THE GENERALIZED HÖLDER AND MORREY-CAMPANATO DIRICHLET PROBLEMS FOR ELLIPTIC SYSTEMS IN THE UPPER-HALF SPACE

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ABSTRACT. We prove well-posedness results for the Dirichlet problem in $\mathbb{R}^n_+$ for homogeneous, second order, constant complex coefficient elliptic systems with boundary data in generalized Hölder spaces $\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and in generalized Morrey-Campanato spaces $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ under certain assumptions on the growth function ω. We also identify a class of growth functions ω for which $\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and for which the aforementioned well-posedness results are equivalent, in the sense that they have the same unique solution, satisfying natural regularity properties and estimates.

CONTENTS

1. Introduction

This paper is devoted to studying the Dirichlet problem for elliptic systems in the upper-half space with data in generalized Hölder spaces and generalized Morrey-Campanato spaces. As a byproduct of the PDE-based techniques developed here, we are able to establish the equivalence of these function spaces. To be more specific requires introducing some notation.

Having fixed $n, M \in \mathbb{N}$ with $n \geq 2$ and $M \geq 1$, consider a homogeneous, constant (complex) coefficient, $M \times M$ second-order system in $\mathbb{R}^n$

$$L := \left(a_{jk}^{\alpha\beta} \partial_j \partial_k \right)_{1 \leq \alpha, \beta \leq M}. \tag{1.1}$$

Here and elsewhere, the summation convention over repeated indices is employed. We make the standing assumption that L is strongly elliptic, in the sense that there exists $\kappa_0 > 0$ such that the following Legendre-Hadamard condition is satisfied:

$$\mathrm{Re} \left[a_{jk}^{\alpha\beta} \xi_j \xi_k \overline{\zeta_\alpha} \zeta_\beta \right] \geq \kappa_0 |\xi|^2 |\zeta|^2 \quad \text{for all}$$

$$\xi = (\xi_j)_{1 \leq j \leq n} \in \mathbb{R}^n \quad \text{and} \quad \zeta = (\zeta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \tag{1.2}$$

Examples include scalar operators, such as the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ or, more generally, operators of the form $\mathrm{div} A \nabla$ with $A = (a_{rs})_{1 \leq r, s \leq n}$ an $n \times n$ matrix with complex entries satisfying the ellipticity condition

$$\inf_{\xi \in S^{n-1}} \mathrm{Re} \left[a_{rs} \xi_r \xi_s \right] > 0, \tag{1.3}$$

(where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$), as well as the complex version of the Lamé system of elasticity in $\mathbb{R}^n$,

$$L := \mu \Delta + (\lambda + \mu) \nabla \mathrm{div}. \tag{1.4}$$

Above, the constants $\lambda, \mu \in \mathbb{C}$ (typically called Lamé moduli), are assumed to satisfy

$$\mathrm{Re} \, \mu > 0 \quad \text{and} \quad \mathrm{Re} \, (2\mu + \lambda) > 0, \tag{1.5}$$

a condition equivalent to the demand that the Lamé system (1.4) satisfies the Legendre-Hadamard ellipticity condition (1.2). While the Lamé system is symmetric, we stress that the results in this paper require no symmetry for the systems involved.

With each system L as in (1.1)-(1.2) one may associate a Poisson kernel, P^L, which is a $\mathbb{C}^{M \times M}$-valued function defined in $\mathbb{R}^{n-1}$ described in detail in Theorem 3.3. This Poisson kernel has played a pivotal role in the treatment of the Dirichlet problem with data in L^p, BMO, VMO and Hölder spaces (see [7, 6]). For now, we make the observation that the Poisson kernel gives rise to a nice approximation to the identity in $\mathbb{R}^{n-1}$ by setting $P_t^L(x') = t^{1-n} P^L(x'/t)$ for every $x' \in \mathbb{R}^{n-1}$ and $t > 0$.

For every point $x \in \mathbb{R}^n$ write $x = (x', t)$, where $x' \in \mathbb{R}^{n-1}$ corresponds to the first $n - 1$ coordinates of x, and $t \in \mathbb{R}$ is the last coordinate of x. As is customary, we shall let $\mathbb{R}_+^n := \{x = (x', t) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, t > 0\}$ denote the upper-half space in $\mathbb{R}^n$, and typically identify its boundary with $(n-1)$-dimensional Euclidean space, via

$\partial\mathbb{R}^n_+ \ni (x',0) \equiv x' \in \mathbb{R}^{n-1}$. The cone with vertex at $x' \in \mathbb{R}^{n-1}$ and aperture $\kappa > 0$ is defined as

$$\Gamma_\kappa(x') := \{y = (y',t) \in \mathbb{R}^n_+ : |x' - y'| < \kappa t\}. \tag{1.6}$$

When $\kappa = 1$ we agree to drop the dependence on aperture and simply write $\Gamma(x')$. Whenever meaningful, the nontangential pointwise trace of a vector-valued function u defined in $\mathbb{R}^n_+$ is given by

$$\left(u\big|^{\text{nt.lim}}_{\partial\mathbb{R}^n_+}\right)(x') := \lim_{\substack{\mathbb{R}^n_+ \ni y \to (x',0) \\ y \in \Gamma_\kappa(x')}} u(y), \quad x' \in \mathbb{R}^{n-1}. \tag{1.7}$$

The unrestricted pointwise trace of a vector-valued function u defined in $\mathbb{R}^n_+$ at each $x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ is taken to be

$$\left(u\big|^{\text{lim}}_{\partial\mathbb{R}^n_+}\right)(x') := \lim_{\mathbb{R}^n_+ \ni y \to (x',0)} u(y), \quad x' \in \mathbb{R}^{n-1}, \tag{1.8}$$

whenever such a limit exists exists.

Definition 1.1. Call a given mapping $\omega : (0,\infty) \to (0,\infty)$ a `growth function` if ω is non-decreasing and $\omega(t) \to 0$ as $t \to 0^+$.

Definition 1.2. Let $E \subset \mathbb{R}^n$ be an arbitrary set (implicitly assumed to have cardinality at least 2) and let ω be a growth function. The `homogeneous` ω`-Hölder space` on E is defined as

$$\dot{\mathscr{C}}^\omega(E, \mathbb{C}^M) := \{u : E \to \mathbb{C}^M : [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} < \infty\}, \tag{1.9}$$

where $[\cdot]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)}$ stands for the seminorm

$$[u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} := \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|u(x) - u(y)|}{\omega(|x - y|)}. \tag{1.10}$$

Let us note that the fact that $\omega(t) \to 0$ as $t \to 0^+$ implies that if $u \in \dot{\mathscr{C}}^\omega(E, \mathbb{C}^M)$ then u is uniformly continuous. The choice $\omega(t) := t^\alpha$ for each $t > 0$, with $\alpha \in (0,1)$, yields the classical scale of Hölder spaces.

Here and elsewhere in the paper, we agree to denote the $(n-1)$-dimensional Lebesgue measure of given Lebesgue measurable set $E \subseteq \mathbb{R}^{n-1}$ by $|E|$. Also, by a cube Q in $\mathbb{R}^{n-1}$ we shall understand a cube with sides parallel to the coordinate axes. Its side-length will be denoted by $\ell(Q)$, and for each $\lambda > 0$ we shall denote by λQ the cube concentric with Q whose side-length is $\lambda\,\ell(Q)$. For every function $h \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ we write

$$h_Q := \fint_Q h(x')\,dx' := \frac{1}{|Q|} \int_Q h(x')\,dx' \in \mathbb{C}^M, \tag{1.11}$$

with the integration performed componentwise.

Definition 1.3. Given a growth function ω along with some integrability exponent $p \in [1,\infty)$, the associated `generalized Morrey-Campanato space` in $\mathbb{R}^{n-1}$ is defined as

$$\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) := \{f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M) : \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} < \infty\}, \tag{1.12}$$

where $\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}$ stands for the seminorm

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \sup_{Q\subset\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left(\fint_Q |f(x') - f_Q|^p \, dx' \right)^{1/p}. \tag{1.13}$$

The choice $\omega(t) := t^\alpha$ with $\alpha \in (0,1)$ corresponds to the classical Morrey-Campanato spaces, while the special case $\omega(t) := 1$ yields the usual space of functions of bounded mean oscillations (BMO). We also define, for every $u \in \mathscr{C}^1(\mathbb{R}^n_+, \mathbb{C}^M)$ and $q \in (0,\infty)$,

$$\|u\|^{(\omega,q)}_{**} := \sup_{Q\subset\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left(\fint_Q \left(\int_0^{\ell(Q)} |(\nabla u)(x',t)|^2 \, t \, dt \right)^{q/2} dx' \right)^{1/q}. \tag{1.14}$$

As far as this seminorm is concerned, there are two reasonable candidates for the endpoint $q = \infty$ (see Proposition 3.4 and Lemma A.1). First, we may consider

$$\|u\|^{(\omega,\exp)}_{**} := \sup_{Q\subset\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left\| \left(\int_0^{\ell(Q)} |(\nabla u)(\cdot,t)|^2 \, t \, dt \right)^{1/2} \right\|_{\exp L, Q} \tag{1.15}$$

where $\|\cdot\|_{\exp L, Q}$ is the version of the norm in the Orlicz space $\exp L$ localized and normalized relative to Q, i.e.,

$$\|f\|_{\exp L, Q} := \inf \left\{ t > 0 : \fint_Q \left(e^{\frac{|f(x')|}{t}} - 1 \right) dx' \leq 1 \right\}. \tag{1.16}$$

Second, corresponding to the limiting case $q = \infty$ we may consider

$$\|u\|^{(\omega,\infty)}_{**} := \sup_{(x',t)\in\mathbb{R}^n_+} \frac{t}{\omega(t)} |(\nabla u)(x',t)|. \tag{1.17}$$

We are ready to describe our main result concerning the Dirichlet problems with data in generalized Hölder and generalized Morrey-Campanato spaces for homogeneous second-order strongly elliptic systems of differential operators with constant complex coefficients (cf. (1.1) and (1.2)). In Section 6 (cf. Theorems 6.1-6.2), we weaken the condition (1.18) and still prove well-posedness for the two Dirichlet problems. The main difference is that in that case they are no longer equivalent as (1.24) might fail (see Example 6.4).

Theorem 1.4. *Consider a strongly elliptic constant complex coefficient second-order $M \times M$ system L, as in (1.1)-(1.2). Also, fix $p \in [1,\infty)$ along with $q \in (0,\infty]$, and let ω be a growth function satisfying, for some finite constant $C_0 \geq 1$,*

$$\int_0^t \omega(s) \frac{ds}{s} + t \int_t^\infty \frac{\omega(s)}{s} \frac{ds}{s} \leq C_0 \, \omega(t) \ \ for \ each \ \ t \in (0,\infty). \tag{1.18}$$

Then the following statements are true.

(a) *The generalized Hölder Dirichlet problem for the system L in $\mathbb{R}^n_+$, i.e.,*

$$\begin{cases} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \quad in \ \ \mathbb{R}^n_+, \\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} < \infty, \\ u\big|^{\mathrm{lim}}_{\partial\mathbb{R}^n_+} = f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad on \ \ \mathbb{R}^{n-1}, \end{cases} \tag{1.19}$$

is well-posed. More specifically, there is a unique solution which is given by

$$u(x', t) = (P^L_t * f)(x'), \qquad \forall\, (x', t) \in \mathbb{R}^n_+, \tag{1.20}$$

where P^L denotes the Poisson kernel for L in $\mathbb{R}^n_+$ from Theorem 3.3. In addition, u belongs to the space $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, satisfies $u\big|_{\partial\mathbb{R}^n_+} = f$, and there exists a finite constant $C = C(n, L, \omega) \geq 1$ such that

$$C^{-1}[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{1.21}$$

(b) *The generalized Morrey-Campanato Dirichlet problem for L in $\mathbb{R}^n_+$, formulated as*

$$\begin{cases} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \quad in \ \ \mathbb{R}^n_+, \\ \|u\|^{(\omega, q)}_{**} < \infty, \\ u\big|^{\mathrm{nt.lim}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{E}^{\omega, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad a.e. \ on \ \ \mathbb{R}^{n-1}, \end{cases} \tag{1.22}$$

is well-posed. More precisely, there is a unique solution (1.22) which is given by (1.20). In addition, u belongs to $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, satisfies $u\big|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$, and there exists a finite constant $C = C(n, L, \omega, p, q) \geq 1$ such that

$$C^{-1}\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \|u\|^{(\omega, q)}_{**} \leq C\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{1.23}$$

Furthermore, all these properties remain true if $\|\cdot\|^{(\omega, q)}_{**}$ is replaced everywhere by $\|\cdot\|^{(\omega, \exp)}_{**}$.

(c) *The following equality between vector spaces holds*

$$\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathscr{E}^{\omega, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \tag{1.24}$$

with equivalent norms, where the right-to-left inclusion is understood in the sense that for each $f \in \mathscr{E}^{\omega, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ there exists a unique $\widetilde{f} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ with the property that $f = \widetilde{f}$ a.e. in $\mathbb{R}^{n-1}$.

As a result, the Dirichlet problems (1.19) and (1.22) are equivalent. Specifically, for any pair of boundary data which may be identified in the sense of (1.24) these problems have the same unique solution (given by (1.20)).

A few comments regarding the previous result. In Lemma 2.1 we shall prove that, for growth functions as in (1.18), each $u \in \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ extends uniquely to a function $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$. Hence, the ordinary restriction $u\big|_{\partial\mathbb{R}^n_+}$ is well-defined in the context of item (a) of Theorem 1.4. In item (b) the situation is slightly different. One can

first show that u extends to a continuous function up to, and including, the boundary. Hence, the non-tangential pointwise trace agrees with the restriction to the boundary everywhere. However, since functions in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ are canonically identified whenever they agree outside of a set of zero Lebesgue measure, the boundary condition in (1.22) is most naturally formulated by asking that the non-tangential boundary trace agrees with the boundary datum almost everywhere. The same type of issue arises when interpreting (1.24). Specifically, while the left-to-right inclusion has a clear meaning, the converse inclusion should be interpreted as saying that each equivalence class in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ (induced by the aforementioned identification) has a unique representative from $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$. We would like to observe that (1.24) extends the well-known result of N.G. Meyers [9] who considered the case $\omega(t) = t^\alpha$, $t > 0$. Here we extend the class of growth functions for which (1.24) holds and our alternative approach is based on PDE.

