A NOTE ON A_∞ ESTIMATES VIA EXTRAPOLATION OF CARLESON MEASURES

STEVE HOFMANN AND JOSÉ MARÍA MARTELL

ABSTRACT. In this note we present a different approach to the A_∞ extrapolation via Carleson measures developed in [HM] and we illustrate the use of this technique by reproving a well known result of [FKP].

1. INTRODUCTION

The extrapolation method for Carleson measures was introduced in [LM] and developed further in [HL], [AHLT], [AHMTT], [HM]. The method is a bootstrapping technique for proving scale invariant estimates on cubes (e.g., reverse Hölder estimates, Carleson measure estimates, BMO estimates), given that (very roughly speaking) the desired estimate holds on those cubes Q for which some controlling Carleson measure μ is sufficiently small in the associated Carleson box R_Q. The exact nature of this control (involving sawtooth subdomains in R_Q) will be made precise later.

In [LM] and [HL] "Carleson $\to A_\infty$" extrapolation was used to obtain reverse Hölder inequalities for some measures associated to PDE which in turn imply solvability of the Dirichlet problem. The Carleson measure condition appears naturally in the quantitative description of the boundary in [LM] and in the control of the coefficients in [HL]. In this latter reference a new proof of the well known result of R. Fefferman, Kenig and Pipher [FKP] is given using the extrapolation method. Roughly speaking, one wants to perturb a given real symmetric second order elliptic operator which is known to be solvable on some Lebesgue space. Assuming that the disagreement between the matrices of the two operators satisfies a Carleson measure condition, the authors show solvability for the perturbed operator on some Lebesgue space L^p with $p < \infty$. We call attention to the fact that the solvability on L^p is equivalent to a reverse Hölder condition for the Poisson kernel (or what is the same, that the harmonic measure is an A_∞ weight with respect to surface measure).

Other extrapolation results appear in [AHLT] and [AHMTT] and involve "Carleson $\to$ Carleson" extrapolation, in which a non-negative measure in the half space $\mathbb{R}^{n+1}_+$ is shown to be a Carleson measure, using properties of another controlling Carleson measure. In [AHLT], the technique was applied to prove the restricted version of the Kato square root conjecture, for divergence form elliptic operators that were small complex perturbations of real symmetric ones. An interesting feature of the "Carleson $\to$ Carleson" extrapolation arguments in [AHLT] and [AHMTT]

is that they were purely real variable in nature —the bootstrapping procedure was separated from the applications to PDE.

A real variable treatment of "Carleson $\rightarrow A_\infty$" extrapolation appears in [HM]. The main result states that in order to show that a given non-negative Borel measure ω satisfies an A_∞ type condition, it suffices to consider cubes for which a controlling Carleson measure is small at all the subscales on some dyadic sawtooth domain, and to verify that the image of ω under a certain projection operator (related to the sawtooth) satisfies an A_∞ condition. This extrapolation result can be used to reprove the main theorem in [FKP]. In doing that, a new version of the "Main Lemma" in [DJK] adapted to discrete sawtooth domains and the projection operators is obtained.

The goal of this note is to give an alternative version, with a different A_∞ type condition, of the main result in [HM]. The class A_∞ can be defined and characterized using different conditions. For instance, $A_\infty = \cup_{p \geq 1} A_p = \cup_{q>1} RH_q$. There are other ways that give quantitative information for the measure induced by the weights in terms of the Lebesgue measure. For instance, if ω is a non-negative Borel regular measure, $\omega \in A_\infty$ if and only if there exist $0 < \alpha, \beta < 1$ such that for every $Q \subset \mathbb{R}^n$

$$E \subset Q, \quad \frac{|E|}{|Q|} > \alpha \quad \implies \quad \frac{\omega(E)}{\omega(Q)} > \beta.$$

One can restrict this condition to subcubes of a given cube Q_0 and this defines $A_\infty(Q_0)$, and consider only dyadic cubes with respect to Q_0 in which case we get $A_\infty^{\mathrm{dyadic}}(Q_0)$ (here one also assumes that ω is dyadically doubling, see below). This A_∞ type condition appears both in the hypotheses (for the projection operator) and also in the conclusion (for the given measure) in the main result in [HM]. In this paper we use yet a different condition for A_∞: $\omega \in A_\infty$ if and only if there exist $0 < \alpha < 1$ and $\beta > 0$ such that for every $Q \subset \mathbb{R}^n$

$$|\{x \in Q : k(x) \leq \beta\, k_Q\}| \leq \alpha\, |Q|,$$

where $k = d\omega/dx$ and k_Q is the average of k on Q. Our extrapolation result (Theorem 2.6) is written in terms of the previous condition (restricted to dyadic cubes of a given cube Q_0, we also allow Q_0 to be $\mathbb{R}^n$) both in the hypotheses (for the projection operator) and also in the conclusion (for ω). As an application of the extrapolation method we modify the new proof of [FKP] given in [HM], in such a way that it can be carried out with this different A_∞ type condition. In passing, we also give some characterizations of the A_∞-dyadic class paying special attention to the dyadically doubling property.

