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QUANTIFYING DEMOCRACY OF WAVELET BASES IN LORENTZ SPACES

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Abstract. We study the efficiency of the greedy algorithm for wavelet bases in Lorentz spaces in order to give the near best approximation. The result is used to give sharp inclusions for the approximation spaces in terms of discrete Lorentz sequence spaces.

1. Introduction

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach (or quasi-Banach) space with a countable unconditional basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$; that is, every $x \in \mathbb{B}$ can be uniquely represented as an unconditionally convergent series $x = \sum_{j\in\mathbb{N}} s_j\, e_j$, for some sequence of scalars $\{s_j : j \in \mathbb{N}\}$.

An approximation algorithm for the space $\mathbb{B}$ and the basis $\mathcal{B}$ is a sequence of maps $F_N : \mathbb{B} \longrightarrow \mathbb{B}, N \in \mathbb{N}$, such that for each $x \in \mathbb{B}$, $F_N(x)$ is a linear combination of at most N elements of the basis $\mathcal{B}$. The most classical algorithm is the one given by the partial sums operators $S_N(x) = \sum_{j=1}^{N} s_j e_j,\ N \in \mathbb{N}$.

The thresholding procedure that is used in image compression and other applications can be modeled using the greedy algorithm. If $x = \sum_{j=1}^{\infty} s_j e_j$ and the basis elements are ordered in such a way that

$$\|s_{j_1} e_{j_1}\|_{\mathbb{B}} \geq \|s_{j_2} e_{j_2}\|_{\mathbb{B}} \geq \|s_{j_3} e_{j_3}\|_{\mathbb{B}} \geq \dots$$

(handling ties arbitrarily) the **greedy algorithm of step N** is defined by the correspondence

$$x = \sum_{j\in\mathbb{N}} s_j e_j \in \mathcal{B} \longmapsto G_N(x) = \sum_{k=1}^{N} s_{j_k} e_{j_k}.$$

For $x \in \mathbb{B}$, the *N-term error of approximation* (with respect to $\mathcal{B}$) is defined by

$$\sigma_N(x)_{\mathbb{B}} = \inf\left\{\|x - y\|_{\mathbb{B}} : y \in \Sigma_N\right\}, \tag{1.1}$$

where Σ_N denotes the set of all elements $y \in \mathbb{B}$ with at most N non-null coefficients in the basis representation. It is clear that $\sigma_N(x)_{\mathbb{B}} \leq \|x - G_N(x)\|_{\mathbb{B}}$. A basis $\mathcal{B}$ is said

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to be **greedy** in $(\mathbb{B}, \|.\|_{\mathbb{B}})$ if there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\|x - G_N(x)\|_{\mathbb{B}} \leq \sigma_N(x)_{\mathbb{B}}, \quad \forall x \in \mathbb{B},\ N = 1,2,3\dots$$

Thus, for such bases the greedy algorithm produces an almost optimal N-term approximation, which leads often to a precise identification of the approximation spaces $\mathcal{A}_q^\alpha(\mathbb{B})$ (see [3] or [5] for precise definitions and results). A result of S.V. Konyagin and V.N. Temlyakov ([11]) gives a characterization of greedy bases in a Banach space $\mathbb{B}$ as those which are unconditional and democratic, that is, for some $c > 0$ we have

$$\Big\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \Big\|_{\mathbb{B}} \leq c \Big\| \sum_{\gamma\in\Gamma'} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}$$

for all finite sets of indices $\Gamma, \Gamma' \in \mathbb{N}$ with the same cardinality.

Wavelets systems are well known examples of greedy bases for many function spaces. In [16], V.N. Temlyakov shows that the Haar basis (and indeed any wavelet system L^p-equivalent to it) is greedy in the Lebesgue spaces $L^p(\mathbb{R}^d)$, $1 < p < \infty$. When wavelets have sufficient smoothness and decay they are also greedy bases for the more general Sobolev and Triebel-Lizorkin classes (see, e.g. [5] and [8].)

In this paper we study the efficiency of wavelet greedy algorithms in the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$. Wavelet bases are unconditional in $L^{p,q}(\mathbb{R}^d)$ for $1 < p < \infty$, $1 \leq q < \infty$ ([14]). When $p = q$, $L^{p,q}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ and by the above mentioned result of V.N. Temlyakov they are also greedy. But when $p \neq q$ the Haar basis is not democratic (hence, not greedy) in $L^{p,q}(\mathbb{R}^d)$ (see [18]). We give in Section 3 a simple proof of the fact that admissible wavelet bases (see the definition below) are not democratic in $L^{p,q}(\mathbb{R}^d)$ when $p \neq q$.

In view of this result it is interesting to ask how far wavelet bases are from being democratic in $L^{p,q}(\mathbb{R}^d)$, $p \neq q$. To quantify democracy of a basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$ we consider

$$h_r(N; \mathbb{B}, \mathcal{B}) = \sup_{Card(\Gamma)=N} \Big\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}$$

and

$$h_l(N; \mathbb{B}, \mathcal{B}) = \inf_{Card(\Gamma)=N} \Big\| \sum_{\gamma\in\Gamma} \frac{e_\gamma}{\|e_\gamma\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}$$

which we call the right and left democracy functions of $\mathcal{B}$ (see [6], [4] and [9].)

The main result of this paper gives a precise estimate for the right and left democracy functions of wavelet admissible bases in $L^{p,q}(\mathbb{R}^d)$ in terms of the exponents p and q.

