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A_∞ ESTIMATES VIA EXTRAPOLATION OF CARLESON MEASURES AND APPLICATIONS TO DIVERGENCE FORM ELLIPTIC OPERATORS

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Abstract. We revisit the "extrapolation method" for Carleson measures, introduced in [LM] to prove A_∞ estimates for certain caloric measures, and we present a purely real variable version of the method suitable for establishing A_∞ estimates. To illustrate the use of this technique, we then reprove a well known result of [FKP].

1. Introduction

In this article we revisit a technique introduced in work of Lewis and Murray [LM], and developed further in [HL], [AHLT], [AHMTT],[†] and which has come to be known as the "extrapolation method" for Carleson measures. The method is a bootstrapping technique for proving scale invariant estimates on cubes (e.g., reverse Hölder estimates, Carleson measure estimates, BMO estimates), given that (very roughly speaking) the desired estimate holds on those cubes Q for which some controlling Carleson measure μ is sufficiently small in the associated Carleson box R_Q. The exact nature of this control (involving sawtooth subdomains in R_Q) will be made precise later.

In [LM], the extrapolation technique was used to prove reverse Hölder estimates for caloric measures in non-cylindrical (i.e., time-varying) domains; in this case μ arose in the quantitative description of the boundary. The results of [LM] were generalized in [HL], where reverse Hölder estimates for parabolic (and elliptic-harmonic) measures were established for variable coefficient parabolic (and elliptic) equations, given appropriate Carleson measure control of the coefficients. In particular, this work included an alternative proof, via the extrapolation method, of a well known result of R. Fefferman, Kenig and Pipher [FKP], that we shall discuss further in Section 3.

The results of [LM] and of [HL] are examples of "Carleson $\to A_\infty$" extrapolation, in which a given non-negative measure ω is shown to belong to A_∞ (or "weak A_∞"), using properties of some controlling Carleson measure μ. The results of [AHLT] and of [AHMTT] involve "Carleson $\to$ Carleson" extrapolation, in which a non-negative measure in the half space $\mathbb{R}^{n+1}_+$ is shown to be a Carleson measure, using properties of another controlling

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[†]We mention also that certain aspects of the arguments in these works are similar in spirit to Carleson's corona construction [Car], and to the David-Semmes Corona Decomposition of a uniformly rectifiable set [DS].

Carleson measure. In [AHLT], the technique was applied to prove the restricted version of the Kato square root conjecture, for divergence form elliptic operators that were small complex perturbations of real symmetric ones. An interesting feature of the "Carleson $\to$ Carleson" extrapolation arguments in [AHLT] and [AHMTT] is that they were purely real variable in nature —the bootstrapping procedure was separated from the applications to PDE.

On the other hand, in [LM] and [HL], the extrapolation arguments were tied specifically to the fact that one was working with harmonic or parabolic measures, and the main goal of this article is to extract the real variable essence of "Carleson $\to A_\infty$" extrapolation.

In this article, we shall present one new result and one new technical innovation. The new result, Theorem 2.1 below, is a purely real variable treatment of "Carleson $\to A_\infty$" extrapolation. The new technical innovation of the present paper is the use of the projection operators $\mathcal{P}_\mathcal{F}$. In retrospect, these are quite natural when working with dyadic sawtooth domains (cf. the "Main Lemma" of [DJK], where indeed a similar construction has appeared).

In order to illustrate the method, and the use of Theorem 2.1, we then show how the latter may be used to reprove the main theorem in [FKP]. To do that we prove some versions of the "Main Lemma" in [DJK] adapted to discrete sawtooth domains (the precise definitions are given below). The first result (cf. Lemma A.1) is written in terms of the projection operators and we use it to reprove the main theorem in [FKP]. The second result (cf. Lemma A.2) is interesting in its own right and is a dyadic analog of the main lemma in [DJK]. The proofs of these results follow the ideas in [DJK], but are technically much simpler, owing to the dyadic setting in which we work here.

An alternative formulation of the extrapolation result is given in [HM]. There we consider a different characterization of A_∞ written in terms of the level sets of the weight, and we discuss some of conditions that equivalently define this class of weights. That approach can also be used to give a new proof of the main theorem in [FKP].

2. Main result

2.1. **Notation.**

- We write $|x-y|_\infty = \max\{|x_i - y_i| : 1 \le i \le n\}$.
- Given a cube $Q \in \mathbb{R}^n$ we denote its center by x_Q and its sidelength by $\ell(Q)$. For any $\tau > 0$ we write τQ for the cube with center x_Q and sidelength $\tau\,\ell(Q)$. By $\mathcal{D}(Q)$ we denote the collection of dyadic subcubes[‡] of Q and $\mathcal{D}(Q)^* = \mathcal{D}(Q) \setminus \{Q\}$. We also write $Q(x,l)$ for the cube centered at x with sidelength l.
- We say that a non-negative Borel measure ω is (concentrically) doubling if for every cube (or ball) Q we have $\omega(2\,Q) \le C_\omega\, \omega(Q)$. It is "dyadically doubling" if $\omega(Q) \le C_\omega\,(Q')$, for every $Q \in \mathcal{D}(Q_0)$, and for every dyadic "child" Q' of Q. Here, Q_0 is either some fixed cube, or $\mathbb{R}^n$.
- Given two dyadically doubling non-negative Borel measures ω and ν, and a fixed cube Q_0 (we allow $Q_0 = \mathbb{R}^n$), we say that $\omega \in A_\infty^{\rm dyadic}(\nu, Q_0)$ if there exist constants $\theta > 0$ and $C < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$ and for all Borel sets $F \subset Q$, we have
$$\frac{\omega(F)}{\omega(Q)} \le C \left(\frac{\nu(F)}{\nu(Q)}\right)^\theta. \tag{2.1}$$

[‡]Note that the term "dyadic" here refers to the grid induced by Q; the cubes in $\mathcal{D}(Q)$ are dyadic cubes of $\mathbb{R}^n$ if and only if Q itself is such.

When ν is Lebesgue measure we shall simply write $\omega \in A_\infty^{\text{dyadic}}(Q_0)$. It is known that A_∞^{dyadic} defines an equivalence relationship (cf. Lemma B.4 in Appendix B), and also that condition (2.1) is equivalent to the following apparently weaker condition (see also (2.5) in the case $\nu =$ Lebesgue measure): there exist $0 < \alpha, \beta < 1$ such that for every $Q \in \mathcal{D}(Q_0)$ and for every Borel set $F \subset Q$, we have that $\nu(F)/\nu(Q) < \alpha$ implies $\omega(F)/\omega(Q) < \beta$, see [GR, Chapter 4] or [HM].

- Given two doubling non-negative Borel measures ω and ν, and a fixed cube Q_0 (we allow $Q_0 = \mathbb{R}^n$), we say that $\omega \in A_\infty(\nu, Q_0)$ if (2.1) holds for all $Q \subset Q_0$ and all Borel sets $F \subset Q$. A_∞ defines an equivalence relationship that can be equivalently defined in terms of the analogous "weaker" condition described above, see [GR, Chapter 4] for more details.
- Given a cube Q we write $\fint_Q f(x)\,dx := \frac{1}{|Q|}\int_Q f(x)\,dx$.
- Let Q be a cube. We denote the associated Carleson box by $R_Q := Q \times (0, \ell(Q))$. We will also at times work with the "short" Carleson box $R_Q^{\text{short}} := Q \times (0, \ell(Q)/2)$, and with the "Whitney box" $W_Q := R_Q \setminus R_Q^{\text{short}} = Q \times [\ell(Q)/2, \ell(Q))$.
- We write $\mathcal{C}$ for the set of Carleson measures in $\mathbb{R}^{n+1}_+$, i.e., the non-negative Borel measures μ on $\mathbb{R}^{n+1}_+$ for which the "Carleson norm"
$$\|\mu\|_{\mathcal{C}} := \sup_{Q \subset \mathbb{R}^n} |Q|^{-1}\,\mu(R_Q) \tag{2.2}$$
is finite; here, the supremum runs over all cubes $Q \subset \mathbb{R}^n$. Analogously, given $Q_0 \subset \mathbb{R}^n$ we write $\mathcal{C}(Q_0)$ for the set of Borel measures that satisfy the previous condition restricted to $Q \in \mathcal{D}(Q_0)$, thus
$$\|\mu\|_{\mathcal{C}(Q_0)} := \sup_{Q \in \mathcal{D}(Q_0)} |Q|^{-1}\,\mu(R_Q).$$
By slight abuse of notation[††], if $Q_0 = \mathbb{R}^n$ we simply write $\mathcal{C} = \mathcal{C}(Q_0)$.
- Given Q and a family of pairwise disjoint dyadic subcubes $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ we define the discrete sawtooth function $\psi_{\mathcal{F}}(x) := \sum_k \ell(Q_k)\,\chi_{Q_k}(x)$. Notice that ψ is a step function supported in $\cup_k Q_k$. We write $\Omega_{\mathcal{F}} = \Omega_{\psi_{\mathcal{F}}}$ for the domain above the graph of $\psi_{\mathcal{F}}$, that is, $\Omega_{\mathcal{F}} := \{(x,t) \in \mathbb{R}^{n+1}_+ : t \geq \psi_{\mathcal{F}}(x)\}$. Notice that $\Omega_{\mathcal{F}} = \mathbb{R}^{n+1}_+ \setminus (\cup_k R_{Q_k})$. We allow $\mathcal{F}$ to be empty in which case $\psi_{\mathcal{F}}(x) = 0$ and $\Omega_{\mathcal{F}} = \mathbb{R}^{n+1}_+$. See Figure 1.
- If μ is a non-negative Borel measure on $\mathbb{R}^{n+1}_+$, then $\mu_{\mathcal{F}} := \mu\,\chi_{\Omega_{\mathcal{F}}}$ will denote its restriction to the dyadic sawtooth $\Omega_{\mathcal{F}}$.
- Given Q and $\mathcal{F}$ as before, we define the projection operator
$$\mathcal{P}_{\mathcal{F}} f(x) := f(x)\,\chi_{\mathbb{R}^n \setminus (\cup_k Q_k)}(x) + \sum_k \Big(\fint_{Q_k} f(y)\,dy\Big)\,\chi_{Q_k}(x).$$
One has that $\mathcal{P}_{\mathcal{F}} \circ \mathcal{P}_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}$, $\mathcal{P}_{\mathcal{F}}$ is selfadjoint and $\|\mathcal{P}_{\mathcal{F}} f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$ for every $1 \leq p \leq \infty$. Observe that if ω is a non-negative Borel measure and $E \subset Q$, then we may naturally define the measure $\mathcal{P}_{\mathcal{F}}\,\omega$ as follows:
$$\mathcal{P}_{\mathcal{F}}\,\omega(E) := \int \mathcal{P}_{\mathcal{F}}\,(\chi_E)\,d\omega = \omega(E \setminus \cup_k Q_k) + \sum_k \frac{|E \cap Q_k|}{|Q_k|}\,\omega(Q_k).$$

[††]Indeed, the abuse is very slight, since one may cover an arbitrary cube Q by a purely dimensional number of dyadic cubes of comparable size, to show that (2.2) is controlled by the analogous supremum taken only over dyadic cubes.

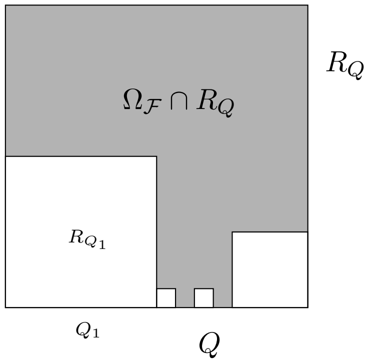

FIGURE 1. Discrete sawtooth $\Omega_{\mathcal{F}}$

In particular, $\mathcal{P}_{\mathcal{F}}\,\omega(Q) = \omega(Q)$. Notice that $\mathcal{P}_{\mathcal{F}}\omega$ is defined in such a way that it coincides with ω in $\mathbb{R}^n \setminus (\cup_k Q_k)$ and in each Q_k we replace ω by $\omega(Q_k)/|Q_k|\,dx$.

- Given Q and $\mathcal{F}$ as before, we introduce a new family $\mathcal{F}'$ consisting of all the dyadic "children" of the cubes in $\mathcal{F}$. Notice that $\mathcal{F}'$ is a family of pairwise disjoint cubes in $\mathcal{D}(Q)$, therefore we define $\mathcal{P}'_{\mathcal{F}} := \mathcal{P}_{\mathcal{F}'}$, which is the projection operator associated with the family $\mathcal{F}'$, and it satisfies the previous properties. We observe that if ω is a non-negative Borel measure and $E \subset Q$, then $\mathcal{P}'_{\mathcal{F}}\omega(E) \le 2^n\,\mathcal{P}_{\mathcal{F}}\omega(E)$. The converse inequality does not hold in general, however if one assumes that ω is dyadically doubling in Q then $\mathcal{P}'_{\mathcal{F}}\omega(E) \approx \mathcal{P}_{\mathcal{F}}\omega(E)$; thus it seems more natural to use $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$.

2.2. A_∞ **estimates via extrapolation of Carleson measures.**

Theorem 2.1. *Let Q_0 be either $\mathbb{R}^n$ or a fixed cube. Given $M_0 > 0$, let $\mu \in \mathcal{C}(Q_0)$ with*

$$|\!|\!|\mu|\!|\!|_{\mathcal{C}(Q_0)} \le M_0$$

and let ω be a non-negative, finite Borel measure in Q_0, for which $\omega(Q) > 0$ for every $Q \in \mathcal{D}(Q_0)$. Suppose that there exists $\delta > 0$ such that for every $Q \in \mathcal{D}(Q_0)$ and every family of pairwise disjoint dyadic subcubes $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ verifying

$$|\!|\!|\mu_{\mathcal{F}}|\!|\!|_{\mathcal{C}(Q)} := \sup_{Q' \in \mathcal{D}(Q)} \frac{\mu(R_{Q'} \cap \Omega_{\mathcal{F}})}{|Q'|} \le \delta\,, \tag{2.3}$$

we have that $\mathcal{P}'_{\mathcal{F}}\,\omega$ satisfies the following property:

$$\forall\,\varepsilon \in (0,1),\ \exists\, C_\varepsilon > 1 \text{ such that } \Big(E \subset Q,\ \ \frac{|E|}{|Q|} \ge \varepsilon \quad \Longrightarrow \quad \frac{\mathcal{P}'_{\mathcal{F}}\,\omega(E)}{\mathcal{P}'_{\mathcal{F}}\,\omega(Q)} \ge \frac{1}{C_\varepsilon}\Big). \tag{2.4}$$

Then, there exist $\eta_0 \in (0,1)$ and $C_0 < \infty$ such that, for every $Q \in \mathcal{D}(Q_0)$,

$$E \subset Q, \quad \frac{|E|}{|Q|} \ge 1 - \eta_0 \quad \Longrightarrow \quad \frac{\omega(E)}{\omega(Q)} \ge \frac{1}{C_0}. \tag{2.5}$$

Furthermore, if ω is dyadically doubling in Q_0 then $\omega \in A_\infty^{\rm dyadic}(Q_0)$.

Remark 2.2. The key hypothesis of the theorem, and the main point that must be verified in applications, is that (2.3) implies (2.4), for sufficiently small δ.

Remark 2.3. We notice that if ω is dyadically doubling in Q_0, then $\mathcal{P}_{\mathcal{F}}\omega \approx \mathcal{P}'_{\mathcal{F}}\omega$ and therefore it suffices to work with the "simpler" projection operator $\mathcal{P}_{\mathcal{F}}$. In that case, we note that the implication (2.3) $\Longrightarrow$ (2.4) is equivalent to the apparently stronger statement that (2.3) $\Longrightarrow$ $\mathcal{P}_{\mathcal{F}}\omega \in A_\infty^{\rm dyadic}(Q)$. Indeed, for every $Q' \in \mathcal{D}(Q)$, we have that

$\|\mu_{\mathcal{F}}\|_{\mathcal{C}(Q')} \leq \|\mu_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \delta$, whence the implication (2.3) $\Longrightarrow$ (2.4) also holds for all such Q' in place of Q. In turn, the fact that (2.4) holds for all $Q' \in \mathcal{D}(Q)$ says precisely that $\mathcal{P}_{\mathcal{F}}\omega \in A_\infty^{\text{dyadic}}(Q)$.

Remark 2.4. One can give an analog of Theorem 2.1 adapted to tents in place of boxes, that is, in (2.3) one can replace $R_{Q'} \cap \Omega_{\mathcal{F}}$ by $T_{Q'} \cap \widetilde{\Omega}_{\mathcal{F}}$ where $T_{Q'}$ is the Carleson tent associated to Q' and $\widetilde{\Omega}_{\mathcal{F}}$ is the domain above the (regular) sawtooth region which is formed by the union of the cones with a fixed aperture and vertices in $\mathbb{R}^{n+1}_+ \setminus \cup_k Q_k$. The proof is almost identical, we only need to apply the original [AHLT, Lemma 3.4] in place of our alternative version contained in Lemma 2.7.

Remark 2.5. The extrapolation theorem is written in such a way that it contains both a global and a local version. We also note the following observations:

- When $Q_0 = \mathbb{R}^n$, if ω is concentrically doubling, then the conclusion of the theorem improves immediately to $\omega \in A_\infty$.
- For the local case, if ω is concentrically doubling, then the conclusion $\omega \in A_\infty^{\text{dyadic}}(Q_0)$ also yields that $\omega \in A_\infty(\frac{1}{2}\, Q_0)$.

Remark 2.6. We notice that in the hypotheses of Theorem 2.1 the attention is restricted to $Q \in \mathcal{D}(Q_0)$ and thus the conclusion (2.5) holds for all $Q \in \mathcal{D}(Q_0)$, which under dyadic doubling implies $\omega \in A_\infty^{\text{dyadic}}(Q)$. If in our hypotheses we consider all cubes $Q \subset Q_0$ then (2.5) holds for all $Q \subset Q_0$. This implies both ω doubling and $\omega \in A_\infty(Q_0)$ (see Section 3.1). For the proof it suffices to change the induction hypotheses (cf. "$H(a)$" below) and consider all cubes $Q \subset Q_0$.