It is illustrative to provide examples of growth functions to which Theorem 1.4 applies. In this vein, we first observe that (1.18) is closely related to the dilation indices of Orlicz spaces studied in [1, 2] in relation to interpolation in Orlicz spaces. Concretely, given a growth function ω set

$$h_\omega(t) := \sup_{s>0} \frac{\omega(st)}{\omega(s)}, \qquad \forall\, t > 0, \tag{1.25}$$

and define the lower and upper dilation indices, respectively, as

$$i_\omega := \sup_{0<t<1} \frac{\log h_\omega(t)}{\log t} \quad \text{and} \quad I_\omega := \inf_{t>1} \frac{\log h_\omega(t)}{\log t}. \tag{1.26}$$

One can see that if $0 < i_\omega \leq I_\omega < 1$ then (1.18) holds. Indeed, it is not hard to check that there exists a constant $C \in (0, \infty)$ with the property that $h_\omega(t) \leq C\, t^{i_\omega/2}$ for every $t \in (0, 1]$, and $h_\omega(t) \leq C\, t^{(I_\omega+1)/2}$ for every $t \in [1, \infty)$. These, in turn, readily yield (1.18).

Now, given $\alpha \in (0, 1)$, if $\omega(t) := t^\alpha$ for each $t > 0$ then $i_\omega = I_\omega = \alpha$ and, hence, (1.18) holds. Note that in that case $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^n_+, \mathbb{C}^M) = \dot{\mathscr{C}}^{\alpha}(\mathbb{R}^n_+, \mathbb{C}^M)$ is the standard homogeneous Hölder space of order α and the particular version of Theorem 1.4 corresponding to this scenario has been established in [6]. This being said, there many examples of interest that are treated here for the first time, such as $\omega(t) = t^\alpha\, (A + \log_+ t)^\theta$ for $A := \max\{1, -\theta/\alpha\}$ and each $t > 0$, or $\omega(t) = t^\alpha\, (A + \log_+(1/t))^\theta$ for $A := \max\{1, \theta/\alpha\}$ and each $t > 0$, with $0 < \alpha < 1$, $\theta \in \mathbb{R}$, and $\log_+(t) := \max\{0, \log t\}$.

In these situations $i_\omega = I_\omega = \alpha$ which guarantees that (1.18) holds. Furthermore, if $\omega(t) := \max\{t^\alpha, t^\beta\}$, or $\omega(t) := \min\{t^\alpha, t^\beta\}$, for each $t > 0$, with $0 < \alpha, \beta < 1$, then in both cases we have $i_\omega = \min\{\alpha, \beta\}$ and $I_\omega = \max\{\alpha, \beta\}$, hence condition (1.18) is verified once again.

The following result, providing a characterization of the generalized Hölder and generalized Morrey-Campanato spaces in terms of the boundary traces of solutions, is a byproduct of the proof of the above theorem.

Corollary 1.5. *Let L be a strongly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$. Fix $p \in [1, \infty)$ along with $q \in (0, \infty)$, and let ω be a growth*

function for which (1.18) *holds. Then for every function* $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ *satisfying* $Lu = 0$ *in* $\mathbb{R}^n_+$ *one has*

$$\|u\|^{(\omega,q)}_{**} \approx \|u\|^{(\omega,\exp)}_{**} \approx \|u\|^{(\omega,\infty)}_{**} \approx [u]_{\dot{\mathscr{C}}^w(\mathbb{R}^n_+, \mathbb{C}^M)} \tag{1.27}$$

where the implicit proportionality constants depend only on L, n, q, *and the constant* C_0 *in* (1.18). *Moreover,*

$$\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) = \left\{ u|_{\partial\mathbb{R}^n_+} : u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \ Lu = 0 \ in \ \mathbb{R}^n_+, \ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} < \infty \right\}$$

$$= \left\{ u|_{\partial\mathbb{R}^n_+} : u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \ Lu = 0 \ in \ \mathbb{R}^n_+, \ \|u\|^{(\omega,q)}_{**} < \infty \right\}$$

$$= \left\{ u|_{\partial\mathbb{R}^n_+} : u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \ Lu = 0 \ in \ \mathbb{R}^n_+, \ \|u\|^{(\omega,\exp)}_{**} < \infty \right\}$$

$$= \left\{ u|_{\partial\mathbb{R}^n_+} : u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \ Lu = 0 \ in \ \mathbb{R}^n_+, \ \|u\|^{(\omega,\infty)}_{**} < \infty \right\}. \tag{1.28}$$

The plan of the paper is as follows. In Section 2 we present some properties of the growth functions and study some of the features of the generalized Hölder and Morrey-Campanato spaces which are relevant to this work. Section 3 is reserved for collecting some known results for elliptic systems, and for giving the proof of Proposition 3.4, where some a priori estimates for the null-solutions of such systems are established. In turn, these estimates allow us to compare the seminorm $\|\cdot\|^{(\omega,q)}_{**}$ (corresponding to various values of q) with $[\cdot]_{\dot{\mathscr{C}}^w(\mathbb{R}^n_+, \mathbb{C}^M)}$. In Section 4 we prove the existence of solutions for the Dirichlet problems with boundary data in $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$. Section 5 contains a Fatou-type result for null-solutions of a strongly elliptic system L belonging to the space $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, which will be a key ingredient when establishing uniqueness for the boundary value problems formulated in Theorem 1.4. Combining the main results of the previous two sections yields two well-posedness results under different assumptions on the growth function: one for boundary data in $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and solutions in $\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$, and another one for boundary data in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and solutions satisfying $\|u\|^{(\omega,q)}_{**} < \infty$ for some $0 < q \leq \infty$, or even in the case where q is replaced by exp. In concert, these two results cover all claims of Theorem 1.4. Finally, in Appendix A we present a John-Nirenberg type inequality of real-variable nature, generalizing some results in [4, 3] by allowing more flexibility due to the involvement of growth functions. This is interesting and useful in its own right. In addition, we are able to show exponential decay for the measure of the associated level sets which, in turn, permits deriving estimates not only in arbitrary L^q spaces but also in the space $\exp L$. Our approach for deriving such results is different from [4, 3], and uses some ideas which go back to a proof of the classical John-Nirenberg exponential integrability for BMO functions due to Calderón. As a matter of fact, our abstract method yields easily Calderón's classical result.

2. Growth Functions, Generalized Hölder and Morrey-Campanato Spaces

We begin by studying some basic properties of growth functions. As explained in the introduction, we ultimately wish to work with growth functions satisfying conditions weaker than (1.18). Indeed, the two mains conditions that we will consider are

$$\int_0^1 \omega(s)\frac{ds}{s} < \infty, \tag{2.1}$$

and

$$t\int_t^\infty \frac{\omega(s)}{s}\frac{ds}{s} \le C_\omega\,\omega(t), \qquad \forall\, t \in (0,\infty), \tag{2.2}$$

for some finite constant $C_\omega \ge 1$. In what follows, C_ω will always denote the constant in (2.2). Clearly, if ω satisfies it satisfies (1.18) then both (2.1) and (2.2) hold but the reverse implication is not true in general (see Example 6.4 in this regard).

Later on, we will need the auxiliary function W defined as

$$W(t) := \int_0^t \omega(s)\frac{ds}{s} \quad \text{for each} \quad t \in (0,\infty). \tag{2.3}$$

Note that (2.1) gives that $W(t) < \infty$ for every $t > 0$. Then (1.18) holds if and only if (2.2) holds and there exists $C \in (0,\infty)$ such that $W(t) \le C\,\omega(t)$ for each $t \in (0,\infty)$.

The following lemma gathers some useful properties on growth functions satisfying condition (2.2).

Lemma 2.1. *Given a growth function ω satisfying* (2.2), *the following statements are true.*

(a) *Whenever $0 < t_1 \le t_2 < \infty$, one has*

$$\frac{\omega(t_2)}{t_2} \le C_\omega\frac{\omega(t_1)}{t_1}. \tag{2.4}$$

(b) *For every $t \in (0,\infty)$ one has*

$$\omega(2t) \le 2C_\omega\,\omega(t). \tag{2.5}$$

(c) *One has $\lim_{t\to\infty}\omega(t)/t = 0$.*

(d) *For each set $E \subset \mathbb{R}^n$ one has $\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M) = \dot{\mathscr{C}}^\omega(\overline{E},\mathbb{C}^M)$, with equivalent norms. More specifically, the restriction map*

$$\dot{\mathscr{C}}^\omega(\overline{E},\mathbb{C}^M) \ni u \longmapsto u\big|_E \in \dot{\mathscr{C}}^\omega(E,\mathbb{C}^M) \tag{2.6}$$

is a linear isomorphism which is continuous in the precise sense that, under the canonically identification of functions $u \in \dot{\mathscr{C}}^\omega(\overline{E},\mathbb{C}^M)$ with $u\big|_E \in \dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)$, one has

$$[u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \le [u]_{\dot{\mathscr{C}}^\omega(\overline{E},\mathbb{C}^M)} \le 2C_\omega[u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \tag{2.7}$$

for each $u \in \dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)$.

Proof. We start observing that for every $t > 0$

$$\frac{\omega(t)}{t} \leq \int_t^\infty \frac{\omega(s)}{s} \frac{ds}{s} \leq C_\omega \frac{\omega(t)}{t}. \tag{2.8}$$

The first inequality uses that ω is non-decreasing and the second is just (2.2). Then, given $t_1 \leq t_2$, we may write

$$\frac{\omega(t_2)}{t_2} \leq \int_{t_2}^\infty \frac{\omega(s)}{s} \frac{ds}{s} \leq \int_{t_1}^\infty \frac{\omega(s)}{s} \frac{ds}{s} \leq C_\omega \frac{\omega(t_1)}{t_1}, \tag{2.9}$$

proving (a). The doubling property in (b) follows at once from (a) by taking $t_2 := 2t_1$ in (2.4). Next, the claim in (c) is justified by passing to limit $t \to \infty$ in the first inequality in (2.8) and using Lebesgue's Dominated Convergence Theorem.

Turning our attention to (d), fix an arbitrary $u \in \mathscr{C}^\omega(E, \mathbb{C}^M)$. As noted earlier, this membership ensures that u is uniformly continuous, hence u extends uniquely to a continuous function v on $\overline{E}$. To show that v belongs to $\dot{\mathscr{C}}^\omega(\overline{E}, \mathbb{C}^M)$ pick two arbitrary distinct points $y, z \in \overline{E}$ and choose two sequences $\{y_k\}_{k\in\mathbb{N}}$, $\{z_k\}_{k\in\mathbb{N}}$ of points in E such that $y_k \to x$ and $z_k \to z$ as $k \to \infty$. By discarding finitely many terms, there is no loss of generality in assuming that $|y_k - z_k| < 2|y - z|$ for each $k \in \mathbb{N}$. Relying on the fact that ω is non-decreasing and (2.5), we may then write

$$|v(y) - v(z)| = \lim_{k\to\infty} |u(y_k) - u(z_k)| \leq [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \limsup_{k\to\infty} \omega(|y_k - z_k|)$$

$$\leq [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \omega(2|y - z|) \leq 2C_\omega [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \omega(|y - z|). \tag{2.10}$$

From this, all claims in (d) follow, completing the proof of the lemma. $\qquad\square$

In the following lemma we treat W defined in (2.3) as a growth function depending on the original ω.

Lemma 2.2. *Let ω be a growth function satisfying (2.1) and (2.2), and let $W(t)$ be defined as in (2.3). Then $W : (0, \infty) \to (0, \infty)$ is a growth function satisfying (2.2) with*

$$C_W \leq 1 + (C_\omega)^2. \tag{2.11}$$

Moreover,

$$\omega(t) \leq C_\omega W(t) \quad for\ each\ \ t \in (0, \infty). \tag{2.12}$$

Proof. By design, W is a non-decreasing function and, thanks to Lebesgue's Dominated Convergence Theorem and (2.1) we have $W(t) \to 0$ as $t \to 0^+$. Also, on account of (2.4), for each $t \in (0, \infty)$ we may write

$$\omega(t) = \int_0^t \frac{\omega(t)}{t} ds \leq C_\omega \int_0^t \frac{\omega(s)}{s} ds = C_\omega W(t), \tag{2.13}$$

proving (2.12). In turn, Fubini's Theorem, (2.2), and (2.12) permit us to estimate

$$
\begin{aligned}
t\int_t^\infty \frac{W(s)}{s}\frac{ds}{s} &= t\int_t^\infty\left(\int_0^s \omega(\lambda)\frac{d\lambda}{\lambda}\right)\frac{ds}{s^2}\\
&= t\int_0^t\left(\int_t^\infty \frac{ds}{s^2}\right)\omega(\lambda)\frac{d\lambda}{\lambda} + t\int_t^\infty\left(\int_\lambda^\infty \frac{ds}{s^2}\right)\omega(\lambda)\frac{d\lambda}{\lambda}\\
&= t\int_0^t \frac{1}{t}\omega(\lambda)\frac{d\lambda}{\lambda} + t\int_t^\infty \frac{\omega(\lambda)}{\lambda}\frac{d\lambda}{\lambda}\\
&\leq W(t) + C_\omega\,\omega(t)\\
&\leq \left(1+(C_\omega)^2\right)W(t),
\end{aligned}
\tag{2.14}
$$

for each $t\in(0,\infty)$. This shows that W satisfies (2.2) with constant $C_W\leq 1+(C_\omega)^2$. $\quad\square$

Moving on, for each given function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$ define the L^p-based mean oscillation of f at a scale $r\in(0,\infty)$ as

$$
\mathrm{osc}_p(f;r) := \sup_{\substack{Q\subset\mathbb{R}^{n-1}\\ \ell(Q)\leq r}}\left(\fint_Q |f(x')-f_Q|^p\,dx'\right)^{1/p}.
\tag{2.15}
$$

The following lemma gathers some results from [6, Lemmas 2.1 and 2.2].