Theorem 1.1. *For $1 < p < \infty$ and $1 \leq q < \infty$, let $\mathcal{B} = \{\psi_Q^l : Q \in \mathcal{D},\ l = 1, \dots, L\}$ be an admissible wavelet basis for $L^{p,q}(\mathbb{R}^d)$. Then*

$$h_l(N; L^{p,q}(\mathbb{R}^d), \mathcal{B}) \approx N^{\frac{1}{\max(p,q)}} \quad \textit{and} \quad h_r(N; L^{p,q}(\mathbb{R}^d), \mathcal{B}) \approx N^{\frac{1}{\min(p,q)}}.$$

The organization of this paper is as follows. Notation and results needed for the proof of Theorem 1.1 will be given in Section 2. Section 3 is devoted to the proof of this theorem. As an application of Theorem 1.1 and Theorems 4 and 5 in [10] we give inclusions for the approximation spaces (see Section 4) in terms of discrete Lorentz spaces (Corollary 4.3) and show that these inclusions are optimal (Lemma 4.8).

2. Preliminaries

2.1. **Lorentz spaces** $L^{p,q}(\mathbb{R}^d)$**.** For $1<p<\infty$ and $1\le q<\infty$ the set of functions f on $\mathbb{R}^d$ such that

$$\|f\|_{p,q}=\Big(q\int_0^\infty t^q|\{x:|f(x)|>t|\}|^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}}<\infty \tag{2.1}$$

defines the Lorentz space $L^{p,q}(\mathbb{R}^d)$. Note that $\|\lambda f\|_{p,q}=|\lambda|\|f\|_{p,q}$. In general, although the expression (2.1) is not a norm, $L^{p,q}(\mathbb{R}^d)$ is a Banach space with a norm equivalent to $\|.\|_{p,q}$ (see [15] or [1] for more information about these spaces). When $1<p<\infty$ and $1\le q<\infty$ the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ are separable rearrangement invariant Banach function spaces. Note that for a measurable set $E\subset\mathbb{R}^d$

$$\|\chi_E\|_{p,q}=\Big(q\int_0^1 t^q|E|^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}}=|E|^{\frac{1}{p}}. \tag{2.2}$$

For future reference we state the following elementary result that gives a discrete characterization of Lorentz spaces.

Proposition 2.1. *Let $1<p<\infty$ and $1\le q<\infty$. For any $a>\max\{1,2^{\frac{1}{p}-\frac{1}{q}}\}$ we have*

$$(i)\ \|f\|_{p,q}\approx\Big(\sum_{k\in\mathbb{Z}}a^{kq}|\{x:|f(x)|\ge a^k\}|^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

and

$$(ii)\ \|f\|_{p,q}\approx\Big(\sum_{k\in\mathbb{Z}}a^{kq}|\{x:a^k\le|f(x)|<a^{k+1}\}|^{\frac{q}{p}}\Big)^{\frac{1}{q}}.$$

2.2. **Wavelet bases for** $L^{p,q}(\mathbb{R}^d)$**.** Let $\mathcal{D}=\{Q_{j,k}=2^{-j}([0,1)^d+k):j\in\mathbb{Z},k\in\mathbb{Z}^d\}$ denote the set of all dyadic cubes in $\mathbb{R}^d$. We say that a finite family of functions $\{\psi^1,\dots,\psi^L\}\subset L^2(\mathbb{R}^d)$ is an **orthonormal wavelet system** if the collection

$$\{\psi^l_{j,k}(x)=2^{\frac{jd}{2}}\psi^l(2^jx-k):j\in\mathbb{Z},k\in\mathbb{Z}^d,L=1,2,\dots,l\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$. For simplicity we write ψ^l_Q for $\psi^l_{j,k}$ when $Q=Q_{j,k}$ is a dyadic cube. The reader is referred to [7], [12] for construction, examples and properties of orthonormal wavelets. Many wavelet families are unconditional bases for $L^p(\mathbb{R}^d)$, $1<p<\infty$. Moreover, there is a characterization for functions $f\in L^p(\mathbb{R}^d)$, $1<p<\infty$, in terms of the wavelet coefficients $\langle f,\psi^l_Q\rangle, Q\in\mathcal{D}, l=1,2\dots,L$, that is,

$$\|f\|_{L^p(\mathbb{R}^d)}\approx\|S_\psi(f)\|_{L^p(\mathbb{R}^d)},\ 1<p<\infty, \tag{2.3}$$

where

$$S_\psi(f)(x)=\Big(\sum_{l=1}^L\sum_{Q\in\mathcal{D}}|\langle f,\psi^l_Q\rangle|^2|Q|^{-1}\chi_Q(x)\Big)^{\frac{1}{2}}. \tag{2.4}$$

The characterization (2.3) holds for the d-dimensional Haar system, for wavelets arising from r-regular multiresolution analyses (see [12], pg. 22), for wavelets belonging to the regularity class R^o (as defined in [7], pg. 64, for d=1), and, in fact, for any orthonormal wavelet in $L^2(\mathbb{R}^d)$ with very mild decay conditions (see [13] and [17]). A wavelet system that satisfies (2.3) for all $p \in (1, \infty)$ is called an **admissible wavelet system**.

P.M Soardi proved in [14] that admissible wavelet systems $\{\psi_Q^l : Q \in \mathcal{D}, l = 1, 2 \dots, L\}$ are unconditional basis for $L^{p,q}(\mathbb{R}^d), 1 < p < \infty, 1 \leq q < \infty$ and there is a characterization similar to (2.3) for these spaces.

Proposition 2.2 ([14]). *Let $1 < p < \infty, 1 \leq q < \infty$ and $\{\psi^l : l = 1, \dots, L\}$ be an admissible wavelet system. Then, any $f \in L^{p,q}(\mathbb{R}^d)$ can be written in the form*

$$f = \sum_{l=1}^{L} \sum_{Q \in \mathcal{D}} \langle f, \psi_Q^l \rangle \psi_Q^l$$

with unconditional convergence in $L^{p,q}(\mathbb{R}^d)$, and moreover

$$\|f\|_{p,q} \approx \|S_\psi(f)\|_{p,q} \tag{2.5}$$

This result was derived from the corresponding wavelet characterization of Lebesgue spaces $L^p(\mathbb{R}^d), 1 < p < \infty$, by applying Boyd's interpolation theorem for sublinear operators.

