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UNIFORM RECTIFIABILITY AND HARMONIC MEASURE II: POISSON KERNELS IN L^p IMPLY UNIFORM RECTIFIABILITY

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ABSTRACT. We present the converse to a higher dimensional, scale invariant version of the classical F. and M. Riesz theorem [RR], proved by the first two authors in [HM]. More precisely, for $n \geq 2$, for an ADR domain $\Omega \subset \mathbb{R}^{n+1}$ which satisfies the Harnack Chain condition plus an interior (but not exterior) Corkscrew condition, we show that absolute continuity of harmonic measure with respect to surface measure on $\partial\Omega$, with scale invariant higher integrability of the Poisson kernel, is sufficient to imply quantitative rectifiability of $\partial\Omega$.

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1. Introduction

This paper is a sequel to the work of the first two named authors [HM], in which we have presented a higher dimensional, scale invariant version of the classical theorem of F. and M. Riesz [RR]. The F. and M. Riesz Theorem states that for a simply connected domain Ω in the complex plane, with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arclength measure on the boundary. In [HM], we proved a scale invariant version of the latter result in higher dimensions. That is, given a domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, satisfying certain quantitative topological properties, whose boundary is rectifiable in an appropriate quantitative sense, we showed that harmonic measure for Ω satisfies a scale invariant version of absolute continuity with respect to surface measure. To be more precise, assuming that Ω satisfies interior "Corkscrew" and "Harnack Chain" conditions (these are scale invariant versions of the topological properties of openness and path connectedness; cf. Definitions 1.2 and 1.4 below), and that $\partial\Omega$ is "Uniformly Rectifiable" (a quantitative, scale invariant version of rectifiability; cf. Definition 1.7), we showed that harmonic measure belongs to weak-A_∞ (cf. Definition 1.16) with respect to surface measure on $\partial\Omega$. Let us note that the weak-A_∞ property implies that the Poisson kernel (i.e., the Radon-Nikodym derivative of harmonic measure with respect to surface measure), satisfies a scale-invariant higher integrability condition (cf. (1.21).)

To put the result of [HM] in context, we note that in previous work establishing absolute continuity of harmonic measure in dimensions $n+1 \geq 3$, other authors have imposed either an extra hypothesis on the geometry of the boundary, stronger than uniform rectifiability, namely that $\partial\Omega$ contains "Big Pieces" (cf. Definition 1.11) of the boundaries of Lipschitz sub-domains of Ω [BL] , or else an extra hypothesis on the geometry of the *exterior* domain $\Omega_{ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ (e.g., an "exterior Corkscrew condition" or "exterior n-disk condition"), which in turn implies the "Big Pieces" condition [DJ], [Se], [Ba]. The absence of such assumptions is the principal advance in [HM]. We refer the reader to the introduction of [HM], for a more detailed historical survey of work related to the results of that paper.

In the present paper, we obtain a converse to the main result of [HM], that is, we show that if $\Omega \subset \mathbb{R}^{n+1}$ satisfies interior Corkscrew and Harnack Chain conditions, if $\partial\Omega$ is n-dimensional "Ahlfors-David Regular" (cf. Definition 1.5), and if harmonic measure ω is absolutely continuous with respect to surface measure σ, with Poisson kernel $k := d\omega/d\sigma$ satisfying the scale invariant higher integrability condition (1.21), then $\partial\Omega$ is uniformly rectifiable (cf. Theorem 1.23). We observe that this result, and our approach to its proof, are in the spirit of the solution of the Painlevé problem ([To], but see also [Ch], [MMV], [Da] and [Vo]), in which analytic information (the existence of non-constant bounded analytic functions) is used to establish rectifiability properties of a set, via the use of Tb theory [NTrV] applied to the Cauchy integral operator. In our case, the use of Tb theory works as follows. To prove quantitative rectifiability of $\partial\Omega$, it suffices to verify a criterion of [DS2], by establishing an L^2 estimate for a square function involving the (twice differentiated) single layer potential operator (cf. (1.8)-(1.10) below, and the accompanying discussion). In turn, this square function bound is obtained via

a so called "local" Tb theorem[1], which reduces matters to checking local, scale-invariant bounds, when the square function acts on an accretive system of testing functions. Using our hypothesis concerning the scale invariant absolute continuity of harmonic measure, we construct these testing functions as normalized Poisson kernels. Eventually, we are able to show that the latter interact well with the square function based upon the layer potential operator, a step which depends in part on extending square function estimates of [DJK] to our setting.

To our knowledge, our result here appears to be the first converse to a theorem of F. and M. Riesz type, at least in the "large constant" setting, although analogous free boundary results have appeared previously in the "small constant" or "regular" setting. More precisely, our work here and in [HM] may be viewed as a "large constant" analogue of the series of papers by Kenig and Toro [KT1, KT2, KT3]. The latter papers say, collectively, that in the presence of a Reifenberg flatness condition and Ahlfors-David regularity, $\log k \in VMO$ iff $\nu \in VMO$, where k is the Poisson kernel with pole at some fixed point, and ν is the unit normal to the boundary. Moreover, under the same background hypotheses, the condition that $\nu \in VMO$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace, thus $\log k \in VMO \iff$ *vanishing UR*, given sufficient Reifenberg flatness. On the other hand, our large constant version "almost" says "$\log k \in BMO \iff UR$", given interior Corkscrews and Harnack Chains. Indeed, it is well known that the A_∞ condition (i.e., weak-A_∞ plus the doubling property) implies that $\log k \in BMO$, while if $\log k \in BMO$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [KT2, KT3] may be viewed as "endpoint" versions of the free boundary results of [AC] and [Je], which say, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal ν (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

Finally, let us mention that in a sequel to this paper and [HM], the first two authors and S. Mayboroda [HMa[2]] have used the results and techniques of [HM] and the present paper to show that $\partial\Omega$ is uniformly rectifiable, given that Ω satisfies interior Corkscrew and Harnack chain conditions, and that $\partial\Omega$ is Ahlfors-David Regular, and assuming the L^2 "Riesz transform bound"

$$\sup_{\varepsilon>0} \int_{\partial\Omega} \left| \int_{y\in\partial\Omega:|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\sigma(y) \right|^2 d\sigma(x) \lesssim \|f\|^2_{L^2(\partial\Omega)}\,. \tag{1.1}$$

In a remarkable paper by Nazarov, Tolsa and Volberg [NToV], the latter result has now been proved in full generality, i.e., assuming only (1.1) and that Ω is the complement of an Ahlfors-David Regular set of co-dimension 1; thus Ω is open but need not satisfy any other topological hypotheses.

1.1. **Notation and Definitions.**

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable

[1] a notion introduced by Christ in [Ch], and implicit in the solution of the Kato square root problem [HMc], [HLMc], [AHLMcT]; the version that we use is proved in [GM], cf. Theorem 2.51 below.

parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Given a domain $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$. A "surface ball" is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$.
- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.
- Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted by $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.
- For $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.
- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\big|_{\partial\Omega}$ denote the "surface measure" on $\partial\Omega$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A. If $A \subset \partial\Omega$, then $\operatorname{int}(A)$ will denote the relative interior, i.e., the largest relatively open set in $\partial\Omega$ contained in A. Thus, for $A \subset \partial\Omega$, the boundary is then well defined by $\partial A := \overline{A} \setminus \operatorname{int}(A)$.
- For a Borel set A, we denote by $C(A)$ the space of continuous functions on A, by $C_c(A)$ the subspace of $C(A)$ with compact support in A, and by $C_b(A)$ the space of bounded continuous functions on A. If A is unbounded, we denote by $C_0(A)$ the space of continuous functions on A converging to 0 at infinity.
- For a Borel subset $A \subset \partial\Omega$, we set $\fint_A f d\sigma := \sigma(A)^{-1} \int_A f d\sigma$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. We use Q to denote a dyadic "cube" on $\partial\Omega$. The latter exist, given that $\partial\Omega$ is ADR (cf. [DS1], [Ch]), and enjoy certain properties which we enumerate in Lemma 1.12 below.

Definition 1.2. (**Corkscrew condition**). Following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "Corkscrew condition" if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a "Corkscrew point" relative to Δ. We note that we may allow $r < C \operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c.

Remark 1.3. We note that, on the other hand, every $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, may be viewed as a Corkscrew point, relative to some surface ball $\Delta \subset \partial\Omega$. Indeed, set $r = K\delta(X)$, with $K > 1$, fix $x \in \partial\Omega$ such that $|X - x| = \delta(X)$, and let $\Delta := \Delta(x, r)$.

Definition 1.4. (**Harnack Chain condition**). Again following [JK], we say that Ω satisfies the Harnack Chain condition if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X)$, $\delta(X') \geq \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1, ..., B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C \operatorname{diam}(B_k)$. The chain of balls is called a "Harnack Chain".

We remark that the Corkscrew condition is a quantitative, scale invariant version of the fact that Ω is open, and the Harnack Chain condition is a scale invariant version of path connectedness.

Definition 1.5. (**Ahlfors-David regular**). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR (or simply ADR) if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \leq H^n(E \cap B(x, r)) \leq C\, r^n, \quad \forall r \in (0, R_0), x \in E, \tag{1.6}$$

where R_0 is the diameter of E (which may be infinite). When $E = \partial\Omega$, the boundary of a domain Ω, we shall sometimes for convenience simply say that "Ω has the ADR property" to mean that $\partial\Omega$ is ADR.

Definition 1.7. (**Uniform Rectifiability**). Following David and Semmes [DS1, DS2], we say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional UR (or simply UR) ("Uniformly Rectifiable"), if it satisfies the ADR condition (1.6), and if for some uniform constant C and for every Euclidean ball $B := B(x_0, r)$, $r \leq \operatorname{diam}(E)$, centered at any point $x_0 \in E$, we have the Carleson measure estimate

$$\iint_B |\nabla^2 \mathcal{S}1(X)|^2 \operatorname{dist}(X, E)\, dX \leq C r^n, \tag{1.8}$$

where $\mathcal{S}f$ is the single layer potential of f, i.e.,

$$\mathcal{S}f(X) := c_n \int_E |X - y|^{1-n} f(y)\, dH^n(y). \tag{1.9}$$

Here, the normalizing constant c_n is chosen so that $\mathcal{E}(X) := c_n|X|^{1-n}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. When $E = \partial\Omega$, the boundary of a domain Ω, we shall sometimes for convenience simply say that "Ω has the UR property" to mean that $\partial\Omega$ is UR.

We note that there are numerous characterizations of uniform rectifiability given in [DS1, DS2]. Let us note that, by "$T1$ reasoning", the Carleson measure condition (1.8) is equivalent to the global L^2 bound

$$\iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S}f(X)|^2\, \delta(X)\, dX \;\leq\; C\, \|f\|^2_{L^2(\partial\Omega)}. \tag{1.10}$$

The condition (1.8) will be most useful for our purposes, and appears in [DS2, Chapter 3, Part III]. We remark that the UR sets are precisely those for which all "sufficiently nice" singular integrals are bounded on L^2 (see [DS1]).

Definition 1.11. ("Big Pieces"). Given a closed set $E \subset \mathbb{R}^{n+1}$ such that E is n-dimensional ADR, and a collection $\mathcal{S}$ of domains in $\mathbb{R}^{n+1}$, we say that E has "big pieces of boundaries of $\mathcal{S}$" (denoted $E \in BP(\partial\mathcal{S})$) if there is a constant $0 < \alpha \leq 1$ such that for every $x \in E$, and $0 < r < \operatorname{diam}(E)$, there is a domain $\Omega' \in \mathcal{S}$ such that

$$H^n(\partial\Omega' \cap B(x,r) \cap E) \geq \alpha H^n(B(x,r) \cap E) \approx \alpha r^n.$$

Lemma 1.12. *(**Existence and properties of the "dyadic grid"**)* [DS1, DS2], [Ch]. *Suppose that $E \subset \mathbb{R}^{n+1}$ satisfies the ADR condition (1.6). Then there exist constants $a_0 > 0$, $\eta > 0$ and $C_1 < \infty$, depending only on dimension and the ADR constants, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(*i*) $E = \cup_j Q_j^k$ *for each* $k \in \mathbb{Z}$

(*ii*) *If* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(*iii*) *For each* (j,k) *and each* $m < k$, *there is a unique* m *such that* $Q_j^k \subset Q_i^m$.

(*iv*) *Diameter* $\left(Q_j^k\right) \leq C_1 2^{-k}$.

(*v*) *Each* Q_j^k *contains some "surface ball"* $\Delta\left(x_j^k, a_0 2^{-k}\right) := B\left(x_j^k, a_0 2^{-k}\right) \cap E$.

(*vi*) $H^n\left(\left\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \tau\, 2^{-k}\right\}\right) \leq C_1\, \tau^\eta\, H^n\left(Q_j^k\right)$, *for all* k, j *and for all* $\tau \in (0, a_0)$.

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Ch]. In that setting, the dyadic parameter $1/2$ should be replaced by some constant $\delta \in (0,1)$. It is a routine matter to verify that one may take $\delta = 1/2$ in the presence of the Ahlfors-David property (1.6) (in this more restrictive context, the result already appears in [DS1, DS2]).
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D} := \cup_k \mathbb{D}_k,$$

  where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.
- Properties (*iv*) and (*v*) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr), \tag{1.13}$$

  for some uniform constant C. We shall denote this ball and surface ball by

$$B_Q := B(x_Q, r)\,, \qquad \Delta_Q := \Delta(x_Q, r), \tag{1.14}$$

  and we shall refer to the point x_Q as the "center" of Q.

- Let us now specialize to the case that $E = \partial\Omega$, with Ω satisfying the Corkscrew condition. Given $Q \in \mathbb{D}(\partial\Omega)$, we shall sometimes refer to a "Corkscrew point relative to Q", which we denote by X_Q, and which we define to be the corkscrew point X_Δ relative to the ball $\Delta := \Delta_Q$ (cf. (1.13), (1.14) and Definition 1.2). We note that

$$\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q). \tag{1.15}$$

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

Definition 1.16. (A_∞, $A_\infty^{\rm dyadic}$ **and weak-**A_∞). Given a surface ball $\Delta = B \cap \partial\Omega$, a Borel measure ω defined on $\partial\Omega$ is said to belong to the class $A_\infty(\Delta)$ if there are positive constants C and θ such that for every $\Delta' = B' \cap \partial\Omega$ with $B' \subseteq B$, and every Borel set $F \subset \Delta'$, we have

$$\omega(F) \leq C \left(\frac{\sigma(F)}{\sigma(\Delta')}\right)^\theta \omega(\Delta'). \tag{1.17}$$

If we replace the surface balls Δ and Δ' by a dyadic cube Q and its dyadic subcubes Q', with $F \subset Q'$, then we say that $\omega \in A_\infty^{\rm dyadic}(Q)$:

$$\omega(F) \leq C \left(\frac{\sigma(F)}{\sigma(Q')}\right)^\theta \omega(Q'). \tag{1.18}$$

Similarly, $\omega \in$ weak-$A_\infty(\Delta)$, with $\Delta = B \cap \partial\Omega$, if for every $\Delta' = B' \cap \partial\Omega$ with $2B' \subseteq B$, we have

$$\omega(F) \leq C \left(\frac{\sigma(F)}{\sigma(\Delta')}\right)^\theta \omega(2\Delta') \tag{1.19}$$

As is well known [CF], [GR], [Sa], the A_∞ (resp. weak-A_∞) condition is equivalent to the property that the measure ω is absolutely continuous with respect to σ, and that its density satisfies a reverse Hölder (resp. weak reverse Hölder) condition. In this paper, we are interested in the case that $\omega = \omega^X$, the harmonic measure with pole at X. In that setting, we let $k^X := d\omega^X/d\sigma$ denote the Poisson kernel, so that (1.17) is equivalent to the reverse Hölder estimate

$$\left(\fint_{\Delta'} \left(k^X\right)^q d\sigma\right)^{1/q} \leq C \fint_{\Delta'} k^X \, d\sigma\,, \tag{1.20}$$

for some $q > 1$ and for some uniform constant C. In particular, when $\Delta' = \Delta$, and $X = X_\Delta$, a Corkscrew point relative to Δ, the latter estimate reduces to

$$\int_\Delta \left(k^{X_\Delta}\right)^q d\sigma \leq C\, \sigma(\Delta)^{1-q}. \tag{1.21}$$

Similarly, (1.19) is equivalent to

$$\left(\fint_{\Delta'} \left(k^X\right)^q d\sigma\right)^{1/q} \leq C \fint_{2\Delta'} k^X \, d\sigma\,. \tag{1.22}$$

Assuming that the latter bound holds with $\Delta' = \Delta$, and with $X = X_\Delta$, then one again obtains (1.21).

1.2. **Statement of the Main Result.** Our main result is as follows. We shall use the terminology that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a **1-sided NTA domain** if it satisfies **interior** (but not necessarily exterior) Corkscrew and Harnack Chain conditions. We remark that such domains are sometimes called "uniform" domains in the literature, but we prefer the designation "1-sided NTA"; first, because the latter is more descriptive and indicates the connection with the notion of an NTA domain introduced in [JK], and second, to avoid confusion with uniform rectifiability of the boundary (a completely different concept). We recall that an NTA domain, as defined in [JK], satisfies both interior and exterior Corkscrew conditions, as well as an interior Harnack Chain condition.

Theorem 1.23. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain, whose boundary is n-dimensional ADR. Suppose also that harmonic measure ω is absolutely continuous with respect to surface measure and that the Poisson kernel $k = d\omega/d\sigma$ satisfies the scale invariant estimate*

$$\int_\Delta \left(k^{X_\Delta}\right)^p \, d\sigma \leq C\, \sigma(\Delta)^{1-p}, \tag{1.24}$$

for some $1 < p < \infty$ and for all surface balls Δ. Then $\partial\Omega$ is UR.

Remark 1.25. As mentioned above (1.24) is (apparently) weaker than ω^{X_Δ} being in weak-$A_\infty(\Delta)$. However, *a posteriori*, we obtain that these two conditions are equivalent. Namely, Theorem 1.23 shows that $\partial\Omega$ is UR and therefore we can apply [HM, Theorem **??**] to obtain that ω^{X_Δ} belongs to weak-$A_\infty(\Delta)$.

Theorem 1.23 leads to an immediate "self-improvement" of itself, in which the hypotheses are assumed to hold only in an appropriate "big pieces" sense.

Theorem 1.26. *Let $E \subset \mathbb{R}^{n+1}$ be a closed set and assume that E is n-dimensional ADR. Suppose further that $E \in BP(\partial\mathcal{S})$ (cf. Definition 1.11), where $\mathcal{S}$ is a collection of 1-sided NTA domains with n-dimensional ADR boundaries, with uniform control of all of the relevant Corkscrew, Harnack Chain and ADR constants. Assume also that there exist $1 < p < \infty$ and $C \geq 1$ such that for every $\Omega' \in \mathcal{S}$ we have that harmonic measure $\omega_{\Omega'}$ is absolutely continuous with respect to surface measure and that the Poisson kernel $k_{\Omega'} = d\omega_{\Omega'}/d\sigma_{\Omega'}$ satisfies the scale invariant estimate*

$$\int_{\Delta'} \left(k_{\Omega'}^{X_{\Delta'}}\right)^p \, d\sigma_{\Omega'} \leq C\, \sigma_{\Omega'}(\Delta')^{1-p}, \tag{1.27}$$

and for all surface balls Δ' on Ω'. Then E is UR.

The proof of this result is as follows. Let $x \in E$ and $0 < r < \operatorname{diam}(E)$. Under the hypotheses of Theorem 1.26, there is Ω' satisfying the hypotheses of Theorem 1.23, with the property that for some $0 < \alpha \leq 1$, we have

$$\sigma\left(\partial\Omega' \cap B(x,r) \cap E\right) \geq \alpha\, \sigma(B(x,r) \cap E). \tag{1.28}$$

By Theorem 1.23, (1.27) implies that $\partial\Omega'$ is UR with uniform control of the constants (since the ADR and 1-sided NTA constants are uniformly controlled, and

also p and C in (1.27) are independent of Ω'). This and (1.28) implies that E has big pieces of UR sets and therefore E is UR, see [DS2].

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2. Proof of Theorem 1.23

2.1. **Preliminaries.** We collect some of the definition and auxiliary results from [HM] that will be used later. In the sequel, $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, will be a connected, open set, ω^X will denote harmonic measure for Ω, with pole at X, and $G(X, Y)$ will be the Green function. At least in the case that Ω is bounded, we may, as usual, define ω^X via the maximum principle and the Riesz representation theorem, after first using the method of Perron-Wiener-Brelot (see, e.g., [He]) to construct a harmonic function "associated" to arbitrary continuous boundary data[2]; indeed, if u is the solution associated to the data f, then the mapping $f \mapsto u(X)$, for each fixed $X \in \Omega$, defines a bounded linear functional on the space of continuous functions on $\partial\Omega$. For unbounded Ω, we may still define harmonic measure via a standard approximation scheme, see, e.g., [HM, Section **??**] for more details. We note for future reference that ω^X is a non-negative, finite, outer regular Borel measure.

