

# THE DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS WITH DATA IN KÖTHE FUNCTION SPACES

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**ABSTRACT.** We show that the boundedness of the Hardy-Littlewood maximal operator on a Köthe function space  $\mathbb{X}$  and on its Köthe dual  $\mathbb{X}'$  is equivalent to the well-posedness of the  $\mathbb{X}$ -Dirichlet and  $\mathbb{X}'$ -Dirichlet problems in  $\mathbb{R}_+^n$  in the class of all second-order, homogeneous, elliptic systems, with constant complex coefficients. As a consequence, we obtain that the Dirichlet problem for such systems is well-posed for boundary data in Lebesgue spaces, variable exponent Lebesgue spaces, Lorentz spaces, Zygmund spaces, as well as their weighted versions. We also discuss a version of the aforementioned result which contains, as a particular case, the Dirichlet problem for elliptic systems with data in the classical Hardy space  $H^1$ , and the Beurling-Hardy space  $HA^p$  for  $p \in (1, \infty)$ . Based on the well-posedness of the  $L^p$ -Dirichlet problem we then prove the uniqueness of the Poisson kernel associated with such systems, as well as the fact that they generate a strongly continuous semigroup in natural settings. Finally, we establish a general Fatou type theorem guaranteeing the existence of the pointwise nontangential boundary trace for null-solutions of such systems.

## 1. INTRODUCTION, STATEMENT OF MAIN RESULTS, AND EXAMPLES

Let  $M \in \mathbb{N}$  be fixed and consider the second-order, homogeneous,  $M \times M$  system, with constant complex coefficients, written (with the usual convention of summation over repeated indices in place) as

$$(1.1) \quad Lu := \left( \partial_r (a_{rs}^{\alpha\beta} \partial_s u_\beta) \right)_{1 \leq \alpha \leq M},$$

when acting on a  $\mathcal{C}^2$  vector-valued function  $u = (u_\beta)_{1 \leq \beta \leq M}$  defined in the upper-half space  $\mathbb{R}_+^n := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$ ,  $n \geq 2$ . A standing

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assumption in this paper is that  $L$  is **elliptic** in the sense that there exists a real number  $\kappa_o > 0$  such that the following Legendre-Hadamard condition is satisfied:

$$(1.2) \quad \begin{aligned} \operatorname{Re} [a_{rs}^{\alpha\beta} \xi_r \xi_s \overline{\eta_\alpha} \eta_\beta] &\geq \kappa_o |\xi|^2 |\eta|^2 \quad \text{for every} \\ \xi &= (\xi_r)_{1 \leq r \leq n} \in \mathbb{R}^n \quad \text{and} \quad \eta = (\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \end{aligned}$$

Two basic examples to keep in mind are the Laplacian  $L := \Delta$  in  $\mathbb{R}^n$ , and the Lamé system

$$(1.3) \quad Lu := \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \quad u = (u_1, \dots, u_n) \in \mathcal{C}^2,$$

where the constants  $\lambda, \mu \in \mathbb{R}$  (typically called Lamé moduli) are assumed to satisfy

$$(1.4) \quad \mu > 0 \quad \text{and} \quad 2\mu + \lambda > 0,$$

a condition actually equivalent to the demand that the Lamé system (1.3) satisfies the Legendre-Hadamard ellipticity condition (1.2).

As is known from the seminal work of S. Agmon, A. Douglis, and L. Nirenberg in [1] and [2], every operator  $L$  as in (1.1)-(1.2) has a Poisson kernel, denoted by  $P^L$ , an object whose properties mirror the most basic characteristics of the classical harmonic Poisson kernel

$$(1.5) \quad P^\Delta(x') := \frac{2}{\omega_{n-1}} \frac{1}{(1 + |x'|^2)^{\frac{n}{2}}}, \quad \forall x' \in \mathbb{R}^{n-1},$$

where  $\omega_{n-1}$  is the area of the unit sphere  $S^{n-1}$  in  $\mathbb{R}^n$ . For details, see Theorem 2.4 below. Here we only wish to note that, using the notation  $P_t(x') := t^{1-n} P(x'/t)$  for each  $t \in (0, \infty)$  and  $x' \in \mathbb{R}^{n-1}$ , where  $P$  is a generic function defined in  $\mathbb{R}^{n-1}$ , it follows that there exists some  $C \in (0, \infty)$  such that

$$(1.6) \quad |P_t^L(x')| \leq C \frac{t}{(t^2 + |x'|^2)^{\frac{n}{2}}}, \quad \forall x' \in \mathbb{R}^{n-1}, \quad \forall t \in (0, \infty).$$

The main goal of this paper is to establish well-posedness results for the Dirichlet problem for a system  $L$ , as above, in  $\mathbb{R}_+^n$  formulated in terms of certain types of function spaces (made precise below).

Prior to formulating the most general result in this paper, some comments on the notation used are in order. The symbol  $\mathcal{M}$  is reserved for the Hardy-Littlewood maximal operator in  $\mathbb{R}^{n-1}$ ; see (2.9). Also, given a function  $u$  defined in  $\mathbb{R}_+^n$ , by  $\mathcal{N}u$  we shall denote the nontangential maximal function of  $u$ ; see (2.3) for a precise definition. Next, by  $u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$  we denote the nontangential limit of the given function  $u$  on the boundary of the upper half-space (canonically identified with  $\mathbb{R}^{n-1}$ ), as defined in (2.4). Going further, denote by  $\mathbb{M}$  the collection of all (equivalence classes of) Lebesgue measurable functions  $f : \mathbb{R}^{n-1} \rightarrow [-\infty, \infty]$  such that  $|f| < \infty$  a.e. in  $\mathbb{R}^{n-1}$ .

Also, call a subset  $\mathbb{Y}$  of  $\mathbb{M}$  a **function lattice** if the following properties hold:

- (i) whenever  $f, g \in \mathbb{M}$  satisfy  $0 \leq f \leq g$  a.e. in  $\mathbb{R}^{n-1}$  and  $g \in \mathbb{Y}$  then necessarily  $f \in \mathbb{Y}$ ;
- (ii)  $0 \leq f \in \mathbb{Y}$  implies  $\lambda f \in \mathbb{Y}$  for every  $\lambda \in (0, \infty)$ ;
- (iii)  $0 \leq f, g \in \mathbb{Y}$  implies  $\max\{f, g\} \in \mathbb{Y}$ .

In passing, note that, granted (i), one may replace (ii)-(iii) above by the condition:  $0 \leq f, g \in \mathbb{Y}$  implies  $f + g \in \mathbb{Y}$ . As usual, we set  $\log_+ t := \max\{0, \ln t\}$  for each  $t \in (0, \infty)$ . Finally, we alert the reader that the notation employed does not always distinguish between vector and scalar valued functions (which should be clear from context).

**Theorem 1.1.** *Let  $L$  be a system as in (1.1)-(1.2), and assume that  $\mathbb{X}, \mathbb{Y}$  are arbitrary collections of measurable functions satisfying*

$$(1.7) \quad \mathbb{X} \subset L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right), \quad \mathbb{Y} \subset L^1\left(\mathbb{R}^{n-1}, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'\right),$$

$$(1.8) \quad \mathbb{Y} \text{ is a function lattice}, \quad \mathcal{M}\mathbb{X} \subset \mathbb{Y}.$$

*Then the  $(\mathbb{X}, \mathbb{Y})$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ ,*

$$(1.9) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \quad \text{in } \mathbb{R}_+^n, \\ \mathcal{N}u \in \mathbb{Y}, \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in \mathbb{X}, \end{cases}$$

*has a unique solution. Moreover, the solution  $u$  of (1.9) is given by*

$$(1.10) \quad u(x', t) = (P_t^L * f)(x') \quad \text{for all } (x', t) \in \mathbb{R}_+^n,$$

*where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$ , and satisfies*

$$(1.11) \quad \mathcal{N}u(x') \leq C \mathcal{M}f(x'), \quad \forall x' \in \mathbb{R}^{n-1},$$

*for some constant  $C \in [1, \infty)$  that depends only on  $L$  and  $n$ .*

Regarding the formulation of Theorem 1.1, we wish to note that the first condition in (1.7) is actually redundant, and we have only included it for its pedagogical value (as it makes the proof of the existence of a solution for (1.9) most natural). Indeed, a more general result of this flavor holds, namely:

$$(1.12) \quad \mathbb{X} \subset \mathbb{M} \text{ and } \mathcal{M}f \not\equiv \infty \text{ for each } f \in \mathbb{X} \implies \mathbb{X} \subset L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right).$$

Granted this, it is clear that the first inclusion in (1.7) is implied by the last condition in (1.8) and the second condition in (1.7). As regards the justification of (1.12), let  $f \in \mathbb{X}$  be arbitrary. Then the hypotheses in (1.12) imply that there exists some  $x'_0 \in \mathbb{R}^{n-1}$  such that  $\mathcal{M}f(x'_0) < \infty$  in which case, for some finite constant  $C = C(n, x'_0) > 0$ , we may estimate

$$(1.13) \quad \int_{\mathbb{R}^{n-1}} |f(x')| \frac{1}{1 + |x'|^n} dx' \leq C \int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1 + |x' - x'_0|^n} dx' \leq C \mathcal{M}f(x'_0) < \infty,$$

where the next-to-last inequality follows from a familiar dyadic annular decomposition argument (in the spirit of (3.18)). Thus, (1.12) is true.

The particular case  $\mathbb{X} = \mathbb{Y}$  holds a special significance (in this vein, see Theorem 1.4 below). Incidentally, in this scenario the first condition in (1.7) is simply implied by the second condition in (1.7) alone. This being said, the case  $\mathbb{X} \neq \mathbb{Y}$  is natural to consider, as it arises commonly in practice. For example, the Dirichlet problem (1.9) is well-posed for any system  $L$  as in (1.1)-(1.2) provided, for a given  $p \in (1, \infty)$ ,

$$(1.14) \quad \mathbb{X} := L^1(\mathbb{R}^{n-1}) \cap L^p(\mathbb{R}^{n-1}) \quad \text{and} \quad \mathbb{Y} := L^{1,\infty}(\mathbb{R}^{n-1}) \cap L^p(\mathbb{R}^{n-1}),$$

since conditions (1.7)-(1.8) are easily verified in this case. We stress that in the formulation of Theorem 1.1 the set  $\mathbb{X}$  is not required to be a function lattice, and this is a relevant observation for the  $(H^1, L^1)$ -Dirichlet problem discussed below in Corollary 1.2 (cf. also Corollary 1.3 for a similar phenomenon).

The proof of Theorem 1.1 in §4 makes strong use of the results established in §3. More specifically, the second inclusion in (1.7) ensures (keeping in mind the function lattice property for  $\mathbb{Y}$ ) the applicability of Theorem 3.2, which yields uniqueness. The first inclusion in (1.7) guarantees the applicability of Theorem 3.1, which shows that the function  $u$  as in (1.10) belongs to  $\mathcal{C}^\infty(\mathbb{R}_+^n)$  and satisfies  $Lu = 0$  in  $\mathbb{R}_+^n$  as well as  $u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f$  and (1.11). Granted the latter property (and bearing in mind the function lattice property for  $\mathbb{Y}$ ), the last condition in (1.8) then guarantees that

$$(1.15) \quad f \in \mathbb{X} \text{ and } u \text{ as in (1.10)} \implies \mathcal{N}u \in \mathbb{Y}.$$

This proves existence in Theorem 1.1. It is worth noting that  $\mathcal{M}\mathbb{X} \subset \mathbb{Y}$  may be replaced in the formulation of Theorem 1.1 (without affecting the conclusions) by the weaker condition (1.15). This is significant, because the latter holds even though the former fails in the important case of the Dirichlet problem with data from the Hardy space, when

$$(1.16) \quad \mathbb{X} := H^1(\mathbb{R}^{n-1}) \quad \text{and} \quad \mathbb{Y} := L^1(\mathbb{R}^{n-1}).$$

This permits us to prove (see §4 for details) the following well-posedness result.

**Corollary 1.2.** *The  $(H^1, L^1)$ -Dirichlet boundary value problem in  $\mathbb{R}_+^n$  is well-posed for each system  $L$  as in (1.1)-(1.2).*

In fact, the weaker condition in the left-hand side of (1.15) is also relevant in other scenarios such as the Dirichlet problem with data from the Beurling-Hardy space, when

$$(1.17) \quad \mathbb{X} := \text{HA}^p(\mathbb{R}^{n-1}) \quad \text{and} \quad \mathbb{Y} := A^p(\mathbb{R}^{n-1}) \quad \text{for some } p \in (1, \infty).$$

Above,  $A^p(\mathbb{R}^{n-1})$  is the classical (convolution) algebra introduced by A. Beurling in [7], while  $\text{HA}^p(\mathbb{R}^{n-1})$  is the Hardy space associated with the Beurling algebra  $A^p(\mathbb{R}^{n-1})$  as in [16] (following work in the complex plane in [8]). For concrete definitions the reader is referred to §4, where the proof of the following well-posedness result may also be found.

**Corollary 1.3.** *For each  $p \in (1, \infty)$ , the  $(\text{HA}^p, A^p)$ -Dirichlet boundary value problem in  $\mathbb{R}_+^n$  is well-posed whenever  $L$  is a system as in (1.1)-(1.2).*

As is apparent from the statement of Theorem 1.1, devising practical ways for checking the validity of the inclusions in (1.7) becomes a significant issue that deserves further attention. One natural, and also general, setting where the named inclusions may be equivalently rephrased as the membership of the intervening weight functions to dual spaces is that of Köthe function spaces. Since the latter class of function spaces plays a significant role for us here, we proceed to summarize their definition and basic properties (more details may be found in Bennett and Sharpley [6] where the terminology employed is that of Banach function spaces; cf. also [12], [22], [43]). Specifically, call a mapping  $\|\cdot\| : \mathbb{M} \rightarrow [0, \infty]$  a **function norm** provided the following properties are satisfied for all  $f, g \in \mathbb{M}$ :

- (1)  $\|f\| = \||f|\|$ , and  $\|f\| = 0$  if and only if  $f = 0$  a.e. in  $\mathbb{R}^{n-1}$ ;
- (2)  $\|f + g\| \leq \|f\| + \|g\|$ , and  $\|\lambda f\| = |\lambda| \|f\|$  for each  $\lambda \in \mathbb{R}$ ;
- (3) if  $|f| \leq |g|$  a.e. in  $\mathbb{R}^{n-1}$  then  $\|f\| \leq \|g\|$ ;
- (4) if  $\{f_k\}_{k \in \mathbb{N}} \subset \mathbb{M}$  is a sequence such that  $|f_k|$  increases to  $|f|$  pointwise a.e. in  $\mathbb{R}^{n-1}$  as  $k \rightarrow \infty$ , then  $\|f_k\|$  increases to  $\|f\|$  as  $k \rightarrow \infty$ ;
- (5) if  $E \subset \mathbb{R}^{n-1}$  is a measurable set of finite measure then its characteristic function  $\mathbf{1}_E$  satisfies  $\|\mathbf{1}_E\| < \infty$ , and  $\int_E |f(x')| dx' \leq C_E \|f\|$  where  $C_E < \infty$  depends on  $E$ , but not on  $f$ .

Given a function norm  $\|\cdot\|$ , the set

$$(1.18) \quad \mathbb{X} := \{f \in \mathbb{M} : \|f\| < \infty\}$$

is referred to as a **Köthe function space** on  $(\mathbb{R}^{n-1}, dx')$ . In such a scenario, we shall write  $\|\cdot\|_{\mathbb{X}}$  in place of  $\|\cdot\|$  in order to emphasize the connection between the function norm  $\|\cdot\|$  and its associated Köthe function space

$\mathbb{X}$ . Then  $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$  is a complete normed vector subspace of  $\mathbb{M}$ , hence a Banach space. It is apparent from the above definitions that many of the classical function spaces in analysis are actually Köthe function spaces. This includes ordinary Lebesgue spaces, variable exponent Lebesgue spaces, Orlicz spaces, Lorentz spaces, mixed-normed spaces, Marcinkiewicz spaces, Morrey spaces, etc. Typically, function spaces whose definitions take into account cancellation or differentiability properties of the functions, such as Hardy spaces, BMO, Sobolev spaces, etc., fail to be Köthe function spaces.

Starting with a Köthe function space  $\mathbb{X}$ , we can define its Köthe dual (also known as its associate space in the terminology of [6]) according to

$$(1.19) \quad \begin{aligned} \mathbb{X}' &:= \{f \in \mathbb{M} : \|f\|_{\mathbb{X}'} < \infty\} \text{ where, for each } f \in \mathbb{M}, \\ \|f\|_{\mathbb{X}'} &:= \sup \left\{ \int_{\mathbb{R}^{n-1}} |f(x') g(x')| dx' : g \in \mathbb{X}, \|g\|_{\mathbb{X}} \leq 1 \right\}. \end{aligned}$$

One can check that  $\|\cdot\|_{\mathbb{X}'}$  is indeed a function norm, hence  $\mathbb{X}'$  is itself a Köthe function space.

An immediate consequence of the above definitions is the generalized Hölder's inequality:

$$(1.20) \quad \int_{\mathbb{R}^{n-1}} |f(x') g(x')| dx' \leq \|f\|_{\mathbb{X}} \|g\|_{\mathbb{X}'}, \quad \text{for all } f \in \mathbb{X}, g \in \mathbb{X}'.$$

In this regard, let us also record here the following characterization of the Köthe dual given in [6, Lemma 2.6, p. 10]:

$$(1.21) \quad \mathbb{X}' = \left\{ g \in \mathbb{M} : \int_{\mathbb{R}^{n-1}} |f(x') g(x')| dx' < \infty \text{ for each } f \in \mathbb{X} \right\}.$$

Moreover,

$$(1.22) \quad (\mathbb{X}')' = \mathbb{X},$$

i.e., the Köthe dual space of  $\mathbb{X}'$  is again  $\mathbb{X}$ . As a consequence, the function norm on  $\mathbb{X}$  may be expressed in terms of the function norm on  $\mathbb{X}'$  according to

$$(1.23) \quad \|f\|_{\mathbb{X}} = \sup \left\{ \int_{\mathbb{R}^{n-1}} |f(x') g(x')| dx' : g \in \mathbb{X}', \|g\|_{\mathbb{X}'} \leq 1 \right\}, \quad \forall f \in \mathbb{X}.$$

For further reference it will be of interest to note that

$$(1.24) \quad \mathbf{1}_E \in \mathbb{X} \cap \mathbb{X}' \text{ if } E \subset \mathbb{R}^{n-1} \text{ is a measurable set of finite measure,}$$

and

$$(1.25) \quad \mathbb{X} \subset L_{\text{loc}}^1(\mathbb{R}^{n-1}), \quad \mathbb{X}' \subset L_{\text{loc}}^1(\mathbb{R}^{n-1}).$$

The key observation is that whenever  $\mathbb{X}, \mathbb{Y}$  are Köthe function spaces then  $\mathbb{Y}$  is a function lattice by design and, thanks to (1.21), the inclusions in (1.7)

are equivalent to the memberships

$$(1.26) \quad \frac{1}{1 + |x'|^n} \in \mathbb{X}' \quad \text{and} \quad \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} \in \mathbb{Y}'.$$

Furthermore, if the last condition in (1.8) is strengthened to

$$(1.27) \quad \mathcal{M} : \mathbb{X} \longrightarrow \mathbb{Y} \quad \text{boundedly,}$$

then by (1.11) and the monotonicity of the function norm in  $\mathbb{Y}$  it follows that there exists a constant  $C = C(n, L, \mathbb{X}, \mathbb{Y}) \in (0, \infty)$  with the property that the solution  $u$  of (1.9) satisfies

$$(1.28) \quad \|\mathcal{N}u\|_{\mathbb{Y}} \leq C \|f\|_{\mathbb{X}}.$$

One convenient practical way of ensuring that (1.26) holds is to check that  $\mathcal{M}$  is bounded on  $\mathbb{X}'$  and  $\mathbb{Y}'$ . This is a consequence of (1.24) and Lemma 2.1, in the body of the paper.

In the important special case of Köthe function spaces satisfying  $\mathbb{X} = \mathbb{Y}$ , the first condition in (1.26) becomes redundant (as it is implied by the second). In this scenario, if

$$(1.29) \quad \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} \in \mathbb{X}' \quad \text{and} \quad \mathcal{M}\mathbb{X} \subset \mathbb{X}$$

then the  $\mathbb{X}$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ , formulated as in (1.9) with  $\mathbb{Y} = \mathbb{X}$ , is well-posed. Moreover,

$$(1.30) \quad \mathcal{M} \text{ bounded on } \mathbb{X} \text{ implies } \|\mathcal{N}u\|_{\mathbb{X}} \leq C \|f\|_{\mathbb{X}}.$$

Let us also note here that, as seen from (1.29) and Lemma 2.1, the first condition in (1.29) may also be expressed in terms of the Hardy-Littlewood maximal operator as

$$(1.31) \quad \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)}) \in \mathbb{X}',$$

where  $\mathcal{M}^{(2)}$  is the two-fold composition of  $\mathcal{M}$  with itself, and where  $B_{n-1}(0', 1)$  denotes the  $(n-1)$ -dimensional Euclidean ball of radius 1 centered at the origin  $0' = (0, \dots, 0) \in \mathbb{R}^{n-1}$ . In particular,

$$(1.32) \quad \text{if } \mathcal{M} \text{ is bounded on } \mathbb{X}' \text{ then the first condition in (1.29) holds.}$$

As a consequence of the above considerations, we have the following notable result showing that the boundedness of the Hardy-Littlewood maximal operator on  $\mathbb{X}$  and  $\mathbb{X}'$  is equivalent to the well-posedness of the  $\mathbb{X}$ -Dirichlet and  $\mathbb{X}'$ -Dirichlet boundary value problems in  $\mathbb{R}_+^n$  for the class of all second-order, homogeneous, elliptic systems, with constant complex coefficients.

**Theorem 1.4.** *Assume that  $L$  is a system as in (1.1)-(1.2), and suppose  $\mathbb{X}$  is a Köthe function space such that*

$$(1.33) \quad \mathcal{M} \text{ is bounded both on } \mathbb{X} \text{ and } \mathbb{X}'.$$

Then the  $\mathbb{X}$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ ,

$$(1.34) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \quad \text{in } \mathbb{R}_+^n, \\ \mathcal{N}u \in \mathbb{X}, \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in \mathbb{X}, \end{cases}$$

is well-posed. In addition, the solution  $u$  of (1.34) is given by  $u(x', t) = (P_t^L * f)(x')$  for all  $(x', t) \in \mathbb{R}_+^n$ , where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$ . Also,

$$(1.35) \quad \|\mathcal{N}u\|_{\mathbb{X}} \approx \|f\|_{\mathbb{X}},$$

where the constants involved depend only on  $\mathbb{X}$ ,  $n$ , and  $L$ .

Moreover, the  $\mathbb{X}'$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ , formulated analogously to (1.34) (with  $\mathbb{X}'$  replacing  $\mathbb{X}$ ) is also well-posed, and the solution enjoys the same type of properties as above.

Finally, the above result is sharp in the sense that the solvability of both the  $\mathbb{X}$ -Dirichlet and the  $\mathbb{X}'$ -Dirichlet boundary value problems for the class of all second-order, homogeneous, elliptic systems, with constant complex coefficients (in the form of convolution with the Poisson kernel) with naturally accompanying bounds (as in (1.35)) is equivalent to the boundedness of the Hardy-Littlewood maximal operator  $\mathcal{M}$  both on  $\mathbb{X}$  and on  $\mathbb{X}'$ .

Assuming Theorem 1.1, the proof of Theorem 1.4 is rather short. Indeed, the discussion preceding its statement gives the well-posedness of the  $\mathbb{X}$ -Dirichlet boundary value problem. Furthermore, since (1.22) entails that the hypothesis (1.33) is stable under replacing  $\mathbb{X}$  by  $\mathbb{X}'$ , the well-posedness of the  $\mathbb{X}'$ -Dirichlet boundary value problem follows as well.

As regards the sharpness claim from the last part of the statement, first assume the solvability of the  $\mathbb{X}$ -Dirichlet boundary value problem for the Laplacian in  $\mathbb{R}_+^n$  (in the form of convolution with the Poisson kernel) with naturally accompanying bounds. Note that, with  $P^\Delta$  as in (1.5), whenever  $0 \leq f \in \mathbb{M}$  we may estimate, for each  $(x', t) \in \mathbb{R}_+^n$ ,

$$(1.36) \quad \begin{aligned} u(x', t) &= (P_t^\Delta * f)(x') = \frac{2}{\omega_{n-1}} \int_{\mathbb{R}^{n-1}} \frac{t}{(t^2 + |x' - y'|^2)^{\frac{n}{2}}} f(y') dy' \\ &\geq C_n \int_{B_{n-1}(x', t)} f(y') dy'. \end{aligned}$$

Hence,

$$(1.37) \quad (\mathcal{N}u)(x') \geq \sup_{t>0} u(x', t) \geq C_n (\mathcal{M}f)(x'), \quad \text{for each } x' \in \mathbb{R}^{n-1},$$



which, together with the upper estimate in (1.35), implies the boundedness of  $\mathcal{M}$  on  $\mathbb{X}$ . Likewise, the solvability of the  $\mathbb{X}'$ -Dirichlet boundary value problem for the Laplacian in  $\mathbb{R}_+^n$  yields the boundedness of  $\mathcal{M}$  on  $\mathbb{X}'$ . This finishes the proof of Theorem 1.4.