Remark 2.3. *For the sake of simplicity, we assume throughout the paper that $L = 1$. Our theorems will remain valid for any $L \geq 1$, since the finite sum appearing in the definition of $S_\psi(f)$ given in (2.4) is harmless in our computations.*

2.3. **A simple Lemma.** We state and prove a simple result that will be used in the proof of Theorem 1.1.

Lemma 2.4. *Let $N_1, N_2 \dots, N_J \in \mathbb{N}$ and $N = N_1 + \dots + N_J$. For any $\alpha > 0$,*

$$\min\{N, N^\alpha\} \leq \sum_{j=1}^{J} N_j^\alpha \leq \max\{N, N^\alpha\}.$$

Proof. If $\alpha \leq 1$, $\sum_{j=1}^J N_j^\alpha \leq \sum_{j=1}^J N_j = N = \max\{N, N^\alpha\}$. On the other hand, $\sum_{j=1}^J N_j^\alpha \geq \left(\sum_{j=1}^J N_j\right)^\alpha = N^\alpha = \min\{N, N^\alpha\}$. If $\alpha > 1$, we have $\sum_{j=1}^J N_j^\alpha \leq \left(\sum_{j=1}^J N_j\right)^\alpha = N^\alpha = \max\{N, N^\alpha\}$. On the other hand, $\sum_{j=1}^J N_j^\alpha \geq \sum_{j=1}^J N_j = N = \min\{N, N^\alpha\}$. □

3. Proof of theorem 1.1

We start by showing that

$$h_r(N; L^{p,q}, \mathcal{B}) \gtrsim N^{\frac{1}{\min(p,q)}} \tag{3.1}$$

and

$$h_l(N; L^{p,q}, \mathcal{B}) \lesssim N^{\frac{1}{\max(p,q)}}. \tag{3.2}$$

This is done by exhibiting two examples of subsets of dyadic cubes Γ_1 and Γ_2, both of cardinality N such that

$$\Big\| \sum_{Q \in \Gamma_1} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \approx N^{\frac{1}{p}} \quad \textit{and} \quad \Big\| \sum_{Q \in \Gamma_2} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \approx N^{\frac{1}{q}}.$$

By (2.5), (2.4) and (2.2), for $Q \in \mathcal{D}$ and $\psi \in L^2(\mathbb{R}^d)$ an orthonormal wavelet

$$\|\psi_Q\|_{p,q} \approx \||Q|^{-\frac{1}{2}}\chi_Q\|_{p,q} = |Q|^{-\frac{1}{2}}|Q|^{\frac{1}{p}} = |Q|^{\frac{1}{p}-\frac{1}{2}}. \tag{3.3}$$

Again by (2.5) we obtain, for any finite set $\Gamma \subset \mathcal{D}$

$$\Big\|\sum_{Q\in\Gamma}\frac{\psi_Q}{\|\psi_Q\|_{p,q}}\Big\|_{p,q} \approx \Big\|\Big(\sum_{Q\in\Gamma}|Q|^{-\frac{2}{p}}\chi_Q\Big)^{\frac{1}{2}}\Big\|_{p,q}. \tag{3.4}$$

Choose $\Gamma_1 = \{Q_1, Q_2 \dots, Q_N\} \subset \mathcal{D}$ a set of disjoint dyadic cubes of the same fixed size $|Q|$. Then, by (3.4) and (2.2)

$$\begin{aligned}\Big\|\sum_{Q\in\Gamma_1}\frac{\psi_Q}{\|\psi_Q\|_{p,q}}\Big\|_{p,q} &\approx \Big\|\sum_{Q\in\Gamma_1}|Q|^{-\frac{1}{p}}\chi_Q\Big\|_{p,q} = \||Q|^{-\frac{1}{p}}\chi_{\bigcup_{Q\in\Gamma_1}Q}\|_{p,q}\\ &= |Q|^{-\frac{1}{p}}\Big|\bigcup_{Q\in\Gamma_1}Q\Big|^{\frac{1}{p}} = N^{\frac{1}{p}}.\end{aligned} \tag{3.5}$$

Choose now $\Gamma_2 = \{\widetilde{Q}_1, \dots, \widetilde{Q}_N\} \subset \mathcal{D}$ a pairwise disjoint family of dyadic cubes all of them of different sizes, say $|\widetilde{Q}_j| = 2^{-jd}$, $j = 1, 2, \dots, N$. By (3.4) we have

$$\Big\|\sum_{\widetilde{Q}\in\Gamma_2}\frac{\psi_{\widetilde{Q}}}{\|\psi_{\widetilde{Q}}\|_{p,q}}\Big\|_{p,q} \approx \Big\|\sum_{j=1}^N|\widetilde{Q}_j|^{-\frac{1}{p}}\chi_{\widetilde{Q}_j}\Big\|_{p,q} = \Big\|\sum_{j=1}^N 2^{\frac{jd}{p}}\chi_{\widetilde{Q}_j}\Big\|_{p,q}.$$