The plan of the paper is as follows. In Section 2 we state our main result. Also, we give the two A_∞-dyadic conditions considered and study the different conditions that equivalently define them. In Section 3 we present the application of our extrapolation method to the perturbation result in [FKP]. We sketch the proof of this application in Section 4 and in Section 5 we prove some results concerning the A_∞-dyadic classes.

2. Main result

2.1. Notation.

- We write $|x - y|_\infty = \max\{|x_i - y_i| : 1 \leq i \leq n\}$.

- We assume that all the cubes are "1/2-open", i.e., they are Cartesian products of intervals closed at the left-hand endpoint, and open on the right. Given a cube $Q \in \mathbb{R}^n$ we denote its center by x_Q and its sidelength by $\ell(Q)$. For any $\tau > 0$ we write τQ for the cube with center x_Q and sidelength $\tau \ell(Q)$. By $\mathcal{D}(Q)$ we
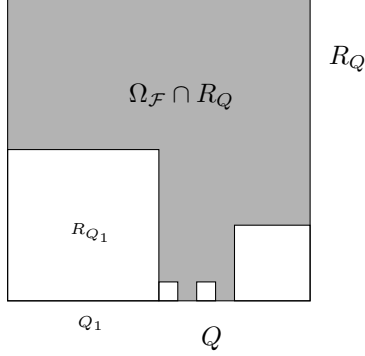
FIGURE 1. Discrete sawtooth $\Omega_\mathcal{F}$

denote the collection of dyadic subcubes[†] of Q and also $\mathcal{D}(Q)^* = \mathcal{D}(Q) \setminus \{Q\}$. We write $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ for the collection of ("classical") dyadic cubes in $\mathbb{R}^n$. We denote by $Q(x,l)$ the cube centered at x with sidelength l.

- Given a cube Q we write $f_Q := \fint_Q f(x)\,dx = \frac{1}{|Q|} \int_Q f(x)\,dx$ for any $f \in L^1(Q)$. Analogously, if ω is a non-negative Borel measure we write $\omega_Q = \omega(Q)/|Q|$. Also, we set $\fint_Q f(x)\,d\nu(x) := \frac{1}{\nu(Q)} \int_Q f(x)\,d\nu(x)$.

- Let Q be a cube. We denote the associated Carleson box by $R_Q := Q \times (0, \ell(Q))$.

- We write $\mathcal{C}$ for the set of Carleson measures in $\mathbb{R}_+^{n+1}$, i.e., the non-negative Borel measures μ on $\mathbb{R}_+^{n+1}$ for which the "Carleson norm"

$$\|\mu\|_\mathcal{C} := \sup_{Q \subset \mathbb{R}^n} |Q|^{-1} \mu(R_Q) \qquad (2.1)$$

is finite; here, the supremum runs over all cubes $Q \subset \mathbb{R}^n$. Analogously, given $Q_0 \subset \mathbb{R}^n$ we write $\mathcal{C}(Q_0)$ for the set of Borel measures that satisfy the previous condition restricted to $Q \in \mathcal{D}(Q_0)$, thus

$$\|\mu\|_{\mathcal{C}(Q_0)} := \sup_{Q \in \mathcal{D}(Q_0)} |Q|^{-1} \mu(R_Q).$$

By slight abuse of notation[‡], if $Q_0 = \mathbb{R}^n$ we simply write $\mathcal{C} = \mathcal{C}(Q_0)$.

- Given Q and a family of pairwise disjoint dyadic subcubes $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ we define the discrete sawtooth function $\psi_\mathcal{F}(x) := \sum_k \ell(Q_k)\, \chi_{Q_k}(x)$. Notice that ψ is a step function supported in $\cup_k Q_k$. We write $\Omega_\mathcal{F} = \Omega_{\psi_\mathcal{F}}$ for the domain above the graph of $\psi_\mathcal{F}$, that is, $\Omega_\mathcal{F} := \{(x,t) \in \mathbb{R}_+^{n+1} : t \geq \psi_\mathcal{F}(x)\}$. Notice that $\Omega_\mathcal{F} = \mathbb{R}_+^{n+1} \setminus (\cup_k R_{Q_k})$. We allow $\mathcal{F}$ to be empty in which case $\psi_\mathcal{F}(x) = 0$ and $\Omega_\mathcal{F} = \mathbb{R}_+^{n+1}$. See Figure 1.