As is apparent from the above proof, the solvability of both the  $\mathbb{X}$ -Dirichlet and the  $\mathbb{X}'$ -Dirichlet boundary value problems for the Laplacian in  $\mathbb{R}_+^n$  (in the form of convolution with the Poisson kernel) with naturally accompanying bounds implies the boundedness of  $\mathcal{M}$  both on  $\mathbb{X}$  and  $\mathbb{X}'$ . As a consequence, the solvability of the  $\mathbb{X}$ -Dirichlet and the  $\mathbb{X}'$ -Dirichlet boundary value problems for the Laplacian in  $\mathbb{R}_+^n$  (in the manner described above) is equivalent to the solvability of the  $\mathbb{X}$ -Dirichlet and the  $\mathbb{X}'$ -Dirichlet boundary value problems in  $\mathbb{R}_+^n$  for all systems  $L$  as in (1.1)-(1.2).

Here we also wish to remark that, under the background assumptions in Theorem 1.4, the first three conditions in (1.34) imply that the non-tangential pointwise trace  $u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}$  exists a.e. in  $\mathbb{R}^{n-1}$ . Indeed, this becomes a consequence of a general Fatou type result established in Theorem 6.1, upon observing that

$$(1.38) \quad \mathbb{X} \subset L^1\left(\mathbb{R}^{n-1}, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'\right).$$

In turn, thanks to (1.21), the latter condition is equivalent to  $\frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} \in \mathbb{X}'$  which is further implied by Lemma 2.1 and (1.24).

At this stage we find it instructive to illustrate the scope of Theorems 1.1-1.4 by providing two examples of interest.

**Example 1: Ordinary Lebesgue spaces.** For  $p \in (1, \infty)$ ,  $\mathbb{X} := L^p(\mathbb{R}^{n-1})$  is a Köthe function space, with Köthe dual  $\mathbb{X}' = L^{p'}(\mathbb{R}^{n-1})$  with  $1/p + 1/p' = 1$ . Hence, in this case (1.33) holds. As such, Theorem 1.4 shows that the  $L^p$ -Dirichlet boundary value problem in  $\mathbb{R}_+^n$ ,

$$(1.39) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}), \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in L^p(\mathbb{R}^{n-1}), \end{cases}$$

is well-posed for any system  $L$  as in (1.1)-(1.2). Moreover, the solution is given by (1.10) and satisfies naturally accompanying bounds. Of course, one can also arrive at the same conclusion using Theorem 1.1 instead, since (1.7)-(1.8) are readily checked for  $\mathbb{X} = \mathbb{Y} := L^p(\mathbb{R}^{n-1})$  with  $p \in (1, \infty)$ .

In §5, the well-posedness of the  $L^p$ -Dirichlet problem (1.39) is then used as a tool for establishing the uniqueness of the (Agmon-Douglis-Nirenberg)

Poisson kernel for the system  $L$  (from Theorem 2.4), and to show that the said kernel satisfies the semigroup property (cf. Theorem 5.1).

**Example 2: Weighted Lebesgue spaces.** Given  $p \in (1, \infty)$ , along with an a.e. positive and finite measurable function  $w$  defined on  $\mathbb{R}^{n-1}$ , let  $L^p(\mathbb{R}^{n-1}, w)$  denote the Lebesgue space of  $p$ -th power integrable functions in the measure space  $(\mathbb{R}^{n-1}, w(x') dx')$ . For a system  $L$  as in (1.1)-(1.2) the  $L^p(\mathbb{R}^{n-1}, w)$ -Dirichlet problem then reads:

$$(1.40) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \quad \text{in } \mathbb{R}_+^n, \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}, w), \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \in L^p(\mathbb{R}^{n-1}, w). \end{cases}$$

Theorem 1.1 may then be invoked in order to show that (with  $A_p(\mathbb{R}^{n-1})$  denoting the class of Muckenhoupt weights, as defined in (2.22))

(1.41) if  $L$  is a system as in (1.1)-(1.2),  $1 < p < \infty$ , and  $w \in A_p(\mathbb{R}^{n-1})$  then the  $L^p(\mathbb{R}^{n-1}, w)$ -Dirichlet problem (1.40) is well-posed, the solution  $u$  is given by (1.10), and satisfies  $\|\mathcal{N}u\|_{L^p(\mathbb{R}^{n-1}, w)} \leq C\|f\|_{L^p(\mathbb{R}^{n-1}, w)}$ .

To see that this is the case, note that  $\mathbb{X} = \mathbb{Y} = L^p(\mathbb{R}^{n-1}, w)$  satisfy (1.8) (taking into account Muckenhoupt's classical result), whereas the second embedding in (1.7) is checked by estimating for every  $h \in L^p(\mathbb{R}^{n-1}, w)$

$$(1.42) \quad \begin{aligned} & \int_{\mathbb{R}^{n-1}} \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} |h(x')| dx' \\ & \leq C \int_{\mathbb{R}^{n-1}} |h(x')| w(x')^{\frac{1}{p}} \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)})(x') w(x')^{-\frac{1}{p}} dx' \\ & \leq C \|h\|_{L^p(\mathbb{R}^{n-1}, w)} \left\| \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)}) \right\|_{L^{p'}(\mathbb{R}^{n-1}, w^{1-p'})} \\ & \leq C \|h\|_{L^p(\mathbb{R}^{n-1}, w)} \left\| \mathbf{1}_{B_{n-1}(0',1)} \right\|_{L^{p'}(\mathbb{R}^{n-1}, w^{1-p'})} \\ & \leq C \|h\|_{L^p(\mathbb{R}^{n-1}, w)} w(B_{n-1}(0',1))^{-\frac{1}{p}}, \end{aligned}$$

where we have used Lemma 2.1 for the first inequality, Hölder's inequality for the second, that  $\mathcal{M}$  is bounded on  $L^{p'}(\mathbb{R}^{n-1}, w^{1-p'})$  since  $w \in A_p(\mathbb{R}^{n-1})$  if and only if  $w^{1-p'} \in A_{p'}(\mathbb{R}^{n-1})$  in the third and, lastly, that  $w \in A_p(\mathbb{R}^{n-1})$ . This takes care of the well-posedness, while the corresponding bound follows from (1.11) and the boundedness of  $\mathcal{M}$  on  $L^p(\mathbb{R}^{n-1}, w)$ .

In this vein, it is worth noting that, as (1.37) shows, the bound in (1.41) in the case when  $L = \Delta$  necessarily places the weight function  $w$  in the Muckenhoupt class  $A_p(\mathbb{R}^{n-1})$ .

One may well wonder whether Theorem 1.4 is also effective in the current setting. However, this is not the case. To illustrate the root of the problem note that, technically speaking,  $L^p(\mathbb{R}^{n-1}, w)$  is not a Köthe function space on  $(\mathbb{R}^{n-1}, dx')$  according to the terminology used earlier. Altering the definition so that  $L^p(\mathbb{R}^{n-1}, w)$  would be a Köthe function space requires working with  $(\mathbb{R}^{n-1}, w(x') dx')$  as the underlying measure space, and such a change affects the manner in which the Köthe dual is computed. Indeed, the Köthe dual of  $L^p(\mathbb{R}^{n-1}, w)$  (which now has to be taken with respect to the measure space  $(\mathbb{R}^{n-1}, w(x') dx')$ ) is  $L^{p'}(\mathbb{R}^{n-1}, w)$ . However,  $\mathcal{M}$  is not necessarily bounded on this space, so (1.33) cannot be ensured.

So far we have seen that the Dirichlet problem with data in ordinary  $L^p$  spaces can be treated by Theorem 1.4, though this theorem ceases to be effective in the case of weighted  $L^p$  spaces. The question now becomes:

(1.43) Is there a suitable version of Theorem 1.4 targeted to more specialized Köthe function spaces, such as rearrangement invariant spaces, devised for the purpose of treating not just  $L^p(\mathbb{R}^{n-1}, w)$ , but a variety of other weighted Köthe spaces?

Recall that a Köthe function space  $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$  is said to be **rearrangement invariant** provided the function norm  $\|f\|_{\mathbb{X}}$  of any  $f \in \mathbb{X}$  may be expressed in terms of the measure of the level sets of that function. The reader is referred to §4 for a more detailed discussion, which also elaborates on the notion of lower and upper Boyd indices, denoted by  $p_{\mathbb{X}}$  and  $q_{\mathbb{X}}$  (our definition ensures that  $p_{\mathbb{X}} = q_{\mathbb{X}} = p$  if  $\mathbb{X} = L^p(\mathbb{R}^{n-1})$ ). Given a weight  $w$  on  $\mathbb{R}^{n-1}$ , if  $f_w^*$  denotes the decreasing rearrangement of  $f$  with respect to the measure  $w(x') dx'$ , the weighted version  $\mathbb{X}(w)$  of the Köthe function space  $\mathbb{X}$  is defined as

$$(1.44) \quad \mathbb{X}(w) := \{f \in \mathbb{M} : \|f_w^*\|_{\overline{\mathbb{X}}} < \infty\}, \quad \|f\|_{\mathbb{X}(w)} := \|f_w^*\|_{\overline{\mathbb{X}}},$$

where  $\overline{\mathbb{X}}$  is the rearrangement invariant function space on  $[0, \infty)$  associated with the original  $\mathbb{X}$  as in Luxemburg's representation theorem. One can check that if  $\mathbb{X} := L^p(\mathbb{R}^{n-1})$ ,  $p \in (1, \infty)$ , then  $\mathbb{X}(w) = L^p(\mathbb{R}^{n-1}, w)$ .

The theorem answering the question posed in (1.43) is as follows.

**Theorem 1.5.** *Let  $L$  be a system as in (1.1)-(1.2), and let  $\mathbb{X}$  be a rearrangement invariant space whose lower and upper Boyd indices satisfy*

$$(1.45) \quad 1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty.$$

Then for every Muckenhoupt weight  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$ , the  $\mathbb{X}(w)$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ ,

$$(1.46) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ \mathcal{N}u \in \mathbb{X}(w), \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in \mathbb{X}(w), \end{cases}$$

has a unique solution. Furthermore, the solution  $u$  of (1.46) is given by  $u(x', t) = (P_t^L * f)(x')$  for all  $(x', t) \in \mathbb{R}_+^n$ , where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$ , and there exists a constant  $C = C(n, L, \mathbb{X}, w) \in (0, \infty)$  with the property that

$$(1.47) \quad \|\mathcal{N}u\|_{\mathbb{X}(w)} \leq C \|f\|_{\mathbb{X}(w)}.$$

As a consequence of the classical result of Lorentz-Shimogaki, given a rearrangement invariant space  $\mathbb{X}$ , condition (1.45) is equivalent to (1.33), i.e., to the fact that  $\mathcal{M}$  is bounded on both  $\mathbb{X}$  and  $\mathbb{X}'$ . Thus, in the class of rearrangement invariant spaces, Theorem 1.5 may be viewed as a weighted version of Theorem 1.4 (to which the latter reduces when the weight is a constant). As was the case with Theorem 1.4, we also have that Theorem 1.5 is sharp; its proof is presented in §4, and the strategy relies on Theorem 1.1. This requires verifying the embedding

$$(1.48) \quad \mathbb{X}(w) \subset L^1\left(\mathbb{R}^{n-1}, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'\right).$$

A direct approach based on duality, along the lines of (1.42), quickly runs into difficulties (due to the general nature of  $\mathbb{X}(w)$ , in contrast to the particular case of  $L^p(\mathbb{R}^{n-1}, w)$  considered in (1.42)). This being said, the fact that (1.42) can be carried out for all weights  $w \in A_p(\mathbb{R}^{n-1})$  eventually allows us to use Rubio de Francia's extrapolation in the context of rearrangement invariant spaces (cf. [12]) in order to derive a similar estimate in  $\mathbb{X}(w)$  (cf. Lemma 4.5 for actual details).

In spite of its elegance and sharpness, Theorem 1.5 is confined to the class of rearrangement invariant spaces. An example of interest, lying outside the latter class, is that of variable exponent Lebesgue spaces. As discussed below, in this setting it is Theorem 1.4 which may be employed in order to treat the corresponding Dirichlet problem.

**Example 3: Variable exponent Lebesgue spaces.** Given a (Lebesgue) measurable function  $p(\cdot) : \mathbb{R}^{n-1} \rightarrow (1, \infty)$ , the variable Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  is defined as the collection of all measurable functions  $f$  such

that, for some  $\lambda > 0$ ,

$$(1.49) \quad \int_{\mathbb{R}^{n-1}} \left( \frac{|f(x')|}{\lambda} \right)^{p(x')} dx' < \infty.$$

Here and elsewhere, we follow the custom of writing  $p(\cdot)$  instead of  $p$  in order to emphasize that the exponent is a function and not necessarily a constant. The set  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  becomes a Köthe function space when equipped with the function norm

$$(1.50) \quad \|f\|_{L^{p(\cdot)}(\mathbb{R}^{n-1})} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^{n-1}} \left( \frac{|f(x')|}{\lambda} \right)^{p(x')} dx' \leq 1 \right\}.$$

This family of spaces generalizes the scale of ordinary Lebesgue spaces. Indeed, if  $p(x') \equiv p_0$ , then  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  equals  $L^{p_0}(\mathbb{R}^{n-1})$ . The Köthe dual space of  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  is  $L^{p'(\cdot)}(\mathbb{R}^{n-1})$ , where the conjugate exponent function  $p'(\cdot)$  is uniquely defined by the demand that

$$(1.51) \quad \frac{1}{p(x')} + \frac{1}{p'(x')} = 1, \quad \forall x' \in \mathbb{R}^{n-1}.$$

Associated to  $p(\cdot)$  we introduce the following natural parameters:

$$(1.52) \quad p_- := \operatorname{ess\,inf}_{\mathbb{R}^{n-1}} p(\cdot) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{\mathbb{R}^{n-1}} p(\cdot).$$

To apply Theorem 1.4 to  $\mathbb{X} := L^{p(\cdot)}(\mathbb{R}^{n-1})$ , we need  $\mathcal{M}$  to be bounded on both  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  and  $L^{p'(\cdot)}(\mathbb{R}^{n-1})$ . This, in turn, is known to imply that  $1 < p_- \leq p_+ < \infty$ ; see [11]. Assuming  $1 < p_- \leq p_+ < \infty$ , it has been shown in [14] that  $\mathcal{M}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  if and only if  $\mathcal{M}$  is bounded on  $L^{p'(\cdot)}(\mathbb{R}^{n-1})$ . Therefore, Theorem 1.4 gives the following result:

$$(1.53) \quad \begin{aligned} &\text{whenever } L \text{ is a second-order system as in (1.1)-(1.2),} \\ &1 < p_- \leq p_+ < \infty, \text{ and if } \mathcal{M} \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^{n-1}), \\ &\text{the } L^{p(\cdot)}(\mathbb{R}^{n-1})\text{-Dirichlet problem for } L \text{ in } \mathbb{R}_+^n \text{ is well-posed.} \end{aligned}$$

Moreover, the sharpness of Theorem 1.4 yields a characterization of the boundedness of the Hardy-Littlewood maximal operator on  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  and on  $L^{p'(\cdot)}(\mathbb{R}^{n-1})$  in terms of the well-posedness of the  $L^{p(\cdot)}(\mathbb{R}^{n-1})$ -Dirichlet and  $L^{p'(\cdot)}(\mathbb{R}^{n-1})$ -Dirichlet problems in  $\mathbb{R}_+^n$ .

Let us further augment the above discussion by noting that, as proved in [11] and [30], the operator  $\mathcal{M}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  if  $p(\cdot)$  satisfies the following log-Hölder continuity conditions: there exist constants  $C \in [0, \infty)$  and  $p_\infty \in [0, \infty)$  such that for each  $x', y' \in \mathbb{R}^{n-1}$ ,

$$(1.54) \quad |p(x') - p(y')| \leq \frac{C}{-\log |x' - y'|} \quad \text{whenever } 0 < |x' - y'| \leq 1/2,$$

and

$$(1.55) \quad |p(x') - p_\infty| \leq \frac{C}{\log(e + |x'|)}.$$

We refer the reader to [10] and [15] for full details and complete references.

Moving on, we discuss two more classes of spaces for which Theorem 1.5 applies.

**Example 4: Weighted Lorentz spaces.** Let  $f^*$  denote the decreasing rearrangement of a function  $f \in \mathbb{M}$  (cf. (4.31)). For  $0 < p, q < \infty$ , define

$$(1.56) \quad \|f\|_{L^{p,q}(\mathbb{R}^{n-1})} := \left( \int_0^\infty f^*(s)^q s^{q/p-1} ds \right)^{1/q},$$

and, corresponding to  $q = \infty$ ,

$$(1.57) \quad \|f\|_{L^{p,\infty}(\mathbb{R}^{n-1})} := \sup_{0 < s < \infty} [f^*(s) s^{1/p}].$$

Then set

$$(1.58) \quad L^{p,q}(\mathbb{R}^{n-1}) := \{f \in \mathbb{M} : \|f\|_{L^{p,q}(\mathbb{R}^{n-1})} < \infty\}.$$

For  $0 < p < \infty$  and  $0 < q \leq \infty$ , the Lorentz spaces just defined are only quasi-normed spaces, but when  $1 < p < \infty$  and  $1 \leq q \leq \infty$ , or when  $p = 1$  and  $1 \leq q < \infty$ , they are equivalent to normed spaces. Also,

if  $1 < p < \infty$  and  $1 \leq q \leq \infty$ , or  $p = 1$  and  $1 \leq q < \infty$ , then

$$(1.59) \quad \mathbb{X} := L^{p,q}(\mathbb{R}^{n-1}) \text{ is a rearrangement invariant function space}$$

with lower and upper Boyd indices given by  $p_{\mathbb{X}} = q_{\mathbb{X}} = p$ .

The spaces  $\mathbb{X}(w)$  are the weighted Lorentz spaces  $L^{p,q}(\mathbb{R}^{n-1}, w(x')dx')$  obtained by replacing  $f^*$  with  $f_w^*$  in (1.56)-(1.58). Granted (1.59), Theorem 1.5 applies and yields the well-posedness of the Dirichlet problem in  $\mathbb{R}_+^n$  for a system  $L$  as in (1.1)-(1.2) with data in  $L^{p,q}(\mathbb{R}^{n-1}, w(x')dx')$  provided  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , and  $w \in A_p(\mathbb{R}^{n-1})$ . In particular, this well-posedness result holds for data in the standard Lorentz spaces  $L^{p,q}(\mathbb{R}^{n-1})$  with  $1 < p < \infty$  and  $1 \leq q \leq \infty$ .

**Example 5: Weighted Orlicz spaces.** Given a Young function  $\Phi$ , define the Orlicz space  $L^\Phi(\mathbb{R}^{n-1})$  to be the function space associated with the Luxemburg norm

$$(1.60) \quad \|f\|_{L^\Phi(\mathbb{R}^{n-1})} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^{n-1}} \Phi \left( \frac{|f(x')|}{\lambda} \right) dx' \leq 1 \right\}.$$

Then  $\mathbb{X} := L^\Phi(\mathbb{R}^{n-1})$  is a rearrangement invariant function space. It turns out that its weighted version  $\mathbb{X}(w)$ , originally defined as in (1.44), may be described as above with the Lebesgue measure replaced by  $w(x') dx'$ . Clearly the Lebesgue spaces are Orlicz spaces with  $\Phi(t) := t^p$ . Other examples

include the Zygmund spaces  $L^p(\log L)^\alpha$ ,  $1 < p < \infty$ ,  $\alpha \in \mathbb{R}$ , which are defined using  $\Phi(t) := t^p \log(e+t)^\alpha$ . In this case,  $p_{\mathbb{X}} = q_{\mathbb{X}} = p$ , so Theorem 1.5 applies and yields the well-posedness of the Dirichlet problem in  $\mathbb{R}_+^n$  for a system  $L$  as in (1.1)-(1.2) with data in the weighted Zygmund spaces  $L^p(\log L)^\alpha(\mathbb{R}^{n-1}, w(x')dx')$ ,  $1 < p < \infty$ ,  $\alpha \in \mathbb{R}$ , and  $w \in A_p(\mathbb{R}^{n-1})$ .

The spaces  $L^p + L^q$  and  $L^p \cap L^q$  can also be treated as Orlicz spaces, with  $\Phi(t) \approx \max\{t^p, t^q\}$  and  $\Phi(t) \approx \min\{t^p, t^q\}$ , respectively. In both cases,  $p_{\mathbb{X}} = \min\{p, q\}$  and  $q_{\mathbb{X}} = \max\{p, q\}$ . Hence, if  $1 < \min\{p, q\}$  and  $\max\{p, q\} < \infty$  then Theorem 1.5 applies. Note that for these and other Orlicz spaces, the Boyd indices can be computed directly from the function  $\Phi$  (see [12, Chapter 4]).

*Remark 1.6.* As the alert reader has perhaps noted, in the applications of Theorem 1.1 (such as those discussed in (1.14), (1.16), Theorem 1.4, Theorem 1.5, as well as in Examples 1-5) we have taken the set  $\mathbb{X}$  to actually be a linear subspace of  $\mathbb{M}$ . This is no accident since, in general, starting with  $\mathbb{X}, \mathbb{Y}$  merely satisfying (1.7)-(1.8), if  $\widehat{\mathbb{X}}$  is the linear span of  $\mathbb{X}$  in  $\mathbb{M}$ , then the pair  $\widehat{\mathbb{X}}, \mathbb{Y}$  continue to satisfy (1.7)-(1.8). Indeed, this is readily seen from the sublinearity of  $\mathcal{M}$  and the fact that  $\mathbb{Y}$  is a function lattice. In particular, for any system  $L$  as in (1.1)-(1.2), the  $(\widehat{\mathbb{X}}, \mathbb{Y})$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$  is uniquely solvable in the same manner as before.

We conclude our list of examples by discussing another significant case when Theorem 1.1 applies.

**Example 6: Morrey spaces.** Recall that the Morrey scale in  $\mathbb{R}^{n-1}$  consists of spaces  $\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})$  defined for each  $p \in (1, \infty)$  and  $\lambda \in (0, n-1)$  according to

$$(1.61) \quad \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}) := \left\{ f \in L_{\text{loc}}^p(\mathbb{R}^{n-1}) : \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})} < \infty \right\}$$

where

$$(1.62) \quad \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})} := \sup_{x' \in \mathbb{R}^{n-1}, r > 0} \left( r^{-\lambda} \int_{B_{n-1}(x', r)} |f(y')|^p dy' \right)^{1/p}.$$

Given a function  $f \in L_{\text{loc}}^p(\mathbb{R}^{n-1})$ , break up

$$(1.63) \quad \int_{\mathbb{R}^{n-1}} |f(x')| \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx' = \sum_{j=0}^{\infty} I_j$$

where

$$(1.64) \quad I_0 := \int_{B_{n-1}(0', 2)} |f(x')| \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'$$

and, for each  $j \in \mathbb{N}$ ,

$$(1.65) \quad I_j := \int_{B_{n-1}(0', 2^{j+1}) \setminus B_{n-1}(0', 2^j)} |f(x')| \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'.$$

Use Hölder's inequality and (1.62) to estimate

$$(1.66) \quad \begin{aligned} I_j &\leq \frac{1 + (j+1) \ln 2}{2^{j(n-1)}} \int_{B_{n-1}(0', 2^{j+1})} |f(x')| dx' \\ &\leq 2^{n-1} \left( \frac{\omega_{n-2}}{n-1} \right)^{\frac{p-1}{p}} \frac{1 + (j+1) \ln 2}{2^{(j+1)(n-1)/p}} \left( \int_{B_{n-1}(0', 2^{j+1})} |f(x')|^p dx' \right)^{1/p} \\ &\leq 2^{n-1} \left( \frac{\omega_{n-2}}{n-1} \right)^{\frac{p-1}{p}} \frac{1 + (j+1) \ln 2}{2^{(j+1)(n-1-\lambda)/p}} \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})}, \quad \forall j \in \mathbb{N}, \end{aligned}$$

and, likewise,

$$(1.67) \quad I_0 \leq 2^{n-1} \left( \frac{\omega_{n-2}}{n-1} \right)^{\frac{p-1}{p}} \frac{1 + \ln 2}{2^{(n-1-\lambda)/p}} \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})}.$$

Bearing in mind that

$$(1.68) \quad \lambda < n-1 \implies \sum_{j=0}^{\infty} \frac{1 + (j+1) \ln 2}{2^{(j+1)(n-1-\lambda)/p}} < \infty,$$

then yields

$$(1.69) \quad \int_{\mathbb{R}^{n-1}} |f(x')| \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx' \leq C_{n,p,\lambda} \|f\|_{\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})}$$

for some finite constant  $C_{n,p,\lambda} > 0$  independent of  $f$ . This proves that if  $p \in (1, \infty)$  and  $\lambda \in (0, n-1)$  then

$$(1.70) \quad \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}) \subset L^1\left(\mathbb{R}^{n-1}, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'\right).$$

In addition, it is clear from (1.61)-(1.62) that

$$(1.71) \quad \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}) \text{ is a function lattice if } 1 < p < \infty \text{ and } 0 < \lambda < n-1,$$

and it has been proved by F. Chiarenza and M. Frasca in [9] that

$$(1.72) \quad \text{the Hardy-Littlewood maximal operator } \mathcal{M} \text{ is bounded on the Morrey space } \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}) \text{ whenever } 1 < p < \infty \text{ and } 0 < \lambda < n-1.$$

Granted (1.70)-(1.72), Theorem 1.1 applies and gives that for any system  $L$  as in (1.1)-(1.2) the  $\mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1})$ -Dirichlet problem

$$(1.73) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ \mathcal{N}u \in \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}), \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \in \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}), \end{cases}$$



is well-posed for arbitrary  $p \in (1, \infty)$  and  $\lambda \in (0, n - 1)$ .

In the last portion of this section we briefly comment on the literature dealing with Dirichlet boundary value problems for elliptic operators in the upper-half space. From the outset it is important to recognize that the nature of these problems is affected not only by the choice of the function space from which the boundary datum  $f$  is selected but also by the means through which the size of the solution  $u$  is measured and the very manner in which its boundary trace is considered. For example, there is an enormous amount of work devoted to the case when the solution  $u$  is sought in various Sobolev spaces in  $\mathbb{R}_+^n$ , the boundary datum  $f$  is assumed to belong to suitable Besov spaces on  $\mathbb{R}^{n-1}$ , and the boundary trace of  $u$  is taken in the generalized sense of Sobolev space theory. Classical references in this regard include [1], [2], [23], [25], [40], and the reader is also invited to consult the literature cited therein.