Lemma 2.3. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathbb{C}^M)$.*

(a) *For every $p,q\in[1,\infty)$ there exists some finite $C=C(p,q,n)>1$ such that*
$$
C^{-1}\,\mathrm{osc}_p(f;r)\leq \mathrm{osc}_q(f;r)\leq C\,\mathrm{osc}_p(f;r),\quad \forall\,r>0.
\tag{2.16}
$$

(b) *For every $\varepsilon>0$,*
$$
\int_1^\infty \mathrm{osc}_1(f;s)\frac{ds}{s^{1+\varepsilon}} < \infty \implies f\in L^1\left(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M.
\tag{2.17}
$$

We augment Lemma 2.3 with similar results involving generalized Morrey-Campanato spaces and generalized Hölder spaces.

Lemma 2.4. *Let ω be a growth function and fix $p\in[1,\infty)$. Then the following properties are valid.*

(a) *If $f\in\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$, then*
$$
\mathrm{osc}_p(f;r)\leq \omega(r)\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\quad \text{for each }\ r\in(0,\infty).
\tag{2.18}
$$

(b) *If ω satisfies (2.2), then for each $f\in\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ one has*
$$
\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\leq \sqrt{n-1}\,C_\omega[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}
\tag{2.19}
$$
and
$$
\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)\subset \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)\subset L^1\left(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^n}\right)^M.
\tag{2.20}
$$

Proof. Note that given any $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and $r > 0$, based on (2.15), the fact that ω is non-decreasing, and (1.13) we may write

$$\mathrm{osc}_p(f; r) = \sup_{\substack{Q \subset \mathbb{R}^{n-1} \\ \ell(Q) \leq r}} \omega(\ell(Q)) \frac{1}{\omega(\ell(Q))} \left(\fint_Q |f(x') - f_Q|^p \, dx' \right)^{1/p}$$

$$\leq \omega(r) \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \tag{2.21}$$

proving (a). Consider next the claims in (b). Given any $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, a combination of (1.13), (1.10), and (2.4) yields

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \sup_{Q \subset \mathbb{R}^{n-1}} \left(\fint_Q \fint_Q \left(\frac{|f(x') - f(y')|}{\omega(\ell(Q))} \right)^p \, dx' \, dy' \right)^{1/p}$$

$$\leq \sup_{Q \subset \mathbb{R}^{n-1}} \frac{\omega(\sqrt{n-1}\ell(Q))}{\omega(\ell(Q))} [f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}$$

$$\leq \sqrt{n-1} C_\omega [f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{2.22}$$

This establishes (2.19), hence also the first inclusion in (2.20). For the second inclusion in (2.20), using Jensen's inequality, (2.18), and (2.2) we may write

$$\int_1^\infty \mathrm{osc}_1(f; s) \frac{ds}{s^2} \leq \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \int_1^\infty \omega(s) \frac{ds}{s^2}$$

$$\leq C_\omega \, \omega(1) \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} < \infty. \tag{2.23}$$

The desired inclusion now follows from this and (2.17) with $\varepsilon := 1$. $\qquad\square$

3. Properties of Elliptic Systems and Their Solutions

The following result is a particular case of more general interior estimates found in [10, Theorem 11.9].

Theorem 3.1. *Let L be a constant complex coefficient system as in* (1.1) *satisfying* (1.2). *Then for every $p \in (0, \infty)$, $\lambda \in (0, 1)$, and $m \in \mathbb{N} \cup \{0\}$ there exists a finite constant $C = C(L, p, m, \lambda, n) > 0$ with the property that for every null-solution u of L in a ball $B(x, R)$, where $x \in \mathbb{R}^n$ and $R > 0$, and every $r \in (0, R)$ one has*

$$\sup_{z \in B(x, \lambda r)} |(\nabla^m u)(z)| \leq \frac{C}{r^m} \left(\fint_{B(x,r)} |u(x)|^p \, dx \right)^{1/p}. \tag{3.1}$$

To proceed, introduce

$$W_{\mathrm{bdd}}^{1,2}(\mathbb{R}_+^n) := \left\{ u \in L_{\mathrm{loc}}^2(\mathbb{R}_+^n) : u, \partial_j u \in L^2\left(\mathbb{R}_+^n \cap B(0, r)\right) \right.$$

$$\left. \text{for each} \ \ j \in \{1, \dots, n\} \ \ \text{and} \ \ r \in (0, \infty) \right\}, \tag{3.2}$$

and define the Sobolev trace Tr, whenever meaningful, as

$$(\mathrm{Tr} \, u)(x') := \lim_{r \to 0^+} \fint_{B((x',0),r) \cap \mathbb{R}_+^n} u(y) \, dy, \qquad x' \in \mathbb{R}^{n-1}. \tag{3.3}$$

The following result is taken from [8, Corollary 2.4].

Proposition 3.2. *Let L be a constant complex coefficient system as in* (1.1) *satisfying* (1.2)*, and suppose $u \in W_{\mathrm{bdd}}^{1,2}(\mathbb{R}_+^n)$ satisfies $Lu = 0$ in $\mathbb{R}_+^n$ and $\mathrm{Tr}\, u = 0$ on $\mathbb{R}^{n-1}$. Then $u \in \mathscr{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$ and there exists a finite constant $C > 0$, independent of u, such that for each $x \in \overline{\mathbb{R}_+^n}$ and each $r > 0$,*

$$\sup_{\mathbb{R}_+^n \cap B(x,r)} |\nabla u| \le \frac{C}{r} \sup_{\mathbb{R}_+^n \cap B(x,2r)} |u|. \tag{3.4}$$

The following theorem is contained in [6, Theorem 2.3 and Proposition 3.1].

Theorem 3.3. *Suppose L is a constant complex coefficient system as in* (1.1)*, satisfying* (1.2)*. Then the following statements are true.*

(a) *There exists a matrix-valued function $P^L = (P_{\alpha\beta}^L)_{1 \le \alpha,\beta \le M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M \times M}$, called the Poisson kernel for L in $\mathbb{R}_+^n$, such that $P^L \in \mathscr{C}^\infty(\mathbb{R}^{n-1})$, there exists some finite constant $C > 0$ such that*

$$|P^L(x')| \le \frac{C}{(1 + |x'|^2)^{n/2}}, \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{3.5}$$

and

$$\int_{\mathbb{R}^{n-1}} P^L(x')\, dx' = I_{M \times M}, \tag{3.6}$$

where $I_{M \times M}$ stands for the $M \times M$ identity matrix. Moreover, if for every $x' \in \mathbb{R}^{n-1}$ and $t > 0$ one defines

$$K^L(x', t) := P_t^L(x') := t^{1-n} P^L(x'/t), \tag{3.7}$$

then $K^L \in \mathscr{C}^\infty\big(\overline{\mathbb{R}_+^n} \setminus B(0, \varepsilon)\big)$ for every $\varepsilon > 0$ and the function $K^L = \big(K_{\alpha\beta}^L\big)_{1 \le \alpha,\beta \le M}$ satisfies

$$LK_{\cdot\beta}^L = 0 \quad \text{in}\ \ \mathbb{R}_+^n \ \ \text{for each}\ \ \beta \in \{1, \dots, M\}, \tag{3.8}$$

where $K_{\cdot\beta}^L := \big(K_{\alpha\beta}^L\big)_{1 \le \alpha \le M}$ is the β-th column in K^L.

(b) *For each function $f = (f_\beta)_{1 \le \beta \le M} \in L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\big)^M$ define, with P^L as above,*

$$u(x', t) := (P_t^L * f)(x'), \qquad \forall\, (x', t) \in \mathbb{R}_+^n. \tag{3.9}$$

Then u is meaningfully defined, via an absolutely convergent integral, and satisfies

$$u \in \mathscr{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \ \ \text{in}\ \ \mathbb{R}_+^n, \quad u\big|_{\partial\mathbb{R}_+^n}^{\mathrm{nt.lim}} = f \ \text{a.e. on}\ \mathbb{R}^{n-1}. \tag{3.10}$$

Furthermore, there exists a finite constant $C > 0$ such that

$$|(\nabla u)(x', t)| \le \frac{C}{t} \int_1^\infty \mathrm{osc}_1(f; st) \frac{ds}{s^2}, \qquad \forall\, (x', t) \in \mathbb{R}_+^n, \tag{3.11}$$

and, for each cube $Q \subset \mathbb{R}^{n-1}$,

$$\left(\int_0^{\ell(Q)} \fint_Q |(\nabla u)(x', t)|^2\, t\, dx'\, dt \right)^{1/2} \le C \int_1^\infty \mathrm{osc}_1(f; s\ell(Q)) \frac{ds}{s^2}. \tag{3.12}$$

Our next proposition contains a number of a priori estimates comparing $\|u\|_{**}^{(\omega,q)}$, corresponding to different values of q, for solutions of $Lu = 0$ in $\mathbb{R}^n_+$. To set the stage, we first state some simple estimates which are true for any function $u \in \mathscr{C}^1(\mathbb{R}^n_+, \mathbb{C}^M)$:

$$\|u\|_{**}^{(\omega,p)} \leq \|u\|_{**}^{(\omega,q)} \leq C \|u\|_{**}^{(\omega,\exp)}, \qquad 0 < p \leq q < \infty, \tag{3.13}$$

where $C = C(q) \geq 1$. Indeed, the first estimate follows at once from Jensen's inequality. The second estimate is a consequence of the fact that $t^{\max\{1,q\}} \leq C(e^t - 1)$ (with $C > 0$ depending on $\max\{1,q\}$) for each $t \in (0,\infty)$ and the definition of $\|\cdot\|_{\exp L,Q}$ (cf. (1.15)).

Proposition 3.4. *Let L be a constant complex coefficient system as in* (1.1) *satisfying the strong ellipticity condition* (1.2), *and let $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M)$ be such that $Lu = 0$ in $\mathbb{R}^n_+$. Then the following statements hold.*

(a) *For every $q \in (0,\infty)$ there there exists a finite constant $C = C(L,n,q) \geq 1$ such that for each $(x',t) \in \mathbb{R}^n_+$ one has*

$$t\,|(\nabla u)(x',t)| \leq C\left(\fint_{|x'-y'|<\frac{t}{2}} \left(\int_{t/2}^{3t/2} |(\nabla u)(y',s)|^2\, s\, ds \right)^{q/2} dy' \right)^{1/q}. \tag{3.14}$$

(b) *There exists a finite constant $C = C(L,n) \geq 1$ such that for each cube $Q \subset \mathbb{R}^{n-1}$ and each $x' \in \mathbb{R}^{n-1}$ one has*

$$\left(\int_0^{\ell(Q)} |(\nabla u)(x',t)|^2\, t\, dt \right)^{1/2} \leq C \left(\int_0^{2\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y',s)\, s|^2\, dy' \frac{ds}{s^n} \right)^{1/2}. \tag{3.15}$$

Furthermore, whenever $2 \leq q < \infty$ there exists a finite constant $C = C(L,n,q) \geq 1$ such that for each cube $Q \subset \mathbb{R}^{n-1}$ and each $x' \in \mathbb{R}^{n-1}$ one has

$$\left(\fint_Q \left(\int_0^{\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y',s)\, s|^2\, dy' \frac{ds}{s^n} \right)^{q/2} dx' \right)^{1/q}$$

$$\leq C \left(\fint_{3Q} \left(\int_0^{3\ell(Q)} |(\nabla u)(x',t)|^2\, t\, dt \right)^{q/2} dx' \right)^{1/q}. \tag{3.16}$$

(c) *There exists a finite constant $C = C(L,n) \geq 1$ such that for each growth function ω one has*

$$\|u\|_{**}^{(\omega,\infty)} \leq C[u]_{\mathscr{C}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)}. \tag{3.17}$$

(d) *For every $q \in (0,\infty)$ there exists a finite constant $C = C(L,n,q) \geq 1$ such that for each growth function ω satisfying* (2.2) *one has*

$$\|u\|_{**}^{(\omega,\infty)} \leq C\, C_\omega \|u\|_{**}^{(\omega,q)}. \tag{3.18}$$

(e) *There exists a finite constant $C = C(L,n) \geq 1$ such that for each growth function ω satisfying* (2.2) *one has*

$$\|u\|_{**}^{(\omega,\exp)} \leq C(C_\omega)^2 \|u\|_{**}^{(\omega,2)}. \tag{3.19}$$

(f) *Let ω be a growth function satisfying* (2.1) *as well as* (2.2), *and define $W(t)$ as in* (2.3). *Then*

$$[u]_{\mathscr{C}^W(\mathbb{R}^n_+, \mathbb{C}^M)} \leq C_\omega(2 + C_\omega) \|u\|_{**}^{(\omega,\infty)}, \tag{3.20}$$

and, if the latter quantity is finite, $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ in the sense of Lemma 2.1(d).

(g) Let ω be a growth function satisfying

$$\int_0^t \omega(s) \frac{ds}{s} \leq C'_\omega \, \omega(t), \qquad \forall \, t \in (0, \infty), \tag{3.21}$$

for some finite constant $C'_\omega > 1$. Then

$$\|u\|_{**}^{(\omega, \exp)} \leq (C'_\omega)^{1/2} \, \|u\|_{**}^{(\omega, \infty)}. \tag{3.22}$$

(h) Let ω be a growth function satisfying (1.18). Then for every $q \in (0, \infty)$

$$\|u\|_{**}^{(\omega, q)} \approx \|u\|_{**}^{(\omega, \exp)} \approx \|u\|_{**}^{(\omega, \infty)} \approx [u]_{\dot{\mathscr{C}}^w(\mathbb{R}^n_+, \mathbb{C}^M)} \tag{3.23}$$

where the implicit constants depend only on L, n, q, and the constant C_0 in (1.18). In particular, if $\|u\|_{**}^{(\omega, q)} < \infty$ for some $q \in (0, \infty]$, or $\|u\|_{**}^{(\omega, \exp)} < \infty$, then $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ in the sense of Lemma 2.1(d).