2.3. **Proof of Theorem 2.1.** As mentioned in the introduction, the proof follows the strategy introduced in [LM], and developed further in [HL], [AHLT] and [AHMTT]. The proof uses an induction argument with continuous parameter. The induction hypothesis is the following: given $a \geq 0$,

$\boxed{H(a)}$ There exist $\eta_a \in (0,1)$ and $C_a < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$ satisfying $\mu(R_Q) \leq a\,|Q|$, it follows that

$$E \subset Q, \quad \frac{|E|}{|Q|} \geq 1 - \eta_a \quad \Longrightarrow \quad \frac{\omega(E)}{\omega(Q)} \geq \frac{1}{C_a}.$$

The induction argument is split in two steps.

Step 1. Show that $H(0)$ holds.

Step 2. Show that there exists $b = b(n,\delta)$ such that for all $0 \leq a \leq M_0$, $H(a)$ implies $H(a+b)$.

Once these steps have been carried out, the proof follows easily: pick $k \geq 1$ such that $(k-1)\,b < M_0 \leq k\,b$ (note that k only depends on $b(n,\delta)$ and M_0). By **Step 1** and **Step 2**, it follows that $H(k\,b)$ holds. Observe that $\|\mu\|_{\mathcal{C}(Q_0)} \leq M_0 \leq k\,b$ implies $\mu(R_Q) \leq k\,b\,|Q|$ for all $Q \subset Q_0$, and by $H(k\,b)$ we conclude (2.5). As observed before, if ω is dyadically doubling the obtained estimate implies $\omega \in A_\infty^{\text{dyadic}}(Q_0)$.

Step 1. $H(0)$ **holds.** If $\mu(R_Q) = 0$ then we take $\mathcal{F}$ to be empty, so that $R_Q \cap \Omega_{\mathcal{F}} = R_Q$, and $\mathcal{P}'_{\mathcal{F}}\,\omega = \omega$. Then (2.3) holds (since $0 \leq \delta$) and therefore we can use (2.4) with ω in place of $\mathcal{P}'_{\mathcal{F}}\,\omega$, which is the desired property.

Step 2. $H(a)$ **implies** $H(a+b)$**.** We will require the following Lemma, which was proved in [AHLT, Lemma 3.4] in the case of regular sawtooth regions (see also [AHMTT]). We recall that R_Q^{short} denotes the "short" Carleson box $Q\times(0,\ell(Q)/2)$.

Lemma 2.7. *Let μ be a non-negative measure on $\mathbb{R}^{n+1}_+$, and let $a\ge 0$, $b>0$. Fix a cube Q such that $\mu(R_Q)\le(a+b)\,|Q|$. Then there exists a family $\mathcal{F}=\{Q_k\}_k$ of non-overlapping dyadic subcubes of Q such that*

$$\|\!|\mu_{\mathcal{F}}|\!\|_{\mathcal{C}(Q)}:=\sup_{Q'\in\mathcal{D}(Q)}\frac{\mu(R_{Q'}\cap\Omega_{\mathcal{F}})}{|Q'|}\le 2^{n+2}\,b\,,\qquad\qquad |B|\le\frac{a+b}{a+2\,b}\,|Q|,\tag{2.6}$$

where B is the union of those Q_k verifying $\mu(R_{Q_k}^{\text{short}})>a\,|Q_k|$.

We postpone the proof until the end of this section. Taking Lemma 2.7 for granted momentarily, we proceed with the proof of Step 2 of the Theorem.

Fix $0\le a\le M_0$ and $Q\in\mathcal{D}(Q_0)$ such that $\mu(R_Q)\le(a+b)\,|Q|$, where we choose b so that $2^{n+2}\,b:=\delta$. We also fix $E\subset Q$ with $|E|\ge(1-\eta)\,|Q|$, where $0<\eta\le\eta_{a,b}$ and $\eta_{a,b}$ is to be chosen. We may now apply the previous lemma to construct the non-overlapping family of cubes $\mathcal{F}$ with the stated properties. Set

$$A=Q\setminus\bigcup_{Q_k\in\mathcal{F}}Q_k,\qquad G=\bigcup_{Q_k\in\mathcal{F}_{\text{good}}}Q_k,\qquad B=\bigcup_{Q_k\in\mathcal{F}\setminus\mathcal{F}_{\text{good}}}Q_k,$$

where $\mathcal{F}_{\text{good}}=\big\{Q_k\in\mathcal{F}:\mu(R_{Q_k}^{\text{short}})\le a\,|Q_k|\big\}$. Then $|B|/|Q|\le(a+b)/(a+2b)$, by (2.6).

We shall also require the following "pigeonhole" lemma, which says that "most" of the cubes Q_k have an ample overlap with E.

Lemma 2.8. *Given $0<\tilde\eta<1$, we set*

$$\mathcal{F}_1=\{Q_k\in\mathcal{F}_{\text{good}}:|E\cap Q_k|\ge(1-\tilde\eta)\,|Q_k|\},\qquad\qquad G_1=\bigcup_{Q_k\in\mathcal{F}_1}Q_k.$$

If $0<\eta\le\eta_1:=\tilde\eta\,\frac{1}{2}\big(1-\frac{M_0+b}{M_0+2\,b}\big)$, then $|A\cup G_1|\ge\eta_1\,|Q|$.

Proof. Take θ such that $|B|=\theta\,|Q|$, and $\theta_0=(M_0+b)/(M_0+2\,b)$. By (2.6) and since $a\le M_0$ we obtain that $\theta\le\theta_0$:

$$\theta\,|Q|=|B|\le\frac{a+b}{a+2\,b}\,|Q|\le\theta_0\,|Q|.$$

We set $B_1=\cup_{Q_k\in\mathcal{F}_{\text{good}}\setminus\mathcal{F}_1}Q_k$ and observe that $B_1\subset G\subset Q\setminus B$. Hence,

$$\begin{aligned}|E\cap B_1|&=\sum_{Q_k\in\mathcal{F}_{\text{good}}\setminus\mathcal{F}_1}|E\cap Q_k|<(1-\tilde\eta)\sum_{Q_k\in\mathcal{F}_{\text{good}}\setminus\mathcal{F}_1}|Q_k|\\&=(1-\tilde\eta)\,|B_1|\le(1-\tilde\eta)\,|Q\setminus B|=(1-\tilde\eta)\,(1-\theta)\,|Q|.\end{aligned}$$

Thus, using that $\theta\le\theta_0$, we have

$$\begin{aligned}(1-\eta)\,|Q|\le|E|&=|E\cap A|+|E\cap B|+|E\cap G_1|+|E\cap B_1|\\&\le|A|+|B|+|G_1|+(1-\tilde\eta)\,(1-\theta)\,|Q|\\&=|A|+|G_1|+\big[\theta+(1-\tilde\eta)\,(1-\theta)\big]\,|Q|\\&\le|A|+|G_1|+\big[1-\tilde\eta\,(1-\theta_0)\big]\,|Q|\end{aligned}$$

and therefore

$$|A\cup G_1|=|A|+|G_1|\ge\big[\tilde\eta\,(1-\theta_0)-\eta\big]\,|Q|\ge\frac{1}{2}\,\tilde\eta\,(1-\theta_0)\,|Q|=\eta_1\,|Q|,$$

where we have used that $\eta\le\tilde\eta\,(1-\theta_0)/2=\eta_1$. □

We now return to the proof of Step 2. To this end, we apply Lemma 2.8. Given $Q_k \in \mathcal{F}_1 \subset \mathcal{F}_{\text{good}}$ we have that $\mu(R^{\text{short}}_{Q_k}) \le a\,|Q_k|$. Moreover,

$$R^{\text{short}}_{Q_k} = \bigcup_{j=1}^{2^n} R_{Q_k^j}, \qquad Q_k^j \in \mathcal{D}(Q_k) \quad \text{with} \quad Q_k = \bigcup_{j=1}^{2^n} Q_k^j, \quad \ell(Q_k^j) = \ell(Q_k)/2;$$

that is, the union runs over the dyadic "children" of Q_k. Then by pigeon-holing, there exists at least one j_0 such that $Q_k^{j_0} =: Q'_k$ satisfies

$$\mu(R_{Q'_k}) \le a\,|Q'_k| \tag{2.7}$$

(there could be more than one j_0 with this property, but we just pick one). We define $\tilde{\mathcal{F}}_1$ to be the collection of those selected "children" Q'_k, with $Q_k \in \mathcal{F}_1$. Then, for each such Q'_k, using the definition of $\mathcal{F}_1$, and taking $0 < \tilde\eta = 2^{-n}\,\eta_a$ (where $0 < \eta_a < 1$ is provided by $H(a)$), we have

$$|Q'_k \setminus E| \le |Q_k \setminus E| \le \tilde\eta\,|Q_k| = \tilde\eta\,2^n |Q'_k| = \eta_a\,|Q'_k|,$$

which yields $|Q'_k \cap E| \ge (1-\eta_a)\,|Q'_k|$. With this estimate and (2.7) in hand, we can use the induction hypothesis $H(a)$ to deduce:

$$\omega(Q'_k \cap E) \ge \frac{1}{C_a}\,\omega(Q'_k), \qquad \forall\, Q'_k \in \tilde{\mathcal{F}}_1. \tag{2.8}$$

On the other hand, if we set $\tilde G_1 = \cup_{Q'_k \in \tilde{\mathcal{F}}_1} Q'_k$, then $|\tilde G_1| = 2^{-n}|G_1|$, by definition of G_1 and $\tilde G_1$. Thus, by Lemma 2.8, having now fixed $\tilde\eta$ above, we have that

$$|A \cup \tilde G_1| = |A| + |\tilde G_1| \ge 2^{-n}\,\eta_1 |Q| =: \eta_2\,|Q|,$$

if $\eta \le \eta_1$, from which it follows that

$$|E \cap (A \cup \tilde G_1)| \ge \frac{1}{2}\eta_2 |Q| =: \eta_{a,b}\,|Q|,$$

if $\eta \le \eta_2/2$, since $|Q \setminus E| \le \eta |Q|$.

We recall that the family $\mathcal{F}$ was constructed using Lemma 2.7 with $2^{n+2}\,b := \delta$. Consequently, by (2.6), we may deduce that (2.3) holds, so in turn, by hypothesis, we can apply (2.4) to the set $E \cap (A \cup \tilde G_1)$, obtaining

$$\frac{\mathcal{P}'_{\mathcal{F}}\,\omega(E \cap (A \cup \tilde G_1))}{\mathcal{P}'_{\mathcal{F}}\,\omega(Q)} \ge \frac{1}{C_{\eta_{a,b}}}.$$

As observed before, $\mathcal{P}'_{\mathcal{F}}\,\omega(Q) = \omega(Q)$. Thus, in order to establish the conclusion of $H(a+b)$, and consequently to complete the proof of Theorem 2.1, it remains only to show that

$$\mathcal{P}'_{\mathcal{F}}\,\omega(E \cap (A \cup \tilde G_1)) \le C\,\omega(E).$$

To this end, we use first the definition of $\mathcal{P}'_{\mathcal{F}}$, and then (2.8) to obtain

$$\begin{aligned}
\mathcal{P}'_{\mathcal{F}}\,\omega(E \cap (A \cup \tilde G_1)) &= \mathcal{P}'_{\mathcal{F}}\,\omega(E \cap A) \,+\, \mathcal{P}'_{\mathcal{F}}\,\omega(E \cap \tilde G_1) \\
&= \omega(E \cap A) \,+ \sum_{Q'_k \in \tilde{\mathcal{F}}_1} \frac{|Q'_k \cap E|}{|Q'_k|}\,\omega(Q'_k) \qquad (2.9)\\
&\le \omega(E) + C_a \sum_{Q'_k \in \tilde{\mathcal{F}}_1} \omega(Q'_k \cap E) \\
&\le C\,\omega(E).
\end{aligned}$$

This concludes the proof of Theorem 2.1, modulo Lemma 2.7.

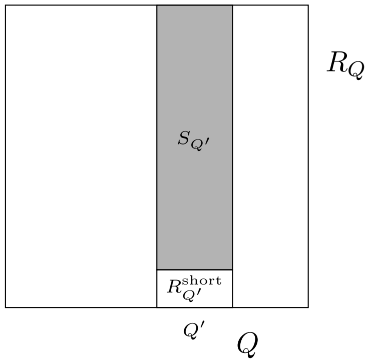

Figure 2. "Stovepipe" $S_{Q'}$

Remark 2.9. As mentioned above, if ω is dyadically doubling one can equivalently work with $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$. Indeed, the proof just presented can be easily adapted to that projection operator: to estimate $\mathcal{P}_{\mathcal{F}} w(E \cap \tilde{G}_1)$, in place of the second term in (2.9) we obtain $\sum_{Q'_k \in \tilde{\mathcal{F}}_1} |Q'_k \cap E| \, |Q_k|^{-1} \, \omega(Q_k)$, and by the doubling condition this quantity is controlled by $C_\omega \sum_{Q'_k \in \tilde{\mathcal{F}}_1} \omega(Q'_k)$.

Proof of Lemma 2.7. The proof is a "Corona" type stopping time argument, following [AHLT, Lemma 3.4] and [AHMTT], although the essential idea appears already in [LM] and [HL].

There are two cases. We recall that $W_Q := Q \times [\ell(Q)/2, \ell(Q))$, and that $R_Q^{\text{short}} := R_Q \setminus W_Q$.

Case 1: $\mu(W_Q) > b|Q|$. In this case $\mu(R_Q^{\text{short}}) \leq a|Q|$, and we may set $\mathcal{F} := \{Q\}$, so that $B = \emptyset = \Omega_{\mathcal{F}} \cap R_Q$ and the desired conclusions follow trivially.

Case 2: $\mu(W_Q) \leq b|Q|$. In this case we perform a dyadic stopping time decomposition, to extract a (possibly empty) family $\mathcal{F} := \{Q_k\}$ of non-overlapping dyadic subcubes of Q which are maximal with respect to the property that

$$\mu(S_{Q_k}) > 2b|Q_k|, \tag{2.10}$$

where for $Q' \in \mathcal{D}(Q)$, $S_{Q'} := Q' \times [\ell(Q')/2, \ell(Q))$ denotes the "stovepipe" above Q' (see Figure 2). We note that $\cup_k \Big(R_{Q_k}^{\text{short}} \cup S_{Q_k}\Big) = \cup_k \big(Q_k \times (0, \ell(Q))\big)$, and that by the maximality of the cubes these unions are comprised of disjoint sets.

We define B to be the union of those $Q_k \in \mathcal{F}$ such that $\mu(R_{Q_k}^{\text{short}}) > a\,|Q_k|$, and we may now readily establish the second estimate in (2.6). Indeed, using (2.10) and the definition of B we have

$$(a + 2\,b)\,|B| \,\leq\, \sum_k \Big(\mu(R_{Q_k}^{\text{short}}) + \mu\big(S_{Q_k}\big)\Big) \,\leq\, \mu(R_Q) \,\leq\, (a+b)\,|Q|.$$

Next, we turn to the first estimate in (2.6). We recall that $\Omega_{\mathcal{F}} := \mathbb{R}^{n+1}_+ \setminus (\cup_k R_{Q_k})$. Fix $Q' \in \mathcal{D}(Q)$. If $Q' \subset Q_k$ for some k then trivially $\mu(R_{Q'} \setminus (\cup_k R_{Q_k})) = 0$. We may therefore suppose that Q' is not contained in any $Q_k \in \mathcal{F}$. We write $A := Q \setminus (\cup_k Q_k)$ and observe that

$$R_{Q'} \setminus (\cup_k R_{Q_k}) = \Big((Q' \cap A) \times \big(0, \ell(Q')\big) \Big) \bigcup \left(\bigcup_{Q_k \subsetneq Q'} \Big(Q_k \times \big[\ell(Q_k), \ell(Q')\big) \Big) \right). \tag{2.11}$$

By the stopping time construction, for every $Q'' \in \mathcal{D}(Q)$ with $Q'' \cap A \neq \emptyset$ we have

$$\mu\big(S_{Q''}\big) \leq 2b\,|Q''|. \tag{2.12}$$

We claim that

$$\sup_{N \in \mathbb{N}} \mu\left((Q' \cap A) \times \big(2^{-N-1}\ell(Q'), \ell(Q')\big) \right) \leq 2b\,|Q'|,$$

and given this claim, by monotone convergence we obtain

$$\mu\left((Q' \cap A) \times \big(0, \ell(Q')\big) \right) \leq 2b\,|Q'|, \tag{2.13}$$

which is the desired bound for the first piece on the right side of (2.11).

We establish the claim as follows. For each $N \in \mathbb{N}$, let $\mathcal{D}_N(Q') \subset \mathcal{D}(Q')$ denote those dyadic subcubes of Q' with sidelength $2^{-N}\ell(Q')$, and let $\mathcal{D}_N(Q', A) \subset \mathcal{D}_N(Q')$ denote those cubes in $\mathcal{D}_N(Q')$ that meet A. Then

$$(Q' \cap A) \times \big(2^{-N-1}\ell(Q'), \ell(Q')\big) \subset \bigcup_{Q'' \in \mathcal{D}_N(Q', A)} S_{Q''},$$

so that

$$\mu\left((Q' \cap A) \times \big(2^{-N-1}\ell(Q'), \ell(Q')\big) \right) \leq \sum_{Q'' \in \mathcal{D}_N(Q', A)} \mu(S_{Q''}) \leq 2b \sum_{Q'' \in \mathcal{D}_N(Q', A)} |Q''| \leq 2b|Q'|,$$

where in the next-to-last inequality we have used (2.12). This proves the claim, and consequently (2.13) also.