Using the discrete characterization of the Lorentz spaces given in Proposition 2.1, part (ii), with $a = 2^{\frac{d}{p}}$ we have

$$\begin{aligned}\Big\|\sum_{\widetilde{Q}\in\Gamma_2}\frac{\psi_{\widetilde{Q}}}{\|\psi_{\widetilde{Q}}\|_{p,q}}\Big\|_{p,q} &\approx \Big(\sum_{k\in\mathbb{Z}}2^{\frac{kqd}{p}}|\{x : 2^{\frac{kd}{p}} \le \sum_{j=1}^N 2^{\frac{jd}{p}}\chi_{\widetilde{Q}_j}(x) < 2^{\frac{(k+1)d}{p}}\}|^{\frac{q}{p}}\Big)^{\frac{1}{q}}\\ &= \Big(\sum_{j=1}^N 2^{\frac{jqd}{p}}|\widetilde{Q}_j|^{\frac{q}{p}}\Big)^{\frac{1}{q}} = N^{\frac{1}{q}}.\end{aligned} \tag{3.6}$$

Observe that (3.5) and (3.6) prove (3.1) and (3.2). Also observe that this shows that wavelet admissible bases cannot be democratic in $L^{p,q}(\mathbb{R}^d)$ for $p \neq q$.

We now show that

$$h_l(N; L^{p,q}, \mathcal{B}) \gtrsim N^{\frac{1}{\max(p,q)}}. \tag{3.7}$$

Let Γ be a subset of $\mathcal{D}$ of cardinality N. By (3.4) and using $\ell^2(\Gamma) \hookrightarrow \ell^\infty(\Gamma)$ we obtain

$$\Big\|\sum_{Q\in\Gamma}\frac{\psi_Q}{\|\psi_Q\|_{p,q}}\Big\|_{p,q} \approx \Big\|\Big(\sum_{Q\in\Gamma}|Q|^{-\frac{2}{p}}\chi_Q(.)\Big)^{\frac{1}{2}}\Big\|_{p,q} \gtrsim \|\sup_{Q\in\Gamma}|Q|^{-\frac{1}{p}}\chi_Q(.)\|_{p,q}. \tag{3.8}$$

Let $F(x) = \sup_{Q\in\Gamma}|Q|^{-\frac{1}{p}}\chi_Q(x)$, which is a finite value function. By part (i) of Proposition 2.1 with $a = 2^{\frac{d}{p}}$

$$\|F\|_{p,q}^q \approx \sum_{j\in\mathbb{Z}}2^{\frac{jqd}{p}}|\{x : F(x) \ge 2^{\frac{jd}{p}}\}|^{\frac{q}{p}}. \tag{3.9}$$

Write $\Gamma = \bigcup_{j=1}^{J} \Gamma_j$ where $\Gamma_j = \{Q \in \Gamma : |Q| = 2^{-dk_j}\}$ with $k_1 > k_2 > \ldots > k_J$. We have $\sum_{j=1}^{J} Card(\Gamma_j) = Card(\Gamma) = N$. Since the disjoint union $\bigcup_{Q\in\Gamma_j} Q$ is contained in $\{x : F(x) \geq 2^{\frac{k_j d}{p}}\}$ we deduce from (3.9)

$$\begin{aligned} \|F\|_{p,q}^q &\gtrsim \sum_{j=1}^{J} 2^{\frac{k_j qd}{p}} \Big| \bigcup_{Q\in\Gamma_j} Q \Big|^{\frac{q}{p}} = \sum_{j=1}^{J} 2^{\frac{k_j qd}{p}} \Big(\sum_{Q\in\Gamma_j} |Q| \Big)^{\frac{q}{p}} \\ &= \sum_{j=1}^{J} (Card(\Gamma_j))^{\frac{q}{p}} \gtrsim \min\{N, N^{\frac{q}{p}}\} \end{aligned} \tag{3.10}$$

where the last inequality is due to Lemma 2.4. From (3.8) and (3.10) we deduce

$$\Big\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \gtrsim \min\{N^{\frac{1}{p}}, N^{\frac{1}{q}}\} = N^{\frac{1}{\max(p,q)}},$$

showing (3.7).

Finally, we need to show the inequality

$$\Big\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \lesssim N^{\frac{1}{\min(p,q)}} \tag{3.11}$$

for any subset Γ of $\mathcal{D}$ of cardinality N. We first linearize the square function. By (3.3) and (2.4)

$$S_\psi\Big(\sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big)(x) \approx S_\psi\Big(\sum_{Q\in\Gamma} |Q|^{\frac{1}{2}-\frac{1}{p}} \psi_Q \Big)(x) = \Big(\sum_{Q\in\Gamma} |Q|^{-\frac{2}{p}} \chi_Q(x) \Big)^{\frac{1}{2}}. \tag{3.12}$$

For each $x \in \bigcup_{Q\in\Gamma} Q$, let Q_x be the smallest dyadic cube in Γ containing x. Then, we have the pointwise estimate

$$\begin{aligned} \Big(\sum_{Q\in\Gamma} |Q|^{-\frac{2}{p}} \chi_Q(x) \Big)^{\frac{1}{2}} &\leq \Big(\sum_{\substack{Q\supset Q_x \\ Q\in\mathcal{D}}} |Q|^{-\frac{2}{p}} \Big)^{\frac{1}{2}} = \Big(\sum_{j=0}^{\infty} (2^{jd}|Q_x|)^{-\frac{2}{p}} \Big)^{\frac{1}{2}} \\ &= |Q_x|^{-\frac{1}{p}} \chi_{Q_x}(x) \Big(\sum_{j=0}^{\infty} 2^{-\frac{2jd}{p}} \Big)^{\frac{1}{2}} \leq c_p |Q_x|^{-\frac{1}{p}} \chi_{Q_x}(x) \\ &\leq c_p \Big(\sum_{Q\in\Gamma} |Q|^{-\frac{2}{p}} \chi_Q(x) \Big)^{\frac{1}{2}}, \end{aligned} \tag{3.13}$$

where the last inequality holds since the right hand side contains at least the cube Q_x (and possibly more). Hence (3.12) and (3.13) show:

$$S_\psi\Big(\sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big)(x) \approx |Q_x|^{-\frac{1}{p}} \chi_{Q_x}(x). \tag{3.14}$$

This linearization procedure has been used by other authors in the context of N-term approximation (see e.g [2], [5], [6], [8]).