Proof. We start by proving (a). Fix $(x', t) \in \mathbb{R}^n_+$ and let $Q_{x',t}$ be the cube in $\mathbb{R}^{n-1}$ centered at x' with side-length t. Then from Theorem 3.1 (presently used with $m := 0$ and $p := \min\{q, 2\}$) and Jensen's inequality we obtain

$$|(\nabla u)(x', t)| \leq C \left(\fint_{|(y', s) - (x', t)| < \frac{t}{2}} |(\nabla u)(y', s)|^p \, dy' \, ds \right)^{1/p}$$

$$\leq C \left(\fint_{|x' - y'| < \frac{t}{2}} \left(\fint_{(t/2, 3t/2)} |(\nabla u)(y', s)|^2 \, ds \right)^{p/2} dy' \right)^{1/p}$$

$$\leq C \left(\fint_{|x' - y'| < \frac{t}{2}} \left(\fint_{(t/2, 3t/2)} |(\nabla u)(y', s)|^2 \, ds \right)^{q/2} dy' \right)^{1/q}$$

$$= C t^{-1} \left(\fint_{|x' - y'| < \frac{t}{2}} \left(\int_{t/2}^{3t/2} |(\nabla u)(y', s)|^2 \, s \, ds \right)^{q/2} dy' \right)^{1/q}, \tag{3.24}$$

proving (3.14). Turning our attention to (b), fix a cube $Q \subset \mathbb{R}^{n-1}$ along with a point $x' \in \mathbb{R}^{n-1}$. First, integrating (3.14) written for $q := 2$ yields

$$\int_0^{\ell(Q)} |(\nabla u)(x', t)|^2 \, t \, dt \leq C \int_0^{\ell(Q)} \frac{1}{t^{n+1}} \int_{t/2}^{3t/2} \int_{|x' - y'| < s} |(\nabla u)(y', s)|^2 \, s \, dy' \, ds \, t \, dt$$

$$\leq C \int_0^{2\ell(Q)} \int_{|x' - y'| < s} |(\nabla u)(y', s)|^2 \int_{2s/3}^{2s} t^{-n} \, dt \, dy' \, s \, ds$$

$$= C \int_0^{2\ell(Q)} \int_{|x' - y'| < s} |(\nabla u)(y', s) \, s|^2 \, dy' \frac{ds}{s^n}, \tag{3.25}$$

and this readily leads to the estimate in (3.15). To justify (3.16), observe that for each nonnegative function $h \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$ we have

$$\fint_Q \left(\int_0^{\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y',s) \, s|^2 \, dy' \frac{ds}{s^n} \right) h(x') \, dx'$$

$$\leq 3^n \fint_{3Q} \int_0^{\ell(Q)} \left(\frac{1}{s^{n-1}} \int_{|y'-x'|<s} h(x') \, dx' \right) |(\nabla u)(y',s)|^2 \, s \, ds \, dy'$$

$$\leq C_n \fint_{3Q} \left(\int_0^{3\ell(Q)} |(\nabla u)(y',s)|^2 \, s \, ds \right) (Mh)(x') \, dx', \qquad (3.26)$$

where M is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$. Note that if $q = 2$ then (3.26) gives at once (3.16) by taking $h = 1$ in Q and using that $Mh \leq 1$. On the other hand, if $q > 2$, we impose the normalization condition $\|h\|_{L^{(q/2)'}(Q,dx'/|Q|)} = 1$ and then rely on (3.26) and Hölder's inequality to write

$$\fint_Q \left(\int_0^{\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y',s) \, s|^2 \, dy' \frac{ds}{s^n} \right) h(x') \, dx'$$

$$\leq C_n \left(\fint_{3Q} \left(\int_0^{3\ell(Q)} |(\nabla u)(y',s)|^2 \, s \, ds \right)^{q/2} dx' \right)^{2/q} \|Mh\|_{L^{(q/2)'}(Q,dx'/|Q|)}$$

$$\leq C \left(\fint_{3Q} \left(\int_0^{3\ell(Q)} |(\nabla u)(y',s)|^2 \, s \, ds \right)^{q/2} dx' \right)^{2/q}, \qquad (3.27)$$

bearing in mind that M is bounded in $L^{(q/2)'}(\mathbb{R}^{n-1})$, given that $q > 2$. Taking now the supremum over all such functions h yields (3.16) on account of Riesz' duality theorem.

As regards (c), fix $(x',t) \in \mathbb{R}^n_+$ and use Theorem 3.1 together with the fact that ω is a non-decreasing function to write

$$|(\nabla u)(x',t)| = \left| \nabla (u(\cdot) - u(x',t))(x',t) \right|$$

$$\leq \frac{C}{t} \fint_{|(y',s)-(x',t)|<t/2} |u(y',s) - u(x',t)| \, dy' \, ds$$

$$\leq C[u]_{\mathscr{C}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \frac{\omega(t)}{t}. \qquad (3.28)$$

In view of (1.17), this readily establishes (3.17).

The claim in (d) is proved by combining (3.14) and (2.5), which permit us to estimate (recall that $Q_{x',t}$ denotes the cube in $\mathbb{R}^{n-1}$ centered at x' with side-length t)

$$\|u\|^{(\omega,\infty)}_{**} \leq C \sup_{(x',t)\in\mathbb{R}^n_+} \frac{1}{\omega(t)} \left(\fint_{(3/2)Q_{x',t}} \left(\int_0^{3t/2} |(\nabla u)(y',s)|^2 \, s \, ds \right)^{q/2} dy' \right)^{1/q}$$

$$\leq C \, C_\omega \, \|u\|^{(\omega,q)}_{**}. \qquad (3.29)$$

Going further, consider the claim in (e). For starters, observe that the convexity of the function $t \mapsto e^t - 1$ readily implies that $2^{n-1}(e^t - 1) \leq e^{2^{n-1}t} - 1$ for every $t > 0$ which, in view of (1.16), allows us to write

$$\|f\|_{\exp L, Q} \leq 2^{n-1} \|f\|_{\exp L, 2Q} \tag{3.30}$$

for each cube Q in $\mathbb{R}^{n-1}$ and each Lebesgue measurable function f on Q.

Turning to the proof of (3.19) in earnest, by homogeneity we may assume that $\|u\|_{**}^{(\omega,2)} = 1$ to begin with. We are going to use Lemma A.1. As a prelude, define

$$F(y', s) := |(\nabla u)(y', s) s|, \qquad \forall (y', s) \in \mathbb{R}_+^n, \tag{3.31}$$

and, for each cube Q in $\mathbb{R}^{n-1}$ and each threshold $N \in (0, \infty)$, consider the set

$$E_{N,Q} := \left\{ x' \in Q : \frac{1}{\omega(\ell(Q))} \left(\int_0^{\ell(Q)} \int_{|x'-y'|<\kappa s} |F(y', s)|^2 \, dy' \frac{ds}{s^n} \right)^{1/2} > N \right\}. \tag{3.32}$$

where $\kappa := 1 + 2\sqrt{n-1}$. Denoting $Q^* := (2\kappa + 1)Q = (3 + 4\sqrt{n-1})Q$, then using Chebytcheff's inequality, and (2.4), for each cube Q in $\mathbb{R}^{n-1}$ and each $N > 0$ we may write

$$
\begin{aligned}
|E_{N,Q}| &\leq \frac{1}{N^2} \frac{1}{\omega(\ell(Q))^2} \int_Q \int_0^{\ell(Q)} \int_{|x'-y'|<\kappa s} |F(y', s)|^2 \, dy' \frac{ds}{s^n} \, dx' \\
&\leq \frac{1}{N^2} \frac{1}{\omega(\ell(Q))^2} \int_{Q^*} \int_0^{\ell(Q)} \left(\int_{|y'-x'|<\kappa s} dx' \right) |F(y', s)|^2 \frac{ds}{s^n} \, dy' \\
&\leq C \frac{1}{N^2} \frac{1}{\omega(\ell(Q))^2} \int_{Q^*} \int_0^{\ell(Q^*)} |(\nabla u)(y', s) s|^2 \frac{ds}{s} \, dy' \\
&\leq C \frac{|Q^*|}{N^2} \frac{\omega(\ell(Q^*))^2}{\omega(\ell(Q))^2} \left(\|u\|_{**}^{(\omega,2)} \right)^2 = C \frac{|Q^*|}{N^2} \left[\frac{\omega(\ell(Q^*))}{\omega(\ell(Q))} \right]^2 \\
&\leq C_0 (C_\omega)^2 \frac{1}{N^2} |Q|, \tag{3.33}
\end{aligned}
$$

for some finite constant $C_0 > 0$. Therefore, taking $N := \sqrt{2C_0} C_\omega > 0$, we conclude that

$$|E_{N,Q}| \leq \frac{1}{2} |Q|. \tag{3.34}$$

This allows us to invoke Lemma A.1 with $\varphi := \omega$, which together with (3.15), (2.5), and (3.30), gives

$$
\begin{aligned}
\|u\|_{**}^{(\omega,\exp)} &\leq C \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left\| \left(\int_0^{\ell(2Q)} \int_{|\cdot - y'|<s} |F(y', s)|^2 \, dy' \frac{ds}{s^n} \right)^{1/2} \right\|_{\exp L, Q} \\
&\leq C (C_\omega)^2. \tag{3.35}
\end{aligned}
$$

This completes the proof of (e).

Turning our attention to (f), fix $x = (x', t)$ and $y = (y', s)$ in $\mathbb{R}^n_+$, and abbreviate $r := |x - y|$. Then,

$$\frac{|u(x) - u(y)|}{W(|x - y|)} \leq \frac{1}{W(r)}|u(x', t) - u(x', t + r)| + \frac{1}{W(r)}|u(x', t + r) - u(y', s + r)|$$

$$+ \frac{1}{W(r)}|u(y', s + r) - u(y', s)|$$

$$=: I + II + III. \tag{3.36}$$

To bound I, we use the Fundamental Theorem of Calculus, (2.4), and (2.3) and obtain

$$I = \frac{1}{W(r)}\left|\int_0^r (\partial_n u)(x', t + \xi)\, d\xi\right| \leq \|u\|_{**}^{(\omega,\infty)} \frac{1}{W(r)}\int_0^r \frac{\omega(t + \xi)}{t + \xi}\, d\xi$$

$$\leq C_\omega \|u\|_{**}^{(\omega,\infty)} \frac{1}{W(r)}\int_0^r \frac{\omega(\xi)}{\xi}\, d\xi = C_\omega \|u\|_{**}^{(\omega,\infty)}. \tag{3.37}$$

Note that III is bounded analogously replacing x' by y' and t by s. For II, we use again the Fundamental Theorem of Calculus, together with (2.4) and (2.12), to write

$$II = \frac{1}{W(r)}\left|\int_0^1 \frac{d}{d\theta}[u(\theta(x', t + r) + (1 - \theta)(y', s + r))]\, d\theta\right|$$

$$= \frac{1}{W(r)}\left|\int_0^1 (x' - y', t - s) \cdot (\nabla u)\big(\theta(x', t + r) + (1 - \theta)(y', s + r)\big)\, d\theta\right|$$

$$\leq \|u\|_{**}^{(\omega,\infty)} \frac{r}{W(r)}\int_0^1 \frac{\omega\big((1 - \theta)s + \theta t + r\big)}{(1 - \theta)s + \theta t + r}\, d\theta$$

$$\leq C_\omega \|u\|_{**}^{(\omega,\infty)} \frac{r}{W(r)}\int_0^1 \frac{\omega(r)}{r}\, d\theta$$

$$\leq (C_\omega)^2 \|u\|_{**}^{(\omega,\infty)}. \tag{3.38}$$

As x and y were chosen arbitrarily, (3.36), (3.37), and (3.38) collectively justify (3.20).

To justify (g), observe that since ω is non-decreasing and satisfies (3.21) we may write

$$\left(\fint_Q \left(\int_0^{\ell(Q)} |(\nabla u)(x', t)|^2\, t\, dt\right)^{q/2} dx'\right)^{1/q} \leq \left(\int_0^{\ell(Q)} \omega(t)^2 \frac{dt}{t}\right)^{1/2} \|u\|_{**}^{(\omega,\infty)}$$

$$\leq (C'_\omega)^{1/2}\omega(\ell(Q)) \|u\|_{**}^{(\omega,\infty)}, \tag{3.39}$$

which readily leads to the desired inequality.

As regards (h), the idea is to combine (3.13), (g), and (d) for the first three equivalences. In concert, (c), the fact that (1.18) gives $W \leq C_0\,\omega$, and (f) also give the last equivalence in (h). The proof of Proposition 3.4 is therefore complete. $\qquad\square$

4. Existence Results

In this section we develop the main tools used to establish the existence of solutions for the boundary value problems formulated in the statement of Theorem 1.4. We start with the generalized Hölder Dirichlet problem.

Proposition 4.1. *Let L be a constant complex coefficient system as in (1.1) satisfying the strong ellipticity condition formulated in (1.2), and let ω be a growth function satisfying (2.2). Given $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, define $u(x', t) := (P_t^L * f)(x')$ for every $(x', t) \in \mathbb{R}_+^n$. Then u is meaningfully defined via an absolutely convergent integral and satisfies*

$$u \in \mathscr{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \ \text{ in } \ \mathbb{R}_+^n, \quad u\big|_{\partial\mathbb{R}_+^n}^{\text{nt.lim}} = f \ \text{ a.e. on } \ \mathbb{R}^{n-1}. \tag{4.1}$$

Moreover, there exists a finite constant $C = C(L, n) > 0$ such that

$$[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}_+^n, \mathbb{C}^M)} \leq C\, C_\omega(1 + C_\omega)[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \tag{4.2}$$

and $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$ with $u\big|_{\partial\mathbb{R}_+^n} = f$.