The Green function may now be constructed by setting

$$G(X, Y) := \mathcal{E}(X - Y) - \int_{\partial\Omega} \mathcal{E}(X - z)\, d\omega^Y(z), \tag{2.1}$$

where $\mathcal{E}(X) := c_n |X|^{1-n}$ is the usual fundamental solution for $\mathcal{L}$, the Laplacian in $\mathbb{R}^{n+1}$. We choose the normalization that makes $\mathcal{E}$ positive. Of course, by construction, $\mathcal{L}_Y G(X, Y) = \delta_X$, the Dirac mass at X, although it is not clear that $G(X, \cdot)$ vanishes on $\partial\Omega$, unless $\partial\Omega$ is regular in the sense of Wiener. In that case, the Green function defined here is the same as the classical one, as defined, e.g., in [GW] (we refer the interested reader to the proof of [HM, Lemma **??**] for verification of this fact).

Lemma 2.2 (Bourgain [Bo])**.** *Suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, such that for every $x \in \partial\Omega$, and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$\omega^Y(\Delta(x, r)) \geq 1/C > 0\,. \tag{2.3}$$

In particular, if Ω satisfies the Corkscrew and Harnack Chain conditions, then for every surface ball Δ, we have

$$\omega^{X_\Delta}(\Delta) \geq 1/C > 0\,. \tag{2.4}$$

We next introduce some terminology.

Definition 2.5. A domain Ω satisfies the **qualitative exterior Corkscrew** condition if there exists $N \gg 1$ such that Ω has exterior corkscrew points at all scales smaller than 2^{-N}. That is, there exists a constant c such that for every surface ball $\Delta = \Delta(x, r)$, with $x \in \partial\Omega$ and $r \leq 2^{-N}$, there is a ball $B(X^{ext}_\Delta, c\, r) \subset B(x, r) \cap \Omega_{ext}$.

[2]Since we have made no assumption as regards Wiener's regularity criterion, our harmonic function is a generalized solution, which may not be continuous up to the boundary.

Remark 2.6. A clarifying comment is in order here: certainly, the existence of Corkscrew points at all scales less than 2^{-N} implies also that there are Corkscrew points at larger scales, but with constants that depend badly on N as the scale grows large. Thus, the adjective "qualitative" refers to the fact that we shall not invoke this condition in a quantitative way, and therefore our estimates will not depend on the parameter N in Definition 2.5. In the sequel, we shall work with certain approximating domains Ω_N satisfying the qualitative exterior Corkscrew condition (for the same N; cf. subsection 2.2 below). We prove quantitative estimates for these domains that are uniform in N, and thus may be transferred to the original domain.

Let us further observe that if Ω satisfies the qualitative exterior Corkscrew condition, then every point in $\partial\Omega$ is regular in the sense of Wiener. Moreover, for 1-sided NTA domains, the qualitative exterior Corkscrew points allow local Hölder continuity at the boundary (albeit with bounds which may depend badly on N).

Lemma 2.7 ([HM, Lemma **??**])**.** *There are positive, finite constants C, depending only on dimension, and $c(n,\theta)$, depending on dimension and $\theta \in (0,1)$, such that the Green function satisfies*

$$G(X,Y) \leq C\,|X-Y|^{1-n} \tag{2.8}$$

$$c(n,\theta)\,|X-Y|^{1-n} \leq G(X,Y)\,, \quad \text{if } |X-Y| \leq \theta\,\delta(X)\,,\ \theta \in (0,1)\,. \tag{2.9}$$

Moreover, if every point on $\partial\Omega$ is regular in the sense of Wiener, then

$$G(X,Y) \geq 0 \tag{2.10}$$

$$G(X,Y) = G(Y,X)\,, \qquad \forall X, Y \in \Omega\,,\ X \neq Y\,. \tag{2.11}$$

Lemma 2.12 ([HM, Lemma **??**])**.** *Let Ω be a 1-sided NTA domain with n-dimensional ADR boundary, and suppose that every $x \in \partial\Omega$ is regular in the sense of Wiener. Fix $B_0 := B(x_0, r_0)$ with $x_0 \in \partial\Omega$, and $\Delta_0 := B_0 \cap \partial\Omega$. Let $B := B(x,r)$, $x \in \partial\Omega$, and $\Delta := B \cap \partial\Omega$, and suppose that $2B \subset B_0$. Then for $X \in \Omega \setminus B_0$ we have*

$$r^{n-1} G(X_\Delta, X) \leq C\omega^X(\Delta). \tag{2.13}$$

If, in addition, Ω satisfies the qualitative exterior corkscrew condition, then

$$\omega^X(\Delta) \leq C r^{n-1} G(X_\Delta, X). \tag{2.14}$$

The constants in (2.13) and (2.14) depend **only** *on dimension and on the constants in the ADR and 1-sided NTA conditions.*

Corollary 2.15 ([HM, Corollary **??**])**.** *Suppose that Ω is a 1-sided NTA domain with n-dimensional ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. Let $B := B(x,r)$, $x \in \partial\Omega$, $\Delta := B \cap \partial\Omega$ and $X \in \Omega \setminus 4B$. Then there is a uniform constant C such that*

$$\omega^X(2\Delta) \leq C\omega^X(\Delta).$$

We next introduce some "discretized" and "geometric" sawtooth and Carleson regions from [HM, Section **??**]. Given a "dyadic cube" $Q \in \mathbb{D}(\partial\Omega)$, the **discretized**

Carleson region $\mathbb{D}_Q$ is defined to be the collection of all dyadic sub-cubes of Q, i.e.,

$$\mathbb{D}_Q := \left\{Q' \in \mathbb{D} : Q' \subseteq Q\right\}. \tag{2.16}$$

Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the **global discretized sawtooth** relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j}, \tag{2.17}$$

i.e., $\mathbb{D}_{\mathcal{F}}$ is the collection of all $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$.

We apologize to the reader for a potentially confusing aspect of this notation: the symbols $\mathbb{D}_Q$, with $Q \in \mathbb{D}$, and $\mathbb{D}_{\mathcal{F}}$, with $\mathcal{F} \subset \mathbb{D}$, denote two very different concepts. In particular, in the case that $\mathcal{F}$ consists of a single cube Q, i.e. $\mathcal{F} = \{Q\}$, the corresponding sawtooth $\mathbb{D}_{\{Q\}}$ is in fact the complement of $\mathbb{D}_Q$.

Given some fixed cube Q, the **local discretized sawtooth** relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q. \tag{2.18}$$

We also introduce the "geometric" Carleson regions and sawtooths. Let us first recall that we write $k = k(Q)$ if $Q \in \mathbb{D}_k$ (cf. Lemma 1.12), and in that case the "length" of Q is denoted by $\ell(Q) = 2^{-k(Q)}$. We also recall that there is a Corkscrew point X_Q, relative to each $Q \in \mathbb{D}$ (in fact, there are many such, but we just pick one). Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω, which satisfy

$$4\,\mathrm{diam}(I) \leq \mathrm{dist}(4I, \partial\Omega) \leq \mathrm{dist}(I, \partial\Omega) \leq 40\,\mathrm{diam}(I), \qquad \forall\, I \in \mathcal{W} \tag{2.19}$$

and also

$$(1/4)\,\mathrm{diam}(I_1) \leq \mathrm{diam}(I_2) \leq 4\,\mathrm{diam}(I_1),$$

whenever I_1 and I_2 touch. Let $\ell(I)$ denote the side length of I, and write $k = k_I$ if $\ell(I) = 2^{-k}$. There are $C_0 \geq 1000\sqrt{n}$ and $m_0 \geq 0$ large enough (depending only on the constants in the Corkscrew condition and in the dyadic cube construction) so that for every cube $Q \in \mathbb{D}$

$$\mathcal{W}_Q := \left\{I \in \mathcal{W} :\ k(Q) - m_0 \ \leq\ k_I \ \leq\ k(Q) + 1\,,\ \text{and}\ \mathrm{dist}(I, Q) \leq C_0\, 2^{-k(Q)}\right\}, \tag{2.20}$$

satisfies that $X_Q \in I$ for some $I \in \mathcal{W}_Q$, and for each dyadic child Q^j of Q, the respective Corkscrew points $X_{Q^j} \in I^j$ for some $I^j \in \mathcal{W}_Q$. Moreover, we may always find an $I \in \mathcal{W}_Q$ with the slightly more precise property that $k(Q) - 1 \leq k_I \leq k(Q)$ and

$$\mathcal{W}_{Q_1} \cap \mathcal{W}_{Q_2} \neq \emptyset, \ \text{whenever}\ 1 \leq \frac{\ell(Q_2)}{\ell(Q_1)} \leq 2, \ \text{and}\ \mathrm{dist}(Q_1, Q_2) \leq 1000\ell(Q_2). \tag{2.21}$$

We introduce some notation: given a subset $A \subset \Omega$, we write $X \to_A Y$ if the interior of A contains all the balls in a Harnack Chain (in Ω), connecting X to Y, and if, moreover, for any point Z contained in any ball in the Harnack Chain, we have $\mathrm{dist}(Z, \partial\Omega) \approx \mathrm{dist}(Z, \Omega \setminus A)$, with uniform control of the implicit constants.

We denote by $X(I)$ the center of a cube $I \in \mathbb{R}^{n+1}$, and we recall that X_Q denotes a designated Corkscrew point relative to Q that can be assumed to be the center of some Whitney cube I such that $I \subset B_Q \cap \Omega$ and $\ell(I) \approx \ell(Q) \approx \operatorname{dist}(I, Q)$.

For each $I \in \mathcal{W}_Q$, we form a Harnack Chain, call it $H(I)$, from the center $X(I)$ to the Corkscrew point X_Q. We now denote by $\mathcal{W}(I)$ the collection of all Whitney cubes which meet at least one ball in the chain $H(I)$, and we set

$$\mathcal{W}_Q^* := \bigcup_{I \in \mathcal{W}_Q} \mathcal{W}(I).$$

We also define, for $\lambda \in (0, 1)$ to be chosen momentarily,

$$U_Q := \bigcup_{\mathcal{W}_Q^*} (1+\lambda) I =: \bigcup_{I \in \mathcal{W}_Q^*} I^* \,. \tag{2.22}$$

By construction, we then have that

$$\mathcal{W}_Q \subset \mathcal{W}_Q^* \subset \mathcal{W} \quad \text{and} \quad X_Q \in U_Q \,,\ X_{Q^j} \in U_Q \,, \tag{2.23}$$

for each child Q^j of Q. It is also clear that there are uniform constants k^* and K_0 such that

$$\begin{gathered} k(Q) - k^* \le k_I \le k(Q) + k^* \,, \quad \forall I \in \mathcal{W}_Q^* \\ X(I) \to_{U_Q} X_Q \,, \quad \forall I \in \mathcal{W}_Q^* \\ \operatorname{dist}(I, Q) \le K_0\, 2^{-k(Q)} \,, \quad \forall I \in \mathcal{W}_Q^* \,, \end{gathered} \tag{2.24}$$

where k^*, K_0 and the implicit constants in the condition $X(I) \to_{U_Q} X_Q$, depend only on the "allowable parameters" (since m_0 and C_0 also have such dependence) and on λ. Thus, by the addition of a few nearby Whitney cubes of diameter also comparable to that of Q, we can "augment" $\mathcal{W}_Q$ so that the Harnack Chain condition holds in U_Q.

We fix the parameter λ so that for any $I, J \in \mathcal{W}$,

$$\begin{gathered} \operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J) \\ \operatorname{int}(I^*) \cap \operatorname{int}(J^*) \neq \emptyset \iff \partial I \cap \partial J \neq \emptyset \end{gathered} \tag{2.25}$$

(the fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathcal{W}$ whose boundaries touch, so that the Harnack Chain property then holds locally, with constants depending upon λ, in $I^* \cup J^*$). By choosing λ sufficiently small, we may also suppose that there is a $\tau \in (1/2, 1)$ such that for distinct $I, J \in \mathcal{W}$,

$$\tau J \cap I^* = \emptyset \,. \tag{2.26}$$

Remark 2.27. We note that any sufficiently small choice of λ (say $0 < \lambda \le \lambda_0$) will do for our purposes.

Of course, there may be some flexibility in the choice of additional Whitney cubes which we add to form the augmented collection $\mathcal{W}_Q^*$, but having made such a choice for each $Q \in \mathbb{D}$, we fix it for all time.

We may then define the **Carleson box** associated to Q by

$$T_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \right). \tag{2.28}$$

Similarly, we may define geometric sawtooth regions as follows. As above, give a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the **global sawtooth** relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F}} := \operatorname{int}\left(\bigcup_{Q'\in\mathbb{D}_{\mathcal{F}}} U_{Q'}\right), \tag{2.29}$$

and again given some fixed $Q \in \mathbb{D}$, the **local sawtooth** relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q'\in\mathbb{D}_{\mathcal{F},Q}} U_{Q'}\right). \tag{2.30}$$

We also define as follows the "Carleson box" T_Δ associated to a surface ball $\Delta := \Delta(x_\Delta, r)$. Let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200\, r \leq 2^{-k}$, and set

$$\mathbb{D}^\Delta := \{Q \in \mathbb{D}_{k(\Delta)} : Q \cap 2\Delta \neq \emptyset\}. \tag{2.31}$$

We then define

$$T_\Delta := \operatorname{int}\left(\bigcup_{Q\in\mathbb{D}^\Delta} \overline{T_Q}\right). \tag{2.32}$$

Given a surface ball $\Delta := \Delta(x, r)$, let $B_\Delta := B(x, r)$, so that $\Delta = B_\Delta \cap \partial\Omega$. Then,

$$\frac{5}{4}B_\Delta \cap \Omega \subset T_\Delta. \tag{2.33}$$

and there exists κ_0 large enough such that

$$\overline{T_\Delta} \subset \kappa_0 B_\Delta \cap \overline{\Omega}. \tag{2.34}$$

Lemma 2.35 (Lemma [HM, **??**]). *Suppose that Ω is a 1-sided NTA domain with an ADR boundary. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$, and $\Omega_{\mathcal{F},Q}$ are also 1-sided NTA domains with ADR boundaries. In all cases, the implicit constants are uniform, and depend only on dimension and on the corresponding constants for Ω.*

Lemma 2.36 (Lemma [HM, **??**]). *Suppose that Ω is a 1-sided NTA domain with an ADR boundary and that Ω also satisfies the qualitative exterior Corkscrew condition. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$, and $\Omega_{\mathcal{F},Q}$ satisfy the qualitative exterior Corkscrew condition. In all cases, the implicit constants are uniform, and depend only on dimension and on the corresponding constants for Ω.*

Corollary 2.37 ([HM, Corollary **??**]). *Suppose that Ω is a 1-sided NTA domain with n-dimensional ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. There is a uniform constant C such that for every pair of surface balls $\Delta := B \cap \partial\Omega$, and $\Delta' := B' \cap \partial\Omega$, with $B' \subseteq B$, and for every $X \in \Omega \setminus 2\kappa_0 B$, where κ_0 is the constant in (2.34) below, we have*

$$\frac{1}{C}\,\omega^{X_\Delta}(\Delta') \leq \frac{\omega^X(\Delta')}{\omega^X(\Delta)} \leq C\,\omega^{X_\Delta}(\Delta').$$

Remark 2.38. Let us recall that the dilation factor λ, defining the fattened Whitney boxes I^* and Whitney regions U_Q, is allowed to be any fixed positive number no larger than some small λ_0 (cf. (2.22) and Remark 2.27). For the rest of the paper we now fix $0 < \lambda \leq \lambda_0/2$, so that, in particular, the previous results apply not only to the Carleson boxes and sawtooths corresponding to U_Q as defined above, but also to those corresponding to "fatter" Whitney regions $U_Q^* = \cup_{W_Q^*} I^{**}$ with $I^{**} = (1 + 2\lambda) I$. We will work with these fatter regions in Sections 2.4 and 2.5 below.

2.2. **Step 1: Passing to the approximating domains.** We first observe that without loss of generality we can assume that $p \leq 2$: If $p > 2$, (1.24) and Hölder's inequality imply the same estimate with $p = 2$.

We define approximating domains as follows. For each large integer N, set $\mathcal{F}_N := \mathbb{D}_N$. We then let $\Omega_N := \Omega_{\mathcal{F}_N}$ denote the usual (global) sawtooth with respect to the family $\mathcal{F}_N$ (cf. (2.24), (2.22) and (2.29).) Thus,

$$\Omega_N = \operatorname{int}\left(\bigcup_{Q\in\mathbb{D}:\, \ell(Q)\geq 2^{-N+1}} U_Q \right), \tag{2.39}$$

so that $\overline{\Omega_N}$ is the union of fattened Whitney cubes $I^* = (1 + \lambda)I$, with $\ell(I) \gtrsim 2^{-N}$, and the boundary of Ω_N consists of portions of faces of I^* with $\ell(I) \approx 2^{-N}$. By virtue of Lemma 2.35, each Ω_N satisfies the ADR, Corkscrew and Harnack Chain properties. We note that, for each of these properties, the constants are uniform in N, and depend only on dimension and on the corresponding constants for Ω.

By construction Ω_N satisfies the qualitative exterior corkscrew condition (Definition 2.5) since it has exterior corkscrew points at all scales $\lesssim 2^{-N}$. By Lemma 2.36 the same statement applies to the Carleson boxes T_Q and T_Δ, and to the sawtooth domains $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q}$ (all of them relative to Ω_N) and even to Carleson boxes within sawtooths.

We write ω_N for the corresponding harmonic measure and k_N for the corresponding Poisson kernel (we know by [DJ] that ω_N is absolutely continuous with respect to surface measure, since Ω_N enjoys a *qualitative* 2-sided Corkscrew condition, i.e., it has exterior Corkscrew points at all scales $\lesssim 2^{-N}$). We are going to show that the scale invariant estimate (1.24) passes uniformly to the approximating domains. To be precise, for all surface balls Δ^N, defined with respect to the approximating domain Ω_N (i.e., $\Delta^N = B \cap \partial\Omega_N$ with B centered at $\partial\Omega_N$), we have

$$\int_{\Delta^N} \left(k_N^{X_{\Delta^N}}\right)^{\tilde{p}} d\sigma_N \leq \widetilde{C}\, \sigma_N(\Delta^N)^{1-\tilde{p}}, \tag{2.40}$$

where $\widetilde{C}$ and $1 < \tilde{p} \leq p$ are independent of N. To prove (2.40), we shall first need to establish several preliminary facts.

Given $X \in \Omega_N \subset \Omega$ we write $\hat{x}$ for a point in $\partial\Omega$ such that $\delta(X) = |X - \hat{x}|$, analogously $\hat{x}_N \in \partial\Omega_N$ with $\delta_N(X) = |X - \hat{x}_N|$ (here δ_N stands for the distance to the boundary of Ω_N). Set $\Delta_X = B(\hat{x}, \delta(X)) \cap \partial\Omega$ and $\Delta_X^N = B(\hat{x}_N, \delta_N(X)) \cap \partial\Omega_N$.

Let $X \in \Omega_N \subset \Omega$ with $\delta_N(X) \approx 2^{-N}$, thus $\delta(X) \approx 2^{-N}$. We claim that for every $Y \in \Omega_N$ we have

$$\frac{\omega_N^Y(\Delta_X^N)}{\sigma_N(\Delta_X^N)} \leq C\, \frac{\omega^Y(\Delta_X)}{\sigma(\Delta_X)}, \tag{2.41}$$

where C is independent of N.

We first consider the case Ω_N bounded. For fixed $Y \in \Omega_N$ we set $u(Y) = \omega_N^Y(\Delta_X^N)$ and $v(Y) = \omega^Y(\Delta_X)$. Note that u, v are harmonic in Ω_N. For every $Y \in \Delta_X^N$, we have $u(Y) = 1 \lesssim \omega^Y(\Delta_X) = v(Y)$ by Lemma 2.2 and the Harnack chain condition. Also, if $Y \in \partial\Omega_N \setminus \Delta_X^N$ we have $u(Y) = 0 \leq \omega^Y(\Delta_X) = v(Y)$. Thus, maximum principle yields that $u(Y) \lesssim v(Y)$ for every $Y \in \Omega_N$ and then we obtain as desired

$$\frac{\omega_N^Y(\Delta_X^N)}{\sigma_N(\Delta_X^N)} \leq C\, \frac{\omega^Y(\Delta_X)}{\sigma(\Delta_X)},$$

where C is independent of N (notice that we have used that $\sigma_N(\Delta_X^N) \approx \delta_N(X)^n \approx \delta(X)^n \approx \sigma(\Delta_X)$.)