Let $\Gamma_{\min} = \{Q \in \Gamma : Q = Q_x \text{ for some } x \in \bigcup_{Q\in\Gamma} Q\}$ be the set of minimal cubes from Γ. The cubes from $\Gamma_{\min}$ are not pairwise disjoint. To have a disjoint family of sets, for each $Q \in \Gamma$ we define, as in [6], $Light(Q) = Q \backslash Shade(Q)$ where

$$Shade(Q) = \bigcup \Big\{ R : R \in \Gamma, R \subsetneq Q \Big\}.$$

Clearly

$$\bigcup_{Q\in\Gamma} Q = \bigcup_{Q\in\Gamma_{\min}} Light(Q)$$

and the sets in the last union are pairwise disjoint. From (3.14) we obtain

$$S_\psi\Big(\sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}}\Big)(x) \approx \sum_{Q\in\Gamma_{\min}} \frac{\chi_{Light(Q)}(x)}{|Q|^{\frac{1}{p}}} \tag{3.15}$$

From (3.15) we deduce

$$\Big\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \approx \Big\| \sum_{Q\in\Gamma_{\min}} |Q|^{-\frac{1}{p}} \chi_{Light(Q)} \Big\|_{p,q}.$$

Write $\Gamma_{\min} = \bigcup_{j=1}^{J} \Gamma_j$ where $\Gamma_j = \{Q \in \Gamma_{\min} : |Q| = 2^{-k_j d}\}$, with $k_1 > k_2 > \ldots > k_J$. We have $\sum_{j=1}^{J} Card(\Gamma_j) = Card(\Gamma_{\min})$. By (ii) of Proposition 2.1 with $a = 2^{\frac{d}{p}}$ we obtain

$$\begin{aligned}
\Big\| \sum_{Q\in\Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} &\approx \Big(\sum_{k\in\mathbb{Z}} 2^{\frac{kqd}{p}} |\{x : 2^{\frac{kd}{p}} \leq \sum_{Q\in\Gamma_{\min}} |Q|^{-\frac{1}{p}} \chi_{Light(Q)}(x) \leq 2^{\frac{(k+1)d}{p}}\}|^{\frac{q}{p}} \Big)^{\frac{1}{q}} \\
&= \Big(\sum_{j=1}^{J} 2^{\frac{k_j qd}{p}} \Big| \bigcup_{Q\in\Gamma_j} Light(Q) \Big|^{\frac{q}{p}} \Big)^{\frac{1}{q}} \leq \Big(\sum_{j=1}^{J} 2^{\frac{k_j qd}{p}} \Big(\sum_{Q\in\Gamma_j} 2^{-k_j d} \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} \\
&= \Big(\sum_{j=1}^{J} (Card(\Gamma_j))^{\frac{q}{p}} \Big)^{\frac{1}{q}} \leq \max\{(Card(\Gamma_{\min})^{\frac{1}{q}}), (Card(\Gamma_{\min})^{\frac{1}{p}}\} \\
&\leq N^{\frac{1}{\min(p,q)}}
\end{aligned}$$

where the next to the last equality is due to Lemma 2.4. This shows (3.11), finishing the proof of Theorem 1.1. □

4. Inclusions for approximation spaces

In this section we will use Theorem 1.1 together with Theorems 4 and 5 of [10] to show inclusions for approximation spaces. We recall the definition of the **approximation spaces**. For a Banach (or quasi-Banach) space $\mathbb{B}$, given $\alpha > 0$ and $0 < r < \infty$ the approximation space is

$$\mathcal{A}_r^\alpha(\mathbb{B}) = \Big\{ x \in \mathbb{B} : \Big[\sum_{N\geq 1} [N^\alpha \sigma_N(x)_{\mathbb{B}}]^r \frac{1}{N} \Big]^{\frac{1}{r}} < \infty \Big\}$$

and

$$\|x\|_{\mathcal{A}_r^\alpha(\mathbb{B})} = \|x\|_{\mathbb{B}} + \Big[\sum_{N\geq 1} [N^\alpha \sigma_N(x)_{\mathbb{B}}]^r \frac{1}{N} \Big]^{\frac{1}{r}},$$

where $\sigma_N(x)_{\mathbb{B}}$ is the N-term error of approximation (see (1.1).) When $r = \infty$ the above definition is modified in the standard way,

$$\mathcal{A}^\alpha_\infty(\mathbb{B}) = \{x \in \mathcal{B} : \sup_{N\geq 1} N^\alpha \sigma_N(x)_{\mathbb{B}} < \infty\}$$

and

$$\|x\|_{\mathcal{A}^\alpha_\infty(\mathbb{B})} = \|x\|_{\mathbb{B}} + \sup_{N\geq 1} N^\alpha \sigma_N(x)_{\mathbb{B}}.$$

The inclusions will be given in terms of sequence spaces over the index set $\mathcal{D}$. Let $\mathcal{C}_o$ be the set of sequences $\mathbf{s} = \{s_Q : Q \in \mathcal{D}\}$ for which we can find an enumeration of the index set $\mathcal{D} = \{(Q_k)_{k=1}^\infty\}$ such that $|s_{Q_1}| \geq |s_{Q_2}| \geq \dots$ and $\lim_{k\longrightarrow\infty} |s_{Q_k}| = 0$. We shall always assume that $\{s_{Q_k}; k \geq 1\}$ corresponds to such an ordering, which coincides with the **non-increasing rearrangement** $\mathbf{s}^*$ of $\mathbf{s}$.

