

$$u(x',t) = (P_t^L * f)(x') = \sum_{j=1}^{\infty} \lambda_j (P_t^L * a_j)(x') = \sum_{j=1}^{\infty} \lambda_j u_j(x',t),$$
(55)

where $u_j(x',t) := (P_t^L * a_j)(x')$ for each $j \in \mathbb{N}$. Consequently, based on the sublinearity of the nontangential maximal operator, the fact that p < 1, the estimate established in (54) (presently used with $a := a_j$), and the quasi-optimality of the atomic decomposition for f, we may write

$$\int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa} u \right)^{p} d\mathcal{L}^{n-1} \leq \int_{\mathbb{R}^{n-1}} \left(\sum_{j=1}^{\infty} \mathcal{N}_{\kappa}(\lambda_{j} u_{j}) \right)^{p} d\mathcal{L}^{n-1}$$

$$\leq \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{\infty} \left(\mathcal{N}_{\kappa}(\lambda_{j} u_{j}) \right)^{p} d\mathcal{L}^{n-1}$$

$$= \sum_{j=1}^{\infty} |\lambda_{j}|^{p} \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa} u_{j} \right)^{p} d\mathcal{L}^{n-1}$$

$$\leq C \sum_{j=1}^{\infty} |\lambda_{j}|^{p} \leq C ||f||_{[H^{p}(\mathbb{R}^{n-1})]^{M}}^{p}.$$
(56)

This proves (44).

Finally, the claims in the first line of (43) are seen by differentiating inside the duality bracket, while the existence of the nontangential boundary trace in the second line of (43) is a consequence of the corresponding result in (33), the estimate in (44), the density of $H^p(\mathbb{R}^{n-1}) \cap L^2(\mathbb{R}^{n-1})$ in $H^p(\mathbb{R}^{n-1})$, and a well-known abstract principle in harmonic analysis (see, e.g., [7, Theorem 2.2, p. 27], [36, Theorem 3.12, p. 60] for results of similar flavor).

Let L be an $M \times M$ system with constant complex coefficients as in (1)-(2) and fix an integrability exponent $p \in (1, \infty)$. From items (b)-(c) in Theorem 2 we then see that the family $T = \{T(t)\}_{t \ge 0}$ where T(0) := I, the identity operator on $[L^p(\mathbb{R}^{n-1})]^M$ and, for each t > 0,

$$T(t): \left[L^{p}(\mathbb{R}^{n-1})\right]^{M} \longrightarrow \left[L^{p}(\mathbb{R}^{n-1})\right]^{M},$$

$$(x'):= \left(P^{L} + f\right)(x') \text{ for all } f \in \left[L^{p}(\mathbb{R}^{n-1})\right]^{M} \quad x' \in \mathbb{R}^{n-1}$$
(57)

$$(T(t)f)(x') := (P_t^L * f)(x') \text{ for all } f \in [L^p(\mathbb{R}^{n-1})]^m, \ x' \in \mathbb{R}^{n-1},$$

is a C_0 -semigroup on $\left[L^p(\mathbb{R}^{n-1})\right]^M$, which satisfies

$$\sup_{t \ge 0} \|T(t)\|_{[L^p(\mathbb{R}^{n-1})]^M \to [L^p(\mathbb{R}^{n-1})]^M} < \infty.$$
(58)

We now proceed to present several Fatou-type theorems and Poisson integral representation formulas for null-solutions of homogeneous constant complex coefficient elliptic second-order systems defined in \mathbb{R}^n_+ and subject to a variety of size conditions.

Theorem 3. Let *L* be an $M \times M$ system with constant complex coefficients as in (1)-(2), and fix some aperture parameter $\kappa > 0$. Suppose $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfies Lu = 0 in \mathbb{R}^n_+ , as well as

$$\int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa}^{(\varepsilon)} u \right) (x') \frac{dx'}{1 + |x'|^{n-1}} < \infty \text{ for each fixed } \varepsilon > 0,$$
(59)

and also assume that there exists $\varepsilon_0 > 0$ such that the following finiteness integral condition holds:

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0 < t < \varepsilon_0} |u(x', t)|}{1 + |x'|^n} \, dx' < \infty.$$
(60)

Then

$$\begin{cases}
\left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right)(x') & exists at \mathcal{L}^{n-1}\text{-}a.e. point \ x' \in \mathbb{R}^{n-1}, \\
u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} & belongs to the space \left[L^{1}\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n}}\right)\right]^{M}, \\
u(x',t) = \left(P_{t}^{L} * \left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right)\right)(x') & for each \ (x',t) \in \mathbb{R}^{n}_{+},
\end{cases}$$
(61)

where P^L is the Agmon-Douglis-Nirenberg Poisson kernel in \mathbb{R}^n_+ associated with the system L as in Theorem 1. In particular, from (17), (61), and (34) it follows that there exists a constant $C = C(L, \kappa) \in (0, \infty)$ with the property that

$$\left|u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right| \leq \mathcal{N}_{\kappa}u \leq C\mathcal{M}\left(u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right) \quad in \quad \mathbb{R}^{n-1}.$$
(62)

It is natural to think of (62), which implies that for almost every point $x' \in \mathbb{R}^{n-1}$ the supremum of the function u in the cone $\Gamma_{\kappa}(x')$ lies in between the (absolute value of the) boundary trace $u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}$ evaluated at x' and a fixed multiple of the Hardy-Littlewood operator acting on the boundary trace $u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}$ at x', as some type of "Pointwise Maximum Principle."

Theorem 3 is optimal from multiple perspectives. First, observe from item (a) of Theorem 2 and item (c) of Theorem 1 that for each $a \in \mathbb{C}^M \setminus \{0\}$ the function

$$u_a(x',t) := t^{1-n} P^L(x'/t) a = K^L(x',t) a, \text{ for each } (x',t) \in \mathbb{R}^n_+,$$
(63)

satisfies $u_a \in \left[\mathcal{C}^{\infty}(\overline{\mathbb{R}^n_+} \setminus \{0\})\right]^M$, $Lu_a = 0$ in \mathbb{R}^n_+ , and $\left(u_a\Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}\right)(x') = 0$ for every aperture parameter $\kappa > 0$ and every point $x' \in \mathbb{R}^{n-1} \setminus \{0'\}$. Moreover, u_a is not identically zero since $\int_{\mathbb{R}^{n-1}} u_a(x',t) dx' = a$ for each t > 0 by (8). As such, the Poisson integral representation formula in the last line of (61) fails for each $a \in \mathbb{C}^M \setminus \{0\}$. Let us also observe that having

$$|u_a(y',t)| \le C|a|t(t^2 + |y'|^2)^{-n/2} \text{ for each } (y',t) \in \mathbb{R}^n_+$$
(64)

entails that for each $\varepsilon > 0$ there exists $C_{a,\varepsilon} \in (0,\infty)$ such that

$$\left(\mathcal{N}_{\kappa}^{(\varepsilon)}u_{a}\right)(x') \leq \frac{C_{a,\varepsilon}}{1+|x'|^{n-1}} \quad \text{for each} \quad x' \in \mathbb{R}^{n-1}.$$
(65)

Hence, condition (59) is presently satisfied by each u_a . In light of Theorem 3 the finiteness integral condition stipulated in (60) then necessarily should fail for each u_a . To check directly that this is the case, recall from (27) that for each $a \in \mathbb{C}^M \setminus \{0\}$ there exists some $\lambda > 0$ such that $\int_{S^{n-2}} |P^L(\lambda \omega)a| d\omega \in (0,\infty)$. In turn, this permits us to estimate

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0 < t < \varepsilon_0} |u_a(x', t)|}{1 + |x'|^n} dx' \ge \int_{B_{n-1}(0',\lambda\varepsilon_0)} \frac{|u_a(x', |x'|/\lambda)|}{1 + |x'|^n} dx'$$
$$= \int_{B_{n-1}(0',\lambda\varepsilon_0)} \frac{(|x'|/\lambda)^{1-n} |P^L(\lambda x'/|x'|)a|}{1 + |x'|^n} dx'$$
$$= \left(\int_{S^{n-2}} |P^L(\lambda\omega)a| d\omega\right) \left(\int_0^{\lambda\varepsilon_0} \frac{\lambda^{n-1}}{\rho(1+\rho^n)} d\rho\right) = \infty,$$
(66)

using (63) and passing to polar coordinates. Thus, (60) fails for each u_a .

Second, the absolute integrability condition (60) may not be in general replaced by membership to the corresponding weak Lebesgue space. For example, in the case $L := \Delta$, the Laplacian in \mathbb{R}^n , if (60) is weakened to the demand that

$$\mathbb{R}^{n-1} \ni x' \mapsto \frac{\sup_{0 < t < \epsilon_0} |u(x',t)|}{1+|x'|^n} \in [0,\infty]$$

is a function belonging to $L^{1,\infty}(\mathbb{R}^{n-1})$ (67)

then Theorem 3 may fail. Indeed, this may be seen by considering the nonzero harmonic function $u(x',t) = t(t^2 + |x'|^2)^{-n/2}$ for each $(x',t) \in \mathbb{R}^n_+$, which satisfies (59) and (67). However, the Poisson integral representation formula in the last line of (61) fails since $\left(u\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}\right)(x') = 0$ for every $x' \in \mathbb{R}^{n-1} \setminus \{0'\}$.

Third, one cannot relax the formulation of the finiteness integral condition (60) by placing the supremum outside the integral sign. To see this, fix an arbitrary $a \in \mathbb{C}^M \setminus \{0\}$

and take u_a as in (63). Then, thanks to (7), we have

$$\sup_{0 < t < \varepsilon_0} \int_{\mathbb{R}^{n-1}} \frac{|u_a(x',t)|}{1+|x'|^n} dx' \le \sup_{0 < t < \varepsilon_0} \int_{\mathbb{R}^{n-1}} |u_a(x',t)| dx'$$
$$= \sup_{0 < t < \varepsilon_0} \int_{\mathbb{R}^{n-1}} \left| P_t^L(x') a \right| dx'$$
$$\le |a| \int_{\mathbb{R}^{n-1}} \left| P^L(x') \right| dx' < \infty.$$
(68)

Yet, again, the Poisson representation formula in the last line of (61) fails.

One notable consequence of Theorem 3 is the Fatou-type theorem and its associated Poisson integral formula presented below.

Theorem 4. Let L be an $M \times M$ system with constant complex coefficients as in (1)-(2). and fix some aperture parameter $\kappa > 0$. Then having

$$\begin{cases}
 u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\
 \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa}u\right)(x') \frac{dx'}{1 + |x'|^{n-1}} < \infty,
\end{cases}$$
(69)

implies that

$$\begin{cases}
\left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)(x') & exists at \mathcal{L}^{n-1}\text{-}a.e. \ point \ x' \in \mathbb{R}^{n-1}, \\
u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} & belongs to the space \left[L^{1}\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\right)\right]^{M}, \\
u(x',t) = \left(P_{t}^{L} * \left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)\right)(x') \ for \ each \ (x',t) \in \mathbb{R}^{n}_{+},
\end{cases}$$
(70)

where P^L is the Agmon-Douglis-Nirenberg Poisson kernel in \mathbb{R}^n_+ associated with the system L as in Theorem 1. In particular, there exists a constant $C = C(L,\kappa) \in (0,\infty)$ such that the following Pointwise Maximum Principle holds:

$$\left|u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right| \leq \mathcal{N}_{\kappa} u \leq C\mathcal{M}\left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right) \quad in \quad \mathbb{R}^{n-1}.$$
(71)

Proof of the fact that Theorem 3 implies Theorem 4. Given u as in (69), we have

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0 < t < \varepsilon_0} |u(x', t)|}{1 + |x'|^n} \, dx' \le C_n \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa} u \right)(x') \, \frac{dx'}{1 + |x'|^{n-1}} < \infty. \tag{72}$$

In view of this and (15), we conclude that the conditions stipulated in (59)-(60) are valid. As such, Theorem 3 guarantees that the properties listed in the first and third lines of (70) hold. In addition, thanks to (17) we now have the membership claimed in the middle line of (70).

A direct, self-contained proof of Theorem 4 (without having to rely on Theorem 3) has been given in [18]. Here we shall indicate how Theorem 4 self-improves to Theorem 3. In the process, we shall need the following weak-* convergence result from [24].