Proof. Let $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and define $u(x', t) := (P_t^L * f)(x')$ for every $(x', t) \in \mathbb{R}_+^n$. By (2.20) and Theorem 3.3(b), u satisfies all properties listed in (4.1). To prove the estimate in (4.2), we first notice that for any $(x', t) \in \mathbb{R}_+^n$, we can write

$$(P_t^L * f)(x') = \int_{\mathbb{R}^{n-1}} P_t^L(x' - y') f(y')\, dy' = \int_{\mathbb{R}^{n-1}} t^{1-n} P^L\left(\frac{x' - y'}{t}\right) f(y')\, dy'$$

$$= \int_{\mathbb{R}^{n-1}} P^L(z') f(x' - tz')\, dz'. \tag{4.3}$$

Fix now $x = (x', t)$ and $y = (y', s)$ arbitrary in $\mathbb{R}_+^n$, and set $r := |x - y|$. By (3.5) and the fact that ω is non-decreasing we obtain

$$|u(x', t) - u(y', s)| = |(P_t^L * f)(x') - (P_s^L * f)(y')|$$

$$\leq C \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z'|^2)^{n/2}} |f(x' - tz') - f(y' - sz')|\, dz'$$

$$\leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z'|^2)^{n/2}} \omega((1 + |z'|)r)\, dz'$$

$$\leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \int_0^\infty \frac{1}{(1 + \lambda^2)^{n/2}} \omega\big((1 + \lambda)r\big) \lambda^{n-1} \frac{d\lambda}{\lambda}$$

$$\leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \left(\int_0^1 \omega(2r)\, \lambda^{n-1} \frac{d\lambda}{\lambda} + \int_1^\infty \frac{\omega(2\lambda r)}{\lambda} \frac{d\lambda}{\lambda}\right)$$

$$= C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \left(\omega(2r) + 2r \int_{2r}^\infty \frac{\omega(\lambda)}{\lambda} \frac{d\lambda}{\lambda}\right)$$

$$\leq C\, C_\omega(1 + C_\omega)\omega(r)[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \tag{4.4}$$

where in the last inequality we have used (2.2) and (2.5). Hence, (4.2) holds. In particular, $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ by Lemma 2.1(d). This and the fact that $u\big|_{\partial\mathbb{R}^n_+}^{\mathrm{nt.lim}} = f$ a.e. in $\mathbb{R}^{n-1}$ with $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then prove that, indeed, $u\big|_{\partial\mathbb{R}^n_+} = f$.   □

The result below is the main tool in the proof of existence of solutions for the generalized Morrey-Campanato Dirichlet problem.

Proposition 4.2. *Let L be a constant complex coefficient system as in (1.1) satisfying the strong ellipticity condition stated in (1.2), and let ω be a growth function satisfying (2.2). Given $1 \le p < \infty$, let $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and define $u(x', t) := (P^L_t * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. Then u is meaningfully defined via an absolutely convergent integral and satisfies*

$$u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \quad Lu = 0 \ \ in \ \ \mathbb{R}^n_+, \quad u\big|_{\partial\mathbb{R}^n_+}^{\mathrm{nt.lim}} = f \ \ a.e. \ on \ \ \mathbb{R}^{n-1}. \tag{4.5}$$

Moreover, for every $q \in (0, \infty]$ there exists a finite constant $C = C(L, n, p, q) > 0$ such that

$$\|u\|_{**}^{(\omega,q)} \le C(C_\omega)^4 \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{4.6}$$

Furthermore, the same is true if $\|\cdot\|_{**}^{(\omega,q)}$ is replaced by $\|\cdot\|_{**}^{(\omega,\exp)}$.

Proof. Given $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, if $u(x', t) := (P^L_t * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$, from (2.20) and Theorem 3.3(b) we see that u satisfies all properties listed in (4.5).

Next, having fixed an arbitrary exponent $q \in (0, \infty)$, based on Proposition 3.4(d), (3.13), Proposition 3.4(e), (3.12), (2.16), (2.18) and (2.2) we may write

$$\begin{aligned}
\|u\|_{**}^{(\omega,\infty)} &\le C\, C_\omega \|u\|_{**}^{(\omega,q)} \le C\, C_\omega \|u\|_{**}^{(\omega,\exp)} \\
&\le C(C_\omega)^3 \|u\|_{**}^{(\omega,2)} \le C(C_\omega)^3 \sup_{t>0} \frac{1}{\omega(t)} \int_1^\infty \mathrm{osc}_1(f, st)\frac{ds}{s^2} \\
&= C(C_\omega)^3 \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \sup_{t>0} \frac{t}{\omega(t)} \int_t^\infty \omega(s)\frac{ds}{s^2} \\
&\le C(C_\omega)^4 \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)},
\end{aligned} \tag{4.7}$$

which proves (4.6) and the corresponding estimate for $\|u\|_{**}^{(\omega,\exp)}$.   □

5. A FATOU-TYPE RESULT AND UNIQUENESS OF SOLUTIONS

We shall now prove a Fatou-type result which is going to be the main ingredient in establishing the uniqueness of solutions for the boundary value problems we are presently considering. More precisely, the following result establishes that any solution in $\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ can be obtained as a convolution of its trace with the associated Poisson kernel.

Proposition 5.1. *Let L be a constant complex coefficient system as in (1.1) satisfying the strong ellipticity condition stated in (1.2), and let ω be a growth function satisfying*

(2.2). *If* $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ *is a function satisfying* $Lu = 0$ *in* $\mathbb{R}^n_+$, *then* $u|_{\partial \mathbb{R}^n_+} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *and*

$$u(x', t) = \left(P^L_t * \left(u|_{\partial \mathbb{R}^n_+} \right) \right)(x'), \qquad \forall \, (x', t) \in \mathbb{R}^n_+. \tag{5.1}$$

Proof. Let $u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfy $Lu = 0$ in $\mathbb{R}^n_+$. By Lemma 2.1(d), it follows that u can be continuously extended to a function (which we call again u) $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$. In particular, the trace $u|_{\partial \mathbb{R}^n_+}$ is well-defined and belongs to the space $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$. To proceed, fix an arbitrary $\varepsilon > 0$ and define $u_\varepsilon = u(\cdot + \varepsilon e_n)$ in $\mathbb{R}^n_+$, where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$. Then, by design, $u_\varepsilon \in \mathscr{C}^\infty(\overline{\mathbb{R}^n_+}, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, $Lu_\varepsilon = 0$ in $\mathbb{R}^n_+$, and $[u_\varepsilon]_{\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)}$. Moreover, using Proposition 3.4(c) and (2.4) we obtain

$$\sup_{(x', t) \in \mathbb{R}^n_+} |(\nabla u_\varepsilon)(x', t)| = \sup_{(x', t) \in \mathbb{R}^n_+} |(\nabla u)(x', t + \varepsilon)|$$

$$\leq C [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \sup_{(x', t) \in \mathbb{R}^n_+} \frac{\omega(t + \varepsilon)}{t + \varepsilon}$$

$$\leq C \, C_\omega [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \frac{\omega(\varepsilon)}{\varepsilon}. \tag{5.2}$$

This implies that ∇u_ε is bounded in $\mathbb{R}^n_+$, hence $u_\varepsilon \in W^{1,2}_{\mathrm{bdd}}(\mathbb{R}^n_+, \mathbb{C}^M)$.

Define next $f_\varepsilon(x') := u(x', \varepsilon) \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and $w_\varepsilon(x', t) := (P^L_t * f_\varepsilon)(x')$ for each $(x', t) \in \mathbb{R}^n_+$. Then, Proposition 4.1 implies that that $w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, $Lw_\varepsilon = 0$ in $\mathbb{R}^n_+$ and $w_\varepsilon|_{\partial \mathbb{R}^n_+} = f_\varepsilon$. Moreover, for every pair of points $x', y' \in \mathbb{R}^{n-1}$ we have, on the one hand,

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)| \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \, \omega(|x' - y'|), \tag{5.3}$$

and, on the other hand, using the Mean Value Theorem and Proposition 3.4(c),

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)|$$

$$\leq |x' - y'| \sup_{z' \in [x', y']} |(\nabla u)(z', \varepsilon)|$$

$$\leq C \, |x' - y'| \, [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \frac{\omega(\varepsilon)}{\varepsilon}. \tag{5.4}$$

Therefore, we conclude that $f_\varepsilon \in \dot{\mathscr{C}}^\Psi(\mathbb{R}^{n-1}, \mathbb{C}^M)$, with norm depending (unfavorably) on the parameter ε, where the growth function Ψ is given by

$$\Psi(t) := \min \left\{ t, \frac{\omega(t)}{\omega(1)} \right\} = \begin{cases} t & \text{if } t \leq 1, \\ \omega(t)/\omega(1) & \text{if } t > 1. \end{cases} \tag{5.5}$$

For every $R > 1$ and $x = (x', t)$, let us now invoke (3.11), (2.16) and (2.18), with Ψ in place of ω, to write

$$
\int_{B(0,R)\cap\mathbb{R}^n_+} |(\nabla w_\varepsilon)(x)|^2 \, dx \leq \int_{B(0,R)\cap\mathbb{R}^n_+} \left(\frac{C}{t} \int_1^\infty \operatorname{osc}_1(f_\varepsilon; st)\frac{ds}{s^2} \right)^2 dx
$$

$$
\leq C \, \|f_\varepsilon\|_{\mathscr{E}^{\Psi,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \int_{B(0,R)\cap\mathbb{R}^n_+} \left(\int_1^\infty \frac{\Psi(st)}{st}\frac{ds}{s} \right)^2 dx
$$

$$
\leq C \, \|f_\varepsilon\|_{\mathscr{E}^{\Psi,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} R^{n-1} \int_0^R \left(\int_t^\infty \frac{\Psi(s)}{s}\frac{ds}{s} \right)^2 dt, \tag{5.6}
$$

and then use (2.2) to observe that

$$
\int_0^R \left(\int_t^\infty \frac{\Psi(s)}{s}\frac{ds}{s} \right)^2 dt \leq \int_0^1 \left(\int_t^1 \frac{ds}{s} + \frac{1}{\omega(1)} \int_1^\infty \frac{\omega(s)}{s}\frac{ds}{s} \right)^2 dt
$$

$$
+ \int_1^R \left(\frac{1}{\omega(1)} \int_1^\infty \frac{\omega(s)}{s}\frac{ds}{s} \right)^2 dt
$$

$$
\leq \int_0^1 \left(\log(1/t) + C_\omega \right)^2 dt + (R-1)(C_\omega)^2 < \infty, \tag{5.7}
$$

Collectively, (5.6) and (5.7) show that $w_\varepsilon \in W^{1,2}_{\mathrm{bdd}}(\mathbb{R}^n_+, \mathbb{C}^M)$.

We now consider $v_\varepsilon := u_\varepsilon - w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M) \cap W^{1,2}_{\mathrm{bdd}}(\mathbb{R}^n_+, \mathbb{C}^M)$, which satisfies $Lv_\varepsilon = 0$ in $\mathbb{R}^n_+$ and $v_\varepsilon|_{\partial\mathbb{R}^n_+} = 0$. Hence, $\operatorname{Tr} v_\varepsilon = 0$ on $\mathbb{R}^{n-1}$ (see (3.3)) and for each $x \in \mathbb{R}^n$ we have

$$
|v_\varepsilon(x)| \leq |v_\varepsilon(x) - v_\varepsilon(0)| + |v_\varepsilon(0)|
$$

$$
\leq \max\left\{ [v_\varepsilon]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}, |v_\varepsilon(0)| \right\}(1 + \omega(|x|)). \tag{5.8}
$$

From this and Proposition 3.2 we then conclude that

$$
\sup_{\mathbb{R}^n_+\cap B(0,r)} |\nabla v_\varepsilon| \leq \frac{C}{r} \sup_{\mathbb{R}^n_+\cap B(0,2r)} |v_\varepsilon| \leq C_\varepsilon \frac{1 + \omega(2r)}{r}, \tag{5.9}
$$

and from Lemma 2.1(c) we see that the right side of (5.9) tends to 0 as $r \to \infty$. This forces $\nabla v_\varepsilon \equiv 0$, and since $v_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M) \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $v_\varepsilon|_{\partial\mathbb{R}^n_+} = 0$ we ultimately conclude that $v_\varepsilon \equiv 0$. Consequently,

$$
u(x', t + \varepsilon) = (P_t^L * f_\varepsilon)(x'), \qquad \forall\, (x', t) \in \mathbb{R}^n_+. \tag{5.10}
$$

Since, as noted earlier, $u|_{\partial\mathbb{R}^n_+} \in \mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, for every $x' \in \mathbb{R}^{n-1}$ and $\varepsilon > 0$ we may now write

$$
\begin{aligned}
\left| u(x', t+\varepsilon) - \left(P_t^L * (u|_{\partial\mathbb{R}^n_+}) \right)(x') \right| &= \left| \left(P_t^L * (f_\varepsilon - u|_{\partial\mathbb{R}^n_+}) \right)(x') \right| \\
&\le \|P_t^L\|_{L^1(\mathbb{R}^{n-1})} \sup_{y' \in \mathbb{R}^{n-1}} \left| f_\varepsilon(y') - u|_{\partial\mathbb{R}^n_+}(y') \right| \\
&= \|P^L\|_{L^1(\mathbb{R}^{n-1})} \sup_{y' \in \mathbb{R}^{n-1}} \left| u(y', \varepsilon) - u|_{\partial\mathbb{R}^n_+}(y') \right| \\
&\le \|P^L\|_{L^1(\mathbb{R}^{n-1})} [u]_{\mathscr{C}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)} \, \omega(\varepsilon).
\end{aligned}
\tag{5.11}
$$

From (3.5) we know that $\|P^L\|_{L^1(\mathbb{R}^{n-1})} < \infty$. Upon letting $\varepsilon \to 0^+$ and using that ω vanishes in the limit at the origin, we see that (5.11) implies (5.1). This finishes the proof of Proposition 5.1. $\square$

6. Well-Posedness Results

We are now ready to prove well-posedness results. We first consider the case in which the boundary data belong to generalized Hölder spaces and we note that, in such a scenario, the only requirement on the growth function is (2.2).