In this paper we are interested in the case when the size of  $u$  is measured in terms of the nontangential maximal function and the trace of  $u$  on the boundary of  $\mathbb{R}_+^n$  is taken in a nontangential pointwise sense (cf. (2.4)). In the particular case when  $L = \Delta$ , the Laplacian in  $\mathbb{R}^n$ , the boundary value problem (1.39) has been treated at length in a number of monographs, including [5], [17], [37], [38], and [39]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel from (1.5), while the uniqueness relies on either the Maximum Principle, or the Schwarz reflection principle for harmonic functions. Neither of the latter techniques may be adapted successfully to prove uniqueness in the case of general systems treated here, so we develop a new approach based on the properties of the Green function for an elliptic system in the upper half-space (reviewed in the appendix). While arguments involving Green functions have been successfully used in the past to prove uniqueness, the novelty here is that we succeed in constructing a Green function whose basic properties are compatible with the very formulation of the original boundary value problem. In our case, a key aspect is the specific manner in which the nontangential maximal function of the derivatives of the said Green function are controlled; cf. (A.29), and other pertinent features from Theorem A.4. It is remarkable that such a detailed analysis may be carried out for the entire class of elliptic systems  $L$  as in (1.1)-(1.2).

There is also a sizable amount of work devoted to studying the (classical) Dirichlet problem for the Laplacian in the upper-half space with a continuous boundary datum  $f$ . In such a scenario, one seeks a harmonic function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n) \cap \mathcal{C}^0(\overline{\mathbb{R}_+^n})$  satisfying  $u|_{\partial\mathbb{R}_+^n} = f$ . As noted by Helms in [18, p. 42 and p. 158], even in the case when the boundary datum  $f$  is a bounded continuous function in  $\mathbb{R}^{n-1}$  the solution  $u$  of this classical Dirichlet problem is not unique. To ensure uniqueness in such a setting one typically specifies

the behavior of  $u(x', t)$  as  $t \rightarrow \infty$ . A case in point is [33], where uniqueness is established in the class of harmonic functions  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n) \cap \mathcal{C}^0(\overline{\mathbb{R}_+^n})$  satisfying  $u(x) = o(|x| \sec^\gamma \theta)$  as  $|x| \rightarrow \infty$  (where  $\theta := \arccos(x_n/|x|)$  and  $\gamma \in \mathbb{R}$  is arbitrary), by proving a Phragmén-Lindelöf principle under the latter growth condition. This builds on the work of [32], [42], and others. In this regard, see also [41]. All these works rely on positivity and specialized properties of the Laplace operator, so the techniques employed do not extend to the considerably more general class of elliptic systems considered in the present paper.

Much attention has also been paid to the case of the  $L^p$ -Dirichlet problem in the upper-half space for variable coefficient scalar elliptic operators in divergence form,  $L = \operatorname{div} A \nabla$ , under various assumptions on the coefficient matrix  $A = A(x', t)$  for  $(x', t) \in \mathbb{R}_+^n$ . For this topic, the interested reader is referred to the excellent exposition in Kenig's monograph [20], as well as the more recent work in [3], [19] and in the references cited there. This body of work crucially relies on the De Giorgi-Nash-Moser theory, an ingredient not available for the type of systems considered in the present paper.

Finally, we wish to mention that in [31] Shen has considered the well-posedness of the Dirichlet problem for elliptic systems  $L$  as in (1.1)-(1.2) in a Lipschitz domain  $\Omega$  with boundary data from Morrey spaces on  $\partial\Omega$ . For this more general class of domains he proved the well-posedness of a boundary value problem formulated as in (1.73) with the upper-half space  $\mathbb{R}_+^n$  replaced by a Lipschitz domain  $\Omega$  but only when  $p = 2$ . In relation to this, the novelty in our paper is the consideration of the full range  $p \in (1, \infty)$ .

## 2. PRELIMINARY MATTERS

Throughout the paper, we let  $\mathbb{N}$  stand for the collection of all strictly positive integers, and set  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . In this way  $\mathbb{N}_0^k$ , where  $k \in \mathbb{N}$ , stands for the set of multi-indices  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $\alpha_j \in \mathbb{N}_0$  for  $1 \leq j \leq k$ . Also, fix  $n \in \mathbb{N}$  with  $n \geq 2$ . We shall work in the upper-half space

$$(2.1) \quad \mathbb{R}_+^n := \{x = (x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\},$$

whose topological boundary  $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$  will be frequently identified with the horizontal hyperplane  $\mathbb{R}^{n-1}$  via  $(x', 0) \equiv x'$ . The origin in  $\mathbb{R}^{n-1}$  is denoted by  $0'$  and we let  $B_{n-1}(x', r)$  stand for the  $(n-1)$ -dimensional Euclidean ball of radius  $r$  centered at  $x' \in \mathbb{R}^{n-1}$ . Fix a number  $\kappa > 0$  and for each boundary point  $x' \in \partial\mathbb{R}_+^n$  introduce the conical nontangential approach region with vertex at  $x'$  as

$$(2.2) \quad \Gamma(x') := \Gamma_\kappa(x') := \{y = (y', t) \in \mathbb{R}_+^n : |x' - y'| < \kappa t\}.$$

Given a vector-valued function  $u : \mathbb{R}_+^n \rightarrow \mathbb{C}^M$ , define the nontangential maximal function of  $u$  by

$$(2.3) \quad (\mathcal{N}u)(x') := (\mathcal{N}_\kappa u)(x') := \sup \{|u(y)| : y \in \Gamma_\kappa(x')\}, \quad x' \in \mathbb{R}^{n-1}.$$

It is well-known that the aperture of the cones used to define the nontangential maximal operator plays only a secondary role; see Proposition A.6 for a concrete result of this flavor. Whenever meaningful, we also define

$$(2.4) \quad u \Big|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}(x') := \lim_{\Gamma_\kappa(x') \ni y \rightarrow (x', 0)} u(y) \quad \text{for } x' \in \mathbb{R}^{n-1}.$$

In the sequel, we shall need to consider a localized version of the nontangential maximal operator. Specifically, given any  $E \subset \mathbb{R}_+^n$ , for each  $u : E \rightarrow \mathbb{C}^M$  we set

$$(2.5) \quad (\mathcal{N}^E u)(x') := (\mathcal{N}_\kappa^E u)(x') := \sup \{|u(y)| : y \in \Gamma_\kappa(x') \cap E\}, \quad x' \in \mathbb{R}^{n-1}.$$

Hence,  $\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa \tilde{u}$  where  $\tilde{u}$  is the extension of  $u$  to  $\mathbb{R}_+^n$  by zero outside  $E$ . In the scenario when  $u$  is originally defined in the entire upper-half space  $\mathbb{R}_+^n$  we may therefore write

$$(2.6) \quad \mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(\mathbf{1}_E u),$$

where  $\mathbf{1}_E$  denotes the characteristic function of  $E$ . Corresponding to the special case when  $E = \{(x', x_n) \in \mathbb{R}_+^n : x_n < \varepsilon\}$ , we simply write  $\mathcal{N}_\kappa^{(\varepsilon)}$  in place of  $\mathcal{N}_\kappa^E$ . That is,

$$(2.7) \quad \mathcal{N}_\kappa^{(\varepsilon)} u(x') := \sup_{\substack{y=(y', y_n) \in \Gamma_\kappa(x') \\ 0 < y_n < \varepsilon}} |u(y)|, \quad x' \in \mathbb{R}^{n-1}.$$

Throughout the paper we use the symbol  $|E|$  to denote the Lebesgue measure of Lebesgue measurable set  $E \subset \mathbb{R}^n$ . The Lebesgue measure itself in  $\mathbb{R}^n$  will be denoted by  $\mathcal{L}^n$ . We let  $Q$  denote open cubes in  $\mathbb{R}^{n-1}$  with sides parallel to the coordinate axes, and employ  $\ell(Q)$  to denote its side-length. We will also use the standard convention  $\lambda Q$ , with  $\lambda > 0$ , for the cube concentric with  $Q$  whose side-length is  $\lambda \ell(Q)$ . For any  $Q$  and any  $h \in L_{\text{loc}}^1(\mathbb{R}^{n-1})$ , we write

$$(2.8) \quad h_Q := \int_Q h d\mathcal{L}^{n-1} := \frac{1}{|Q|} \int_Q h(x') dx'.$$

If the function  $h$  is  $\mathbb{C}^M$ -valued, the average is taken componentwise. The Hardy-Littlewood maximal operator on  $\mathbb{R}^{n-1}$  is defined as

$$(2.9) \quad \mathcal{M}f(x') := \sup_{Q \ni x'} \int_Q |f(y')| dy', \quad x' \in \mathbb{R}^{n-1}.$$

Also, we write

$$(2.10) \quad \mathcal{M}^{(2)} := \mathcal{M} \circ \mathcal{M}$$

for the two-fold composition of  $\mathcal{M}$  with itself. We follow the customary notation  $A \approx B$  in order to indicate that each quantity  $A, B$  is dominated by a fixed multiple of the other (via constants independent of the essential parameters intervening in  $A, B$ ).

**Lemma 2.1.** *For  $x' \in \mathbb{R}^{n-1}$  one has*

$$(2.11) \quad \mathcal{M}(\mathbf{1}_{B_{n-1}(0',1)})(x') \approx \frac{1}{1 + |x'|^{n-1}},$$

and

$$(2.12) \quad \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)})(x') \approx \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}},$$

where the implicit constants depend only on  $n$ .

*Proof.* The proof of (2.11) is elementary but we include it for completeness. Note first that for every  $x' \in \mathbb{R}^{n-1}$ , if we denote by  $Q_{x'}$  the cube in  $\mathbb{R}^{n-1}$  centered at the origin and with side-length  $2(|x'| + 1)$ , then  $x' \in Q_{x'}$  and  $B_{n-1}(0', 1) \subset Q_{x'}$ . Thus, we easily obtain

$$(2.13) \quad \begin{aligned} \mathcal{M}(\mathbf{1}_{B_{n-1}(0',1)})(x') &\geq \int_{Q_{x'}} \mathbf{1}_{B_{n-1}(0',1)}(y') dy' \\ &= \frac{|B_{n-1}(0',1)|}{|Q_{x'}|} \geq \frac{C_n}{1 + |x'|^{n-1}}. \end{aligned}$$

To obtain the converse inequality we first observe that, clearly,

$$(2.14) \quad \mathcal{M}(\mathbf{1}_{B_{n-1}(0',1)})(x') \leq 1 \leq \frac{C_n}{1 + |x'|^{n-1}}, \quad \text{whenever } |x'| \leq 2.$$

Suppose next that  $|x'| > 2$ . Notice that if  $x' \in Q \subset \mathbb{R}^{n-1}$  and there is some  $y' \in Q \cap B_{n-1}(0', 1)$  then

$$(2.15) \quad |x'| \leq |x' - y'| + |y'| \leq \sqrt{n} \ell(Q) + 1 \leq \sqrt{n} \ell(Q) + |x'|/2.$$

Therefore  $\ell(Q) > |x'|/(2\sqrt{n})$ , which entails

$$(2.16) \quad \int_Q \mathbf{1}_{B_{n-1}(0',1)}(y') dy' \leq \frac{|B_{n-1}(0',1)|}{|Q|} \leq \frac{C_n}{|x'|^{n-1}} \leq \frac{C_n}{1 + |x'|^{n-1}}.$$

The same inequality trivially holds in the case when  $Q$  is disjoint from  $B_{n-1}(0', 1)$ . Taking the supremum of the most extreme sides of (2.16) over all cubes  $Q$  containing  $x'$  then yields the upper estimate in (2.11) in the case when  $|x'| > 2$ . This finishes the proof of (2.11).

Turning to the proof of (2.12), we first invoke an auxiliary estimate whose proof can be found in [13]:

$$(2.17) \quad \begin{aligned} \mathcal{M}^{(2)}f(x') &\approx \mathcal{M}_{L \log L} f(x') := \sup_{Q \ni x'} \|f\|_{L \log L, Q} \\ &\text{uniformly for } f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}) \text{ and } x' \in \mathbb{R}^{n-1}, \end{aligned}$$

where  $\|\cdot\|_{L \log L, Q}$  stands for the localized and normalized Luxemburg norm

$$(2.18) \quad \|f\|_{L \log L, Q} := \inf \left\{ \lambda > 0 : \int_Q \Phi \left( \frac{|f(x')|}{\lambda} \right) dx' \leq 1 \right\},$$

with  $\Phi(t) := t \log(e+t)$ ,  $t \geq 0$ . Defining  $\varphi(t) := (\Phi^{-1}(t^{-1}))^{-1}$  for  $t \in (0, \infty)$  and  $\varphi(0) := 0$ , easy calculations lead to

$$(2.19) \quad \|1_{B_{n-1}(0',1)}\|_{L \log L, Q} = \varphi \left( \frac{|B_{n-1}(0',1) \cap Q|}{|Q|} \right) = \varphi \left( \int_Q 1_{B_{n-1}(0',1)}(y') dy' \right).$$

Using then (2.17), (2.19), the fact that  $\varphi$  is a continuous strictly increasing function in  $[0, \infty)$ , and (2.11), we conclude that

$$(2.20) \quad \begin{aligned} \mathcal{M}^{(2)}(1_{B_{n-1}(0',1)})(x') &\approx \sup_{Q \ni x'} \varphi \left( \int_Q 1_{B_{n-1}(0',1)}(y') dy' \right) \\ &= \varphi \left( \sup_{Q \ni x'} \int_Q 1_{B_{n-1}(0',1)}(y') dy' \right) \\ &= \varphi(\mathcal{M}(1_{B_{n-1}(0',1)})(x')) \approx \varphi \left( \frac{1}{1 + |x'|^{n-1}} \right), \end{aligned}$$

uniformly for  $x' \in \mathbb{R}^{n-1}$ . Thus, to complete the proof of (2.12) we only need to find a suitable estimate for the last term above. To this end, one can easily check that  $\Phi^{-1}(t) \approx t / \log(e+t)$  which gives that  $\varphi(t) \approx t \log(e+t^{-1})$ . This and (2.20) then yield

$$(2.21) \quad \mathcal{M}^{(2)}(1_{B_{n-1}(0',1)})(x') \approx \frac{1}{1 + |x'|^{n-1}} \log(e + 1 + |x'|^{n-1}) \approx \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}},$$

uniformly for  $x' \in \mathbb{R}^{n-1}$ , as desired.  $\square$

We next introduce the class of **Muckenhoupt weights**. Call a real-valued function  $w$  defined on  $\mathbb{R}^{n-1}$  a **weight** if it is non-negative and measurable. Given a weight  $w$  and  $p \in [1, \infty]$ , we write  $L^p(\mathbb{R}^{n-1}, w) = L^p(\mathbb{R}^{n-1}, w dx')$ . If  $1 < p < \infty$ , a weight  $w$  belongs to the Muckenhoupt class  $A_p = A_p(\mathbb{R}^{n-1})$  if

$$(2.22) \quad [w]_{A_p} := \sup_{Q \subset \mathbb{R}^{n-1}} \left( \int_Q w(x') dx' \right) \left( \int_Q w(x')^{1-p'} dx' \right)^{p-1} < \infty,$$

where  $p' = p/(p-1)$  denotes the conjugate exponent of  $p$ . Corresponding to  $p = 1$ , the class  $A_1 = A_1(\mathbb{R}^{n-1})$  is then defined as the collection of all weights  $w$  in  $\mathbb{R}^{n-1}$  for which

$$(2.23) \quad [w]_{A_1} := \sup_{Q \subset \mathbb{R}^{n-1}} \left( \operatorname{ess\,inf}_Q w \right)^{-1} \left( \int_Q w(x') dx' \right) < \infty.$$

In particular,

$$(2.24) \quad \int_Q w(y') dy' \leq [w]_{A_1} w(x') \quad \text{for a.e. } x' \in Q,$$

for every cube  $Q \subset \mathbb{R}^{n-1}$ . Equivalently,

$$(2.25) \quad \mathcal{M}w(x') \leq [w]_{A_1} w(x') \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}.$$

Finally, corresponding to  $p = \infty$ , we let  $A_\infty$  stand for  $\bigcup_{1 \leq p < \infty} A_p$ .

We summarize a number of well-known facts which are relevant for us here. See, e.g., [17] for a more detailed discussion, including the following basic properties:

- (i) given  $1 < p < \infty$  and a weight  $w$ , then  $w \in A_p$  if and only if  $\mathcal{M}$  is bounded on  $L^p(\mathbb{R}^{n-1}, w)$ ;
- (ii) given  $1 < p < \infty$  and a weight  $w$ , then  $w \in A_p$  if and only if  $w^{1-p'} \in A_{p'}$ , and  $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{p'-1}$ ;
- (iii) if  $w_1, w_2 \in A_1$  and  $1 \leq p < \infty$ , then  $w_1 w_2^{1-p} \in A_p$  and  $[w_1 w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$ ;
- (iv) the classes  $A_p$ ,  $1 \leq p < \infty$ , may be equivalently defined using balls in  $\mathbb{R}^{n-1}$  (in place of cubes), in which scenario  $[w]_{A_p}^{\text{balls}} \approx [w]_{A_p}$  with implicit constants depending only on  $n$  and  $p$ .

In the last part of this section we discuss the notion of Poisson kernel in  $\mathbb{R}_+^n$  for an operator  $L$  as in (1.1)-(1.2).

**Definition 2.2** (Poisson Kernel for  $L$  in  $\mathbb{R}_+^n$ ). *Let  $L$  be a second-order elliptic system with complex coefficients as in (1.1)-(1.2). A Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  is a matrix-valued function  $P^L = (P_{\alpha\beta}^L)_{1 \leq \alpha, \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{M \times M}$  such that the following conditions hold:*

- (a) *there exists  $C \in (0, \infty)$  such that*

$$(2.26) \quad |P^L(x')| \leq \frac{C}{(1 + |x'|^2)^{\frac{n}{2}}} \quad \text{for each } x' \in \mathbb{R}^{n-1};$$

- (b) *the function  $P^L$  is Lebesgue measurable and  $\int_{\mathbb{R}^{n-1}} P^L(x') dx' = I_{M \times M}$ , the  $M \times M$  identity matrix;*

- (c) *if  $K^L(x', t) := P_t^L(x') := t^{1-n} P^L(x'/t)$ , for each  $x' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$ , then the function  $K^L = (K_{\alpha\beta}^L)_{1 \leq \alpha, \beta \leq M}$  satisfies (in the sense of distributions)*

$$(2.27) \quad LK_{\beta}^L = 0 \quad \text{in } \mathbb{R}_+^n \quad \text{for each } \beta \in \{1, \dots, M\},$$

where  $K_{\cdot\beta}^L := (K_{\alpha\beta}^L)_{1 \leq \alpha \leq M}$ .

*Remark 2.3.* The following comments pertain to Definition 2.2.

- (i) Condition (a) ensures that the integral in part (b) is absolutely convergent.
- (ii) Condition (c) and the ellipticity of the operator  $L$  ensure (cf. [26, Theorem 10.9, p. 318]) that  $K^L \in \mathcal{C}^\infty(\mathbb{R}_+^n)$ . In particular, (2.27) holds in a pointwise sense. Also, given that  $P^L(x') = K^L(x', 1)$  for each  $x' \in \mathbb{R}^{n-1}$ , we deduce that  $P^L \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$ .
- (iii) Condition (b) is equivalent to  $\lim_{t \rightarrow 0^+} P_t^L(x') = \delta_{0'}(x') I_{M \times M}$  in  $\mathcal{D}'(\mathbb{R}^{n-1})$ , where  $\delta_{0'}$  is Dirac's distribution with mass at the origin  $0'$  of  $\mathbb{R}^{n-1}$ .
- (iv) For all  $x \in \mathbb{R}_+^n$  and  $\lambda > 0$  we have  $K^L(\lambda x) = \lambda^{1-n} K^L(x)$ .

Poisson kernels for elliptic boundary value problems in a half-space have been studied extensively in [1], [2], [21, §10.3], [34], [35], [36]. Here we record a corollary of more general work done by S. Agmon, A. Douglis, and L. Nirenberg in [2].

**Theorem 2.4.** *Any elliptic differential operator  $L$  as in (1.1)-(1.2) has a Poisson kernel  $P^L$  in the sense of Definition 2.2, which has the additional property that the function*

$$(2.28) \quad K^L(x', t) := P_t^L(x') \quad \text{for all } (x', t) \in \mathbb{R}_+^n,$$

*satisfies  $K^L \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n} \setminus B(0, \varepsilon))$  for every  $\varepsilon > 0$ .*

*Remark 2.5.* As a consequence of part (iv) in Remark 2.3 and the regularity of  $K$  stated in Theorem 2.4, we have that for each multi-index  $\alpha \in \mathbb{N}_0^n$  there exists  $C_\alpha \in (0, \infty)$  with the property that

$$(2.29) \quad |(\partial^\alpha K^L)(x)| \leq C_\alpha |x|^{1-n-|\alpha|}, \quad \text{for every } x \in \overline{\mathbb{R}_+^n} \setminus \{0\}.$$

In this respect, we wish to note that this estimate is stronger than what a direct application of the properties of Poisson kernels listed in Definition 2.2 would imply. Specifically, as noted in part (ii) of Remark 2.3, we have  $K^L \in \mathcal{C}^\infty(\mathbb{R}_+^n)$  which, in concert with part (iv) of Remark 2.3, shows that (2.29) holds for  $x \in \Gamma_\kappa(0')$ , for each  $\kappa > 0$ , with a constant also depending on the parameter  $\kappa$ .

### 3. TOOLS FOR EXISTENCE AND UNIQUENESS

This section is devoted to proving the results stated in Theorems 3.1-3.2 below. Here and elsewhere, the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

**Theorem 3.1** (Main Tool for the Existence Part). *Let  $L$  be a system as in (1.1)-(1.2). Given a Lebesgue measurable function  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^M$  satisfying*

$$(3.1) \quad \int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1 + |x'|^n} dx' < \infty,$$

*set*

$$(3.2) \quad u(x', t) := (P_t^L * f)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n,$$

*where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.4. Then  $u$  is meaningfully defined via an absolutely convergent integral,*

$$(3.3) \quad u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \quad Lu = 0 \quad \text{in } \mathbb{R}_+^n, \quad u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \quad \text{a.e. in } \mathbb{R}^{n-1}$$

*(convergence holds, for instance, in the set of Lebesgue points of  $f$ ), and there exists a constant  $C = C(n, L) \in (0, \infty)$  with the property that*

$$(3.4) \quad \mathcal{N}u(x') \leq C \mathcal{M}f(x'), \quad \forall x' \in \mathbb{R}^{n-1}.$$

**Theorem 3.2** (Main Tool for the Uniqueness Part). *Let  $L$  be a system as in (1.1)-(1.2). Assume that  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n)$  is such that  $Lu = 0$  in  $\mathbb{R}_+^n$ , its nontangential maximal function  $\mathcal{N}u$  satisfies*

$$(3.5) \quad \int_{\mathbb{R}^{n-1}} \mathcal{N}u(x') \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx' < \infty,$$

*and that  $u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = 0$  a.e. in  $\mathbb{R}^{n-1}$ . Then  $u \equiv 0$  in  $\mathbb{R}_+^n$ .*

In preparation to presenting the proof of Theorem 3.1 we first deal with a purely real variable lemma pertaining to the stability of the first weighted  $L^1$  space appearing in (1.7) under convolutions with a fixed (matrix-valued) function whose size is controlled by the harmonic Poisson kernel. In the same context, we also deal with nontangential maximal function estimates and nontangential limits.

**Lemma 3.3.** *Let  $P = (P_{\alpha\beta})_{1 \leq \alpha, \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{M \times M}$  be a Lebesgue measurable function satisfying, for some  $c \in (0, \infty)$ ,*

$$(3.6) \quad |P(x')| \leq \frac{c}{(1 + |x'|^2)^{\frac{n}{2}}} \quad \text{for each } x' \in \mathbb{R}^{n-1},$$

*and recall that  $P_t(x') := t^{1-n} P(x'/t)$  for each  $x' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$ . Then, for each  $t \in (0, \infty)$  fixed, the operator*

$$(3.7) \quad L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right) \ni f \mapsto P_t * f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)$$



is well-defined, linear and bounded, with operator norm controlled by  $C(t+1)$ . Moreover, for every  $\kappa > 0$  there exists a finite constant  $C_\kappa > 0$  with the property that for each  $x' \in \mathbb{R}^{n-1}$ ,

$$(3.8) \quad \sup_{|x'-y'| < \kappa t} |(P_t * f)(y')| \leq C_\kappa \mathcal{M}f(x'), \quad \forall f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right).$$

Finally, given any function

$$(3.9) \quad f = (f_\beta)_{1 \leq \beta \leq M} \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right) \subset L^1_{\text{loc}}(\mathbb{R}^{n-1}),$$

at every Lebesgue point  $x'_0 \in \mathbb{R}^{n-1}$  of  $f$  there holds

$$(3.10) \quad \lim_{\substack{(x', t) \rightarrow (x'_0, 0) \\ |x' - x'_0| < \kappa t}} (P_t * f)(x') = \left( \int_{\mathbb{R}^{n-1}} P(x') dx' \right) f(x'_0),$$

and the function

$$(3.11) \quad \mathbb{R}_+^n \ni (x', t) \mapsto (P_t * f)(x') \in \mathbb{C}^M \quad \text{is locally integrable in } \mathbb{R}_+^n.$$

*Proof.* Pick a function  $f$  as in (3.9) and fix some  $t \in (0, \infty)$ . First, consider the issue whether  $P_t * f$  is well-defined, via an absolutely convergent integral. In this regard, note that for any  $x', y' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$  one has  $|y'| \leq (1 + |x'|/t)(t + |x' - y'|)$  and  $1 \leq (1/t)(t + |x' - y'|)$ , hence

$$(3.12) \quad 1 + |y'| \leq (1 + |x'|/t + 1/t)(t + |x' - y'|).$$

Thus, for each fixed  $x' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$ , we have

$$(3.13) \quad \int_{\mathbb{R}^{n-1}} \frac{t}{(t + |x' - y'|)^n} |f(y')| dy' \\ \leq C t (1 + |x'|/t + 1/t)^n \int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^n} dy' < \infty,$$

which, in light of (3.6), shows that  $P_t * f$  is meaningfully defined via an absolutely convergent integral. To proceed, observe from (3.6) and (1.5) that there exists some  $C \in (0, \infty)$  with the property that

$$(3.14) \quad |P_t(x')| \leq C P_t^\Delta(x') \quad \text{for all } x' \in \mathbb{R}^{n-1}, t \in (0, \infty).$$

Consequently,

$$(3.15) \quad \int_{\mathbb{R}^{n-1}} \frac{|(P_t * f)(x')|}{1 + |x'|^n} dx' \leq C \int_{\mathbb{R}^{n-1}} (P_t^\Delta * |f|)(x') P_1^\Delta(x') dx' \\ = C \left( (P_t^\Delta * |f|) * P_1^\Delta \right)(0') \\ = C \left( (P_t^\Delta * P_1^\Delta) * |f| \right)(0') \\ = C (P_{t+1}^\Delta * |f|)(0')$$

$$\leq C(t+1) \int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1+|y'|^n} dy',$$

where we have used the semigroup property for the harmonic Poisson kernel (cf., e.g., [37, (vi), p.62]), and where the last inequality follows from (3.13) written with  $t+1$  in place of  $t$  and  $x' = 0'$ . Now all desired conclusions concerning (3.7) are seen from (3.15).