Let us treat the case Ω_N unbounded. Let $M \gg 1$ and set $B_M^N = B(\hat{x}_N, M\,\delta_N(X))$ and $\Delta_M^N = B_M^N \cap \partial\Omega_N$. Take $\Omega_{N,M} = T_{\Delta_M^N}$ where $T_{\Delta_M^N} \subset \Omega_N$ denotes the Carleson box corresponding to Δ_M^N for the domain Ω_N. By (2.33) and (2.34) (with Ω replaced by Ω_N) we have

$$\frac{5}{4}B_N^M \cap \Omega_N \subset \Omega_{N,M}, \qquad\qquad \overline{\Omega_{N,M}} \subset \kappa_0 B_N^M \cap \overline{\Omega_N}.$$

These and the fact that $M \gg 1$ allow us to obtain that $X \in \Omega_{N,M}$, $\hat{x}_N \in \partial\Omega_N \cap \partial\Omega_{N,M}$, $\delta_{\Omega_{N,M}}(X) = \delta_N(X) \approx 2^{-N}$, and

$$\Delta_X^N = B(\hat{x}_N, \delta_N(X)) \cap \partial\Omega_N = B(\hat{x}_N, \delta_N(X)) \cap \partial\Omega_{N,M}.$$

Then, we proceed as in the previous case and consider $u_M(Y) = \omega_{\Omega_{N,M}}^Y(\Delta_X^N)$ and $v(Y) = \omega^Y(\Delta_X)$ for $Y \in \Omega_{N,M}$. The same argument above and the maximum principle in the bounded domain $\Omega_{N,M}$ yields $u_M(Y) \lesssim v(Y)$ for every $Y \in \Omega_{N,M}$ with constant independent on N, M and X. On the other hand notice that the solutions $u_M(Y) = \omega_{\Omega_{N,M}}^Y(\Delta_X^N)$ are monotone increasing on any fixed Ω_{N,M_0}, as $M_0 \leq M \to \infty$, by the maximum principle. We then obtain that $u_M(Y) \to u(Y) := \omega_N^X(\Delta_X^N)$, uniformly on compacta, by Harnack's convergence theorem. Thus, $u(Y) \lesssim v(Y)$ for every $Y \in \Omega_N$ and then the previous argument leads to the desired estimate.

Proposition 2.42. *There exists $1 < \tilde{p} \leq p$ such that for all $N \gg 1$ if we let $\Delta^N = B \cap \Omega_N$ be a surface ball with $r(B) \approx 2^{-N}$, then*

$$\left(\fint_{\Delta^N} \left(k_N^{X_{\Delta^N}} \right)^{\tilde{p}}\, d\sigma_N \right)^{\frac{1}{\tilde{p}}} \lesssim \sigma_N(\Delta^N)^{-1} \tag{2.43}$$

where the constant is uniform in N. Furthermore, for every $Y \in \Omega_N \setminus 2\kappa_0 B$ we have

$$\left(\fint_{\Delta^N} \left(k_N^{Y} \right)^{\tilde{p}}\, d\sigma_N \right)^{\frac{1}{\tilde{p}}} \lesssim \frac{\omega_N^Y(\Delta^N)}{\sigma_N(\Delta^N)}, \tag{2.44}$$

where the constant is uniform in N.

Proof. Let us momentarily assume (2.43) and we obtain (2.44). As observed above Ω_N is a 1-sided NTA domain with ADR boundary and satisfies the qualitative exterior corkscrew condition. Thus we can use Corollary 2.37: we take $\Delta^N = B \cap \Omega_N$ with $r(B) \approx 2^{-N}$ and for every $Y \in \Omega_N \setminus 2\kappa_0 B$ we have

$$\fint_{(\Delta^N)'} k_N^Y \, d\sigma_N \approx \omega_N^Y(\Delta^N) \fint_{(\Delta^N)'} k_N^{X_{\Delta^N}} \, d\sigma_N, \qquad B' \subset B.$$

Then for σ_N-a.e. $x \in \Delta^N$ we take $B' = B(x, r)$ and let $r \to 0$ to obtain $k_N^Y(x) \approx \omega_N^Y(\Delta^N)\, k_N^{X_{\Delta^N}}(x)$ with constants that do not depend on N. Consequently, we have

$$\left(\fint_{\Delta^N} \left(k_N^Y\right)^{\tilde{p}} \, d\sigma_N\right)^{\frac{1}{\tilde{p}}} \lesssim \omega_N^Y(\Delta^N) \left(\fint_{\Delta^N} \left(k_N^{X_{\Delta^N}}\right)^{\tilde{p}} \, d\sigma_N\right)^{\frac{1}{\tilde{p}}}, \qquad Y \in \Omega_N \setminus 2\kappa_0 B.$$

This and (2.43) clearly give (2.44).

We next show (2.43). Write $\Delta^N = \Delta^N(x_N, r) = B(x_N, r) \cap \partial\Omega_N$ with $x_N \in \partial\Omega_N$ and $r \approx 2^{-N}$. We take $x \in \partial\Omega$ such that $\delta(x_N) = |x_N - x| \approx 2^{-N}$ and set $\Delta = \Delta(x, M\, 2^{-N})$ with M (independent of N) large enough. Note that

$$B(x_N, 4r) \cap \Omega_N \subset \cup_I I^*, \tag{2.45}$$

where the union runs over a collection of Whitney boxes $I \in \mathcal{W}(\Omega)$, of uniformly bounded cardinality, all with $\mathrm{int}(I^*) \subset \Omega_N$, $\ell(I) \approx 2^{-N}$, and $B(x_N, 4r) \cap I^* \neq \emptyset$. Notice that we have $B(x_N, 4r) \cap \partial\Omega_N \subset \cup_I \partial I^*$. For each such I, there is a $Q_I \in \mathbb{D}_{\mathcal{F}_N}$ such that

$$\ell(I) \approx \ell(Q_I) \approx 2^{-N} \approx \mathrm{dist}(I, Q_I),$$

so by definition $I \in \mathcal{W}^*_{Q_I}$. Thus, for any such I we have that

$$|x_{Q_I} - x| \lesssim \ell(Q_I) + d(Q_I, I) + \ell(I) + 4r + |x_N - x| \leq M\, 2^{-N}, \tag{2.46}$$

for M chosen large enough, and therefore $x_{Q_I} \in \Delta$.

Let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < M\, 2^{-N} \leq 2^{-k}$ (below we may need to make M larger). We define

$$\mathbb{P} := \mathrm{int} \left(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{\Omega_{\mathcal{F}_N, Q}} \right).$$

where $\mathbb{D}^\Delta$ is defined in (2.31) (note that $k(\Delta)$ is slightly different.)

Let us make some observations about this set $\mathbb{P}$. Note first that $\mathbb{D}_{\mathcal{F}_N}$ is the collection of all dyadic cubes whose side length are at least 2^{-N+1}. Thus for every $Q \in \mathbb{D}^\Delta$, we have that $\ell(Q) = 2^{-k(\Delta)}$ and $\mathbb{D}_{\mathcal{F}_N, Q}$ consists of all dyadic subcubes of Q whose side length is at least 2^{-N+1}, i.e., those $Q' \in \mathbb{D}_Q$ with $2^{-N+1} \leq \ell(Q) \leq 2^{-k(\Delta)}$ and these are a finite number. Note also that the cardinality of $\mathbb{D}^\Delta$ is finite and depends only on M and the ADR constants, and therefore $\overline{\mathbb{P}}$ is the union of a finite number (the cardinality depends only on M and on the parameters that appear in the definition of $\mathcal{W}^*_Q$) of fattened Whitney boxes that are "close" to Δ and have size comparable to 2^{-N}. Note also, that $\mathbb{P} \subset \Omega_N$. It is easy to see that $B(x_N, 4r) \cap \Omega_N \subset \mathbb{P}$. Namely, for every $Y \in B(x_N, 4r) \cap \Omega_N$ we have shown above using (2.45) that there exists $I^* \ni Y$ with $\ell(I) \approx 2^{-N}$, $I \in \mathcal{W}^*_{Q_I}$ and $x_{Q_I} \in \Delta$. Therefore there is $Q^* \in \mathbb{D}^\Delta$ such that $Q_I \subset Q^*$. Thus, $Y \in \overline{\Omega_{\mathcal{F}_N, Q_I}} \subset \bigcup_{Q \in \mathbb{D}^\Delta} \overline{\Omega_{\mathcal{F}_N, Q}}$.

Notice that $B(x_N, 4r) \cap \Omega_N$ is open so there exists a ball B_Y centered at y that is contained in this set. For every $Z \in B_Y$ we have just shown that $z \in \bigcup_{Q\in\mathbb{D}^\Delta} \overline{\Omega_{\mathcal{F}_N,Q}}$, this eventually leads to $Y \in \mathbb{P}$.

Lemma 2.47. *$\mathbb{P}$ is an NTA domain with ADR boundary, that is, it satisfies the interior* **and** *exterior corkscrew conditions, the (interior) Harnack chain condition, and $\partial\mathbb{P}$ is n-dimensional ADR; moreover, all of the corresponding constants are independent of N. Consequently, $\omega_{\mathbb{P}}$, the harmonic measure for the domain $\mathbb{P}$, is absolutely continuous with respect to surface measure and its Poisson kernel, $k_{\mathbb{P}}$, satisfies the scale invariant estimate*

$$\int_{\Delta^{\mathbb{P}}} \left(k_{\mathbb{P}}^{X_{\Delta^{\mathbb{P}}}}\right)^{\hat{p}} d\sigma_{\mathbb{P}} \leq C\, \sigma_{\mathbb{P}}(\Delta^{\mathbb{P}})^{1-\hat{p}}, \tag{2.48}$$

for some $1 < \hat{p} < \infty$ and for all surface balls $\Delta^{\mathbb{P}}$, where C and $\hat{p}$ are independent of N.

The proof of this result is given below.

Fix $y \in \Delta^N = B(x_N, r) \cap \partial\Omega_N$. Then $y \in \partial I^*$ with $I \in \mathcal{W}(\Omega)$, $\ell(I) \approx 2^{-N}$ and $\text{int}(I^*) \subset \Omega_N$. We recall that $\overline{\mathbb{P}}$ is a finite union of fattened Whitney boxes $J^* \in \mathcal{W}(\Omega)$, and we define $\mathbb{P}_{I^*}$ to be the interior of the union of those boxes $J^* \subset \mathbb{P}$ that overlap I^*. It is easy to see that $I \subset \mathbb{P}_{I^*} \subset \mathbb{P}$. We shall see from the proof of Lemma 2.47 that $\mathbb{P}_{I^*}$ is an NTA domain with constants independent of N. Moreover, we claim that taking M above larger we have that $\mathbb{P}_{I^*}$ is *strictly* contained in $\mathbb{P}$, and in fact $\mathbb{P} \setminus \mathbb{P}_{I^*}$ contains some Whitney box $J \in \mathcal{W}(\Omega)$. We may verify this claim as follows. We fix $Q_I \in \mathcal{F}_N$ with $\ell(Q_I) \approx \ell(I) \approx 2^{-N}$ and $\text{dist}(I, Q_I) \approx 2^{-N}$. Notice that, for M large enough, there is some $Q \in \mathbb{D}^\Delta$ such that $Q_I \subset Q$ (cf. (2.45)-(2.46).) For this Q, take $J \in \mathcal{W}^*_Q$ which is clearly in $\mathbb{P}$. Note that

$$\ell(J) \approx \ell(Q) \approx M\, 2^{-N} \approx M\, \ell(I)$$

and therefore for M large enough we have that $I^* \cap J^* = \emptyset$, by properties of Whitney cubes (since otherwise $\ell(I) \approx \ell(J)$). This shows that $\mathbb{P}_{I^*} \subsetneq \mathbb{P}$.

Set $u(Z) = G_N(Z, X_{\Delta^N})$ and $v(Z) = G_{\mathbb{P}}(Z, X_{\Delta^N})$ where G_N, $G_{\mathbb{P}}$ are respectively the Green functions associated with the domains Ω_N and $\mathbb{P}$. Notice that u and v are non-negative harmonic functions in $\mathbb{P}_{I^*}$ (since $\mathbb{P}_{I^*} \subset \mathbb{P} \subset \Omega_N$) and we can assume that $X_{\Delta^N} \notin \mathbb{P}_{I^*}$: indeed, by the Harnack chain condition, and our observation above that $\mathbb{P}_{I^*} \subsetneq \mathbb{P}$, we may assume that X_{Δ^N} is the center of a Whitney box $J \subset \mathbb{P} \setminus \mathbb{P}_{I^*}$. Notice that $u\big|_{2\,\Delta^N} = v\big|_{2\,\Delta^N} = 0$ (since $2\,\Delta^N \subset \partial\Omega_N$ and since $B(x_N, 4\,r) \cap \Omega_N \subset \mathbb{P}$ implies that $2\,\Delta^N = B(x_N, 2\,r) \cap \partial\mathbb{P}$). Therefore $u = v \equiv 0$ in $B(y, 2^{-N-N_1}) \cap \partial\mathbb{P}_{I^*}$ (here N_1 is a fixed big integer such that $B(y, 2^{-N-N_1}) \cap \partial\mathbb{P}_{I^*} \subset 2\Delta^N$. As mentioned before, $\mathbb{P}_{I^*}$ is an NTA domain with constants independent of N and we have the (full) comparison principle (see [JK, Lemma 4.10]):

$$\frac{u(Z)}{v(Z)} \approx \frac{u(X(I))}{v(X(I))},$$

for every $Z \in B(y, 2^{-N-N_1}/C) \cap \mathbb{P}_{I^*}$ with C large independent of N. We remind the reader that $X(I)$ stands for the center of the Whitney box I. Notice that

$$|X(I) - X_{\Delta^N}| \approx \text{dist}(X(I), \partial\mathbb{P}) \approx \text{dist}(X_{\Delta^N}, \partial\mathbb{P}) \approx 2^{-N}\,.$$

Thus, the Harnack chain condition and the size estimates for the Green functions imply that $u(X(I))/v(X(I)) \approx 1$. Hence $u(Z) \approx v(Z)$ for all $Z \in B(y, 2^{-N-N_1}/C) \cap \mathbb{P}_{I^*}$.

As observed above Ω_N is a 1-sided NTA with ADR boundary and also satisfies the qualitative exterior corkscrew condition, so that, in particular, we may apply Lemma 2.12. We take $s \ll 2^{-N}$ and write $\Delta^N(y,s) = B(y,s) \cap \Omega_N$ to obtain

$$\frac{u(X_{\Delta^N(y,s)})}{s} = \frac{G_N(X_{\Delta^N(y,s)}, X_{\Delta^N})}{s} \approx \frac{\omega_N^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_N(\Delta^N(y,s))}.$$

The same can be done with $G_{\mathbb{P}}$ (indeed $\mathbb{P}$ is NTA) after observing that $\Delta^N(y,s)$ is also a surface ball for $\mathbb{P}$ and that after using Harnack if needed $X_{\Delta^N(y,s)}$ is a corkscrew point with respect to $\mathbb{P}$ for that surface ball. Then,

$$\frac{v(X_{\Delta^N(y,s)})}{s} = \frac{G_{\mathbb{P}}(X_{\Delta^N(y,s)}, X_{\Delta^N})}{s} \approx \frac{\omega_{\mathbb{P}}^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_{\mathbb{P}}(\Delta^N(y,s))}$$

Note that if s is small enough then $Z = X_{\Delta^N(y,s)} \in B(y, 2^{-N-N_1}/C) \cap \mathbb{P}_{I^*}$. This allows us to gather the previous estimates and conclude that

$$\frac{\omega_N^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_N(\Delta^N(y,s))} \approx \frac{\omega_{\mathbb{P}}^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_{\mathbb{P}}(\Delta^N(y,s))}.$$

Next we use the Lebesgue differentiation theorem and obtain that $k_N^{X_{\Delta^N}}(y) \approx k_{\mathbb{P}}^{X_{\Delta^N}}(y)$ for a.e. $y \in \Delta^N$. Then if we set $\tilde{p} = \min\{p, \hat{p}\}$ and use (2.48) we obtain

$$\begin{aligned}\left(\fint_{\Delta^N} \left(k_N^{X_{\Delta^N}}\right)^{\tilde{p}} d\sigma_N\right)^{\frac{1}{\tilde{p}}} &\lesssim \left(\fint_{\Delta^N} \left(k_{\mathbb{P}}^{X_{\Delta^N}}\right)^{\tilde{p}} d\sigma_{\mathbb{P}}\right)^{\frac{1}{\tilde{p}}} \\ &\leq \left(\fint_{\Delta^N} \left(k_{\mathbb{P}}^{X_{\Delta^N}}\right)^{\hat{p}} d\sigma_{\mathbb{P}}\right)^{\frac{1}{\hat{p}}} \lesssim \sigma_{\mathbb{P}}(\Delta^N)^{-1} \approx \sigma_N(\Delta^N)^{-1}.\end{aligned}$$

This completes the proof of Proposition 2.42. □

Proof of Lemma 2.47. The proof is very elementary. The corkscrew conditions are as follows. Fix $\Delta_{\mathbb{P}} = \Delta_{\mathbb{P}}(x,r)$ with $0 < r < \operatorname{diam}(\partial\mathbb{P}) \approx 2^{-N}$. Since $x \in \partial\mathbb{P}$, which is a finite union of fattened boxes of size comparable to 2^{-N}, $x \in \partial I^*$ with $\ell(I) \approx 2^{-N}$. The fact that I^* is a cube yields easily that you can always find a corkscrew point in the segment that joints x and $X(I)$ (the center of I) with constants independent of N and this is an interior corkscrew point for $\mathbb{P}$. For the exterior corkscrew condition we notice that ∂I^* can be covered by the union of the Whitney boxes that are neighbors of I (i.e., whose boundaries touch). Then $x \in J$ with $\partial I \cap \partial J \neq \emptyset$. Note that $J \subset \operatorname{int}(J^*)$ and hence J^* cannot be one of the Whitney boxes that define $\mathbb{P}$. Take J' a neighbor of J: if $\operatorname{int}((J')^*) \subset \mathbb{P}$, then $(J')^*$ "bites" a small portion of J (since $(J')^*$ is a small dilation of J'); otherwise $(J')^*$ does not "bite" J. Eventually, we see that the part of J minus all these "bites" is contained in $\mathbb{P}^c$. Note that x is in one of this portions and therefore we can find a corkscrew point in the segment between x and the center of J. This is possible since $r \lesssim 2^{-N}$.

We show the Harnack chain condition. Let $X_1, X_2 \in \mathbb{P}$. Then, for each $i = 1, 2$ there exists $Q_i \in \mathbb{D}^{\Delta}$ and $Q_i' \in \mathbb{D}_{\mathcal{F}_N, Q_i}$ with $X_i \in \operatorname{int}(I_i^*)$ where $I_i \in \mathcal{W}_{Q_i'}^*$. Note that $\ell(Q_i) \approx \ell(Q_i') \approx \ell(I_i) \approx 2^{-N}$. If I_1^*, I_2^* overlap then the condition is clear.

Otherwise, $\operatorname{dist}(I_1^*, I_2^*) \approx \operatorname{dist}(I_1, I_2) \approx 2^{-N}$ and $|X_1 - X_2| \approx 2^{-N}$. Moreover, we can construct an appropriate chain in each fattened box between X_i and $X(I_i)$. In fact, if $\delta_{\mathbb{P}}(X_i) \approx 2^{-N}$ the number of balls is bounded by a dimensional constant, whereas if $\delta_{\mathbb{P}}(X_i) \ll 2^{-N}$ then we simply use that each cube I_i^* has the Harnack chain property with uniform constants. Therefore, it suffices to assume that X_i is the center of I_i^*. In such a case $\delta_{\mathbb{P}}(X_i) \approx 2^{-N}$, $i = 1, 2$, $|X_1 - X_2| \approx 2^{-N}$ and therefore we need to find a Harnack chain whose cardinality is uniformly bounded (in N).

Therefore we fix two fattened Whitney boxes I_1^*, I_2^* and we want to find a Harnack chain between the centers of such boxes. As just mentioned, if I_1^*, I_2^* overlap we can easily construct a finite (depending only on λ) chain between $X(I_1)$ and $X(I_2)$ (with respect to the domain $I_1^* \cup I_2^*$) such that $\operatorname{diam}(B_i) \approx 2^{-N}$, $\operatorname{dist}(B_i, \partial(I_1^* \cup I_2^*)) \approx 2^{-N}$. Let us observe that in such a case

$$\operatorname{dist}(B_i, \partial\mathbb{P}) \leq \operatorname{diam}(\mathbb{P}) \approx 2^{-N} \approx \operatorname{dist}(B_i, \partial(I_1^* \cup I_2^*)) \leq \operatorname{dist}(B_i, \partial\mathbb{P})$$

Thus the constructed Harnack chain is valid for the domain $\mathbb{P}$. If I_1^*, I_2^* do not overlap it suffices to find a chain of fattened boxes $J_1^*, \dots, J_K^*$ in $\mathbb{P}$ such that $J_1^* = I_1^*$, $J_K^* = I_2^*$ and with the property that J_k^* and J_{k+1}^* overlap. Notice that K is uniformly bounded since $\mathbb{P}$ is a finite union of fattened Whitney boxes with cardinality bounded uniformly in N. Notice that if Q_1, $Q_2 \in \mathbb{D}^\Delta$ we have $\ell(Q_1) = \ell(Q_2)$ and $\operatorname{dist}(Q_1, Q_2) \leq 4\,\ell(Q_2)$. Therefore, the construction of the sets $\mathcal{W}_Q$ guarantees that $\mathcal{W}_{Q_1} \cap \mathcal{W}_{Q_2}$ is non-empty (cf. (2.21)) and there exists $I \in \mathcal{W}_{Q_1} \cap \mathcal{W}_{Q_2}$. This leads to show that $I \subset \operatorname{int}(U_{Q_1}) \cap \operatorname{int}(U_{Q_2})$ which in turn gives $I \subset \Omega_{\mathcal{F}_N, Q_1} \cap \Omega_{\mathcal{F}_N, Q_2}$. To conclude we just observe that each $\Omega_{\mathcal{F}_N, Q_i}$ enjoys the Harnack chain property by Lemma 2.35 and this means that we can connect any box in $\Omega_{\mathcal{F}_N, Q_i}$ with I as before.