Theorem 6.1. *Let L be a constant complex coefficient $M \times M$ system as in (1.1) satisfying the strong ellipticity condition (1.2). Also, let ω be a growth function satisfying (2.2). Then the generalized Hölder Dirichlet problem for L in $\mathbb{R}^n_+$, formulated as*

$$
\begin{cases}
u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\
Lu = 0 \quad in \ \ \mathbb{R}^n_+, \\
[u]_{\mathscr{C}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} < \infty, \\
u|^{\mathrm{lim}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) \ \ on \ \ \mathbb{R}^{n-1},
\end{cases}
\tag{6.1}
$$

is well-posed. More specifically, there exists a unique solution which is given by

$$
u(x', t) = (P_t^L * f)(x'), \qquad \forall\, (x', t) \in \mathbb{R}^n_+,
\tag{6.2}
$$

where P^L denotes the Poisson kernel for the system L in $\mathbb{R}^n_+$ from Theorem 3.3. In addition, u extends to a function in $\mathscr{C}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$, and there exists a finite constant $C = C(n, L, \omega) \ge 1$ such that

$$
C^{-1}[f]_{\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \le [u]_{\mathscr{C}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \le C[f]_{\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)}.
\tag{6.3}
$$

Proof. Given $f \in \mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, define u as in (6.2). Proposition 4.1 then implies that u satisfies all conditions in (6.1). Also, u extends to a function in $\mathscr{C}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$, and the second inequality in (6.3) holds. Moreover, (2.7) yields

$$
[f]_{\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} = [u|_{\partial\mathbb{R}^n_+}]_{\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)} \le [u]_{\mathscr{C}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)} \le 2C_\omega[u]_{\mathscr{C}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)},
\tag{6.4}
$$

so that the first inequality in (6.3) follows.

It remains to prove that the solution is unique. However, this follows at once from Proposition 5.1. Indeed, the first three conditions in (6.1) imply (5.1) and since $u|_{\partial\mathbb{R}^n_+} = f$ we conclude that necessarily $u(x', t) = \left(P^L_t * f\right)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. $\square$

Here is the well-posedness for the generalized Morrey-Campanato Dirichlet problem. In this case, the growth function is assumed to satisfy both (2.1) and (2.2).

Theorem 6.2. *Let L be a constant complex coefficient $M \times M$ system as in (1.1) satisfying the strong ellipticity condition (1.2). Fix $p \in [1, \infty)$ along with $q \in (0, \infty]$, and let ω be a growth function satisfying (2.1) and (2.2). Then the generalized Morrey-Campanato Dirichlet problem for L in $\mathbb{R}^n_+$, namely*

$$\begin{cases} u \in \mathscr{C}^\infty(\mathbb{R}^n_+, \mathbb{C}^M), \\ Lu = 0 \quad in \quad \mathbb{R}^n_+, \\ \|u\|^{(\omega,q)}_{**} < \infty, \\ u\big|^{\mathrm{nt.lim}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad a.e. \ on \ \mathbb{R}^{n-1}, \end{cases} \tag{6.5}$$

is well-posed. More specifically, there exists a unique solution which is given by

$$u(x', t) = (P^L_t * f)(x'), \qquad \forall\, (x', t) \in \mathbb{R}^n_+, \tag{6.6}$$

where P^L denotes the Poisson kernel for the system L in $\mathbb{R}^n_+$ from Theorem 3.3. Moreover, with W defined as in (2.3), the solution u extends to a function in $\dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$, and there exists a finite constant $C = C(n, L, \omega, p, q) \geq 1$ for which

$$C^{-1}\|f\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \|u\|^{(\omega,q)}_{**} \leq C\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \tag{6.7}$$

Furthermore, all results remain valid if $\|\cdot\|^{(\omega,q)}_{**}$ is replaced everywhere by $\|\cdot\|^{(\omega,\exp)}_{**}$.

Proof. Having fixed $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, if u is defined as in (6.6) then Proposition 4.2 implies the validity of all conditions in (6.5) and also of the second inequality in (6.7) (even replacing q by exp). In the case $q = \infty$ we invoke Proposition 3.4(f) to obtain that $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ in the sense of Lemma 2.1(d). Note that we also have

$$[u|_{\partial\mathbb{R}^n_+}]_{\dot{\mathscr{C}}^W(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq [u]_{\mathscr{C}^W(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)} \leq 2C_W[u]_{\dot{\mathscr{C}}^W(\mathbb{R}^n_+, \mathbb{C}^M)} \leq C(C_\omega)^4\|u\|^{(\omega,\infty)}_{**} \tag{6.8}$$

thanks to (2.7) (for the growth function W), Lemma 2.2, and (3.20).

Given that, on the one hand, $u|_{\partial\mathbb{R}^n_+} = u\big|^{\mathrm{nt.lim}}_{\partial\mathbb{R}^n_+}$ everywhere in $\mathbb{R}^{n-1}$ due to the fact that $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, and that, on the other hand, $u\big|^{\mathrm{nt.lim}}_{\partial\mathbb{R}^n_+} = f$ a.e. in $\mathbb{R}^{n-1}$, we conclude that $u|_{\partial\mathbb{R}^n_+} = f$ a.e. in $\mathbb{R}^{n-1}$. In addition, (2.19) (applied to W), Lemma 2.2, and (6.8) permit us to estimate

$$\|f\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|u|_{\partial\mathbb{R}^n_+}\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \sqrt{n-1}\, C_W[u|_{\partial\mathbb{R}^n_+}]_{\dot{\mathscr{C}}^W(\mathbb{R}^{n-1}, \mathbb{C}^M)}$$

$$\leq C(C_\omega)^6\|u\|^{(\omega,\infty)}_{**} \leq C(C_\omega)^7\|u\|^{(\omega,q)}_{**} \leq C(C_\omega)^7\|u\|^{(\omega,\exp)}_{**}, \tag{6.9}$$

where $0 < q < \infty$ and where we have also used Proposition 3.4(d) and (3.13).

To prove that the solution is unique, we note that having $\|u\|_{**}^{(\omega,q)} < \infty$ for a given $q \in (0,\infty]$, or even $\|u\|_{**}^{(\omega,\exp)} < \infty$, implies that $\|u\|_{**}^{(\omega,\infty)} < \infty$ by Proposition 3.4(d) and (3.13). Having established this, Proposition 3.4(f) applies and yields that $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$. Consequently, $u|_{\partial\mathbb{R}_+^n} = u|_{\partial\mathbb{R}_+^n}^{\mathrm{nt.lim}}$ everywhere in $\mathbb{R}^{n-1}$, and if we also take into account the boundary condition from (6.5), we conclude that $u|_{\partial\mathbb{R}_+^n} = f$ a.e. on $\mathbb{R}^{n-1}$. Moreover, since Lemma 2.2 ensures that W is a growth function satisfying (2.2), we may invoke Proposition 5.1 to write

$$u(x', t) = \left(P_t^L * \left(u|_{\partial\mathbb{R}_+^n}\right)\right)(x') = \left(P_t^L * f\right)(x'), \qquad \forall\, (x', t) \in \mathbb{R}_+^n. \tag{6.10}$$

The proof of the theorem is therefore finished. $\qquad\square$

Remark 6.3. Theorems 6.1 and 6.2 are closely related. To elaborate in this, fix a growth function ω satisfying (2.1) and (2.2). From (2.19) and Proposition 4.2 it follows that, given any $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$, the unique solution of the boundary value problem (6.1), i.e., $u(x', t) = (P_t^L * f)(x')$ for $(x', t) \in \mathbb{R}_+^n$, also solves (6.5), regarding now f as a function in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ (cf. (2.20)) with $p \in [1,\infty)$ and $q \in (0,\infty]$ arbitrary (and even with $\|\cdot\|_{**}^{(\omega,q)}$ replaced by $\|\cdot\|_{**}^{(\omega,\exp)}$). As such, u satisfies (6.7) whenever (2.1) holds.

This being said, the fact that $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ does not guarantee, in general, that the corresponding solution satisfies $u \in \dot{\mathscr{C}}^\omega(\mathbb{R}_+^n, \mathbb{C}^M)$, even though we have established above that the solution to the boundary value problem (6.5) belongs to $\dot{\mathscr{C}}^W(\mathbb{R}_+^n, \mathbb{C}^M)$. Note that, as seen from (1.9)-(1.10) and (2.12), the space $\dot{\mathscr{C}}^W(\mathbb{R}_+^n, \mathbb{C}^M)$ contains $\dot{\mathscr{C}}^\omega(\mathbb{R}_+^n, \mathbb{C}^M)$.

This aspect is fully clarified with the help of Example 6.4 discussed further below, where we construct some growth function ω satisfying (2.1), (2.2), and for which the space $\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C})$ is strictly bigger than $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C})$. Its relevance for the issue at hand is as follows. Consider the boundary problem (6.5) formulated with L being the Laplacian in $\mathbb{R}^n$ and with $f \in \mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C}) \setminus \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C})$ as boundary datum. Its solution u then necessarily satisfies $u \notin \dot{\mathscr{C}}^\omega(\mathbb{R}_+^n, \mathbb{C})$, for otherwise Lemma 2.1(d) would imply $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$ and since $u|_{\partial\mathbb{R}_+^n} = u|_{\partial\mathbb{R}_+^n}^{\mathrm{nt.lim}} = f$ a.e. on $\mathbb{R}^{n-1}$ and f is continuous in $\mathbb{R}^{n-1}$ we would conclude that f coincides everywhere with $u|_{\partial\mathbb{R}_+^n} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C})$, a contradiction.

In spite of the previous remark, Theorem 1.4 states that the boundary problems (6.1) and (6.5) are actually equivalent under the stronger assumption (1.18) on the growth function. Here is the proof of Theorem 1.4.

Proof of Theorem 1.4. We start with the observation that (1.18) and Lemma 2.2 yield $C_0^{-1} W(t) \le \omega(t) \le C_0 W(t)$ for each $t \in (0,\infty)$. Therefore,

$$\dot{\mathscr{C}}^\omega(\mathbb{R}_+^n, \mathbb{C}^M) = \dot{\mathscr{C}}^W(\mathbb{R}_+^n, \mathbb{C}^M), \quad \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}_+^n}, \mathbb{C}^M) = \dot{\mathscr{C}}^W(\overline{\mathbb{R}_+^n}, \mathbb{C}^M), \tag{6.11}$$

and

$$\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathscr{E}^{W,p}(\mathbb{R}^{n-1}, \mathbb{C}^M), \tag{6.12}$$

as vector spaces, with equivalent norms.

Having made these identifications, we now proceed to observe that (a) follows directly from Theorem 6.1, while (b) is implied by Theorem 6.2 with the help of (6.11) and (6.12). To deal with (c), we first observe that the left-to-right inclusion follows from Lemma 2.4(b), whereas (2.19) provides the accompanying estimate for the norms. For the converse inclusion, fix $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and set $u(x', t) := (P_t^L * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. Theorem 6.2 and (6.11) then imply that $u \in \dot{\mathscr{C}}^{\omega}(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$. Introduce $\widetilde{f} := u|_{\partial\mathbb{R}^n_+}$ and note that $\widetilde{f} \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ with $\widetilde{f} = f$ a.e. on $\mathbb{R}^{n-1}$. Then $u(x', t) = (P_t^L * \widetilde{f})(x')$ and, thanks to (6.3), (3.23), and (6.7), we have

$$[\widetilde{f}]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C[u]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^n_+, \mathbb{C}^M)} \leq C \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \, . \tag{6.13}$$

This completes the treatment of (c), and finishes the proof of Theorem 1.4.   □

We are now in a position to give the proof of Corollary 1.5.

Proof of Corollary 1.5. We start by observing that (1.27) is a direct consequence of Proposition 3.4(h). In particular, the last three equalities in (1.28) follow at once. Also, the fact that the second set in the first line of (1.28) is contained in $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ is a consequence of Lemma 2.1(d). Finally, given any $f \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, if u is the solution of (1.19) corresponding to this choice of boundary datum, then $u|_{\partial\mathbb{R}^n_+} = f$ and u also satisfies the required conditions to be an element in the second set displayed in (1.28).   □

The following example shows that conditions (2.1) and (2.2) do not imply (1.24).

Example 6.4. Fix two real numbers $\alpha, \beta \in (0, 1)$ and consider the growth function $\omega : (0, \infty) \to (0, \infty)$ defined for each $t > 0$ as

$$\omega(t) := \begin{cases} t^{\alpha}, & \text{if } t \leq 1, \\ 1 + (\log t)^{\beta}, & \text{if } t > 1. \end{cases} \tag{6.14}$$

Clearly, ω satisfies (2.1), and we also claim that ω satisfies (2.2). Indeed, for $t \leq 1$,

$$\int_t^{\infty} \frac{\omega(s)}{s} \frac{ds}{s} = \int_t^1 s^{\alpha-1} \frac{ds}{s} + \int_1^{\infty} \frac{1 + (\log s)^{\beta}}{s^2} \, ds \leq C(t^{\alpha-1} + 1) \leq 2C \, t^{\alpha-1}. \tag{6.15}$$

For $t \in [1, \infty)$, define

$$F(t) := \frac{t \displaystyle\int_t^{\infty} \frac{\omega(s)}{s} \frac{ds}{s}}{\omega(t)} = \frac{\displaystyle\int_t^{\infty} \frac{1 + (\log s)^{\beta}}{s^2 \, ds}}{\dfrac{1 + (\log t)^{\beta}}{t}}, \tag{6.16}$$

which is a continuous function in $[1, \infty)$ and satisfies $F(1) < \infty$. Moreover, using L'Hôpital's rule,

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{-(1 + (\log t)^{\beta})}{\beta(\log t)^{\beta-1} - (1 + (\log t)^{\beta})} = 1. \tag{6.17}$$

Hence, F is bounded, which amounts to having ω satisfy (2.2). The function W, defined as in (2.3), is currently given by

$$W(t) = \begin{cases} \dfrac{1}{\alpha}t^\alpha, & \text{if } t \leq 1, \\[2mm] \dfrac{1}{\alpha} + \dfrac{1}{\beta+1}(\log t)^{\beta+1} + \log t, & \text{if } t > 1. \end{cases} \tag{6.18}$$

Since (1.18) would imply $W(t) \leq C\omega(t)$ which is not the case for t sufficiently large, we conclude that the growth function ω satisfies (2.1) and (2.2) but it does not satisfy (1.18).