Before proceeding with the rest of the proof, let us momentarily digress in order to note that, once some  $\kappa > 0$  has been fixed, (3.6) self-improves in the sense that there exists  $C_\kappa \in (0, \infty)$  such that, for every  $x' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$ ,

$$(3.16) \quad |P_t(x' - y')| \leq C_\kappa \frac{t}{(t^2 + |x'|^2)^{\frac{n}{2}}} \quad \text{whenever } |y'| < \kappa t.$$

Indeed, this follows from the fact that  $|x'| \leq \max\{1, \kappa\}(t + |x' - y'|)$  whenever  $x', y' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$  are such that  $|y'| < \kappa t$  which, in turn, is easily justified by the triangle inequality.

To deal with (3.8), pick a function  $f$  as in (3.9). Also, fix  $x' \in \mathbb{R}^{n-1}$  and let  $y' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$  satisfy  $|x' - y'| < \kappa t$ . Granted (3.16), this implies

$$(3.17) \quad |P_t(y' - z')| \leq C_\kappa \frac{t}{(t^2 + |x' - z'|^2)^{\frac{n}{2}}} \quad \text{for every } z' \in \mathbb{R}^{n-1}.$$

Based on (3.17) we may then estimate

$$(3.18) \quad \begin{aligned} |(P_t * f)(y')| &\leq \int_{\mathbb{R}^{n-1}} |P_t(y' - z')| |f(z')| dz' \\ &\leq C_\kappa \int_{\mathbb{R}^{n-1}} \frac{t}{(t^2 + |x' - z'|^2)^{\frac{n}{2}}} |f(z')| dz' \\ &\leq C_\kappa \int_{B_{n-1}(x', t)} |f(z')| dz' \\ &\quad + \sum_{j=0}^{\infty} \int_{B_{n-1}(x', 2^{j+1}t) \setminus B_{n-1}(x', 2^j t)} \frac{t}{(t^2 + |x' - z'|^2)^{\frac{n}{2}}} |f(z')| dz' \\ &\leq C_\kappa \sum_{j=0}^{\infty} 2^{-j} \int_{B_{n-1}(x', 2^j t)} |f(z')| dz' \leq C_\kappa \mathcal{M}f(x'), \end{aligned}$$

from which (3.8) follows.

Let us now deal with (3.10). To this end, abbreviate

$$(3.19) \quad A := \int_{\mathbb{R}^{n-1}} P(x') dx' \in \mathbb{C}^{M \times M}.$$

Also, select a function  $f$  as in (3.9) and introduce

$$(3.20) \quad u(x', t) := (P_t * f)(x') \quad \text{for each } (x', t) \in \mathbb{R}_+^n.$$

From what we have proved already, this function is well-defined by an absolutely convergent integral. In the remainder of the proof, we shall adapt the argument in [37, p. 198], where the case  $L = \Delta$  and  $f \in L^p(\mathbb{R}^{n-1})$ ,  $1 \leq p \leq \infty$ , has been treated. Specifically, fix a Lebesgue point  $x'_0 \in \mathbb{R}^{n-1}$  of  $f$  and let  $\varepsilon > 0$  be arbitrary. Then there exists  $\delta > 0$  such that

$$(3.21) \quad \int_{B_{n-1}(0', r)} |f(z' + x'_0) - f(x'_0)| dz' < \varepsilon, \quad \forall r \in (0, \delta].$$

In particular, if we set

$$(3.22) \quad g := [f(\cdot + x'_0) - f(x'_0)] \mathbf{1}_{B_{n-1}(0', \delta)} \quad \text{in } \mathbb{R}^{n-1},$$

then (3.21) implies (for some dimensional constant  $c_n > 0$ )

$$(3.23) \quad \mathcal{M}g(0') \leq c_n \varepsilon.$$

Then, bearing in mind (3.19), for each  $y' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$  we may write

$$(3.24) \quad \begin{aligned} u(y' + x'_0, t) - Af(x'_0) &= \int_{\mathbb{R}^{n-1}} P_t(y' + x'_0 - z') [f(z') - f(x'_0)] dz' \\ &= \int_{\mathbb{R}^{n-1}} P_t(y' - z') [f(z' + x'_0) - f(x'_0)] dz'. \end{aligned}$$

In turn, this and (3.16) then imply that, under the assumption that  $y' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$  satisfy  $|y'| < \kappa t$ , we have

$$(3.25) \quad \begin{aligned} &|u(y' + x'_0, t) - Af(x'_0)| \\ &\leq C_\kappa \int_{\{z' \in \mathbb{R}^{n-1} : |z'| < \delta\}} \frac{t}{(t^2 + |z'|^2)^{\frac{n}{2}}} |f(z' + x'_0) - f(x'_0)| dz' \\ &\quad + C_\kappa \int_{\{z' \in \mathbb{R}^{n-1} : |z'| \geq \delta\}} \frac{t}{(t^2 + |z'|^2)^{\frac{n}{2}}} |f(z' + x'_0) - f(x'_0)| dz' \\ &=: I_1 + I_2. \end{aligned}$$

Note that thanks to (3.22), (1.5), and (3.8) (used with  $P = P^\Delta$ ,  $f = g$ , and  $x' = y' = 0$ ), for some constant  $C_\kappa \in (0, \infty)$  independent of  $\varepsilon$  and  $f$  we have

$$(3.26) \quad \begin{aligned} I_1 &= C_\kappa \int_{\mathbb{R}^{n-1}} \frac{t}{(t^2 + |z'|^2)^{\frac{n}{2}}} |g(z')| dz' \\ &= C_\kappa (P_t^\Delta * |g|)(0') \leq C_\kappa \mathcal{M}g(0') \leq C_\kappa \varepsilon, \end{aligned}$$

where the last inequality is (3.23). As regards  $I_2$ , we first observe that if  $|z'| \geq \delta$  then

$$(3.27) \quad 1 + |z' + x'_0| \leq (1 + |x'_0|)(1 + |z'|) \leq (1 + |x'_0|)(1 + \delta^{-1})|z'|.$$

Thus,

$$(3.28) \quad \begin{aligned} I_2 &\leq C_\kappa t \int_{\{z' \in \mathbb{R}^{n-1}: |z'| \geq \delta\}} \frac{1}{|z'|^n} |f(z' + x'_0) - f(x'_0)| dz' \\ &\leq Ct \left( \int_{\{z' \in \mathbb{R}^{n-1}: |z'| \geq \delta\}} \frac{|f(z' + x'_0)|}{(1 + |z' + x'_0|)^n} dz' + \frac{|f(x'_0)|}{\delta} \right) \\ &\leq Ct \left( \int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1 + |x'|^n} dx' + |f(x'_0)| \right), \end{aligned}$$

where  $C$  depends only on  $n, \kappa, x'_0$ , and  $\delta$ . Hence  $\lim_{t \rightarrow 0^+} I_2 = 0$ . This, (3.26), and (3.25) then imply

$$(3.29) \quad \limsup_{|y'| < \kappa t, t \rightarrow 0^+} |u(y' + x'_0, t) - Af(x'_0)| \leq C_\kappa \varepsilon,$$

for some  $C_\kappa \in (0, \infty)$  independent of  $\varepsilon$  and  $f$ . Now the claim in (3.10) is clear from (3.29) and (3.19)-(3.20). Finally, (3.13) implies  $u \in L^1_{\text{loc}}(\mathbb{R}_+^n)$ , and this takes care of (3.11).  $\square$

After these preparations, the proof of Theorem 3.1 is short and straightforward.

*Proof of Theorem 3.1.* That  $u$  in (3.2) is well-defined and satisfies (3.4) as well as  $u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f$  a.e. in  $\mathbb{R}^{n-1}$  follows immediately from Lemma 3.3, Theorem 2.4, and the normalization of the Poisson kernel (cf. part (b) in Definition 2.2). Next, given a multi-index  $\alpha \in \mathbb{N}_0^n$ , from (2.29) if  $|\alpha| \geq 1$  and from (2.28) combined with part (a) in Definition 2.2 if  $|\alpha| = 0$  we see that there exists a constant  $C_\alpha \in (0, \infty)$  with the property that

$$(3.30) \quad |(\partial^\alpha K^L)(x', t)| \leq C_\alpha t^{-|\alpha|} \frac{t}{(t + |x'|)^n}, \quad \forall (x', t) \in \mathbb{R}_+^n.$$

In concert with (3.13), this justifies differentiation under the integral defining  $u$  so, ultimately,  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n)$ . Moreover,  $Lu = 0$  in  $\mathbb{R}_+^n$  by (3.2), part (c) in Definition 2.2, and part (ii) in Remark 2.3.  $\square$

*Remark 3.4.* In the proof of Theorem 3.1, the construction of a function  $u$  satisfying (3.3) is based on the formula (3.2) in which  $P^L$  is the Agmon-Douglis-Nirenberg Poisson kernel for  $L$  from Theorem 2.4. Such a choice ensured that (3.30) holds which, as noted in Remark 2.5, is not immediately clear for a “generic” Poisson kernel in the sense of Definition 2.2. Later on, in Theorem 5.1, we shall actually show that there exists precisely one

Poisson kernel for the system  $L$  in the sense of Definition 2.2, so this issue will eventually become a moot point. This being said, in the proof of Theorem 5.1 it is important to know that

$$(3.31) \quad \text{for any } P^L \text{ as in Definition 2.2, properties (3.3) and (3.4) remain valid for } u \text{ as in (3.1)-(3.2).}$$

To see that this is indeed the case, assume that  $P^L$  is as in Definition 2.2. Then, if  $u$  is as in (3.1)-(3.2), it follows from Lemma 3.3 that  $u \in L^1_{\text{loc}}(\mathbb{R}_+^n)$ ,  $u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f$  a.e. in  $\mathbb{R}^{n-1}$ , and  $\mathcal{N}u \leq CMf$ . As such, there remains to show that  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n)$  and  $Lu = 0$  in  $\mathbb{R}_+^n$ . The strategy is to prove that  $Lu = 0$  in the sense of distributions in  $\mathbb{R}_+^n$ , which then forces  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n)$  by elliptic regularity (cf. [26, Theorem 10.9, p.318]). With this goal in mind, pick an arbitrary vector-valued test function  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+^n)$  and, with  $L^\top$  denoting the transposed of  $L$ , compute

$$(3.32) \quad \begin{aligned} & \int_{\mathbb{R}_+^n} \langle u(x), (L^\top \varphi)(x) \rangle dx \\ &= \int_{\mathbb{R}_+^n} \left\langle \int_{\mathbb{R}^{n-1}} P_t^L(x' - y') f(y') dy', (L^\top \varphi)(x', t) \right\rangle dx' dt \\ &= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}_+^n} \left\langle P_t^L(x' - y') f(y'), (L^\top \varphi)(x', t) \right\rangle dx' dt \right) dy' \\ &= \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}_+^n} \left\langle K^L(x) f(y'), [L^\top(\varphi(\cdot + (y', 0)))](x) \right\rangle dx \right) dy' \\ &= 0. \end{aligned}$$

Above, the first equality uses (3.2), the second one is based on Fubini's theorem (whose applicability is ensured by (3.13)), the third employs the definition of  $K^L$  and a natural change of variables, while the fourth one follows from (2.27). Hence,  $Lu = 0$  in the sense of distributions in  $\mathbb{R}_+^n$ , and the proof of (3.31) is complete.

We now turn to the task of proving Theorem 3.2, which is the key technical result of this paper. In the process, we shall make use of all the auxiliary results from Appendix A, which the reader is invited to review at this stage.

*Proof of Theorem 3.2.* Fix  $\kappa > 0$  and let  $u = (u_\beta)_{1 \leq \beta \leq M} \in \mathcal{C}^\infty(\mathbb{R}_+^n)$  be such that  $Lu = 0$  in  $\mathbb{R}_+^n$ ,  $\mathcal{N}_\kappa u$  satisfies (3.5), and  $u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = 0$  a.e. in  $\mathbb{R}^{n-1}$ . The goal is to show that  $u \equiv 0$  in  $\mathbb{R}_+^n$ . To this end, fix an arbitrary point  $x^* \in \mathbb{R}_+^n$  and consider the Green function  $G = G(\cdot, x^*)$  in  $\mathbb{R}_+^n$  with pole at  $x^*$  for  $L^\top$ , the transposed of the operator  $L$  (cf. Definition A.3 and Theorem A.4

for details on this matter). By design, this is a matrix-valued function, say  $G = (G_{\alpha\gamma})_{1 \leq \alpha, \gamma \leq M}$ .

We shall apply Theorem A.1 to a suitably chosen vector field and compact set. To set the stage, consider the compact set

$$(3.33) \quad K_\star := \overline{B(x^\star, r)} \subset \mathbb{R}_+^n, \quad \text{where } r := \frac{3}{4} \text{dist}(x^\star, \partial\mathbb{R}_+^n).$$

Also, consider a function

$$(3.34) \quad \begin{aligned} \psi &\in \mathcal{C}^\infty(\mathbb{R}) \text{ with the property that } 0 \leq \psi \leq 1, \\ \psi(t) &= 0 \text{ for } t \leq 1, \text{ and } \psi(t) = 1 \text{ for } t \geq 2. \end{aligned}$$

Fix (for now) some  $\varepsilon \in (0, r/4)$ , and define

$$(3.35) \quad \psi_\varepsilon(x) := \psi(x_n/\varepsilon) \text{ for each } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

In particular, the conditions on  $\varepsilon$  and  $r$  ensure that

$$(3.36) \quad \psi_\varepsilon(x^\star) = 1.$$

To proceed, fix  $\gamma \in \{1, \dots, M\}$  and define in  $\mathbb{R}_+^n$  (as usual, using the summation convention over repeated indices)

$$(3.37) \quad \vec{F} := \left( \psi_\varepsilon G_{\alpha\gamma} a_{jk}^{\alpha\beta} \partial_k u_\beta - \psi_\varepsilon u_\alpha a_{kj}^{\beta\alpha} \partial_k G_{\beta\gamma} - u_\beta G_{\alpha\gamma} a_{kj}^{\alpha\beta} \partial_k \psi_\varepsilon \right)_{1 \leq j \leq n}.$$

From (A.39), (A.41), (A.42), and (3.37) it follows that  $\vec{F} \in L_{\text{loc}}^1(\mathbb{R}_+^n, \mathbb{C}^n)$ , and a direct calculation shows that  $\text{div } \vec{F}$  (considered in the sense of distributions in  $\mathbb{R}_+^n$ ) is given by

$$(3.38) \quad \begin{aligned} \text{div } \vec{F} &= (\partial_j \psi_\varepsilon) G_{\alpha\gamma} a_{jk}^{\alpha\beta} \partial_k u_\beta + \psi_\varepsilon (\partial_j G_{\alpha\gamma}) a_{jk}^{\alpha\beta} \partial_k u_\beta \\ &\quad + \psi_\varepsilon G_{\alpha\gamma} a_{jk}^{\alpha\beta} (\partial_j \partial_k u_\beta) - (\partial_j \psi_\varepsilon) u_\alpha a_{kj}^{\beta\alpha} \partial_k G_{\beta\gamma} \\ &\quad - \psi_\varepsilon (\partial_j u_\alpha) a_{kj}^{\beta\alpha} \partial_k G_{\beta\gamma} - \psi_\varepsilon u_\alpha a_{kj}^{\beta\alpha} (\partial_j \partial_k G_{\beta\gamma}) \\ &\quad - (\partial_j u_\beta) G_{\alpha\gamma} a_{kj}^{\alpha\beta} \partial_k \psi_\varepsilon - u_\beta (\partial_j G_{\alpha\gamma}) a_{kj}^{\alpha\beta} \partial_k \psi_\varepsilon \\ &\quad - u_\beta G_{\alpha\gamma} a_{kj}^{\alpha\beta} (\partial_j \partial_k \psi_\varepsilon) \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9, \end{aligned}$$

where the last equality defines the  $I_i$ 's. Let us analyze some of these terms. Changing variables  $j' = k$  and  $k' = j$  in  $I_1$  yields

$$(3.39) \quad I_1 = (\partial_{k'} \psi_\varepsilon) G_{\alpha\gamma} a_{k'j'}^{\alpha\beta} \partial_{j'} u_\beta = -I_7.$$

For  $I_2$  we change variables  $j' = k$ ,  $k' = j$ ,  $\alpha' = \beta$ ,  $\beta' = \alpha$  in order to write

$$(3.40) \quad I_2 = \psi_\varepsilon (\partial_{k'} G_{\beta'\gamma}) a_{k'j'}^{\beta'\alpha'} \partial_{j'} u_{\alpha'} = -I_5.$$

As regards  $I_3$ , we have

$$(3.41) \quad I_3 = \psi_\varepsilon G_{\alpha\gamma} (Lu)_\alpha = 0,$$

by the assumptions on  $u$ . For  $I_6$  we observe that (with  $G_{\cdot\gamma} := (G_{\mu\gamma})_\mu$ )

$$(3.42) \quad I_6 = -\psi_\varepsilon u_\alpha (L^\top G_{\cdot\gamma})_\alpha = -\psi_\varepsilon u_\alpha \delta_{\alpha\gamma} \delta_{x^*} = -\psi_\varepsilon u_\gamma \delta_{x^*},$$

thanks to (A.26) where we recall that  $G = G(\cdot, x^*)$  is the Green function for  $L^\top$  with pole at  $x^*$ . Collectively, these equalities permit us to conclude that

$$(3.43) \quad \operatorname{div} \vec{F} = -\psi_\varepsilon u_\gamma \delta_{x^*} - (\partial_j \psi_\varepsilon) u_\alpha a_{kj}^{\beta\alpha} \partial_k G_{\beta\gamma} \\ - u_\beta (\partial_j G_{\alpha\gamma}) a_{kj}^{\alpha\beta} \partial_k \psi_\varepsilon - u_\beta G_{\alpha\gamma} a_{kj}^{\alpha\beta} (\partial_j \partial_k \psi_\varepsilon) \quad \text{in } \mathcal{D}'(\mathbb{R}_+^n).$$

Notice that the first term in the right-hand side is a distribution supported at the singleton  $\{x^*\}$  and therefore is in  $\mathcal{E}'_{K_\star}(\mathbb{R}_+^n)$ . The remaining terms are in  $L^1(\mathbb{R}_+^n)$ , as seen from estimates (3.57), (3.59) established below. Thus, condition (a) in Theorem A.1 holds.

To verify condition (c) in Theorem A.1 we first observe that  $\psi_\varepsilon \equiv 0$  in the horizontal strip  $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_n < \varepsilon\}$ . In light of (3.37), this clearly implies that

$$(3.44) \quad \vec{F}|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = 0 \quad \text{everywhere on } \partial \mathbb{R}_+^n.$$

Let us now turn our attention to condition (b) in Theorem A.1. This is a purely qualitative membership, so bounds depending on  $\varepsilon$  and  $x^*$  are permissible. We first observe from (3.37) that there exists  $C \in (0, \infty)$  such that

$$(3.45) \quad |\vec{F}| \leq C \mathbf{1}_{\{x_n \geq \varepsilon\}} |G| |\nabla u| + C \mathbf{1}_{\{x_n \geq \varepsilon\}} |u| |\nabla G| + C \varepsilon^{-1} \mathbf{1}_{\{\varepsilon \leq x_n \leq 2\varepsilon\}} |u| |G|,$$

in  $\mathbb{R}_+^n$ . Pick a point  $x' \in \mathbb{R}^{n-1}$ , and select  $y = (y', y_n) \in \Gamma_{\kappa/4}(x')$  with  $y_n \geq \varepsilon$  (where  $\kappa > 0$  was fixed above). Let  $\rho := \min\{1/4, 9\kappa/(16 + 4\kappa)\}$ . We claim that

$$(3.46) \quad B(y, \rho\varepsilon) \subset \Gamma_\kappa(x').$$

Indeed, if  $z = (z', z_n) \in B(y, \rho\varepsilon)$  then  $z_n > 3\varepsilon/4$  and  $y_n < z_n + \varepsilon\rho$ . Hence,  $y_n < (4\rho/3 + 1)z_n$ . Since  $|y' - x'| < (\kappa/4)y_n$  also holds, we obtain

$$(3.47) \quad |z' - x'| \leq |z - y| + |y' - x'| < \rho\varepsilon + (\kappa/4)y_n < (4\rho/3 + \kappa\rho/3 + \kappa/4)z_n \leq \kappa z_n,$$

ultimately proving (3.46). Using this and the interior estimates from Theorem A.5 we may therefore write

$$(3.48) \quad |\nabla u(y)| \leq C(\rho\varepsilon)^{-1} \int_{B(y, \rho\varepsilon)} |u(z)| dz \\ \leq C\varepsilon^{-1} \sup_{z \in B(y, \rho\varepsilon)} |u(z)| \leq C\varepsilon^{-1} \mathcal{N}_\kappa u(x').$$

Next, consider a point  $y = (y', y_n) \in \Gamma_{\kappa/4}(x') \setminus K_\star$  with  $y_n \geq \varepsilon$ . Then, as before, (3.46) holds. Let us also note that any  $z \in B(y, 2\rho\varepsilon)$  satisfies

$|z - x^\star| > 3r/4$  since, upon recalling that  $\varepsilon < r/2$  and  $\rho \leq 1/4$ , we may estimate

$$(3.49) \quad r < |y - x^\star| \leq |y - z| + |z - x^\star| < 2\rho\varepsilon + |z - x^\star| \leq r/4 + |z - x^\star|.$$

Thus,

$$(3.50) \quad B(y, 2\rho\varepsilon) \cap \overline{B(x^\star, 3r/4)} = \emptyset.$$

In particular,  $L^\top G = 0$  in  $B(y, 2\rho\varepsilon)$ . As such, we can use interior estimates for  $G$  (cf. Theorem A.5) in this ball and (A.28) in order to write (with the help of (3.46) and (3.50))

$$(3.51) \quad |\nabla G(y)| \leq C(\rho\varepsilon)^{-1} \int_{B(y, \rho\varepsilon)} |G(z)| dz \leq C\varepsilon^{-1} \sup_{z \in B(y, \rho\varepsilon)} |G(z)| \\ \leq C\varepsilon^{-1} \mathcal{N}_\kappa^{K^c} G(x') \leq C\varepsilon^{-1} \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}},$$

where  $K := \overline{B(x^\star, 3r/4)} \subset \mathbb{R}_+^n$ .

From (3.45), (3.48), (3.51), and (A.28) we deduce that

$$(3.52) \quad \mathcal{N}_{\kappa/4}^{K^c} \vec{F}(x') \leq C_{\varepsilon, \kappa, x^\star} \mathcal{N}_\kappa u(x') \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}}, \quad \forall x' \in \mathbb{R}^{n-1}.$$

Consequently, based on (3.52) and the assumption on  $\mathcal{N}_\kappa u$  in (3.5), we obtain that

$$(3.53) \quad \int_{\mathbb{R}^{n-1}} \mathcal{N}_{\kappa/4}^{K^c} \vec{F}(x') dx' \leq C_{\varepsilon, \kappa, x^\star} \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa u(x') \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx' < \infty.$$

The above estimate shows that  $\mathcal{N}_{\kappa/4}^{K^c} \vec{F} \in L^1(\partial\mathbb{R}_+^n)$  which, together with (A.56), implies  $\mathcal{N}_\kappa^{K^c} \vec{F} \in L^1(\partial\mathbb{R}_+^n)$ . Hence, condition (b) in Theorem A.1 holds as well.