- If μ is a non-negative Borel measure on $\mathbb{R}_+^{n+1}$, then $\mu_\mathcal{F} := \mu\,\chi_{\Omega_\mathcal{F}}$ will denote its restriction to the dyadic sawtooth $\Omega_\mathcal{F}$.

- Given Q and $\mathcal{F}$ as before, we define the projection operator

$$\mathcal{P}_\mathcal{F} f(x) := f(x)\,\chi_{\mathbb{R}^n \setminus (\cup_k Q_k)}(x) + \sum_k \left(\fint_{Q_k} f(y)\,dy \right) \chi_{Q_k}(x).$$

[†]Note that the term "dyadic" here refers to the grid induced by Q; the cubes in $\mathcal{D}(Q)$ are dyadic cubes of $\mathbb{R}^n$ if and only if Q itself is such.

[‡]Indeed, the abuse is very slight, since one may cover an arbitrary cube Q by a purely dimensional number of dyadic cubes of comparable size, to show that (2.1) is controlled by the analogous supremum taken only over dyadic cubes.

One has that $\mathcal{P}_{\mathcal{F}} \circ \mathcal{P}_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}$, $\mathcal{P}_{\mathcal{F}}$ is selfadjoint and $\|\mathcal{P}_{\mathcal{F}} f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$ for every $1 \leq p \leq \infty$. Observe that if ω is a non-negative Borel measure and $E \subset Q$, then we may naturally define the measure $\mathcal{P}_{\mathcal{F}} \omega$ as follows:

$$\mathcal{P}_{\mathcal{F}} \omega(E) := \int \mathcal{P}_{\mathcal{F}}(\chi_E) \, d\omega = \omega(E \setminus \cup_k Q_k) + \sum_k \omega_{Q_k} |E \cap Q_k|.$$

In particular, $\mathcal{P}_{\mathcal{F}} \omega(Q) = \omega(Q)$. If $\omega \ll dx$ and we write $k = d\omega/dx$ for its Radon-Nikodym derivative it follows that $\mathcal{P}_{\mathcal{F}} \omega \ll dx$ and $d(\mathcal{P}_{\mathcal{F}} \omega)/dx = \mathcal{P}_{\mathcal{F}} k$.

- Given Q and $\mathcal{F}$ as before, we introduce a new family $\mathcal{F}'$ consisting of all the dyadic "children" of the cubes in $\mathcal{F}$. Notice that $\mathcal{F}'$ is a family of pairwise disjoint cubes in $\mathcal{D}(Q)$, therefore we define $\mathcal{P}'_{\mathcal{F}} := \mathcal{P}_{\mathcal{F}'}$, which is the projection operator associated with the family $\mathcal{F}'$, and it satisfies the previous properties. We observe that if ω is a non-negative Borel measure and $E \subset Q$, then $\mathcal{P}'_{\mathcal{F}} \omega(E) \leq 2^n \mathcal{P}_{\mathcal{F}} \omega(E)$. The converse inequality does not hold in general, however if one assumes that ω is dyadically doubling in Q (see the definition below) then $\mathcal{P}'_{\mathcal{F}} \omega(E) \approx \mathcal{P}_{\mathcal{F}} \omega(E)$; thus it seems more natural to use $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$.

2.2. A_∞^{dyadic} **weights.** We introduce two different A_∞^{dyadic} conditions and give characterizations of them. Under doubling the results that we present here are classical (see [CF], [GR]). For the sake of completeness and since we want to pay special attention to the non-doubling case we include the proofs (that follow the classical ideas as well) in Section 5. In what follows all the measures are assumed to be non-negative, regular and Borel. For such a measure ω, we also assume that $0 < \omega(Q) < \infty$ for all $Q \in \mathcal{D}(Q_0)$ with Q_0 being either a fixed cube or $\mathbb{R}^n$.

Definition 2.1. Let Q_0 be either $\mathbb{R}^n$ or a fixed cube and let ω, ν be two non-negative regular Borel measures on Q_0. Assume that ν is "dyadically doubling", that is, $\nu(Q) \leq C_\nu (Q')$, for every $Q \in \mathcal{D}(Q_0)$, and for every dyadic "child" Q' of Q.