Turning to the remaining piece on the right side of (2.11), we note that

$$Q_k \times \big[\ell(Q_k), \ell(Q')\big) \,\subset\, Q_k^* \times \big[\ell(Q_k^*)/2, \ell(Q')\big) \,\subset\, S_{Q_k^*},$$

where Q_k^* denotes the dyadic "parent" of Q_k. Therefore, by the maximality of Q_k, we have

$$\begin{aligned} \sum_{Q_k \subsetneq Q'} \mu\Big(Q_k \times \big[\ell(Q_k), \ell(Q')\big) \Big) &\leq \sum_{Q_k \subsetneq Q'} \mu(S_{Q_k^*}) \leq 2\,b \sum_{Q_k \subsetneq Q'} |Q_k^*| \\ &= 2^{n+1}\,b \sum_{Q_k \subsetneq Q'} |Q_k| \,\leq\, 2^{n+1}\,b|Q'|. \end{aligned}$$

□

3. Application to second order elliptic boundary value problems

3.1. **Additional Notation.**

- Given $X \in \mathbb{R}^{n+1}_+$ we write $X = (x, \varrho(X))$, that is, $\varrho(X) = \operatorname{dist}(X, \partial\mathbb{R}^{n+1}_+)$.
- For any $X, Y \in \mathbb{R}^{n+1}_+$, we write $|X - Y|_\infty = \max\{|x - y|_\infty, |\varrho(X) - \varrho(Y)|\}$, notice that this is the ℓ^∞-distance in $\mathbb{R}^{n+1}_+$. In this way, for any $X \in \mathbb{R}^{n+1}_+$ and $0 < r \leq 2\,\varrho(X)$, we write $R(X, r) = \{Y \in \mathbb{R}^{n+1}_+ : |Y - X|_\infty < r/2\}$ which is the cube in $\mathbb{R}^{n+1}_+$ with center X and sidelength r (that is, radius $r/2$).
- If R is a cube in $\mathbb{R}^{n+1}_+$, we denote its center by X_R and its sidelength by $\ell(R)$ such that $R = R(X_R, \ell(R))$. Notice that $R \subset \mathbb{R}^{n+1}_+$ yields $\ell(R) \leq 2\,\varrho(X_R)$. Given τ we denote by $\tau\,R$ the τ-dilation of R, that is, the cube with center X_R and with sidelength $\tau\,\ell(R)$.
- Given a cube $Q \subset \mathbb{R}^n$ we set $X_Q = (x_Q, 4\,\ell(Q))$ and $A_Q = (x_Q, \ell(Q))$.

- A weight w is a non-negative locally integrable function. A weight induces a Borel measure as follows: for any measurable set E we write $w(E) := \int_E w(x)\,dx$.
- Given a weight w and $1 < p < \infty$ we say that $w \in RH_p$ if there exists a constant C_p such that for every Q
$$\Big(\fint_Q w(x)^p\,dx\Big)^{\frac{1}{p}} \le C_p \fint_Q w(x)\,dx.$$
Given a cube Q_0, if the previous condition holds for any cube $Q \subset Q_0$ we write $w \in RH_p(Q_0)$.
- Let A_∞ be the set of Muckenhoupt weights in $\mathbb{R}^n$. That is, given ω a non-negative Borel measure on $\mathbb{R}^n$ we say that $\omega \in A_\infty$ if there exist $0 < \alpha, \beta < 1$ such that for every cube Q and for every measurable set $E \subset Q$ we have
$$\frac{|E|}{|Q|} < \alpha \quad \Longrightarrow \quad \frac{\omega(E)}{\omega(Q)} < \beta.$$
It is easy to see that this yields that ω is doubling —one estimates $\omega(\lambda\,Q \setminus Q)/\omega(\lambda\,Q)$ for λ sufficiently close to 1 and then iterates. This condition implies that ω is absolutely continuous with respect to the Lebesgue measure (we use the standard notation $\omega \ll dx$) and that its Radon-Nikodym derivative $k = d\omega/dx$ (which is a weight) satisfies $k \in RH_p$, see [GR, Chapter 4] for details. Indeed one can alternatively define A_∞ as the class of non-negative Borel measures absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivatives in $\cup_q RH_q$. Also, as mentioned above, A_∞ can be defined in terms of the estimates (2.1) with ν being the Lebesgue measure.
- Given $Q_0 \subset \mathbb{R}^n$, we have that $R_{Q_0} = \cup_{Q\in\mathcal{D}(Q_0)^*} U_Q$ where $\mathcal{D}(Q_0)^* = \mathcal{D}(Q_0) \setminus \{Q_0\}$, and for every cube Q we write $U_Q = Q \times [\ell(Q), 2\,\ell(Q))$. Notice that this is a Whitney decomposition of R_{Q_0} with respect to the distance to the boundary $\mathbb{R}^n$. Observe that the sets U_Q are pairwise disjoint. See Figure 3. To avoid confusion, we point out that the Whitney boxes U_Q used here differ slightly from the boxes W_Q used in the previous section; this is merely a matter of technical convenience.
- Given $Q_0 \subset \mathbb{R}^n$, we decompose R_{Q_0} into Whitney boxes $R_{Q_0} = \cup_{Q\in\mathcal{D}(Q_0)^*} U_Q$. For every $f \in L^1(Q_0)$ we define the dyadic averaging operator
$$P_s^{Q_0} f(y) := \sum_{Q\in\mathcal{D}(Q_0)^*} \Big(\fint_Q f(z)\,dz\Big)\,\chi_{U_Q}(y,s).$$
Note that in the sum there is at most one non-zero term since the sets U_Q are a disjoint partition of R_{Q_0}. We can alternatively define $P_s^{Q_0} f(y) := \fint_Q f(z)\,dz$ where $Q = Q(y,s)$ is the unique dyadic cube in $\mathcal{D}^*(Q_0)$ such that $y \in Q$ and $s/2 < \ell(Q) \le s$. This definition extends trivially to non-negative Borel measures.

3.2. **Introduction.** We work with real symmetric second order elliptic operators: $Lf(X) = -\operatorname{div}(A(X)\,\nabla f(X))$, $X \in \mathbb{R}^{n+1}_+$, with $A(X) = (a_{i,j}(X))_{1\le i,j\le n+1}$ being a real, symmetric $(n+1)\times(n+1)$ matrix such that $a_{i,j} \in L^\infty(\mathbb{R}^{n+1}_+)$ for $1 \le i,j \le n+1$, and A is uniformly elliptic, that is, there exists $0 < \lambda \le 1$ such that
$$\lambda\,|\xi|^2 \le A(X)\,\xi\cdot\xi \le \lambda^{-1}\,|\xi|^2,$$
for all $\xi \in \mathbb{R}^{n+1}$ and almost every $X \in \mathbb{R}^{n+1}_+$.

Some of the material below is taken from [Ken, Chapter 1], the reader might find convenient to have this reference handy.

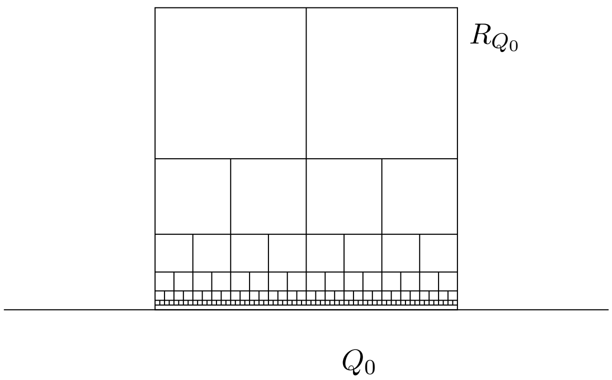

FIGURE 3. Whitney decomposition of R_{Q_0}

The solutions of the Dirichlet problem are represented by the harmonic measure. Namely, there exists a family of regular Borel probability measures $\{\omega_L^X\}_{X\in\mathbb{R}^{n+1}_+}$ in $\mathbb{R}^n$ such that for every $f \in C_0(\mathbb{R}^n)$, the function

$$u(X) = \int_{\mathbb{R}^n} f(y)\, d\omega_L^X(y)$$

is a classical solution of the Dirichlet problem

$$\begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ u\big|_{\mathbb{R}^n} = f \end{cases} \tag{3.1}$$

This family $\{\omega_L^X\}_{X\in\mathbb{R}^{n+1}_+}$ is called the L-harmonic measure. Sometimes, we will drop the subindex L. For a fixed $X_0 \in \mathbb{R}^{n+1}_+$ we let $\omega = \omega^{X_0}$ and abusing of the notation ω is called the harmonic measure.

If $\omega_L^X \ll dx$, we write the Poisson kernel as k_L^X, that is, $k_L^X = d\omega_L^X/dx$ is the Radon-Nikodym derivative of ω_L^X with respect to dx. Again for a fixed $X_0 \in \mathbb{R}^{n+1}_+$ we let $k = k^{X_0}$ and k is called the Poisson kernel (notice that for every $X \in \mathbb{R}^{n+1}_+$, ω^X and ω are mutually absolutely continuous).

We recall the fundamental relationship between solvability of the Dirichlet problem with L^p data, and higher integrability of the Poisson kernel, essentially as stated in [Ken, Theorem 1.7.3].

Theorem 3.1. *Given an operator L as above and $1 < p < \infty$, the following statements are equivalent:*

(a) *If $u \in C_0(\mathbb{R}^{n+1}_+)$ is a classical solution of the Dirichlet problem (3.1) with data $f \in C_0(\mathbb{R}^n)$ then*

$$\|u^*\|_{L^{p'}(\mathbb{R}^n)} \le C\, \|f\|_{L^{p'}(\mathbb{R}^n)}, \tag{3.2}$$

where $u^*(x) = \sup_{Y\in\Gamma_\alpha(x)} |u(Y)|$ with $\Gamma_\alpha(x) = \{Y \in \mathbb{R}^{n+1}_+ : |x-y|_\infty < \alpha\, \varrho(Y)\}$, $\alpha > 0$.

(*b*) *$\omega \in RH_p$; by this we mean that $\omega \ll dx$ and for each cube $Q \subset \mathbb{R}^n$, we have that the Poisson kernel satisfies $k^{X_Q} \in RH_p(Q)$, uniformly in Q.*[‡‡] *That is, there exists a uniform constant C_0 such that for all $Q \subset \mathbb{R}^n$,*

$$\Big(\fint_{Q'} k^{X_Q}(y)^p\,dy\Big)^{1/p} \le C_0 \fint_{Q'} k^{X_Q}(y)\,dy, \qquad \forall\, Q' \subset Q. \tag{3.3}$$

(*c*) *$\omega \ll dx$, and there is a uniform constant C_0 such that for every Q in $\mathbb{R}^n$, we have the scale invariant L^p estimate*

$$\int_Q k^{X_Q}(y)^p\,dy \le C_0 |Q|^{1-p}. \tag{3.4}$$

When (a) occurs we say that $(D)_{p'}$ is solvable for L or that L is solvable in $L^{p'}$. In such case, for every $f \in L^{p'}(\mathbb{R}^n)$ there exists a unique u such that $Lu = 0$ in $\mathbb{R}^{n+1}_+$, (3.2) holds and u converges non-tangentially to f a.e..

Given two operators L_0 and L as above with associated matrices A_0 and A, we define their disagreement as

$$a(X) := \sup_{|X-Y|_\infty < \varrho(X)/2} |\mathcal{E}(Y)|, \qquad \mathcal{E}(Y) = A(Y) - A_0(Y).$$

3.3. **Main application.** In this section, to illustrate the use of Theorem 2.1, we present an alternative proof of a well known result of [FKP].

Theorem 3.2 ([FKP]). *Let L_0 and L be two operators as above with a being their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that*

$$\sup_{Q \in \mathbb{R}^n} \frac{1}{|Q|} \int_{R_Q} \frac{a(X)^2}{\varrho(X)}\,dX < \infty. \tag{3.5}$$

Then, we have that $\omega_0 \in A_\infty$ implies $\omega \in A_\infty$. More precisely, if L_0 is solvable in some $L^{p'}$, $1 < p' < \infty$, there exists $1 < q' < \infty$ such that L is solvable in $L^{q'}$.

We prove this result by using the extrapolation of Carleson measures Theorem 2.1. We take $d\mu(X) = \frac{a(X)^2}{\varrho(X)}\,dX$, that is, $d\mu(x,t) = a(x,t)^2\,\frac{dt}{t}\,dx$ and (3.5) gives $\mu \in \mathcal{C}$. Therefore, to show that the harmonic measure $\omega \in A_\infty$, it suffices to fix Q and a family $\mathcal{F}$ such that (2.3) holds and prove that $\mathcal{P}_\mathcal{F}\,\omega$ satisfies the A_∞ condition in (2.4). We will introduce some intermediate operators that allow us to pass from L_0 to L. Since the smallness in (2.3) is guaranteed above the discrete sawtooth region, we first introduce L_1 such that the disagreement with L_0 lives in that region (this is done in the first step). Once we have the solvability of L_1 we will be changing this operator in subsequent steps and in the end we will end up with L.

Let us call the reader's attention to the fact that in any given step we work with L_i and L_{i+1} in such a way that L_i is the "known" and L_{i+1} is the "unknown" in the sense that we have some nice properties for L_i and we want to infer them to L_{i+1}. For any of these operators L_i we write ω_i for the harmonic measure and, when it exists, k_i for the Poisson kernel.

[‡‡]In [Ken], condition (b) is stated in slightly different form, involving a global reverse Hölder estimate for harmonic measure with one fixed pole; it is well known that the present version of (b), as well as (c), are also equivalent to condition (a).

3.4. **Auxiliary results.** We summarize some well known results for divergence form elliptic equations that we will use in the sequel. The reader is referred to [Ken, Chapter 1] and the references therein for full details.

Theorem 3.3. *There exists a unique function* $G : \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$, $G \geq 0$ *such that*

(i) $G(\cdot, Y) \in W^2_1(\mathbb{R}^{n+1}_+ \setminus R(Y,r)) \cap \dot{W}^1_{1,0}(\mathbb{R}^{n+1}_+)$ *for each* $Y \in \mathbb{R}^{n+1}_+$ *and* $r > 0$.

(ii) $LG(\cdot, Y) = -\delta_Y$ *for each* $Y \in \mathbb{R}^{n+1}_+$

(iii) $G(X,Y) = G(Y,X)$ *for each* $X, Y \in \mathbb{R}^{n+1}_+$.

Remark 3.4. It is well known that the Green function enjoys several other properties, but we shall make explicit use only of those listed above.

Lemma 3.5 (Caccioppoli)**.** *Let* $Q \subset \mathbb{R}^n$ *and let* R *be a cube in* $\mathbb{R}^{n+1}_+$ *such that* $\overline{\tau R} \subset R_Q$ *with* $\tau > 1$. *If* $Lu = 0$ *in* R_Q, *then*

$$\fint_R |\nabla u(Y)|^2 \, dY \leq C_{\lambda,n,\tau} \, \ell(R)^{-2} \fint_{\tau R} u(Y)^2 \, dY. \tag{3.6}$$

Lemma 3.6 (Comparison Principle)**.** *Given* $Q \subset \mathbb{R}^n$, *let* u, v *be two non-negative functions such that* u, $v \in W^2_1(R_{2Q})$; u, $v \in C(\overline{R_{2Q}})$; $u|_{2Q} = v|_{2Q} = 0$; *and* $Lu = Lv = 0$ *in* R_{2Q}. *Then there is a* $C = C_{n,\lambda}$ *such that for every* $X \in R_Q$,

$$C^{-1} \frac{u(A_Q)}{v(A_Q)} \leq \frac{u(X)}{v(X)} \leq C \frac{u(A_Q)}{v(A_Q)}, \tag{3.7}$$

where $A_Q = (x_Q, \ell(Q))$, *and* x_Q *is the center of* Q.

Lemma 3.7 (Doubling)**.** *There exists* $C = C(\lambda, n)$ *such that for every cube* $Q \in \mathbb{R}^n$

$$\omega^X(2\,Q) \leq C\,\omega^X(Q).$$

Lemma 3.8 (Caffarelli-Fabes-Mortola-Salsa)**.** *There exists a constant* $C = C_{n,\lambda} < \infty$ *such that for every cube* Q, *we have*

$$\omega^X(Q) \geq 1/C, \qquad \forall X \in 4\,Q \times [\ell(Q), 5\,\ell(Q)]. \tag{3.8}$$

Moreover, given $X, Y \in \mathbb{R}^{n+1}_+$ *such that* $|X - Y|_\infty > 2\,\varrho(Y)$ *we have*

$$G(X,Y) \approx \frac{\omega^X(Q(y, \varrho(Y))}{\varrho(Y)^{n-1}}, \tag{3.9}$$

where the implicit constants depend only on dimension and ellipticity.

Lemma 3.9. *Given* $Q \subset \mathbb{R}^n$, *let* L_1 *and* L_2 *be elliptic operators such that* $L_1 \equiv L_2$ *in* R_Q. *If the corresponding harmonic measures* ω_1, ω_2 *are absolutely continuous with respect to the Lebesgue measure (we write* k_1 *and* k_2 *for the Poisson kernels), then*

$$k_1^{X_Q}(y) \approx k_2^{X_Q}(y), \qquad \text{for a.e. } y \in \tfrac{1}{2}\,Q.$$

Proof. The result is standard, and may be proved by a routine application of the comparison principle (Lemma 3.6) to the respective Green functions. We leave the details to the interested reader. □

Lemma 3.10. *Let* $Q \subset Q_0$ *and set* $X_0 = (x_{Q_0}, 4\,\ell(Q_0))$, $X_Q = (x_Q, 4\,\ell(Q))$ *where* x_{Q_0} *and* x_Q *are respectively the centers of* Q_0 *and* Q. *If* $\omega \ll dx$ *then*

$$k^{X_Q}(y) \approx \frac{k^{X_0}(y)}{\omega^{X_0}(Q)}, \qquad \text{for a.e. } y \in Q. \tag{3.10}$$

Proof. By [Ken, Corollary 1.3.8], we have that for every cube $\tilde{Q} \subset Q$,

$$\omega^{X_Q}(\tilde{Q}) \approx \frac{\omega^{X_0}(\tilde{Q})}{\omega^{X_0}(Q)}.$$

The conclusion follows by Lebesgue's differentiation theorem, as $\tilde{Q} \downarrow y$. □

For an elliptic operator L, given u such that $Lu = 0$ in $\mathbb{R}^{n+1}_+$, we define the square function

$$S_\alpha u(x) = \Big(\iint_{\Gamma_\alpha(x)} |\nabla u(x,t)|^2 \, t^{1-n} \, dt \Big)^{\frac{1}{2}},$$

where

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\}$$

is the cone with vertex x and aperture α. We then have the following:

Theorem 3.11 (Dahlberg-Jerison-Kenig [DJK]*)**.** *Suppose that for some $p' \in (1,\infty)$, $(D)_{p'}$ is solvable for L. Then, if u is a solution of the Dirichlet problem with data $f \in L^{p'}(\mathbb{R}^n)$, we have, for all $\alpha > 0$,*

$$\|S_\alpha u\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)},$$

where the implicit constant depends on dimension, ellipticity, α, and on the constants in the L^p estimates for the Poisson kernel of L.