Having verified all hypotheses in Theorem A.1, from (A.4), (3.44), (3.43), and (3.36), we obtain that

$$(3.54) \quad 0 = - \int_{\partial\mathbb{R}_+^n} e_n \cdot (\vec{F}|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}) d\mathcal{L}^{n-1} = (\mathcal{C}_b^\infty(\mathbb{R}_+^n))^* \langle \operatorname{div} \vec{F}, 1 \rangle_{\mathcal{C}_b^\infty(\mathbb{R}_+^n)} \\ = -u_\gamma(x^\star) - \int_{\mathbb{R}_+^n} (\partial_j \psi_\varepsilon) u_\alpha a_{kj}^{\beta\alpha} \partial_k G_{\beta\gamma} d\mathcal{L}^n \\ - \int_{\mathbb{R}_+^n} u_\beta (\partial_j G_{\alpha\gamma}) a_{kj}^{\alpha\beta} \partial_k \psi_\varepsilon d\mathcal{L}^n - \int_{\mathbb{R}_+^n} u_\beta G_{\alpha\gamma} a_{kj}^{\alpha\beta} (\partial_j \partial_k \psi_\varepsilon) d\mathcal{L}^n.$$

We claim that the three integrals in the rightmost side of (3.54) converge to zero as  $\varepsilon \rightarrow 0^+$ . This, in turn, will imply that  $u(x^\star) = 0$  and since  $x^\star \in \mathbb{R}_+^n$  is an arbitrary point we may ultimately conclude that  $u \equiv 0$  in  $\mathbb{R}_+^n$ , as desired.



An inspection of the aforementioned integrals reveals that we need to prove that

$$(3.55) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u| |\nabla G| d\mathcal{L}^n = 0,$$

and

$$(3.56) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^2} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u| |G| d\mathcal{L}^n = 0.$$

From (A.35), and the Mean Value Theorem we have

$$(3.57) \quad \begin{aligned} & \frac{1}{\varepsilon^2} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u| |G| d\mathcal{L}^n \\ &= \frac{1}{\varepsilon^2} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u(x', x_n)| |G(x', x_n) - G(x', 0)| dx \\ &\leq \frac{C}{\varepsilon} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u(x', x_n)| \left( \sup_{0 < t < 2\varepsilon} |(\partial_{x_n} G)(x', t)| \right) dx \\ &\leq \frac{C}{\varepsilon} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u(x', x_n)| \left( \sup_{0 < t < 2\varepsilon} |(\nabla G)(x', t)| \right) dx. \end{aligned}$$

Note that the integral in (3.55) can also be controlled by the last quantity in (3.57). Therefore, matters have been reduced to showing that

$$(3.58) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u(x', x_n)| \left( \sup_{0 < t < 2\varepsilon} |(\nabla G)(x', t)| \right) dx = 0.$$

In this regard, by (A.29) we have

$$(3.59) \quad \begin{aligned} & \frac{C}{\varepsilon} \int_{\{x=(x', x_n) \in \mathbb{R}^n : \varepsilon < x_n < 2\varepsilon\}} |u(x', x_n)| \left( \sup_{0 < t < 2\varepsilon} |(\nabla G)(x', t)| \right) dx \\ &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa^{(2\varepsilon)} u(x') \mathcal{N}_\kappa^{(2\varepsilon)} (\nabla G)(x') dx' \\ &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa^{(2\varepsilon)} u(x') \mathcal{N}_\kappa^{K^c} (\nabla G)(x') dx' \\ &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa^{(2\varepsilon)} u(x') \frac{1}{1 + |x'|^{n-1}} dx' \\ &\leq C \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa^{(2\varepsilon)} u(x') \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx', \end{aligned}$$

where  $\mathcal{N}_\kappa^{(2\varepsilon)}$  is the truncated nontangential maximal function defined as in (2.7). Notice that  $\lim_{\varepsilon \rightarrow 0^+} \mathcal{N}_\kappa^{(2\varepsilon)} u(x') = 0$  for a.e.  $x' \in \mathbb{R}^{n-1}$  by the fact that

$u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = 0$ . On the other hand,  $0 \leq \mathcal{N}_\kappa^{(2\varepsilon)}u(x') \leq \mathcal{N}_\kappa u(x')$  for each  $x' \in \mathbb{R}^{n-1}$ , and therefore the integrand is uniformly controlled by an  $L^1(\mathbb{R}^{n-1})$  function thanks to our assumption in (3.5). Thus, the desired formula (3.58) follows on account of (3.59), (3.5), and Lebesgue's Dominated Convergence Theorem. This finishes the proof of Theorem 3.2.  $\square$

We conclude this section with a remark which, in particular, shows that in the special case when  $L = \Delta$  the hypotheses in Theorem 3.2 may be slightly relaxed.

*Remark 3.5.* If the system  $L$  (assumed to be as in (1.1)-(1.2)) is such that its fundamental solution  $E^L$  from Theorem A.2 is a radial function when restricted to  $\mathbb{R}^n \setminus \{0\}$ , then the logarithm in (3.5) may be omitted. This is seen by inspecting the proof of Theorem 3.2 and making use of part (1) in Theorem A.4.

#### 4. WELL-POSEDNESS FOR THE DIRICHLET PROBLEM

In this section, Theorems 3.1 and 3.2 will be used to prove Theorem 1.1, Corollary 1.2, Corollary 1.3, and Theorem 1.5.

*Proof of Theorem 1.1.* For existence, invoke Theorem 3.1 (whose applicability is ensured by the first condition in (1.7)) and note that if  $u$  is as in (3.2) then the first, second, and last conditions in (1.9) are satisfied. In addition, (1.11) is simply (3.4). Together, (1.8) and (1.11) then permit us to conclude that the third condition in (1.9) is also satisfied. Hence,  $u$  solves (1.9). For uniqueness, assume that both  $u_1$  and  $u_2$  solve (1.9) for the same datum  $f$  and set  $u := u_1 - u_2 \in \mathcal{C}^\infty(\mathbb{R}_+^n)$ . Then  $Lu = 0$  in  $\mathbb{R}_+^n$  and, since  $\mathcal{N}u_1, \mathcal{N}u_2 \in \mathbb{Y}$ , the estimate  $0 \leq \mathcal{N}u \leq \mathcal{N}u_1 + \mathcal{N}u_2 \leq 2 \max\{\mathcal{N}u_1, \mathcal{N}u_2\}$  forces  $\mathcal{N}u \in \mathbb{Y}$  by the properties of the function lattice  $\mathbb{Y}$ . Granted this, Theorem 3.2 applies (thanks to the second condition in (1.7)) and gives that  $u \equiv 0$  in  $\mathbb{R}_+^n$ , hence  $u_1 = u_2$  as wanted.  $\square$

Before presenting the proof of Corollary 1.2, some comments are in order. Having fixed some  $q \in (1, \infty)$  (whose actual choice is ultimately immaterial), one may define the **Hardy space**  $H^1(\mathbb{R}^{n-1})$  as

$$(4.1) \quad H^1(\mathbb{R}^{n-1}) := \left\{ f \in L^1(\mathbb{R}^{n-1}) : f = \sum_{j \in \mathbb{N}} \lambda_j a_j \quad \text{a.e. in } \mathbb{R}^{n-1}, \right. \\ \left. \text{for some } (1, q)\text{-atoms } \{a_j\}_{j \in \mathbb{N}} \text{ and scalars } \{\lambda_j\}_{j \in \mathbb{N}} \in \ell^1 \right\}.$$

For each  $f \in H^1(\mathbb{R}^{n-1})$  we then set  $\|f\|_{H^1(\mathbb{R}^{n-1})} := \inf \sum_{j \in \mathbb{N}} |\lambda_j|$  with the infimum taken over all atomic representations of  $f$  as  $\sum_{j \in \mathbb{N}} \lambda_j a_j$ .

Recall that a Lebesgue measurable function  $a : \mathbb{R}^{n-1} \rightarrow \mathbb{C}$  is said to be an  $(1, q)$ -atom if, for some cube  $Q \subset \mathbb{R}^{n-1}$ , one has

$$(4.2) \quad \text{supp } a \subset Q, \quad \|a\|_{L^q(\mathbb{R}^{n-1})} \leq |Q|^{1/q-1}, \quad \int_{\mathbb{R}^{n-1}} a(y') dy' = 0.$$

*Proof of Corollary 1.2.* Having already established Theorem 1.1, we only need to check that the implication in (1.15) holds if  $\mathbb{X} = H^1(\mathbb{R}^{n-1})$  and  $\mathbb{Y} = L^1(\mathbb{R}^{n-1})$  (recall that we have  $H^1(\mathbb{R}^{n-1}) \subset L^1(\mathbb{R}^{n-1})$ , thus (1.7) and the first condition in (1.8) are clear for this choice). To this end, assume first that

$$(4.3) \quad u(x', t) := (P_t * a)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n,$$

where  $a : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^M$  is a Lebesgue measurable function (whose scalar components are) as in (4.2). Then (3.4), Hölder's inequality, the  $L^q$ -boundedness of the Hardy-Littlewood maximal function, and the normalization of the atom permit us to write

$$(4.4) \quad \begin{aligned} \int_{2\sqrt{n}Q} \mathcal{N}_\kappa u d\mathcal{L}^{n-1} &\leq C \int_{2\sqrt{n}Q} \mathcal{M}a d\mathcal{L}^{n-1} \\ &\leq C|Q|^{1/q'} \left( \int_{2\sqrt{n}Q} (\mathcal{M}a)^q d\mathcal{L}^{n-1} \right)^{1/q} \\ &\leq C|Q|^{1/q'} \left( \int_{\mathbb{R}^{n-1}} (\mathcal{M}a)^q d\mathcal{L}^{n-1} \right)^{1/q} \\ &\leq C|Q|^{1/q'} \|a\|_{L^q(\mathbb{R}^{n-1})} \leq C, \end{aligned}$$

for some constant  $C \in (0, \infty)$  depending only on  $n, L, \kappa$ . To proceed, fix an arbitrary point  $x' \in \mathbb{R}^{n-1} \setminus 2\sqrt{n}Q$ . If  $\ell(Q)$  and  $x'_Q$  are, respectively, the side-length and center of the cube  $Q$ , this choice entails

$$(4.5) \quad |z' - x'_Q| \leq \max\{\kappa, 2\}(t + |z' - \xi'|), \quad \forall (z', t) \in \Gamma_\kappa(x'), \quad \forall \xi' \in Q.$$

Indeed, if  $(z', t) \in \Gamma_\kappa(x')$  and  $\xi' \in Q$  then, first,  $|z' - x'_Q| \leq |z' - \xi'| + |\xi' - x'_Q|$  and, second,

$$(4.6) \quad |\xi' - x'_Q| \leq \frac{\sqrt{n}}{2}\ell(Q) \leq \frac{1}{2}|x' - x'_Q| \leq \frac{1}{2}(|x' - z'| + |z' - x'_Q|) \leq \frac{1}{2}(\kappa t + |z' - x'_Q|),$$

from which (4.5) follows. Next, using (2.28), the vanishing moment condition for the atom, the Mean Value Theorem together with (3.30) and (4.5), Hölder's inequality and, finally, the support and normalization of the atom, for each  $(z', t) \in \Gamma_\kappa(x')$  we may estimate

$$(4.7) \quad |(P_t^L * a)(z')| = \left| \int_{\mathbb{R}^{n-1}} [K^L(z' - y', t) - K^L(z' - x'_Q, t)] a(y') dy' \right|$$

$$\begin{aligned}
&\leq \int_Q |K^L(z' - y', t) - K^L(z' - x'_Q, t)| |a(y')| dy' \\
&\leq C \frac{\ell(Q)}{(t + |z' - x'_Q|)^n} \int_Q |a(y')| dy' \\
&\leq C \frac{\ell(Q)}{(t + |z' - x'_Q|)^n} |Q|^{1/q'} \|a\|_{L^q(\mathbb{R}^{n-1})} \\
&\leq \frac{C\ell(Q)}{(t + |z' - x'_Q|)^n}.
\end{aligned}$$

In turn, (4.7) implies that for each  $x' \in \mathbb{R}^{n-1} \setminus 2\sqrt{n}Q$  we have

$$\begin{aligned}
(4.8) \quad (\mathcal{N}_\kappa u)(x') &= \sup_{(z', t) \in \Gamma_\kappa(x')} |(P_t^L * a)(z')| \\
&\leq \sup_{(z', t) \in \Gamma_\kappa(x')} \frac{C\ell(Q)}{(t + |z' - x'_Q|)^n} \leq \frac{C\ell(Q)}{|x' - x'_Q|^n},
\end{aligned}$$

hence

$$(4.9) \quad \int_{\mathbb{R}^{n-1} \setminus 2\sqrt{n}Q} \mathcal{N}_\kappa u d\mathcal{L}^{n-1} \leq C \int_{\mathbb{R}^{n-1} \setminus 2\sqrt{n}Q} \frac{\ell(Q)}{|x' - x'_Q|^n} dx' = C,$$

for some constant  $C \in (0, \infty)$  depending only on  $n, L$ . From (4.4) and (4.9) we deduce that whenever  $u$  is as in (4.3) then

$$(4.10) \quad \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa u d\mathcal{L}^{n-1} \leq C,$$

for some constant  $C \in (0, \infty)$  independent of the atom.

To conclude, for each  $p \in [1, \infty)$  define the tent spaces

$$(4.11) \quad \mathcal{T}^p(\mathbb{R}_+^n) := \{u : \mathbb{R}_+^n \rightarrow \mathbb{C}^M : u \text{ measurable and } \mathcal{N}_\kappa u \in L^p(\mathbb{R}^{n-1})\}$$

equipped with the norm  $\|u\|_{\mathcal{T}^p(\mathbb{R}_+^n)} := \|\mathcal{N}_\kappa u\|_{L^p(\mathbb{R}^{n-1})}$ . It may be actually checked that the pair  $(\mathcal{T}^p(\mathbb{R}_+^n), \|\cdot\|_{\mathcal{T}^p(\mathbb{R}_+^n)})$  is a Köthe function space, relative to the background measure space  $(\mathbb{R}_+^n, \mathcal{L}^n)$ . In this context, with  $q \in (1, \infty)$  the exponent intervening in (4.1), consider the assignment

$$\begin{aligned}
(4.12) \quad T : L^q(\mathbb{R}^{n-1}) &\longrightarrow \mathcal{T}^q(\mathbb{R}_+^n) \text{ given by} \\
Tf &:= u, \text{ where } u \text{ is associated with } f \text{ as in (3.2).}
\end{aligned}$$

Thanks to Theorem 3.1,  $T$  is a well-defined linear and bounded operator and, given what we have just proved in (4.10), it has the property that  $\|Ta\|_{\mathcal{T}^1(\mathbb{R}^{n-1})} \leq C$  for every  $(1, q)$ -atom  $a$ , for some constant  $C \in (0, \infty)$  independent of the atom in question. Granted these, it follows (see [4] for very general results of this nature) that  $T$  extends as a linear and bounded operator from  $H^1(\mathbb{R}^{n-1})$  into  $\mathcal{T}^1(\mathbb{R}^{n-1})$ . In light of (4.12), this shows that the implication in (1.15) is indeed true if  $\mathbb{X}, \mathbb{Y}$  are as in (1.16). This proves that,

for each system  $L$  as in (1.1)-(1.2), the corresponding  $(H^1, L^1)$ -Dirichlet boundary value problem in  $\mathbb{R}_+^n$ ,

$$(4.13) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ \mathcal{N}_\kappa u \in L^1(\mathbb{R}^{n-1}), \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in H^1(\mathbb{R}^{n-1}), \end{cases}$$

has a unique solution. Moreover, the above argument also shows that the following naturally accompanying estimate holds

$$(4.14) \quad \|\mathcal{N}_\kappa u\|_{L^1(\mathbb{R}^{n-1})} \leq C \|f\|_{H^1(\mathbb{R}^{n-1})},$$

for some  $C = C(n, L, \kappa) \in (0, \infty)$ . Hence, (4.13) is well-posed.

In closing, we wish to note that one can give a proof of (4.14), and also of (1.15), which avoids working with tent spaces by reasoning directly as follows (incidentally, this is also going to be useful later on, in the proof of Corollary 1.3). Let  $f \in H^1(\mathbb{R}^{n-1})$  and consider a quasi-optimal atomic decomposition, say  $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$  with

$$(4.15) \quad \frac{1}{2} \sum_{j \in \mathbb{N}} |\lambda_j| \leq \|f\|_{H^1(\mathbb{R}^{n-1})} \leq \sum_{j \in \mathbb{N}} |\lambda_j|.$$

For each  $N \in \mathbb{N}$ , write  $f_N := \sum_{j=1}^N \lambda_j a_j$ . Clearly,  $f_N \rightarrow f$  in  $L^1(\mathbb{R}^{n-1})$  as  $N \rightarrow \infty$ , hence also a.e. after eventually passing to a subsequence. To prove (4.14) in the case when  $u$  is defined as in (1.10) we proceed as follows. For each  $N \in \mathbb{N}$ , introduce  $u_N(x', t) := (P_t^L * f_N)(x')$  for all  $(x', t) \in \mathbb{R}_+^n$  (this makes sense since  $f_N \in L^1(\mathbb{R}^{n-1})$ ). Then, using (3.4), we may write

$$(4.16) \quad \begin{aligned} \|\mathcal{N}_\kappa u - \mathcal{N}_\kappa(u_N)\|_{L^1, \infty(\mathbb{R}^{n-1})} &\leq \|\mathcal{N}_\kappa(u - u_N)\|_{L^1, \infty(\mathbb{R}^{n-1})} \\ &\leq C \|\mathcal{M}(f - f_N)\|_{L^1, \infty(\mathbb{R}^{n-1})} \\ &\leq C \|f - f_N\|_{L^1(\mathbb{R}^{n-1})} \longrightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Thus,  $\mathcal{N}_\kappa(u_N) \rightarrow \mathcal{N}_\kappa u$  in  $L^{1, \infty}(\mathbb{R}^{n-1})$  as  $N \rightarrow \infty$  and, by passing to a subsequence  $\{N_j\}_{j \in \mathbb{N}}$ , we may ensure that  $\mathcal{N}_\kappa(u_{N_j}) \rightarrow \mathcal{N}_\kappa u$  pointwise a.e. in  $\mathbb{R}^{n-1}$  as  $j \rightarrow \infty$  (cf., e.g., the discussion in [28, Example 6, pp. 4776-4777]). In turn, if for each  $j \in \mathbb{N}$  we set  $v_j(x', t) := (P_t^L * a_j)(x')$  for  $(x', t) \in \mathbb{R}_+^n$ , this readily gives

$$(4.17) \quad \mathcal{N}_\kappa u \leq \sum_{j \in \mathbb{N}} |\lambda_j| \mathcal{N}_\kappa(v_j) \text{ a.e. in } \mathbb{R}^{n-1}.$$

From (4.17), (4.10), and (4.15) we then conclude that

$$(4.18) \quad \|\mathcal{N}_\kappa u\|_{L^1(\mathbb{R}^{n-1})} \leq C \sum_{j \in \mathbb{N}} |\lambda_j| \leq C \|f\|_{H^1(\mathbb{R}^{n-1})},$$

finishing the alternative proof of (4.14) and the implication in (1.15).  $\square$

As a preamble to the proof of Corollary 1.3 we first properly define the spaces intervening in (1.17). Given  $p \in (1, \infty)$  define the **Beurling space**  $A^p(\mathbb{R}^{n-1})$  as the collection of  $p$ -th power locally integrable functions  $f$  in  $\mathbb{R}^{n-1}$  satisfying (with  $p'$  denoting the Hölder conjugate exponent of  $p$ )

$$(4.19) \quad \|f\|_{A^p(\mathbb{R}^{n-1})} := \sum_{k=0}^{\infty} 2^{k(n-1)/p'} \|f \mathbf{1}_{C_k}\|_{L^p(\mathbb{R}^{n-1})} < \infty,$$

where  $C_0 := B_{n-1}(0', 1)$  and  $C_k := B_{n-1}(0', 2^k) \setminus \overline{B_{n-1}(0', 2^{k-1})}$  for each  $k \in \mathbb{N}$ . This readily implies that

$$(4.20) \quad (A^p(\mathbb{R}^{n-1}), \|\cdot\|_{A^p(\mathbb{R}^{n-1})}) \text{ is a Banach space, which is a function lattice, and embeds continuously into } L^1(\mathbb{R}^{n-1}).$$

Next, call a function  $a \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$  a **central**  $(1, p)$ -atom provided there exists a cube  $Q$  in  $\mathbb{R}^{n-1}$ , centered at the origin and having side-length  $\ell(Q) \geq 1$  such that

$$(4.21) \quad \text{supp } a \subseteq Q, \quad \|a\|_{L^p(\mathbb{R}^{n-1})} \leq |Q|^{1/p-1} \text{ and } \int_{\mathbb{R}^{n-1}} a(x') dx' = 0.$$

Then, following [16], we define the Beurling-Hardy space as

$$(4.22) \quad \text{HA}^p(\mathbb{R}^{n-1}) := \left\{ f \in L^1(\mathbb{R}^{n-1}) : f = \sum_{j \in \mathbb{N}} \lambda_j a_j \text{ a.e. in } \mathbb{R}^{n-1}, \text{ for some} \right. \\ \left. \text{central } (1, p)\text{-atoms } \{a_j\}_{j \in \mathbb{N}} \text{ and } \{\lambda_j\}_{j \in \mathbb{N}} \in \ell^1 \right\},$$

and for each  $f \in \text{HA}^p(\mathbb{R}^{n-1})$  set  $\|f\|_{\text{HA}^p(\mathbb{R}^{n-1})} := \inf \sum_{j \in \mathbb{N}} |\lambda_j|$  with the infimum taken over representations of  $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$  as in (4.22). Various alternative characterizations of  $\text{HA}^p(\mathbb{R}^{n-1})$  may be found in [16, Theorem 3.1, p. 505]. Here we only wish to note that, as is apparent from definitions,

$$(4.23) \quad \text{HA}^p(\mathbb{R}^{n-1}) \hookrightarrow H^1(\mathbb{R}^{n-1}).$$

Given a system  $L$  as in (1.1)-(1.2) and having fixed some  $\kappa > 0$  and  $p \in (1, \infty)$ , the  $(\text{HA}^p, A^p)$ -Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$  is

then formulated as

$$(4.24) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \quad \text{in } \mathbb{R}_+^n, \\ \mathcal{N}_\kappa u \in A^p(\mathbb{R}^{n-1}), \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = f \in \text{HA}^p(\mathbb{R}^{n-1}). \end{cases}$$

We are now ready to present the proof of Corollary 1.3 dealing with the well-posedness of (4.24) for each  $p \in (1, \infty)$ .

*Proof of Corollary 1.3.* Fix  $p \in (1, \infty)$ . Granted Theorem 1.1, we are left with verifying that the implication in (1.15) holds if  $\mathbb{X} = \text{HA}^p(\mathbb{R}^{n-1})$  and  $\mathbb{Y} = A^p(\mathbb{R}^{n-1})$  (since (1.7) and the first condition in (1.8) are clear for this choice, thanks to (4.20)). With this goal in mind, pick a Lebesgue measurable function  $a : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^M$  whose scalar components are as in (4.21) and define  $u$  as in (4.3). Also, with the cube  $Q \subset \mathbb{R}^{n-1}$  centered at the origin and side-length  $\ell(Q) \geq 1$  as in (4.21), let  $N_a$  be the smallest nonnegative integer which is larger than or equal to  $\log_2(n\ell(Q))$ . Using (3.4), the  $L^p$ -boundedness of the Hardy-Littlewood maximal function, and the normalization of the central  $(1, p)$ -atom we obtain

$$(4.25) \quad \begin{aligned} \sum_{k=0}^{N_a} 2^{k(n-1)/p'} \|(\mathcal{N}_\kappa u) \mathbf{1}_{C_k}\|_{L^p(\mathbb{R}^{n-1})} &\leq C \|\mathcal{M}a\|_{L^p(\mathbb{R}^{n-1})} \sum_{k=0}^{N_a} 2^{k(n-1)/p'} \\ &\leq C \|a\|_{L^p(\mathbb{R}^{n-1})} 2^{N_a(n-1)/p'} \\ &\leq C |Q|^{1/p-1} (2^{\log_2(n\ell(Q))})^{(n-1)/p'} \\ &= C < \infty, \end{aligned}$$

where the constant  $C$  is independent of the central  $(1, p)$ -atom  $a$ . Next, fix an arbitrary integer  $k \geq N_a + 1$  along with some point  $x' \in C_k$ . This choice entails  $|x'| > 2^{k-1} \geq 2^{N_a} \geq n\ell(Q)$  which, in turn, forces  $x' \in \mathbb{R}^{n-1} \setminus 2\sqrt{n}Q$ . Granted this, the same type of estimates as in (4.7)-(4.8) (this time with  $x'_Q = 0'$ ) lead to the conclusion that there exists a constant  $C \in (0, \infty)$  depending only on  $n, L, \kappa$  with the property that

$$(4.26) \quad (\mathcal{N}_\kappa u)(x') \leq \frac{C\ell(Q)}{|x'|^n} \quad \text{whenever } x' \in C_k \text{ with } k \geq N_a + 1.$$

Having established this we may then estimate

$$(4.27) \quad \sum_{k=N_a+1}^{\infty} 2^{k(n-1)/p'} \|(\mathcal{N}_\kappa u) \mathbf{1}_{C_k}\|_{L^p(\mathbb{R}^{n-1})}$$

$$\begin{aligned}
&\leq C\ell(Q) \sum_{k=N_a+1}^{\infty} 2^{k(n-1)/p'} \left( \int_{C_k} |x'|^{-np} dx' \right)^{1/p} \\
&\leq C\ell(Q) \sum_{k=N_a+1}^{\infty} 2^{k(n-1)/p'} 2^{-n(k-1)} 2^{k(n-1)/p} \\
&\leq C\ell(Q) \sum_{k=N_a+1}^{\infty} 2^{-k} \\
&\leq C\ell(Q) 2^{-N_a} = C < \infty,
\end{aligned}$$

for some constant  $C$  independent of  $a$ . In concert, (4.25) and (4.27) prove that there exists some constant  $C \in (0, \infty)$  such that whenever  $u$  is as in (4.3) for some central  $(1, p)$ -atom  $a$  then

$$(4.28) \quad \|\mathcal{N}_\kappa u\|_{A^p(\mathbb{R}^{n-1})} \leq C.$$

Going further, we shall make use of (4.28) in order to show that, if for an arbitrary function  $f \in \text{HA}^p(\mathbb{R}^{n-1})$  we set  $u(x', t) := (P_t^L * f)(x')$  for all  $(x', t) \in \mathbb{R}_+^n$ , then

$$(4.29) \quad \|\mathcal{N}_\kappa u\|_{A^p(\mathbb{R}^{n-1})} \leq C \|f\|_{\text{HA}^p(\mathbb{R}^{n-1})},$$

for some finite constant  $C > 0$  independent of  $f$ . Specifically, (4.29) is justified with the help of (4.28), (4.23), and (4.20) by reasoning almost verbatim as in (4.15)-(4.18). This proves the implication in (1.15) in the current context, and shows that (4.24) is well-posed.  $\square$

The proof of Theorem 1.5 requires some prerequisites, and we begin by discussing rearrangement invariant spaces. To set the stage, let  $\mu_f$  denote the distribution function of a given  $f \in \mathbb{M}$ , i.e.,

$$(4.30) \quad \mu_f(\lambda) := |\{x' \in \mathbb{R}^{n-1} : |f(x')| > \lambda\}|, \quad \forall \lambda \geq 0.$$

Call two functions  $f, g \in \mathbb{M}$  equimeasurable provided  $\mu_f = \mu_g$ . A **rearrangement invariant space** (or, r.i. space for short) is a Köthe function space  $\mathbb{X}$  with the property that equimeasurable functions have the same function norm in  $\mathbb{X}$  (i.e., if  $\|f\|_{\mathbb{X}} = \|g\|_{\mathbb{X}}$  for all  $f, g \in \mathbb{X}$  such that  $\mu_f = \mu_g$ ). In particular, if  $\mathbb{X}$  is an r.i. space, one can check that its Köthe dual space  $\mathbb{X}'$  is also rearrangement invariant.