To show that $\partial\mathbb{P}$ is ADR we proceed as follows. Notice that $\partial\mathbb{P} \subset \cup_{Q \in \mathbb{D}^\Delta} \partial\Omega_{\mathcal{F}_N, Q}$ where each $\partial\Omega_{\mathcal{F}_N, Q}$ is ADR by Lemma 2.35 and the cardinality of $\mathbb{D}^\Delta$ is finite and depends only on M and the ADR constants of Ω. Thus the upper bound of the ADR condition for $\partial\mathbb{P}$ follows at once with constants that are independent of N. For the lower bound, let $B = B(x, r)$ with $x \in \partial\mathbb{P}$ and $r \leq \operatorname{diam}\mathbb{P} \approx 2^{-N}$. As $\partial\mathbb{P}$ is comprised of partial faces of fattened Whitney cubes I^* with $\ell(I) \approx 2^{-N}$, then x lies in a subset F of a (closed) face of I^*, with $F \subset B \cap \partial\mathbb{P}$, and $H^n(F \cap B) = H^n(F) \gtrsim r^n$, as desired. Again the constants are independent of N.

Having shown that $\mathbb{P}$ is an NTA domain with ADR boundary, with constants independent of N, we may invoke [DJ] to deduce absolute continuity of harmonic measure, along with the desired scale invariant estimate (2.48) for the Poisson kernel, with constants independent of N. □

With these preliminaries in hand, we now turn to the proof of (2.40). To this end, we fix Δ^N, and consider three cases. If $r(\Delta^N) \approx 2^{-N}$ then (2.43) gives the desired estimate. Assume now that $r(\Delta^N) \gg 2^{-N}$. We cover Δ^N by those dyadic cubes $Q^N \in \mathbb{D}_N(\partial\Omega_N)$ that meet Δ^N (we remind the reader that $\mathbb{D}_N(\partial\Omega_N)$ are the dyadic cubes associated to the ADR set $\partial\Omega_N$ with $\ell(Q_N) = 2^{-N}$). Note that for each of those cubes we have $X_{\Delta^N} \in \Omega_N \setminus 2\kappa_0 B(x_{Q^N}, C\,2^{-N})$ and then we can use

(2.44) with $Y = X_{\Delta^N}$ and the surface ball $\Delta^N(x_{Q^N}, C\,2^{-N})$:

$$\begin{aligned}\int_{\Delta^N}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N &\leq \sum_{Q^N}\sigma_N(Q^N)\fint_{Q^N}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N\\ &\lesssim \sum_{Q^N}\sigma_N(Q^N)\fint_{\Delta^N(x_{Q^N},C\,2^{-N})}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N\\ &\lesssim \sum_{Q^N}\sigma_N(Q^N)\left(\frac{\omega_N^{X_{\Delta^N}}\big(\Delta^N(x_{Q^N},C\,2^{-N})\big)}{\sigma_N\big(\Delta^N(x_{Q^N},C\,2^{-N})\big)}\right)^{\tilde p}\\ &\lesssim \sum_{Q^N}\sigma_N(Q^N)\left(\frac{\omega_N^{X_{\Delta^N}}(\Delta^N_{X_{Q^N}})}{\sigma_N(\Delta^N_{X_{Q^N}})}\right)^{\tilde p},\end{aligned}$$

where in the last estimate we have used that ω_N is doubling: let us observe that $\Delta^N(x_{Q^N}, C\,2^{-N})$ and $\Delta^N_{X_{Q^N}}$ are balls whose radii are comparable to 2^{-N} and the distance of their centers is controlled by 2^{-N}, and that $\delta_N(X_{\Delta^N}) \approx r(\Delta^N) \gg 2^{-N}$. Let us observe that $X_{Q^N} \in \Omega_N$ and $\delta(X_{Q^N}) \approx \delta_N(X_{Q^N}) \approx 2^{-N} \ll \delta_N(X_{\Delta^N}) \approx \delta(X_{\Delta^N})$. Then we use (2.41) to obtain

$$\int_{\Delta^N}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N \lesssim \sum_{Q^N}\sigma_N(Q^N)\left(\frac{\omega^{X_{\Delta^N}}(\Delta_{X_{Q^N}})}{\sigma(\Delta_{X_{Q^N}})}\right)^{\tilde p} \tag{2.49}$$

Given Q^N, its center x_{Q^N} is in $\partial\Omega_N \subset \Omega$ and therefore there exist $\tilde x_{Q^N} \in \partial\Omega$ such that $\delta(x_{Q^N}) = |x_{Q^N} - \tilde x_{Q^N}| \approx 2^{-N}$. Then using the dyadic cube structure in $\partial\Omega$, there exists a unique $R(Q^N) \in \mathbb{D}_N(\partial\Omega)$ that contains $\tilde x_{Q^N}$. We note that $\Delta_{X_{Q^N}}$ and $\Delta_{R(Q^N)}$ are surface balls in $\partial\Omega$ with radii comparable to 2^{-N} and whose centers are separated at most $C\,2^{-N}$. This means that $\Delta_{X_{Q^N}} \subset C\,\Delta_{R(Q^N)}$. On the other hand, one can show that $C\,\Delta_{R(Q^N)} \subset C'\,\Delta_{X_{\Delta^N}}$ by using that Q^N meets Δ^N and that $r(\Delta^N) \gg 2^{-N}$.

Next we want to control the overlapping of the family $\{C\,\Delta_{R(Q^N)}\}_{Q^N}$. It is easy to see that if $C\,\Delta_{R(Q^N)} \cap C\,\Delta_{R(Q_1^N)} \neq \emptyset$ then $|x_{Q^N} - x_{Q_1^N}| \lesssim 2^{-N} = \ell(Q_1^N) = \ell(Q^N)$. Thus the fact that the cubes Q^N are all dyadic disjoint with side length 2^{-N} yields

$$\#\{Q^N : C\,\Delta_{R(Q^N)} \cap C\,\Delta_{R(Q_1^N)} \neq \emptyset\} \leq \#\{Q^N : |x_{Q^N} - x_{Q_1^N}| \lesssim 2^{-N}\} \leq C.$$

Gathering the previous facts we obtain that

$$\sum_{Q^N}\chi_{\Delta_{X_{Q^N}}} \leq \sum_{Q^N}\chi_{C\,\Delta_{R(Q^N)}} \leq C\,\chi_{C'\,\Delta_{X_{\Delta^N}}}.$$

Set $\Delta_0 = C'\,\Delta_{X_{\Delta^N}}$ and observe that by the Harnack Chain property $\omega^{X_{\Delta^N}} \approx \omega^{X_{\Delta_0}}$. Thus plugging these estimates into (2.49) we conclude that

$$\int_{\Delta^N}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N \lesssim \sum_{Q^N}\sigma_N(Q^N)\left(\frac{\omega^{X_{\Delta_0}}(\Delta_{X_{Q^N}})}{\sigma(\Delta_{X_{Q^N}})}\right)^{\tilde p} \lesssim \sum_{Q^N}\int_{\Delta_{X_{Q^N}}}\left(k^{X_{\Delta_0}}\right)^{\tilde p}\,d\sigma$$

$$\lesssim \int_{\Delta_0} \left(k^{X_{\Delta_0}}\right)^{\tilde{p}}\, d\sigma \leq \left(\int_{\Delta_0} \left(k^{X_{\Delta_0}}\right)^{p}\, d\sigma\right)^{\frac{\tilde{p}}{p}} \sigma(\Delta_0)^{\frac{p-\tilde{p}}{p}} \lesssim \sigma(\Delta_0)^{1-\tilde{p}} \approx \sigma_N(\Delta^N)^{1-\tilde{p}},$$

where we have used $\tilde{p} \leq p$ and our hypothesis (1.24) and where the involved constants are independent of N. This completes the case $r(\Delta^N) \gg 2^{-N}$.

To complete Step 1, we need to establish (2.40) for $r := r(\Delta^N) \ll 2^{-N}$. Set $\Delta^N = B(x_N, r) \cap \partial\Omega_N$. Given $\Delta^N(x_N, C\,2^{-N})$ we consider the associated $\mathbb{P}$ as before. For a fixed $y \in \Delta^N = B(x_N, r) \cap \partial\Omega_N$, we have that $y \in \partial I^*$ with $\ell(I) \approx 2^{-N}$ and $\text{int}(I^*) \subset \Omega_N$. We define $\mathbb{P}_{I^*}$ as before and recall that it is an NTA domain with constants independent of N. Let us note that $5\Delta^N \subset \partial\Omega_N \cap \partial\mathbb{P}_{I^*}$, since $r \ll 2^{-N}$. We let Y_{Δ^N} denote a Corkscrew point with respect to Δ^N, for the domain $\mathbb{P}_{I^*}$. Let $c < 1$ be small enough such that $Y_{\Delta^N} \notin T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$, where $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ is the Carleson box with respect to $\Delta^N(y, 2\,c\,r)$ relative to the domain $\mathbb{P}_{I^*}$ (note that c depends on the various geometric constants of $\mathbb{P}_{I^*}$ which are all independent of N, see (2.34)). Notice that by Lemma 2.35, $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ inherits the (interior) Corkscrew and Harnack Chain conditions and also the ADR property from $\mathbb{P}_{I^*}$. Moreover, $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ also inherits the exterior corkscrew condition, with uniform bounds. Indeed, consider a surface ball $\Delta_\star(z,\rho) \subset \partial T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$, with $z \in \partial T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$, and $\rho \lesssim r \approx \text{diam}(T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)})$. If $\text{dist}(z, \partial\mathbb{P}_{I^*}) \leq \rho/100$, then $B(z,\rho) \cap \left(\mathbb{R}^{n+1} \setminus T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}\right)$ contains an exterior Corkscrew point inherited from $\mathbb{P}_{I^*}$. On the other hand, if $\text{dist}(z, \partial\mathbb{P}_{I^*}) > \rho/100$, then z lies on a face of some fattened Whitney cube $J^* \in \mathcal{W}(\mathbb{P}_{I^*})$, of side length $\ell(J^*) \gtrsim \rho$, for which there is some adjoining Whitney cube J' containing an exterior Corkscrew point in $B(z,\rho) \cap \left(\mathbb{P}_{I^*} \setminus T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}\right)$. Consequently $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ is an NTA domain with constants independent of N, r and y.

Set $u(Z) = G_N(Z, Y_{\Delta^N})$ and $v(Z) = G_{\mathbb{P}_{I^*}}(Z, Y_{\Delta^N})$ where G_N, $G_{\mathbb{P}_{I^*}}$ are respectively the Green functions associated with the domains Ω_N and $\mathbb{P}_{I^*}$. Then u and v are non-negative harmonic functions in $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ (since $Y_{\Delta^N} \notin T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$). Notice that for c small, $r \ll 2^{-N}$, and $y \in \Delta^N = \Delta^N(x_N, r)$ we have that

$$\Delta^N_1 := \Delta^N(y, cr) := \left(B(y, c\,r) \cap \partial\Omega_N\right) \subset \left(\partial\Omega_N \cap \partial\mathbb{P}_{I^*} \cap \partial T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}\right).$$

In particular, $u\big|_{\Delta^N_1} = v\big|_{\Delta^N_1} = 0$. As noted above, $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ is an NTA domain with constants independent of N and thus by [JK, Lemma 4.10]) we have the comparison principle:

$$\frac{u(Z)}{v(Z)} \approx \frac{u(X_0)}{v(X_0)}, \qquad \forall Z \in B(y, cr/C) \cap T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)},$$

with C sufficiently large but independent of N, and where $X_0 \in T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$ is a corkscrew point associated to the surface ball Δ^N_1, relative to each of the domains Ω_N, $\mathbb{P}_{I^*}$ and $T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$. By the Harnack chain condition in $\mathbb{P}_{I^*}$ and since

$$|X_0 - Y_{\Delta^N}| \approx \text{dist}(X_0, \partial\mathbb{P}_{I^*}) \approx \text{dist}(Y_{\Delta^N}, \partial\mathbb{P}_{I^*}) \approx r,$$

one can show that $u(X_0)/v(X_0) \approx 1$ which eventually leads to $u(Z) \approx v(Z)$ for all $Z \in B(y, cr/C) \cap T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$.

We can now apply Lemma 2.12 (since, as observed above, Ω_N satisfies the required "qualitative assumption"), so that if we take $s \ll r$ and write $\Delta^N(y,s) = B(y,s)\cap\Omega_N$, we have

$$\frac{u(X_{\Delta^N(y,s)})}{s} = \frac{G_N(X_{\Delta^N(y,s)}, X_{\Delta^N})}{s} \approx \frac{\omega_N^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_N(\Delta^N(y,s))}.$$

The same can be done with $G_{\mathbb{P}_{I^*}}$ (indeed $\mathbb{P}_{I^*}$ is NTA) after observing that $\Delta^N(y,s)$ is also a surface ball for $\mathbb{P}_{I^*}$ and that, after using Harnack if needed, $X_{\Delta^N(y,s)}$ may be taken to be a corkscrew point with respect to $\mathbb{P}_{I^*}$ for that surface ball. Then,

$$\frac{v(X_{\Delta^N(y,s)})}{s} = \frac{G_{\mathbb{P}_{I^*}}(X_{\Delta^N(y,s)}, X_{\Delta^N})}{s} \approx \frac{\omega_{\mathbb{P}_{I^*}}^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_{\mathbb{P}_{I^*}}(\Delta^N(y,s))}$$

Note that if s is small enough then $Z = X_{\Delta^N(y,s)} \in B(y, cr/C)\cap T^{\mathbb{P}_{I^*}}_{\Delta^N(y,2\,c\,r)}$. This allows us to gather the previous estimates and conclude that

$$\frac{\omega_N^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_N(\Delta^N(y,s))} \approx \frac{\omega_{\mathbb{P}_{I^*}}^{X_{\Delta^N}}(\Delta^N(y,s))}{\sigma_{\mathbb{P}_{I^*}}(\Delta^N(y,s))}.$$

Next we use the Lebesgue differentiation theorem and obtain that $k_N^{X_{\Delta^N}}(y) \approx k_{\mathbb{P}_{I^*}}^{X_{\Delta^N}}(y)$ for a.e. $y\in\Delta^N$. Then

$$\begin{aligned}\left(\fint_{\Delta^N}\left(k_N^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_N\right)^{\frac{1}{\tilde p}} &\lesssim \left(\fint_{\Delta^N}\left(k_{\mathbb{P}_{I^*}}^{X_{\Delta^N}}\right)^{\tilde p}\,d\sigma_{\mathbb{P}_{I^*}}\right)^{\frac{1}{\tilde p}}\\ &\le \left(\fint_{\Delta^N}\left(k_{\mathbb{P}_{I^*}}^{X_{\Delta^N}}\right)^{\hat p}\,d\sigma_{\mathbb{P}_{I^*}}\right)^{\frac{1}{\hat p}} \lesssim \sigma_{\mathbb{P}_{I^*}}(\Delta^N)^{-1} \approx \sigma_N(\Delta^N)^{-1}\end{aligned}$$

where we have used that $\tilde p \le \hat p$ and (2.48). This completes the proof of (2.40)

Remark. We notice that to obtain (2.48), in place of using [DJ], we could have invoked [VV]: $\mathbb{P}$ is a polyhedral domain (its boundary consists of a finite number of flat "faces"), is NTA and has the ADR property, thus $k_{\mathbb{P}}$ is a RH_2 weight. Thus $\hat p = 2$ and $\tilde p = \min\{p,2\}$.

2.3. **Step 2: Local Tb theorem for square functions.** Having already established Step 1, we want to show that Ω_N has the UR property with uniform bounds. In the last step we shall show that this ultimately implies that Ω inherits this property. Therefore, in all the remaining steps, but the last one, we drop the already fixed subindex/superindex N everywhere and write Ω to denote the corresponding approximating domain Ω_N. Our main assumption is that (2.40) holds, and this rewrites as

$$\int_\Delta \left(k^{X_\Delta}\right)^{\tilde p}\,d\sigma \le C\,\sigma(\Delta)^{1-\tilde p}, \tag{2.50}$$

where $\tilde p$ and C are independent of N.

We introduce some notation. We have already defined $\mathcal{W} = \mathcal{W}(\Omega)$ the collection of Whitney boxes of Ω. We also define $\mathcal{W}^{ext} = \mathcal{W}(\Omega_{ext})$ the collection of Whitney boxes of $\Omega_{ext} = \mathbb{R}^{n+1}\setminus\overline{\Omega}$. Set $\mathcal{W}_0 = \mathcal{W}\cup\mathcal{W}^{ext}$ —notice that we can get directly this collection by taking the Whitney decomposition of $\mathbb{R}^{n+1}\setminus\partial\Omega$. We

can then consider as before $\mathcal{W}_Q$ with the same fixed C_0 as before (this guarantees among other things that $\mathcal{W}_Q \neq \emptyset$). With that fixed C_0 we define analogously $\mathcal{W}_Q^{ext}$ (using only I's in $\mathcal{W}^{ext}$). Notice that we have not assumed exterior corkscrew points for Ω_{ext} and therefore $\mathcal{W}_Q^{ext}$ might be the null set. We then define

$$V_Q = \bigcup_{I\in\mathcal{W}_Q} I, \qquad \Lambda(x) = \bigcup_{x\in Q\in\mathbb{D}} V_Q \qquad \Lambda_{Q_0}(x) = \bigcup_{x\in Q\in\mathbb{D}_{Q_0}} V_Q$$

and analogously

$$V_Q^{ext} = \bigcup_{I\in\mathcal{W}_Q^{ext}} I, \qquad \Lambda^{ext}(x) = \bigcup_{x\in Q\in\mathbb{D}} V_Q^{ext} \qquad \Lambda_{Q_0}^{ext}(x) = \bigcup_{x\in Q\in\mathbb{D}_{Q_0}} V_Q^{ext},$$

where $x \in \partial\Omega$ and $Q_0 \in \mathbb{D}$. We also consider the "two-sided" cones $\tilde{\Lambda}(x) = \Lambda(x) \cup \Lambda^{ext}(x)$ and $\tilde{\Lambda}_{Q_0}(x) = \Lambda_{Q_0}(x) \cup \Lambda_{Q_0}^{ext}(x)$.

In this step we are going to apply the following local Tb theorem for square functions in [GM][3]:

Theorem 2.51 (Local Tb theorem for square functions on ADR sets, [GM])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected open set whose boundary $\partial\Omega$ is ADR. We assume that there is an exponent $q \in (1, 2]$, and a finite constant $\mathbb{A}_0 > 1$ such that for every $Q \in \mathbb{D}$ there exists a function b_Q satisfying*

$$\int_{\partial\Omega} |b_Q|^q \, d\sigma \leq \mathbb{A}_0 \, \sigma(Q) \tag{2.52}$$

$$\left| \int_Q b_Q \, d\sigma \right| \geq \frac{1}{\mathbb{A}_0} \, \sigma(Q) \tag{2.53}$$

$$\int_Q \left(\iint_{\tilde{\Lambda}_Q(x)} |\nabla^2 \mathcal{S} b_Q(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{q}{2}} d\sigma(x) \leq \mathbb{A}_0 \, \sigma(Q) \tag{2.54}$$

Then,

$$\iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S} f(Y)|^2 \, \delta(Y) \, dY \leq C \, \|f\|^2_{L^2(\partial\Omega)} \tag{2.55}$$

We observe that we can easily show

$$\int_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S} f(Y)|^2 \, \delta(Y) \, dY \approx \int_{\partial\Omega} \int_{\tilde{\Lambda}(x)} |\nabla^2 \mathcal{S} f(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \, d\sigma(x)$$

which is nothing but the comparability of the "vertical" and "conical" square functions. In this way, we see (2.54) as an L^q-testing condition for the local (conical) square function and the conclusion states the L^2 boundedness of the (conical or vertical) square function.

[3] In fact, the theorem of [GM] treats more general square functions, whose kernels satisfy some appropriate point-wise smoothness and decay conditions, but we have specialized the result here to suit our purposes.

2.3.1. *Outline of the remainder of the proof.* Let us now outline our approach to the remainder of the proof of our main theorem. We first note that the conclusion of Theorem 2.51, namely estimate (2.55), is simply a re-statement of (1.10), which by [DS2] implies uniform rectifiability of $\partial\Omega$ (cf. Definition 1.7). We shall establish (2.55) with $\Omega = \Omega_N$, but with bounds that are uniform in N. Thus, we shall obtain that $\partial\Omega_N$ is UR, uniformly in N. In subsection 2.6, we then "pass to the limit", to deduce that $\partial\Omega$ is UR, for our original domain Ω.

Thus, our main goal is to construct a system of accretive testing functions $\{b_Q\}$, verifying the hypotheses of Theorem 2.51. We shall construct these testing functions as normalized Poisson kernels, which will satisfy the size condition (2.52), and the accretivity condition (2.53), by virtue of the higher integrability estimate (2.50), and Bourgain's estimate (2.4), respectively. The principal difficulty is to verify condition (2.54). In the reminder of this subsection, we carry out this program, eventually reducing the proof of (2.54) to a certain localized square function estimate for the Green function, namely (2.62). In turn, we shall establish the latter by first extending the square function/non-tangential maximal function estimates of [DJK] to the setting of a 1-sided NTA domain with ADR boundary, and then proving bounds for the localized non-tangential maximal function of the gradient of the Green function. These last two steps are carried out in subsections 2.4 and 2.5, respectively.