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Lemma 1. Suppose $\{f_j\}_{j\in\mathbb{N}} \subseteq L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})$ is a sequence of functions satisfying

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{j \in \mathbb{N}} |f_j(x')|}{1 + |x'|^n} \, dx' < \infty.$$
(73)

Then there exist $f \in L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})$ and a sub-sequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ with the property that

$$\lim_{k \to \infty} \int_{\mathbb{R}^{n-1}} \phi(y') f_{j_k}(y') \frac{dy'}{1+|y'|^n} = \int_{\mathbb{R}^{n-1}} \phi(y') f(y') \frac{dy'}{1+|y'|^n}$$
(74)

for every function ϕ belonging to $\mathcal{C}_b^0(\mathbb{R}^{n-1})$, the space of \mathbb{C} -valued continuous and bounded functions in \mathbb{R}^{n-1} .

We are ready to provide a proof of Theorem 3 which relies on Theorem 4.

Proof of Theorem 3. For each $\varepsilon > 0$ define $u_{\varepsilon}(x', t) := u(x', t + \varepsilon)$ for each $(x', t) \in \overline{\mathbb{R}^n_+}$. Also, set $f_{\varepsilon}(x') := u(x', \varepsilon)$ for each $x' \in \mathbb{R}^{n-1}$. Since we have $\mathcal{N}_{\kappa} u_{\varepsilon} \leq \mathcal{N}_{\kappa}^{(\varepsilon)} u$ on \mathbb{R}^{n-1} , we conclude that

$$u_{\varepsilon} \in \left[\mathcal{C}^{\infty}(\overline{\mathbb{R}^{n}_{+}})\right]^{M}, \quad u_{\varepsilon}\big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} = f_{\varepsilon} \quad \mathrm{on} \quad \mathbb{R}^{n-1}, \quad Lu_{\varepsilon} = 0 \quad \mathrm{in} \quad \mathbb{R}^{n}_{+},$$

$$\int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa}u_{\varepsilon}\right)(x') \frac{dx'}{1+|x'|^{n-1}} \leq \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa}^{(\varepsilon)}u\right)(x') \frac{dx'}{1+|x'|^{n-1}} < \infty.$$

$$(75)$$

As such, Theorem 4 applies to each u_{ε} , and the Poisson integral representation formula in the last line of (70) presently guarantees that for each $\varepsilon > 0$ we have

$$u(x',t+\varepsilon) = u_{\varepsilon}(x',t) = \left(P_t^L * f_{\varepsilon}\right)(x')$$
$$= \int_{\mathbb{R}^{n-1}} P_t^L(x'-y') f_{\varepsilon}(y') \, dy' \text{ for each } (x',t) \in \mathbb{R}^n_+.$$
(76)

On the other hand, property (60) entails

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0<\varepsilon<\varepsilon_0} |f_\varepsilon(x')|}{1+|x'|^n} \, dx' = \int_{\mathbb{R}^{n-1}} \frac{\sup_{0<\varepsilon<\varepsilon_0} |u(x',\varepsilon)|}{1+|x'|^n} \, dx' < \infty.$$
(77)

Granted this finiteness property, the weak-* convergence result recalled in Lemma 1 may be used for the sequence $\{f_{\varepsilon_0/(2j)}\}_{j\in\mathbb{N}} \subset [L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})]^M$ to conclude that there exist some function $f \in [L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})]^M$ and some sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0)$ which converges to zero, such that

$$\lim_{k \to \infty} \int_{\mathbb{R}^{n-1}} \phi(y') f_{\varepsilon_k}(y') \frac{dy'}{1 + |y'|^n} = \int_{\mathbb{R}^{n-1}} \phi(y') f(y') \frac{dy'}{1 + |y'|^n}$$
(78)

for every continuous bounded function $\phi : \mathbb{R}^{n-1} \to \mathbb{C}^{M \times M}$. Estimate (7) ensures for each fixed point $(x', t) \in \mathbb{R}^n_+$ the assignment

$$\mathbb{R}^{n-1} \ni y' \mapsto \phi(y') := (1 + |y'|^n) P_t^L(x' - y') \in \mathbb{C}^{M \times M}$$

$$(79)$$

is a bounded, continuous, $\mathbb{C}^{M \times M}$ -valued function.

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At this stage, from (76) and (78) used for the function ϕ defined in (79) we obtain (bearing in mind that u is continuous in \mathbb{R}^n_+) that

$$u(x',t) = \int_{\mathbb{R}^{n-1}} P_t^L(x'-y')f(y')\,dy' \quad \text{for each} \quad x = (x',t) \in \mathbb{R}^n_+.$$
(80)

With this in hand, and recalling that $f \in \left[L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right)\right]^M$, we may invoke Theorem 2 (cf. (33)) to conclude that

$$u\Big|_{\partial \mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}$$
 exists and equals f at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} . (81)

With this in hand, all conclusions in (61) are implied by (80)-(81). $\hfill \Box$

Moving on, we consider two families of semi-norms on the class of continuous functions in \mathbb{R}^n_+ , namely

$$||u||_{*,\rho} := \rho^{-1} \cdot \sup_{B(0,\rho) \cap \mathbb{R}^n_+} |u| \in [0, +\infty] \text{ for each } \rho \in (0,\infty),$$
(82)

and

$$\|u\|_{*,\varepsilon,\rho} := \rho^{-1} \cdot \sup_{\substack{\varepsilon < t < \rho \\ |x'| < \rho}} |u(x',t)| \quad \text{whenever} \quad 0 < \varepsilon < \rho < \infty.$$
(83)

Whenever $\liminf_{\rho \to \infty} \|u\|_{*,\rho} = 0$ we shall say that u has subcritical growth. In this vein, it is worth observing that

a continuous function
$$u : \mathbb{R}^n_+ \to \mathbb{C}$$
 has $\lim_{\rho \to \infty} ||u||_{*,\rho} = 0$
if and only if u is bounded on any bounded subset of
 \mathbb{R}^n_+ and $u(x) = o(|x|)$ as $x \in \mathbb{R}^n_+$ satisfies $|x| \to \infty$.
(84)

In relation to the family of seminorms introduced in (82)-(83), let us also observe that for each continuous function $u : \mathbb{R}^n_+ \to \mathbb{C}$ we have

$$\|u\|_{*,\rho} = \frac{1}{\log 2} \int_{\rho}^{2\rho} \|u\|_{*,\rho} \frac{dt}{t} \le \frac{2}{\log 2} \int_{\rho}^{2\rho} \|u\|_{*,t} \frac{dt}{t}$$
(85)

for each $\rho \in (0, \infty)$, and that

$$\|u\|_{*,\varepsilon,\rho} \le \sqrt{2} \|u\|_{*,\sqrt{2}\rho} \le \rho^{-1} \|u\|_{L^{\infty}(\mathbb{R}^{n}_{+})} \quad \text{whenever} \quad 0 < \varepsilon < \rho < \infty.$$
(86)

Our next major theorem is a novel Fatou-type result (plus a naturally accompanying Poisson integral representation formula), recently established in [24], of the sort discussed below.

Theorem 5. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Let $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ be such that Lu = 0 in \mathbb{R}^n_+ . In addition, assume

$$\liminf_{\rho \to \infty} \|u\|_{*,\varepsilon,\rho} = 0 \quad for \ each \ fixed \ \varepsilon > 0, \tag{87}$$

and suppose that there exists $\varepsilon_0 > 0$ such that the following finiteness integral condition holds:

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0 < t < \varepsilon_0} |u(x', t)|}{1 + |x'|^n} \, dx' < \infty.$$
(88)

Then, for each aperture parameter $\kappa > 0$,

$$\begin{pmatrix} u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} \end{pmatrix} (x') \quad exists \ at \ \mathcal{L}^{n-1} \text{-}a.e. \ point \ x' \in \mathbb{R}^{n-1},$$

$$u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} \quad belongs \ to \ the \ space \ \left[L^{1} \left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n}} \right) \right]^{M},$$

$$u(x',t) = \left(P_{t}^{L} * \left(u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} \right) \right) (x') \quad for \ each \ (x',t) \in \mathbb{R}^{n}_{+}.$$

$$(89)$$

As a consequence, there exists a constant $C = C(L, \kappa) \in (0, \infty)$ with the property that the following Pointwise Maximum Principle holds:

$$\left|u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right| \leq \mathcal{N}_{\kappa} u \leq C \mathcal{M}\left(u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}}\right) \quad in \quad \mathbb{R}^{n-1}.$$
(90)

The Fatou-type result established in Theorem 5 is optimal from a multitude of perspectives. First, the mildly weaker version of the subcritical growth condition stated in (87) cannot be relaxed. Indeed, fix $a \in \mathbb{C}^M \setminus \{0\}$ and consider the function u(x',t) := tafor each $(x',t) \in \partial \mathbb{R}^n_+$. Then $||u||_{*,\varepsilon,\rho} = |a| > 0$, hence (87) fails while (88) and the first two properties listed in (89) hold. Nonetheless, the Poisson integral representation formula claimed in the last line of (89) fails (since $u|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.} = 0$ everywhere on \mathbb{R}^{n-1} whereas u is nonzero).

Second, the finiteness integral condition (88) may not be dropped. To justify this claim, bring in the Poisson kernel $P^L : \mathbb{R}^{n-1} \to \mathbb{C}^{M \times M}$ associated with the system L as in Theorem 1 and, having fixed some $a \in \mathbb{C}^M \setminus \{0\}$, consider the function u_a defined as in (63). In addition to the properties this function enjoys mentioned earlier, for each $\varepsilon > 0$ fixed we have $||u_a||_{*,\varepsilon,\rho} \leq C|a|\rho^{-1}\varepsilon^{1-n} \to 0$ as $\rho \to \infty$. However, the Poisson integral representation formula claimed in the last line of (89) obviously fails. The source of the failure is the fact that (88) does not presently materialize (as already noted in (66)).

Third, as seen from (68), one cannot relax the formulation of the finiteness integral condition (88) by placing the supremum outside the integral sign.

In particular, Theorem 5 implies a uniqueness result, to the effect that whenever L is an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n one has

$$u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \right]^{M}, \ Lu = 0 \quad \text{in} \quad \mathbb{R}^{n}_{+} \\ u \text{ satisfies both (87) and (88)} \\ u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} = 0 \quad \text{at} \ \mathcal{L}^{n-1}\text{-a.e. point in} \ \mathbb{R}^{n-1} \end{array} \right\} \Longrightarrow u \equiv 0 \quad \text{in} \quad \mathbb{R}^{n}_{+}.$$
(91)

This should be compared with the following uniqueness result within the class of nullsolutions of the system L exhibiting subcritical growth (also established in [24]). **Theorem 6.** Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n and fix an aperture parameter $\kappa > 0$. Assume $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ is such that Lu = 0 in \mathbb{R}^n_+ , u satisfies the subcritical growth condition

$$\liminf_{\rho \to \infty} \|u\|_{*,\rho} = 0, \tag{92}$$

and that $u\Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.} = 0$ at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} . Then $u \equiv 0$ in \mathbb{R}^n_+ .