- We say that $\omega \preceq \nu$ if there exist $0 < \alpha, \beta < 1$ such that for every $Q \in \mathcal{D}(Q_0)$ we have

$$E \subset Q, \quad \frac{\nu(E)}{\nu(Q)} < \alpha \quad \Longrightarrow \quad \frac{\omega(E)}{\omega(Q)} < \beta. \tag{2.2}$$

- We say that $\omega \in A_\infty^{\text{dyadic},*}(Q_0, \nu)$ if $\omega \preceq \nu$.

- We say that $\omega \in A_\infty^{\text{dyadic}}(Q_0, \nu)$ if ω is dyadically doubling and $\omega \preceq \nu$.

When $\nu = dx$ (which is dyadically doubling for any dyadic grid), we simply write $A_\infty^{\text{dyadic}}(Q_0)$ or $A_\infty^{\text{dyadic},*}(Q_0)$

Proposition 2.2. *Let Q_0 be either $\mathbb{R}^n$ or a fixed cube, and let ω, ν be a non-negative regular Borel measures on Q_0. Assume that ν is dyadically doubling. The following statements are equivalent:*

(a) *$\omega \in A_\infty^{\text{dyadic},*}(Q_0, \nu)$, that is, $\omega \preceq \nu$.*

(b) *There exist $0 < \theta < \infty$ and $1 \leq C_0 < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$ and for all Borel sets $E \subset Q$ we have*

$$\frac{\omega(E)}{\omega(Q)} \leq C_0 \left(\frac{\nu(E)}{\nu(Q)} \right)^\theta.$$

(c) *$\omega \ll \nu$ and if we write $k_\omega = d\omega/d\nu$ for its Radon-Nikodym derivative, we have that there exist $0 < \alpha < 1$ and $0 < \beta < \infty$ such that for all $Q \in \mathcal{D}(Q_0)$*

$$\nu\{x \in Q : k_\omega(x) \leq \beta \textstyle\fint_Q k_\omega \, d\nu\} \leq \alpha \, \nu(Q).$$

(d) $\omega \ll \nu$ and if we write $k_\omega = d\omega/d\nu$ for its Radon-Nikodym derivative, there exist $0 < \beta, C_1 < \infty$ such that for all $Q \in \mathcal{D}(Q_0)$ and all $\lambda > \fint_Q k_\omega \, d\nu$

$$\omega\{x \in Q : k_\omega(x) \geq \lambda\} \leq C_1 \, \lambda \, \nu\{x \in Q : k_\omega(x) > \beta \, \lambda\}.$$

(e) $\omega \ll \nu$ and if we write $k_\omega = d\omega/d\nu$ for its Radon-Nikodym derivative, there exists $0 < \delta < \infty$ such that $k \in RH_{1+\delta}^{\mathrm{dyadic}}(Q_0, \nu)$, that is, there is $1 \leq C_2 < \infty$ such that for all $Q \in \mathcal{D}(Q_0)$

$$\left(\fint_Q k_\omega(x)^{1+\delta} \, d\nu(x) \right)^{\frac{1}{1+\delta}} \leq C_2 \fint_Q k_\omega(x) \, d\nu(x).$$

Remark 2.3. Let us observe that the fact that $\omega \ll \nu$ is only assumed in (c), (d) and (e): one needs this property to state the corresponding conditions. Notice that (b) easily implies that $\omega \ll \nu$. In the proof, we see that (a) (that is, $\omega \preceq \nu$) also yields the absolute continuity.

Proposition 2.4. *Let Q_0 be either $\mathbb{R}^n$ or a fixed cube. Let ω and ν be a non-negative regular Borel measures.*

(i) *If both ω and ν are dyadically doubling, then $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0, \nu)$, if and only if, $\nu \in A_\infty^{\mathrm{dyadic}}(Q_0, \omega)$.*

(ii) *$A_\infty^{\mathrm{dyadic}}(Q_0, \cdot)$ defines an equivalence relationship on the set of dyadically doubling measures.*

Remark 2.5. Notice that the set of $A_\infty^{\mathrm{dyadic},*}(Q_0, \nu)$ measures that are dyadically doubling coincides with $A_\infty^{\mathrm{dyadic}}(Q_0, \nu)$, and therefore statements (b)–(e) characterize $A_\infty^{\mathrm{dyadic}}(Q_0, \nu)$ (in the presence of a dyadic doubling hypothesis). Also, by (i) it follows that if both measures ω and ν are dyadically doubling then in any of the properties (a)–(e) in Proposition 2.2 one can switch ω and ν. In particular, if $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0, \nu)$ there exist $0 < \theta, \theta' < \infty$ and $1 \leq C_0 < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$ and for all Borel sets $E \subset Q$ we have

$$C_0^{-1} \left(\frac{\nu(E)}{\nu(Q)} \right)^{\theta'} \leq \frac{\omega(E)}{\omega(Q)} \leq C_0 \left(\frac{\nu(E)}{\nu(Q)} \right)^{\theta}.$$