Given  $f \in \mathbb{M}$ , the **decreasing rearrangement** of  $f$  with respect to the Lebesgue measure in  $\mathbb{R}^{n-1}$  is the function  $f^*$ , with domain  $[0, \infty)$ , defined by

$$(4.31) \quad f^*(t) := \inf \{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad 0 \leq t < \infty.$$



Relative to the original function  $f$ , its decreasing rearrangement satisfies, for each  $\lambda \geq 0$ ,

$$(4.32) \quad |\{x' \in \mathbb{R}^{n-1} : |f(x')| > \lambda\}| = |\{t \in [0, \infty) : f^*(t) > \lambda\}|.$$

Applying the Luxemburg representation theorem yields the following: given an r.i. space  $\mathbb{X}$ , there exists a unique r.i. space  $\overline{\mathbb{X}}$  on  $[0, \infty)$  such that for each  $f \in \mathbb{M}$  one has  $f \in \mathbb{X}$  if and only if  $f^* \in \overline{\mathbb{X}}$  and, in this case,  $\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}}$ . Furthermore,  $(\overline{\mathbb{X}})' = \overline{\mathbb{X}'}$ , and so  $\|f\|_{\mathbb{X}'} = \|f^*\|_{\overline{\mathbb{X}'}}$  for every  $f \in \mathbb{M}$ .

Using this representation we can now introduce the Boyd indices of an r.i. space  $\mathbb{X}$ . Given  $f \in \overline{\mathbb{X}}$ , consider the dilation operator  $D_t$ ,  $0 < t < \infty$ , by setting  $D_t f(s) := f(s/t)$  for each  $s \geq 0$ . Writing

$$(4.33) \quad h_{\mathbb{X}}(t) := \sup \{ \|D_t f\|_{\overline{\mathbb{X}}} : f \in \overline{\mathbb{X}} \text{ with } \|f\|_{\overline{\mathbb{X}}} \leq 1 \}, \quad t \in (0, \infty),$$

the lower and upper Boyd indices may, respectively, be defined as

$$(4.34) \quad p_{\mathbb{X}} := \lim_{t \rightarrow \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \sup_{1 < t < \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)},$$

and

$$(4.35) \quad q_{\mathbb{X}} := \lim_{t \rightarrow 0^+} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \inf_{0 < t < 1} \frac{\log t}{\log h_{\mathbb{X}}(t)}.$$

By design,  $1 \leq p_{\mathbb{X}} \leq q_{\mathbb{X}} \leq \infty$ . The Boyd indices for  $\mathbb{X}$  are related to those for  $\mathbb{X}'$  via

$$(4.36) \quad p_{\mathbb{X}'} = (q_{\mathbb{X}})' \quad \text{and} \quad q_{\mathbb{X}'} = (p_{\mathbb{X}})'.$$

*Remark 4.1.* Some authors (including [6]) define the Boyd indices as the reciprocals of  $p_{\mathbb{X}}$  and  $q_{\mathbb{X}}$  defined above. We have chosen the present definition since it yields  $p_{\mathbb{X}} = q_{\mathbb{X}} = p$  if  $\mathbb{X} = L^p(\mathbb{R}^{n-1})$ .

The importance of Boyd indices stems from the fact that they play a significant role in interpolation (see, e.g., [6, Chapter 3]). For example, the classical result of Lorentz-Shimogaki states that the Hardy-Littlewood maximal operator  $\mathcal{M}$  is bounded on an r.i. space  $\mathbb{X}$  if and only if  $p_{\mathbb{X}} > 1$ . Additionally, Boyd's theorem asserts that the Hilbert transform is bounded on an r.i. space  $\mathbb{X}$  on  $\mathbb{R}$  if and only if  $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$ . See [6, Chapter 3] for the precise statements and complete references.

Given an r.i. space  $\mathbb{X}$  on  $\mathbb{R}^{n-1}$ , we wish to introduce a weighted version  $\mathbb{X}(w)$  of  $\mathbb{X}$  via an analogous definition in which the underlying measure in  $\mathbb{R}^{n-1}$  now is  $d\mu(x') := w(x') dx'$ . These spaces appeared in [13] as an abstract generalization of a variety of weighted function spaces. Specifically, fix a weight  $w \in A_{\infty}(\mathbb{R}^{n-1})$  (in particular,  $0 < w < \infty$  a.e.). Given  $f \in \mathbb{M}$ , let  $w_f$  denote the distribution function of  $f$  with respect to the measure  $w(x') dx'$ :

$$(4.37) \quad w_f(\lambda) := w(\{x' \in \mathbb{R}^{n-1} : |f(x')| > \lambda\}), \quad \lambda \geq 0.$$

We also let  $f_w^*$  denote the decreasing rearrangement of  $f$  with respect to the measure  $w(x') dx'$ , i.e.,

$$(4.38) \quad f_w^*(t) := \inf \{ \lambda \geq 0 : w_f(\lambda) \leq t \}, \quad 0 \leq t < \infty.$$

Granted these, define the weighted space  $\mathbb{X}(w)$  by

$$(4.39) \quad \mathbb{X}(w) := \{ f \in \mathbb{M} : \|f_w^*\|_{\mathbb{X}} < \infty \}.$$

This may be viewed as a Köthe function space, but with underlying measure  $w(x') dx'$ , and with the function norm

$$(4.40) \quad \|f\|_{\mathbb{X}(w)} := \|f_w^*\|_{\mathbb{X}}.$$

Note that if  $\mathbb{X}$  is the Lebesgue space  $L^p(\mathbb{R}^{n-1})$ ,  $p \in (1, \infty)$ , it follows that  $\mathbb{X}(w) = L^p(\mathbb{R}^{n-1}, w)$ , the Lebesgue space of  $p$ -th power integrable functions in the measure space  $(\mathbb{R}^{n-1}, w(x') dx')$ .

The boundedness of the Hardy-Littlewood maximal operator on these weighted r.i. spaces was considered in [13], [12], from which we quote the following result.

**Lemma 4.2** ([13]). *Let  $\mathbb{X}$  be an r.i. space whose lower Boyd index satisfies  $p_{\mathbb{X}} > 1$ . Then for every  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$ , the Hardy-Littlewood maximal operator  $\mathcal{M}$  is bounded on  $\mathbb{X}(w)$ .*

At the heart of the proof of Theorem 1.5 there is an analog of (1.42) valid in the context of weighted rearrangement invariant function spaces. We are going to derive this in Lemma 4.5 below, by relying on the following Rubio de Francia extrapolation for r.i. spaces obtained in [12]:

**Theorem 4.3** ([12]). *Let  $\mathcal{F}$  be a given family of pairs  $(f, g)$  of non-negative, measurable functions that are not identically zero. Suppose that for some fixed exponent  $p_0 \in [1, \infty)$  and every weight  $w_0 \in A_{p_0}(\mathbb{R}^{n-1})$ , one has*

$$(4.41) \quad \int_{\mathbb{R}^{n-1}} f(x')^{p_0} w_0(x') dx' \leq C_{w_0} \int_{\mathbb{R}^{n-1}} g(x')^{p_0} w_0(x') dx', \quad \forall (f, g) \in \mathcal{F}.$$

*Then if  $\mathbb{X}$  is an r.i. space such that  $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$ , it follows that for each weight  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$  there holds*

$$(4.42) \quad \|f\|_{\mathbb{X}(w)} \leq C_w \|g\|_{\mathbb{X}(w)}, \quad \forall (f, g) \in \mathcal{F}.$$

*Remark 4.4.* As discussed in [12], inequalities of the form (4.41) or (4.42) (both in hypotheses and in the conclusion) are assumed to hold for any  $(f, g) \in \mathcal{F}$  for which the left-hand side is finite.

**Lemma 4.5.** *Let  $\mathbb{X}$  be an r.i. space with the property that its lower and upper Boyd indices satisfy  $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$ . For every  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$ ,*

there exists  $C = C(n, \mathbb{X}, w) \in (0, \infty)$  such that for each  $h \in \mathbb{X}(w)$  there holds

$$(4.43) \quad \int_{\mathbb{R}^{n-1}} |h(x')| \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)})(x') dx' \leq C \|\mathbf{1}_{B_{n-1}(0',1)}\|_{\mathbb{X}(w)}^{-1} \|h\|_{\mathbb{X}(w)}.$$

In particular, from (4.43) and Lemma 2.1 one has the continuous inclusion

$$(4.44) \quad \mathbb{X}(w) \hookrightarrow L^1\left(\mathbb{R}^{n-1}, \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}} dx'\right).$$

*Proof.* We obtain this result via extrapolation using Theorem 4.3. Fix a sufficiently large integer  $N$  and, for every  $h \in \mathbb{M}$ , set  $h_N := h \mathbf{1}_{\{x' \in B_{n-1}(0',N): |h(x')| \leq N\}}$ . In particular,

$$(4.45) \quad I_N(h) := \int_{\mathbb{R}^{n-1}} |h_N(x')| \mathcal{M}^{(2)}(\mathbf{1}_{B_{n-1}(0',1)})(x') dx' \leq N |B_{n-1}(0', N)| = C_N < \infty.$$

We now consider the family of pairs:

$$(4.46) \quad \mathcal{F}_N := \{(F_1, F_2) = (I_N(h) \mathbf{1}_{B_{n-1}(0',1)}, |h_N|) : h \in \mathbb{M}\}.$$

Given  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^{n-1})$ , there exists  $C = C(n, p, w) \in (0, \infty)$  such that for every  $N \geq 1$  we may write (as in (1.42) with  $h_N$  replacing  $h$ )

$$(4.47) \quad I_N(h) \leq C \|h_N\|_{L^p(\mathbb{R}^{n-1}, w)} w(B_{n-1}(0', 1))^{-\frac{1}{p}}.$$

Thus, for every  $(F_1, F_2) \in \mathcal{F}_N$  we have

$$(4.48) \quad \begin{aligned} \int_{\mathbb{R}^{n-1}} F_1(x')^p w(x') dx' &= I_N(h)^p w(B_{n-1}(0', 1)) \leq C \|h_N\|_{L^p(\mathbb{R}^{n-1}, w)}^p \\ &= C \int_{\mathbb{R}^{n-1}} F_2(x')^p w(x') dx', \end{aligned}$$

where  $C \in (0, \infty)$  is independent of  $N$ . Notice that the left-hand side of the previous estimate is finite thanks to (4.45). Granted this, we may invoke Theorem 4.3 to conclude that for  $\mathbb{X}$  as in the statement and every  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$  we have

$$(4.49) \quad \begin{aligned} I_N(h) \|\mathbf{1}_{B_{n-1}(0',1)}\|_{\mathbb{X}(w)} &= \|F_1\|_{\mathbb{X}(w)} \leq C \|F_2\|_{\mathbb{X}(w)} \\ &= C \|h_N\|_{\mathbb{X}(w)} \leq C \|h\|_{\mathbb{X}(w)}, \end{aligned}$$

with  $C \in (0, \infty)$  independent of  $N$ . Note that this estimate holds for every  $h \in \mathbb{M}$  since the left-hand side is always finite by (4.45). Consequently, (4.43) follows from (4.49) and Lebesgue's Monotone Convergence Theorem upon letting  $N \rightarrow \infty$ .  $\square$

We are finally ready to present the proof of Theorem 1.5.

*Proof of Theorem 1.5.* The idea is to invoke Theorem 1.1 with  $\mathbb{X} = \mathbb{Y} = \mathbb{X}(w)$ . Note that (4.44) takes care of the second embedding in (1.7) from which, as pointed out before, the first embedding in (1.7) also follows. The two conditions in (1.8) are verified upon noting that, by design,  $\mathbb{X}(w)$  is a function lattice, and by referencing Lemma 4.2. As such, Theorem 1.1 applies and yields existence and uniqueness for the Dirichlet problem (1.46) in the desired manner. To complete the proof of Theorem 1.5 there remains to observe that the bound in (1.47) is a direct consequence of (1.11) and Lemma 4.2.  $\square$

## 5. RETURN TO THE POISSON KERNEL

One aspect left open by Theorem 2.4 is the uniqueness of the Agmon-Douglis-Nirenberg Poisson kernel in the conceivably larger class of such kernels outlined by Definition 2.2. The goal here is to address this issue and also establish the semigroup property for this unique Poisson kernel.

**Theorem 5.1.** *Let  $L$  be a second-order elliptic system with complex coefficients as in (1.1)-(1.2). Then there exists a unique Poisson kernel  $P^L$  for  $L$  in  $\mathbb{R}_+^n$  in the sense of Definition 2.2. Moreover, this Poisson kernel satisfies the semigroup property*

$$(5.1) \quad P_{t_1}^L * P_{t_2}^L = P_{t_1+t_2}^L \quad \text{for every } t_1, t_2 > 0.$$

The convolution between the two matrix-valued functions in (5.1) is understood in a natural fashion, taking into account the algebraic multiplication of matrices. On this note, one significant consequence of identity (5.1) is the commutativity of the convolution product for the matrix-valued functions  $P_{t_1}^L$  and  $P_{t_2}^L$ , i.e.,  $P_{t_1}^L * P_{t_2}^L = P_{t_2}^L * P_{t_1}^L$  for each  $t_1, t_2 > 0$ . We shall further elaborate on this topic after discussing the proof of Theorem 5.1. Here we only wish to remark that, in the classical case  $L = \Delta$ , the semigroup property (5.1) is proved in [37, (vi) p. 62] making use of the explicit formula for the Fourier transform of  $P^\Delta$ . Instead, in the case of an arbitrary system  $L$  as in (1.1)-(1.2), our strategy is to rely on the well-posedness of the  $L^p$ -Dirichlet problem (1.39).

*Proof of Theorem 5.1.* Let  $P^L$  stand for the Agmon-Douglis-Nirenberg Poisson kernel for  $L$  from Theorem 2.4 and assume that  $Q^L$  is another Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  in the sense of Definition 2.2. Fix an arbitrary vector-valued function  $f \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$  and define for each  $(x', t) \in \mathbb{R}_+^n$

$$(5.2) \quad u_1(x', t) := (P_t^L * f)(x') \quad \text{and} \quad u_2(x', t) := (Q_t^L * f)(x').$$

Then Theorem 3.1 and (3.31) in Remark 3.4 imply that, for any given  $p \in (1, \infty)$ , both  $u_1$  and  $u_2$  solve the  $L^p$ -Dirichlet boundary value problem in  $\mathbb{R}_+^n$  as formulated in (1.39). The well-posedness of this boundary value problem

(cf. the discussion in Example 1 in §1) then forces  $u_1 = u_2$  in  $\mathbb{R}_+^n$  which further translates into  $(P_t^L * f)(x') = (Q_t^L * f)(x')$  for all  $(x', t) \in \mathbb{R}_+^n$  and all  $f \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ . In turn, this yields  $P^L = Q^L$  a.e. in  $\mathbb{R}^{n-1}$ , hence everywhere by the continuity of  $P^L$  and  $Q^L$  (see part (ii) in Remark 2.3). This finishes the proof of the first claim in the statement of theorem.

Consider now the semigroup property (5.1). To get started, fix  $t_2 > 0$  and pick an arbitrary vector-valued function  $f \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ . Let  $P^L$  be the unique Poisson kernel for  $L$  and, for each  $(x', t) \in \mathbb{R}_+^n$ , define this time

$$(5.3) \quad u_1(x', t) := (P_t^L * (P_{t_2}^L * f))(x') \quad \text{and} \quad u_2(x', t) := (P_{t+t_2}^L * f)(x').$$

Fix some  $p \in (1, \infty)$  and observe that  $P_{t_2}^L * f \in L^p(\mathbb{R}^{n-1})$  by (3.8). Finally, consider the  $L^p$ -Dirichlet boundary value problem

$$(5.4) \quad \begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \\ Lu = 0 \quad \text{in } \mathbb{R}_+^n, \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}), \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = P_{t_2}^L * f \in L^p(\mathbb{R}^{n-1}). \end{cases}$$

From the discussion in Example 1 in §1 we know that  $u_1$  is the unique solution of (5.4) and we claim that  $u_2$  also solves (5.4). Assuming this momentarily, it follows that  $u_1 = u_2$  in  $\mathbb{R}_+^n$ , hence  $((P_t^L * P_{t_2}^L) * f)(x') = (P_{t+t_2}^L * f)(x')$  for all  $x' \in \mathbb{R}^{n-1}$ , all  $t \in (0, \infty)$ , and each  $f \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ . Much as before, this readily implies  $P_t^L * P_{t_2}^L = P_{t+t_2}^L$  in  $\mathbb{R}^{n-1}$  for each  $t \in (0, \infty)$ , and (5.1) follows from this by taking  $t := t_1$ .

To finish the proof, there remains to check that, as claimed,  $u_2$  from (5.3) is a solution of (5.4). To this end, introduce  $v(x', t) := (P_t^L * f)(x')$  for  $(x', t) \in \mathbb{R}_+^n$  and note that, by Theorem 3.1,  $v$  satisfies

$$(5.5) \quad v \in \mathcal{C}^\infty(\mathbb{R}_+^n), \quad Lv = 0 \quad \text{in } \mathbb{R}_+^n, \quad \mathcal{N}v(x') \leq C \mathcal{M}f(x') \quad \text{for all } x' \in \mathbb{R}^{n-1}.$$

Since, by design,  $u_2(x', t) = v(x', t + t_2)$  for all  $(x', t) \in \mathbb{R}_+^n$ , we easily deduce from (5.5) that

$$(5.6) \quad u_2 \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n}), \quad Lu_2 = 0 \quad \text{in } \mathbb{R}_+^n, \quad \mathcal{N}u_2(x') \leq C \mathcal{M}f(x') \quad \text{for all } x' \in \mathbb{R}^{n-1}.$$

Hence,  $\mathcal{N}u_2 \in L^p(\mathbb{R}^{n-1})$  and since for each  $x' \in \mathbb{R}^{n-1}$  we have

$$(5.7) \quad (u_2|_{\partial \mathbb{R}_+^n}^{\text{n.t.}})(x') = u_2(x', 0) = v(x', t_2) = (P_{t_2}^L * f)(x'),$$

it follows that  $u_2$  solves (5.4). This finishes the proof of Theorem 5.1.  $\square$

Theorem 5.1 has several consequences of independent interest, and here we wish to single out the following result.

**Corollary 5.2.** *Let  $L$  be a homogeneous second-order elliptic system with (complex) constant coefficients, and let  $P^L$  denote its unique Poisson kernel in  $\mathbb{R}_+^n$  (cf. Theorem 5.1). Also, let  $\mathbb{X}$  be a Köthe function space with the property that  $\mathcal{M}$  is bounded on  $\mathbb{X}$ . Then the family  $\{T_t\}_{t>0}$ , where for each  $t \in (0, \infty)$ ,*

$$(5.8) \quad T_t : \mathbb{X} \rightarrow \mathbb{X}, \quad T_t f(x') := (P_t^L * f)(x') \quad \text{for every } f \in \mathbb{X}, x' \in \mathbb{R}^{n-1},$$

*is a semigroup of bounded linear operators on  $\mathbb{X}$  which satisfies*

$$(5.9) \quad \sup_{t>0} \|T_t\|_{\mathcal{L}(\mathbb{X})} < \infty,$$

*where  $\mathcal{L}(\mathbb{X})$  is the Banach space of linear and bounded operators on  $\mathbb{X}$ .*

*Furthermore, under the additional assumption that the function norm in  $\mathbb{X}$  is absolutely continuous, meaning that for any given  $f \in \mathbb{X}$  there holds*

$$(5.10) \quad \left. \begin{array}{l} (A_j)_{j \in \mathbb{N}} \text{ measurable subsets of } \mathbb{R}^{n-1} \\ \text{with } \mathbf{1}_{A_j} \rightarrow 0 \text{ a.e. in } \mathbb{R}^{n-1} \text{ as } j \rightarrow \infty \end{array} \right\} \implies \lim_{j \rightarrow \infty} \| |f| \cdot \mathbf{1}_{A_j} \|_{\mathbb{X}} = 0,$$

*it follows that  $\{T_t\}_{t>0}$  is a strongly continuous semigroup in the sense that*

$$(5.11) \quad \lim_{t \rightarrow 0^+} T_t f = f \quad \text{in } \mathbb{X}, \quad \text{for each } f \in \mathbb{X}.$$

*Proof.* From (1.12), (1.25), the assumptions on  $\mathbb{X}$ , and (3.7)-(3.8) in Lemma 3.3 it follows that for each  $t \in (0, \infty)$  the operator  $T_t : \mathbb{X} \rightarrow \mathbb{X}$  is well-defined, linear, and bounded. Moreover, there exists a finite constant  $C > 0$  with the property that for each  $x' \in \mathbb{R}^{n-1}$ ,

$$(5.12) \quad \sup_{t>0} |(T_t f)(x')| \leq C \mathcal{M}f(x'), \quad \forall f \in \mathbb{X} \subseteq L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right).$$

Bearing in mind the assumptions on  $\mathbb{X}$ , this readily gives (5.9). The semigroup property for the family  $\{T_t\}_{t>0}$  is then a consequence of (5.1), (1.12), and (3.7).

Concerning the strong continuity property of the semigroup  $\{T_t\}_{t>0}$ , fix an arbitrary  $f \in \mathbb{X}$  and note that, as a consequence of the last condition in (3.3), we have  $T_t f \rightarrow f$  a.e. in  $\mathbb{R}^{n-1}$  as  $t \rightarrow 0^+$ . In addition,  $|T_t f| \leq C \mathcal{M}f \in \mathbb{X}$  by (5.12) and the assumptions on  $\mathbb{X}$ . From these and Lebesgue's Dominated Convergence Theorem in  $\mathbb{X}$  (itself equivalent to the absolute continuity of the function norm in  $\mathbb{X}$ ; cf. [6, Proposition 3.6, p. 16]), it follows that (5.11) holds.  $\square$

For a thorough discussion pertaining to the absolute continuity of the function norm in a Köthe  $\mathbb{X}$  the interested reader is referred to [6, Chapter 1, §3]. We conclude by giving a list of examples of scales of spaces

satisfying all hypotheses in Corollary 5.2 (i.e., Köthe spaces with an absolutely continuous function norm on which the Hardy-Littlewood maximal operator is bounded):

- (i) Ordinary Lebesgue spaces  $L^p(\mathbb{R}^{n-1})$  with  $p \in (1, \infty)$ .
- (ii) Variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbb{R}^{n-1})$  on which the Hardy-Littlewood maximal operator is bounded (see [10, Theorem 2.62, p. 47] for the absolute continuity of the function norm in this setting).
- (iii) Lorentz spaces  $L^{p,q}(\mathbb{R}^{n-1})$  with  $1 < p, q < \infty$  (which in this range are reflexive, hence have absolutely continuous function norms by [6, Chapter 1, § 4]).
- (iv) Orlicz spaces  $L^\Phi(\mathbb{R}^{n-1})$ , where  $\Phi$  is a Young function.

For technical reasons, the weighted Lebesgue spaces  $L^p(\mathbb{R}^{n-1}, w(x') dx')$ , with  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^{n-1})$ , do not fall directly under the scope of Corollary 5.2 (since they fail to be Köthe spaces in the ordinary sense adopted in this paper, i.e., with respect to the background measure space  $(\mathbb{R}^{n-1}, dx')$ ). Nonetheless, the same type of conclusions as in Corollary 5.2 hold, and this is actually the case for a more general scale of weighted spaces. Specifically, consider

(5.13)

a rearrangement invariant space  $\mathbb{X}$  with lower Boyd index  $p_{\mathbb{X}} > 1$

and also fix some Muckenhoupt weight  $w \in A_{p_{\mathbb{X}}}(\mathbb{R}^{n-1})$ .

Finally, recall the weighted version  $\mathbb{X}(w)$  of  $\mathbb{X}$  defined in (4.39), and consider the condition that for every  $f \in \mathbb{X}(w)$  one has

(5.14)

$$\left. \begin{array}{l} (A_j)_{j \in \mathbb{N}} \text{ measurable subsets of } \mathbb{R}^{n-1} \\ \text{with } \mathbf{1}_{A_j} \rightarrow 0 \text{ a.e. in } \mathbb{R}^{n-1} \text{ as } j \rightarrow \infty \end{array} \right\} \implies \lim_{j \rightarrow \infty} \| |f| \cdot \mathbf{1}_{A_j} \|_{\mathbb{X}(w)} = 0.$$

Then a cursory inspection of the proof of [6, Proposition 3.6, p. 16] reveals that (5.14) implies Lebesgue's Dominated Convergence Theorem in  $\mathbb{X}(w)$ . Based on this, Lemma 3.3, and Lemma 4.2, the same type of reasoning as in the proof of Corollary 5.2 works and yields the following result.