We wish to note that the subcritical growth condition (92) is sharp. Concretely, while $\lim_{\rho \to \infty} \|u\|_{*,\rho} \text{ always exists and is a non-negative number, its failure to vanish does not$ $force u to be identically zero. Indeed, for any <math>a \in \mathbb{C}^M \setminus \{0\}$ the function u(x',t) := tasatisfies $u \in \left[\mathcal{C}^{\infty}(\overline{\mathbb{R}^n_+})\right]^M$, Lu = 0 in \mathbb{R}^n_+ , and $u\Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.} = 0$ everywhere in \mathbb{R}^{n-1} (for any aperture parameter $\kappa > 0$). This being said, $\sup_{B(0,\rho) \cap \mathbb{R}^n_+} |u| = |a|\rho$ for each $\rho > 0$, hence

$$\liminf_{\rho \to \infty} \left(\rho^{-1} \sup_{B(0,\rho) \cap \mathbb{R}^n_+} |u| \right) = |a| > 0.$$
(93)

A comment on the genesis of the subcritical growth condition (92) is in also order. Suppose $L := \Delta$ (the Laplacian in \mathbb{R}^n) and one is interested in establishing a uniqueness result in the class of functions

$$u \in \mathcal{C}^{\infty}(\mathbb{R}^n_+) \cap \mathcal{C}^0(\overline{\mathbb{R}^n_+}) \quad \text{with} \quad \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n_+, \tag{94}$$

to the effect that the boundary trace $u|_{\partial \mathbb{R}^n_+}$ determines u. Since u(x',t) = t for each $(x',t) \in \mathbb{R}^n_+$ is a counterexample, a further demand must be imposed, in addition to (94), to rule out this pathological example. To identify this demand, consider a function u as in (94) which satisfies $u|_{\partial \mathbb{R}^n_+} = 0$. Then Schwarz's reflection principle ensures that

$$\widetilde{u}(x',t) := \begin{cases} u(x',t) & \text{if } t \ge 0, \\ -u(x',-t) & \text{if } t > 0, \end{cases} \quad \forall (x',t) \in \mathbb{R}^n,$$
(95)

is a harmonic function in \mathbb{R}^n . Interior estimates then imply the existence of a dimensional constant $C_n \in (0, \infty)$ with the property that

$$\left| (\nabla \widetilde{u})(x) \right| \le C_n \rho^{-1} \cdot \sup_{B(x,\rho)} \left| \widetilde{u} \right| \text{ for each } x \in \mathbb{R}^n \text{ and } \rho > 0.$$
(96)

In this context, it is clear that the subcritical growth condition (92) is a quantitatively optimal property guaranteeing the convergence to zero of the right-hand side of the inequality in (96) as $\rho \to \infty$, for each $x \in \mathbb{R}^n$ fixed. And this is precisely what is needed here since this further implies $\nabla \tilde{u} \equiv 0$ in \mathbb{R}^n , which ultimately forces $u \equiv 0$ in \mathbb{R}^n .

The new challenges in Theorem 6 stem from the absence of a Schwarz's reflection principle in the more general class of systems we are currently considering, and the lack of continuity of the function at boundary points. Our proof of Theorem 6 circumvents these obstacles by making use of Agmon-Douglis-Nirenberg estimates near the boundary. In turn, Theorem 5 is established using Theorem 6. bounded function; in fact,

$$\liminf_{\rho \to \infty} \rho^{-1} \|f\|_{[L^{\infty}(B_{n-1}(0',\rho))]^M} \le \liminf_{\rho \to \infty} \|u\|_{*,\rho} = 0.$$
(97)

As such, in the context of boundary value problems for the system L in the upper halfspace, the subcritical growth condition (92) is most relevant whenever the formulation of the problem in question involves boundary data functions which are locally bounded (more precisely, satisfying the condition formulated in (97)). On the other hand, having a function $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfying the finiteness integral condition (88) and such that $f := u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}$ exists at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} , guarantees that f belongs to the weighted Lebesgue space $\left[L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})\right]^M$. This membership (which is the most general condition allowing one to define null-solutions to the system L by taking the convolution with the Poisson kernel P^L described in Theorem 1) does not force f to be locally bounded.

Comparing the uniqueness statements from Theorem 6 and (91), it is worth noting that the subcritical growth condition (92) appearing in Theorem 6 decouples into (87) and

$$\liminf_{\rho \to \infty} \left[\rho^{-1} \sup_{\substack{0 < t < \varepsilon \\ |x'| < \rho}} |u(x', t)| \right] = 0 \quad \text{for each fixed} \quad \varepsilon > 0.$$
(98)

By way of contrast, in (91) in place of (98) we are employing the finiteness integral condition (88).

In relation to the Fatou-type results discussed so far we wish to raise the following issue.

Open Question 1. Can the format of the Fatou-type result from Theorem 3 be reconciled with that of Theorem 5? In other words, are these two seemingly distinct results particular manifestations of a more general, inclusive phenomenon?

Moving on, we say that a Lebesgue measurable function $f : \mathbb{R}^{n-1} \to \mathbb{C}$ belongs to the class of functions with subcritical growth, denoted $SCG(\mathbb{R}^{n-1})$, provided

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^n} \, dx' < \infty \quad \text{and} \quad \lim_{\rho \to \infty} \left[\rho^{-1} \|f\|_{L^{\infty}(B_{n-1}(0',\rho))} \right] = 0. \tag{99}$$

As indicated in the corollary below (which appears in [24]), there is a Fatou-type result in the context of Theorem 6 provided we slightly strengthen the condition demanded in (92).

Corollary 1. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Assume $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfies Lu = 0 in \mathbb{R}^n_+ , as well as

the following Dini type condition at infinity:

$$\int_{1}^{\infty} \|u\|_{*,t} \frac{dt}{t} = \int_{1}^{\infty} \left(\sup_{B(0,t)\cap\mathbb{R}^{n}_{+}} |u|\right) \frac{dt}{t^{2}} < \infty.$$
(100)

Then, for each aperture parameter $\kappa > 0$,

$$\begin{aligned} & \left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)(x') \quad exists \ at \ \mathcal{L}^{n-1}-a.e. \ point \ x' \in \mathbb{R}^{n-1}, \\ & u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} \quad belongs \ to \quad \left[\mathrm{SCG}(\mathbb{R}^{n-1})\right]^{M} \subset \left[L^{1}\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n}}\right)\right]^{M}, \end{aligned}$$
(101)
$$& u(x',t) = \left(P_{t}^{L} * \left(u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)\right)(x') \quad for \ each \ point \ (x',t) \in \mathbb{R}^{n}_{+}. \end{aligned}$$

In particular, there exists a constant $C = C(L,\kappa) \in (0,\infty)$ for which the following Pointwise Maximum Principle holds:

$$\left|u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\left|\leq\mathcal{N}_{\kappa}u\leq C\mathcal{M}\left(u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)\quad in \quad \mathbb{R}^{n-1}.$$
(102)

Indeed, from (85), (86), (100), and Lebesgues Dominated Convergence Theorem it follows that (87) holds. Also, based on a dyadic decomposition argument and (85) one can show that

$$\int_{\mathbb{R}^{n-1}} \frac{\sup_{0 \le t \le 1} |u(x',t)|}{1+|x'|^n} \, dx' \le C_n \int_1^\infty \|u\|_{*,t} \, \frac{dt}{t}.$$
(103)

In view of (100), this means that (88) holds. As a result, Theorem 5 applies and gives (89). Together with the fact that the subcritical growth property is hereditary, we then conclude that all claims in (101) are true.

3. Well-Posedness of Boundary Value Problems

In this section we shall use the Poisson kernels and Fatou-type theorems from $\S 2$ as tools for establishing the well-posedness of a variety of boundary value problems in the upper half-space \mathbb{R}^n_+ for second-order, homogeneous, constant complex coefficient, elliptic systems in \mathbb{R}^n .

3.1. The Dirichlet problem with boundary data from weighted L^1 . The template of the Fatou-type result from Theorem 4 prefigures the format of the well-posedness result discussed in the theorem below.

Theorem 7. Let L be an $M \times M$ system with constant complex coefficients as in (1)-(2), and fix an aperture parameter $\kappa > 0$. Then for each function

$$f: \mathbb{R}^{n-1} \to \mathbb{C}^M \quad Lebesgue \ measurable$$

and $\mathcal{M}f \in L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}})$ (104)

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(recall that \mathcal{M} is the Hardy-Littlewood maximal operator in \mathbb{R}^{n-1}) the boundary value problem

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa}u\right)(x') \frac{dx'}{1+|x'|^{n-1}} < \infty, \\ u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \ \mathbb{R}^{n-1}, \end{cases}$$
(105)

is uniquely solvable. Moreover, the solution u of (105) is given by (32) and satisfies

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^{n-1}} \, dx' \leq \int_{\mathbb{R}^{n-1}} \left(\mathcal{N}_{\kappa} u \right) (x') \, \frac{dx'}{1+|x'|^{n-1}} \\ \leq C \int_{\mathbb{R}^{n-1}} \left(\mathcal{M} f \right) (x') \, \frac{dx'}{1+|x'|^{n-1}} \tag{106}$$

for some constant $C = C(n, L, \kappa) \in (0, \infty)$ independent of f.

For each f as in (104), the membership of $\mathcal{M}f$ to $L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}})$ implies that $\mathcal{M}f < \infty$ at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} , which further entails $f \in [L^1_{\text{loc}}(\mathbb{R}^{n-1})]^M$. Granted this, Lebesgue's Differentiation Theorem applies and gives $|f| \leq \mathcal{M}f$ at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} . From this, the fact that f is Lebesgue measurable, and the last property in (104), we ultimately conclude that

$$f \in \left[L^1 \left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}} \right) \right]^M \subset \left[L^1 \left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n} \right) \right]^M.$$
(107)

In particular, it is meaningful to define u as in (32), and this ensures that the properties claimed in the first and last lines of (105) hold. Also, (17) and (34) imply (106) which, in turn, validates the finiteness condition in the second line of (105). This proves existence for the boundary value problem (105), and uniqueness follows from Theorem 4.

3.2. The Dirichlet problem with data from L^p and other related spaces. The well-posedness of the L^p -Dirichlet boundary value problem was established in [21]. As noted in [24], our earlier results yield an alternative approach.

Theorem 8. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n and fix an aperture parameter $\kappa > 0$. For any $p \in (1, \infty)$ the L^p -Dirichlet boundary value problem for L in \mathbb{R}^n_+ , i.e.,

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ \mathcal{N}_{\kappa}u \quad belongs \ to \ the \ space \quad L^{p}(\mathbb{R}^{n-1}), \\ u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \ \mathbb{R}^{n-1}, \end{cases}$$
(108)

has a unique solution for each $f \in [L^p(\mathbb{R}^{n-1})]^M$. Moreover, the solution u of (108) is given by (32) and satisfies

$$\|f\|_{[L^{p}(\mathbb{R}^{n-1})]^{M}} \leq \|\mathcal{N}_{\kappa}u\|_{L^{p}(\mathbb{R}^{n-1})} \leq C\|f\|_{[L^{p}(\mathbb{R}^{n-1})]^{M}}$$
(109)

for some constant $C \in [1, \infty)$ that depends only on L, n, p, and κ .

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Indeed, since

$$L^p(\mathbb{R}^{n-1}) \hookrightarrow L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\right)$$
 for each $p \in [1, \infty),$ (110)

and since the Hardy-Littlewood maximal operator

$$\mathcal{M}: L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$$
 is bounded for each $p \in (1, \infty]$, (111)

we may regard (108) as a "sub-problem" of (105). As such, Theorem 7 ensures existence (in the specified format) and uniqueness. The estimates claimed in (109) are implied by (17), (34), and (111).

Remark 1. A multitude of other important "sub-problems" of (105) present themselves. For example, if for each $p \in (1, \infty)$ and each Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1})$ (cf., e.g., [11]) we let $L^p_w(\mathbb{R}^{n-1})$ denote the space of Lebesgue measurable *p*-th power integrable functions in \mathbb{R}^{n-1} with respect to the measure $w\mathcal{L}^{n-1}$, then the fact that (cf. [21])

$$L^p_w(\mathbb{R}^{n-1}) \hookrightarrow L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}})$$
 and
 $\mathcal{M}: L^p_w(\mathbb{R}^{n-1}) \to L^p_w(\mathbb{R}^{n-1})$ boundedly,
(112)

ultimately implies that for each integrability exponent $p \in (1, \infty)$, each weight $w \in A_p(\mathbb{R}^{n-1})$, and each aperture parameter $\kappa > 0$, the L^p_w -Dirichlet boundary value problem for L in \mathbb{R}^n_+ , i.e.,

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \text{ in } \mathbb{R}^{n}_{+}, \\ \mathcal{N}_{\kappa}u \text{ belongs to the space } L^{p}_{w}(\mathbb{R}^{n-1}), \\ u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \text{ at } \mathcal{L}^{n-1}\text{-a.e. point in } \mathbb{R}^{n-1}, \end{cases}$$
(113)

has a unique solution for each $f \in [L_w^p(\mathbb{R}^{n-1})]^M$, and the solution u of (108) (which continues to be given by (32)) satisfies

$$\|f\|_{[L^p_w(\mathbb{R}^{n-1})]^M} \le \|\mathcal{N}_{\kappa} u\|_{L^p_w(\mathbb{R}^{n-1})} \le C \|f\|_{[L^p_w(\mathbb{R}^{n-1})]^M}.$$
(114)

Similarly, since for the Lorentz spaces $L^{p,q}(\mathbb{R}^{n-1})$ with $p \in (1,\infty)$, $q \in (0,\infty]$, we also have (again, see [21])

$$L^{p,q}(\mathbb{R}^{n-1}) \hookrightarrow L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}})$$
 and
 $\mathcal{M}: L^{p,q}(\mathbb{R}^{n-1}) \to L^{p,q}(\mathbb{R}^{n-1})$ boundedly, (115)

we also conclude that the version of the Dirichlet problem (108) naturally formulated in such a setting continues to be well-posed. To offer yet another example, recall the scale of Morrey spaces $\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})$ in \mathbb{R}^{n-1} , defined for each $p \in (1,\infty)$ and $\lambda \in (0, n-1)$ according to

$$\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^{n-1}) : \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})} < \infty \right\}$$
(116)

where

$$||f||_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})} := \sup_{x' \in \mathbb{R}^{n-1}, r > 0} \left(r^{-\lambda} \int_{B_{n-1}(x',r)} |f|^p \, d\mathcal{L}^{n-1} \right)^{1/p}.$$
 (117)

Given that

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^{n-1}) \subset L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\right)$$
provided $1 and $0 < \lambda < n-1$,
$$(118)$$$

and since (cf., e.g., [5])

the Hardy-Littlewood operator
$$\mathcal{M}$$
 is
bounded on $\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})$ if $1 and (119) $0 < \lambda < n-1$,$

we once again conclude that the version of the Dirichlet problem (108) naturally formulated in terms of Morrey spaces becomes well-posed. For more examples of this nature and further details the reader is referred to [21].