2.3.2. *Verification of* (2.52)-(2.54). In order to apply Theorem 2.51, and to define the functions b_Q, we shall require some geometric preliminaries. Given $Q \in \mathbb{D}$, we recall that there exists a surface ball $\Delta_Q = \Delta(x_Q, r_Q)$ such that $\Delta_Q \subset Q \subset C\,\Delta_Q = B(x_Q, C\,r_Q) \cap \partial\Omega$ with $r_Q \approx \ell(Q)$. Let κ_1 be large enough such that $C\,r_Q < \kappa_1 \ell(Q)$ and also with $\kappa_1 > \kappa_0$ where κ_0 is given in (2.64) below. Set $\tilde{B}_Q = B(x_Q, \kappa_1\,\ell(Q))$. We notice that if $x \in Q$ and $Y \in \Lambda_Q(x)$ then $Y \in V_{Q'}$ with $x \in Q' \in \mathbb{D}_Q$ and $Y \in I \in \mathcal{W}_{Q'}$. Thus,

$$|Y - x_Q| \leq \ell(I) + d(I, Q') + \ell(Q') + |x - x_Q| \lesssim C_0\ell(Q') + \ell(Q) \lesssim C_0\ell(Q).$$

The same can be done for $Y \in \Lambda_Q^{ext}(x)$ and therefore by taking κ_1 sufficiently large (depending on K_0) we have

$$\bigcup_{x\in Q} \tilde{\Lambda}(x) \subset \tilde{B}_Q. \tag{2.56}$$

Next we set $\hat{B}_Q = \kappa_2\,\tilde{B}_Q$, with κ_2 large enough so that $\hat{X}_Q$, the corkscrew point relative to $\hat{\Delta}_Q = \hat{B}_Q \cap \Omega$, satisfies $\hat{X}_Q \notin 6\,\tilde{B}_Q$ (it suffices to take $\kappa_2 = 6/c$ with c the constant that appears in the corkscrew condition).

Bearing in mind the previous considerations, we are ready to define our functions b_Q as follows: we set $b_Q = \sigma(Q)\,\eta_Q\,k^{\hat{X}_Q}$ where η_Q is a smooth cut-off (defined in $\mathbb{R}^{n+1}$) with $0 \leq \eta_Q \leq 1$, $\operatorname{supp}\eta_Q \subset 5\,\hat{B}_Q$, $\eta_Q \equiv 1$ in $4\,\hat{B}_Q$, and $\|\nabla\eta_Q\|_\infty \lesssim \ell(Q)^{-1}$. We also take $q = \tilde{p}$ and we recall that $1 < \tilde{p} \leq p \leq 2$.

Using Harnack's inequality, that $q = \tilde{p}$ and (2.50) we obtain (2.52):

$$\int_{\partial\Omega} |b_Q|^q\,d\sigma \leq \sigma(Q)^q \int_{5\,\hat{\Delta}_Q} \left(k^{\hat{X}_Q}\right)^q\,d\sigma \lesssim \sigma(Q)^q \int_{5\,\hat{\Delta}_Q} \left(k^{X_{5\hat{\Delta}_Q}}\right)^q\,d\sigma$$

$$\lesssim \tilde{C}\,\sigma(Q)^q\,\sigma(5\,\hat{\Delta}_Q)^{1-q} \lesssim \sigma(Q).$$

Regarding, (2.53) we have by Lemma 2.2 and the Harnack chain condition that

$$\left|\int_Q b_Q\,d\sigma\right| = \sigma(Q)\int_Q \eta_Q\,k^{\hat{X}_Q}\,d\sigma \gtrsim \sigma(Q)\,\omega^{X_Q}(\Delta_Q) \geq \sigma(Q)/C.$$

Next we show (2.54). Let $X \in \hat{B}_Q \cap \Omega_{ext}$. Then

$$\begin{aligned}
\nabla_X^2 \mathcal{S}b_Q(X) &= \int_{\partial\Omega} \nabla_X^2 \mathcal{E}(X-y)\,b_Q(y)\,d\sigma(y)\\
&= \sigma(Q)\int_{\partial\Omega} \nabla_X^2 \mathcal{E}(X-y)\,\eta_Q(y)\,d\omega^{\hat{X}_Q}(y)\\
&= \sigma(Q)\int_{\partial\Omega} \nabla_X^2 \mathcal{E}(X-y)\,(\eta_Q(y)-1)\,d\omega^{\hat{X}_Q}(y)\\
&\qquad + \sigma(Q)\int_{\partial\Omega} \nabla_X^2 \mathcal{E}(X-y)\,d\omega^{\hat{X}_Q}(y)\\
&= I_1(X) + I_2(X).
\end{aligned}$$

For I_2 we observe that $u(Z) = \nabla_X^2 \mathcal{E}(X-Z)$ is harmonic in Ω and $C^2(\bar{\Omega})$ since $X \in \Omega_{ext}$. Thus, for every $Z \in \Omega$ we have

$$\nabla_X^2 \mathcal{E}(X-Z) = u(Z) = \int_{\partial\Omega} u(y)\,d\omega^Z(y) = \int_{\partial\Omega} \nabla_X^2 \mathcal{E}(X-y)\,d\omega^Z(y). \tag{2.57}$$

We would like to point out that we have implicitly used uniqueness of the solution which follows from the maximum principle even for an unbounded domain since $\nabla_X^2 \mathcal{E}(X-Z) \to 0$ as $Z \to \infty$. We apply (2.57) with $Z = \hat{X}_Q$ and note that since $|X - \hat{X}_Q| \approx \ell(Q)$ for every $X \in \hat{B}_Q \cap \Omega_{ext}$, we therefore obtain

$$|I_2(X)| = \sigma(Q)|\nabla_X^2 \mathcal{E}(X-\hat{X}_Q)| \lesssim \sigma(Q)\,|X-\hat{X}_Q|^{-(n+1)} \approx \ell(Q)^{-1}.$$

For I_1, we observe that Ω is an approximating domain whose boundary consists of portions of faces of fattened Whitney cubes of size comparable to 2^{-N}. Thus, its (outward) unit normal ν is well defined a.e. on $\partial\Omega$, and we can apply the divergence theorem to obtain

$$\begin{aligned}
I_1(X) &= \sigma(Q)\,\nabla_X^2 \int_{\partial\Omega} \mathcal{E}(X-y)\,(\eta_Q(y)-1)\,d\omega^{\hat{X}_Q}(y)\\
&= \sigma(Q)\,\nabla_X^2 \int_{\partial\Omega} \mathcal{E}(X-y)\,(\eta_Q(y)-1)\,\nabla_Y G(\hat{X}_Q,y)\cdot\nu(y)\,d\sigma(y)\\
&= \sigma(Q)\,\nabla_X^2 \iint_\Omega \mathrm{div}_Y\left(\mathcal{E}(X-Y)\,(\eta_Q(Y)-1)\,\nabla_Y G(\hat{X}_Q,Y)\right)dY\\
&= \sigma(Q)\,\nabla_X^2 \iint_\Omega \nabla_Y\big(\mathcal{E}(X-Y)\big)\,(\eta_Q(Y)-1)\,\nabla_Y G(\hat{X}_Q,Y)\,dY\\
&\qquad + \sigma(Q)\,\nabla_X^2 \iint_\Omega \mathcal{E}(X-Y)\nabla\eta_Q(Y)\,\nabla_Y G(\hat{X}_Q,Y)\,dY\\
&\qquad\qquad + \sigma(Q)\,\nabla_X^2 \iint_\Omega \mathcal{E}(X-Y)(\eta_Q(Y)-1)\,\mathrm{div}_Y(\nabla_Y G(\hat{X}_Q,Y))\,dY\\
&=: I_{11}(X) + I_{12}(X) + 0\,,
\end{aligned}$$

where we have used that the term in the next-to-last line vanishes, since $\eta_Q - 1$ is supported in $\mathbb{R}^{n+1} \setminus 4\,\hat{B}_Q$, and since $\hat{X}_Q \in \hat{B}_Q$ implies that $G(\hat{X}_Q, \cdot)$ is harmonic in $\Omega \setminus \hat{B}_Q$. Notice that the integration by parts can be justified even when Ω is unbounded, since $\mathcal{E}$ and G have sufficient decay at infinity.

We estimate the terms I_{11} and I_{12} in turn. First, notice that if $X \in \hat{B}_Q$ and $Y \in \Omega \setminus 4\,\hat{B}_Q$ we have $r(\hat{B}_Q) \lesssim |Y - x_Q| \approx |Y - \hat{X}_Q| \approx |Y - X|$. Then, writing $S_k(Q) = 2^{k+1}\,\hat{B}_Q \setminus 2^k\,\hat{B}_Q$, $k \geq 2$, and using that $\eta_Q - 1$ is supported in $\mathbb{R}^{n+1} \setminus 4\,\hat{B}_Q$, we have that for every $X \in \hat{B}_Q$,

$$
\begin{aligned}
|I_{11}(X)| &\leq \sigma(Q) \iint_\Omega |\nabla_X^2 \nabla_Y \mathcal{E}(X-Y)|\,|\eta_Q(Y) - 1|\,|\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&\lesssim \sigma(Q) \iint_{\Omega\setminus 4\,\hat{B}_Q} |Y - X|^{-(n+2)}\,|\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&\lesssim \sigma(Q) \sum_{k=2}^{\infty} \iint_{\Omega\cap S_k(Q)} |Y - x_Q|^{-(n+2)}\,|\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&\lesssim \ell(Q)^{-2} \sum_{k=2}^{\infty} 2^{-(n+2)\,k} \iint_{\Omega\cap S_k(Q)} |\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&=: \ell(Q)^{-2} \sum_{k=2}^{\infty} 2^{-(n+2)\,k}\,\mathcal{I}_k.
\end{aligned}
\tag{2.58}
$$

To estimate $\mathcal{I}_k$ we cover $S_k(Q)$ by a purely dimensional number of balls meeting $S_k(Q)$ and whose radii are $2^{k-5}\,r(\hat{B}_Q)$. Then, it suffices to get an estimate with the integral restricted to such a ball B_k. We may assume without loss of generality that $2\,B_k \not\subset \Omega_{ext}$ for otherwise we have that $\Omega \cap S_k(Q) \cap B_k = \emptyset$. We then have two cases: $2\,B_k \subset \Omega$ and $2\,B_k \not\subset \Omega$. In the second case, since $2\,B_k$ is neither contained in Ω nor in Ω_{ext}, there exists $y_k \in \partial\Omega \cap 2\,B_k$. We then set $\tilde{B}_k = B(y_k, 3\,r(B_k))$, and observe that this ball is centered on $\partial\Omega$ and contains B_k. Since $G(\hat{X}_Q, \cdot)$ is harmonic in $2\tilde{B}_k \cap \Omega$ and vanishes on $\partial\Omega$, and since Ω is an approximating domain in which the Gauss/Green theorem holds, we can use Caccioppoli's inequality at the boundary to write

$$
\begin{aligned}
\mathcal{I}_{k,B_k} &:= \iint_{\Omega\cap S_k(Q)\cap B_k} |\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&\leq \iint_{\Omega\cap\tilde{B}_k} |\nabla_Y G(\hat{X}_Q, Y)|\,dY \\
&\lesssim |\tilde{B}_k|^{\frac{1}{2}} \left(\iint_{\Omega\cap\tilde{B}_k} |\nabla_Y G(\hat{X}_Q, Y)|^2\,dY \right)^{1/2} \\
&\lesssim (2^k\,\ell(Q))^{\frac{n+1}{2}}\,r(\tilde{B}_k)^{-1} \left(\iint_{\Omega\cap 2\,\tilde{B}_k} |G(\hat{X}_Q, Y)|^2\,dY \right)^{1/2} \\
&\lesssim (2^k\,\ell(Q))^{\frac{n+1}{2}-1} (2^k\,\ell(Q))^{1-n}\,|\tilde{B}_k|^{\frac{1}{2}} \\
&= 2^k\,\ell(Q)\,,
\end{aligned}
$$

by the size estimates for the Green function plus the fact that $|\hat{X}_Q - Y| \geq 2^{k-2}\, r(\hat{B}_Q)$ for $Y \in \Omega \cap 2\,\tilde{B}_k$.

In the other case, i.e., when $2\,B_k \subset \Omega$, the situation is simpler: we just use the (interior) Caccioppoli inequality and repeat the same computations with B_k in place of $\tilde{B}_k$. Summing over the collection of B_k's covering $S_k(Q)$, which for fixed k is a family of uniformly bounded cardinality, we conclude that

$$\mathcal{I}_k \lesssim 2^k\, \ell(Q). \tag{2.59}$$

We plug this estimate into (2.58) to obtain

$$|I_{11}(X)| \lesssim \ell(Q)^{-2} \sum_{k=2}^{\infty} 2^{-(n+2)\,k}\, \mathcal{I}_k \lesssim \ell(Q)^{-1}.$$

Let us estimate $I_{12}(X)$. We notice that $\nabla\eta_Q$ is supported in $5\,\hat{B}_Q \setminus 4\,\hat{B}_Q$. Take $Y \in 5\,\hat{B}_Q \setminus 4\,\hat{B}_Q$. If $X \in \hat{B}_Q$ then $|X - Y| \approx |\hat{X}_Q - Y| \approx \ell(Q)$ which yields $|\nabla^2_X \mathcal{E}(X - Y)| \lesssim |X - Y|^{-(n+1)} \approx \ell(Q)^{-(n+1)}$. Then using (2.59) with $k = 2$ we obtain

$$|I_{12}(X)| \lesssim \ell(Q)^{-2} \iint_{\Omega \cap 5\,\hat{B}_Q \setminus 4\,\hat{B}_Q} |\nabla_Y G(\hat{X}_Q, Y)|\, dY \lesssim \ell(Q)^{-2}\, \mathcal{I}_2 \lesssim \ell(Q)^{-1}.$$

Collecting our estimates for I_{11} and I_{12} we have shown that $|I_1(X)| \lesssim \ell(Q)^{-1}$. Thus, for all $X \in \hat{B}_Q \cap \Omega_{ext}$ we have

$$|\nabla^2_X \mathcal{S} b_Q(X)| \leq C\, \ell(Q)^{-1}.$$

Remark 2.60. In the previous argument, for the estimate of I_1 we have only used that $X \in \hat{B}_Q$ and we have not used that $X \in \Omega_{ext}$.

Next we let $X \in \hat{B}_Q \cap \Omega$, and suppose first that $\delta(X) \gtrsim \ell(Q)$. If $y \in 5\hat{\Delta}_Q$ we have $|\nabla^2_X \mathcal{E}(X - y)| \lesssim |X - y|^{-(n+1)} \leq \delta(X)^{-(n+1)} \lesssim \ell(Q)^{-(n+1)}$. Consequently,

$$|\nabla^2_X \mathcal{S} b_Q(X)| = \left| \int_{\partial\Omega} \nabla^2_X \mathcal{E}(X - y)\, b_Q(y)\, d\sigma(y) \right| \lesssim \ell(Q)^{-1} \omega^{\hat{X}_Q}(5\,\hat{\Delta}_Q) \leq \ell(Q)^{-1}.$$

It remains to treat the case $X \in \hat{B}_Q \cap \Omega$, with $\delta(X) \leq c\ell(Q)$, where c is to be chosen. Notice that

$$2\,\ell(Q)/C \leq \delta(\hat{X}_Q) \leq |\hat{X}_Q - X| + \delta(X) \leq |\hat{X}_Q - X| + c\,\ell(Q).$$

Then we pick $c < C^{-1}$ we obtain $|\hat{X}_Q - X| > \ell(Q)/C$. Next, we write as before $\nabla^2_X \mathcal{S} b_Q(X) = I_1(X) + I_2(X)$. For I_1, since $X \in \hat{B}_Q$, by Remark 2.60 we conclude that $|I_1(X)| \lesssim \ell(Q)^{-1}$. For I_2 we observe that (2.1) gives

$$|I_2(X)| = \sigma(Q) \left| \int_{\partial\Omega} \nabla^2_X \mathcal{E}(X - y)\, d\omega^{\hat{X}_Q}(y) \right| = \sigma(Q) \left| \nabla^2_X \big(\mathcal{E}(X - \hat{X}_Q) - G(X, \hat{X}_Q) \big) \right|$$
$$\lesssim \sigma(Q)\, |X - \hat{X}_Q|^{-(n+1)} + \sigma(Q)\, |\nabla^2_X G(X, \hat{X}_Q)| \lesssim \ell(Q)^{-1} + \sigma(Q)\, |\nabla^2_X G(X, \hat{X}_Q)|.$$

Therefore, collecting all of our estimates, we have shown that for every $X \in \hat{B}_Q$,

$$|\nabla^2_X \mathcal{S} b_Q(X)| \leq C\, \ell(Q)^{-1} + \sigma(Q)\, |\nabla^2_X G(X, \hat{X}_Q)| \chi_{\hat{B}_Q \cap \Omega \cap \{\delta(X) \leq c\ell(Q)\}}(X). \tag{2.61}$$

Notice that if $x \in \partial\Omega$ then

$$\iint_{\tilde{\Lambda}_Q(x)} \frac{dY}{\delta(Y)^{n-1}} \leq \sum_{x\in Q'\in\mathbb{D}_Q} \sum_{I\in\mathcal{W}_{Q'}\cup\mathcal{W}^{ext}_{Q'}} \iint_I \frac{dY}{\delta(Y)^{n-1}}$$

$$\approx \sum_{x\in Q'\in\mathbb{D}_Q} \sum_{I\in\mathcal{W}_{Q'}\cup\mathcal{W}^{ext}_{Q'}} \frac{\ell(I)^{n+1}}{\ell(I)^{n-1}} \lesssim \sum_{x\in Q'\in\mathbb{D}_Q} \ell(Q')^2 \lesssim \ell(Q)^2$$

where we have used that $\#(\mathcal{W}_{Q'}\cup\mathcal{W}^{ext}_{Q'}) \lesssim C_{C_0}$. Therefore, using (2.56) and (2.61) we have

$$\int_Q \left(\iint_{\tilde{\Lambda}_Q(x)} |\nabla^2 \mathcal{S}b_Q(Y)|^2 \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{q}{2}} d\sigma(x)$$

$$\lesssim \ell(Q)^{-q} \int_Q \left(\iint_{\tilde{\Lambda}_Q(x)} \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{q}{2}} d\sigma(x)$$

$$+ \sigma(Q)^q \int_Q \left(\iint_{\Lambda_Q(x)\cap\{\delta(Y)\leq c\ell(Q)\}} |\nabla^2_Y G(Y,\hat{X}_Q)|^2 \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{q}{2}} d\sigma(x)$$

$$\lesssim \sigma(Q) + \sigma(Q)^q \int_Q S_Q u(x)^q \, d\sigma(x)$$

where we set $u(Y) = \nabla_Y G(Y,\hat{X}_Q)$ and

$$S_Q u(x) = \left(\iint_{\Gamma_Q(x)} |\nabla u(Y)|^2 \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{1}{2}}, \qquad \Gamma_Q(x) := \bigcup_{x\in Q'\in\mathbb{D}_Q} U_{Q'}$$

(note then that $S_Q u$ involves $\nabla^2 G$). We claim that

$$\int_Q S_Q u(x)^q \, d\sigma(x) \lesssim \sigma(Q)^{1-q}. \tag{2.62}$$

Assuming this momentarily we obtain as desired (2.54):

$$\int_Q \left(\iint_{\tilde{\Lambda}_Q(x)} |\nabla^2 \mathcal{S}b_Q(Y)|^2 \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{q}{2}} d\sigma(x) \lesssim \sigma(Q).$$

Modulo the claim (2.62), we have now verified all the hypotheses of Theorem 2.51 and we therefore conclude that (2.55) holds. The latter in turn implies that $\partial\Omega$ is UR (see [DS2, p. 44]).

To complete this stage of our proof, it remains to establish (2.62). We do this in Section 2.5, after first proving a bound for the square function in terms of the non-tangential maximal function in Section 2.4. As mentioned before, at present, Ω is actually an approximating domain (i.e., Ω stands for Ω_N with N large enough). Thus, the conclusion that we have obtained in the current step is that for every $N \gg 1$, we have that Ω_N has the UR property with uniform constants. In Section 2.6 we show that this property may be transmitted to Ω, thereby completing the proof of Theorem 1.23.

2.4. **Step 3: Good-λ inequality for the square function and the non-tangential maximal function.** We recall that $U_Q = \bigcup_{W_Q^*} I^*$ with $I^* = (1+\lambda)I$, where $0 < \lambda \leq \lambda_0/2$ was fixed above. As mentioned in Remark 2.38, since $0 < 2\lambda \leq \lambda_0$, the fattened Whitney boxes $I^{**} = (1+2\lambda)I$, and corresponding Whitney regions $U_Q^* := \bigcup_{W_Q^*} I^{**}$ enjoy the same properties as do I and U_Q.