**Corollary 5.3.** *Assuming (5.13) and that the system  $L$  is as in (1.1)-(1.2), the family  $\{T_t\}_{t>0}$ , where for each  $t \in (0, \infty)$ ,*

$$(5.15) \quad T_t : \mathbb{X}(w) \rightarrow \mathbb{X}(w)$$

and

$$(5.16) \quad T_t f(x') := (P_t^L * f)(x') \text{ for every } f \in \mathbb{X}(w), x' \in \mathbb{R}^{n-1},$$

is a semigroup of bounded linear operators on  $\mathbb{X}(w)$ , satisfying

$$(5.17) \quad \sup_{t>0} \|T_t\|_{\mathcal{L}(\mathbb{X}(w))} < \infty.$$

Moreover, under the additional assumption that (5.14) holds, this semigroup is strongly continuous.

Of course, Corollary 5.3 contains as particular cases the scale of weighted Lebesgue spaces  $L^p(\mathbb{R}^{n-1}, w)$  with  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^{n-1})$ , as well as the scales of weighted Lorentz spaces that are reflexive (cf. [6, Corollary 4.4, p. 23]) and weighted Orlicz spaces, discussed in the last part of §1.

## 6. A FATOU TYPE THEOREM

The goal in this section is to use the tools developed in §3 in order to prove the following Fatou type result.

**Theorem 6.1.** *Let  $L$  be a system as in (1.1)-(1.2) and let  $P^L$  be its Poisson kernel in  $\mathbb{R}_+^n$ . Assume that*

$$(6.1) \quad \begin{aligned} &u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \quad Lu = 0 \text{ in } \mathbb{R}_+^n, \quad \mathcal{N}u \in L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right), \\ &\text{and there exists a sequence } \{t_j\}_{j \in \mathbb{N}} \subset (0, \infty) \text{ satisfying } \lim_{j \rightarrow \infty} t_j = 0 \\ &\text{and such that } \mathcal{M}u(\cdot, t_j) \in L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right) \text{ for every } j \in \mathbb{N}. \end{aligned}$$

Then

$$(6.2) \quad \begin{aligned} &u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} \text{ exists a.e. in } \mathbb{R}^{n-1}, \\ &u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} \in L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right), \\ &u(x', t) = \left(P_t^L * (u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}})\right)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \end{aligned}$$

In particular, the conclusions in (6.2) hold whenever

$$(6.3) \quad u \in \mathcal{C}^\infty(\mathbb{R}_+^n), \quad Lu = 0 \text{ in } \mathbb{R}_+^n, \quad \mathcal{M}(\mathcal{N}u) \in L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right).$$

Prior to presenting the proof of Theorem 6.1, we isolate a useful weak compactness result. To state it, denote by  $\mathcal{C}_{\text{van}}(\mathbb{R}^{n-1})$  the space of continuous functions in  $\mathbb{R}^{n-1}$  vanishing at infinity.

**Lemma 6.2.** *Let  $v : \mathbb{R}^{n-1} \rightarrow (0, \infty)$  be a Lebesgue measurable function. Consider a sequence  $\{f_j\}_{j \in \mathbb{N}} \subset L^1(\mathbb{R}^{n-1}, v)$  such that  $F := \sup_{j \in \mathbb{N}} |f_j| \in L^1(\mathbb{R}^{n-1}, v)$ . Then there exists a subsequence  $\{f_{j_k}\}_{k \in \mathbb{N}}$  of  $\{f_j\}_{j \in \mathbb{N}}$  and a function  $f \in L^1(\mathbb{R}^{n-1}, v)$  with the property that*

$$(6.4) \quad \int_{\mathbb{R}^{n-1}} f_{j_k}(x') \varphi(x') v(x') dx' \longrightarrow \int_{\mathbb{R}^{n-1}} f(x') \varphi(x') v(x') dx' \text{ as } k \rightarrow \infty,$$



for every  $\varphi \in \mathcal{C}_{\text{van}}(\mathbb{R}^{n-1})$ .

*Proof.* Set  $\tilde{f}_j := f_j v$  for each  $j \in \mathbb{N}$ , and  $\tilde{F} := Fv$ . Then

$$(6.5) \quad |\tilde{f}_j| \leq \tilde{F} \in L^1(\mathbb{R}^{n-1}) \quad \text{for each } j \in \mathbb{N}.$$

Let  $\mathcal{M}$  be the space of finite Borel regular measures in  $\mathbb{R}^{n-1}$ , viewed as a Banach space when equipped with the norm induced by the total variation. Then

$$(6.6) \quad L^1(\mathbb{R}^{n-1}) \hookrightarrow \mathcal{M} = (\mathcal{C}_{\text{van}}(\mathbb{R}^{n-1}))^*.$$

From (6.5)-(6.6) and Alaoglu's theorem it follows that there exists a subsequence  $\{\tilde{f}_{j_k}\}_{k \in \mathbb{N}}$  and some  $\mu \in \mathcal{M}$  with the property that

$$(6.7) \quad \int_{\mathbb{R}^{n-1}} \tilde{f}_{j_k}(x') \varphi(x') dx' \longrightarrow \int_{\mathbb{R}^{n-1}} \varphi(x') d\mu(x') \quad \text{as } k \rightarrow \infty,$$

for every  $\varphi \in \mathcal{C}_{\text{van}}(\mathbb{R}^{n-1})$ . We claim that

$$(6.8) \quad \mu \ll \mathcal{L}^{n-1}.$$

To justify this claim, fix a Lebesgue measurable set  $E_0 \subset \mathbb{R}^{n-1}$  with  $\mathcal{L}^{n-1}(E_0) = 0$ . Given the goals we have in mind, there is no loss of generality in assuming that  $E_0$  is bounded. To proceed, pick an arbitrary  $\varepsilon > 0$ . Since  $\tilde{F}$  is a nonnegative function in  $L^1(\mathbb{R}^{n-1})$ , there exists  $\delta > 0$  such that

$$(6.9) \quad \int_U \tilde{F} d\mathcal{L}^{n-1} < \varepsilon, \quad \text{for each measurable set } U \subset \mathbb{R}^{n-1} \text{ with } \mathcal{L}^{n-1}(U) < \delta.$$

By the outer regularity of  $\mathcal{L}^{n-1}$ , there exists an open and bounded subset  $U_0$  of  $\mathbb{R}^{n-1}$  containing  $E_0$  and such that  $\mathcal{L}^{n-1}(U_0) < \delta$ . For any  $\varphi \in \mathcal{C}(\mathbb{R}^{n-1})$  supported in  $U_0$  we may then use (6.7) and (6.9) to estimate

$$(6.10) \quad \left| \int_{\mathbb{R}^{n-1}} \varphi d\mu \right| = \lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^{n-1}} \tilde{f}_{j_k} \varphi d\mathcal{L}^{n-1} \right| \\ \leq \|\varphi\|_{L^\infty(\mathbb{R}^{n-1})} \int_{U_0} \tilde{F} d\mathcal{L}^{n-1} \leq \|\varphi\|_{L^\infty(\mathbb{R}^{n-1})} \varepsilon.$$

In turn, this forces  $|\mu|(U_0) \leq \varepsilon$ , hence  $|\mu|(E_0) \leq \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, we conclude that  $|\mu|(E_0) = 0$  and (6.8) follows. Next, from (6.8) and the Radon-Nikodym Theorem we conclude that there exists  $\tilde{f} \in L^1(\mathbb{R}^{n-1})$  such that

$$(6.11) \quad d\mu = \tilde{f} d\mathcal{L}^{n-1}.$$

At this stage, (6.4) follows with  $f := \tilde{f}/v \in L^1(\mathbb{R}^{n-1}, v)$  based on (6.7) and (6.11).  $\square$

We are now ready to tackle the proof of Theorem 6.1.

*Proof of Theorem 6.1.* By assumption, the function  $u$  satisfies

$$(6.12) \quad \mathcal{N}u \in L^1(\mathbb{R}^{n-1}, v) \quad \text{where} \quad v(x') := \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}}, \quad \forall x' \in \mathbb{R}^{n-1}.$$

For each  $j \in \mathbb{N}$  consider the function  $u_j$  defined by  $u_j(x', t) := u(x', t + t_j)$  for each  $(x', t) \in \mathbb{R}_+^n$ . Observe that, for each  $j \in \mathbb{N}$ , the function  $u_j$  belongs to  $\mathcal{C}^\infty(\overline{\mathbb{R}_+^n})$ , thus

$$(6.13) \quad f_j := u_j|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = u_j|_{\partial \mathbb{R}_+^n} = u(\cdot, t_j)$$

exists and satisfies

$$(6.14) \quad |f_j| \leq \mathcal{N}u_j \leq \mathcal{N}u \quad \text{in } \mathbb{R}^{n-1}.$$

In particular, we conclude from (6.12)-(6.14) that

$$(6.15) \quad f_j, \mathcal{N}u_j \in L^1(\mathbb{R}^{n-1}, v) \quad \text{for each } j \in \mathbb{N}.$$

Keeping in mind (6.12)-(6.15), we may then invoke Lemma 6.2 (with  $F \leq \mathcal{N}u$ ) to conclude that there exists a subsequence  $\{f_{j_k}\}_{k \in \mathbb{N}}$  of  $\{f_j\}_{j \in \mathbb{N}}$  and a function  $f \in L^1(\mathbb{R}^{n-1}, v)$  with the property that

$$(6.16) \quad \int_{\mathbb{R}^{n-1}} f_{j_k}(x') \varphi(x') v(x') dx' \longrightarrow \int_{\mathbb{R}^{n-1}} f(x') \varphi(x') v(x') dx' \quad \text{as } k \rightarrow \infty,$$

for every  $\varphi \in \mathcal{C}_{\text{van}}(\mathbb{R}^{n-1})$ .

To proceed, let us observe that for each  $k \in \mathbb{N}$  the function  $f_{j_k}$  clearly satisfies (3.1), by (6.12) and (6.15). Next for each  $k \in \mathbb{N}$  define

$$(6.17) \quad U_k(x', t) := (P_t^L * f_{j_k})(x'), \quad \forall (x', t) \in \mathbb{R}_+^n.$$

Note that, thanks to Theorem 3.1, this entails

$$(6.18) \quad \begin{aligned} U_k &\in \mathcal{C}^\infty(\mathbb{R}_+^n), \quad LU_k = 0 \quad \text{in } \mathbb{R}_+^n, \\ U_k|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} &= f_{j_k} \quad \text{a.e. in } \mathbb{R}^{n-1}, \quad \mathcal{N}U_k \in L^1(\mathbb{R}^{n-1}, v), \end{aligned}$$

where the last condition is a consequence of (6.17), (3.4), (6.13), and the last line in (6.1). On the other hand, for each  $k \in \mathbb{N}$ , the function  $u_{j_k}$  satisfies the same quartet of conditions as  $U_k$  in (6.18) (where, this time, the condition  $\mathcal{N}u_{j_k} \in L^1(\mathbb{R}^{n-1}, v)$  is seen straight from (6.15)). As such, Theorem 3.2 applies to the difference  $u_{j_k} - U_k$  and yields  $u_{j_k} = U_k$  in  $\mathbb{R}_+^n$  for each  $k \in \mathbb{N}$ . Hence,

$$(6.19) \quad u(x', t + t_{j_k}) = u_{j_k}(x', t) = (P_t^L * f_{j_k})(x'), \quad \forall (x', t) \in \mathbb{R}_+^n, \quad \forall k \in \mathbb{N}.$$

Going further, fix  $(x', t) \in \mathbb{R}_+^n$  and consider the function

$$(6.20) \quad \varphi_{x', t}(y') := \frac{1}{v(y')} P_t^L(x' - y') = \frac{1 + |x'|^{n-1}}{1 + \log_+ |x'|} P_t^L(x' - y'), \quad \forall y' \in \mathbb{R}^{n-1},$$

and note that, from part (a) in Definition 2.2 and part (ii) in Remark 2.3, it follows that  $\varphi_{x',t} \in \mathcal{C}_{\text{van}}(\mathbb{R}^{n-1})$ . Granted this, by combining (6.19), (6.20), (6.16), and also bearing in mind that  $u$  is continuous in  $\mathbb{R}_+^n$  and  $\lim_{k \rightarrow \infty} t_{j_k} = 0$ , we conclude that for each  $x' \in \mathbb{R}^{n-1}$  and each  $t \in (0, \infty)$ ,

$$\begin{aligned}
 (6.21) \quad u(x', t) &= \lim_{k \rightarrow \infty} u(x', t + t_{j_k}) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^{n-1}} P_t^L(x' - y') f_{j_k}(y') dy' \\
 &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^{n-1}} \varphi_{x',t}(y') f_{j_k}(y') v(y') dy' \\
 &= \int_{\mathbb{R}^{n-1}} \varphi_{x',t}(y') f(y') v(y') dy' = (P_t^L * f)(x').
 \end{aligned}$$

Hence,  $u(x', t) = (P_t^L * f)(x')$  for each  $(x', t) \in \mathbb{R}_+^n$  for some  $f \in L^1(\mathbb{R}^{n-1}, v)$ . Having established this, Lemma 3.3 (with  $P = P^L$ ) yields that the non-tangential limit of  $u$  on  $\partial\mathbb{R}_+^n$  exists and equals  $f$ , proving the first conclusion in (6.2). The second conclusion in (6.2) is immediate from (6.21) and (3.9)-(3.10), keeping in mind that  $L^1(\mathbb{R}^{n-1}, v) \subseteq L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)$ . Finally, the third conclusion in (6.2) is implicit in (6.21).

There remains to show that (6.3) implies (6.1) (parenthetically, we note that  $\mathcal{M}$  acts in a meaningful way on  $\mathbb{M}$ , hence on the lower semicontinuous function  $\mathcal{N}u$ ). Indeed, the membership in (6.3) implies that  $\mathcal{M}(\mathcal{N}u) < \infty$  a.e. in  $\mathbb{R}^{n-1}$ , which further entails  $\mathcal{N}u \in L_{\text{loc}}^1(\mathbb{R}^{n-1})$ . From this and Lebesgue's Differentiation Theorem we then deduce that  $\mathcal{N}u \leq \mathcal{M}(\mathcal{N}u)$  a.e. in  $\mathbb{R}^{n-1}$  which, in light of the last condition in (6.3), ultimately yields the membership in the first line of (6.1). Moreover, the fact that  $|u(\cdot, t)| \leq \mathcal{N}u$  in  $\mathbb{R}^{n-1}$  for each  $t \in (0, \infty)$  implies  $\mathcal{M}u(\cdot, t) \leq \mathcal{M}(\mathcal{N}u)$  in  $\mathbb{R}^{n-1}$  for each  $t \in (0, \infty)$ , so the last condition in (6.1) also follows from (6.3).  $\square$

It is clear that the Fatou-type result from Theorem 6.1 (cf. (6.3), in particular) is valid in the class of null-solutions  $u$  of  $L$  for which  $\mathcal{N}u$  belongs to weighted Lebesgue spaces as in Example 2, variable exponent Lebesgue spaces as in Example 3, weighted Lorentz spaces as in Example 4, as well as weighted Orlicz spaces as in Example 5. Indeed, the discussion in §1 shows that the Fatou type result from Theorem 6.1 holds in the settings of Theorem 1.4 and Theorem 1.5. The case of ordinary Lebesgue spaces deserves special mention, and a precise statement, which also includes the end-point case  $p = 1$ , is presented below.

**Corollary 6.3.** *Assume the system  $L$  is as in (1.1)-(1.2). Then for each  $p \in [1, \infty)$ ,*

$$(6.22) \quad \left. \begin{array}{l} u \in \mathcal{C}^\infty(\mathbb{R}_+^n) \\ Lu = 0 \text{ in } \mathbb{R}_+^n \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}) \end{array} \right\} \implies \left\{ \begin{array}{l} u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} \text{ exists a.e. in } \mathbb{R}^{n-1}, \\ \text{belongs to } L^p(\mathbb{R}^{n-1}), \\ \text{and } u(x', t) = \left( P_t^L * (u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}) \right)(x'), \\ \text{for every } (x', t) \in \mathbb{R}_+^n, \end{array} \right.$$

where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$ .

*Proof.* For  $p \in (1, \infty)$ , the desired conclusion follows directly from the implication (6.3) $\implies$ (6.2) in Theorem 6.1, the boundedness of the Hardy-Littlewood maximal operator on  $L^p(\mathbb{R}^{n-1})$ , and the fact that  $L^p(\mathbb{R}^{n-1}) \subset L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right)$  by Hölder's inequality. There remains to treat the case  $p = 1$ , and this will follow from the implication (6.1) $\implies$ (6.2) in Theorem 6.1 as soon as we check that, under the current assumptions,  $\mathcal{M}u(\cdot, t) \in L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right)$  for every fixed  $t \in (0, \infty)$ . To this end, from interior estimates (cf. Theorem A.5) we first deduce that  $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C_{L,n,t} \|\mathcal{N}u\|_{L^1(\mathbb{R}^{n-1})}$ . Since we also have  $|u(\cdot, t)| \leq \mathcal{N}u \in L^1(\mathbb{R}^{n-1})$ , it ultimately follows that  $u(\cdot, t) \in L^\infty(\mathbb{R}^{n-1}) \cap L^1(\mathbb{R}^{n-1}) \subset L^2(\mathbb{R}^{n-1})$ . Hence, given that  $\mathcal{M}$  is bounded on  $L^2(\mathbb{R}^{n-1})$ , we conclude that  $\mathcal{M}u(\cdot, t) \in L^2(\mathbb{R}^{n-1}) \subset L^1\left(\mathbb{R}^{n-1}, \frac{1+\log_+|x'|}{1+|x'|^{n-1}} dx'\right)$ , as wanted.  $\square$

## APPENDIX A. AUXILIARY RESULTS

We begin by recording a suitable version of the divergence theorem recently obtained in [27]. To state it requires a few preliminaries which we dispense with first. As usual, let  $\mathcal{D}'(\mathbb{R}_+^n)$  denote the space of distributions in  $\mathbb{R}_+^n$  and write  $\mathcal{E}'(\mathbb{R}_+^n)$  for the space of distributions in  $\mathbb{R}_+^n$  that are compactly supported. Hence,

$$(A.1) \quad \mathcal{E}'(\mathbb{R}_+^n) \hookrightarrow \mathcal{D}'(\mathbb{R}_+^n) \quad \text{and} \quad L_{\text{loc}}^1(\mathbb{R}_+^n) \hookrightarrow \mathcal{D}'(\mathbb{R}_+^n).$$

For each compact set  $K \subset \mathbb{R}_+^n$ , define  $\mathcal{E}'_K(\mathbb{R}_+^n) := \{u \in \mathcal{E}'(\mathbb{R}_+^n) : \text{supp } u \subset K\}$  and consider

$$(A.2) \quad \mathcal{E}'_K(\mathbb{R}_+^n) + L^1(\mathbb{R}_+^n) := \{u \in \mathcal{D}'(\mathbb{R}_+^n) : \exists v_1 \in \mathcal{E}'_K(\mathbb{R}_+^n) \text{ and } \exists v_2 \in L^1(\mathbb{R}_+^n) \text{ such that } u = v_1 + v_2 \text{ in } \mathcal{D}'(\mathbb{R}_+^n)\}.$$

Also, introduce  $\mathcal{C}_b^\infty(\mathbb{R}_+^n) := \mathcal{C}^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$  and let  $(\mathcal{C}_b^\infty(\mathbb{R}_+^n))^*$  denote its algebraic dual. Moreover, we let  $(\mathcal{C}_b^\infty(\mathbb{R}_+^n))^* \langle \cdot, \cdot \rangle_{\mathcal{C}_b^\infty(\mathbb{R}_+^n)}$  denote the natural

duality pairing between these spaces. It is useful to observe that for every compact set  $K \subset \mathbb{R}_+^n$  one has

$$(A.3) \quad \mathcal{E}'_K(\mathbb{R}_+^n) + L^1(\mathbb{R}_+^n) \subset (\mathcal{C}_b^\infty(\mathbb{R}_+^n))^*.$$

**Theorem A.1** ([27]). *Assume that  $K \subset \mathbb{R}_+^n$  is a compact set and that  $\vec{F} \in L_{\text{loc}}^1(\mathbb{R}_+^n, \mathbb{C}^n)$  is a vector field satisfying the following conditions:*

- (a)  $\operatorname{div} \vec{F} \in \mathcal{E}'_K(\mathbb{R}_+^n) + L^1(\mathbb{R}_+^n)$ , where the divergence is taken in the sense of distributions;
- (b)  $\mathcal{N}_\kappa^{K^c} \vec{F} \in L^1(\mathbb{R}^{n-1})$ , where  $\kappa > 0$  and  $K^c := \mathbb{R}_+^n \setminus K$ ;
- (c) there exists  $\vec{F}|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$  a.e. in  $\mathbb{R}^{n-1}$ .

Then, with  $e_n := (0, \dots, 0, 1) \in \mathbb{R}^n$  and “dot” denoting the standard inner product in  $\mathbb{R}^n$ ,

$$(A.4) \quad (\mathcal{C}_b^\infty(\mathbb{R}_+^n))^* \langle \operatorname{div} \vec{F}, 1 \rangle_{\mathcal{C}_b^\infty(\mathbb{R}_+^n)} = - \int_{\mathbb{R}^{n-1}} e_n \cdot (\vec{F}|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}) d\mathcal{L}^{n-1}.$$

The theorem below summarizes properties of a distinguished fundamental solution for constant (complex) coefficient, homogeneous systems. A proof of the present formulation may be found in [26, Theorem 11.1, pp. 347-348] and [26, Theorem 7.54, pp. 270-271] (cf. also [29] and the references therein). Below,  $S^{n-1}$  is the unit sphere centered at the origin in  $\mathbb{R}^n$ ,  $\sigma$  is its canonical surface measure, and  $\omega_{n-1} := \sigma(S^{n-1})$  denotes its area.

**Theorem A.2.** *Fix  $n, m, M \in \mathbb{N}$  with  $n \geq 2$ , and consider an  $M \times M$  system of homogeneous differential operators of order  $2m$ ,*

$$(A.5) \quad \mathfrak{L} := \sum_{|\alpha|=2m} A_\alpha \partial^\alpha,$$

with matrix coefficients  $A_\alpha \in \mathbb{C}^{M \times M}$ . Assume that  $\mathfrak{L}$  satisfies the weak ellipticity condition

$$(A.6) \quad \det[\mathfrak{L}(\xi)] \neq 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

where

$$(A.7) \quad \mathfrak{L}(\xi) := \sum_{|\alpha|=2m} \xi^\alpha A_\alpha \in \mathbb{C}^{M \times M}, \quad \forall \xi \in \mathbb{R}^n.$$

Then the  $M \times M$  matrix  $E$  defined at each  $x \in \mathbb{R}^n \setminus \{0\}$  by

$$(A.8) \quad E(x) := \frac{1}{4(2\pi i)^{n-1}(2m-1)!} \Delta_x^{(n-1)/2} \int_{S^{n-1}} (x \cdot \xi)^{2m-1} \operatorname{sgn}(x \cdot \xi) [\mathfrak{L}(\xi)]^{-1} d\sigma(\xi)$$

if  $n$  is odd, and

$$(A.9) \quad E(x) := \frac{-1}{(2\pi i)^n (2m)!} \Delta_x^{n/2} \int_{S^{n-1}} (x \cdot \xi)^{2m} \ln |x \cdot \xi| [\mathfrak{L}(\xi)]^{-1} d\sigma(\xi)$$

if  $n$  is even, satisfies the following properties.

- (1) Each entry in  $E$  is a tempered distribution in  $\mathbb{R}^n$ , and a real-analytic function in  $\mathbb{R}^n \setminus \{0\}$  (hence, in particular, it belongs to  $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ ). Moreover,

$$(A.10) \quad E(-x) = E(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

- (2) If  $I_{M \times M}$  is the  $M \times M$  identity matrix, then for each  $y \in \mathbb{R}^n$

$$(A.11) \quad \mathfrak{L}_x[E(x-y)] = \delta_y(x) I_{M \times M}$$

in the sense of tempered distributions in  $\mathbb{R}^n$ , where the subscript  $x$  denotes the fact that the operator  $\mathfrak{L}$  in (A.11) is applied to each column of  $E$  in the variable  $x$ .

- (3) Define the  $M \times M$  matrix-valued function

$$(A.12) \quad \mathcal{P}(x) := \frac{-1}{(2\pi i)^n (2m-n)!} \int_{S^{n-1}} (x \cdot \xi)^{2m-n} [\mathfrak{L}(\xi)]^{-1} d\sigma(\xi), \quad \forall x \in \mathbb{R}^n.$$

Then the entries of  $\mathcal{P}$  are identically zero when either  $n$  is odd or  $n > 2m$ , and are homogeneous polynomials of degree  $2m-n$  when  $n \leq 2m$ . Moreover, there exists a  $\mathbb{C}^{M \times M}$ -valued function  $\Phi$ , with entries in  $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ , that is positive homogeneous of degree  $2m-n$  such that

$$(A.13) \quad E(x) = \Phi(x) + (\ln|x|)\mathcal{P}(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

- (4) For each  $\beta \in \mathbb{N}_0^n$  with  $|\beta| \geq 2m-1$ , the restriction to  $\mathbb{R}^n \setminus \{0\}$  of the matrix distribution  $\partial^\beta E$  is of class  $\mathcal{C}^\infty$  and positive homogeneous of degree  $2m-n-|\beta|$ .