2.3. A_∞ estimates via extrapolation of Carleson measures.

Theorem 2.6. *Let Q_0 be either $\mathbb{R}^n$ or a fixed cube. Given $M_0 > 0$, let $\mu \in \mathcal{C}(Q_0)$ with*

$$\|\mu\|_{\mathcal{C}(Q_0)} \leq M_0$$

and let ω be a non-negative Borel measure in Q_0. Assume that $\omega \ll dx$ and write $k = d\omega/dx$ for its Radon-Nikodym derivative. Suppose that there exists $\delta > 0$ such that for every $Q \in \mathcal{D}(Q_0)$ and every family of pairwise disjoint dyadic subcubes $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ verifying

$$\|\mu_\mathcal{F}\|_{\mathcal{C}(Q)} := \sup_{Q' \in \mathcal{D}(Q)} \frac{\mu(R_{Q'} \cap \Omega_\mathcal{F})}{|Q'|} \leq \delta, \tag{2.3}$$

we have that $\mathcal{P}'_\mathcal{F} \omega$ satisfies the following property: for all $0 < \alpha < 1$ there exists $\beta > 0$ such that

$$\left| \{x \in Q : \mathcal{P}'_\mathcal{F} k(x) \leq \beta \, (\mathcal{P}'_\mathcal{F} \omega)_Q\} \right| = \left| \{x \in Q : \mathcal{P}'_\mathcal{F} k(x) \leq \beta \, \omega_Q\} \right| \leq \alpha \, |Q|. \tag{2.4}$$

Then, there exist $0 < \alpha_0 < 1$ and $\beta_0 > 0$ such that for every cube $Q \in \mathcal{D}(Q_0)$

$$\left| \{x \in Q : k(x) \leq \beta_0 \, \omega_Q\} \right| \leq \alpha_0 \, |Q|. \tag{2.5}$$

Consequently $\omega \in A_\infty^{\mathrm{dyadic},*}(Q_0)$.

Remark 2.7. This result should be compare with the main theorem in [HM] where it is not assumed that $\omega \ll dx$ and the $A_\infty^{\mathrm{dyadic}}$ type conditions (2.4) and (2.5) are given in terms of (2.2) —indeed, the equivalent conditions with "$\geq$" in place of "$<$".

Remark 2.8. The key hypothesis of the theorem, and the main point that must be verified in applications, is that (2.3) implies (2.4), for sufficiently small δ.

Remark 2.9. We note that the implication (2.3) $\implies$ (2.4) is equivalent to the apparently stronger statement that (2.3) $\implies \mathcal{P}_\mathcal{F}\omega \in A_\infty^{\mathrm{dyadic},*}(Q)$. Indeed, for every $Q' \in \mathcal{D}(Q)$, we have that $\|\mu_\mathcal{F}\|_{\mathcal{C}(Q')} \leq \|\mu_\mathcal{F}\|_{\mathcal{C}(Q)} \leq \delta$, whence the implication (2.3) $\implies$ (2.4) holds also for all such Q' in place of Q. In turn, the fact that (2.4) holds for all $Q' \in \mathcal{D}(Q)$ says precisely that $\mathcal{P}_\mathcal{F}\omega \in A_\infty^{\mathrm{dyadic},*}(Q)$. We also notice that if ω is dyadically doubling in Q_0, then $\mathcal{P}_\mathcal{F}\omega \approx \mathcal{P}_\mathcal{F}'\omega$ and therefore it suffices to work with the "simpler" projection operator $\mathcal{P}_\mathcal{F}$. In such a case the conclusion is $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$.

Remark 2.10. One can give an analog of Theorem 2.6 adapted to tents in place of boxes, that is, in (2.3) one can replace $R_{Q'} \cap \Omega_\mathcal{F}$ by $T_{Q'} \cap \widetilde{\Omega}_\mathcal{F}$ where $T_{Q'}$ is the Carleson tent associated to Q' and $\widetilde{\Omega}_\mathcal{F}$ is the domain above the (regular) sawtooth region which is formed by the union of the cones with a fixed aperture and vertices in $\mathbb{R}_+^{n+1} \setminus \cup_k Q_k$. The proof is almost identical, we only need to apply the original [AHLT, Lemma 3.4] in place of our alternative version contained in Lemma 2.13.