For each $0 < \tau, r < \infty$ we define the **discrete Lorentz space** by

$$\ell^{\tau,r} = \{\mathbf{s} \in \mathcal{C}_o : \|\mathbf{s}\|_{l^{\tau,r}} = \Big[\sum_{k\geq 1} (k^{\frac{1}{\tau}} |s_{Q_k}|)^r \frac{1}{k}\Big]^{\frac{1}{r}} < \infty\}$$

and if $r = \infty$

$$\ell^{\tau,\infty} = \{\mathbf{s} \in \mathcal{C}_o : \|\mathbf{s}\|_{l^{\tau,\infty}} = \sup_{k\geq 1} k^{\frac{1}{\tau}} |s_{Q_k}| < \infty\}.$$

Let $f \in L^{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$, and write $f = \sum_{Q\in\mathcal{D}} \langle f, \psi_Q\rangle \psi_Q$. Then, define $\ell^{\tau,r}(L^{p,q})$, as the set of all $f \in L^{p,q}(\mathbb{R}^d)$ such that the sequence $\{\|\langle f, \psi_{Q_k}\rangle \psi_{Q_k}\|_{p,q} : k \geq 1\} \in \ell^{\tau,r}$ and

$$\|f\|_{\ell^{\tau,r}(L^{p,q})} = \big\|\|\langle f, \psi_{Q_k}\rangle \psi_{Q_k}\|_{p,q}\big\|_{\ell^{\tau,r}}$$

where as before $\|\langle f, \psi_{Q_1}\rangle \psi_{Q_1}\|_{p,q} \geq \|\langle f, \psi_{Q_2}\rangle \psi_{Q_2}\|_{p,q} \geq \dots$. By (3.3) we can write

$$\|f\|_{l^{\tau,r}(L^{p,q})} \approx \||\langle f, \psi_{Q_k}\rangle|\, |Q_k|^{\frac{1}{p}-\frac{1}{2}}\|_{\ell^{\tau,r}} = \Big[\sum_{k\geq 1} (k^{\frac{1}{\tau}} |Q_k|^{\frac{1}{p}-\frac{1}{2}} |\langle f, \psi_{Q_k}\rangle|)^r \frac{1}{k}\Big]^{\frac{1}{r}}$$

with the obvious modifications if $r = \infty$.

For the reader's convenience we write below the statements of Theorems 4 and 5 of [10] adapted to our setting and our notation.

Theorem 4.1. *Let $0 < p < \infty$, let $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$ be an unconditional basis of $\mathbb{B}$. The following statements are equivalent.*

1. *For all $0 < q < p$, and $s = \frac{1}{q} - \frac{1}{p}$, there exists $C_s < \infty$ such that*
$$\sigma_N(x) \leq C_s \|x\|_{\ell^{q,q}(\mathbb{B})} N^{-s} \qquad \text{for all} \quad x \in \ell^{q,q}(\mathbb{B})\,. \tag{4.1}$$
2. *There exists q, $0 < q < p$, such that (4.1) holds for $s = \frac{1}{q} - \frac{1}{p}$.*
3. *There exists C_1, such that, for all $\Gamma \subset \mathbb{N}$*
$$\Big\|\sum_{j\in\Gamma} \frac{e_j}{\|e_j\|_{\mathbb{B}}}\Big\|_{\mathbb{B}} \leq C_1 |\Gamma|^{1/p}\,.$$
4. *For all $0 < q < p$, and for all $0 < r \leq \infty$, $\ell^{q,r}(\mathbb{B}) \hookrightarrow \mathcal{A}^s_r(\mathbb{B})$ with $s = \frac{1}{q} - \frac{1}{p}$.*

Theorem 4.2. *Let $0 < p < \infty$, let $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$ be an unconditional basis of $\mathbb{B}$. The following statements are equivalent.*

1. *For all $0 < q < p$, and $s = \frac{1}{q} - \frac{1}{p}$, there exists $C_s < \infty$ such that*
$$\|x\|_{\ell^{q,q}(\mathbb{B})} \leq C_s \|x\|_{\mathbb{B}} N^s \qquad \text{for all} \quad x \in \Sigma_N\,. \tag{4.2}$$

2. *There exists q, $0 < q < p$, such that (4.2) holds for $s = \frac{1}{q} - \frac{1}{p}$.*
3. *There exists C_1, such that, for all $\Gamma \subset \mathbb{N}$*

$$\frac{1}{C_1}|\Gamma|^{1/p} \leq \Big\| \sum_{j\in\Gamma} \frac{e_j}{\|e_j\|_{\mathbb{B}}} \Big\|_{\mathbb{B}}.$$

4. *For all $0 < q < p$, and for all $0 < r \leq \infty$, $\mathcal{A}^s_r(\mathbb{B}) \hookrightarrow \ell^{q,r}(\mathbb{B})$ with $s = \frac{1}{q} - \frac{1}{p}$.*

Corollary 4.3. *Let $\mathcal{B} = \{\psi^l_Q : Q \in \mathcal{D}, l = 1, 2, \dots, L\}$ be an admissible wavelet basis for $L^{p,q}(\mathbb{R}^d)$, $1 < p < \infty$, $1 \leq q < \infty$. For every $\alpha > 0$ and $0 < r \leq \infty$ we have*

$$\ell^{\tau^-,r}(L^{p,q}) \hookrightarrow \mathcal{A}^\alpha_r(L^{p,q}) \hookrightarrow \ell^{\tau^+,r}(L^{p,q}) \tag{4.3}$$

where $\frac{1}{\tau^-} = \alpha + \frac{1}{\min(p,q)}$ and $\frac{1}{\tau^+} = \alpha + \frac{1}{\max(p,q)}$.