For this choice of ω, we now proceed to check that $\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C}) \neq \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C})$. To this end, consider the function

$$f(x) := \log_+ |x_1|, \qquad \forall\, x = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \tag{6.19}$$

where $\log_+ t := \max\{0, \log t\}$. With $e_1 =: (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$ we then have

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x-y|)} \geq \lim_{x_1 \to \infty} \frac{|f(x_1 e_1) - f(e_1)|}{\omega(|x_1 e_1 - e_1|)} = \lim_{x_1 \to \infty} \frac{\log x_1}{1 + (\log(x_1 - 1))^\beta} = \infty, \tag{6.20}$$

since $\beta < 1$. This means that $f \notin \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C})$. To prove that $f \in \mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C})$, consider $\widetilde{Q} := (a, b) \times Q \subset \mathbb{R}^{n-1}$, where Q is an arbitrary cube in $\mathbb{R}^{n-2}$ and $a, b \in \mathbb{R}$ are arbitrary numbers satisfying $a < b$. Then,

$$\|f\|_{\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C})} \leq \sup_{\widetilde{Q} \subset \mathbb{R}^{n-1}} \frac{1}{\omega(\ell(\widetilde{Q}))} \fint_{\widetilde{Q}} \fint_{\widetilde{Q}} |f(x) - f(y)| \, dx \, dy$$

$$\leq \sup_{a < b} \frac{1}{\omega(b-a)} H(a, b), \tag{6.21}$$

where

$$H(a, b) := \fint_a^b \fint_a^b \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1. \tag{6.22}$$

We shall now prove that the right-hand side of (6.21) is finite considering several different cases.

Case I: $1 \leq a < b$. In this scenario, define

$$G(\lambda) := 1 + 2\lambda - 2\lambda(\lambda+1)\log\left(1 + \frac{1}{\lambda}\right), \quad \forall\, \lambda > 0. \tag{6.23}$$

Note that G is continuous in $(0, \infty)$, $G(0) = 1$, and by L'Hôpital's rule, $\lim_{\lambda \to \infty} G(\lambda) = 0$, hence G is bounded. Also,

$$H(a, b) = \frac{b^2 - a^2 - 2ab\log(b/a)}{(b-a)^2} = G\big(a/(b-a)\big). \tag{6.24}$$

Consequently, whenever $b - a \geq 1$ we have

$$H(a, b) = G\big(a/(b-a)\big) \leq C \leq C\big(1 + (\log(b-a))^\beta\big) = C\omega(b-a). \tag{6.25}$$

Again by L'Hôpital's rule, $\lim_{\lambda\to\infty} \lambda^\alpha G(\lambda) = 0$, hence $\lambda^\alpha G(\lambda) \leq C$ for every $\lambda > 0$. Therefore, whenever $0 < b - a < 1$ we may write

$$H(a,b) = G\big(a/(b-a)\big) \leq C\left(\frac{b-a}{a}\right)^\alpha \leq C(b-a)^\alpha = C\omega(b-a). \tag{6.26}$$

All these show that $H(a,b) \leq C\omega(b-a)$ in this case.

Case II: $a < b \leq -1$. This case is analogous to the previous one by symmetry.

Case III: $-1 \leq a < b \leq 1$. This case is straightforward since $H(a,b) = 0$, given that $\log_+ |x_1| = \log_+ |y_1| = 0$ whenever $a < x_1, y_1 < b$.

Case IV: $-1 < a < 1 < b$. In this case we obtain

$$H(a,b) = \frac{1}{(b-a)^2} \int_1^b \int_1^b |\log x_1 - \log y_1|\, dx_1\, dy_1$$

$$+ \frac{1}{(b-a)^2} \int_a^1 \int_1^b \log x_1\, dx_1\, dy_1 + \frac{1}{(b-a)^2} \int_1^b \int_a^1 \log y_1\, dx_1\, dy_1$$

$$\leq \frac{(b-1)^2}{(b-a)^2} H(1,b) + 2\frac{(1-a)(b\log b - b + 1)}{(b-a)^2}. \tag{6.27}$$

For the first term in the right-hand side of (6.27), we use (6.25) and (6.26) (written with $a := 1$) and obtain, keeping in mind that in this case $a < 1$,

$$\frac{(b-1)^2}{(b-a)^2} H(1,b) \leq C\omega(b-1)\left(\frac{b-1}{b-a}\right)^2 \leq C\omega(b-a). \tag{6.28}$$

To bound the second term in the right-hand side of (6.27), we first use the fact that $1 - a < 2$ and $\log t \leq t - 1$ for every $t \geq 1$ to obtain

$$\frac{(1-a)(b\log b - b + 1)}{(b-a)^2} \leq 2\frac{b(b-1) - b + 1}{(b-a)^2} \leq 2\left(\frac{b-1}{b-a}\right)^2 \leq 2$$

$$\leq 2\big(1 + (\log(b-a))^\beta\big) = 2\omega(b-a), \tag{6.29}$$

whenever $b - a \geq 1$. To study the case when $b - a < 1$, bring in the auxiliary function

$$\widetilde{G}(\lambda) := \frac{\lambda\log\lambda - \lambda + 1}{(\lambda-1)^{1+\alpha}}, \qquad \forall \lambda > 1. \tag{6.30}$$

By L'Hôpital's rule, $\lim_{\lambda\to 1+} \widetilde{G}(\lambda) = 0$, hence $\widetilde{G}(\lambda) \leq C$ for each $\lambda \in (1,2]$. If $b - a < 1$, we clearly have $1 < b \leq 2$ which, in turn, permits us to estimate

$$\frac{(1-a)(b\log b - b + 1)}{(b-a)^2} = \frac{(1-a)(b-1)^{1+\alpha}G(b)}{(b-a)^2}$$

$$\leq \frac{C(b-1)^{1+\alpha}}{b-a} \leq C(b-a)^\alpha = C\omega(b-a). \tag{6.31}$$

Consequently, we have obtained that $H(a,b) \leq C\omega(b-a)$ in this case as well.

Case V: $a < -1 < b < 1$. This is analogue to Case IV, again by symmetry.

Case VI: $a < -1$, $b > 1$. We break the interval (a, b) into two intervals $(a, 0)$ and $(0, b)$ to obtain

$$H(a, b) \leq \frac{1}{(b-a)^2}(I + II), \tag{6.32}$$

where, using Case IV and Case V,

$$I := \int_a^0 \int_a^0 \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1 + \int_0^b \int_0^b \big| \log_+ x_1 - \log_+ y_1 \big| \, dx_1 \, dy_1$$

$$= H(a, 0)(0 - a)^2 + H(0, b)(b - 0)^2 \leq C|a|^2 \omega(|a|) + C b^2 \omega(b)$$

$$\leq 2C(b-a)^2 \omega(b-a). \tag{6.33}$$

Similarly, by Case IV,

$$II := \int_a^0 \int_0^b \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1 + \int_0^b \int_a^0 \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1$$

$$\leq 2\big(\max\{|a|, b\} - 0 \big)^2 H\big(\max\{|a|, b\}, 0 \big)$$

$$\leq 2C \max\{|a|, b\}^2 \, \omega(\max\{|a|, b\})$$

$$\leq 2C(b-a)^2 \, \omega(b-a). \tag{6.34}$$

Thus, $H(a, b) \leq C\omega(b-a)$ in this case also.

Collectively, the results in Cases I-VI prove that $f \in \mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C})$.

Appendix A. John-Nirenberg's Inequality Adapted to Growth Functions

In what follows we assume that all cubes are half-open, that is, they can be written in the form $Q = [a_1, a_1 + \ell(Q)) \times \cdots \times [a_{n-1}, a_{n-1} + \ell(Q))$ with $a_i \in \mathbb{R}^{n-1}$ and $\ell(Q) > 0$. Notice that since ∂Q has Lebesgue measure zero the half-open assumption is harmless. Subdividing dydically yields the collection of (half-open) dyadic-subcubes of a given cube Q, which we shall denote by $\mathbb{D}_Q$. For the following statement, and with the aim of considering global results, it is also convenient to consider the case $Q = \mathbb{R}^{n-1}$ in which scenario we take $\mathbb{D}_Q$ to be the classical dyadic grid generated by $[0, 1)^{n-1}$, or any other dyadic grid. Let us also recall the definition of the dyadic Hardy-Littlewood maximal function localized to a given cube Q, i.e.,

$$\big(M_Q^d f\big)(x) := \sup_{x \in Q' \in \mathbb{D}_Q} \fint_{Q'} |f(y')| \, dy', \qquad x \in Q, \tag{A.1}$$

for each $f \in L^1(Q)$. The following result is an extension of the John-Nirenberg inequality obtained in [4, 3] (when $\omega \equiv 1$) adapted to our growth function.

Lemma A.1. *Let $F \in L^2_{\text{loc}}(\mathbb{R}^n_+)$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function. Let $Q_0 \subset \mathbb{R}^{n-1}$ be an arbitrary half-open cube, or $Q_0 = \mathbb{R}^{n-1}$. Assume that there are numbers $\alpha \in (0, 1)$ and $N \in (0, \infty)$ such that*

$$\left| \left\{ x' \in Q : \frac{1}{\varphi(\ell(Q))} \left(\int_0^{\ell(Q)} \int_{|x'-y'| < \kappa s} |F(y', s)|^2 \, dy' \frac{ds}{s^n} \right)^{1/2} > N \right\} \right| \leq \alpha|Q| \tag{A.2}$$

for every cube $Q \in \mathbb{D}_{Q_0}$ and $\kappa := 1 + 2\sqrt{n-1}$. Then, for every $t > 0$

$$\sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{|Q|} \left| \left\{ x' \in Q : \frac{1}{\varphi(\ell(Q))} \left(\int_0^{\ell(Q)} \int_{|x'-y'|<\kappa s} |F(y',s)|^2 \, dy' \frac{ds}{s^n} \right)^{1/2} > t \right\} \right|$$

$$\leq \frac{1}{\alpha} e^{-\frac{\log(\alpha^{-1})}{N} t}. \quad \text{(A.3)}$$

Hence, for each $q \in (0, \infty)$ there exists a finite constant $C = C(\alpha, q) \geq 1$ such that

$$\sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\varphi(\ell(Q))} \left(\fint_Q \left(\int_0^{\ell(Q)} \int_{|x'-y'|<s} |F(y',s)|^2 \, dy' \frac{ds}{s^n} \right)^{q/2} dx' \right)^{1/q} \leq CN. \quad \text{(A.4)}$$

Moreover, there exists some finite $C = C(\alpha) \geq 1$ such that

$$\sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\varphi(\ell(Q))} \left\| \left(\int_0^{\ell(Q)} \int_{|\cdot-y'|<s} |F(y',s)|^2 \, dy' \frac{ds}{s^n} \right)^{1/2} \right\|_{\exp L, Q} \leq CN. \quad \text{(A.5)}$$

The previous result can be proved using the arguments in [4, 3] with appropriate modifications. Here we present an alternative abstract argument based on ideas that go back to Calderón, as presented in [11] (see also [12, 5]). This also contains as a particular case the classical John-Nirenberg result concerning the exponential integrability of BMO functions.

Proposition A.2. *Let $Q_0 \subset \mathbb{R}^{n-1}$ be an arbitrary half-open cube, or $Q_0 = \mathbb{R}^{n-1}$. For every $Q \in \mathbb{D}_{Q_0}$ assume that there exist two non-negative functions G_Q, $H_Q \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$ such that*

$$G_Q(x') \leq H_Q(x') \quad \text{for almost every } x' \in Q, \quad \text{(A.6)}$$

and, for every $Q' \in \mathbb{D}_Q \setminus \{Q\}$,

$$G_Q(x') \leq G_{Q'}(x') + H_Q(y') \quad \text{for a.e. } x \in Q' \text{ and for a.e. } y' \in \widehat{Q'}, \quad \text{(A.7)}$$

where $\widehat{Q'}$ is the dyadic parent of Q'. For each $\alpha \in (0,1)$ define

$$\mathfrak{m}_\alpha := \sup_{Q \in \mathbb{D}_{Q_0}} \inf \left\{ \lambda > 0 : |\{ x' \in Q : H_Q(x') > \lambda \}| \leq \alpha |Q| \right\}. \quad \text{(A.8)}$$

Then, for every $\alpha \in (0,1)$ one has

$$\sup_{Q \in \mathbb{D}_{Q_0}} \frac{|\{ x \in Q : G_Q(x') > t \}|}{|Q|} \leq \frac{1}{\alpha} e^{-\log(\alpha^{-1}) \frac{t}{\mathfrak{m}_\alpha}}, \qquad \forall \, t > 0. \quad \text{(A.9)}$$

As a consequence,

$$\sup_{Q \in \mathbb{D}_{Q_0}} \|G_Q\|_{\exp L, Q} \leq \frac{1 + \alpha^{-1}}{\log(\alpha^{-1})} \mathfrak{m}_\alpha \quad \text{(A.10)}$$

and for every $q \in (0, \infty)$ there exists a finite constant $C = C(q) > 0$ such that

$$\sup_{Q \in \mathbb{D}_{Q_0}} \left(\fint_Q G_Q(x')^q \, dx' \right)^{1/q} \leq C_q \frac{1}{\alpha^{1/q} \log(\alpha^{-1})} \mathfrak{m}_\alpha. \quad \text{(A.11)}$$