Later on, in Theorem 13, we shall see that in fact the end-point $p = \infty$ is permissible in the context of Theorem 8; that is, the L^{∞} -Dirichlet problem is well-posed. At the other end of the spectrum, i.e., for p = 1, the very nature of (108) changes. Indeed, at least when $L = \Delta$, the Laplacian in \mathbb{R}^n , from [35, Proposition 1, p. 119] we know that for any harmonic function u in \mathbb{R}^n_+ with $\mathcal{N}_{\kappa} u \in L^1(\mathbb{R}^{n-1})$ there exists $f \in H^1(\mathbb{R}^{n-1})$ such that $u(x',t) = (P_t^{\Delta} * f)(x')$ for each $(x',t) \in \mathbb{R}^n_+$. In concert with Theorem 4 and the observation that $H^1(\mathbb{R}^{n-1})$ is a subspace of $L^1(\mathbb{R}^{n-1})$, this implies that any harmonic function u in \mathbb{R}^n_+ with $\mathcal{N}_{\kappa} u \in L^1(\mathbb{R}^{n-1})$ (for some $\kappa > 0$) has a nontangential boundary trace $u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}$ at \mathcal{L}^{n-1} -a.e. point in \mathbb{R}^{n-1} which actually belongs to the Hardy space $H^1(\mathbb{R}^{n-1})$. Thus, the boundary data are necessarily in a Hardy space in this case. This feature accounts for the manner in which we now formulate the following well-posedness result.

Theorem 9. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n and fix an aperture parameter $\kappa > 0$. Then the (H^1, L^1) -Dirichlet boundary value problem for L in \mathbb{R}^n_+ , i.e.,

$$\begin{cases}
 u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\
 \mathcal{N}_{\kappa}u \quad belongs \ to \ the \ space \quad L^{1}(\mathbb{R}^{n-1}), \\
 u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \ \mathbb{R}^{n-1},
\end{cases}$$
(120)

has a unique solution for each f belonging to the Hardy space $[H^1(\mathbb{R}^{n-1})]^M$. In addition, the solution u of (120) is given by (32) and satisfies

$$\|\mathcal{N}_{\kappa}u\|_{L^{1}(\mathbb{R}^{n-1})} \leq C \|f\|_{[H^{1}(\mathbb{R}^{n-1})]^{M}}$$
(121)

for some constant $C \in (0, \infty)$ which depends only on L, n, and κ .

Theorem 9 has been originally established in [21], and the present work yields an alternative proof. Indeed, existence follows from item (e) of Theorem 2, while uniqueness is implied by Theorem 4.

In relation to the work discussed so far in this section we wish to formulate several open questions. We start by formulating a question which asks for allowing more general operators in the statement of [35, Proposition 1, p. 119].

Open Question 2. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Suppose $0 and fix some <math>\kappa > 0$. Also, consider a function $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfying Lu = 0 in \mathbb{R}^n_+ . Show that $\mathcal{N}_{\kappa} u \in L^p(\mathbb{R}^{n-1})$ if and only if there exists $f \in [H^p(\mathbb{R}^{n-1})]^M$ such that $u(x',t) = (P_t^L * f)(x')$ for each $(x',t) \in \mathbb{R}^n_+$. Moreover, show that $\|\mathcal{N}_{\kappa} u\|_{L^p(\mathbb{R}^{n-1})} \approx \|f\|_{[H^p(\mathbb{R}^{n-1})]^M}$.

Theorem 8 provides an answer to this question in the range $p \in (1, \infty)$, while Theorem 13 (discussed later on) addresses the case $p = \infty$. Also, item (e) of Theorem 2 is directly relevant to the issue at hand in the range $p \in \left(\frac{n-1}{n}, 1\right]$.

Our next question asks for allowing more general operators in the formulation of [11, Theorem 4.23, p. 190].

Open Question 3. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Suppose $0 and fix some <math>\kappa > 0$. Also, consider a function $u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^n_+)\right]^M$ satisfying Lu = 0 in \mathbb{R}^n_+ and $\mathcal{N}_{\kappa}u \in L^p(\mathbb{R}^{n-1})$. Show that $f := \lim_{t\to 0^+} u(\cdot, t)$ exists in the sense of tempered distributions in \mathbb{R}^{n-1} , i.e., in $\left[\mathcal{S}'(\mathbb{R}^{n-1})\right]^M$.

The following question pertains to the well-posedness of a brand of Dirichlet problem in which the boundary trace is taken in a weak, distributional sense.

Open Question 4. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Suppose $0 and fix some <math>\kappa > 0$. Show that for each $f \in [H^p(\mathbb{R}^{n-1})]^M$ the following boundary value problem is uniquely solvable and a naturally accompanying estimate holds:

$$\begin{aligned}
 & \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \right]^{M}, \quad Lu = 0 \quad \text{in} \quad \mathbb{R}^{n}_{+}, \\
 & \mathcal{N}_{\kappa}u \quad \text{belongs to the space} \quad L^{p}(\mathbb{R}^{n-1}), \\
 & \lim_{t \to 0^{+}} u(\cdot, t) = f \quad \text{in} \quad \left[\mathcal{S}'(\mathbb{R}^{n-1}) \right]^{M}.
\end{aligned} \tag{122}$$

Our earlier work shows that (122) is indeed well-posed if $p \in [1, \infty]$.

The question below has to do with the solvability of the so-called Regularity problem. This is a brand of Dirichlet problem in which the boundary data is selected from Sobolev spaces (L^p -based, of order one) and, as a result, stronger regularity is demanded of the solution.

Open Question 5. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Fix an integrability exponent $p \in (1, \infty)$ along with an aperture parameter $\kappa > 0$. Also, pick an arbitrary f in the Sobolev space $[W^{1,p}(\mathbb{R}^{n-1})]^M$.

Find additional conditions, either on the system L or the boundary datum f, guaranteeing that the Regularity problem formulated as

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \text{ in } \mathbb{R}^{n}_{+}, \\ \mathcal{N}_{\kappa}u, \, \mathcal{N}_{\kappa}(\nabla u) \text{ belong to } L^{p}(\mathbb{R}^{n-1}), \\ u\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \text{ at } \mathcal{L}^{n-1}\text{-.a.e. point in } \mathbb{R}^{n-1}, \end{cases}$$
(123)

is solvable and a naturally accompanying estimate holds.

Work relevant to this question may be found in [20] where a large class of systems L, including scalar operators (such as the Laplacian) as well as the Lamé system (4), has been identified with the property that the Regularity problem (123) is uniquely solvable for each $f \in [W^{1,p}(\mathbb{R}^{n-1})]^M$ with 1 . Also, in [22] the following link betweenthe solvability of the Regularity problem (123), and the domain of the infinitesimal $generator of the <math>C_0$ -semigroup $T = \{T(t)\}_{t\geq 0}$ associated with L as in (57), has been established.

Theorem 10. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Fix some $p \in (1, \infty)$ and consider the C_0 -semigroup $T = \{T(t)\}_{t\geq 0}$ on $[L^p(\mathbb{R}^{n-1})]^M$ associated with L as in (57). Denote by \mathbf{A} the infinitesimal generator of T, with domain $D(\mathbf{A})$. Then $D(\mathbf{A})$ is a dense linear subspace of $[W^{1,p}(\mathbb{R}^{n-1})]^M$ and, in fact,

$$D(\mathbf{A}) = \left\{ f \in \left[W^{1,p}(\mathbb{R}^{n-1}) \right]^M : the \ problem \ (123) \ with \\ boundary \ datum \ f \ is \ solvable \right\}.$$
(124)

In particular, $D(\mathbf{A}) = \left[W^{1,p}(\mathbb{R}^{n-1})\right]^M$ if and only if the Regularity problem (123) is solvable for arbitrary data $f \in \left[W^{1,p}(\mathbb{R}^{n-1})\right]^M$.

Moving on, we present a Fatou-type theorem from [24] which refines work in [21, Theorem 6.1] and [21, Corollary 6.3].

Theorem 11. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Assume that $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfies Lu = 0 in \mathbb{R}^n_+ . If $\mathcal{N}_{\kappa}u \in L^p(\mathbb{R}^{n-1})$ for some $p \in [1, \infty]$ and $\kappa > 0$, then

$$\begin{pmatrix} u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} \end{pmatrix} (x') \quad exists \ at \ \mathcal{L}^{n-1} \text{-}a.e. \ point \ x' \in \mathbb{R}^{n-1},$$

$$u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} \quad belongs \ to \ \left[L^{p}(\mathbb{R}^{n-1}) \right]^{M} \subset \left[L^{1}\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n}} \right) \right]^{M},$$

$$u(x',t) = \left(P_{t}^{L} * \left(u \big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} \right) \right) (x') \quad for \ each \ point \ (x',t) \in \mathbb{R}^{n}_{+}.$$

$$(125)$$

In view of (110), when $p \in [1, \infty)$ all claims are direct consequences of Theorem 4 (also bearing (17) in mind). The result corresponding to the end-point $p = \infty$ is no longer implied by Theorem 4 as the finiteness condition in the second line of (69) fails in general

for bounded functions (the best one can say in such a scenario is that $\mathcal{N}_{\kappa} u \in L^{\infty}(\mathbb{R}^{n-1})$). Nonetheless, Corollary 1 applies and all desired conclusions now follow from this.