Proof. For the left hand side inclusion of (4.3) use $h_r(N; L^{p,q}(\mathbb{R}^d), \mathcal{B}) \approx N^{\frac{1}{\min(p,q)}}$ from Theorem 1.1 and apply the implication 3 $\Rightarrow$ 4 of Theorem 4 in [10] (see Theorem 4.1) with $\mathbb{B} = L^{p,q}(\mathbb{R}^d)$, and an admissible wavelet basis. Similarly, use $h_l(N; L^{p,q}(\mathbb{R}^d), \mathcal{B}) \approx N^{\frac{1}{\max(p,q)}}$ from Theorem 1.1 and apply the implication 3 $\Rightarrow$ 4 of Theorem 5 in [10] (see Theorem 4.2) to obtain the right hand inclusion. □

Corollary 4.4. *(**Jackson's inequalities**) Let $\mathcal{B} = \{\psi^l_Q : Q \in \mathcal{D}, l = 1, 2, \dots L\}$ be an admissible wavelet basis for $L^{p,q}(\mathbb{R}^d)$, $1 < p < \infty$, $1 \leq q < \infty$. Let $\alpha > 0$, $0 < r \leq \infty$, and $\frac{1}{\tau^-} = \alpha + \frac{1}{\min(p,q)}$. There exists $C > 0$ such that for all $f \in \ell^{\tau^-,r}(L^{p,q})$,*

$$\sigma_N(f)_{L^{p,q}} \leq C N^{-\alpha} \|f\|_{\ell^{\tau^-,r}(L^{p,q})} \quad \textit{for all } N \geq 1\,. \tag{4.4}$$

Proof. Notice that $\mathcal{A}^\alpha_r(L^{p,q}) \hookrightarrow \mathcal{A}^\alpha_\infty(L^{p,q})$ for all $0 < r \leq \infty$. By the left hand side of inclusion (4.3) from Corollary 4.3, $\ell^{\tau^-,r}(L^{p,q}) \hookrightarrow \mathcal{A}^\alpha_\infty(L^{p,q})$ and this inclusion is equivalent to (4.4). □

Remark 4.5. *The strongest inequality in (4.4) is obtained when $r = \infty$. Notice that for all $r > \tau^-$, inequality (4.4) is stronger than the one obtained in statement 1 of Theorem 4 in* [10] *when adapted to our setting (see Theorem 4.1).*

Corollary 4.6. *(**Bernstein's inequalities**) Let $\mathcal{B} = \{\psi^l_Q : Q \in \mathcal{D}, l = 1, 2, \dots L\}$ be an admissible wavelet basis for $L^{p,q}(\mathbb{R}^d)$, $1 < p < \infty$, $1 \leq q < \infty$. Let $\alpha > 0$, $0 < r \leq \infty$, and $\frac{1}{\tau^+} = \alpha + \frac{1}{\max(p,q)}$. There exists $C > 0$ such that for all $f \in \Sigma_N \subset L^{p,q}$,*

$$\|f\|_{\ell^{\tau^+,r}(L^{p,q})} \leq C \|f\|_{p,q} N^\alpha \quad \textit{for all } N \geq 1. \tag{4.5}$$

Proof. If $f \in \Sigma_N$ we have $\sigma_k(f)_{L^{p,q}} = 0$ if $k \geq N$ and $\sigma_k(f)_{L^{p,q}} \leq \|f\|_{p,q}$ if $1 \leq k < N$. Hence

$$\|f\|_{\mathcal{A}^\alpha_r(L^{p,q})} \leq \|f\|_{p,q} + \|f\|_{p,q} \left(\sum_{k=1}^{N-1} k^{\alpha r} \frac{1}{k} \right)^{1/r} \approx \|f\|_{p,q} N^\alpha\,.$$

This inequality, together with the right hand side inclusion in (4.3) gives the result. □

Remark 4.7. *Notice that for all $r < \tau^+$, inequality (4.5) is stronger than the one obtained in statement 1 of Theorem 5 in* [10] *when adapted to our setting (see Theorem 4.2).*

The inclusions in Corollary 4.3 are best possible in the sense described in the following Lemma.

Lemma 4.8. *Let $\mathcal{B} = \{\psi_Q^l : Q \in \mathcal{D}, l = 1, 2, \dots L\}$ be an admissible wavelet basis for $L^{p,q}(\mathbb{R}^d)$, $1 < p < \infty$, $1 \le q < \infty$. For every $\alpha > 0$ and $0 < r \le \infty$ we have the following:*

(i) *If the inclusion $\ell^{\widetilde{\tau},r}(L^{p,q}) \hookrightarrow \mathcal{A}_r^\alpha(L^{p,q})$ holds for some $\widetilde{\tau} > 0$, we must have $\widetilde{\tau} \le \tau^-$ where $\frac{1}{\tau^-} = \alpha + \frac{1}{\min(p,q)}$.*

(ii) *If the inclusion $\mathcal{A}_r^\alpha(L^{p,q}) \hookrightarrow \ell^{\widetilde{\tau},r}(L^{p,q})$ holds for some $\widetilde{\tau} > 0$, we must have $\tau^+ \le \widetilde{\tau}$ where $\frac{1}{\tau^+} = \alpha + \frac{1}{\max(p,q)}$.*