Lemma 3.12. *Let μ be a Carleson measure and Q_0 be a cube in $\mathbb{R}^n$. For every $1 < p < \infty$ we have*

$$\iint_{R_{Q_0}} P_s^{Q_0} f(y)^p \, d\mu(y,s) \lesssim \|\mu\|_{\mathcal{C}(Q_0)} \int_{Q_0} f(y)^p \, dy \tag{3.11}$$

where

$$\|\mu\|_{\mathcal{C}(Q_0)} := \sup_{Q \in \mathcal{D}(Q_0)} \frac{\mu(R_Q)}{|Q|}.$$

Proof. For every $\lambda > 0$, we set $E_\lambda = \{x \in Q_0 : M^d_{Q_0} f(x) > \lambda\}$, where $M^d_{Q_0}$ is the dyadic Hardy-Littlewood maximal function with respect to Q_0. If $\lambda \le \lambda_0 := ⨍_{Q_0} f(z)\, dz$ we have

$$\mu\{(y,s) \in R_{Q_0} : P_s^{Q_0} f(y) > \lambda\} \le \mu(R_{Q_0}) \le \|\mu\|_{\mathcal{C}(Q_0)} \, |Q_0| \le \|\mu\|_{\mathcal{C}(Q_0)} \frac{1}{\lambda} \int_{Q_0} f(z)\, dz.$$

On the other hand, if $\lambda > \lambda_0$ we can perform the Calderón-Zygmund decomposition to construct a family of maximal (thus pairwise disjoint) cubes $\{Q_j\}_j \subset \mathcal{D}(Q_0)$ such that $E_\lambda = \cup_j Q_j$. Notice that $Q_j \subsetneq Q_0$, otherwise Q_0 is maximal and then $\lambda_0 = ⨍_{Q_0} f(z)\, dz > \lambda$, which is a contradiction.

Let $(y,s) \in R_{Q_0}$ satisfy $P_s^{Q_0} f(y) = ⨍_Q f(z)\, dz > \lambda$, where $Q := Q(y,s) \in \mathcal{D}(Q_0)^*$ is the unique cube with $(y,s) \in U_Q$. By maximality there exists j such that $Q \subset Q_j$. Then $y \in Q \subset Q_j$ and also $s < 2\,\ell(Q) \le 2\,\ell(Q_j)$, therefore $(y,s) \in R_{Q_j^*}$ where Q_j^* is the dyadic "parent" of Q_j. As observed $Q_j \subsetneq Q_0$ and then $Q_j^* \in \mathcal{D}(Q_0)$. Consequently,

$$\begin{aligned}\mu\{(y,s) \in R_{Q_0} : P_s^{Q_0} f(y) > \lambda\} &\le \mu\big(\cup_j R_{Q_j^*}\big) \le \sum_j \mu(R_{Q_j^*}) \le \|\mu\|_{\mathcal{C}(Q_0)} \sum_j |Q_j^*| \\ &= 2^n \, \|\mu\|_{\mathcal{C}(Q_0)} \, |E_\lambda|.\end{aligned}$$

*In fact, the theorem in [DJK] is somewhat more general than the result stated here, but we do not require the full version.

Combining the two estimates obtained above, we conclude

$$\begin{aligned}\iint_{R_{Q_0}} P_s^{Q_0} f(y)^p \, d\mu(y,s) &= \int_0^\infty p\,\lambda^p\,\mu\{(y,s)\in R_{Q_0} : P_s^{Q_0} f(y) > \lambda\}\,\frac{d\lambda}{\lambda} \\ &\lesssim \|\mu\|_{\mathcal{C}(Q_0)} \int_{Q_0} f(z)\,dz \int_0^{\lambda_0} \lambda^{p-1}\,\frac{d\lambda}{\lambda} + \|\mu\|_{\mathcal{C}(Q_0)} \int_{\lambda_0}^\infty \lambda^p\,|E_\lambda|\,\frac{d\lambda}{\lambda} \\ &\lesssim \|\mu\|_{\mathcal{C}(Q_0)} \left(\int_{Q_0} f(z)\,dz\,\lambda_0^{p-1} + \|M_{Q_0}^d f\|_{L^p(Q_0)}^p \right) \\ &\lesssim \|\mu\|_{\mathcal{C}(Q_0)} \|M_{Q_0}^d f\|_{L^p(Q_0)}^p \lesssim \|\mu\|_{\mathcal{C}(Q_0)} \int_{Q_0} f(z)^p\,dz.\end{aligned}$$

□

4. Proof of Theorem 3.2

We want to apply Theorem 2.1 with the Carleson measure $d\mu(X) = \frac{a(X)^2}{\varrho(X)}\,dX$. Given $\delta > 0$ to be chosen, we fix Q_0 and a family of pairwise disjoint subcubes $\mathcal{F} = \{Q_k\}_k \in \mathcal{D}(Q_0)$ such that

$$\sup_{Q\in\mathcal{D}(Q_0)} \frac{\mu(R_Q \cap \Omega_{\mathcal{F}})}{|Q|} \le \delta. \tag{4.1}$$

Set $X_0 = (x_0, 4\,\ell(Q_0))$ with x_0 being the center of Q_0.

As L_0 is solvable in some space $L^{p'}$ then $\omega_{L_0}^{X_0} = \omega_0^{X_0} \in RH_p(Q_0)$ uniformly in Q_0. This means that $\omega_0^{X_0} \ll dx$ and $k_0^{X_0} \in RH_p(Q_0)$ uniformly in Q_0. Without loss of generality we can assume that $1 < p < 2$ (as $RH_{p_1} \subset RH_{p_2}$ for $p_2 < p_1$). As $\omega_L^{X_0}$ is doubling, it suffices to work with $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$, thus our goal is to show that $\mathcal{P}_{\mathcal{F}}\,\omega_L^{X_0}$ satisfies (2.4), with uniform constants. Notice that for a Borel set E, from the definition we have

$$\mathcal{P}_{\mathcal{F}}\,\omega_L^{X_0}(E) = \int_{\mathbb{R}^n} \mathcal{P}_{\mathcal{F}}(\chi_E)(x)\,d\omega_L^{X_0}(x) = u(X_0),$$

where u is a solution of the Dirichlet problem with data $\mathcal{P}_{\mathcal{F}}(\chi_E)$.

4.1. **Step 0.** We first make a reduction that allows us to use qualitative properties of the unknown harmonic measure.

We define $A_\gamma(x,t) = A(x,t)$ for $t > \gamma$ and $A_\gamma(x,t) = A_0(x,t)$ for $0 \le t \le \gamma$. In the following steps we work with L_γ in place of L. We note that the ellipticity constants of A_γ are controlled by those of A and A_0, uniformly in γ. Also, $|A_0(X) - A_\gamma(X)| \le |A_0(X) - A(X)|$ and thus the Carleson condition is controlled independently of γ. Notice that $L_\gamma = L_0$ in the strip $\{(x,t) : 0 \le t < \gamma\}$ and then in every step, by the comparison principle, we can use that all the harmonic measures are in RH_p (that is, they are absolutely continuous with respect to dx and the Poisson kernels are in RH_p). Notice that the constants will depend on γ but in our arguments we will only use this qualitatively and not quantitatively. In particular in Step 1 we have *a priori* that $\omega_1^{X_0} \ll dx$ and that $k_1^{X_0} \in L^p(Q_0)$ (this depends on γ, but we only use this in a qualitative way). Therefore, we can carry out the whole argument and in the end we shall establish the reverse Hölder inequality (4.13) below for k_{L_γ} with q and C_0 independent of γ. One may then pass to the limit as follows: by [Ken, p. 41] for any smooth function φ we have $\langle \varphi, \omega_{L_\gamma}^{X_0}\rangle \longrightarrow \langle \varphi, \omega_L^{X_0}\rangle$ as $\gamma \to 0^+$. For any cube Q_0, and for every smooth function φ in $L^{q'}(Q_0)$ with $\|\varphi\|_{L^{q'}(Q_0)} = 1$ we have

$$|\langle \varphi, \omega_L^{X_0}\rangle| = \lim_{\gamma\to 0^+} |\langle \varphi, \omega_{L_\gamma}^{X_0}\rangle| \le \sup_{\gamma>0} \|k_{L_\gamma}^{X_0}\|_{L^q(Q_0)}\,\|\varphi\|_{L^{q'}(Q_0)} \le C_0 \sup_\gamma |Q_0|^{-1/q'}\,\omega_{L_\gamma}^{X_0}(Q_0)$$

$$\leq C_0\, |Q_0|^{-1/q'}.$$

Thus, $\Lambda_{\omega_L^{X_0}}(\varphi) := \langle \varphi, \omega_L^{X_0} \rangle$ is a functional in $(L^{q'}(Q))^*$, so $\omega_L^{X_0} \ll dx$ in Q_0 and $k_L^{X_0}$ verifies (3.4) with p replaced by q. This in turn implies as desired that L is solvable in $L^{q'}$ by Theorem 3.1.

Taking this reduction into account we can assume without loss of generality that all the harmonic measures below are absolutely continuous with respect to the Lebesgue measure and also that the Poisson kernels satisfy (qualitatively) RH_p.

4.2. **Step 1.** We introduce the operator L_1 defined as $L_1 = L$ in

$$\Omega_0 := R_{Q_0} \cap \Omega_{\mathcal{F}} = R_{Q_0} \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k})$$

and $L_1 = L_0$ otherwise (see Figure 4). More precisely, L_1 is the divergence form elliptic operator with associated matrix $A_1 = A$ in Ω_0 and $A_1 = A_0$ otherwise. We set $\mathcal{E}_1(Y) = A_1(Y) - A_0(Y) = \mathcal{E}(Y)\ \chi_{\Omega_0}(Y)$. In what follows we write $\omega_0 = \omega_{L_0}$, $\omega_1 = \omega_{L_1}$, $G_1 = G_{L_1}$.

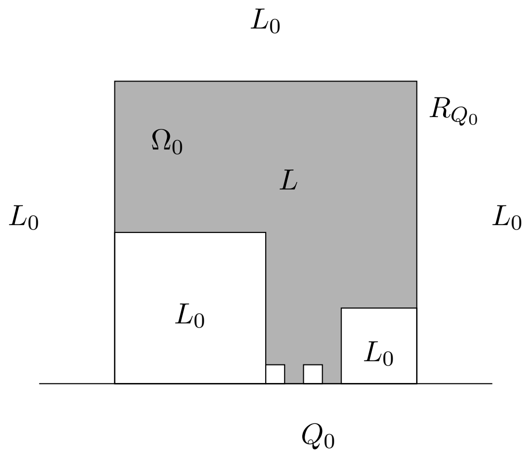

FIGURE 4. Definition of L_1

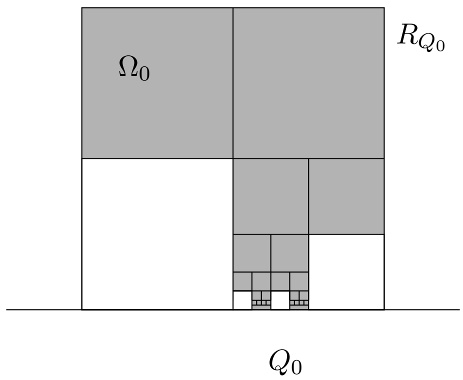

FIGURE 5. Whitney decomp. of Ω_0

We recall that $k_0^{X_0} \in RH_p(Q_0)$, and in particular we have

$$\int_{Q_0} k_0^{X_0}(y)^p\, dy \leq C_0 |Q_0|^{1-p}. \tag{4.2}$$

Our immediate goal in Step 1 is to show that (4.2) remains true (with a different but uniform constant, independent of Q_0), when $k_0^{X_0}$ is replaced by $k_1^{X_0}$, the Poisson kernel for the operator L_1 defined above.

To this end, let $g \geq 0$ be a smooth function supported on Q_0, such that $\|g\|_{L^{p'}(Q_0)} = 1$, and let u_0 and u_1 be the corresponding solutions to the Dirichlet problems for L_0 and L_1 with boundary data g. Then, following [FKP], we have

$$\begin{aligned} F_1(X_0) := |u_1(X_0) - u_0(X_0)| &= \Big| \int_{\mathbb{R}^{n+1}_+} \nabla_Y G_1(X_0, Y)\, \mathcal{E}_1(Y)\, \nabla u_0(Y)\, dY \Big| \\ &\leq \int_{\Omega_0} |\nabla_Y G_1(X_0, Y)|\, |\mathcal{E}(Y)|\, |\nabla u_0(Y)|\, dY. \end{aligned}$$

We perform a Whitney decomposition of R_{Q_0} with respect to the distance to the boundary $\mathbb{R}^n$ such that $R_{Q_0} = \cup_{Q \in \mathcal{D}(Q_0)^*} U_Q$ (see Figure 3). Since $\Omega_0 = R_{Q_0} \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k})$ we have that $\Omega_0 = \cup_{Q\in\mathcal{F}_1} U_Q$ where $\mathcal{F}_1 = \mathcal{D}(Q_0)^* \setminus \big(\bigcup_{Q_k\in\mathcal{F}} \mathcal{D}(Q_k)^*\big)$, see Figure 5. Then,

$$\begin{aligned} F_1(X_0) &\le \sum_{Q\in\mathcal{F}_1} \int_{U_Q} |\nabla_Y G_1(X_0,Y)|\,|\mathcal{E}(Y)|\,|\nabla u_0(Y)|\,dY \\ &\le \sum_{Q\in\mathcal{F}_1} \sup_{U_Q} |\mathcal{E}| \Big(\int_{U_Q} |\nabla_Y G_1(X_0,Y)|^2\,dY\Big)^{\frac12} \Big(\int_{U_Q} |\nabla u_0(Y)|^2\,dY\Big)^{\frac12}. \end{aligned} \tag{4.3}$$

By definition of X_0, we have that $v(Y) = G_1(X_0,Y)$ is a non-negative solution of $L_1 v = 0$ in R_{2Q_0} (as $X_0 \notin R_{2Q_0}$) and $\overline{2\,U_Q} \subset R_{2Q_0}$. Hence, we can apply Caccioppoli's inequality (Lemma 3.5) to obtain

$$\int_{U_Q} |\nabla_Y G_1(X_0,Y)|^2\,dY \le C_{\lambda,n}\,\ell(Q)^{-2} \int_{2\,U_Q} G_1(X_0,Y)^2\,dY \lesssim \int_{2\,U_Q} \frac{G_1(X_0,Y)^2}{\varrho(Y)^2}\,dY,$$

since $\ell(U_Q) = \ell(Q) \approx \varrho(Y)$ for every $Y \in 2\,U_Q$. By (3.9), for every $Y \in 2\,U_Q$ we have

$$\frac{G_1(X_0,Y)}{\varrho(Y)} \approx \frac{\omega_1^{X_0}(Q)}{|Q|}. \tag{4.4}$$

Thus,

$$\begin{aligned} \int_{U_Q} |\nabla_Y G_1(X_0,Y)|^2\,dY &\lesssim \Big(\frac{\omega_1^{X_0}(Q)}{|Q|}\Big)^2 |2\,U_Q| \\ &\approx \Big(\frac{\omega_1^{X_0}(Q)}{|Q|}\Big)^{2-p} \int_{\frac14 U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p\,dy\,ds, \end{aligned}$$

where $P_s^{Q_0}$ is the dyadic averaging operator defined above.