As a corollary, we note that, given any aperture parameter $\kappa > 0$ along with an integrability exponent $p \in (1, \infty]$, Theorem 11 implies (together with (17), (34), and (111)) the following L^p -styled Maximum Principle:

$$\begin{aligned} \|u\|_{\partial\mathbb{R}^n_+}^{\kappa-n.t.}\|_{[L^p(\mathbb{R}^{n-1})]^M} &\approx \|\mathcal{N}_{\kappa}u\|_{L^p(\mathbb{R}^{n-1})} \text{ uniformly in} \\ \text{the class of functions } u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^n_+)\right]^M \text{ satisfying} \\ Lu &= 0 \text{ in } \mathbb{R}^n_+ \text{ as well as } \mathcal{N}_{\kappa}u \in L^p(\mathbb{R}^{n-1}). \end{aligned}$$
(126)

Theorem 11 is sharp, in the sense that the corresponding result fails for $p \in \left(\frac{n-1}{n}, 1\right)$. To see that this is the case, fix some vector $a \in \mathbb{C}^M \setminus \{0\}$ along with some point $z' \in \mathbb{R}^{n-1} \setminus \{0'\}$ and consider the function

$$u_{\star}(x',t) := K^{L}(x',t)a - K^{L}(x'-z',t)a \text{ for each } (x',t) \in \mathbb{R}^{n}_{+}.$$
 (127)

Then u_{\star} belongs to the space $\left[\mathcal{C}^{\infty}(\overline{\mathbb{R}^{n}_{+}} \setminus \{(0',0),(z',0)\})\right]^{M}$, satisfies $Lu_{\star} = 0$ in \mathbb{R}^{n}_{+} , and $\left(u_{\star}\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)(x') = 0$ for every aperture parameter $\kappa > 0$ and every point $x' \in \mathbb{R}^{n-1} \setminus \{0', z'\}$. In addition, we may choose a, z' such that u_{\star} is not identically zero (otherwise this would force $K^{L}(x', t)$ to be independent of x', a happenstance precluded by, e.g., (29) and (8)-(9)). Hence, on the one hand, the Poisson integral representation formula in the last line of (125) presently fails. On the other hand, from the well-known fact that

$$\delta_{0'} - \delta_{z'} \in H^p(\mathbb{R}^{n-1}) \text{ for each } p \in \left(\frac{n-1}{n}, 1\right), \tag{128}$$

it follows that

$$f := (\delta_{0'} - \delta_{z'})a \in \left[H^p(\mathbb{R}^{n-1})\right]^M \text{ for each } p \in \left(\frac{n-1}{n}, 1\right).$$
(129)

Moreover, f is related to the function u from (127) via $u_{\star}(x',t) = (P_t^L * f)(x')$ at each point $(x',t) \in \mathbb{R}^n_+$, with the convolution understood as in (42). As such, (44) implies that for each aperture parameter $\kappa > 0$ we have

$$\mathcal{N}_{\kappa} u_{\star} \in L^{p}(\mathbb{R}^{n-1}) \text{ for each } p \in \left(\frac{n-1}{n}, 1\right).$$
 (130)

Parenthetically, we wish to pint out that the membership in (130) may also be justified directly based on (127) and the estimates for the kernel function K^L from item (a) in Theorem 2 which, collectively, show that

for
$$x' \in \mathbb{R}^{n-1}$$
, the nontangential maximal function
 $(\mathcal{N}_{\kappa}u_{\star})(x')$ behaves like $|x'|^{1-n}$ if x' is near 0', like $|x'-z'|^{1-n}$
if x' is near z' , like $|x'|^{-n}$ if x' is near infinity, and is otherwise bounded.
(131)

Granted this, it follows that $\mathcal{N}_{\kappa}u_{\star} \in L^{p}(\mathbb{R}^{n-1})$ if and only if p(n-1) < n-1 and pn > n-1, a set of conditions equivalent to $p \in \left(\frac{n-1}{n}, 1\right)$.

$$\begin{cases}
 u_{\star} \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu_{\star} = 0 \quad \text{in} \quad \mathbb{R}^{n}_{+}, \\
 \mathcal{N}_{\kappa}u_{\star} \in L^{p}(\mathbb{R}^{n-1}) \quad \text{for each} \quad p \in \left(\frac{n-1}{n}, 1\right), \\
 u_{\star}\Big|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.} = 0 \quad \text{at} \ \mathcal{L}^{n-1}\text{-a.e. point in} \quad \mathbb{R}^{n-1}, \\
 \text{the function } u_{\star} \text{ is not identically zero in } \mathbb{R}^{n}_{+}.
\end{cases}$$
(132)

This is in sharp contrast to Theorem 11, and points to the fact that when p < 1 the pointwise nontangential boundary trace of a null-solution of the system L no longer characterizes the original function.

3.3. The subcritical growth Dirichlet problem. Recall that $SCG(\mathbb{R}^{n-1})$ stands for the class of functions exhibiting subcritical growth in \mathbb{R}^{n-1} , defined as in (99). In relation to this class, we have the following well-posedness result from [24].

Theorem 12. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Also, fix an aperture parameter $\kappa > 0$. Then the subcritical growth Dirichlet boundary value problem for L in \mathbb{R}^n_+ , formulated as

$$\begin{cases}
 u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\
 \lim_{\rho \to \infty} \|u\|_{*,\rho} = 0, \\
 u\Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \text{ point in } \mathbb{R}^{n-1},
\end{cases}$$
(133)

has a unique solution for each $f \in [SCG(\mathbb{R}^{n-1})]^M$. Moreover, the solution u of (133) is given by (32) and satisfies the following Weak Local Maximum Principle:

$$\sup_{B(0,\rho)\cap\mathbb{R}^{n}_{+}}|u| \leq C\left(\|f\|_{[L^{\infty}(B_{n-1}(0',2\rho))]^{M}} + \int_{\mathbb{R}^{n-1}\setminus B_{n-1}(0',2\rho)}\frac{\rho|f(y')|}{\rho^{n}+|y'|^{n}}\,dy'\right)$$
(134)

for each $\rho \in (0,\infty)$, where $C \in [1,\infty)$ depends only on L and n.

Note that having $f \in [\operatorname{SCG}(\mathbb{R}^{n-1})]^M$ ensures that f satisfies (31) which, in turn, allows us to define the solution u via the convolution with the Poisson kernel (cf. item (c) of Theorem 2). Uniqueness follows at once from Theorem 6. To close, we remark that the second condition imposed on f in (99), which amounts to saying that f has subcritical growth, is natural in the context of (133). Indeed, whenever u satisfies $\lim_{\rho \to \infty} \|u\|_{*,\rho} = 0$ and $f := u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-n.t.}$ exists \mathcal{L}^{n-1} -a.e. in \mathbb{R}^{n-1} it is not difficult to see that

$$\rho^{-1} \|f\|_{[L^{\infty}(B_{n-1}(0',\rho))]^{M}} \le \|u\|_{*,\rho} \to 0 \quad \text{as} \quad \rho \to \infty,$$
(135)

which ultimately implies the second condition in (99).

3.4. The L^{∞} -Dirichlet boundary value problem. Here we revisit Theorem 8 and consider the (initially forbidden) end-point $p = \infty$. Our result below is well-known in the particular case when $L = \Delta$, the Laplacian in \mathbb{R}^n , but all known proofs (e.g., that of [11, Theorem 4.8, p. 174], or that of [34, Proposition 1, p. 199]) make use of specialized properties of harmonic functions. Following [24], here we are able to treat the L^{∞} -Dirichlet boundary value problem in \mathbb{R}^n_+ for any homogeneous constant complex coefficient elliptic second-order system in a conceptually simple manner, relying on our more general result from Theorem 12.

Theorem 13. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n and fix an aperture parameter $\kappa > 0$. Then the L^{∞} -Dirichlet boundary value problem for L in \mathbb{R}^n_+ ,

$$\begin{cases}
 u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap L^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \\
 Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\
 u\Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \text{ point in } \mathbb{R}^{n-1},
\end{cases}$$
(136)

has a unique solution for each $f \in [L^{\infty}(\mathbb{R}^{n-1})]^M$. Moreover, the solution u of (136) is given by (32) and satisfies the Weak Maximum Principle

$$\|f\|_{[L^{\infty}(\mathbb{R}^{n-1})]^{M}} \le \|u\|_{[L^{\infty}(\mathbb{R}^{n}_{+})]^{M}} \le C \|f\|_{[L^{\infty}(\mathbb{R}^{n-1})]^{M}},$$
(137)

for some constant $C \in [1, \infty)$ that depends only on L and n.

Since

$$L^{\infty}(\mathbb{R}^{n-1}) \subset \mathrm{SCG}(\mathbb{R}^{n-1}), \tag{138}$$

and since (134) readily implies (137), we may regard (136) as a "sub-problem" of (133). This ensures existence (in the specified format), uniqueness, as well as the estimate claimed in (137).

3.5. The classical Dirichlet boundary value problem. Given $E \subseteq \mathbb{R}^m$, for some $m \in \mathbb{N}$, define $\mathcal{C}_b^0(E)$ to be the space of \mathbb{C} -valued functions defined on E which are continuous and bounded. The theorem below appears in [24]. The particular case when $L = \Delta$, the Laplacian in \mathbb{R}^n is a well-known, classical result (see, e.g., [3, Theorem 7.5, p. 148], or [11, Theorem 4.4, p. 170]), so the novelty here is the consideration of much more general operators.

Theorem 14. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n . Then the classical Dirichlet boundary value problem for L in \mathbb{R}^n_+ ,

$$\begin{aligned} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \mathcal{C}^{0}_{b}(\overline{\mathbb{R}^{n}_{+}}) \right]^{M}, \\ Lu &= 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ u \Big|_{\partial \mathbb{R}^{n}_{+}} &= f \quad in \quad \mathbb{R}^{n-1}, \end{aligned}$$

$$(139)$$

has a unique solution for each $f \in [\mathcal{C}^0_b(\mathbb{R}^{n-1})]^M$. Moreover, the solution u of (139) is given by (32) and satisfies the Weak Maximum Principle

$$\sup_{\mathbb{R}^{n-1}} |f| \le \sup_{\overline{\mathbb{R}^n_+}} |u| \le C \sup_{\mathbb{R}^{n-1}} |f| \tag{140}$$

for some constant $C \in [1, \infty)$ that depends only on L and n.

Existence is a consequence of item (c) of Theorem 2, uniqueness is implied by Theorem 13, and (140) follows from (137).

The nature of the constant C appearing in the Weak Maximum Principle (140) (as well as other related inequalities) has been studied by G. Kresin and V. Maz'ya in [13].

Open Question 6. In the context of Theorem 14, if the boundedness requirement is dropped both for the boundary datum and for the solution, does the resulting boundary value problem continue to be solvable?

This is known to be the case when $L = \Delta$, the Laplacian in \mathbb{R}^n ; see, e.g. [3, Theorem 7.11, p. 150].

3.6. The sublinear growth Dirichlet problem. Given a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ and $\theta \in [0, 1)$ we define the space of sublinear growth functions of order θ , denoted by $\mathrm{SLG}_{\theta}(E)$, as the collection of Lebesgue measurable functions $w : E \to \mathbb{C}$ satisfying

$$||w||_{\mathrm{SLG}_{\theta}(E)} := \mathrm{ess\,sup}_{x\in E} \frac{|w(x)|}{1+|x|^{\theta}} < \infty.$$
 (141)

Hence, $\operatorname{SLG}_0(E) = L^{\infty}(E)$. Also, it clear from definitions that for each continuous function $u \in \operatorname{SLG}_{\theta}(\mathbb{R}^n_+)$ we have

$$\|u\|_{*,\rho} \le \frac{1+\rho^{\theta}}{\rho} \|u\|_{\mathrm{SLG}_{\theta}(\mathbb{R}^{n}_{+})} \quad \text{for each} \quad \rho \in (0,\infty).$$

$$(142)$$

As a consequence, any continuous function in $SLG_{\theta}(\mathbb{R}^{n}_{+})$ with $\theta \in [0, 1)$ has subcritical growth, i.e.,

$$\lim_{\rho \to \infty} \|u\|_{*,\rho} = 0 \text{ for each } u \in \mathcal{C}^0(\mathbb{R}^n_+) \cap \mathrm{SLG}_\theta(\mathbb{R}^n_+) \text{ with } \theta \in [0,1).$$
(143)

In fact, for each continuous function $u: \mathbb{R}^n_+ \to \mathbb{C}$ we have

$$\|u\|_{\mathrm{SLG}_{\theta}(\mathbb{R}^{n}_{+})} = \sup_{\rho>0} \frac{\rho}{1+\rho^{\theta}} \|u\|_{*,\rho}.$$
(144)

Indeed, the right-pointing inequality is clear from (142), while the left-pointing inequality in (144) may be justified by writing

$$\frac{|u(x)|}{1+|x|^{\theta}} \leq \frac{|x|}{1+|x|^{\theta}} \Big(|x|^{-1} \cdot \sup_{B(0,|x|)} |u| \Big) = \frac{|x|}{1+|x|^{\theta}} ||u||_{*,|x|} \\
\leq \sup_{\rho>0} \frac{\rho}{1+\rho^{\theta}} ||u||_{*,\rho} \quad \text{for each} \quad x \in \mathbb{R}^{n}_{+},$$
(145)

and then taking the supremum over all $x \in \mathbb{R}^n_+$. Finally, we wish to note that

$$\operatorname{SLG}_{\theta}(\mathbb{R}^{n-1}) \subset \operatorname{SCG}(\mathbb{R}^{n-1})$$
 whenever $\theta \in [0,1).$ (146)

The following result from [24] extends Theorem 13 (which corresponds to the case when $\theta = 0$).