Let us set some notation: for fixed $Q_0 \in \mathbb{D}$, and for $x \in Q_0$,

$$S_{Q_0}u(x) = \left(\iint_{\Gamma_{Q_0}(x)} |\nabla u(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{1}{2}}, \qquad \tilde{N}_{Q_0,*}u(x) = \sup_{Y \in \tilde{\Gamma}_{Q_0}(x)} |u(Y)|$$

where

$$\Gamma_{Q_0}(x) := \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q, \qquad \tilde{\Gamma}_{Q_0}(x) := \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q^*.$$

We also define a localized dyadic maximal operator

$$\mathcal{M}_{Q_0} f(x) = \sup_{x \in Q \in \mathbb{D}_{Q_0}} \fint_Q |f(y)| \, d\sigma(y).$$

Given any $Q \in \mathbb{D}_{Q_0}$, for any family $\mathcal{F} \subset \mathbb{D}_{Q_0}$ of disjoint dyadic cubes we define the fattened versions of the Carleson box T_Q and the local sawtooth $\Omega_{\mathcal{F},Q_0}$ by

$$\tilde{T}_{Q_0} = \operatorname{int}\left(\bigcup_{Q \in \mathbb{D}_{Q_0}} U_Q^* \right), \qquad \tilde{\Omega}_{\mathcal{F},Q_0} = \operatorname{int}\left(\bigcup_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} U_Q^* \right). \tag{2.63}$$

It is straightforward to show that there exists κ_0 (depending on K_0) large enough such that

$$\overline{\tilde{T}_{Q_0}} \subset B(x_{Q_0}, \kappa_0\, \ell(Q_0)) \cap \overline{\Omega}. \tag{2.64}$$

Let us write $B_{Q_0}^{\kappa_0} = B(x_{Q_0}, \kappa_0\, \ell(Q_0))$ and note that in particular, for every $x \in Q_0$ and every pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0}$, we have

$$\overline{\Gamma_{Q_0}(x)} \subset \overline{\tilde{\Gamma}_{Q_0}(x)} \subset \overline{\tilde{T}_{Q_0}} \subset B_{Q_0}^{\kappa_0} \cap \overline{\Omega}, \qquad \overline{\tilde{\Omega}_{\mathcal{F},Q_0}} \subset \overline{\tilde{T}_{Q_0}} \subset B_{Q_0}^{\kappa_0} \cap \overline{\Omega}. \tag{2.65}$$

Proposition 2.66. *Given $Q_0 \in \mathbb{D}$, let u be harmonic in $2B_{Q_0}^{\kappa_0} \cap \Omega$. Then, for every $1 < q < \infty$ we have*

$$\|S_{Q_0}u\|_{L^q(Q_0)} \leq C\, \|\tilde{N}_{Q_0,*}u\|_{L^q(Q_0)}, \tag{2.67}$$

provided the left hand side is finite.

Remark. Let us emphasize that in this part of the argument, u is allowed to be any harmonic function in $2\,B_{Q_0}^{\kappa_0} \cap \Omega$, not only $\nabla G(\cdot, \hat{X}_Q)$ as above.

Proof. The proof is based on the standard "good-λ" argument of [DJK] (see also [Br]), but adjusted to our setting. Fix $1 < q < \infty$ and assume (qualitatively) that the left hand side of (2.67) is finite. In particular we have that

$$\fint_{Q_0} S_{Q_0}u \, d\sigma \leq \left(\fint_{Q_0} (S_{Q_0}u)^q \, d\sigma \right)^{\frac{1}{q}} < \infty.$$

We consider first the case that $\lambda > \fint_{Q_0} S_{Q_0} u \, d\sigma$. Set

$$E_\lambda = \{x \in Q_0 : \mathcal{M}_{Q_0}(S_{Q_0} u)(x) > \lambda\}.$$

Then by the usual Calderón-Zygmund decomposition argument, there exists a pairwise disjoint family of cubes $\{P_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$, which are maximal with respect to the property that $\fint_{P_j} S_{Q_0} u \, d\sigma > \lambda$, such that $E_\lambda = \cup_j P_j$. Let $\tilde{P}_j$ denote the dyadic parent of P_j. Then $\tilde{P}_j$ must contain a subset of positive σ-measure on which $S_{Q_0} u(x) \leq \lambda$, by maximality of P_j. Fix a cube in $\{P_j\}_j$, say P_0, and set

$$F_\lambda = \{x \in P_0 : S_{Q_0} u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*} u(x) \leq \gamma\,\lambda\}$$

where $\beta > 1$ is to be chosen and $0 < \gamma < 1$. We are going to show that

$$\sigma(F_\lambda) \leq C\,\gamma^\theta\,\sigma(P_0).$$

Assume that $\sigma(F_\lambda) > 0$ otherwise there is nothing to prove. We claim that if β is large enough, depending only on the ADR constants, then $\sigma(F_\lambda) < \sigma(P_0)$. Otherwise, we would have $S_{Q_0} u(x) > \beta\,\lambda$ for a.e. $x \in P_0$ and then by the maximality of P_0 and the fact that $\partial\Omega$ is ADR,

$$\beta\,\lambda < \fint_{P_0} S_{Q_0} u d\sigma \leq C_1 \fint_{\tilde{P}_0} S_{Q_0} u d\sigma \leq C_1\,\lambda.$$

Choosing $\beta > C_1$, we obtain a contradiction. Thus, $\sigma(F_\lambda) < \sigma(P_0)$, so by the inner regularity of σ, there exists a compact set $F \subset F_\lambda \subset P_0$ such that

$$0 < \frac{1}{2}\,\sigma(F_\lambda) \leq \sigma(F) \leq \sigma(F_\lambda) < \sigma(P_0).$$

Since $\sigma(F) < \sigma(P_0)$, it follows that $\mathrm{int}(P_0) \setminus F$ is a non-empty open set. By a standard stopping time procedure, we may then subdivide P_0 dyadically, to extract a pairwise disjoint family of cubes $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{P_0} \setminus \{P_0\}$, which are maximal with respect to the property that $Q_j \cap F = \emptyset$.

We see next that $F = P_0 \setminus (\cup_{\mathcal{F}} Q_j)$. If $x \in F \subset P_0$ and $x \in Q_j$ then $x \in Q_j \cap F$ which contradicts the fact that $Q_j \cap F = \emptyset$. On the other hand, let $x \in P_0 \setminus (\cup_{\mathcal{F}} Q_j)$. Pick $Q_k^x \in \mathbb{D}_{P_0}$ the unique cube containing x with $\ell(Q_k^x) = 2^{-k}\,\ell(P_0)$, $k \geq 1$. Note that $Q_k^x \cap F \neq \emptyset$, otherwise $Q_k^x \subset Q_j$ for some j and we would have $x \in Q_j$. Let $x_k \in Q_k^x \cap F$. Then $|x_k - x| \lesssim \ell(Q_k^x) = 2^{-k}\,\ell(P_0)$ and therefore $x_k \to x$ as $k \to \infty$. Since F is closed and $x_k \in F$ we conclude that $x \in F$ as desired.

Notice that for any $x \in F = P_0 \setminus (\cup_{\mathcal{F}} Q_j)$ and for any $z \in \tilde{P}_0$ we have that

$$\Gamma_{Q_0}(x) = \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q = \left(\bigcup_{x \in Q \in \mathbb{D}_{Q_0}, Q \subset P_0} U_Q\right) \bigcup \left(\bigcup_{x \in Q \in \mathbb{D}_{Q_0}, Q \supsetneq P_0} U_Q\right)$$

$$= \Gamma_{P_0}(x) \bigcup \left(\bigcup_{Q \in \mathbb{D}_{Q_0}, Q \supset \tilde{P}_0} U_Q\right) \subset \Gamma_{P_0}(x) \bigcup \left(\bigcup_{z \in Q \in \mathbb{D}_{Q_0}} U_Q\right) = \Gamma_{P_0}(x) \cup \Gamma_{Q_0}(z).$$

Let us recall that P_0 is a Calderón-Zygmund cube in E_λ and that we can pick $z_0 \in \tilde{P}_0$ with $S_{Q_0} u(z_0) \leq \lambda$ (indeed we know that this happens in a set of positive

measure in $\tilde{P}_0$.) Then for any $x \in F \subset F_\lambda$ we have

$$\begin{aligned}(2.68)\quad \beta\,\lambda < S_{Q_0}u(x) &= \left(\iint_{\Gamma_{Q_0}(x)} |\nabla u(Y)|^2\,\frac{dY}{\delta(Y)^{n-1}}\right)^{\frac12}\\ &\le \left(\iint_{\Gamma_{P_0}(x)} |\nabla u(Y)|^2\,\frac{dY}{\delta(Y)^{n-1}}\right)^{\frac12} + \left(\iint_{\Gamma_{Q_0}(z_0)} |\nabla u(Y)|^2\,\frac{dY}{\delta(Y)^{n-1}}\right)^{\frac12}\\ &= S_{P_0}u(x) + S_{Q_0}u(z_0)\\ &\le S_{P_0}u(x) + \lambda\end{aligned}$$

and therefore $F \subset \{x \in P_0 : S_{P_0}u(x) > (\beta-1)\,\lambda\}$.

Next we claim that

$$(2.69)\qquad \bigcup_{x\in F}\Gamma_{P_0}(x) \subset \bigcup_{Q\in\mathbb{D}_{\mathcal{F},P_0}} U_Q \subset \tilde{\Omega}_{\mathcal{F},P_0}$$

where $\tilde{\Omega}_{\mathcal{F},P_0}$ is the fattened version of $\Omega_{\mathcal{F},P_0}$ defined in (2.63). The second containment in (2.69) is trivial (since $U_Q \subset U_Q^*$), so let us verify the first. We take $Y \in \Gamma_{P_0}(x)$ with $x \in F$. Then, $Y \in U_Q$ where $x \in Q \in \mathbb{D}_{P_0}$. Since $x \in F = P_0 \setminus (\cup_{\mathcal{F}} Q_j)$ we must have $Q \in \mathbb{D}_{\mathcal{F}}$, for otherwise $Q \subset Q_j$ for some $Q_j \in \mathcal{F}$ (cf. (2.17)), and this would imply that $x \in Q_j$. Therefore $Q \in \mathbb{D}_{\mathcal{F},P_0}$ which gives the first inclusion.

Write $\tilde{\omega}_\star = \tilde{\omega}_\star^{X_0}$ to denote the harmonic measure for the domain $\tilde{\Omega}_{\mathcal{F},P_0}$ where $X_0 = A_{P_0}$ is a Corkscrew point in $\tilde{\Omega}_{\mathcal{F},P_0}$, at scale $\operatorname{diam}(P_0)$, whose existence is given in [HM, Proposition **??**] and [HM, Corollary **??**] (which we may apply to $\tilde{\Omega}_{\mathcal{F},P_0}$ in place of $\Omega_{\mathcal{F},P_0}$ since $0 < 2\,\lambda \le \lambda_0$, see Remark 2.38). Let us also write $\tilde{\delta}_\star(Y)$ to denote the distance from Y to $\partial\tilde{\Omega}_{\mathcal{F},P_0}$, and $\tilde{G}_\star$ to denote the corresponding Green function. Given $Y \in \tilde{\Omega}_{\mathcal{F},P_0}$, we choose $y_Y \in \partial\tilde{\Omega}_{\mathcal{F},P_0}$ such that $|Y - y_Y| = \tilde{\delta}_\star(Y)$. By definition, for $x \in F$ and $Y \in \Gamma_{P_0}(x)$, there is a $Q \in \mathbb{D}_{P_0}$ such that $Y \in U_Q$ and $x \in Q$. Thus, by the triangle inequality, and the definition of U_Q, we have that for $Y \in \Gamma_{P_0}(x)$,

$$(2.70)\qquad |x - y_Y| \le |x - Y| + \tilde{\delta}_\star(Y) \approx \delta(Y) + \tilde{\delta}_\star(Y) \approx \tilde{\delta}_\star(Y)\,,$$

where in the last step we have used that

$$(2.71)\qquad \delta(Y) \approx \tilde{\delta}_\star(Y) \quad \text{for } Y \in \bigcup_{Q\in\mathbb{D}_{\mathcal{F},P_0}} U_Q\,.$$

Then, since $F = P_0 \setminus (\cup_{\mathcal{F}} Q_j) \subset \partial\Omega \cap \partial\tilde{\Omega}_{\mathcal{F},P_0}$, see [HM, Proposition **??**], we have

$$\begin{aligned}(2.72)\quad \tilde{\omega}_\star(F)\,(\lambda\,(\beta-1))^2 &= \int_F (\lambda\,(\beta-1))^2\,d\tilde{\omega}_\star \le \int_F S_{P_0}u(x)^2\,d\tilde{\omega}_\star(x)\\ &= \int_F \iint_{\Gamma_{P_0}(x)} |\nabla u(Y)|^2\,\frac{dY}{\delta(Y)^{n-1}}\,d\tilde{\omega}_\star(x)\\ &\lesssim \iint_{\bigcup_{Q\in\mathbb{D}_{\mathcal{F},P_0}} U_Q} |\nabla u(Y)|^2\,\delta(Y)\left(\frac{1}{\delta(Y)^n}\int_{F\cap B(y_Y,C\,\tilde{\delta}_\star(Y))} d\tilde{\omega}_\star(x)\right)dY\end{aligned}$$

$$\lesssim \iint_{\tilde{\Omega}_{\mathcal{F},P_0}} |\nabla u(Y)|^2\, \tilde{\delta}_\star(Y)\, \frac{\tilde{\omega}_\star^{X_0}(\tilde{\Delta}_\star(y_Y, C\,\tilde{\delta}_\star(Y))}{(C\,\tilde{\delta}_\star(Y))^n}\, dY,$$

where we have used (2.70), (2.69)and (2.71).

We now claim that for $Y \in \tilde{\Omega}_{\mathcal{F},P_0}$, we have

$$\frac{\tilde{\omega}_\star^{X_0}(\tilde{\Delta}_\star(y_Y, C\,\tilde{\delta}_\star(Y))}{(C\,\tilde{\delta}_\star(Y))^n} \lesssim \frac{\tilde{G}_\star(X_0, Y)}{\tilde{\delta}_\star(Y)}. \tag{2.73}$$

Indeed, if $\tilde{\delta}_\star(Y) < \tilde{\delta}_\star(X_0)/(2\,C)$, then (2.73) is immediate by Lemma 2.12 and (2.11). Otherwise, we have $\tilde{\delta}_\star(Y) \approx \tilde{\delta}_\star(X_0) \approx \ell(P_0) \gtrsim |X_0 - Y|$, whence (2.73) follows directly from (2.9) and the Harnack Chain condition, and the fact that harmonic measure is a probability measure.

We recall that by hypothesis, u is harmonic in $2B_{Q_0}^{\kappa_0} \cap \Omega \supset \tilde{\Omega}_{\mathcal{F},P_0}$ (cf. (2.65)). Let $\mathcal{L} := \nabla \cdot \nabla$ denote the usual Laplacian in $\mathbb{R}^{n+1}$, so that $\mathcal{L}(u^2) = 2\,|\nabla u|^2$ in $\tilde{\Omega}_{\mathcal{F},P_0}$. Combining these observations with (2.72)-(2.73), we see that

$$\begin{aligned}
\tilde{\omega}_\star(F)\,(\lambda\,(\beta-1))^2 &\lesssim \iint_{\tilde{\Omega}_{\mathcal{F},P_0}} |\nabla u(Y)|^2\, \tilde{G}_\star(X_0, Y)\, dY \\
&= \frac{1}{2} \iint_{\tilde{\Omega}_{\mathcal{F},P_0}} \mathcal{L}(u^2)(Y)\, \tilde{G}_\star(X_0, Y)\, dY \\
&= -\frac{1}{2} u(X_0)^2 + \frac{1}{2} \int_{\partial\tilde{\Omega}_{\mathcal{F},P_0}} u(y)^2\, d\tilde{\omega}_\star^{X_0}(y).
\end{aligned} \tag{2.74}$$

where the last step is a well known identity obtained using properties of the Green function.

Let $Y \in \tilde{\Omega}_{\mathcal{F},P_0}$, so that $Y \in U_Q^*$ for some $Q \in \mathbb{D}_{\mathcal{F},P_0}$. By definition of $\mathbb{D}_{\mathcal{F},P_0}$, this Q cannot be contained in any $Q_j \in \mathcal{F}$. Therefore, $Q \cap F \neq \emptyset$. Indeed, otherwise $Q \cap F = \emptyset$, which by maximality of the cubes in $\mathcal{F}$, would imply that $Q \subset Q_j$ for some $Q_j \in \mathcal{F}$, a contradiction. Thus, there is some $x \in Q \cap F$ which then satisfies $x \in P_0 \setminus (\cup_{\mathcal{F}} Q_j)$ and $x \in Q \in \mathbb{D}_{P_0}$. Hence, $Y \in U_Q^* \subset \tilde{\Gamma}_{P_0}(x)$ with $x \in F$. Since $F \subset F_\lambda$ we have that

$$|u(Y)| \leq \sup_{Z \in \tilde{\Gamma}_{P_0}(x)} |u(Z)| \leq \tilde{N}_{Q_0,*} u(x) \leq \gamma\,\lambda.$$

Thus, $|u(Y)| \leq \gamma\,\lambda$ and in particular $u(Y) \geq -\gamma\,\lambda$, for all $Y \in \tilde{\Omega}_{\mathcal{F},P_0}$. Next, we apply [JK, Theorem 6.4] to the (qualitative) NTA domain $\tilde{\Omega}_{\mathcal{F},P_0}$ (we recall that, in the present stage of the argument, Ω is actually Ω_N for some large N, which satisfies the qualitative exterior Corkscrew condition (Definition 2.5); thus, the bounded domain $\tilde{\Omega}_{\mathcal{F},P_0}$ enjoys an exterior Corkscrew condition with constants that may depend very badly on N.) Of course, the interior Corkscrew and Harnack chain constants, as well as the ADR constants, are controlled uniformly in N). Consequently, we obtain that u has non-tangential limit $\tilde{\omega}_\star$-a.e.. Since $|u(Y)| \leq \gamma\,\lambda$ for every $Y \in \tilde{\Omega}_{\mathcal{F},P_0}$, its non-tangential limit therefore satisfies this same bound $\tilde{\omega}_\star$-a.e., i.e., for $\tilde{\omega}_\star$-a.e. $y \in \partial\tilde{\Omega}_{\mathcal{F},P_0}$ we have $|u(y)| \leq \gamma\,\lambda$. Consequently, by (2.74), we conclude that $\tilde{\omega}_\star(F)\,(\lambda\,(\beta-1))^2 \lesssim (\gamma\,\lambda)^2$ and then $\tilde{\omega}_\star(F) \lesssim \gamma^2$.

Let us now recall some notions used in [HM] that will allow us to apply [HM, Lemma **??**], a dyadic version of the "sawtooth lemma" of [DJK]. Given the family $\mathcal{F} = \{Q_j\}_j$ chosen above, and a Borel measure μ defined on $\partial\Omega$, we define the measure $\mathcal{P}_{\mathcal{F}}\,\mu$ as follows:

$$\mathcal{P}_{\mathcal{F}}\,\mu(A) := \mu(A \setminus \cup_{\mathcal{F}} Q_j) + \sum_{\mathcal{F}} \frac{\sigma(A \cap Q_j)}{\sigma(Q_j)}\,\mu(Q_j).$$

In particular, we have that $\mathcal{P}_{\mathcal{F}}\,\mu(Q) = \mu(Q)$, for every $Q \in \mathbb{D}_{\mathcal{F}}$ (i.e., for Q not contained in any $Q_j \in \mathcal{F}$), and also that $\mathcal{P}_{\mathcal{F}}\,\mu(Q_j) = \mu(Q_j)$ for every $Q_j \in \mathcal{F}$. Recall that $\tilde{\omega}_\star := \tilde{\omega}_\star^{X_0}$, and set $\omega := \omega^{X_0}$ (these are harmonic measures for the domains $\tilde{\Omega}_{\mathcal{F},P_0}$ and Ω respectively). Let $\nu = \nu^{X_0}$ be the measure defined by

$$\nu(F) = \tilde{\omega}_\star\left(F \setminus (\cup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}} \frac{\omega(F \cap Q_j)}{\omega(Q_j)}\,\tilde{\omega}_\star(R_j), \qquad F \subset Q_0, \tag{2.75}$$

where $R_j \subset \tilde{\Omega}_{\mathcal{F},P_0}$ is the n-dimensional cube constructed in [HM, Proposition **??**], satisfying

$$\ell(R_j) \approx \ell(Q_j) \approx \operatorname{dist}(R_j, Q_j) \approx \operatorname{dist}(R_j, \partial\Omega)\,.$$

Observe that $\mathcal{P}_{\mathcal{F}}\nu$ depends only on $\omega_\star$ and not on ω; more precisely,

$$\mathcal{P}_{\mathcal{F}}\nu(F) = \tilde{\omega}_\star\left(F \setminus (\cup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)}\,\tilde{\omega}_\star(R_j), \qquad F \subset Q_0. \tag{2.76}$$

The "dyadic sawtooth lemma" [HM, Lemma **??**] says that $\mathcal{P}_{\mathcal{F}}\nu$ is equivalent to $\mathcal{P}_{\mathcal{F}}\omega$ in the sense of dyadic A_∞.