Remark 2.11. The extrapolation theorem is written in such a way that it contains both a global and a local version. We note also the following observations:

- When $Q_0 = \mathbb{R}^n$, if ω is concentrically doubling, then the conclusion of the theorem improves immediately to $\omega \in A_\infty$ (see the precise definition in Section 3.1).

- For the local case, if ω is concentrically doubling, then the conclusion $\omega \in A_\infty^{\mathrm{dyadic},*}(Q_0)$ yields also that $\omega \in A_\infty(\frac{1}{2}Q_0)$ (see the precise definition in Section 3.1).

Remark 2.12. We notice that in the hypotheses of Theorem 2.6 the attention is restricted to $Q \in \mathcal{D}(Q_0)$ and thus the conclusion (2.5) holds for all $Q \in \mathcal{D}(Q_0)$. If in our hypotheses we consider all cubes $Q \subset Q_0$ then (2.5) holds for all $Q \subset Q_0$. This implies both ω doubling and $\omega \in A_\infty(Q_0)$. For the proof it suffices to change the induction hypotheses (cf. "$H(a)$" below) and consider all cubes $Q \subset Q_0$.

2.4. **Proof of Theorem 2.6.** As mentioned in the introduction, the proof is a modification of the argument in [HM] which in turn follows the strategy introduced in [LM], and developed further in [HL], [AHLT] and [AHMTT]. The proof uses an induction argument with continuous parameter. The induction hypothesis is the following: given $a \geq 0$,

$$\boxed{H(a)} \quad \boxed{\begin{array}{l} \text{There exist } \alpha_a \in (0,1) \text{ and } \beta_a > 0 \text{ such that for every } Q \in \\ \mathcal{D}(Q_0) \text{ satisfying } \mu(R_Q) \leq a\,|Q|, \text{ it follows that} \\ \qquad \left|\left\{x \in Q : k(x) \leq \beta_a\,\omega_Q\right\}\right| \leq \alpha_a\,|Q|. \end{array}}$$

The induction argument is split in two steps.

Step 1. Show that $H(0)$ holds.

Step 2. Show that there exists $b = b(n,\delta)$ such that for all $0 \leq a \leq M_0$, $H(a)$ implies $H(a + b)$.

Once these steps have been carried out, the proof follows easily: pick $k \geq 1$ such that $(k-1)\,b < M_0 \leq k\,b$ (note that k only depends on $b(n,\delta)$ and M_0). By **Step 1** and **Step 2**, it follows that $H(k\,b)$ holds. Observe that $\|\mu\|_{\mathcal{C}(Q_0)} \leq M_0 \leq k\,b$ implies $\mu(R_Q) \leq k\,b\,|Q|$ for all $Q \subset Q_0$, and by $H(k\,b)$ we conclude (2.5).

Step 1. $H(0)$ **holds.** If $\mu(R_Q) = 0$ then we take $\mathcal{F}$ to be empty, so that $R_Q \cap \Omega_{\mathcal{F}} = R_Q$, and $\mathcal{P}'_{\mathcal{F}} \omega = \omega$. Then (2.3) holds (since $0 \leq \delta$) and therefore we can use (2.4) with ω and k in place of $\mathcal{P}'_{\mathcal{F}} \omega$ and $\mathcal{P}'_{\mathcal{F}} k$, which is the desired property.

Step 2. $H(a)$ **implies** $H(a + b)$**.** We will require the following Lemma from [HM] (and we refer the reader to that paper for the proof). An earlier variant appeared in [AHLT, Lemma 3.4], in the case of regular sawtooth regions (see also [AHMTT]). Let R_Q^{short} denote the "short" Carleson box $Q \times (0, \ell(Q)/2)$.

Lemma 2.13. *Let μ be a non-negative measure on $\mathbb{R}^{n+1}_+$, and let $a \geq 0$, $b > 0$. Fix a cube Q such that $\mu(R_Q) \leq (a + b)\,|Q|$. Then there exists a family $\mathcal{F} = \{Q_k\}_k$ of non-overlapping dyadic subcubes of Q such that*

$$\|\mu_{\mathcal{F}}\|_{\mathcal{C}(Q)} := \sup_{Q' \in \mathcal{D}(Q)} \frac{\mu(R_{Q'} \cap \Omega_{\mathcal{F}})}{|Q'|} \leq 2^{n+2}\,b, \qquad |B| \leq \frac{a + b}{a + 2\,b}\,|Q|, \quad (2.6)$$

where B is the union of those Q_k verifying $\mu(R_{Q_k}^{\mathrm{short}}) > a\,|Q_k|$.