Theorem 15. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n , and fix an aperture parameter $\kappa > 0$ along with some exponent $\theta \in [0, 1)$. Then the sublinear growth Dirichlet boundary value problem for L in \mathbb{R}^n_+ , formulated as

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \mathrm{SLG}_{\theta}(\mathbb{R}^{n}_{+})\right]^{M}, \\ Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ u\Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \quad \mathbb{R}^{n-1}, \end{cases}$$
(147)

has a unique solution for each $f \in [SLG_{\theta}(\mathbb{R}^{n-1})]^M$. Moreover, the solution u of (147) is given by (32) and satisfies

$$\|f\|_{[\operatorname{SLG}_{\theta}(\mathbb{R}^{n-1})]^{M}} \le \|u\|_{[\operatorname{SLG}_{\theta}(\mathbb{R}^{n}_{+})]^{M}} \le C\|f\|_{[\operatorname{SLG}_{\theta}(\mathbb{R}^{n-1})]^{M}}$$
(148)

for some constant $C \in [1, \infty)$ depending only on L, n, and θ .

Thanks to (146) plus the fact that (134) and (144) readily imply (148), we may regard (147) as a "sub-problem" of (133). Such a point of view then guarantees existence (in the class of solutions specified in (147)), uniqueness, and also the estimate claimed in (148).

The linear function u(x',t) = ta for each $(x',t) \in \mathbb{R}^n_+$ (where $a \in \mathbb{C}^M \setminus \{0\}$ is a fixed vector) serves as a counterexample to the version of Theorem 15 corresponding to $\theta = 1$. Thus, restricting the exponent θ to [0,1) is optimal.

As first noted in [24], we also have a Fatou-type result in the context of functions with sublinear growth (extending the case $p = \infty$ of Theorem 11 which corresponds to $\theta = 0$). This reads as follows:

Theorem 16. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n and fix an aperture parameter $\kappa > 0$. Assume $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfies Lu = 0 in \mathbb{R}^n_+ . If $u \in [SLG_{\theta}(\mathbb{R}^n_+)]^M$ for some $\theta \in [0, 1)$ then

$$\begin{pmatrix} u \\ \partial \mathbb{R}^{n}_{+} \end{pmatrix} (x') \quad exists \ at \ \mathcal{L}^{n-1} \text{-} a.e. \ point \ x' \in \mathbb{R}^{n-1}, \\ u \\ \partial \mathbb{R}^{n}_{+} \quad belongs \ to \quad \left[\operatorname{SLG}_{\theta}(\mathbb{R}^{n-1}) \right]^{M} \subset \left[L^{1} \left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n}} \right) \right]^{M},$$

$$u(x',t) = \left(P_{t}^{L} * \left(u \\ \partial \mathbb{R}^{n}_{+} \right) \right) (x') \quad for \ each \ point \ (x',t) \in \mathbb{R}^{n}_{+}.$$

$$(149)$$

As a consequence of this and Theorem 15, in the present setting the following version of the Maximum Principle holds:

$$\left\|u\right|_{\partial\mathbb{R}^{n}_{+}}^{\kappa-n.t.}\left\|_{\left[\mathrm{SLG}_{\theta}(\mathbb{R}^{n-1})\right]^{M}}\approx\|u\|_{\left[\mathrm{SLG}_{\theta}(\mathbb{R}^{n}_{+})\right]^{M}}.$$
(150)

Since thanks to (142) we have

$$\int_{1}^{\infty} \|u\|_{*,t} \frac{dt}{t} \le \|u\|_{[\mathrm{SLG}_{\theta}(\mathbb{R}^{n}_{+})]^{M}} \int_{1}^{\infty} \frac{1+t^{\theta}}{t^{2}} dt < \infty,$$
(151)

we may invoke Corollary 1 to conclude that the properties listed in (101) hold. It remains to check that the second item in (149) holds, and this may be seen directly from definitions.

Once again, the linear function u(x',t) = ta for each $(x',t) \in \mathbb{R}^n_+$ (where $a \in \mathbb{C}^M \setminus \{0\}$ is a fixed vector) becomes a counterexample to the version of Theorem 16 corresponding to the end-point case $\theta = 1$. As such, restricting the exponent θ to [0,1) is sharp.

3.7. The Dirichlet problem with boundary data in Hölder spaces. Given $E \subset \mathbb{R}^m$ (for some $m \in \mathbb{N}$) and $\theta > 0$, we define the homogeneous Hölder space of order θ on E, denoted by $\dot{\mathcal{C}}^{\theta}(E)$, as the collection of functions $w : E \to \mathbb{C}$ satisfying

$$\|w\|_{\dot{\mathcal{C}}^{\theta}(E)} := \sup_{\substack{x,y\in E\\x\neq y}} \frac{|w(x) - w(y)|}{|x - y|^{\theta}} < \infty.$$
(152)

Also, define the inhomogeneous Hölder space of order θ on E as

$$\mathcal{C}^{\theta}(E) := \left\{ w \in \dot{\mathcal{C}}^{\theta}(E) : \sup_{E} |w| < \infty \right\},\tag{153}$$

and set $||w||_{\mathcal{C}^{\theta}(E)} := ||w||_{\dot{\mathcal{C}}^{\theta}(E)} + \sup_{E} |w|$ for each $w \in \mathcal{C}^{\theta}(E)$. Clearly,

$$\mathcal{C}^{\theta}(E) \subseteq \dot{\mathcal{C}}^{\theta}(E) \subseteq \mathrm{SLG}_{\theta}(E) \text{ for each } \theta > 0.$$
 (154)

In particular, together with (143) this implies that any function in $\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n}_{+})$ with $\theta \in (0, 1)$ has subcritical growth.

The well-posedness of the $\dot{\mathcal{C}}^{\theta}$ -Dirichlet problem was studied in [23] (see also [19]). Here we follow the approach in [24] which uses item (d) in Theorem 2 and Theorem 16 to give an alternative, conceptually simpler proof.

Theorem 17. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n , and fix $\theta \in (0,1)$. Then the $\dot{\mathcal{C}}^{\theta}$ -Dirichlet boundary value problem for L in \mathbb{R}^n_+ , formulated as

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}}) \right]^{M}, \\ Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ u \big|_{\partial \mathbb{R}^{n}_{+}} = f \quad on \quad \mathbb{R}^{n-1}, \end{cases}$$
(155)

has a unique solution for each $f \in [\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^M$. The solution u of (155) is given by (32) and there exists a constant $C = C(n, L, \theta) \in [1, \infty)$ with the property that

$$\|f\|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}} \le \|u\|_{[\dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}})]^{M}} \le C \|f\|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}}.$$
(156)

To prove existence, consider $p := \left(1 + \frac{\theta}{n-1}\right)^{-1}$ and note that this further implies $p \in \left(\frac{n-1}{n}, 1\right)$ and $\theta = (n-1)\left(\frac{1}{p}-1\right)$. In particular, with \sim denoting the equivalence relation identifying any two functions which differ by a constant (cf., e.g., [11, Theorem 5.30,

p.307]), we have $(H^p(\mathbb{R}^{n-1}))^* = \dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})/\sim$. Next, given an arbitrary function $f = (f_{\beta})_{1 \leq \beta \leq M} \in [\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^M$, it is meaningful to define $u(x',t) := (P_t^L * f)(x')$ for all $(x',t) \in \mathbb{R}^n_+$. Then $u \in [\mathcal{C}^{\infty}(\mathbb{R}^n_+)]^M$ satisfies Lu = 0 in \mathbb{R}^n_+ , and for each $j \in \{1,\ldots,n\}$ and each $(x',t) \in \mathbb{R}^n_+$ we have

$$t^{1-\theta}(\partial_j u)(x',t) = \left\{ \left\langle t^{1-(n-1)(\frac{1}{p}-1)}(\partial_j K^L_{\alpha\beta})(x'-\cdot,t), \left[f_\beta\right] \right\rangle \right\}_{1 \le \alpha \le M}$$
(157)

where $\langle \cdot, \cdot \rangle$ is the pairing between distributions belonging to the Hardy space $H^p(\mathbb{R}^{n-1})$ and equivalence classes (modulo constants) of functions belonging to $\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})$. In turn, based on (157), for each $(x',t) \in \mathbb{R}^n_+$ and $j \in \{1,\ldots,n\}$ we may estimate

$$|t^{1-\theta}(\partial_{j}u)(x',t)|$$

$$\leq ||t^{1-(n-1)(\frac{1}{p}-1)}(\partial_{j}K^{L})(x'-\cdot,t)||_{[H^{p}(\mathbb{R}^{n-1})]^{M\times M}}||f||_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}}.$$

$$(158)$$

In view of (37), this further entails the existence of a constant $C \in (0, \infty)$ with the property that

$$\sup_{(x',t)\in\mathbb{R}^n_+}\left\{t^{1-\theta}\big|(\nabla u)(x',t)\big|\right\} \le C\|f\|_{[\dot{\mathcal{C}}^\theta(\mathbb{R}^{n-1})]^M}.$$
(159)

On the other hand, a well-known elementary argument (of a purely real-variable nature, based solely on the Mean-Value Theorem; see, e.g., [23, §6, Step 4]) implies that, for some constant $C = C(n, \theta) \in (0, \infty)$,

$$\|u\|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n}_{+})]^{M}} \leq C \sup_{(x',t)\in\mathbb{R}^{n}_{+}} \Big\{ t^{1-\theta} \big| (\nabla u)(x',t) \big| \Big\}.$$
(160)

At this stage, (156) follows by combining (159) with (160), keeping in mind the natural identification $\dot{C}^{\theta}(\mathbb{R}^{n}_{+}) \equiv \dot{C}^{\theta}(\overline{\mathbb{R}^{n}_{+}})$. This finishes the proof of the existence for the problem (155), and the justification of (156). In view of (154), uniqueness for the problem (155) follows from Theorem 16.

As a byproduct of the above argument, we see that for each $\theta \in (0, 1)$ we have

$$\|u\|_{[\dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}})]^{M}} \approx \sup_{(x',t)\in\mathbb{R}^{n}_{+}} \left\{ t^{1-\theta} | (\nabla u)(x',t) | \right\} \approx \|u\|_{\partial\mathbb{R}^{n}_{+}} \|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}}$$

uniformly for $u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}}) \right]^{M}$ with $Lu = 0$ in \mathbb{R}^{n}_{+} . (161)

In this regard, let us also remark that for each exponent $\theta \in (0, 1)$ and each aperture parameter $\kappa > 0$ we also have

$$\|u\|_{[\dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}})]^{M}} \approx \sup_{x' \in \mathbb{R}^{n-1}} \|u\|_{[\dot{\mathcal{C}}^{\theta}(\Gamma_{\kappa}(x'))]^{M}} \text{ uniformly for}$$

$$u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \dot{\mathcal{C}}^{\theta}(\overline{\mathbb{R}^{n}_{+}})\right]^{M} \text{ with } Lu = 0 \text{ in } \mathbb{R}^{n}_{+}.$$
(162)

To justify this, let the function u be as in the last line above and set $f := u |_{\partial \mathbb{R}^n_+}$. Then, given any $x', y' \in \mathbb{R}^{n-1}$, if z is the point in $\overline{\Gamma_{\kappa}(x')} \cap \overline{\Gamma_{\kappa}(y')}$ closest to $\partial \mathbb{R}^n_+$ we may estimate

$$|f(x') - f(y')| \leq |u(x') - u(z)| + |u(y') - u(z)|$$

$$\leq ||u||_{[\dot{\mathcal{C}}^{\theta}(\Gamma_{\kappa}(x'))]^{M}} |x' - z|^{\theta} + ||u||_{[\dot{\mathcal{C}}^{\theta}(\Gamma_{\kappa}(y'))]^{M}} |y' - z|^{\theta}$$

$$\leq C \Big(\sup_{\xi \in \mathbb{R}^{n-1}} ||u||_{[\dot{\mathcal{C}}^{\theta}(\Gamma_{\kappa}(\xi))]^{M}} \Big) |x' - y'|^{\theta}, \qquad (163)$$

for some $C = C(\kappa, \theta) \in (0, \infty)$. Hence,

$$\|u\|_{\partial\mathbb{R}^{n}_{+}}\|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}} = \|f\|_{[\dot{\mathcal{C}}^{\theta}(\mathbb{R}^{n-1})]^{M}} \le C \sup_{x'\in\mathbb{R}^{n-1}} \|u\|_{[\dot{\mathcal{C}}^{\theta}(\Gamma_{\kappa}(x'))]^{M}}$$
(164)

which, together with (161), establishes the left-pointing inequality in the first line of (162). Since the right-pointing inequality is trivial, this concludes the proof of (162).