Next, we note that since P_0 is not contained in any $Q_j \in \mathcal{F}$,

$$\begin{aligned}
\mathcal{P}_{\mathcal{F}}\nu(P_0) &= \tilde{\omega}_\star^{X_0}\left(P_0 \setminus (\cup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq Q} \frac{\sigma(P_0 \cap Q_j)}{\sigma(Q_j)}\,\tilde{\omega}_\star^{X_0}(R_j) \\
&= \tilde{\omega}_\star^{X_0}\left(P_0 \setminus (\cup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq P_0} \tilde{\omega}_\star^{X_0}(R_j) \\
&\gtrsim \tilde{\omega}_\star^{X_0}\left(P_0 \setminus (\cup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}, Q_j \subsetneq Q} \tilde{\omega}_\star^{X_0}\left(B(x_j^\star, r_j) \cap \partial\Omega_{\mathcal{F},P_0}\right) \\
&\geq \tilde{\omega}_\star^{X_0}(\tilde{\Delta}_\star^{P_0}),
\end{aligned}$$

where in the third line we have used the doubling property of $\tilde{\omega}_\star^{X_0}$ (plus a subdivision and Harnack Chain argument if $\ell(Q_j) \approx \ell(P_0)$), and in the last line we have used [HM, Proposition **??**], along with [HM, Proposition **??**] and [HM, Proposition **??**] and the doubling property to ignore the difference between $Q \setminus (\cup_{\mathcal{F}} Q_j)$ and $Q \cap \partial\Omega_{\mathcal{F},Q_0}$. Also, $\tilde{\Delta}_\star^{P_0} = \tilde{\Delta}_\star^{P_0}(\tilde{x}_{P_0}^\star, t_{P_0})$ is a surface ball such that $\tilde{x}_{P_0}^\star \in \partial\tilde{\Omega}_{\mathcal{F},P_0}$, $t_{P_0} \approx \ell(P_0)$, $\operatorname{dist}(P_0, \tilde{\Delta}_\star^{P_0}) \lesssim \ell(P_0)$, and the implicit constants may depend upon K_0, see [HM, Proposition **??**]. Using Lemma 2.2, Harnack chain and [HM, Corollary **??**] it is immediate that $\tilde{\omega}_\star^{X_0}(\tilde{\Delta}_\star^{P_0}) \geq C$ and therefore $\mathcal{P}_{\mathcal{F}}\nu(P_0) \geq C$. On the other hand, since $F = P_0 \setminus (\cup_{\mathcal{F}} Q_j)$ we conclude that

$$\frac{\mathcal{P}_{\mathcal{F}}\nu(F)}{\mathcal{P}_{\mathcal{F}}\nu(P_0)} \lesssim \mathcal{P}_{\mathcal{F}}\nu(F) = \tilde{\omega}_\star^{X_0}(F) \lesssim \gamma^2.$$

Thus Harnack chain, [HM, Corollary **??**] and [HM, Lemma **??**] imply

(2.77)
$$\omega^{X_{\hat\Delta_{P_0}}}(F) \le \frac{\omega^{X_{\hat\Delta_{P_0}}}(F)}{\omega^{X_{\hat\Delta_{P_0}}}(P_0)} \approx \frac{\omega^{X_0}(F)}{\omega^{X_0}(P_0)} = \frac{\mathcal{P}_{\mathcal{F}}\omega^{X_0}(F)}{\mathcal{P}_{\mathcal{F}}\omega^{X_0}(P_0)} \lesssim \left(\frac{\mathcal{P}_{\mathcal{F}}\nu(F)}{\mathcal{P}_{\mathcal{F}}\nu(P_0)}\right)^{\frac{1}{\theta}} \lesssim \gamma^{2/\theta},$$

where we recall that $\Delta_{P_0} \subset P_0 \subset \hat\Delta_{P_0}$.

We are now nearly ready to derive a "good-lambda" estimate on P_0 (cf. (2.80) below). We first need the following auxiliary result, whose proof we defer for the moment.

Lemma 2.78. *Assume (2.50). Then, given a surface ball $\Delta_0 = B_0 \cap \partial\Omega$ for every $\Delta = B \cap \partial\Omega$ with $B \subset B_0$ we have*

$$\left(\fint_\Delta \left(k^{X_{\Delta_0}}\right)^{\tilde p}\, d\sigma\right)^{1/\tilde p} \le C \fint_\Delta k^{X_{\Delta_0}}\, d\sigma. \tag{2.79}$$

Notice that this result says that $k^{X_{\Delta_0}} \in RH_{\tilde p}(\Delta_0) \subset A_\infty(\Delta_0)$ for all Δ_0 and the constants are uniform in Δ_0. Since $A_\infty(\Delta_0)$ defines an equivalence relationship (on the set of doubling measures on Δ_0) we obtain that $\sigma \in A_\infty(\Delta_0, \omega^{X_{\Delta_0}})$ for all Δ_0 and the constants are uniform in Δ_0. Thus, there exist positive constants C and ϑ such that for every $\Delta_0 = B_0 \cap \partial\Omega$, $\Delta = B \cap \partial\Omega$ with $B \subset B_0$ and every Borel set E we have

$$\frac{\sigma(E)}{\sigma(\Delta)} \le C \left(\frac{\omega^{X_{\Delta_0}}(E)}{\omega^{X_{\Delta_0}}(\Delta)}\right)^{\vartheta}$$

We apply this with $\Delta = \Delta_0 = \hat\Delta_{P_0}$ and with $E = F \subset P_0 \subset \hat\Delta_{P_0}$. Then, Lemma 2.2 and (2.77) imply

$$\frac{\sigma(F)}{\sigma(P_0)} \approx \frac{\sigma(F)}{\sigma(\hat\Delta_{P_0})} \lesssim \left(\frac{\omega^{X_{\hat\Delta_{P_0}}}(F)}{\omega^{X_{\hat\Delta_{P_0}}}(\hat\Delta_{P_0})}\right)^{\vartheta} \lesssim \omega^{X_{\hat\Delta_{P_0}}}(F)^{\vartheta} \lesssim \gamma^{2\,\vartheta/\theta}.$$

Since $\sigma(F_\lambda) \le 2\,\sigma(F)$, we therefore have obtained

$$\sigma\big(\{x \in P_0 : S_{Q_0}u(x) > \beta\,\lambda, \tilde N_{Q_0,*}u(x) \le \gamma\,\lambda\}\big) \le C\,\gamma^{2\,\vartheta/\theta}\sigma(P_0), \tag{2.80}$$

where all the constants are independent of P_0.

We recall that P_0 is an arbitrary cube in $\{P_j\}_j$, which is a family of Calderón-Zygmund cubes associated with E_λ for $\lambda > \fint_{Q_0} S_{Q_0}u\, d\sigma$. In addition, for σ-a.e. $x \in Q_0$ such that $S_{Q_0}u(x) > \beta\,\lambda$, we have $\lambda < \beta\,\lambda < S_{Q_0}u(x) \le \mathcal{M}_{Q_0}\left(S_{Q_0}u\right)(x)$ and therefore $x \in E_\lambda$. Using these observations and (2.80) in each P_j, we obtain

$$\begin{aligned}
&\sigma\big(\{x \in Q_0 : S_{Q_0}u(x) > \beta\,\lambda, \tilde N_{Q_0,*}u(x) \le \gamma\,\lambda\}\big)\\
&\qquad = \sigma\big(\{x \in Q_0 : S_{Q_0}u(x) > \beta\,\lambda, \tilde N_{Q_0,*}u(x) \le \gamma\,\lambda\} \cap E_\lambda\big)\\
&\qquad = \sum_j \sigma\big(\{x \in P_j : S_{Q_0}u(x) > \beta\,\lambda, \tilde N_{Q_0,*}u(x) \le \gamma\,\lambda\}\big)\\
&\qquad \le C\,\gamma^{2\,\vartheta/\theta}\sum_j \sigma(P_j)\\
&\qquad = C\,\gamma^{2\,\vartheta/\theta}\sigma(E_\lambda)
\end{aligned}$$

$$= C\,\gamma^{2\,\vartheta/\theta}\sigma\big(\{x\in Q_0 : \mathcal{M}_{Q_0}(S_{\,Q_0}u) > \lambda\}\big).$$

Thus, we have shown that for every $\lambda > \fint_{Q_0} S_{\,Q_0}u\,d\sigma$ we have

$$\sigma\big(\{x\in Q_0 : S_{\,Q_0}u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*}u(x)\le \gamma\,\lambda\}\big) \le C\,\gamma^{\theta'}\sigma\big(\{x\in Q_0 : \mathcal{M}_{Q_0}(S_{\,Q_0}u) > \lambda\}\big). \tag{2.81}$$

Let us now consider the case $\lambda \le \fint_{Q_0} S_{\,Q_0}u\,d\sigma$. We are going to show that

$$\sigma\big(\{x\in Q_0 : S_{\,Q_0}u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*}u(x)\le \gamma\,\lambda\}\big) \le C\,\gamma^{\theta'}\sigma(Q_0). \tag{2.82}$$

We repeat the previous computations with $P_0 = Q_0$ with the main difference that P_0 is no longer a maximal Calderón-Zygmund cube. We take the same set F_λ and we assume that $\sigma(F_\lambda) > 0$, otherwise the desired estimate is trivial. Before we used the maximality of P_0 to show that $\sigma(F_\lambda) < \sigma(P_0)$ for β large enough and this was only used to obtain that $\sigma(F) < \sigma(P_0)$. Here we cannot do that and we proceed as follows. By the inner regularity of σ we can find a compact set $\tilde{F}$, such that $\emptyset \ne \tilde{F} \subset F_\lambda \subset P_0$ and

$$0 < \frac{1}{2}\,\sigma(F_\lambda) \le \sigma(\tilde{F}) \le \sigma(F_\lambda) \le \sigma(P_0).$$

If $\sigma(F_\lambda) < \sigma(P_0)$ or $\sigma(\tilde{F}) < \sigma(F_\lambda)$ we set $F = \tilde{F}$ and we have $\sigma(F) < \sigma(P_0)$. Otherwise, i.e., if $\sigma(\tilde{F}) = \sigma(F_\lambda) = \sigma(P_0)$, we use the following lemma:

Lemma 2.83. *Let Ω have the ADR property with constant C_1, i.e.,*

$$C_1^{-1}\,r^n \le \sigma(\Delta(x,r)) \le C_1\,r^n$$

for all $x\in\partial\Omega$ and $0 < r < \operatorname{diam}\partial\Omega$. Let $Q\in\mathbb{D}$ and recall that there exists $\Delta_Q = \Delta(x_Q, r_Q)$ with $r_Q \approx \ell(Q)$ with the property that $\Delta_Q \subset Q \subset \Delta(x_Q, C_2\,r_Q)$ for some uniform constant $C_2 \ge 1$. Take $\tau_0 = (2\,C_1^2)^{-1/n}$ and set $\Delta_Q^{\tau_0} = \Delta(x_Q, \tau_0\,r_Q)$. Then,

$$\overline{\Delta_Q^{\tau_0}} \subset \Delta_Q \subset Q \qquad \text{and} \qquad (2\,C_1^4\,C_2^n)^{-1}\sigma(Q) \le \sigma\big(\overline{\Delta_Q^{\tau_0}}\big) \le \frac{3}{4}\,\sigma(Q).$$

Assuming this momentarily (the proof is given below), we apply this result to P_0. Set $F = \tilde{F}\cap\overline{\Delta_{P_0}^{\tau_0}} \subset \tilde{F}$ which is a compact set. As we have $\tilde{F}\subset P_0$ and $\sigma(\tilde{F}) = \sigma(F_\lambda) = \sigma(P_0)$ we obtain

$$\begin{aligned}(2\,C_1^4\,C_2^n)^{-1}\sigma(F_\lambda) = (2\,C_1^4\,C_2^n)^{-1}\sigma(P_0) &\le \sigma\big(\overline{\Delta_{P_0}^{\tau_0}}\big) = \sigma\big(\overline{\Delta_{P_0}^{\tau_0}}\cap\tilde{F}\big) + \sigma\big(\overline{\Delta_{P_0}^{\tau_0}}\setminus\tilde{F}\big)\\ &\le \sigma(F) + \sigma(P_0\setminus\tilde{F}) = \sigma(F) \le \sigma\big(\overline{\Delta_{P_0}^{\tau_0}}\big) \le \frac{3}{4}\,\sigma(P_0) = \frac{3}{4}\,\sigma(F_\lambda).\end{aligned}$$

Thus, we have found a compact set F such that $\emptyset \ne F\subset F_\lambda\subset P_0$ and

$$0 < (2\,C_1^4\,C_2^n)^{-1}\sigma(F_\lambda) \le \sigma(F) < \sigma(F_\lambda) = \sigma(P_0).$$

This allows us to run the stopping time argument and find the family $\mathcal{F}$ as before. We then continue the argument and notice that in (2.68), we used the maximality of P_0. Here, the analogous estimate is trivial: since $P_0 = Q_0$, it follows that for any $x\in F\subset F_\lambda$, we have $S_{\,P_0}u(x) = S_{\,Q_0}(x) > \beta\,\lambda > (\beta-1)\,\lambda$ and therefore $F\subset\{x\in P_0 : S_{\,P_0}u(x) > (\beta-1)\,\lambda\}$. From this point the proof continues without

change (except for the fact that we have $\sigma(F_\lambda) \lesssim \sigma(F)$ in place of $\sigma(F_\lambda) \leq 2\,\sigma(F)$), so that (2.80) holds, and in the present case this is our desired estimate (2.82).

Let $1 < q < \infty$ and write $a_{Q_0} = \fint_{Q_0} S_{Q_0} u \, d\sigma < \infty$. Then

$$\begin{aligned}
\|S_{Q_0} u\|_{L^q(Q_0)}^q &= \beta^q \int_0^\infty q\,\lambda^q \sigma\big(\{x \in Q_0 : S_{Q_0} u(x) > \beta\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\
&\leq \beta^q \int_0^\infty q\,\lambda^q \sigma\big(\{x \in Q_0 : S_{Q_0} u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*} u(x) \leq \gamma\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\
&\quad + (\beta/\gamma)^q \,\|\tilde{N}_{Q_0,*} u\|_{L^q(Q_0)}^q \\
&= \beta^q \int_0^{a_{Q_0}} \cdots \frac{d\lambda}{\lambda} + \beta^q \int_{a_{Q_0}}^\infty \cdots \frac{d\lambda}{\lambda} + (\beta/\gamma)^q \,\|\tilde{N}_{Q_0,*} u\|_{L^q(Q_0)}^q \\
&= I + II + (\beta/\gamma)^q \,\|\tilde{N}_{Q_0,*} u\|_{L^q(Q_0)}^q.
\end{aligned}$$

For I we use (2.82) and Jensen's inequality:

$$I \leq C\,\gamma^{\theta'} \beta^q\, \sigma(Q_0) \int_0^{a_{Q_0}} q\,\lambda^q \,\frac{d\lambda}{\lambda} = C\,\gamma^{\theta'} \beta^q\, \sigma(Q_0)\, a_{Q_0}^q \leq C\,\gamma^{\theta'}\,\beta^q \,\|S_{Q_0} u\|_{L^q(Q_0)}^q.$$

For II we use (2.81) and the fact that $\mathcal{M}_{Q_0}$ is bounded on $L^q(Q_0)$ (notice that $\mathcal{M}_{Q_0}$ is the localized dyadic Hardy-Littlewood maximal function and we can insert the characteristic function of Q_0 in the argument for free):

$$\begin{aligned}
II &\leq C\,\gamma^{\theta'}\,\beta^q \int_{a_{Q_0}}^\infty q\,\lambda^q\,\sigma\big(\{x \in Q_0 : \mathcal{M}_{Q_0}(S_{Q_0} u) > \lambda\}\big)\,\frac{d\lambda}{\lambda} \\
&= C\,\gamma^{\theta'}\,\beta^q \,\|\mathcal{M}_{Q_0}(S_{Q_0} u)\|_{L^q(Q_0)}^q \leq C\,\gamma^{\theta'}\,\beta^q \,\|S_{Q_0} u\|_{L^q(Q_0)}^q.
\end{aligned}$$

Gathering the obtained estimates we conclude that

$$\|S_{Q_0} u\|_{L^q(Q_0)}^q \leq C\,\gamma^{\theta'}\,\beta^q \,\|S_{Q_0} u\|_{L^q(Q_0)}^q + (\beta/\gamma)^q \,\|\tilde{N}_{Q_0,*} u\|_{L^q(Q_0)}^q.$$

Choosing γ small enough so that $C\,\gamma^{\theta'}\,\beta^q < \frac{1}{2}$ the first term in the right hand side can be hidden in the left hand side (since it is finite by assumption) and we conclude as desired (2.67). □

Proof of Lemma 2.78. Notice that in proving (2.79), we may suppose that the balls B and B_0 have respective radii $r_B \ll r_{B_0}$; otherwise, if $r_B \approx r_{B_0}$, then (2.79) reduces immediately to (2.50). We now may proceed as in the first part of the proof of Proposition 2.42 in order to use Corollary 2.37: we take $\Delta = B \cap \Omega$ and for every $Y \in \Omega \setminus 2\kappa_0 B$ we have

$$\fint_{\Delta'} k^Y \, d\sigma \approx \omega^Y(\Delta) \fint_{\Delta'} k^{X_\Delta} \, d\sigma \qquad B' \subset B.$$

By the Harnack chain condition this estimate holds for $Y = X_{\Delta_0}$. Then for σ-a.e. $x \in \Delta$ we take $B' = B(x, r)$ and let $r \to 0$ to obtain $k^{X_{\Delta_0}}(x) \approx \omega^{X_{\Delta_0}}(\Delta)\, k^{X_\Delta}(x)$. Consequently, by (2.50) we have

$$\left(\fint_\Delta \left(k^{X_{\Delta_0}}\right)^{\tilde{p}} \, d\sigma\right)^{\frac{1}{\tilde{p}}} \lesssim \omega^{X_{\Delta_0}}(\Delta) \left(\fint_\Delta \left(k^{X_\Delta}\right)^{\tilde{p}} \, d\sigma\right)^{\frac{1}{\tilde{p}}} \lesssim \frac{\omega^{X_{\Delta_0}}(\Delta)}{\sigma(\Delta)} = \fint_\Delta k^{X_{\Delta_0}} \, d\sigma.$$

□

Proof of Lemma 2.83. The proof is almost trivial. What $\overline{\Delta_Q^{\tau_0}} \subset \Delta_Q$ follows at once from the fact that $\tau_0 < 1$. On the other hand, let us notice that $\overline{\Delta_Q^{\tau_0}} \subset \Delta(x_Q, (3/2)^{1/n}\,\tau_0\, r_Q)$ and therefore the ADR property gives

$$\sigma\big(\overline{\Delta_Q^{\tau_0}}\big) \leq \sigma\big(\Delta(x_Q, (3/2)^{1/n}\,\tau_0\, r_Q)\big) \leq C_1\,\frac{3}{2}\,\tau_0^n\, r_Q^n = \frac{3}{4}\,C_1^{-1}\, r_Q^n \leq \frac{3}{4}\,\sigma(\Delta_Q)$$

and

$$\sigma(Q) \leq \sigma(\Delta(x_Q, C_2\, r_Q)) \leq C_1\, C_2^n\, r^n \leq C_1^2\, C_2^n\, \tau_0^{-n}\sigma(\Delta_Q^{\tau_0}) \leq 2\, C_1^4\, C_2^n\, \sigma\big(\overline{\Delta_Q^{\tau_0}}\big).$$

□

2.4.1. *Good-λ inequality for truncated cones.* In order to apply Proposition 2.66 we need to know a priori that $\|S_{Q_0} u\|_{L^q(Q_0)} < \infty$ (qualitatively), which can be verified as follows. Since Ω_N satisfies the exterior corkscrew condition at small scales, we may invoke [JK] to control the square function by the non-tangential maximal operator with constants that may depend on N (we recall that at this stage $\Omega = \Omega_N$), but this gives, in particular, qualitative finiteness of the square function. The results of [JK] apply to square functions and non-tangential maximal functions defined via the classical cones $\widetilde{\Gamma}_\alpha(x) := \{Y \in \Omega : |Y - x| < (1+\alpha)\,\delta(Y)\}$, but our dyadic cones may be compared to the classical cones by varying the aperture parameter α. Thus, we may infer also that the our square function has (qualitatively) finite L^p norm, given finiteness of our non-tangential maximal function, so we may then apply Proposition 2.66 to obtain that (2.67) holds with uniform bounds.

There is, however, a different approach that consists of working with cones that are "truncated" so that they stay away from the boundary. This in turns implies that the truncated square function is bounded and therefore that the corresponding left hand side is finite. Passing to the limit we conclude that the a priori finiteness hypothesis can be removed from Proposition 2.66. We present this argument for the sake of self-containment, and because the argument is of independent interest.

Before stating the precise result we introduce some notation. Given Q_0, we take $k \geq k(Q_0) + 2$ (recall that $k(Q_0)$ is defined in such a way $\ell(Q_0) = 2^{-k(Q_0)}$) a large enough integer (eventually, $k \uparrow \infty$). We define $\Gamma^k_{Q_0}(x)$ to be the truncated cone where the cubes in the union satisfy additionally $\ell(Q) \geq 2^{-k}$. The corresponding truncated square function is written as $S^k_{Q_0}$.

Proposition 2.84. *Given $Q_0 \in \mathbb{D}$, let u be harmonic in $2B^{\kappa_0}_{Q_0} \cap \Omega$. Then, for every $1 < q < \infty$ and for every $k \geq k(Q_0) + 2$ we have*

$$\|S^k_{Q_0} u\|_{L^q(Q_0)} \leq C\,\|\tilde{N}_{Q_0,*} u\|_{L^q(Q_0)}, \tag{2.85}$$

where the constant C is independent of k and consequently (2.67) holds whether or not the left hand side is finite.