Before proving this result and Lemma A.1, let us illustrate how Proposition A.2 yields the classical John-Nirenberg result regarding the exponential integrability of BMO functions. Concretely, pick $f \in \mathrm{BMO}(\mathbb{R}^{n-1})$. Fix an arbitrary cube Q_0 and for every $Q \in \mathbb{D}_{Q_0}$ define $G_Q := |f - f_Q|$ and $H_Q := 2^{n-1} M_Q^d \big(|f - f_Q| \big)$ (cf. (A.1)). Clearly, (A.6) holds by Lebesgue's Differentiation Theorem. Moreover, for every $Q' \in \mathbb{D}_Q \setminus Q$, $x' \in Q'$, and $y' \in \widehat{Q'}$ we have

$$G_Q(x') \leq |f(x') - f_{Q'}| + |f'_Q - f_Q| \leq G_{Q'}(x') + \fint_{Q'} |f(z') - f_Q|\, dz'$$

$$\leq G_{Q'}(x') + 2^{n-1} \fint_{\widehat{Q'}} |f(z') - f_Q|\, dz' \leq G_{Q'}(x') + H_{Q'}(y'), \quad \text{(A.12)}$$

and (A.7) follows. Going further, by the weak-type $(1,1)$ of the dyadic Hardy-Littlewood maximal function, for every $\lambda > 0$ we may write

$$|\{x' \in Q : H_Q(x') > \lambda\}| \leq \frac{2^{n-1}}{\lambda} \int_Q |f(y') - f_Q|\, dy' \leq \frac{2^{n-1} \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}{\lambda} |Q|. \quad \text{(A.13)}$$

In particular, choosing for instance $\alpha := e^{-1}$, if we use the previous estimate with $\lambda := 2^{n-1} \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}/\alpha$ we obtain $\mathfrak{m}_\alpha \leq 2^{n-1} \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}/\alpha$. Thus, (A.9) yields

$$\frac{|\{x \in Q_0 : |f(x') - f_{Q_0}| > t\}|}{|Q_0|} \leq \frac{1}{\alpha}\, e^{-\frac{\alpha \log(\alpha^{-1})}{2^{n-1}} \frac{t}{\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}}$$

$$= e \cdot e^{-\frac{1}{2^{n-1}e} \frac{t}{\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}} \quad \text{(A.14)}$$

while (A.10) gives

$$\|f - f_{Q_0}\|_{\exp L, Q_0} \leq (1 + e)\, e\, 2^{n-1} \|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})} \quad \text{(A.15)}$$

which are the well-known John-Nirenberg inequalities.

We now turn to the proof of Lemma A.1.

Proof of Lemma A.1. Let F, α, and N be fixed as in the statement of the lemma. For every $Q \in \mathbb{D}_{Q_0}$ and $x' \in \mathbb{R}^{n-1}$, define

$$G_Q(x') := \frac{1}{\varphi(\ell(Q))} \Big(\int_0^{\ell(Q)} \int_{|x'-z'|<s} |F(z',s)|^2\, dz' \frac{ds}{s^n} \Big)^{1/2} \quad \text{(A.16)}$$

and

$$H_Q(x') = \frac{1}{\varphi(\ell(Q))} \Big(\int_0^{\ell(Q)} \int_{|x'-z'|<\kappa s} |F(z',s)|^2\, dz' \frac{ds}{s^n} \Big)^{1/2}. \quad \text{(A.17)}$$

Note that (A.6) is trivially verified since $\kappa > 1$. To proceed, fix $Q' \in \mathbb{D}_Q$ along with $x' \in Q'$ and $y' \in \widehat{Q'}$. If $|x' - z'| < s$ with $\ell(Q') \leq s \leq \ell(Q)$ then

$$|y' - z'| \leq |y' - x'| + |x' - z'| < 2\sqrt{n-1}\, \ell(Q') + s \leq \kappa s. \quad \text{(A.18)}$$

Therefore, since φ is non-decreasing,

$$G_Q(x') \leq \frac{\varphi(\ell(Q'))}{\varphi(\ell(Q))} G_{Q'}(x') + \frac{1}{\varphi(\ell(Q))} \Big(\int_{\ell(Q')}^{\ell(Q)} \int_{|x'-z'|<s} |F(z',s)|^2 \, dz' \frac{ds}{s^n} \Big)^{1/2}$$
$$\leq G_{Q'}(x') + H_Q(y'), \quad \text{(A.19)}$$

establishing (A.7). Moreover, (A.2) gives immediately that $\mathfrak{m}_\alpha \leq N$. Granted this, (A.9), (A.11), and (A.10), (with $\alpha \in (0,1)$ given by (A.2)) prove, respectively (A.3), (A.4), and (A.5). $\qquad\square$

Finally, we give the proof of Proposition A.2.

Proof of Proposition A.2. We start by introducing some notation. Set

$$\Xi(t) := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{|E_Q(t)|}{|Q|} := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{|\{x' \in Q : G_Q(x') > t\}|}{|Q|}, \qquad 0 < t < \infty. \quad \text{(A.20)}$$

Fix $\alpha \in (0,1)$, let $\varepsilon > 0$ be arbitrary, and write $\lambda_\varepsilon = \mathfrak{m}_\alpha + \varepsilon$. From (A.8) it follows that

$$|F_{Q,\varepsilon}| := |\{x' \in Q : H_Q(x') > \lambda_\varepsilon\}| \leq \alpha |Q|, \qquad \forall Q \in \mathbb{D}_{Q_0}. \quad \text{(A.21)}$$

Fix now $Q \in \mathbb{D}_{Q_0}$, $\beta \in (\alpha, 1)$ (we will eventually let $\beta \to 1^+$) and set

$$\Omega_Q := \{x' \in Q : M_Q^d(1_{F_{Q,\varepsilon}})(x') > \beta\}. \quad \text{(A.22)}$$

Note that (A.21) ensures that

$$\fint_Q 1_{F_{Q,\varepsilon}}(y') \, dy' = \frac{|F_{Q,\varepsilon}|}{|Q|} \leq \alpha < \beta, \quad \text{(A.23)}$$

hence we can extract a family of pairwise disjoint stopping-time cubes $\{Q_j\}_j \subset \mathbb{D}_Q \setminus \{Q\}$ so that $\Omega_Q = \cup_j Q_j$ and for every j

$$\frac{|F_{Q,\varepsilon} \cap Q_j|}{|Q_j|} > \beta, \quad \frac{|F_{Q,\varepsilon} \cap Q'|}{|Q'|} \leq \beta, \quad Q_j \subsetneq Q' \in \mathbb{D}_Q. \quad \text{(A.24)}$$

Let $t > \lambda_\varepsilon$ and note that (A.6) gives

$$\lambda_\varepsilon < t < G_Q(x') \leq H_Q(x') \quad \text{for a.e.} \quad x' \in E_Q(t). \quad \text{(A.25)}$$

which implies that

$$\beta < 1 = \mathbf{1}_{F_{Q,\varepsilon}}(x') \leq M_Q^d\big(\mathbf{1}_{F_{Q,\varepsilon}}\big)(x') \quad \text{for a.e.} \quad x' \in E_Q(t). \quad \text{(A.26)}$$

Hence,

$$|E_Q(t)| = |E_Q(t) \cap \Omega_Q| = \sum_j |E_Q(t) \cap Q_j|. \quad \text{(A.27)}$$

For every j, by the second estimate in (A.24) applied to $\widehat{Q}_j$, the dyadic parent of Q_j, we have $|F_{Q,\varepsilon} \cap \widehat{Q}_j|/|\widehat{Q}_j| \leq \beta < 1$, therefore $|\widehat{Q}_j \setminus F_{Q,\varepsilon}|/|\widehat{Q}_j| > 1 - \beta > 0$. In particular, (A.7) guarantees that we can find $\widehat{x}_j' \in \widehat{Q}_j \setminus F_{Q,\varepsilon}$, such that for a.e. $x' \in Q_j$ we have

$$G_Q(x') \leq G_{Q_j}(x') + H_Q(\widehat{x}_j') \leq G_{Q_j}(x') + \lambda_\varepsilon. \quad \text{(A.28)}$$

Consequently, $G_{Q_j}(x') > t - \lambda_\varepsilon$ for a.e. $x' \in E_Q(t) \cap Q_j$ which further implies

$$|E_Q(t) \cap Q_j| \leq |\{x' \in Q_j : G_{Q_j}(x') > t - \lambda_\varepsilon\}| \leq \Xi(t - \lambda_\varepsilon)|Q_j|. \qquad (A.29)$$

In turn, this permits us to estimate

$$|E_Q(t)| = \sum_j |E_Q(t) \cap Q_j| \leq \Xi(t - \lambda_\varepsilon)\sum_j |Q_j| \leq \Xi(t - \lambda_\varepsilon)\frac{1}{\beta}\sum_j |F_{Q,\varepsilon} \cap Q_j|$$

$$\leq \Xi(t - \lambda_\varepsilon)\frac{1}{\beta}|F_{Q,\varepsilon}| \leq \Xi(t - \lambda_\varepsilon)\frac{\alpha}{\beta}|Q|, \quad (A.30)$$

where we have used (A.24), that the cubes $\{Q_j\}_j$ are pairwise disjoint and, finally, (A.21). Dividing by $|Q|$ and taking the supremum over all $Q \in \mathbb{D}_{Q_0}$ we arrive at

$$\Xi(t) \leq \frac{\alpha}{\beta}\Xi(t - \lambda_\varepsilon), \qquad t > \lambda_\varepsilon. \qquad (A.31)$$

Since this is valid for all $\beta \in (\alpha, 1)$, we can now let $\beta \to 1^+$, iterate the previous expression, and use the fact that $\Xi(t) \leq 1$ to conclude that

$$\Xi(t) \leq \frac{1}{\alpha}\alpha^{\frac{t}{\lambda_\varepsilon}} = \frac{1}{\alpha}e^{-\log(\alpha^{-1})\frac{t}{\lambda_\varepsilon}}, \qquad t > 0. \qquad (A.32)$$

Recalling that $\lambda_\varepsilon = \mathfrak{m}_\alpha + \varepsilon$ and letting $\varepsilon \to 0^+$ establishes (A.9).

We shall next indicate how (A.9) implies (A.10). Concretely, if we take $t := \frac{1+\alpha^{-1}}{\log(\alpha^{-1})}\mathfrak{m}_\alpha$ we see that (A.9) gives

$$\fint_Q \left(e^{\frac{G_Q(x')}{t}} - 1\right) dx' = \int_0^\infty \frac{|\{x' \in Q : G_Q(x')/t > \lambda\}|}{|Q|}e^\lambda \, d\lambda$$

$$\leq \frac{1}{\alpha}\int_0^\infty e^{-\log(\alpha^{-1})\frac{\lambda t}{\mathfrak{m}_\alpha}}e^\lambda \, d\lambda = \frac{1}{\alpha}\int_0^\infty e^{-\alpha^{-1}\lambda} \, d\lambda = 1. \quad (A.33)$$

With this in hand, (A.10) follows with the help of (1.16).

At this stage, there remains to justify (A.11). This can be done invoking again (A.9):

$$\fint_Q G_Q(x')^q \, dx' = \int_0^\infty \frac{|\{x' \in Q : G_Q(x') > \lambda\}|}{|Q|}q \, \lambda^q \frac{d\lambda}{\lambda} \leq \frac{1}{\alpha}\int_0^\infty e^{-\log(\alpha^{-1})\frac{\lambda}{\mathfrak{m}_\alpha}}q \, \lambda^q \frac{d\lambda}{\lambda}$$

$$= \frac{1}{\alpha}\left(\frac{\mathfrak{m}_\alpha}{\log(\alpha^{-1})}\right)^q \int_0^\infty e^{-\lambda}q \, \lambda^q \frac{d\lambda}{\lambda} = C_q\frac{1}{\alpha}\left(\frac{\mathfrak{m}_\alpha}{\log(\alpha^{-1})}\right)^q. \quad (A.34)$$

This completes the proof of Proposition A.2. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

REFERENCES

[1] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, Vol. 129, Academic Press Inc., Boston, MA, 1988.

[2] A. Fiorenza and M. Krbec, *Indices of Orlicz spaces and some applications*, Comment. Math. Univ. Carolin., **38** (1997), no. 3, 433–451.

[3] S. Hofmann, J.M. Martell, and S. Mayboroda, *Transference of scale-invariant estimates from Lipschitz to non-tangentially accessible to uniformly rectifiable domains*, arXiv:1904.13116.

[4] S. Hofmann and S. Mayboroda, *Hardy and* BMO *spaces associated to divergence form elliptic operators*, Math. Ann., **344** (2009), no. 1, 37–116.

[5] A.K. Lerner and C. Pérez, *Self-improving properties of generalized Poincaré type inequalities through rearrangements*, Math. Scand., **97** (2005), no. 2, 217–234.

[6] J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *The* BMO*-Dirichlet problem for elliptic systems in the upper-half space and quantitative characterizations of* VMO, Anal. PDE. **12** (2019), no. 3, 605–720.

[7] ———, *The Dirichlet problem for elliptic systems with data in Köthe function spaces*, Rev. Mat. Iberoam., **32** (2016), no. 3, 913–970.

[8] V. Maz′ya, M. Mitrea, and T. Shaposhnikova, *The Dirichlet problem in Lipschitz domains with boundary data in Besov spaces for higher order elliptic systems with rough coefficients*, Journal d'Analyse Mathématique, **110** (2010), no. 1, 167–239.

[9] N.G. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc., **15** (1964), no. 5, 717–721.

[10] D. Mitrea, *Distributions, Partial Differential Equations, and Harmonic Analysis*, Universitext, Springer, New York, NY, 2013.

[11] U. Neri, *Some properties of functions with bounded mean oscillation*, Studia Math., **61** (1977), no. 1, 63–75. MR 0445210

[12] C. Pérez, *Calderón-Zygmund theory related to Poincaré-Sobolev inequalities, fractional integrals and singular integral operators*, in J. Lukes and L. Pick, editors, Function Spaces, Nonlinear Analysis and Applications. Lecture notes of the Spring Lectures in analysis, Charles University and Academy of Sciences, 1999, pp. 31–94. Available at http://grupo.us.es/anaresba/trabajos/carlosperez/31.pdf.

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