As an immediate consequence of Theorem 17 and Theorem 13 we obtain the following well-posedness result for the Dirichlet problem with boundary data from *inhomogeneous* Hölder spaces.

Corollary 2. Let L be an $M \times M$ homogeneous constant complex coefficient elliptic second-order system in \mathbb{R}^n , and fix $\theta \in (0,1)$. Then the \mathcal{C}^{θ} -Dirichlet boundary value problem for L in \mathbb{R}^n_+ , formulated as

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+}) \cap \mathcal{C}^{\theta}(\overline{\mathbb{R}^{n}_{+}}) \right]^{M}, \\ Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ u \big|_{\partial \mathbb{R}^{n}_{+}} = f \quad on \quad \mathbb{R}^{n-1}, \end{cases}$$
(165)

has a unique solution for each $f \in [\mathcal{C}^{\theta}(\mathbb{R}^{n-1})]^M$. The solution u of (165) is given by (32) and there exists a constant $C = C(n, L, \theta) \in [1, \infty)$ with the property that

$$\|f\|_{[\mathcal{C}^{\theta}(\mathbb{R}^{n-1})]^{M}} \le \|u\|_{[\mathcal{C}^{\theta}(\overline{\mathbb{R}^{n}_{+}})]^{M}} \le C \|f\|_{[\mathcal{C}^{\theta}(\mathbb{R}^{n-1})]^{M}}.$$
(166)

As mentioned earlier in the narrative (cf. (4)), the Lamé system of elasticity fits into the general framework considered in this paper and, as such, all results so far apply to this special system. In this vein, it is of interest to raise the following issue:

Open Question 7. Formulate and prove Fatou-type theorems and well-posedness results for various versions of the Dirichlet problem in the upper half-space, of the sort discussed in this paper, for the Stokes system of hydrodynamics.

3.8. The Dirichlet problem with data in BMO and VMO. In his ground breaking 1971 article [8], C. Fefferman writes "The main idea in proving [that the dual of the Hardy space H^1 is the John-Nirenberg space BMO] is to study the [harmonic] Poisson integral of a function in BMO." For example, the key PDE result announced by C. Fefferman

in [8] states that

a measurable function
$$f$$
 with $\int_{\mathbb{R}^{n-1}} |f(x')| (1+|x'|)^{-n} dx' < +\infty$ belongs to the space $BMO(\mathbb{R}^{n-1})$ if and only
if its Poisson integral $u : \mathbb{R}^n_+ \to \mathbb{R}$, with respect to the Laplace operator in \mathbb{R}^n , satisfies
$$\sup_{x'\in\mathbb{R}^{n-1}} \sup_{r>0} \left\{ r^{1-n} \int_{|x'-y'|< r} \int_0^r |(\nabla u)(y',t)|^2 t \, dt \, dx' \right\} < +\infty.$$
(167)

One of the primary aims in [23] was to advance this line of research by developing machinery capable of dealing with the scenario in which the Laplacian in (167) is replaced by much more general second-order elliptic systems with complex coefficients. To review the relevant results in this regard, some notation is needed.

A Borel measure μ in \mathbb{R}^n_+ is said to be a Carleson measure in \mathbb{R}^n_+ provided

$$\|\mu\|_{\mathcal{C}(\mathbb{R}^{n}_{+})} := \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{\mathcal{L}^{n-1}(Q)} \int_{0}^{\ell(Q)} \int_{Q} d\mu(x',t) < \infty,$$
(168)

where the supremum runs over all cubes Q in \mathbb{R}^{n-1} (with sides parallel to the coordinate axes), and $\ell(Q)$ is the side-length of Q. Call a Borel measure μ in \mathbb{R}^n_+ a vanishing Carleson measure whenever μ is a Carleson measure to begin with and, in addition,

$$\lim_{r \to 0^+} \left(\sup_{Q \subset \mathbb{R}^{n-1}, \, \ell(Q) \le r} \frac{1}{\mathcal{L}^{n-1}(Q)} \int_0^{\ell(Q)} \int_Q d\mu(x', t) \right) = 0.$$
(169)

Next, the Littlewood-Paley measure associated with a continuously differentiable function u in \mathbb{R}^n_+ is $|\nabla u(x',t)|^2 t dx' dt$, and we set

$$\|u\|_{**} := \sup_{Q \subset \mathbb{R}^{n-1}} \left(\frac{1}{\mathcal{L}^{n-1}(Q)} \int_0^{\ell(Q)} \int_Q |\nabla u(x',t)|^2 t \, dx' dt \right)^{\frac{1}{2}}.$$
 (170)

In particular, for a continuously differentiable function u in \mathbb{R}^n_+ we have

$$||u||_{**} < \infty \iff |\nabla u(x',t)|^2 t \, dx' dt \text{ is a Carleson measure in } \mathbb{R}^n_+.$$
(171)

The John-Nirenberg space $BMO(\mathbb{R}^{n-1})$, of functions of bounded mean oscillations in \mathbb{R}^{n-1} , is defined as the collection of complex-valued functions $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$ satisfying

$$||f||_{\text{BMO}(\mathbb{R}^{n-1})} := \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{\mathcal{L}^{n-1}(Q)} \int_{Q} \left| f(x') - f_{Q} \right| dx' < \infty,$$
(172)

where $f_Q := \frac{1}{\mathcal{L}^{n-1}(Q)} \int_Q f \, d\mathcal{L}^{n-1}$ for each cube Q in \mathbb{R}^{n-1} , and with the supremum taken over all such cubes Q. It turns out (cf., e.g., [9]) that

$$BMO(\mathbb{R}^{n-1}) \subset L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right)$$
(173)

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which opens the door for considering the convolution of the Poisson kernel from Theorem 1 with BMO functions in \mathbb{R}^{n-1} (cf. item (c) in Theorem 2). Clearly, for every $f \in L^1_{loc}(\mathbb{R}^{n-1})$ we have

$$\|f\|_{BMO(\mathbb{R}^{n-1})} = \|f + C\|_{BMO(\mathbb{R}^{n-1})}, \quad \forall C \in \mathbb{C}, \\\|f\|_{BMO(\mathbb{R}^{n-1})} = \|\tau_{z'}f\|_{BMO(\mathbb{R}^{n-1})}, \quad \forall z' \in \mathbb{R}^{n-1}, \\\|f\|_{BMO(\mathbb{R}^{n-1})} = \|\delta_{\lambda}f\|_{BMO(\mathbb{R}^{n-1})}, \quad \forall \lambda \in (0,\infty),$$
(174)

where $\tau_{z'}$ is the operator of translation by z', i.e., $(\tau_{z'}f)(x') := f(x' + z')$ for every $x' \in \mathbb{R}^{n-1}$, and δ_{λ} is the operator of dilation by λ , i.e., $(\delta_{\lambda}f)(x') := f(\lambda x')$ for every $x' \in \mathbb{R}^{n-1}$.

As visible from the first line of (174), it happens that $\|\cdot\|_{BMO(\mathbb{R}^{n-1})}$ is only a seminorm. Indeed, for every $f \in L^1_{loc}(\mathbb{R}^{n-1})$ we have $\|f\|_{BMO(\mathbb{R}^{n-1})} = 0$ if and only if f is a constant (in \mathbb{C}) at \mathcal{L}^{n-1} -a.e. in \mathbb{R}^{n-1} . Occasionally, we find it useful to mod out its null-space, in order to render the resulting quotient space Banach. Specifically, for two \mathbb{C} -valued Lebesgue measurable functions f, g defined in \mathbb{R}^{n-1} we say that $f \sim g$ provided f - g is constant \mathcal{L}^{n-1} -a.e. in \mathbb{R}^{n-1} . This is an equivalence relation and we let

$$[f] := \left\{ g : \mathbb{R}^{n-1} \to \mathbb{C} : g \text{ measurable and } f \sim g \right\}$$
(175)

denote the equivalence class of any given \mathbb{C} -valued Lebesgue measurable function f defined in \mathbb{R}^{n-1} . In particular, the quotient space

$$\widetilde{BMO}(\mathbb{R}^{n-1}) := \left\{ [f] : f \in BMO(\mathbb{R}^{n-1}) \right\}.$$
(176)

becomes complete (hence Banach) when equipped with the norm

$$\| [f] \|_{\widetilde{BMO}(\mathbb{R}^{n-1})} := \| f \|_{BMO(\mathbb{R}^{n-1})} \text{ for each } f \in BMO(\mathbb{R}^{n-1}).$$
(177)

Moving on, the Sarason space of \mathbb{C} -valued functions of vanishing mean oscillations in \mathbb{R}^{n-1} is defined by

$$VMO(\mathbb{R}^{n-1}) := \left\{ f \in BMO(\mathbb{R}^{n-1}) : \\ \lim_{r \to 0^+} \left(\sup_{Q \subset \mathbb{R}^{n-1}, \, \ell(Q) \le r} \frac{1}{\mathcal{L}^{n-1}(Q)} \int_Q \left| f(x') - f_Q \right| dx' \right) = 0 \right\}.$$
(178)

The space VMO(\mathbb{R}^{n-1}) turns out to be a closed subspace of BMO(\mathbb{R}^{n-1}). In fact, if UC(\mathbb{R}^{n-1}) stands for the space of \mathbb{C} -valued uniformly continuous functions in \mathbb{R}^{n-1} , then a well-known result of Sarason [30, Theorem 1, p. 392] implies that, in fact,

$$f \in BMO(\mathbb{R}^{n-1})$$
 belongs to the space $VMO(\mathbb{R}^{n-1})$ if and
only if there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset UC(\mathbb{R}^{n-1}) \cap$
 $BMO(\mathbb{R}^{n-1})$ such that $||f - f_j||_{BMO(\mathbb{R}^{n-1})} \longrightarrow 0$ as $j \to \infty$. (179)

Another characterization of $VMO(\mathbb{R}^{n-1})$ due to Sarason (cf. [30, Theorem 1, p. 392]) is as follows:

a given function
$$f \in BMO(\mathbb{R}^{n-1})$$
 actually belongs to
the space $VMO(\mathbb{R}^{n-1})$ if and only if $\lim_{\mathbb{R}^{n-1}\ni z'\to 0'} \|\tau_{z'}f - f\|_{BMO(\mathbb{R}^{n-1})} = 0.$ (180)

We are now ready to recall the first main result from [23]. This concerns the wellposedness of the BMO-Dirichlet problem in the upper half-space for systems L as in (1)-(2). The existence of a unique solution is established in the class of functions u satisfying a Carleson measure condition (expressed in terms of the finiteness of (170)). The formulation of the theorem emphasizes the fact that this contains as a "sub-problem" the VMO-Dirichlet problem for L in \mathbb{R}^n_+ (in which scenario u satisfies a vanishing Carleson measure condition).

Theorem 18. Let L be an $M \times M$ elliptic constant complex coefficient system as in (1)-(2), and fix an aperture parameter $\kappa > 0$. Then the BMO-Dirichlet boundary value problem for L in \mathbb{R}^n_+ , namely

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ \left|\nabla u(x',t)\right|^{2} t \, dx' dt \ is \ a \ Carleson \ measure \ in \ \mathbb{R}^{n}_{+}, \\ u\right|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \ \mathbb{R}^{n-1}, \end{cases}$$
(181)

has a unique solution for each $f \in [BMO(\mathbb{R}^{n-1})]^M$. Moreover, this unique solution satisfies the following additional properties:

(i) With P^L denoting the Poisson kernel for L in \mathbb{R}^n_+ from Theorem 1, one has the Poisson integral representation formula

$$u(x',t) = (P_t^L * f)(x'), \quad \forall (x',t) \in \mathbb{R}^n_+.$$
 (182)

(ii) The size of the solution is comparable to the size of the boundary datum, i.e., there exists $C = C(n, L) \in (1, \infty)$ with the property that

$$C^{-1} \|f\|_{[BMO(\mathbb{R}^{n-1})]^M} \le \|u\|_{**} \le C \|f\|_{[BMO(\mathbb{R}^{n-1})]^M}.$$
(183)

(iii) There exists a constant $C = C(n, L) \in (0, \infty)$ independent of u with the property that the following uniform BMO estimate holds:

$$\sup_{\varepsilon>0} \|u(\cdot,\varepsilon)\|_{[\mathrm{BMO}(\mathbb{R}^{n-1})]^M} \le C \|u\|_{**}.$$
(184)

Moreover, u satisfies a vanishing Carleson measure condition in \mathbb{R}^n_+ if and only if u converges to its boundary datum vertically in $[BMO(\mathbb{R}^{n-1})]^M$, i.e.,

$$\lim_{\varepsilon \to 0^+} \|u(\cdot,\varepsilon) - f\|_{[BMO(\mathbb{R}^{n-1})]^M} = 0 \iff \begin{cases} |\nabla u(x',t)|^2 t \, dx' dt \ is \\ a \ vanishing \ Carleson \\ measure \ in \ \mathbb{R}^n_+. \end{cases}$$
(185)

(iv) The following regularity results hold:

$$f \in \left[\text{VMO}(\mathbb{R}^{n-1}) \right]^M \iff \begin{cases} \left| \nabla u(x',t) \right|^2 t \, dx' dt \text{ is a vanishing} \\ Carleson \text{ measure in } \mathbb{R}^n_+ \end{cases}$$
(186)

$$\iff \lim_{\mathbb{R}^n_+ \ni z \to 0} \|\tau_z u - u\|_{**} = 0, \tag{187}$$

where $(\tau_z u)(x) := u(x+z)$ for each $x, z \in \mathbb{R}^n_+$.