Proof. It is straightforward to see that the truncated cones are away from the boundary: the cubes that define the cones have side length at least 2^{-k} and thus the corresponding Whitney boxes have side length at lest $C\,2^{-k}$, thus $\operatorname{dist}(\Gamma^k_{Q_0}, \partial\Omega) \gtrsim 2^{-k}$. Note that u is harmonic in $2\,B^{\kappa_0}_{Q_0} \cap \Omega$ and therefore smooth in $\frac{3}{2}\,B^{\kappa_0}_{Q_0} \cap \{Y \in$

$\Omega : \delta(Y) \gtrsim 2^{-k}\}$. Thus, (2.65) implies $|\nabla u| \leq C_{Q_0,k,u}$ in $\Gamma^k_{Q_0}(x)$, $x \in Q_0$, and consequently for every $x \in Q_0$ we have $S^k_{Q_0} u \in L^\infty(Q_0)$. The bound may depend of k, Q_0 but we will use this qualitatively and a not quantitatively. Note also, that if $Q \in \mathbb{D}_{Q_0}$ with $\ell(Q) = 2^{-k}$ then $\Gamma^k_{Q_0}(x)$ does not depend on $x \in Q$ (i.e. $\Gamma^k_{Q_0}(x) = \Gamma^k_{Q_0}(x')$ for every x, $x' \in Q$) and thus $S^k_{Q_0} u$ is constant on Q.

Let us write $\mathcal{M}^k_{Q_0}$ for the truncated dyadic maximal function where the cubes in the sup have side length at least 2^{-k}. Associated to $S^k_{Q_0} u$, and for each $\lambda > \fint_{Q_0} S^k_{Q_0}\, d\sigma$ we define the corresponding E_λ; we clearly have $E_\lambda = \cup_j P_j$ with P_j being maximal cubes as before satisfying further $\ell(P_j) \geq 2^{-k}$ (note that as we have observed that $S^k_{Q_0}$ is constant on cubes of size 2^{-k} we could have taken $\mathcal{M}_{Q_0}$ obtaining the same family $\{P_j\}$ with the same properties.) Fix one of these cubes P_0 and define F_λ with the truncated square function $S^k_{Q_0}$ replacing S_{Q_0} —we do not truncate the non-tangential operator. Then we proceed as before, assume that $\sigma(F_\lambda) > 0$ and find the corresponding set F, and a family $\mathcal{F}$ with the same properties as before. We can easily see that $\Gamma^k_{Q_0}(x) \subset \Gamma^k_{P_0}(x) \cup \Gamma^k_{Q_0}(z)$ for every $x \in F$ and $z \in \tilde{P}_0$. Then, since P_0 is a Calderón-Zygmund cube in E_λ, we can pick $z_0 \in \tilde{P}_0$ with $S^k_{Q_0} u(z) \leq \lambda$ (indeed we know that this happens in a set of positive measure in $\tilde{P}_0$.) Then for any $x \in F \subset F_\lambda$, we have

$$
\begin{aligned}
\beta\,\lambda < S^k_{Q_0} u(x) &= \left(\iint_{\Gamma^k_{Q_0}(x)} |\nabla u(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{1}{2}} \\
&\leq \left(\iint_{\Gamma^k_{P_0}(x)} |\nabla u(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{1}{2}} + \left(\iint_{\Gamma^k_{Q_0}(z_0)} |\nabla u(Y)|^2 \, \frac{dY}{\delta(Y)^{n-1}} \right)^{\frac{1}{2}} \\
&= S^k_{P_0} u(x) + S^k_{Q_0} u(z_0) \\
&\leq S_{P_0} u(x)^k + \lambda
\end{aligned}
\tag{2.86}
$$

and therefore $F \subset \{x \in P_0 : S^k_{P_0} u(x) > (\beta - 1)\,\lambda\}$.

Next, using that $S^k_{P_0} u(x) \leq S_{P_0} u(x)$ we easily obtain (2.72) and from this point the argument goes through without change. Thus, we obtain the following analog of (2.80)

$$
\sigma(\{x \in P_0 : S^k_{Q_0} u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*} u(x) \leq \gamma\,\lambda\}) \leq C\,\gamma^{2\,\vartheta/\theta} \sigma(P_0), \tag{2.87}
$$

where all the constants are independent of P_0 and k.

Let us consider the case $\lambda \leq \fint_{Q_0} S^k_{Q_0} u\, d\sigma$. The same argument as before works in this case and we easily obtain

$$
\sigma(\{x \in Q_0 : S^k_{Q_0} u(x) > \beta\,\lambda, \tilde{N}_{Q_0,*} u(x) \leq \gamma\,\lambda\}) \leq C\,\gamma^{\theta'} \sigma(Q_0). \tag{2.88}
$$

Gathering (2.87) and (2.88) and repeating the computations above we conclude that

$$
\|S^k_{Q_0} u\|^q_{L^q(Q_0)} \leq C\,\gamma^{\theta'}\,\beta^q\,\|S^k_{Q_0} u\|^q_{L^q(Q_0)} + (\beta/\gamma)^q\,\|\tilde{N}_{Q_0,*} u\|^q_{L^q(Q_0)},
$$

where all the constants are independent of k. As observed, the fact that we are working with truncated square functions gives us that $S^k_{Q_0} u \in L^\infty(Q_0)$ (we use this in a qualitative way but not quantitatively). Thus, choosing γ small enough so that

$C\,\gamma^{\theta'}\,\beta^q < \frac{1}{2}$ the first term in the right hand side can be hidden in the left hand side. Hence, we obtain as desired (2.85)

$$\|S^k_{Q_0} u\|^q_{L^q(Q_0)} \leq C\,\beta^q\,\|\tilde{N}_{Q_0,*} u\|^q_{L^q(Q_0)}$$

where C is independent of k. Letting $k \uparrow \infty$, the monotone convergence theorem gives (2.67) without assuming that the left hand side is finite. Thus we have proved a version of Proposition 2.66 where there is no need to assume that the left hand side is finite. □

2.5. **Step 4: Proof of (2.62).** In order to obtain (2.62) we shall use Proposition 2.66, applied to $u = \nabla G$, with our exponent $q = \tilde{p}$. Thus, we need to study the non-tangential maximal function of u, and it suffices to do this for each component of u, i.e., for any given partial derivative of G. More precisely, we set $u(Y) = \partial_{Y_j} G(Y, \hat{X}_Q)$, where $\hat{X}_Q$ is the corkscrew point associated to $\hat{\Delta}_Q$ defined in Section 2.3. As mentioned there, $\hat{X}_Q \notin 6\,\tilde{B}_Q$ with $\tilde{B}_Q = B(x_Q, \kappa_1\,\ell(Q))$ and $\kappa_1 > \kappa_0$. In particular $\hat{X}_Q \notin 6\,B_Q^{\kappa_0}$, so that u is harmonic in $2B_Q^{\kappa_0} \cap \Omega$.

Proposition 2.89. *We have*

$$\|\tilde{N}_{Q,*} u\|^q_{L^q(Q)} \leq C\,\sigma(Q)^{1-q}\,.$$

Proof. Let $x \in Q$ and $Y \in \tilde{\Gamma}_Q(x)$. By (2.65) applied to Q in place of Q_0 we have $\tilde{\Gamma}_Q(x) \subset B_Q^{\kappa_0} \cap \Omega$ and therefore u is harmonic in $B(Y, 3\,\delta(Y)/4) \subset \frac{7}{4} B_Q^{\kappa_0}$. We then obtain by the mean value property of harmonic functions, Caccioppoli's inequality, the Harnack chain condition and Lemma 2.12, that

$$\begin{aligned} |u(Y)| &\leq \fint_{B(Y,\delta(Y)/2)} |\nabla_Z G(Z, \hat{X}_Q)|\,dZ \lesssim \delta(Y)^{-1} \left(\fint_{B(Y,\frac{3}{4}\delta(Y))} G(Z, \hat{X}_Q)^2\,dZ \right)^{\frac{1}{2}} \\ &\lesssim \frac{G(Y, \hat{X}_Q)}{\delta(Y)} \approx \frac{G(X_{\Delta_Y}, \hat{X}_Q)}{\delta(Y)} \approx \frac{\omega^{\hat{X}_Q}(\Delta_Y)}{\delta(Y)^n} \end{aligned}$$

where $\Delta_Y = \Delta(y, \delta(Y))$, with $y \in \partial\Omega$ such that $|Y - y| = \delta(Y)$. It is easy to see that we can find C large enough so that $x \in \Delta(y, C\,\delta(Y))$ and $\Delta_Y \subset C\,\hat{\Delta}_Q$. As usual, we let $X_{C\,\hat{\Delta}_Q}$ denote a Corkscrew point with respect to $C\,\hat{\Delta}_Q$. Then,

$$\begin{aligned} |u(Y)| &\lesssim \frac{1}{\delta(Y)^n} \int_{\Delta_Y} k^{\hat{X}_Q}\,d\sigma \leq \frac{1}{\delta(Y)^n} \int_{\Delta(y,C\,\delta(Y))} k^{\hat{X}_Q}\,\chi_{C\,\hat{\Delta}_Q}\,d\sigma \\ &\lesssim \frac{1}{\delta(Y)^n} \int_{\Delta(y,C\,\delta(Y))} k^{X_{C\,\hat{\Delta}_Q}}\,\chi_{C\,\hat{\Delta}_Q}\,d\sigma \lesssim M(k^{X_{C\,\hat{\Delta}_Q}}\,\chi_{C\,\hat{\Delta}_Q})(x)\,, \end{aligned}$$

where in the next-to-last inequality we have used the Harnack Chain condition. Therefore, using that $q = \tilde{p}$ and (2.50), we obtain

$$\begin{aligned} \|\tilde{N}_{Q,*} u\|^q_{L^q(Q)} &\lesssim \|M(k^{X_{C\,\hat{\Delta}_Q}}\,\chi_{C\,\hat{\Delta}_Q})\|^q_{L^q(Q)} \lesssim \int_{C\,\hat{\Delta}_Q} \left(k^{X_{C\,\hat{\Delta}_Q}}\right)^q\,d\sigma \\ &\lesssim \sigma(C\,\hat{\Delta}_Q)^{1-q} \lesssim \sigma(Q)^{1-q}. \end{aligned}$$

□

Now we are ready to establish (2.62). Given $Q \in \mathbb{D}$, we first apply Proposition 2.84 (or else Proposition 2.66) with q, with Q in place of Q_0, and with $u = \partial_{Y_j} G(\cdot, \hat{X}_Q)$ as above, since the latter is harmonic in $2B_Q^{\kappa_0} \cap \Omega$. Then, using Proposition 2.89, we obtain as desired that

$$\|S_Q u\|_{L^q(Q)}^q \leq C \, \|\tilde{N}_{Q,*} u\|_{L^q(Q)}^q \leq C \, \sigma(Q)^{1-q} \, .$$

2.6. **UR for Ω.** To conclude the proof of our main result we see that the UR property for the approximating domains Ω_N with uniform bounds passes to Ω. As observed before, as a consequence of the Tb theorem Theorem 2.51 we have obtained that (2.55) holds for Ω_N with uniform bounds. This in turn implies that $\partial\Omega_N$ is UR (see [DS2, p. 44]) with uniform bounds. To obtain that this property is preserved when passing to the limit we use an argument, based on ideas of Guy David, along the lines of [HM, Appendix **??**] (indeed the present situation is easier) where we have to switch the roles of Ω and Ω_N.

To show that Ω has the UR property we use the singular integral characterization. We recall that a closed, n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is UR if and only if for all singular kernels K as below, and corresponding truncated singular integrals T_ε, we have that

$$\sup_{\varepsilon>0} \int_E |T_\varepsilon f|^2 \, dH^n \leq C_K \int_E |f|^2 \, dH^n. \tag{2.90}$$

Here,

$$T_{E,\varepsilon} f(x) = T_\varepsilon f(x) := \int_E K_\varepsilon(x-y) \, f(y) \, dH^n(y) \, ,$$

where $K_\varepsilon(x) := K(x) \, \Phi(|x|/\varepsilon)$, with $0 \leq \Phi \leq 1$, $\Phi(\rho) \equiv 1$ if $\rho \geq 2$, $\Phi(\rho) \equiv 0$ if $\rho \leq 1$, and $\Phi \in C^\infty(\mathbb{R})$, and where the singular kernel K is an odd function, smooth on $\mathbb{R}^{n+1} \setminus \{0\}$, and satisfying

$$|K(x)| \, \leq \, C \, |x|^{-n} \tag{2.91}$$

$$|\nabla^m K(x)| \, \leq \, C_m \, |x|^{-n-m} \, , \qquad \forall m = 1, 2, 3, \ldots \, . \tag{2.92}$$

We also introduce the following extension of these operators

$$\mathcal{T}_E f(X) := \int_E K(X-y) \, f(y) \, dH^n(y) \, , \qquad X \in \mathbb{R}^{n+1} \setminus E. \tag{2.93}$$

We define non-tangential approach regions $\Upsilon_\tau^E(x)$ as follows. Let $\mathcal{W}_E$ denote the collection of cubes in the Whitney decomposition of $\mathbb{R}^{n+1} \setminus E$, and set $\mathcal{W}_\tau(x) := \{I \in \mathcal{W}_E : \operatorname{dist}(I, x) < \tau \ell(I)\}$. We then define

$$\Upsilon_\tau^E(x) := \bigcup_{I \in \mathcal{W}_\tau(x)} I^*$$

(thus, roughly speaking, τ is the "aperture" of $\Upsilon_\tau^E(x)$). For $F \in C(\mathbb{R}^{n+1} \setminus E)$ we may then also define the non-tangential maximal function

$$N_{*,\tau}^E(F)(x) := \sup_{Y \in \Upsilon_\tau^E(x)} |F(Y)|.$$

Let us recall [HM, Lemma **??**] which states that if $E \subset \mathbb{R}^{n+1}$ is n-dimensional UR, we then have

$$\int_E \left(N^E_{*,\tau}(\mathcal{T}_E f)\right)^2 \, dH^n \leq C_{\tau,K} \int_E |f|^2 dH^n. \tag{2.94}$$

for every $0 < \tau < \infty$ and with $C_{\tau,K}$ depending only on n, τ, K and the UR constants.

After these preliminaries, in order to show that $\partial\Omega$ is UR we take one of the previous kernels K and form the corresponding operators $T_\varepsilon = T_{\partial\Omega,\varepsilon}$. Fix $\varepsilon > 0$ and $N \geq 1$ such that $\varepsilon \gg 2^{-N}$. We write $\partial\Omega = \cup_j Q_j$ with $Q_j \in \mathbb{D}_N(\partial\Omega)$. For fixed j, if we write $\tilde{Q}_j$ for the dyadic parent of Q_j, we have by construction that $X_{\tilde{Q}_j} \in \text{int}(U_{\tilde{Q}_j}) \subset \Omega_N$. Then in the segment joining x_j (the center of Q_j) and $X_{\tilde{Q}_j}$ there exists a point $\hat{x}_j \in \partial\Omega_N$. Next we take $Q_j(N)$ the unique cube in $\mathbb{D}_N(\partial\Omega_N)$ such that $\hat{x}_j \in Q_j(N)$. Note that we have $\text{dist}(Q_j, Q_j(N)) \approx 2^{-N}$. Since $\partial\Omega_N$ is ADR with uniform bounds in N, any given $Q(N) \in \mathbb{D}(\partial\Omega_N)$ can serve in this way for at most a bounded number of $Q_j \in \mathbb{D}(\partial\Omega)$. Thus we have

$$\sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} 1_{Q_j(N)}(x) \leq C\,, \qquad \forall x \in \partial\Omega_N. \tag{2.95}$$

As usual, we set $\sigma := H^n|_{\partial\Omega}$, and we now also let $\sigma_N := H^n|_{\partial\Omega_N}$. For τ large enough, we have that if $x' \in Q_j(N)$ and $x \in Q_j$, then $x \in \Upsilon_\tau^{\partial\Omega_N}(x')$. Thus,

$$\begin{aligned}
\int_{\partial\Omega} |\mathcal{T}_{\partial\Omega_N} f|^2 \, d\sigma &= \sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} \int_{Q_j} |\mathcal{T}_{\partial\Omega_N} f|^2 \, d\sigma \\
&= \sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} \frac{1}{\sigma_N(Q_j(N))} \int_{Q_j(N)} \int_{Q_j} |\mathcal{T}_{\partial\Omega_N} f(x)|^2 \, d\sigma(x)\, d\sigma_N(x') \\
&\lesssim \sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} \int_{Q_j(N)} \left(N^{\partial\Omega_N}_{*,\tau}\left(\mathcal{T}_{\partial\Omega_N} f\right)\right)^2 \, d\sigma_N \leq C_{\tau,K} \int_{\partial\Omega_N} |f|^2 d\sigma_N\,,
\end{aligned}$$

where in the last line we have used first the ADR properties of $\partial\Omega_N$ and $\partial\Omega$, and then (2.95) and (2.94) with $E = \partial\Omega_N$ (as we may do, since $\partial\Omega_N$ is UR with uniform bounds). In particular we observe that $C_{\tau,K}$ is independent of N.

Thus we have shown that $\mathcal{T}_{\partial\Omega_N} : L^2(\partial\Omega_N) \to L^2(\partial\Omega)$. Since the kernel K is odd, we therefore obtain by duality that

$$\mathcal{T}_{\partial\Omega} : L^2(\partial\Omega) \to L^2(\partial\Omega_N). \tag{2.96}$$

Let us now observe that K_ε is odd, smooth away the origin and satisfies (2.91), (2.92) uniformly in ε. Thus (2.96) applies to the corresponding operator $\mathcal{T}_{\partial\Omega,\varepsilon}$ defined by means of K_ε, that is, we have $\mathcal{T}_{\partial\Omega,\varepsilon} : L^2(\partial\Omega) \to L^2(\partial\Omega_N)$ with bounds that are uniform on N and ε. Then we proceed as in Case 2 of [HM, Appendix **??**] and write

$$\begin{aligned}
\int_{\partial\Omega} |T_{\partial\Omega,\varepsilon} f|^2 \, d\sigma &= \sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} \int_{Q_j} |T_{\partial\Omega,\varepsilon} f|^2 \, d\sigma \\
&= \sum_{Q_j \in \mathbb{D}_N(\partial\Omega)} \frac{1}{\sigma_N(Q_j(N))} \int_{Q_j(N)} \int_{Q_j} |T_{\partial\Omega,\varepsilon} f(x)|^2 \, d\sigma(x)\, d\sigma_N(x')
\end{aligned}$$

$$\lesssim \sum_{Q_j\in\mathbb{D}_N(\partial\Omega)} \frac{1}{\sigma_N(Q_j(N))}\int_{Q_j(N)}\int_{Q_j} |T_{\partial\Omega,\varepsilon}f(x)-\mathcal{T}_{\partial\Omega,\varepsilon}f(x')|^2\,d\sigma(x)\,d\sigma_N(x')$$
$$+ \sum_{Q_j\in\mathbb{D}_N(\partial\Omega)}\int_{Q_j(N)} |\mathcal{T}_{\partial\Omega,\varepsilon}f(x')|^2\,d\sigma_N(x'),$$
$$= I + II,$$

where we have use the ADR property for both $\partial\Omega$ and $\partial\Omega_N$. To estimate II we use (2.95) and (2.96) applied to $\mathcal{T}_{\partial\Omega,\varepsilon}$:

$$II \lesssim \int_{\partial\Omega_N} |\mathcal{T}_{\partial\Omega,\varepsilon}f|^2\,d\sigma_N \le C_{\tau,K}\int_{\partial\Omega}|f|^2\,d\sigma.$$

On the other hand, standard Calderón-Zygmund arguments that are left to the interested reader give that (2.91) and (2.92) imply

$$|T_{\partial\Omega,\varepsilon}f(x)-\mathcal{T}_{\partial\Omega,\varepsilon}f(x')| \lesssim M_{\partial\Omega}f(x), \qquad x\in Q_j,\ x'\in Q_j(N),$$

where $M_{\partial\Omega}$ is the Hardy-Littlewood maximal operator on $\partial\Omega$ and the constants are uniform in ε and N, and where we have used that $2^{-N}\ll\varepsilon$. Hence,

$$I \lesssim \sum_{Q_j\in\mathbb{D}_N(\partial\Omega)}\int_{Q_j} M_{\partial\Omega}f(x)^2\,d\sigma(x) = \int_{\partial\Omega} M_{\partial\Omega}f(x)^2\,d\sigma(x) \lesssim \int_{\partial\Omega}|f(x)|^2\,d\sigma(x).$$

Gathering our estimates, we conclude that

$$\int_{\partial\Omega}|T_{\partial\Omega,\varepsilon}f|^2\,d\sigma \lesssim \int_{\partial\Omega}|f|^2\,d\sigma,$$

where the implicit constants are independent of ε and N, which gives at once that $\partial\Omega$ is UR. □

Remark 2.97. Gathering together this section with [HM, Appendix **??**] we conclude that $\partial\Omega$ is UR if and only if Ω_N is UR with uniform bounds for all $N\gg 1$: here we have shown the right-to-left implication and the converse argument is given in [HM, Appendix **??**].

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