Next we see that $\sup_{U_Q} |\mathcal{E}| \le a(Y)$ for every $Y \in \frac14 U_Q$, by a routine geometric argument that we leave to the reader. Hence, we obtain

$$\begin{aligned} &\sup_{U_Q} |\mathcal{E}| \Big(\int_{U_Q} |\nabla_Y G_1(X_0,Y)|^2\,dY\Big)^{\frac12} \\ &\qquad \lesssim \Big(\frac{\omega_1^{X_0}(Q)}{|Q|}\Big)^{\frac{2-p}{2}} \Big(\int_{\frac14 U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p a(y,s)^2\,dy\,ds\Big)^{\frac12} \\ &\qquad \approx \ell(Q)^{\frac12} \Big(\frac{\omega_1^{X_0}(Q)}{|Q|}\Big)^{\frac{2-p}{2}} \Big(\int_{\frac14 U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p \frac{a(y,s)^2}{s}\,dy\,ds\Big)^{\frac12}. \end{aligned}$$

We plug this estimate into (4.3):

$$\begin{aligned} F_1(X_0) &\lesssim \sum_{Q\in\mathcal{F}_1} \ell(Q)^{\frac12} \Big(\frac{\omega_1^{X_0}(Q)}{|Q|}\Big)^{\frac{2-p}{2}} \Big(\int_{\frac14 U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p \frac{a(y,s)^2}{s}\,dy\,ds\Big)^{\frac12} \\ &\qquad\qquad \times \Big(\int_{U_Q} |\nabla u_0(Y)|^2\,dY\Big)^{\frac12} \\ &\le \Big(\sum_{Q\in\mathcal{F}_1} \int_{\frac14 U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p \frac{a(y,s)^2}{s}\,dy\,ds\Big)^{\frac12} \\ &\qquad\qquad \times \Big(\sum_{Q\in\mathcal{F}_1} \left(\frac{\omega_1^{X_0}(Q)}{|Q|}\right)^{2-p} \int_{U_Q} |\nabla u_0(y,s)|^2\,s\,dy\,ds\Big)^{\frac12} \\ &=: I \cdot II. \end{aligned}$$

We estimate each factor in turn. For I, we define

$$d\tilde{\mu}(y,s) = \chi_{\Omega_0}(y,s)\, d\mu(y,s) = \chi_{\Omega_0}(y,s)\, a(y,s)^2 \frac{dy\, ds}{s}$$

so by the dyadic Carleson Embedding Lemma 3.12, we have

$$I^2 \leq \int_{R_{Q_0}} \Big(P_s^{Q_0} k_1^{X_0}(y)\Big)^p\, d\tilde{\mu}(y,s) \lesssim \|\tilde{\mu}\|_{\mathcal{C}(Q_0)} \int_{Q_0} k_1^{X_0}(y)^p\, dy,$$

and therefore by (4.1) we obtain

$$I \lesssim \delta^{\frac{1}{2}}\, \|k_1^{X_0}\|_{L^p(Q_0)}^{\frac{p}{2}}.$$

We now estimate II:

$$\begin{aligned}
II^2 &= \sum_{Q\in\mathcal{F}_1} \frac{1}{|Q|}\int_Q \left[\left(\frac{\omega_1^{X_0}(Q)}{|Q|}\right)^{2-p} \int_{U_Q} |\nabla u_0(y,s)|^2\, s\, dy\, ds\right] dx \\
&\lesssim \sum_{Q\in\mathcal{F}_1} \int_Q \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \int_{U_Q} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx \\
&= \sum_{Q\in\mathcal{F}_1} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \int_Q |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx \\
&\lesssim \sum_{Q\in\mathcal{F}_1} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \int_{|x-y|<\alpha s} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx, \\
&= \sum_{Q\in\mathcal{F}_1} \iint_{U_Q} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \int_{|x-y|<\alpha s} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx
\end{aligned}$$

for a sufficiently large choice of α. In turn, the last expression is bounded by

$$\begin{aligned}
&\iint_{R_{Q_0}} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \int_{|x-y|<\alpha s} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx \\
&\qquad \leq \int_{\mathbb{R}^n} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \left(\iint_{|x-y|<\alpha s} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\right) dx \\
&\qquad = \int_{\mathbb{R}^n} \Big(M(k_1^{X_0}\,\chi_{Q_0})(x)\Big)^{2-p} \, (S_\alpha(u_0)(x))^2\, dx.
\end{aligned}$$

Since we have assumed that $1 < p < 2$ we can use Hölder's inequality with exponent $p'/2 > 1$ to obtain

$$II \leq \|S_\alpha u_0\|_{L^{p'}}\, \|M(k_1^{X_0}\,\chi_{Q_0})\|_{L^p}^{\frac{2-p}{2}} \lesssim \|g\|_{L^{p'}(Q_0)}\, \|k_1^{X_0}\|_{L^p(Q_0)}^{\frac{2-p}{2}} = \|k_1^{X_0}\|_{L^p(Q_0)}^{\frac{2-p}{2}},$$

where we have used Theorem 3.11 (and the fact that $(D)_{p'}$ is solvable for L_0). Collecting our estimates for I and II we conclude

$$F_1(X_0) = |u_1(X_0) - u_0(X_0)| \lesssim \delta^{\frac{1}{2}}\, \|k_1^{X_0}\|_{L^p(Q_0)}. \tag{4.5}$$

Since $k_0^{X_0}$ satisfies (4.2), we may therefore obtain (4.2) for $k_1^{X_0}$ by taking a supremum over all g as above, and then hiding the error in (4.5) for δ small enough (here we use the qualitative estimate $\|k_1^{X_0}\|_{L^p(Q_0)} < \infty$, see Step 0.)

4.2.1. *Self-improvement of Step 1.* So far we have only proved that $k_1^{X_0}$ satisfies a scale invariant L^p estimate on the cube Q_0 (cf. (4.2)). In order to carry out Step 2, we will first need to extend (4.2) to obtain a reverse Hölder estimate on every dyadic subcube of Q_0. The key fact that will allow us to do so is that, in (4.1), the sup is taken with respect to all such cubes. The idea of the proof is to repeat the previous argument for a fixed $Q \in \mathcal{D}(Q_0)$ to obtain the analogue of (4.2) on Q, for the Poisson kernel associated to L_1, which is now defined with respect to

$$\Omega_Q := R_Q \cap \Omega_{\mathcal{F}} = R_Q \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k}).$$

The definition of the operator L_1 will depend on Q, but we will address this issue by use of the comparison principle.

We now fix $Q \in \mathcal{D}(Q_0)$. Let $X_Q = (x_Q, 4\,\ell(Q))$ where x_Q is the center of Q. Let us define a new operator $L_1^Q = L$ in Ω_Q and $L_1^Q = L_0$ otherwise in $\mathbb{R}^{n+1}_+$, and let $k_{L_1^Q}^{X_Q}$ denote the Poisson kernel for L_1^Q with pole at X_Q. We claim that

$$\int_Q k_{L_1^Q}^{X_Q}(x)^p dx \le C_1 |Q|^{1-p}, \tag{4.6}$$

for some C_1 independent of Q. Indeed, if $Q \subset Q_k$ for some $Q_k \in \mathcal{F}$ then we obtain that $\Omega_Q = \emptyset$ and $L_1^Q \equiv L_0$ in $\mathbb{R}^{n+1}_+$. In that case, (4.6) holds by hypothesis. Otherwise, since trivially $\|\mu\|_{\mathcal{C}(Q)} \le \|\mu\|_{\mathcal{C}(Q_0)}$ for every $Q \in \mathcal{D}(Q_0)$, we have that the analogue of (4.1) obviously holds on Q, for the same family $\mathcal{F}$ (or to be more precise, for the family $\mathcal{F}_Q$ defined as the family of cubes in $\mathcal{F}$ that meet Q). Consequently, if Q is not contained in any $Q_k \in \mathcal{F}$, then we may simply repeat the previous argument with respect to Q, and we obtain (4.6) exactly as before. This proves the claim.

Now by (3.8), we have that $\int_Q k_{L_1^Q}^{X_Q}(x)dx \ge 1/C$, and combining this estimate with (4.6) we obtain

$$\left(\fint_Q k_{L_1^Q}^{X_Q}(x)^p\, dx\right)^{\frac{1}{p}} \le CC_1 \fint_Q k_{L_1^Q}^{X_Q}(x)\, dx. \tag{4.7}$$

Next, we want to pass from $k_{L_1^Q}^{X_Q}$ to $k_{L_1}^{X_Q}$. Notice that $L_1 \equiv L_1^Q$ in R_Q, therefore Lemma 3.9 yields that

$$k_1^{X_Q}(y) = k_{L_1}^{X_Q}(y) \approx k_{L_1^Q}^{X_Q}(y), \qquad \text{for a.e. } y \in \tfrac{1}{2}\,Q.$$

The latter fact, (4.7) and the doubling property imply that

$$\left(\fint_{\frac{1}{2}Q} k_1^{X_Q}(x)^p\, dx\right)^{\frac{1}{p}} \lesssim \left(\fint_Q k_{L_1^Q}^{X_Q}(x)^p\, dx\right)^{\frac{1}{p}} \lesssim \fint_Q k_{L_1^Q}^{X_Q}(x)\, dx \lesssim \fint_{\frac{1}{2}Q} k_1^{X_Q}(x)\, dx. \tag{4.8}$$

Consequently, by Lemma 3.10 we have

$$\left(\fint_{\frac{1}{2}Q} k_1^{X_0}(x)^p\, dx\right)^{\frac{1}{p}} \lesssim \fint_{\frac{1}{2}Q} k_1^{X_0}(x)\, dx, \qquad \forall\, Q \in \mathcal{D}(Q_0). \tag{4.9}$$

Then we use Lemma B.7 to obtain the following:

Conclusion (Step 1)**.** *There exists $1 < r < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$,*

$$\left(\fint_Q k_1^{X_0}(x)^r\, dx\right)^{\frac{1}{r}} \le C \fint_Q k_1^{X_0}(x)\, dx. \tag{4.10}$$

That is, $\omega_1^{X_0} \in A_\infty^{\mathrm{dyadic}}(Q_0)$. Hence we deduce that the same is true for $\mathcal{P}_{\mathcal{F}}\,\omega_1^{X_Q}$, by the following lemma.

Lemma 4.1. *Suppose that* $\omega \in A_\infty^{\rm dyadic}(Q)$*, for some fixed cube* Q*, and suppose that* $\mathcal{F} = \{Q_k\} \subset \mathcal{D}(Q)$ *is a non-overlapping family. Then also* $\mathcal{P}_\mathcal{F}\,\omega \in A_\infty^{\rm dyadic}(Q)$*.*

Sketch of proof. The proof is a straightforward consequence of the definition of $\mathcal{P}_\mathcal{F}$, plus a simplified version of Lemma 2.8, using the apparently weaker definition of $A_\infty^{\rm dyadic}(Q)$ in (B.6) (for a different argument see [HM]). We omit the details. □

4.3. **Step 2.** We define the operator L_2 such that the disagreement with L_1 lives inside the Carleson boxes corresponding to the family $\mathcal{F}$. That is, set $L_2 = L$ in $R_{Q_0} \setminus \Omega_\mathcal{F} = \cup_{Q_k\in\mathcal{F}} R_{Q_k}$ and $L_2 = L_1$ otherwise (see Figure 6). We write $\omega_1 = \omega_{L_1}^{X_0}$ and $\omega_2 = \omega_{L_2}^{X_0}$ for the corresponding harmonic measures for L_1 and L_2 in $\mathbb{R}^{n+1}_+$ with fixed pole at $X_0 = (x_0, 4\,\ell(Q_0))$. We also let $\nu_1 = \nu_1^{X_0}$ and $\nu_2 = \nu_2^{X_0}$ denote the harmonic measures of L_1 and L_2 with pole at X_0, with respect to the domain $\Omega_\mathcal{F} = \mathbb{R}^{n+1}_+ \setminus \cup_{Q_k\in\mathcal{F}} R_{Q_k}$. We notice that $L_1 = L_2$ in $\Omega_\mathcal{F}$ and therefore $\nu_1 = \nu_2$.

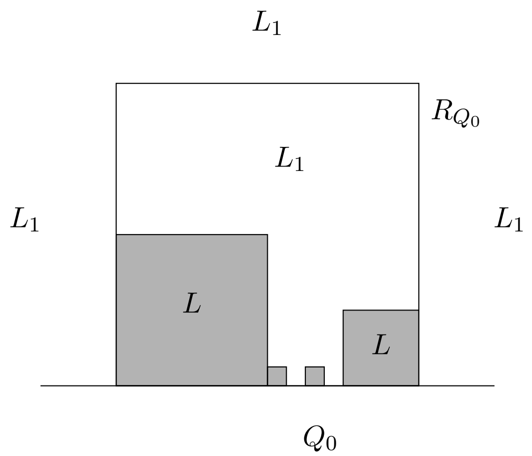

FIGURE 6. Definition of L_2

We apply the sawtooth lemma for projections (see Lemma A.1 in Appendix A below) to both L_1 and L_2 and then we obtain that for all $Q \subset \mathcal{D}(Q_0)$ and $F \subset Q$

$$\left(\frac{\mathcal{P}_\mathcal{F}\,\omega_i(F)}{\mathcal{P}_\mathcal{F}\,\omega_i(Q)}\right)^{\theta_i} \lesssim \frac{\mathcal{P}_\mathcal{F}\,\bar\nu_i(F)}{\mathcal{P}_\mathcal{F}\,\bar\nu_i(Q)} \lesssim \frac{\mathcal{P}_\mathcal{F}\,\omega_i(F)}{\mathcal{P}_\mathcal{F}\,\omega_i(Q)}, \qquad i = 1, 2\,;$$

that is, $\mathcal{P}_\mathcal{F}\,\omega_i \in A_\infty^{\rm dyadic}(\mathcal{P}_\mathcal{F}\,\bar\nu_i, Q_0)$, for $i = 1, 2$ —here we use that $\mathcal{P}_\mathcal{F}\,\omega_i$ and $\mathcal{P}_\mathcal{F}\,\bar\nu_i$ are dyadically doubling by Lemmas B.1 and B.2. As observed above, $\nu_1 = \nu_2$ and therefore (A.2) implies that $\mathcal{P}_\mathcal{F}\,\bar\nu_1 = \mathcal{P}_\mathcal{F}\,\bar\nu_2$. Since $A_\infty^{\rm dyadic}(Q_0)$ defines an equivalence relationship, and since we showed in Step 1 that $\mathcal{P}_\mathcal{F}\,\omega_1 \in A_\infty^{\rm dyadic}(Q_0)$ (with respect to Lebesgue measure), we also conclude that $\mathcal{P}_\mathcal{F}\,\omega_2 \in A_\infty^{\rm dyadic}(Q_0)$:

Conclusion (Step 2)**.** *There exists* θ*,* $\theta' > 0$ *such that*

$$\left(\frac{|F|}{|Q|}\right)^{\theta} \lesssim \frac{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(F)}{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(Q)} \lesssim \left(\frac{|F|}{|Q|}\right)^{\theta'}, \qquad Q \in \mathcal{D}(Q_0), \quad F \subset Q.$$

4.4. **Step 3.** To complete the proof it remains to change the operator outside R_{Q_0}. Thus, we define $L_3 = L_2$ in R_{Q_0} and $L_3 = L$ otherwise (see Figure 7). Let us observe that $L_3 = L$ in $\mathbb{R}^{n+1}_+$.

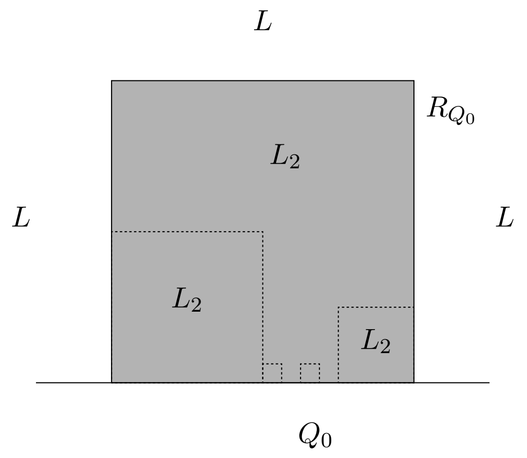

FIGURE 7. Definition of L_3

We want to show that (2.4) holds with $\mathcal{P}_\mathcal{F}$ in place of $\mathcal{P}'_\mathcal{F}$. We fix $0 < \varepsilon < 1$ and take $E \subset Q_0$ with $|E|/|Q_0| \geq \varepsilon$. Let us observe that we can disregard the trivial case $\mathcal{F} = \{Q_0\}$ since we have

$$\frac{\mathcal{P}_\mathcal{F}\omega_3^{X_0}(E)}{\mathcal{P}_\mathcal{F}\omega_3^{X_0}(Q_0)} = \frac{\frac{|E|}{|Q_0|}\,\omega_3^{X_0}(Q_0)}{\frac{|Q_0|}{|Q_0|}\,\omega_3^{X_0}(Q_0)} = \frac{|E|}{|Q_0|} \geq \varepsilon.$$

We take $j \geq 2$ large enough such that $2^{-j+1} < 1-(1-\varepsilon/2)^{1/n}$. We set $\tilde{Q}_0 = (1-2^{-j+1})\,Q_0$ and observe that $Q_0 \setminus \tilde{Q}_0 = \cup_\Lambda Q$ where $\Lambda \subset \mathcal{D}(Q_0)$ and $\ell(Q) = 2^{-j}\,\ell(Q_0)$ for every $Q \in \Lambda$. Notice that Λ consists of all dyadic cubes in $\mathcal{D}(Q_0)$ with sidelength $2^{-j}\,\ell(Q_0)$ which are adjacent to the boundary of Q_0. We write $F = E \cap \tilde{Q}_0$ and observe that

$$\varepsilon\,|Q_0| \leq |E| \leq |F| + |Q_0 \setminus \tilde{Q}_0| \leq |F| + (1-(1-2^{-j+1})^n)\,|Q_0| < |F| + \frac{\varepsilon}{2}\,|Q_0|$$

and therefore $|F|/|Q_0| \geq \varepsilon/2$. Then, using the conclusion of Step 2 we obtain

$$\frac{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(F)}{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(Q_0)} \geq C\left(\frac{|F|}{|Q_0|}\right)^\theta \geq C\left(\frac{\varepsilon}{2}\right)^\theta.$$

We notice that $\mathcal{P}_\mathcal{F}\omega_2^{X_0}(Q_0) = \omega_2^{X_0}(Q_0) \geq C$ by Lemma 3.8 and $\mathcal{P}_\mathcal{F}\omega_3^{X_0}(Q_0) = \omega_3^{X_0}(Q_0) \leq 1$. We claim that

$$\mathcal{P}_\mathcal{F}\omega_3^{X_0}(F) \geq C_\varepsilon\,\mathcal{P}_\mathcal{F}\omega_2^{X_0}(F). \tag{4.11}$$

Assuming this for the moment and gathering the obtained estimates we conclude that

$$\frac{\mathcal{P}_\mathcal{F}\omega_3^{X_0}(E)}{\mathcal{P}_\mathcal{F}\omega_3^{X_0}(Q_0)} \geq \mathcal{P}_\mathcal{F}\omega_3^{X_0}(F) \geq C_\varepsilon\,\mathcal{P}_\mathcal{F}\omega_2^{X_0}(F) \geq C'_\varepsilon\,\frac{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(F)}{\mathcal{P}_\mathcal{F}\omega_2^{X_0}(Q_0)} \geq C'_\varepsilon\left(\frac{\varepsilon}{2}\right)^\theta.$$

We show (4.11). Notice that $L_2 \equiv L_3$ in R_{Q_0}, then as in Lemma 3.9 by the comparison principle we have that $k_2^{X_0}(y) \approx k_3^{X_0}(y)$ for a.e. $y \in \tilde{Q}_0$ where the constants depend on j and hence on ε. This implies that $\omega_2^{X_0}(F \setminus (\cup_{Q_k\in\mathcal{F}}Q_k)) \approx \omega_3^{X_0}(F \setminus (\cup_{Q_k\in\mathcal{F}}Q_k))$ and then,

$$\begin{aligned}\mathcal{P}_\mathcal{F}\omega_3^{X_0}(F) &= \omega_3^{X_0}(F \setminus (\cup_{Q_k\in\mathcal{F}}Q_k)) + \sum_{Q_k\in\mathcal{F}} \frac{|F\cap Q_k|}{|Q_k|}\omega_3^{X_0}(Q_k) \\ &\geq C_\epsilon\,\omega_2^{X_0}(F \setminus (\cup_{Q_k\in\mathcal{F}}Q_k)) + \sum_{Q_k\in\mathcal{F}} \frac{|F\cap Q_k|}{|Q_k|}\omega_3^{X_0}(Q_k)\end{aligned}$$

and it remains to estimate the second term. Note that in the sum we can restrict ourselves to those cubes in $\mathcal{F}$ that meet F, therefore we pick such a cube Q_k.