Taking this lemma for granted, we return to the proof of Step 2. Fix $0 \leq a \leq M_0$ and $Q \in \mathcal{D}(Q_0)$ such that $\mu(R_Q) \leq (a+b)\,|Q|$, where we choose b so that $2^{n+2}\,b := \delta$. We may now apply the previous lemma to construct the non-overlapping family of cubes $\mathcal{F}$ with the stated properties. Set

$$A = Q \setminus \bigcup_{Q_k \in \mathcal{F}} Q_k, \qquad G = \bigcup_{Q_k \in \mathcal{F}_{\mathrm{good}}} Q_k, \qquad B = \bigcup_{Q_k \in \mathcal{F} \setminus \mathcal{F}_{\mathrm{good}}} Q_k,$$

where $\mathcal{F}_{\mathrm{good}} = \{Q_k \in \mathcal{F} : \mu(R_{Q_k}^{\mathrm{short}}) \leq a\,|Q_k|\}$. Set $1 - \theta_0 := (M_0 + b)/(M_0 + 2\,b)$ and then $|B| \leq (1 - \theta_0)\,|Q|$ by (2.6) and since $a \leq M_0$. Thus, $|A \cup G| \geq \theta_0\,|Q|$.

Given $Q_k \in \mathcal{F}_{\mathrm{good}}$ we have that $\mu(R_{Q_k}^{\mathrm{short}}) \leq a\,|Q_k|$. Moreover,

$$R_{Q_k}^{\mathrm{short}} = \bigcup_{j=1}^{2^n} R_{Q_k^j}, \qquad Q_k^j \in \mathcal{D}(Q_k) \quad \text{with} \quad Q_k = \bigcup_{j=1}^{2^n} Q_k^j, \quad \ell(Q_k^j) = \ell(Q_k)/2;$$

that is, the union runs over the dyadic "children" of Q_k. Then by pigeon-holing, there exists at least one j_0 such that $Q_k^{j_0} =: Q'_k$ satisfies

$$\mu(R_{Q'_k}) \leq a\,|Q'_k| \qquad (2.7)$$

(there could be more than one j_0 with this property, but we just pick one). We write $\tilde{\mathcal{F}}_{\mathrm{good}}$ for the collection of those selected "children" Q'_k, with $Q_k \in \mathcal{F}_{\mathrm{good}}$, and $\tilde{G} = \cup_{Q'_k \in \tilde{\mathcal{F}}_{\mathrm{good}}} Q'_k$. Then, it follows that

$$|A \cup \tilde{G}| = |A| + |\tilde{G}| = |A| + 2^{-n}\,|G| \geq 2^{-n}\,|A \cup G| \geq 2^{-n}\,\theta_0\,|Q|.$$

By (2.6), we may deduce that (2.3) follows, so in turn, by hypothesis, for $0 < \alpha < 1$ to be chosen, there exists $\beta > 0$ such that (2.4) holds. Let us define

$$\mathcal{F}_1 = \{Q'_k \in \tilde{\mathcal{F}}_{\mathrm{good}} : \omega_{Q'_k} \leq \beta\,\omega_Q\}, \qquad G_1 = \cup_{Q'_k \in \mathcal{F}_1} Q'_k.$$

Let $0 < \beta_0 < \beta \min\{1, \beta_a\}$ (β_a is given by $H(a)$) and set $E_{\beta_0} = \{x \in Q : k(x) \leq \beta_0\,\omega_Q\}$. By (2.7) we can use $H(a)$ for every Q'_k and then

$$\begin{aligned}
|E_{\beta_0} \cap (\tilde{G} \setminus G_1)| &= \sum_{Q'_k \in \tilde{\mathcal{F}}_{\mathrm{good}} \setminus \mathcal{F}_1} \left|\{x \in Q'_k : k(x) \leq \beta_0\,\omega_Q\}\right| \\
&\leq \sum_{Q'_k \in \tilde{\mathcal{F}}_{\mathrm{good}} \setminus \mathcal{F}_1} \left|\{x \in Q'_k : k(x) \leq \beta_a\,\omega_{Q'_k}\}\right| \\
&\leq \alpha_a \sum_{Q'_k \in \tilde{\mathcal{F}}_{\mathrm{good}} \setminus \mathcal{F}_1} |Q'_k| \leq \alpha_a\,|A \cup \tilde{G}|.
\end{aligned}$$