As a consequence, the VMO-Dirichlet boundary value problem for L in \mathbb{R}^n_+ , i.e.,

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M}, \quad Lu = 0 \quad in \quad \mathbb{R}^{n}_{+}, \\ \left|\nabla u(x',t)\right|^{2} t \, dx' dt \ is \ a \ vanishing \ Carleson \ measure \ in \ \mathbb{R}^{n}_{+}, \\ u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} = f \quad at \ \mathcal{L}^{n-1}\text{-}a.e. \ point \ in \ \mathbb{R}^{n-1}, \end{cases}$$
(188)

has a unique solution for each $f \in [VMO(\mathbb{R}^{n-1})]^M$. Moreover, its solution is given by (182), satisfies (183)-(184), and

$$\lim_{\varepsilon \to 0^+} \|u(\cdot,\varepsilon) - f\|_{[\mathrm{BMO}(\mathbb{R}^{n-1})]^M} = 0.$$
(189)

It is reassuring to remark that replacing the original boundary datum f by f + Cwhere $C \in \mathbb{C}^M$ in (181) changes the solution u into u + C (given that convolution with the Poisson kernel reproduces constants from \mathbb{C}^M ; cf. (8)). As such, the BMO-Dirichlet problem for L in \mathbb{R}^n_+ is also well-posed, if uniqueness of the solution is now understood modulo constants from \mathbb{C}^M .

The proof of Theorem 18 given in [23] employs a quantitative Fatou-type theorem, which includes a Poisson integral representation formula along with a characterization of BMO in terms of boundary traces of null-solutions of elliptic systems in \mathbb{R}^n_+ . A concrete statement is given below in Theorem 19. Among other things, the said theorem shows that the demands formulated in the first two lines of (181) imply that the pointwise nontangential limit considered in the third line of (181) is always meaningful, and that the boundary datum should necessarily be selected from the space BMO. This theorem also highlights the fact that it is natural to seek a solution of the BMO Dirichlet problem by taking the convolution of the boundary datum with the Poisson kernel P^L associated with the system L. Finally, Theorem 19 readily implies the uniqueness of solution for the BMO-Dirichlet problem (181).

Theorem 19. Let L be an $M \times M$ elliptic system with constant complex coefficients as in (1)-(2) and consider P^L , the Poisson kernel in \mathbb{R}^n_+ associated with L as in Theorem 1. Also, fix an aperture parameter $\kappa > 0$. Then there exists a constant $C = C(L, n, \kappa) \in$ $(1,\infty)$ with the property that

$$\begin{cases} u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M} \\ Lu = 0 \quad in \quad \mathbb{R}^{n}_{+} \\ and \quad \|u\|_{**} < \infty \end{cases}$$

$$\implies \begin{cases} u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.} exists \ a.e. \ in \quad \mathbb{R}^{n-1}, \ lies \ in \quad \left[\mathrm{BMO}(\mathbb{R}^{n-1})\right]^{M}, \\ u(x',t) = \left(P_{t}^{L} * \left(u\Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.}\right)\right)(x') \ for \ all \ (x',t) \in \mathbb{R}^{n}_{+}, \\ and \ C^{-1}\|u\|_{**} \leq \|u\|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-n.t.}\|_{[\mathrm{BMO}(\mathbb{R}^{n-1})]^{M}} \leq C\|u\|_{**}. \end{cases}$$

$$(190)$$

In fact, the following characterization of $[BMO(\mathbb{R}^{n-1})]^M$, adapted to the system L, holds:

$$\left[\mathrm{BMO}(\mathbb{R}^{n-1})\right]^M = \left\{ u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} : u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^n_+) \right]^M, \ Lu = 0 \ in \ \mathbb{R}^n_+, \ \|u\|_{**} < \infty \right\}.$$
(191)

Moreover,

$$\mathrm{LMO}(\mathbb{R}^n_+) := \left\{ u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^n_+) \right]^M : Lu = 0 \text{ in } \mathbb{R}^n_+ \text{ and } \|u\|_{**} < \infty \right\}$$
(192)

is a linear space on which $\|\cdot\|_{**}$ is a seminorm with null-space \mathbb{C}^M , the quotient space $\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M$ becomes complete (hence Banach) when equipped with $\|\cdot\|_{**}$, and the nontangential pointwise trace operator acting on equivalence classes in the context

$$\mathrm{LMO}(\mathbb{R}^{n}_{+})/\mathbb{C}^{M} \ni [u] \longmapsto \left[u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\mathrm{n.t.}} \right] \in \left[\widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}) \right]^{M}$$
(193)

is a well-defined linear isomorphism between Banach spaces, where [u] in (193) denotes the equivalence class of u in $\mathrm{LMO}(\mathbb{R}^n_+)/\mathbb{C}^M$ and $\left[u\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}\right]$ is interpreted as in (175).

There is also a counterpart of the Fatou-type result stated as Theorem 19 emphasizing the space VMO in place of BMO. Specifically, the following theorem was proved in [23].

Theorem 20. Let L be an $M \times M$ elliptic system with constant complex coefficients as in (1)-(2) and consider P^L , the associated Poisson kernel for L in \mathbb{R}^n_+ from Theorem 1. Also, fix an aperture parameter $\kappa > 0$. Then for any function

$$u \in \left[\mathcal{C}^{\infty}(\mathbb{R}^{n}_{+})\right]^{M} \text{ satisfying } Lu = 0 \text{ in } \mathbb{R}^{n}_{+} \text{ and } \|u\|_{**} < \infty$$
(194)

one has

$$\left| \nabla u(x',t) \right|^2 t \, dx' dt \text{ is} \\ a \text{ vanishing Carleson} \\ measure \text{ in } \mathbb{R}^n_+ \end{array} \right\} \Longrightarrow \begin{cases} u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-\text{n.t.}} exists \text{ a.e. in } \mathbb{R}^{n-1}, \text{ and} \\ u \Big|_{\partial \mathbb{R}^n_+}^{\kappa-\text{n.t.}} \text{ is in } \left[\text{VMO}(\mathbb{R}^{n-1}) \right]^M. \end{cases}$$
(195)

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Furthermore, the following characterization of the space $\left[\text{VMO}(\mathbb{R}^{n-1})\right]^M$, adapted to the system L, holds:

$$\left[\operatorname{VMO}(\mathbb{R}^{n-1})\right]^{M} = \left\{ u \Big|_{\partial \mathbb{R}^{n}_{+}}^{\kappa-\operatorname{n.t.}} : u \in \operatorname{LMO}(\mathbb{R}^{n}_{+}) \text{ and } \left| \nabla u(x',t) \right|^{2} t \, dx' dt \\ \text{ is a vanishing Carleson measure in } \mathbb{R}^{n}_{+} \right\}.$$

$$(196)$$

There is yet another version of the space of functions of vanishing mean oscillations which we would like to recall. To set the stage, let $C_0^0(\mathbb{R}^{n-1})$ be the space of all continuous functions in \mathbb{R}^{n-1} which vanish at infinity, equipped with the supremum norm. Also, let $\{R_j\}_{1\leq j\leq n-1}$ be the family of Riesz transforms in \mathbb{R}^{n-1} . Define $\text{CMO}(\mathbb{R}^{n-1})$ as the collection of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$ which may be expressed as

$$f = f_0 + \sum_{j=1}^{n-1} R_j f_j \text{ in } \mathbb{R}^{n-1} \text{ in } \mathbb{R}^{n-1}$$

for some $f_0, f_1, \dots, f_{n-1} \in \mathcal{C}_0^0(\mathbb{R}^{n-1}),$ (197)

and set

$$||f||_{\mathrm{CMO}(\mathbb{R}^{n-1})} := \inf \left\{ ||f_0||_{L^{\infty}(\mathbb{R}^{n-1})} + \sum_{j=1}^{n-1} ||f_j||_{L^{\infty}(\mathbb{R}^{n-1})} \right\}$$
(198)

where the infimum is taken over all possible representations of f as in (197). Then $\text{CMO}(\mathbb{R}^{n-1})$ becomes a Banach space, which may be alternatively characterized as the pre-dual of the Hardy space $H^1(\mathbb{R}^{n-1})$ (cf. [28, (2.0'), p. 185]; see also [4] and [6] for more on this topic). One may also show that $\text{CMO}(\mathbb{R}^{n-1})$ is a closed subspace of $\text{BMO}(\mathbb{R}^{n-1})$, and $\mathcal{C}_0^0(\mathbb{R}^{n-1})$ is dense in $\text{CMO}(\mathbb{R}^{n-1})$. Hence,

$$\operatorname{CMO}(\mathbb{R}^{n-1})$$
 is the closure of $\mathcal{C}_0^0(\mathbb{R}^{n-1})$ in $\operatorname{BMO}(\mathbb{R}^{n-1})$. (199)

However, the Sarason space VMO(\mathbb{R}^{n-1}) (from (178)) is strictly larger than CMO(\mathbb{R}^{n-1}). In relation to the latter version of the space of functions of vanishing mean oscillations we wish to pose the following question.

Open Question 8. Formulate and prove a well-posedness result for the Dirichlet problem in the upper half-space, for an $M \times M$ elliptic second-order homogeneous constant complex coefficient system L, with boundary data from $[\text{CMO}(\mathbb{R}^{n-1})]^M$. Also, prove a Fatou-type theorem for null-solutions of L in \mathbb{R}^n_+ , which naturally accompanies the said well-posedness result.

To address these issues, a new brand of Carleson measure must be identified.

We close by recording the following result proved in [23]. The first item can be thought of as an analogue of Fefferman's theorem, characterizing BMO as in (167), in the case of elliptic systems with complex coefficients. The second item may be viewed as a characterization of VMO in the spirit of Fefferman's original result.

Theorem 21. Let L be an $M \times M$ elliptic system with constant complex coefficients as in (1)-(2) and consider the Poisson kernel P^L in \mathbb{R}^n_+ associated with the system L as in Theorem 1. Also, assume $f: \mathbb{R}^{n-1} \to \mathbb{C}^M$ is a Lebesgue measurable function satisfying

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1+|x'|^n} \, dx' < \infty.$$
(200)

Finally, let u be the Poisson integral of f in \mathbb{R}^n_+ with respect to the system L, i.e., $u : \mathbb{R}^n_+ \to \mathbb{C}^M$ is given by $u(x',t) := (P^L_t * f)(x')$ for each $(x',t) \in \mathbb{R}^n_+$. Then the following statements are true.

- (a) The function f belongs to the space $[BMO(\mathbb{R}^{n-1})]^M$ if and only if $|\nabla u(x',t)|^2 t \, dx' dt$ is a Carleson measure in \mathbb{R}^n_+ (or, equivalently, $||u||_{**} < \infty$; cf. (171)).
- (b) The function f belongs to the space $\left[\text{VMO}(\mathbb{R}^{n-1}) \right]^M$ if and only if $|\nabla u(x',t)|^2 t \, dx' dt$ is a vanishing Carleson measure in \mathbb{R}^n_+ .

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