Case 1: $Q_k \subset \tilde{Q}_0$. As in the previous computations $\omega_3^{X_0}(Q_k) \geq C_\varepsilon\, \omega_2^{X_0}(Q_k)$.

Case 2: $Q_k \not\subset \tilde{Q}_0$. This means that $Q_k \setminus \tilde{Q}_0 \neq \emptyset$ and then there is $Q' \in \Lambda$ with $Q_k \cap Q' \neq \emptyset$. This yields that $Q' \subsetneq Q_k$ (otherwise, $Q_k \subset Q'$ which implies that $Q_k \subset Q_0 \setminus \tilde{Q}_0$ contradicting the fact that $F \cap Q_k \neq \emptyset$ since $F \subset \tilde{Q}_0$.) Since Q' is adjacent to the boundary of Q_0 then so is Q_k. We notice that there exists $\bar{Q}_k \in \mathcal{D}(Q_k)$ with $\ell(\bar{Q}_k) = \ell(Q_k)/2$ (i.e., $\bar{Q}_k$ is a dyadic "child" of Q_k) that it is not adjacent to ∂Q_0 (we have $Q_k \subsetneq Q_0$ since the case $\mathcal{F} = \{Q_0\}$ was disregarded). In this case we necessarily have $\bar{Q}_k \subset \tilde{Q}_0$: if $\bar{Q}_k$ meets $Q_0 \setminus \tilde{Q}_0$ then there is $Q'' \in \Lambda$ with $\bar{Q}_k \cap Q'' \neq \emptyset$ and then either $Q'' \subset \bar{Q}_k$ which implies that $\bar{Q}_k$ is adjacent to the boundary of Q_0 leading to a contradiction, or $\bar{Q}_k \subsetneq Q''$ which implies $Q_k \subset Q'' \subset Q_0 \setminus \tilde{Q}_0$ contradicting the fact that $F \cap Q_k \neq \emptyset$ since $F \subset \tilde{Q}_0$. Given this, since $\omega_2^{X_0}$ is doubling we have

$$\omega_3^{X_0}(Q_k) \geq \omega_3^{X_0}(\bar{Q}_k) \geq C_\varepsilon\, \omega_2^{X_0}(\bar{Q}_k) \geq C_\varepsilon\, \omega_2^{X_0}(Q_k).$$

Thus in both cases we can conclude as desired

$$\mathcal{P}_{\mathcal{F}}\omega_3^{X_0}(F) \geq C_\epsilon\, \omega_2^{X_0}(F \setminus (\cup_{Q_k \in \mathcal{F}} Q_k)) + C_\varepsilon \sum_{Q_k \in \mathcal{F}} \frac{|F \cap Q_k|}{|Q_k|} \omega_2^{X_0}(Q_k) = C_\epsilon\, \mathcal{P}_{\mathcal{F}}\omega_2^{X_0}(F).$$

Let us summarize what we have obtained so far (we recall that $L_3 \equiv L$):

Conclusion (Step 3)**.** *There exists $\delta > 0$ for which the following statement holds: given $\varepsilon \in (0,1)$, there is $C_\epsilon < \infty$ such that for every $Q_0 \subset \mathbb{R}^n$, if $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q_0)$ is a pairwise disjoint collection of dyadic subcubes of Q_0 satisfying $\|\mu_{\mathcal{F}}\|_{\mathcal{C}(Q_0)} < \delta$, then*

$$F \subset Q_0, \quad \frac{|F|}{|Q_0|} \geq \varepsilon \quad \Longrightarrow \quad \frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(Q_0)} \geq \frac{1}{C_\varepsilon}.$$

4.5. **Step 4.** In order to apply the extrapolation result we need to be able to fix the pole relative to a given cube Q_0, and show that the conclusion of Step 3 still applies to dyadic subcubes of Q_0.

Proposition 4.2. *There exists $\delta > 0$ for which the following statement holds: given $\varepsilon \in (0,1)$, there is $C_\epsilon < \infty$ such that for every $Q_0 \subset \mathbb{R}^n$ and for all $Q \in \mathcal{D}(Q_0)$, if $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ is a pairwise disjoint collection of dyadic subcubes of Q satisfying*

$$\sup_{Q' \in \mathcal{D}(Q)} \frac{\mu(R_{Q'} \cap \Omega_{\mathcal{F}})}{|Q'|} \leq \delta, \tag{4.12}$$

then

$$F \subset Q, \quad \frac{|F|}{|Q|} \geq \varepsilon \quad \Longrightarrow \quad \frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(Q)} \geq \frac{1}{C_\varepsilon}.$$

Consequently, $\omega^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ uniformly in Q_0. In particular, there exist $1 < q < \infty$ and a uniform constant C_0 such that we have the following reverse Hölder inequalities for all $Q_0 \subset \mathbb{R}^n$,

$$\Big(\fint_{Q_0} k_L^{X_{Q_0}}(y)^q\, dy\Big)^{\frac{1}{q}} \leq C_0 \fint_{Q_0} k_L^{X_{Q_0}}(y)\, dy \approx \frac{1}{|Q_0|}. \tag{4.13}$$

Proof. Take an arbitrary $\varepsilon \in (0,1)$ and let $\delta > 0$ and C_ϵ be given by the conclusion of Step 3. We fix $Q_0 \subset \mathbb{R}^n$ and $Q \in \mathcal{D}(Q_0)$. Let $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ be such that (4.12) holds. Then, we use Lemma 3.10 and for every $F \subset Q$ we obtain

$$\begin{aligned}\mathcal{P}_{\mathcal{F}}\omega_L^{X_Q}(F) &= \omega_L^{X_Q}(F \setminus (\cup_{Q_k\in\mathcal{F}} Q_k)) + \sum_{Q_k\in\mathcal{F}} \frac{|F\cap Q_k|}{|Q_k|}\,\omega_L^{X_Q}(Q_k) \\ &\approx \frac{\omega_L^{X_{Q_0}}(F \setminus (\cup_{Q_k\in\mathcal{F}} Q_k))}{\omega_L^{X_{Q_0}}(Q)} + \sum_{Q_k\in\mathcal{F}} \frac{|F\cap Q_k|}{|Q_k|}\,\frac{\omega_L^{X_{Q_0}}(Q_k)}{\omega_L^{X_{Q_0}}(Q)} \\ &= \frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q)} = \frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(Q)}.\end{aligned}$$

Given $F \subset Q$ with $|F|/|Q| \geq \varepsilon$ we apply the previous estimate and the conclusion of Step 3 with cube Q in place of Q_0 to conclude that

$$\frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}\omega_L^{X_{Q_0}}(Q)} \approx \mathcal{P}_{\mathcal{F}}\omega_L^{X_Q}(F) \approx \frac{\mathcal{P}_{\mathcal{F}}\omega_L^{X_Q}(F)}{\mathcal{P}_{\mathcal{F}}\omega_L^{X_Q}(Q)} \geq \frac{1}{C_\varepsilon}.$$

Next, by the extrapolation result Theorem 2.1, there exist $\eta_0 \in (0,1)$ and $C_0 < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$,

$$F \subset Q, \quad \frac{|F|}{|Q|} \geq 1 - \eta_0 \quad \Longrightarrow \quad \frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q)} \geq \frac{1}{C_0}.$$

This fact plus the classical result in [CF] (see the proof of Lemma B.4 below) imply the existence of $q = q_L$ and a uniform constant C_1 such that for all $Q \in \mathcal{D}(Q_0)$,

$$\Big(\fint_Q k_L^{X_{Q_0}}(y)^q\,dy\Big)^{\frac{1}{q}} \leq C_1 \fint_Q k_L^{X_{Q_0}}(y)\,dy.$$

If we specify this estimate to $Q = Q_0$ we obtain as desired (4.13). We notice that the previous estimate and the fact that $\omega_L^{X_{Q_0}}$ is doubling imply $k_L^{X_{Q_0}} \in RH_q(Q_0)$. $\square$

From this result, we see that (4.13) and Theorem 3.1 yield as desired that L is solvable in $L^{q'}$.

Appendix A. Discrete Sawtooth Lemmas

We present some versions of the main lemma in [DJK] which are valid for discrete sawtooth regions based on dyadic cubes. The first result involves the projection operators and was used in Step 2 above. The second result (cf. Lemma A.2) is interesting in its own right and is a dyadic analog of the main lemma in [DJK]. For both lemmas, the proofs follow the idea of the argument in [DJK], but are technically much simpler, owing to the dyadic setting in which we work here.

Lemma A.1 (Discrete sawtooth lemma for projections)**.** *Let Q_0 be a fixed cube in $\mathbb{R}^n$, let $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q_0)$ be a family of pairwise disjoint dyadic cubes and let $\mathcal{P}_{\mathcal{F}}$ be the corresponding projection operator. Set $\Omega_{\mathcal{F}} = \mathbb{R}^{n+1}_+ \setminus (\cup_{Q_k\in\mathcal{F}} R_{Q_k})$. We write $\omega = \omega^{X_0}$ and $\nu = \nu^{X_0}$ for the harmonic measures of L with fixed pole at $X_0 = (x_{Q_0}, 4\,\ell(Q_0))$ with respect to the domains $\mathbb{R}^{n+1}_+$ and $\Omega_{\mathcal{F}}$. Let $\bar{\nu} = \bar{\nu}^{X_0}$ be the measure defined by*

$$\bar{\nu}(F) = \nu(F \setminus (\cup_{Q_k\in\mathcal{F}} R_{Q_k})) + \sum_{Q_k\in\mathcal{F}} \frac{\omega(F\cap Q_k)}{\omega(Q_k)}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}), \qquad F \subset Q_0. \tag{A.1}$$

We observe that $\mathcal{P}_{\mathcal{F}}\bar{\nu}$ depends only on ν and not on ω since

$$\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)=\nu(F\setminus(\cup_{Q_k\in\mathcal{F}}R_{Q_k}))+\sum_{Q_k\in\mathcal{F}}\frac{|F\cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}}),\qquad F\subset Q_0. \tag{A.2}$$

Then, there exists $\theta>0$ such that for all $Q\in\mathcal{D}(Q_0)$ and $F\subset Q$, we have

$$\left(\frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}\right)^{\theta}\lesssim\frac{\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)}{\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)}\lesssim\frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}. \tag{A.3}$$

Proof. Set $E_0=Q_0\setminus(\cup_{Q_k\in\mathcal{F}}Q_k)$. We first observe that (A.2) follows from the definitions of $\mathcal{P}_{\mathcal{F}}$ and $\bar{\nu}$: given $F\subset Q_0$,

$$\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)=\bar{\nu}(F\cap E_0)+\sum_{Q_k\in\mathcal{F}}\frac{|F\cap Q_k|}{|Q_k|}\,\bar{\nu}(Q_k)=\nu(F\cap E_0)+\sum_{Q_k\in\mathcal{F}}\frac{|F\cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}}),$$

where we have used that the cubes in $\mathcal{F}$ are disjoint and therefore $\bar{\nu}(Q_k)=\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})$.

We first show the righthand side inequality in (A.3). Let us fix $Q\in\mathcal{D}(Q_0)$, $F\subset Q$.

Case 1: There exists $Q_k\in\mathcal{F}$ such that $Q\subset Q_k$. Note that by (A.2) we have

$$\frac{\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)}{\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)}=\frac{\frac{|F\cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})}{\frac{|Q\cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})}=\frac{|F|}{|Q|}=\frac{\frac{|F\cap Q_k|}{|Q_k|}\,\omega(Q_k)}{\frac{|Q\cap Q_k|}{|Q'|}\,\omega(Q_k)}=\frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)}.$$

Case 2: Q is not contained in any cube of $\mathcal{F}$. Notice that if $Q_k\in\mathcal{F}$ with $Q_k\cap Q\neq\emptyset$, then $Q_k\subsetneq Q$. Using (A.2) we observe that

$$\begin{aligned}\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)&=\nu(Q\cap E_0)+\sum_{Q_k\in\mathcal{F},Q_k\subsetneq Q}\frac{|Q\cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})\\&=\nu(Q\cap E_0)+\sum_{Q_k\in\mathcal{F},Q_k\subsetneq Q}\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})=\nu(\overline{R_Q}\cap\partial\Omega_{\mathcal{F}}).\end{aligned} \tag{A.4}$$

Pick $A_Q=(x_Q,\ell(Q))$ and notice that $d(A_Q,\partial\Omega_{\mathcal{F}})\approx d(A_Q,\mathbb{R}^n)\approx\ell(Q)$ (here we are using that $Q_k\subsetneq Q$ for all $Q_k\in\mathcal{F}$ such that $Q_k\cap Q\neq\emptyset$) thus A_Q is a corkscrew point for Q with respect to both domains. Then, we can use [Ken, Lemma 1.3.8] (as $X_0\notin R_{2Q}$) to obtain that for any Borel set $G\subset Q$,

$$\omega^{A_Q}(G)\approx\frac{\omega^{X_0}(G)}{\omega^{X_0}(Q)}=\frac{\omega(G)}{\omega(Q)}. \tag{A.5}$$

The same occurs for ν and ν^{A_Q} and for any $G\subset\overline{R_Q}\cap\partial\Omega_{\mathcal{F}}$:

$$\nu^{A_Q}(G)\approx\frac{\nu^{X_0}(G)}{\nu^{X_0}(\overline{R_Q}\cap\partial\Omega_{\mathcal{F}})}=\frac{\nu(G)}{\nu(\overline{R_Q}\cap\partial\Omega_{\mathcal{F}})}. \tag{A.6}$$

Using (A.4) and (A.6) we obtain

$$\begin{aligned}\frac{\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)}{\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)}&=\frac{\nu(F\cap E_0)}{\nu(\overline{R_Q}\cap\partial\Omega_{\mathcal{F}})}+\sum_{Q_k\in\mathcal{F},Q_k\subsetneq Q}\frac{|F\cap Q_k|}{|Q_k|}\,\frac{\nu(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})}{\nu(\overline{R_Q}\cap\partial\Omega_{\mathcal{F}})}\\&\approx\nu^{A_Q}(F\cap E_0)+\sum_{Q_k\in\mathcal{F},Q_k\subsetneq Q}\frac{|F\cap Q_k|}{|Q_k|}\,\nu^{A_Q}(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}}).\end{aligned}$$

We claim that the following estimates hold (the proof is given below)

$$\nu^{A_Q}(F\cap E_0)\lesssim\omega^{A_Q}(F\cap E_0),\qquad\nu^{A_Q}(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})\lesssim\omega^{A_Q}(Q_k). \tag{A.7}$$

These and (A.5) imply

$$\begin{aligned}\frac{\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)}{\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)} &\lesssim \omega^{A_Q}(F\cap E_0) + \sum_{Q_k\in\mathcal{F}, Q_k\subsetneq Q} \frac{|F\cap Q_k|}{|Q_k|}\,\omega^{A_Q}(Q_k) \\ &\approx \frac{\omega(F\cap E_0)}{\omega(Q)} + \sum_{Q_k\in\mathcal{F}, Q_k\subsetneq Q} \frac{|F\cap Q_k|}{|Q_k|}\,\frac{\omega(Q_k)}{\omega(Q)} = \frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\omega(Q)} = \frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)},\end{aligned}$$

where in the last equality we have used that $\mathcal{P}_{\mathcal{F}}\omega(Q) = \omega(Q)$.