- (5) For each  $\beta \in \mathbb{N}_0^n$  there exists  $C_\beta \in (0, \infty)$  such that the estimate

$$(A.14) \quad |\partial^\beta E(x)| \leq \begin{cases} \frac{C_\beta}{|x|^{n-2m+|\beta|}} & \text{if either } n \text{ is odd, or } n > 2m, \text{ or if } |\beta| > 2m-n, \\ \frac{C_\beta(1+|\ln|x||)}{|x|^{n-2m+|\beta|}} & \text{if } 0 \leq |\beta| \leq 2m-n, \end{cases}$$

holds for each  $x \in \mathbb{R}^n \setminus \{0\}$ .

- (6) When restricted to  $\mathbb{R}^n \setminus \{0\}$ , the entries of  $\widehat{E}$  (with “hat” denoting the Fourier transform) are  $\mathcal{C}^\infty$  functions and, moreover,

$$(A.15) \quad \widehat{E}(\xi) = (-1)^m [\mathfrak{L}(\xi)]^{-1} \quad \text{for each } \xi \in \mathbb{R}^n \setminus \{0\}.$$

(7) Writing  $E^{\mathfrak{L}}$  in place of  $E$  to emphasize the dependence on  $\mathfrak{L}$ , the fundamental solution  $E^{\mathfrak{L}}$  with entries as in (A.8)-(A.9) satisfies

$$(A.16) \quad \begin{aligned} (E^{\mathfrak{L}})^{\top} &= E^{\mathfrak{L}\top}, \quad \overline{E^{\mathfrak{L}}} = E^{\overline{\mathfrak{L}}}, \quad (E^{\mathfrak{L}})^* = E^{\mathfrak{L}*}, \\ \text{and } E^{\lambda\mathfrak{L}} &= \lambda^{-1} E^{\mathfrak{L}} \text{ for each } \lambda \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

where  $\mathfrak{L}^{\top}$ ,  $\overline{\mathfrak{L}}$ , and  $\mathfrak{L}^* = \overline{\mathfrak{L}^{\top}}$  denote the transposed, the complex conjugate, and the Hermitian adjoint of  $\mathfrak{L}$ , respectively.

(8) Any fundamental solution  $\mathbb{E}$  of the system  $\mathfrak{L}$  in  $\mathbb{R}^n$ , whose entries are tempered distributions in  $\mathbb{R}^n$ , is of the form  $\mathbb{E} = E + Q$  where  $E$  is as in (A.8)-(A.9) and  $Q$  is an  $M \times M$  matrix whose entries are polynomials in  $\mathbb{R}^n$  and whose columns,  $Q_k$ ,  $k \in \{1, \dots, M\}$ , satisfy the pointwise equations  $\mathfrak{L}Q_k = 0 \in \mathbb{C}^M$  in  $\mathbb{R}^n$  for each  $k \in \{1, \dots, M\}$ .

(9) In the particular case when  $M = 1$  and  $m = 1$ , i.e., in the situation when  $\mathfrak{L} = \operatorname{div} A \nabla$  for some matrix  $A = (a_{jk})_{1 \leq j, k \leq n} \in \mathbb{C}^{n \times n}$ , and when in place of (A.6) the strong ellipticity condition

$$(A.17) \quad \operatorname{Re} \left[ \sum_{j,k=1}^n a_{jk} \xi_j \xi_k \right] \geq C |\xi|^2, \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

is imposed, the fundamental solution  $E$  of  $\mathfrak{L}$  from (A.8)-(A.9) takes the explicit form

$$(A.18) \quad E(x) = \begin{cases} -\frac{1}{(n-2)\omega_{n-1}\sqrt{\det(A_{\text{sym}})}} \left[ ((A_{\text{sym}})^{-1}x) \cdot x \right]^{\frac{2-n}{2}} & \text{if } n \geq 3, \\ \frac{1}{4\pi\sqrt{\det(A_{\text{sym}})}} \log \left[ ((A_{\text{sym}})^{-1}x) \cdot x \right] & \text{if } n = 2. \end{cases}$$

Here,  $A_{\text{sym}} := \frac{1}{2}(A + A^{\top})$  stands for the symmetric part of the coefficient matrix  $A = (a_{rs})_{1 \leq r, s \leq n}$  and  $\log$  denotes the principal branch of the complex logarithm function (defined by the requirement that  $z^t = e^{t \log z}$  for all  $z \in \mathbb{C} \setminus (-\infty, 0]$  and all  $t \in \mathbb{R}$ ).

Before introducing the notion of Green function we discuss several pieces of notation. First,  $\operatorname{diag} := \{(x, x) : x \in \mathbb{R}_+^n\}$  denotes the diagonal in the Cartesian product  $\mathbb{R}_+^n \times \mathbb{R}_+^n$ . Second, given a function  $G(\cdot, \cdot)$  of two vector variables,  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \operatorname{diag}$ , for each  $k \in \{1, \dots, n\}$  we agree to write  $\partial_{X_k} G$  and  $\partial_{Y_k} G$ , respectively, for the partial derivative of  $G$  with respect to  $x_k$ , and  $y_k$  (the  $k$ -th components of  $x$  and  $y$ , respectively). This convention may be iterated, lending a natural meaning to  $\partial_X^\alpha \partial_Y^\beta G$ , for each pair of multi-indices  $\alpha, \beta \in \mathbb{N}_0^n$ . Also, we shall interpret  $\nabla_X G$ , and  $\nabla_Y G$ , as the gradients of  $G$  with respect to  $x$ , and  $y$ . Third, for each point  $y \in \mathbb{R}_+^n$  define

$$(A.19) \quad B_y := B(y, \tfrac{1}{2} \operatorname{dist}(y, \partial \mathbb{R}_+^n)) \quad \text{and, as usual, set } B_y^c := \mathbb{R}_+^n \setminus B_y.$$

Given a function  $u$  which is absolutely integrable over bounded subsets of  $\mathbb{R}_+^n$ , define (whenever meaningful) the Sobolev trace as

$$(A.20) \quad (\text{Tr } u)(x') := \lim_{r \rightarrow 0^+} \int_{B((x', 0), r) \cap \mathbb{R}_+^n} u \, d\mathcal{L}^n, \quad x' \in \partial\mathbb{R}^{n-1}.$$

For each  $p \in (1, \infty)$  let  $W^{1,p}(\mathbb{R}_+^n)$  be the classical  $L^p$ -based Sobolev space of order one in  $\mathbb{R}_+^n$ , and use the symbol  $\dot{W}^{1,p}(\mathbb{R}_+^n)$  for the closure of  $\mathcal{C}_0^\infty(\mathbb{R}_+^n)$  in  $W^{1,p}(\mathbb{R}_+^n)$ . Then for each function  $u \in W^{1,p}(\mathbb{R}_+^n)$ ,  $1 < p < \infty$ , the trace  $\text{Tr } u$  exists a.e. on  $\partial\mathbb{R}_+^n$  and belongs to  $B_{1-1/p}^{p,p}(\mathbb{R}^{n-1})$ , where for each  $p \in (1, \infty)$  and  $s \in (0, 1)$  the Besov space  $B_s^{p,p}(\mathbb{R}^{n-1})$  is defined as the collection of all measurable functions  $f$  in  $\mathbb{R}^{n-1}$  with the property that

$$(A.21) \quad \|f\|_{B_s^{p,p}(\mathbb{R}^{n-1})} := \|f\|_{L^p(\mathbb{R}^{n-1})} + \left( \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x') - f(y')|^p}{|x' - y'|^{n-1+sp}} dx' dy' \right)^{1/p} < \infty.$$

In fact, for each  $p \in (1, \infty)$  the operator

$$(A.22) \quad \text{Tr} : W^{1,p}(\mathbb{R}_+^n) \longrightarrow B_{1-1/p}^{p,p}(\mathbb{R}^{n-1})$$

is well-defined, linear and bounded, and has a linear and bounded right-inverse.

**Definition A.3.** Let  $L$  be a constant coefficient, second-order, elliptic differential operator as in (1.1). Call  $G(\cdot, \cdot) : \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag} \rightarrow \mathbb{C}^{M \times M}$  a **Green function** for  $L$  in  $\mathbb{R}_+^n$  provided for each  $y \in \mathbb{R}_+^n$  the following properties hold:

$$(A.23) \quad G(\cdot, y) \in L_{\text{loc}}^1(\mathbb{R}_+^n),$$

$$(A.24) \quad G(\cdot, y)|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = 0 \quad \text{a.e. in } \mathbb{R}^{n-1},$$

$$(A.25) \quad \mathcal{N}^{B_y^c} G(\cdot, y) \in \bigcup_{1 < p < \infty} L^p(\mathbb{R}^{n-1}),$$

$$(A.26) \quad L[G(\cdot, y)] = \delta_y I_{M \times M} \quad \text{in } \mathcal{D}'(\mathbb{R}_+^n),$$

where  $L$  acts in the “dot” variable on the columns of  $G$ .

We remark that, in the context of Definition A.3, we always have

$$(A.27) \quad G(\cdot, y) \in \mathcal{C}^\infty(\mathbb{R}_+^n \setminus \{y\}) \quad \text{for each } y \in \mathbb{R}_+^n,$$

by (A.23), (A.26), and elliptic regularity (cf. [26, Theorem 10.9, p.318]). Other basic properties of the Green function are collected in our next result.

**Theorem A.4** ([24]). Assume that  $L$  is a constant (complex) coefficient, second-order, elliptic differential operator as in (1.1). Then there exists a unique Green function  $G(\cdot, \cdot) = G^L(\cdot, \cdot)$  for  $L$  in  $\mathbb{R}_+^n$ , in the sense of



*Definition A.3.* Moreover, this Green function also satisfies the following additional properties:

- (1) Given  $\kappa > 0$ , for each  $y \in \mathbb{R}_+^n$  and each compact neighborhood  $K$  of  $y$  in  $\mathbb{R}_+^n$  there exists a finite constant  $C = C(n, L, \kappa, K, y) > 0$  such that for every  $x' \in \mathbb{R}^{n-1}$  there hold

$$(A.28) \quad \mathcal{N}_\kappa^{K^c}(G(\cdot, y))(x') \leq C \frac{1 + \log_+ |x'|}{1 + |x'|^{n-1}}$$

(if the fundamental solution  $E^L$  of  $L$  from Theorem A.2 is a radial function in  $\mathbb{R}^n \setminus \{0\}$ , then the logarithm in (A.28) may actually be omitted). Moreover, for any multi-indices  $\alpha, \beta \in \mathbb{N}_0^n$  such that  $|\alpha| + |\beta| > 0$ , there exists  $C = C(n, L, \kappa, \alpha, \beta, K, y) \in (0, \infty)$  such that

$$(A.29) \quad \mathcal{N}_\kappa^{K^c}((\partial_X^\alpha \partial_Y^\beta G)(\cdot, y))(x') \leq \frac{C}{1 + |x'|^{n-2+|\alpha|+|\beta|}}.$$

In particular,

$$(A.30) \quad \mathcal{N}_\kappa^{K^c}((\partial_X^\alpha \partial_Y^\beta G^L)(\cdot, y)) \in \bigcap_{1 < p \leq \infty} L^p(\mathbb{R}^{n-1}), \quad \forall \alpha, \beta \in \mathbb{N}_0^n.$$

- (2) For each fixed  $y \in \mathbb{R}_+^n$ , there holds

$$(A.31) \quad G^L(\cdot, y) \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n} \setminus B(y, \varepsilon)) \text{ for every } \varepsilon > 0.$$

- (3) The function  $G^L(\cdot, \cdot)$  is translation invariant in the tangential variables in the sense that

$$(A.32) \quad G^L(x - (z', 0), y - (z', 0)) = G^L(x, y) \text{ for every } (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag} \text{ and } z' \in \mathbb{R}^{n-1}.$$

- (4) With  $\text{Tr}$  denoting the Sobolev trace on  $\partial\mathbb{R}_+^n$  (cf. (A.20)-(A.22)), one has

$$(A.33) \quad G^L(\cdot, y) \in \bigcap_{k \in \mathbb{N}} \bigcap_{\frac{n}{n-1} < p < \infty} W^{k,p}(\mathbb{R}_+^n \setminus K) \text{ and } \text{Tr}[G^L(\cdot, y)] = 0, \\ \text{for every } y \in \mathbb{R}_+^n \text{ and any compact } K \subset \mathbb{R}_+^n \text{ with } y \in K^\circ.$$

- (5) If  $G^{L^\top}(\cdot, \cdot)$  denotes the (unique, by the first part of the statement) Green function for  $L^\top$  in  $\mathbb{R}_+^n$ , then

$$(A.34) \quad G^L(x, y) = \left[ G^{L^\top}(y, x) \right]^\top, \quad \forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag}.$$

Hence, as a consequence of (A.34), (A.24), and (A.31), for each fixed  $x \in \mathbb{R}_+^n$  and  $\varepsilon > 0$ ,

$$(A.35) \quad G^L(x, \cdot) \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n} \setminus B(x, \varepsilon)) \text{ and } G^L(x, \cdot) \Big|_{\partial\mathbb{R}_+^n} = 0 \text{ on } \mathbb{R}^{n-1}.$$

(6) If  $E^L$  denotes the fundamental solution of  $L$  from Theorem A.2, then the matrix-valued function

$$(A.36) \quad R_L(x, y) := E^L(x - y) - G^L(x, y), \quad \forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag},$$

extends to a function  $R_L(\cdot, \cdot) \in \mathcal{C}^\infty(\mathbb{R}_+^n \times \mathbb{R}_+^n)$  which satisfies the following estimate: for any multi-indices  $\alpha, \beta \in \mathbb{N}_0^n$  there exists a finite constant  $C_{\alpha\beta} > 0$  with the property that for every  $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ ,

$$(A.37) \quad |(\partial_X^\alpha \partial_Y^\beta R_L)(x, y)| \leq \begin{cases} C_{\alpha\beta} |x - \bar{y}|^{2-n-|\alpha|-|\beta|} & \text{if } |\alpha| + |\beta| > 0, \text{ or } n \geq 3, \\ C + C|\ln |x - \bar{y}|| & \text{if } |\alpha| = |\beta| = 0 \text{ and } n = 2, \end{cases}$$

where  $C \in (0, \infty)$ , and  $\bar{y} := (y', -y_n)$  if  $y = (y', y_n) \in \mathbb{R}_+^n$ .

(7) For any multi-indices  $\alpha, \beta \in \mathbb{N}_0^n$  there exists a finite constant  $C_{\alpha\beta} > 0$  such that

$$(A.38) \quad |(\partial_X^\alpha \partial_Y^\beta G^L)(x, y)| \leq C_{\alpha\beta} |x - y|^{2-n-|\alpha|-|\beta|},$$

$\forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag}, \text{ if either } n \geq 3, \text{ or } |\alpha| + |\beta| > 0,$

and, corresponding to  $|\alpha| = |\beta| = 0$  and  $n = 2$ , there exists  $C \in (0, \infty)$  such that

$$(A.39) \quad |G^L(x, y)| \leq C + C|\ln |x - \bar{y}||, \quad \forall (x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \setminus \text{diag}.$$

(8) For each  $\alpha, \beta \in \mathbb{N}_0^n$  one has

$$(A.40) \quad \sup_{y \in \mathbb{R}_+^n} \|(\partial_X^\alpha \partial_Y^\beta G^L)(\cdot, y)\|_{L^{\frac{n}{n-2+|\alpha|+|\beta|}, \infty}(\mathbb{R}_+^n)} < +\infty,$$

if either  $n \geq 3$ , or  $|\alpha| + |\beta| > 0$ .

In particular,

$$(A.41) \quad G^L(\cdot, y) \in L^{\frac{n}{n-2}, \infty}(\mathbb{R}_+^n), \quad \text{uniformly in } y \in \mathbb{R}_+^n, \quad \text{if } n \geq 3,$$

$$(A.42) \quad \nabla_X G^L(\cdot, y), \nabla_Y G^L(\cdot, y) \in L^{\frac{n}{n-1}, \infty}(\mathbb{R}_+^n), \quad \text{uniformly in } y \in \mathbb{R}_+^n,$$

and

$$(A.43) \quad \nabla_X^2 G^L(\cdot, y), \nabla_X \nabla_Y G^L(\cdot, y), \quad \text{and} \quad \nabla_Y^2 G^L(\cdot, y)$$

belong to  $L^{1, \infty}(\mathbb{R}_+^n)$ , uniformly in  $y \in \mathbb{R}_+^n$ .

(9) If  $p \in [1, \frac{n}{n-1})$ , then for each  $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  one has

$$(A.44) \quad \zeta G^L(\cdot, y) \in \dot{W}^{1, p}(\mathbb{R}_+^n) \quad \text{for each } y \in \mathbb{R}_+^n$$

and  $\sup_{y \in \mathbb{R}_+^n} \|\zeta G^L(\cdot, y)\|_{W^{1, p}(\mathbb{R}_+^n)} < \infty.$

(10) If the fundamental solution  $E^L$  for  $L$  from Theorem A.2 is a radial function in  $\mathbb{R}^n \setminus \{0\}$ , then (with  $\bar{y} \in \mathbb{R}_-^n$  denoting the reflection of  $y \in \mathbb{R}_+^n$  across  $\partial\mathbb{R}_+^n$ )

$$(A.45) \quad G^L(x, y) = E^L(x - y) - E^L(x - \bar{y}), \quad \forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \text{diag}.$$

(11) If  $n \geq 3$ , then for every  $x = (x', t) \in \mathbb{R}_+^n$  and every  $y \in \mathbb{R}_+^n \setminus \{x\}$  one has (with  $P^L$  denoting the Agmon-Douglis-Nirenberg Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.4)

$$(A.46) \quad G^L(x, y) = E^L(x - y) - P_t^L * \left( [E^L(\cdot - y)]|_{\partial\mathbb{R}_+^n} \right)(x'),$$

with the convolution applied to each column of the matrix inside the round parentheses.

(12) The Agmon-Douglis-Nirenberg Poisson kernel  $P^L = (P_{\gamma\alpha}^L)_{1 \leq \gamma, \alpha \leq M}$  for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.4 is related to the Green function  $G^L$  for  $L$  in  $\mathbb{R}_+^n$  according to the formula

$$(A.47) \quad P_{\gamma\alpha}^L(z') = -a_{nn}^{\beta\alpha}(\partial_{Y_n} G_{\gamma\beta}^L)((z', 1), 0), \quad \forall z' \in \mathbb{R}^{n-1},$$

for each  $\alpha, \gamma \in \{1, \dots, M\}$ .

In particular, formulas (A.47) and (A.45) imply that whenever the fundamental solution  $E^L = (E_{\gamma\beta}^L)_{1 \leq \gamma, \beta \leq M}$  of  $L$  from Theorem A.2 is a radial function then for each  $\alpha, \gamma \in \{1, \dots, M\}$  one has

$$(A.48) \quad P_{\gamma\alpha}^L(z') = 2a_{nn}^{\beta\alpha}(\partial_n E_{\gamma\beta}^L)(z', 1), \quad \forall z' \in \mathbb{R}^{n-1}.$$

We shall now record the following versatile version of interior estimates for higher-order elliptic systems. A proof may be found in [26, Theorem 11.9, p. 364].

**Theorem A.5.** Consider a homogeneous, constant coefficient, higher-order system  $\mathfrak{L}$  as in (A.5), satisfying the weak ellipticity condition (A.6). Then for each null-solution  $u$  of  $\mathfrak{L}$  in a ball  $B(x, R)$  (where  $x \in \mathbb{R}^n$  and  $R > 0$ ),  $0 < p < \infty$ ,  $\lambda \in (0, 1)$ ,  $\ell \in \mathbb{N}_0$ , and  $0 < r < R$ , one has

$$(A.49) \quad \sup_{z \in B(x, \lambda r)} |\nabla^\ell u(z)| \leq \frac{C}{r^\ell} \left( \int_{B(x, r)} |u|^p d\mathcal{L}^n \right)^{1/p},$$

where  $C = C(L, p, \ell, \lambda, n) > 0$  is a finite constant.

Finally, we discuss the dependence of the size of the nontangential maximal function, corresponding to various apertures, in weighted  $L^p$  spaces.

**Proposition A.6.** For every  $\kappa, \kappa' > 0$ ,  $p \in (0, \infty)$  and  $w \in A_\infty(\mathbb{R}^{n-1})$ , there exist finite constants  $C_0, C_1 > 0$  such that

$$(A.50) \quad C_0 \|\mathcal{N}_\kappa u\|_{L^p(\mathbb{R}^{n-1}, w)} \leq \|\mathcal{N}_{\kappa'} u\|_{L^p(\mathbb{R}^{n-1}, w)} \leq C_1 \|\mathcal{N}_\kappa u\|_{L^p(\mathbb{R}^{n-1}, w)},$$

for each function  $u : \mathbb{R}_+^n \rightarrow \mathbb{C}$ .

*Proof.* As in the unweighted case, the proof of this result is based on a point-of-density argument. Fix  $\lambda > 0$  and for every  $\kappa > 0$  write

$$(A.51) \quad O_\kappa = \{x' \in \mathbb{R}^{n-1} : (\mathcal{N}_\kappa u)(x') > \lambda\}.$$

It is easy to show that  $O_\kappa$  is open. Pick  $0 < \gamma < 1$  so that  $1 - (\kappa/(\kappa + \kappa'))^{n-1} < \gamma < 1$  and write  $A_\kappa := \mathbb{R}^{n-1} \setminus O_\kappa$ . Also, for every  $\gamma \in (0, 1)$  introduce

$$(A.52) \quad A_\kappa^\gamma := \left\{x' \in \mathbb{R}^{n-1} : \frac{|A_\kappa \cap B_{n-1}(x', r)|}{|B_{n-1}(x', r)|} \geq \gamma \text{ for each } r > 0\right\}.$$

We are going to show that  $O_{\kappa'} \subset \mathbb{R}^{n-1} \setminus A_\kappa^\gamma$ . Given  $x' \in O_{\kappa'}$ , we can take  $(y', t) \in \Gamma_{\kappa'}(x')$  such that  $|u(y', t)| > \lambda$ . Note that  $B_{n-1}(y', \kappa t) \subset B_{n-1}(x', (\kappa + \kappa')t)$ . On the other hand, we have that  $B_{n-1}(y', \kappa t) \subset O_\kappa$ : if  $z' \in B_{n-1}(y', \kappa t)$  then  $(y', t) \in \Gamma_\kappa(z')$  and, therefore,  $\mathcal{N}_\kappa u(z') \geq |u(y', t)| > \lambda$ . All these show that  $B_{n-1}(y', \kappa t) \subset O_\kappa \cap B_{n-1}(x', (\kappa + \kappa')t)$ . This implies that

$$(A.53) \quad \frac{|B_{n-1}(x', (\kappa + \kappa')t) \cap A_\kappa|}{|B_{n-1}(x', (\kappa + \kappa')t)|} = 1 - \frac{|B_{n-1}(x', (\kappa + \kappa')t) \cap O_\kappa|}{|B_{n-1}(x', (\kappa + \kappa')t)|} \\ \leq 1 - \frac{|B_{n-1}(y', \kappa t)|}{|B_{n-1}(x', (\kappa + \kappa')t)|} = 1 - \left(\frac{\kappa}{\kappa + \kappa'}\right)^{n-1} < \gamma,$$

which forces  $x' \notin A_\kappa^\gamma$  in light of (A.52). In turn, this shows that

$$(A.54) \quad O_{\kappa'} \subset \mathbb{R}^{n-1} \setminus A_\kappa^\gamma \subset \{x' \in \mathbb{R}^{n-1} : \mathcal{M}(\mathbf{1}_{O_\kappa})(x') \geq c_n(1 - \gamma)\},$$

for some dimensional constant  $c_n \in (0, \infty)$  (whose appearance is due to the fact that the Hardy-Littlewood maximal operator has been defined in (2.9) using cubes rather than balls). Since  $w \in A_\infty(\mathbb{R}^{n-1})$ , we can take  $q \in (1, \infty)$  such that  $w \in A_q(\mathbb{R}^{n-1})$ . Thus,  $\mathcal{M}$  is bounded on  $L^q(\mathbb{R}^{n-1}, w)$  and, consequently,

$$(A.55) \quad w(O_{\kappa'}) \leq w\left(\{x' \in \mathbb{R}^{n-1} : \mathcal{M}(\mathbf{1}_{O_\kappa})(x') \geq c_n(1 - \gamma)\}\right) \\ \leq [c_n(1 - \gamma)]^{-q} \|\mathcal{M}(\mathbf{1}_{O_\kappa})\|_{L^q(\mathbb{R}^{n-1}, w)}^q \leq C w(O_\kappa),$$

where  $C \in (0, \infty)$  depends only on  $n, \kappa, \kappa', q, w$ . The level set estimate just derived readily yields (A.50).  $\square$

It follows from Proposition A.6 and (2.6) that, for every  $\kappa, \kappa' > 0$  and  $p \in (0, \infty)$ , there exist finite constants  $C_0, C_1 > 0$  such that

$$(A.56) \quad C_0 \|\mathcal{N}_\kappa^E u\|_{L^p(\mathbb{R}^{n-1})} \leq \|\mathcal{N}_{\kappa'}^E u\|_{L^p(\mathbb{R}^{n-1})} \leq C_1 \|\mathcal{N}_\kappa^E u\|_{L^p(\mathbb{R}^{n-1})},$$

$$(A.57) \quad C_0 \|\mathcal{N}_\kappa^{(\varepsilon)} u\|_{L^p(\mathbb{R}^{n-1})} \leq \|\mathcal{N}_{\kappa'}^{(\varepsilon)} u\|_{L^p(\mathbb{R}^{n-1})} \leq C_1 \|\mathcal{N}_\kappa^{(\varepsilon)} u\|_{L^p(\mathbb{R}^{n-1})},$$

for each function  $u$ , set  $E \subset \mathbb{R}^n$ , and number  $\varepsilon > 0$ .

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