Proof. (i) Let Γ_1 be a collection of $2N$, $N \in \mathbb{N}$, pairwise disjoint dyadic cubes of equal size $|Q|$ and set $f_1 = \sum_{Q\in\Gamma_1} \frac{\psi_Q}{\|\psi_Q\|_{p,q}}$. From Lemma 1 in [10], for $k = 1, 2, \dots, 2N-1$ we have

$$\sigma_k(f_1) = \inf_{\Gamma'\subset\Gamma_1,\ |\Gamma'|=2N-k} \Big\| \sum_{Q\in\Gamma'} \frac{\psi_Q}{\|\psi_Q\|_{p,q}} \Big\|_{p,q} \approx (2N-k)^{\frac{1}{p}} \tag{4.6}$$

where the last equivalence is due to (3.5). From (4.6) we deduce

$$\|f_1\|_{\mathcal{A}_r^\alpha(L^{p,q})} \gtrsim \Big\{ \sum_{k=1}^{N} [k^\alpha (2N-k)^{\frac{1}{p}}]^r \frac{1}{k} \Big\}^{\frac{1}{r}} \ge N^{\frac{1}{p}} \Big\{ \sum_{k=1}^{N} k^{\alpha r} \frac{1}{k} \Big\}^{\frac{1}{r}} \gtrsim$$
$$\gtrsim N^{\frac{1}{p}} N^\alpha = N^{\alpha+\frac{1}{p}} \tag{4.7}$$

On the other hand,

$$\|f_1\|_{\ell^{\tilde\tau,r}(L^{p,q})} = \Big\{ \sum_{k=1}^{2N} (k^{\frac{1}{\tilde\tau}} 1)^r \frac{1}{k} \Big\}^{\frac{1}{r}} \approx (2N)^{\frac{1}{\tilde\tau}} \approx N^{\frac{1}{\tilde\tau}} \tag{4.8}$$

From the inclusion $\ell^{\widetilde{\tau},r} \hookrightarrow \mathcal{A}_r^\alpha(L^{p,q})$, (4.7) and (4.8) we deduce $N^{\alpha+\frac{1}{p}} \lesssim N^{\frac{1}{\tilde\tau}}$ for all $N = 1, 2, \dots$. Thus

$$\alpha + \frac{1}{p} \le \frac{1}{\tilde\tau}\,. \tag{4.9}$$

Choose now $\Gamma_2 = \{\tilde Q, \dots, \tilde Q_{2N}\} \subset \mathcal{D}$ be a collection of pairwise disjoint dyadic cubes all of them of different sizes, say $|Q| = 2^{-jd}$, $j = 1, 2, \dots, 2N$. Set $f_2 = \sum_{\tilde Q\in\Gamma_2} \frac{\psi_{\tilde Q}}{\|\psi_{\tilde Q}\|_{p,q}}$. Using (3.6) in place of (3.5) and arguing as in (4.6) we obtain

$$\sigma_k(f_2)_{L^{p,q}} \approx (2N-k)^{\frac{1}{q}}, \quad k = 1, 2, \dots, 2N-1\,.$$

This results gives $\|f_2\|_{\mathcal{A}_r^\alpha(L^{p,q})} \gtrsim N^{\alpha+\frac{1}{q}}$ with an argument similar to that of (4.7). On the other hand $\|f_2\|_{\ell^{\tilde\tau,r}(L^{p,q})} \approx N^{\frac{1}{\tilde\tau}}$. The assumed inclusion then produces $N^{\alpha+\frac{1}{q}} \lesssim N^{\frac{1}{\tilde\tau}}$ for all $N = 1, 2, \dots$. Thus

$$\alpha + \frac{1}{q} \le \frac{1}{\tilde\tau}. \tag{4.10}$$

From (4.9) and (4.10) we deduce $\frac{1}{\tau^-} = \alpha + \frac{1}{\min(p,q)} \le \frac{1}{\tilde\tau}$ proving part (i) of Lemma 4.8.

(ii) Take Γ_1 as in the proof of part (i). Then, by (4.6)

$$\|f_1\|_{\mathcal{A}_r^\alpha(L^{p,q})} \lesssim N^{\frac{1}{p}} + \Big\{ \sum_{k=1}^{2N-1} [k^\alpha (2N-k)^{\frac{1}{p}}]^r \frac{1}{k} \Big\}^{\frac{1}{r}} \lesssim N^{\frac{1}{p}} + N^{\frac{1}{p}} \Big\{ \sum_{k=1}^{2N-1} k^{\alpha r} \frac{1}{k} \Big\}^{\frac{1}{r}} \lesssim$$

$$\lesssim N^{\frac{1}{p}+\alpha}. \tag{4.11}$$

From the inclusion $\mathcal{A}_r^\alpha(L^{p,q}) \hookrightarrow \ell^{\tilde{\tau},r}(L^{p,q})$, (4.11) and (4.8) we deduce $N^{\frac{1}{\tilde{\tau}}} \lesssim N^{\frac{1}{p}+\alpha}$, for all $N = 1, 2, \dots$. Thus

$$\frac{1}{\tilde{\tau}} \leq \alpha + \frac{1}{p}. \tag{4.12}$$

Using now the function f_2 of the proof of part i), and arguing similarly as above we must have $N^{\frac{1}{\tilde{\tau}}} \lesssim N^{\alpha+\frac{1}{q}}$ for all $N = 1, 2, \dots$. Thus

$$\frac{1}{\tilde{\tau}} \leq \alpha + \frac{1}{q}. \tag{4.13}$$

The inequalities (4.12) and (4.13) give

$$\frac{1}{\tilde{\tau}} \leq \alpha + \frac{1}{\max(p,q)} = \alpha + \frac{1}{\tau^+}$$

proving the desired result. □

Remark 4.9. *A similar result as the one in Lemma 4.8 can be obtained from Theorem 1.1 and* 4 ⇒ 3 *of Theorems 4 and 5 in* [10] *(see Theorems 4.1 and 4.2) assuming* $\alpha < 1/\tilde{\tau}$ *in statements (i) and (ii) of the Lemma. The direct and simple proof given above does not need to assume this condition.*

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