Once we have shown the righthand side inequality in (A.3) we apply Lemma B.4 and the fact that $\mathcal{P}_{\mathcal{F}}\omega$ and $\mathcal{P}_{\mathcal{F}}\bar{\nu}$ are dyadically doubling by Lemmas B.1 and B.2 to conclude that for all $Q\in\mathcal{D}(Q_0)$ and $F\subset Q$

$$\frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)} \lesssim \left(\frac{\mathcal{P}_{\mathcal{F}}\bar{\nu}(F)}{\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q)}\right)^{1/\theta}.$$

To complete the proof we need to show the estimates claimed in (A.7). We start with the first one. We write $u(Z)=\omega^Z(F\cap E_0)$ and $\tilde{u}(Z)=\nu^Z(F\cap E_0)$. We have the following: $u,\ \tilde{u}\ge 0$, $Lu=0$ in $\mathbb{R}^{n+1}_+$, $L\tilde{u}=0$ in $\Omega_{\mathcal{F}}$, $u|_{\mathbb{R}^n}=\chi_{F\cap E_0}$, $\tilde{u}|_{\partial\Omega_{\mathcal{F}}}=\chi_{F\cap E_0}$. For every $Z\in\partial\Omega_{\mathcal{F}}$ we notice that $\tilde{u}(Z)\le u(Z)$ —if $Z\in F\cap E_0$, $\tilde{u}(Z)=u(Z)=1$; if $Z\notin F\cap E_0$, $\tilde{u}(Z)=0\le u(Z)$. Also, $L\tilde{u}=Lu=0$ in $\Omega_{\mathcal{F}}$. Thus, the maximum principle yields that $\tilde{u}(Z)\le u(Z)$ for all $Z\in\Omega_{\mathcal{F}}$. We use that $A_Q\in\Omega_{\mathcal{F}}$ since Q is not contained in any cube of $\mathcal{F}$ to conclude as desired

$$\nu^{A_Q}(F\cap E_0)=\tilde{u}(A_Q)\le u(A_Q)=\omega^{A_Q}(F\cap E_0).$$

Next we show the second estimate in (A.7). For every $Q_k\in\mathcal{F}$, $Q_k\subsetneq Q$ we write $A_{Q_k}=(x_{Q_k},\ell(Q_k))$ and observe that $A_{Q_k}\in\partial\Omega_{\mathcal{F}}$. Also notice that $\overline{R_{Q_k}}\subset R(A_{Q_k},3\,\ell(Q_k))$ which is the $\mathbb{R}^{n+1}$-cube centered at A_{Q_k} and with sidelength $3\,\ell(Q_k)$. Thus, by the doubling property for ν^{A_Q} we have

$$\nu^{A_Q}(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})\le\nu^{A_Q}(R(A_{Q_k},3\,\ell(Q_k))\cap\partial\Omega_{\mathcal{F}})\lesssim\nu^{A_Q}(R(A_{Q_k},\ell(Q_k)/2)\cap\partial\Omega_{\mathcal{F}}).$$

We write $S_{Q_k}=R(A_{Q_k},\ell(Q_k)/2)\cap\partial\Omega_{\mathcal{F}}$ and observe that this set lives on the upper face of R_{Q_k}. Consider $u(Z)=\omega^Z(Q_k)$, $\tilde{u}(Z)=\nu^Z(S_{Q_k})$. Notice that $u,\ \tilde{u}\ge 0$, $Lu=0$ in $\mathbb{R}^{n+1}_+$, $L\tilde{u}=0$ in $\Omega_{\mathcal{F}}$, $u|_{\mathbb{R}^n}=\chi_{Q_k}$, $\tilde{u}|_{\partial\Omega_{\mathcal{F}}}=\chi_{S_{Q_k}}$. We observe that if $Z\in\partial\Omega_{\mathcal{F}}$ then $\tilde{u}(Z)\lesssim u(Z)$: indeed, if $Z\in S_{Q_k}$ then $\tilde{u}(Z)=1\approx\omega^Z(Q_k)=u(Z)$ and if $Z\notin S_{Q_k}$, $\tilde{u}(Z)=0\le u(Z)$. Also $Lu=L\tilde{u}=0$ in $\Omega_{\mathcal{F}}$. Therefore, the maximum principle yields that $\tilde{u}(Z)\lesssim u(Z)$ for all $Z\in\Omega_{\mathcal{F}}$. Then, proceeding as before we conclude

$$\nu^{A_Q}(\overline{R_{Q_k}}\cap\partial\Omega_{\mathcal{F}})\lesssim\nu^{A_Q}(S_{Q_k})=\tilde{u}(A_Q)\lesssim u(A_Q)=\omega^{A_Q}(Q_k).$$

□

Lemma A.2 (Discrete sawtooth lemma). *Let Q_0 be a fixed cube in $\mathbb{R}^n$ and let $\mathcal{F}=\{Q_k\}_k\subset\mathcal{D}(Q_0)$ be a family of pairwise disjoint dyadic cubes. Set $\Omega_{\mathcal{F}}=\mathbb{R}^{n+1}_+\setminus(\cup_{Q_k\in\mathcal{F}}R_{Q_k})$. We write $\omega=\omega^{X_0}$ and $\nu=\nu^{X_0}$ for the harmonic measures of L with pole at $X_0=(x_{Q_0},4\,\ell(Q_0))$ with respect to the domains $\mathbb{R}^{n+1}_+$ and $\Omega_{\mathcal{F}}$. Let $\bar{\nu}=\bar{\nu}^{X_0}$ be the measure defined by* (A.1). *Then, there exists $\theta>0$ such that for all $Q\in\mathcal{D}(Q_0)$ and $F\subset Q$, we have*

$$\left(\frac{\omega(F)}{\omega(Q)}\right)^{\theta}\lesssim\frac{\bar{\nu}(F)}{\bar{\nu}(Q)}\lesssim\frac{\omega(F)}{\omega(Q)}. \tag{A.8}$$

In particular, if $F \subset Q \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k})$, we have

$$\left(\frac{\omega(F)}{\omega(Q)}\right)^\theta \lesssim \frac{\nu(F)}{\nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}})} \lesssim \frac{\omega(F)}{\omega(Q)}. \tag{A.9}$$

Proof. We proceed as in the proof of Lemma A.1 and fix $Q \in \mathcal{D}(Q_0)$ and $F \subset Q$. Set $E_0 = Q_0 \setminus (\cup_{Q_k \in \mathcal{F}} Q_k)$.

Case 1: There exists $Q_k \in \mathcal{F}$ such that $Q \subset Q_k$. We use the definition of $\bar{\nu}$ to conclude as desired

$$\frac{\bar{\nu}(F)}{\bar{\nu}(Q)} = \frac{\frac{\omega(F\cap Q_k)}{\omega(Q_k)}\, \nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}})}{\frac{\omega(Q\cap Q_k)}{\omega(Q_k)}\, \nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}})} = \frac{\omega(F)}{\omega(Q)}.$$

Case 2: Q is not contained in any cube of $\mathcal{F}$. Notice that if $Q_k \in \mathcal{F}$ with $Q_k \cap Q \neq \emptyset$, then $Q_k \subsetneq Q$ and

$$\begin{aligned}
\bar{\nu}(Q) &= \nu(Q \cap E_0) + \sum_{Q_k \in \mathcal{F}, Q_k \subsetneq Q} \frac{\omega(Q \cap Q_k)}{\omega(Q_k)}\, \nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}) \\
&= \nu(Q \cap E_0) + \sum_{Q_k \in \mathcal{F}, Q_k \subsetneq Q} \nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}) = \nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}}).
\end{aligned} \tag{A.10}$$

Then we use (A.5), (A.6) and (A.7) to conclude that

$$\begin{aligned}
\frac{\bar{\nu}(F)}{\bar{\nu}(Q)} &= \frac{\nu(F \cap E_0)}{\nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}})} + \sum_{Q_k \in \mathcal{F}, Q_k \subsetneq Q} \frac{\omega(F \cap Q_k)}{\omega(Q_k)} \frac{\nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}})}{\nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}})} \\
&\approx \nu^{A_Q}(F \cap E_0) + \sum_{Q_k \in \mathcal{F}, Q_k \subsetneq Q_1} \frac{\omega^{A_Q}(F \cap Q_k)}{\omega^{A_Q}(Q_k)}\, \nu^{A_Q}(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}) \\
&\lesssim \omega^{A_Q}(F \cap E_0) + \sum_{Q_k \in \mathcal{F}, Q_k \subsetneq Q} \omega^{A_Q}(F \cap Q_k) \\
&= \omega^{A_Q}(F) \approx \frac{\omega(F)}{\omega(Q)},
\end{aligned}$$

and this completes Case 2.

Once we have shown the righthand side inequality in (A.8) we apply Lemma B.4 and the fact that ω and $\bar{\nu}$ are dyadically doubling in Q_0 (see Lemma B.2 below) to conclude that for all $Q \in \mathcal{D}(Q_0)$ and $F \subset Q$

$$\frac{\omega(F)}{\omega(Q)} \lesssim \left(\frac{\bar{\nu}(F)}{\bar{\nu}(Q)}\right)^{1/\theta}.$$

To show (A.9) we observe that if $F \subset E_0$ then $\bar{\nu}(F) = \nu(F)$. Also, notice that we cannot be in Case 1 unless $F = \emptyset$: we would have $Q \subset Q_k \in \mathcal{F}$ which gives $Q \subset Q_0 \setminus E_0$, and $F \subset Q \cap E_0$. This means that we can use (A.10). Gathering the obtained estimates we obtain (A.9):

$$\left(\frac{\omega(F)}{\omega(Q)}\right)^\theta \lesssim \frac{\bar{\nu}(F)}{\bar{\nu}(Q)} = \frac{\nu(F)}{\nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}})} \lesssim \frac{\omega(F)}{\omega(Q)}.$$

□

Appendix B. Dyadically doubling and Muckenhoupt weights

Fixed a cube Q_0, in what follows we work with Borel measures ω such that $0 < \omega(Q) < \infty$ for every $Q \in \mathcal{D}(Q_0)$. We say that ω is dyadically doubling in Q_0 if there exists C_ω such that $\omega(Q) \le C_\omega\, \omega(Q') < \infty$ for every $Q \in \mathcal{D}(Q_0)$, and for every dyadic "child" Q' of Q. It is not difficult to show that $C_\omega \ge 2^n$ (since Q is the union of its 2^n dyadic "children").

Lemma B.1. *Fix Q_0. Let ω be a dyadically doubling measure in Q_0 with constant C_ω. Then for every family $\mathcal{F} \subset \mathcal{D}(Q_0)$ of pairwise disjoint dyadic cubes, $\mathcal{P}_\mathcal{F}\omega$ is dyadically doubling in Q_0, indeed $\mathcal{P}_\mathcal{F}\omega(Q) \le C_\omega\, \mathcal{P}_\mathcal{F}\omega(Q')$ for every $Q \in \mathcal{D}(Q_0)$, and for every dyadic "child" Q' of Q.*

Proof. We fix $Q \in \mathcal{D}(Q_0)$ and one of its dyadic "children" Q'. We consider different cases.

Case 1: There exists $Q_k \in \mathcal{F}$ with $Q \subset Q_k$. The estimate is trivial in this case:

$$\mathcal{P}_\mathcal{F}\omega(Q) = \frac{|Q|}{|Q_k|}\,\omega(Q_k) = 2^n\,\frac{|Q'|}{|Q_k|}\,\omega(Q_k) = 2^n\,\mathcal{P}_\mathcal{F}\omega(Q') \le C_\omega\,\mathcal{P}_\mathcal{F}\omega(Q') < \infty.$$

Case 2: $Q' \in \mathcal{F}$. Notice that $\mathcal{P}_\mathcal{F}\omega(Q') = \omega(Q')$. Let $\mathcal{F}_1$ be the family of cubes $Q_k \in \mathcal{F}$ with $Q_k \cap Q \neq \emptyset$ and observe that if $Q_k \in \mathcal{F}_1$ then $Q_k \subsetneq Q$. Thus,

$$\begin{aligned}\mathcal{P}_\mathcal{F}\omega(Q) &= \omega(Q \setminus (\cup_{Q_k\in\mathcal{F}} Q_k)) + \sum_{Q_k\in\mathcal{F}_1} \frac{|Q_k \cap Q|}{|Q_k|}\omega(Q_k) = \omega(Q \setminus (\cup_{Q_k\in\mathcal{F}} Q_k)) + \sum_{Q_k\in\mathcal{F}_1} \omega(Q_k)\\ &= \omega(Q) \le C_\omega\,\omega(Q') = C_\omega\,\mathcal{P}_\mathcal{F}\omega(Q') < \infty.\end{aligned}$$

Case 3: None of the conditions in the previous cases occur. We take the same set $\mathcal{F}_1$ and observe that if $Q_k \in \mathcal{F}_1$ then $Q_k \subsetneq Q$ (otherwise we are driven to Case 1). Let $\mathcal{F}_2$ be the family of cubes $Q_k \in \mathcal{F}$ with $Q_k \cap Q' \neq \emptyset$. Notice that if $Q_k \in \mathcal{F}_2$ then $Q_k \subsetneq Q'$: otherwise, either $Q_k = Q'$ which leads us to Case 2, or $Q' \subsetneq Q_k$ which implies $Q \subset Q_k$ and this is Case 1. Then proceeding as in the previous case one obtains that $\mathcal{P}_\mathcal{F}\omega(Q) = \omega(Q)$ and $\mathcal{P}_\mathcal{F}\omega(Q') = \omega(Q')$ which in turn imply

$$\mathcal{P}_\mathcal{F}\omega(Q) = \omega(Q) \le C_\omega\,\omega(Q') = C_\omega\,\mathcal{P}_\mathcal{F}\omega(Q') < \infty.$$

□

Lemma B.2. *Under the hypotheses of Lemma A.1, $\bar\nu$ and $\mathcal{P}_\mathcal{F}\bar\nu$ are dyadically doubling in Q_0.*

Proof. We first consider $\bar\nu$. Let us fix $Q \in \mathcal{D}(Q_0)$ and one of its dyadic "children" Q'.

Case 1: There exists $Q_k \in \mathcal{F}$ with $Q \subset Q_k$. The estimate is trivial in this case since ω is dyadically doubling:

$$\bar\nu(Q) = \frac{\omega(Q)}{\omega(Q_k)}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F}) \le C_\omega\,\frac{\omega(Q')}{\omega(Q_k)}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F}) = C_\omega\,\bar\nu(Q') < \infty.$$

Case 2: $Q' \in \mathcal{F}$. Notice that $\bar\nu(Q') = \nu(\overline{R_{Q'}} \cap \partial\Omega_\mathcal{F})$. Let $\mathcal{F}_1$ be the family of cubes $Q_k \in \mathcal{F}$ with $Q_k \cap Q \neq \emptyset$ and observe that if $Q_k \in \mathcal{F}_1$ then $Q_k \subsetneq Q$. Thus,

$$\begin{aligned}\bar\nu(Q) &= \nu(Q \setminus (\cup_{Q_k\in\mathcal{F}} Q_k)) + \sum_{Q_k\in\mathcal{F}_1} \frac{\omega(Q_k \cap Q)}{\omega(Q_k)}\nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F})\\ &= \nu(Q \setminus (\cup_{Q_k\in\mathcal{F}} Q_k)) + \sum_{Q_k\in\mathcal{F}_1} \nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F})\\ &= \nu(\overline{R_Q} \cap \partial\Omega_\mathcal{F}).\end{aligned}$$

Note that $A_{Q'} = (x_{Q'}, \ell(Q')) \in \partial\Omega_{\mathcal{F}}$ since $Q' \in \mathcal{F}$ and also that $\overline{R_Q} \subset R(A_{Q'}, 4\,\ell(Q'))$ which is the $\mathbb{R}^{n+1}$-cube centered at $A_{Q'}$ with sidelength $4\,\ell(Q')$. Thus, we have

$$\begin{aligned}\bar{\nu}(Q) &= \nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}}) \le \nu(R(A_{Q'}, 4\,\ell(Q')) \cap \partial\Omega_{\mathcal{F}}) \lesssim C_\nu\, \nu(R(A_{Q'}, \ell(Q')/2) \cap \partial\Omega_{\mathcal{F}}) \\ &\le C_\nu\, \nu(\overline{R_{Q'}} \cap \partial\Omega_{\mathcal{F}}) = \bar{\nu}(Q'),\end{aligned}$$

where we have used that $\nu = \nu^{X_0}$ is doubling.

Case 3: None of the conditions in the previous cases occur. We take the same set $\mathcal{F}_1$ and observe that if $Q_k \in \mathcal{F}_1$ then $Q_k \subsetneq Q$ (otherwise we are driven to Case 1). Let $\mathcal{F}_2$ be the family of cubes $Q_k \in \mathcal{F}$ with $Q_k \cap Q' \neq \emptyset$. Notice that if $Q_k \in \mathcal{F}_2$ then $Q_k \subsetneq Q'$: otherwise, either $Q_k = Q'$ which leads us to Case 2, or $Q' \subsetneq Q_k$ which implies $Q \subset Q_k$ and this is Case 1. Then proceeding as in the previous case one obtains that $\bar{\nu}(Q) = \nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}})$ and $\bar{\nu}(Q') = \nu(\overline{R_{Q'}} \cap \partial\Omega_{\mathcal{F}})$. Set $Y_{Q'} = (x_{Q'}, t_{Q'})$ such that $Y_{Q'} \in \partial\Omega_{\mathcal{F}}$ (notice that $0 \le t_{Q'} \le \ell(Q')/2$) and observe that $\overline{R_Q} \subset R(Y_{Q'}, 5\,\ell(Q'))$ which is the $\mathbb{R}^{n+1}$-cube centered at $Y_{Q'}$ with sidelength $5\,\ell(Q')$. Then,

$$\begin{aligned}\bar{\nu}(Q) &= \nu(\overline{R_Q} \cap \partial\Omega_{\mathcal{F}}) \le \nu(R(Y_{Q'}, 5\,\ell(Q')) \cap \partial\Omega_{\mathcal{F}}) \lesssim C_\nu\, \nu(R(Y_{Q'}, \ell(Q')/2) \cap \partial\Omega_{\mathcal{F}}) \\ &\le C_\nu\, \nu(\overline{R_{Q'}} \cap \partial\Omega_{\mathcal{F}}) = \bar{\nu}(Q'),\end{aligned}$$

where we have used that $\nu = \nu^{X_0}$ is doubling. This completes the proof for $\bar{\nu}$.

What $\mathcal{P}_{\mathcal{F}}\bar{\nu}$ is dyadically doubling follows from Lemma B.1 in which case the constant would depend on ω and ν. This is not the right approach as we have already observed that $\mathcal{P}_{\mathcal{F}}\bar{\nu}$ does not depend on ω. Following the previous scheme we can see that the doubling constant does not depend on ω: In *Cases 2, 3* we have that $\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q) = \bar{\nu}(Q)$ and $\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q') = \bar{\nu}(Q')$ and the doubling condition follows at once from the previous computations. In Case 1 we obtain

$$\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q) = \frac{|Q|}{|Q_k|}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}) = 2^n\,\frac{|Q'|}{|Q_k|}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_{\mathcal{F}}) = 2^n\,\mathcal{P}_{\mathcal{F}}\bar{\nu}(Q') < \infty.$$

□

Remark B.3. Notice that the doubling constant of $\bar{\nu}$ can be controlled by the maximum of the following quantities:

$$\sup_{Q \subset 3\,Q_0} \frac{\omega^{X_0}(Q)}{\omega^{X_0}(\frac{1}{3}\,Q)}, \qquad \sup_{X,s} \frac{\nu^{X_0}(R(X, 5\,s) \cap \Omega_{\mathcal{F}})}{\nu^{X_0}(R(X, s/2) \cap \Omega_{\mathcal{F}})},$$

where the second sup runs over $X \in \Omega_{\mathcal{F}}$ and $s \le \ell(Q_0)/2$. On the other hand, the doubling constant of $\mathcal{P}_{\mathcal{F}}\bar{\nu}$ can be controlled by 2^n and the second sup right above.

Next we give a version of the classical result in [CF] valid in our dyadic case. The proof of this result follows the standard arguments in [GR] although one has to adapt the ideas to the dyadic and local setting considered here. We give the proof for completeness.

Lemma B.4. *Let Q_0 be a fixed cube and let ω_1, ω_2 be two dyadically doubling measures in Q_0. Assume that there exist positive constants C_0, θ_0 such that for all $Q \in \mathcal{D}(Q_0)$ and $F \subset Q$,*

$$\frac{\omega_2(F)}{\omega_2(Q)} \le C_0 \left(\frac{\omega_1(F)}{\omega_1(Q)}\right)^{\theta_0}. \tag{B.1}$$

Then, there exist positive constants C_1, θ_1 such that for all $Q \in \mathcal{D}(Q_0)$ and $F \subset Q$,

$$\frac{\omega_1(F)}{\omega_1(Q)} \le C_1 \left(\frac{\omega_2(F)}{\omega_2(Q)}\right)^{\theta_1}. \tag{B.2}$$

To prove this result we need a local Calderón-Zygmund decomposition for dyadically doubling weights. The proof is standard and we leave it to the interested reader.

Lemma B.5. *Given Q_0 and ω a dyadically doubling measure in Q_0 with constant C_ω, we consider the local dyadic Hardy-Littlewood maximal function with respect to ω:*

$$\mathcal{M}_\omega f(x) = \sup_{x\in Q\in\mathcal{D}(Q_0)} \frac{1}{\omega(Q)} \int_Q |f(y)|\,d\omega(y).$$

For any $0 \le f \in L^1(Q_0,\omega)$ and $\lambda \ge \frac{1}{\omega(Q_0)} \int_{Q_0} |f(y)|\,d\omega(y)$, there exists a collection of maximal and therefore disjoint dyadic cubes $\{Q_j\}_j \subset \mathcal{D}(Q_0)$ such that

$$\Omega_\lambda = \{x\in Q_0 : \mathcal{M}_\omega f(x) > \lambda\} = \bigcup_j Q_j, \tag{B.3}$$

$$f(x) \le \lambda, \quad \textit{for } \omega\textit{-a.e. } x\notin \Omega_\lambda \tag{B.4}$$

$$\lambda < \frac{1}{\omega(Q_j)} \int_{Q_j} f(y)\,d\omega(y) \le C_\omega\,\lambda. \tag{B.5}$$

Proof of Lemma B.4. Pick $0<\alpha<1$ and $\beta = 1-\big(\frac{1-\alpha}{C_0}\big)^{1/\theta_0}$, and notice that $0<\beta<1$ since $C_0\ge 1$. Then (B.1) applied to $Q\setminus F$ implies that for every $Q\in\mathcal{D}(Q_0)$,

$$F\subset Q, \quad \frac{\omega_2(F)}{\omega_2(Q)} < \alpha \quad \Longrightarrow \quad \frac{\omega_1(F)}{\omega_1(Q)} < \beta. \tag{B.6}$$

We see that this (apparently) weaker condition implies the desired conclusion. Assume momentarily that $\omega_1 \ll \omega_2$. Then the Radon-Nikodym derivative $h = d\omega_1/d\omega_2$ satisfies that $h\in L^1(Q_0,\omega_2)$ and $0\le h(x)<\infty$ for ω_2-a.e. $x\in Q_0$.

Fixed $Q\in\mathcal{D}(Q_0)$ we write $\tau = C_{\omega_2}/\alpha$,

$$\lambda_0 = \frac{1}{\omega_2(Q)}\int_Q h(x)\,d\omega_2(x) = \frac{\omega_1(Q)}{\omega_2(Q)}$$

and $\lambda_k = \tau^k\,\lambda_0$. Notice that $\lambda_0<\lambda_1<\lambda_2<\cdots$ since $\tau > C_{\omega_2}\ge 1$. For every $k\ge 0$ we apply Lemma B.5 in Q to h with dyadically doubling measure ω_2: let $\{Q_j^k\}_j \subset \mathcal{D}(Q)\subset \mathcal{D}(Q_0)$ be the corresponding collection of cubes such that $\Omega_k = \Omega_{\lambda_k} = \bigcup_j Q_j^k$. Fix $Q_{j_0}^k$ and observe that if $Q_{j_0}^k \cap Q_j^{k+1} \ne \emptyset$, then $Q_j^{k+1}\subset Q_{j_0}^k$. Otherwise we would have $Q_{j_0}^k \subsetneq Q_j^{k+1}$; by (B.5) we observe that $\frac{1}{\omega_2(Q_j^{k+1})}\int_{Q_j^{k+1}} h\,d\omega_2 > \lambda_{k+1} > \lambda_k$, and then $Q_{j_0}^k$ would not be maximal. Then using (B.3) and (B.5) we obtain

$$\begin{aligned}\omega_2(Q_{j_0}^k\cap\Omega_{k+1}) &= \sum_{j:Q_j^{k+1}\subset Q_{j_0}^k} \omega_2(Q_j^{k+1}) < \frac{1}{\lambda_{k+1}} \sum_{j:Q_j^{k+1}\subset Q_{j_0}^k} \int_{Q_j^{k+1}} h\,d\omega_2\\ &\le \frac{1}{\lambda_{k+1}}\int_{Q_{j_0}^k} h\,d\omega_2 \le \frac{C_{\omega_2}\lambda_k}{\lambda_{k+1}}\,\omega_2(Q_{j_0}^k) = \alpha\,\omega_2(Q_{j_0}^k).\end{aligned}$$

This estimate allows us to use (B.6) which in turn gives $\omega_1(Q_{j_0}^k\cap\Omega_{k+1}) < \beta\,\omega_1(Q_{j_0}^k)$. Next we sum on j_0 and conclude that $\omega_1(\Omega_{k+1}) < \beta\,\omega_1(\Omega_k)$ since $\Omega_{k+1}\subset\Omega_k$. By iterating this expression we obtain $\omega_1(\Omega_k) < \beta^k\,\omega_1(\Omega_0)$. Similarly, $\omega_2(\Omega_k) < \alpha^k\,\omega_1(\Omega_0)$, which implies

$$\omega_2\Big(\bigcap_k \Omega_k\Big) = \lim_{k\to\infty}\omega_2(\Omega_k) = 0.$$

Let $0<\epsilon<-\log\beta/\log\tau$. Then $0<\tau^\epsilon\,\beta<1$, and by (B.4)

$$\frac{1}{\omega_2(Q)}\int_Q h(x)^{1+\epsilon}\,d\omega_2(x)$$

$$
\begin{aligned}
&= \frac{1}{\omega_2(Q)} \int_{Q\setminus\Omega_0} h(x)^{1+\epsilon}\, d\omega_2(x) + \frac{1}{\omega_2(Q)} \sum_{k=0}^{\infty} \int_{\Omega_k\setminus\Omega_{k+1}} h(x)^{1+\epsilon}\, d\omega_2(x) \\
&\le \lambda_0^\epsilon \frac{1}{\omega_2(Q)} \int_Q h(x)\, d\omega_2(x) + \frac{1}{\omega_2(Q)} \sum_{k=0}^{\infty} \lambda_{k+1}^\epsilon \int_{\Omega_k} h(x)\, d\omega_2(x) \\
&= \lambda_0^\epsilon \frac{\omega_1(Q)}{\omega_2(Q)} + \frac{1}{\omega_2(Q)} \sum_{k=0}^{\infty} \lambda_{k+1}^\epsilon\, \omega_1(\Omega_k) \\
&\le \lambda_0^\epsilon \frac{\omega_1(Q)}{\omega_2(Q)} + \lambda_0^\epsilon \frac{\omega_1(\Omega_0)}{\omega_2(Q)} \sum_{k=0}^{\infty} \tau^{(k+1)\,\epsilon}\, \beta^k \\
&\le \lambda_0^\epsilon \frac{\omega_1(Q)}{\omega_2(Q)} \left(1 + \tau^\epsilon\, (1-\tau^\epsilon\, \beta)^{-1}\right) \\
&= \left(\frac{\omega_1(Q)}{\omega_2(Q)}\right)^{1+\epsilon} C_1^{1+\epsilon}.
\end{aligned}
\tag{B.7}
$$

This estimate implies that for all $F\subset Q$,

$$
\begin{aligned}
\frac{\omega_1(F)}{\omega_2(Q)} &= \frac{1}{\omega_2(Q)} \int_Q \chi_F\; h\, d\omega_2 \le \left(\frac{1}{\omega_2(Q)} \int_Q h^{1+\epsilon}\, d\omega_2\right)^{\frac{1}{1+\epsilon}} \left(\frac{\omega_2(F)}{\omega_2(Q)}\right)^{\frac{1}{(1+\epsilon)'}} \\
&\le \frac{\omega_1(Q)}{\omega_2(Q)}\, C_1 \left(\frac{\omega_2(F)}{\omega_2(Q)}\right)^{\frac{1}{(1+\epsilon)'}},
\end{aligned}
$$

which is (B.2) with $\theta_1 = 1/(1+\epsilon)'$. Notice that ϵ and C_1 depend only on α, β and C_{ω_2}.

Next we see how to proceed in the general case starting from (B.6). We define a new measure $\tilde\omega_2 = \omega_2 + \delta\, \omega_1$ with $\delta>0$. It is clear that $\omega_1 \ll \tilde\omega_2$ and also that $\tilde\omega_2$ is dyadically doubling in Q_0 with constant $C_{\tilde\omega_2} = C_{\omega_1} + C_{\omega_2}$. We claim that setting $\tilde\beta = 1-\min\{1-\beta, \alpha/2\}$, $\tilde\alpha = \alpha/2$ we have for every $Q\in\mathcal{D}(Q_0)$,

$$
F\subset Q, \quad \frac{\tilde\omega_2(F)}{\tilde\omega_2(Q)} < \tilde\alpha \quad\Longrightarrow\quad \frac{\omega_1(F)}{\omega_1(Q)} < \tilde\beta.
\tag{B.8}
$$

Assuming this (B.6) holds for ω_1, $\tilde\omega_2$. By the previous case, since $\omega_1 \ll \tilde\omega_2$, there exist $\tilde\epsilon$, $\tilde C_1$ such that for every $Q\in\mathcal{D}(Q_0)$, $F\subset Q$ we have

$$
\frac{\omega_1(F)}{\omega_1(Q)} \le \tilde C_1 \left(\frac{\tilde\omega_2(F)}{\tilde\omega_2(Q)}\right)^{\frac{1}{(1+\tilde\epsilon)'}}.
$$

As mentioned above $\tilde\epsilon$, $\tilde C_1$ depend only on $\tilde\alpha$, $\tilde\beta$, $C_{\tilde\omega_2}$ and these are ultimately given in terms of α, β, C_{ω_1}, C_{ω_2}. Next we see that $\omega_1 \ll \omega_2$: given $F\subset Q_0$ with $\omega_2(F)=0$, the previous inequality applied to $Q=Q_0$ gives as desired

$$
0 \le \frac{\omega_1(F)}{\omega_1(Q)} \le \tilde C_1 \left(\frac{\delta\, \omega_1(F)}{\tilde\omega_2(Q_0)}\right)^{\frac{1}{(1+\tilde\epsilon)'}} \le \tilde C_1 \left(\delta\, \frac{\omega_1(F)}{\omega_2(Q_0)}\right)^{\frac{1}{(1+\tilde\epsilon)'}} \longrightarrow 0, \qquad \text{as } \delta\to 0^+.
$$

Thus, we get back to the first case and obtain (B.7) which eventually leads to (B.2) with C_1 and θ_1 as stated above.

To complete the proof we obtain (B.8). Given F as there, it follows that $\tilde\omega_2(Q\setminus F)/\tilde\omega_2(Q) > 1-\alpha/2$. We see that $\omega_1(Q\setminus F)/\omega_1(Q) > \min\{1-\beta, \alpha/2\}$, which yields as desired $\omega_1(F)/\omega_1(Q) < \tilde\beta$. If this were not the case then we would have $\omega_1(Q\setminus F)/\omega_1(Q) \le \alpha/2$ and also that $\omega_1(F)/\omega_1(Q)\ge\beta$. By (B.6) the latter gives $\omega_2(F)/\omega_2(Q)\ge\alpha$ and therefore

$\omega_2(Q\setminus F)/\omega_2(Q) \le 1-\alpha$. Gathering these estimates we get a contradiction

$$\frac{\tilde\omega_2(Q\setminus F)}{\tilde\omega_2(Q)} = \frac{\omega_2(Q\setminus F)}{\tilde\omega_2(Q)} + \delta\,\frac{\omega_1(Q\setminus F)}{\tilde\omega_2(Q)} \le \frac{\omega_2(Q\setminus F)}{\omega_2(Q)} + \frac{\omega_1(Q\setminus F)}{\omega_1(Q)} \le 1-\alpha/2.$$

□

Remark B.6. Let us observe that (B.7) can be equivalently written as

$$\left(\frac{1}{\omega_2(Q)}\int_Q h(x)^{1+\epsilon}\,d\omega_2(x)\right)^{\frac{1}{1+\epsilon}} \le C_1\,\frac{1}{\omega_2(Q)}\int_Q h(x)\,d\omega_2(x)$$

and this shows that $h \in RH^{\rm dyadic}_{1+\epsilon}(Q_0,\omega_2)$

Lemma B.7. *Let Q be a cube and let v be a concentrically doubling weight in Q, that is, $0<v<\infty$ a.e. in Q, $v\in L^1(Q)$ and there is $C_0>1$ such that $v(Q')\le C_0\,v(\frac12\,Q')$ for all $Q'\subset Q$. Assume that there exist $C_1\ge 1$ and $1<p<\infty$ such that*

$$\left(\fint_{\frac12\,Q'} v(x)^p\,dx\right)^{\frac1p} \le C_1 \fint_{\frac12\,Q'} v(x)\,dx, \qquad \forall\, Q'\in\mathcal{D}(Q). \tag{B.9}$$

Then $v\in RH^{\rm dyadic}_r(Q)$, that is, there exist $1<r<\infty$ and $C\ge 1$ depending on n,p,C_0,C_1 such that

$$\left(\fint_{Q'} v(x)^r\,dx\right)^{\frac1r} \le C \fint_{Q'} v(x)\,dx, \qquad \forall\, Q'\in\mathcal{D}(Q). \tag{B.10}$$

Furthermore, if v is a doubling weight in $2\,Q$, then (B.10) *holds for every $Q'\subset Q$ (with a different constant), thus $v\in RH_r(Q)$.*

Proof. We first observe that (B.9) and Hölder's inequality imply that for all $Q'\in\mathcal{D}(Q)$ and $F\subset \frac12\,Q'$

$$\frac{v(F)}{v(\frac12\,Q')} \le C_1\left(\frac{|F|}{|\frac12\,Q'|}\right)^{\frac{1}{p'}}. \tag{B.11}$$

We pick $0<\alpha<(C_1^{p'}\,2^n)^{-1}$. Let $E\subset Q'\in\mathcal{D}(Q)$ be such that $|E|/|Q'|>1-\alpha$. Set $E_0=E\cap\frac12\,Q'$ and $F_0=\frac12\,Q'\setminus E$. We observe that

$$(1-\alpha)\,2^n\,|\tfrac12\,Q'| < |E| \le |E_0| + |Q'\setminus \tfrac12\,Q'| = |E_0| + (2^n-1)\,|\tfrac12\,Q'|.$$

Then $|E_0|/|\frac12\,Q'|>1-2^n\,\alpha$ and so $|F_0|/|\frac12\,Q'|<2^n\,\alpha$. We apply (B.11) to conclude that $v(F_0)/v(\frac12\,Q')<C_1\,(2^n\,\alpha)^{\frac{1}{p'}}$ which in turn gives $v(E_0)/v(\frac12\,Q')>1-C_1\,(2^n\,\alpha)^{\frac{1}{p'}}$. This and the fact that v is doubling imply

$$\frac{v(E)}{v(Q')} \ge \frac{v(E_0)}{v(\frac12\,Q')}\,\frac{v(\frac12\,Q')}{v(Q')} > \frac{1-C_1\,(2^n\,\alpha)^{\frac{1}{p'}}}{C_0} = 1-\beta$$

with $0<\beta<1$. We have obtained that there exist $0<\alpha,\beta<1$ such that for every $Q'\in\mathcal{D}(Q)$

$$E\subset Q', \quad \frac{|E|}{|Q'|}>1-\alpha \quad\Longrightarrow\quad \frac{v(E)}{v(Q')}>1-\beta. \tag{B.12}$$

Passing to the complement, this implies (B.6) with $d\omega_1 = v\,dx$ and $d\omega_2=dx$. Then, we can follow the proof of Lemma B.4 (notice that ω_1,ω_2 are dyadically doubling in Q and that $h=v$) to obtain (B.7) which by Remark B.6 is (B.10) with $r=1+\epsilon$.

What (B.10) extends to all cubes $Q'\subset Q$ under doubling is standard, details are left to the interested reader. □

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