On the other hand, by the definition of $\mathcal{P}'_{\mathcal{F}}$ it follows that

$$
\begin{aligned}
|G_1| &= \sum_{Q'_k \in \mathcal{F}_1} |Q'_k| \\
&= \sum_{Q'_k \in \mathcal{F}_1} \left| \{ x \in Q'_k : \omega_{Q'_k} \le \beta\, \omega_Q \} \right| \\
&= \left| \{ x \in \tilde{G} : \mathcal{P}'_{\mathcal{F}}\, k(x) \le \beta\, \mathcal{P}'_{\mathcal{F}}\, \omega_Q \} \right|.
\end{aligned}
$$

and also that

$$
|E_{\beta_0} \cap A| \le \left| \{ x \in A : k(x) \le \beta\, \omega_Q \} \right| = \left| \{ x \in A : \mathcal{P}'_{\mathcal{F}}\, k(x) \le \beta\, \mathcal{P}'_{\mathcal{F}}\, \omega_Q \} \right|.
$$

Then, (2.4) yields

$$
\begin{aligned}
|E_{\beta_0} \cap (A \cup \tilde{G})| &\le |E_{\beta_0} \cap A| + |G_1| + |E_{\beta_0} \cap (\tilde{G} \setminus G_1)| \\
&\le \left| \{ x \in Q : \mathcal{P}'_{\mathcal{F}}\, k(x) \le \beta\, \mathcal{P}'_{\mathcal{F}}\, \omega_Q \} \right| + \alpha_a\, |A \cup \tilde{G}| \\
&\le \alpha\, |Q| + \alpha_a\, |A \cup \tilde{G}|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|E_{\beta_0}| &\le |E_{\beta_0} \cap (A \cup \tilde{G})| + |Q \setminus (A \cup \tilde{G})| \le (\alpha + 1)\, |Q| - (1 - \alpha_a)\, |A \cup \tilde{G}| \\
&\le (\alpha + 1 - 2^{-n}\, \theta_0\, (1 - \alpha_a))\, |Q| =: \alpha_0\, |Q|.
\end{aligned}
$$

To complete the proof it suffices to take $0 < \alpha < 2^{-n}\, \theta_0\, (1 - \alpha_a)$ and this guarantees that $0 < \alpha_0 < 1$. $\qquad\square$

Remark 2.14. As mentioned above, if ω is dyadically doubling one can equivalently work with $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$. Indeed, the proof just presented can be easily adapted to that projection operator with the following modifications: The new collection $\mathcal{F}_1$ consists of those $Q'_k \in \tilde{\mathcal{F}}_{\text{good}}$ so that $\omega_{Q_k} \le \beta\, \omega_Q$. That ω is dyadically doubling implies $\omega_{Q_k} \le C_\omega\, 2^{-n}\, \omega_{Q'_k}$. Using this, one obtains that $|E_{\beta_0} \cap (\tilde{G} \setminus G_1)| \le \alpha_a\, |A \cup \tilde{G}|$ provided $0 < \beta_0 < \beta\, C_\omega^{-1}\, 2^n\, \beta_a$. On the other hand, one easily estimates $|G_1|$ and $|E_{\beta_0} \cap A|$ taking into account the definitions of $\mathcal{F}_1$ and $\mathcal{P}_{\mathcal{F}}$.

3. Application to second order elliptic boundary value problems

3.1. **Additional Notation.**

- Given $X \in \mathbb{R}^{n+1}_+$ we write $X = (x, \varrho(X))$, that is, $\varrho(X) = \operatorname{dist}(X, \partial\mathbb{R}^{n+1}_+)$.

- For any $X, Y \in \mathbb{R}^{n+1}_+$, we write $|X - Y|_\infty = \max\{|x - y|_\infty, |\varrho(X) - \varrho(Y)|\}$, notice that this is the ℓ^∞-distance in $\mathbb{R}^{n+1}_+$. In this way, for any $X \in \mathbb{R}^{n+1}_+$ and $0 < r \le 2\, \varrho(X)$, we write $R(X, r) = \{Y \in \mathbb{R}^{n+1}_+ : |Y - X|_\infty < r/2\}$ which is the cube in $\mathbb{R}^{n+1}_+$ with center X and sidelength r (that is, radius $r/2$).

- If R is a cube in $\mathbb{R}^{n+1}_+$, we denote its center by X_R and its sidelength by $\ell(R)$ such that $R = R(X_R, \ell(R))$. Notice that $R \subset \mathbb{R}^{n+1}_+$ yields $\ell(R) \le 2\, \varrho(X_R)$. Given τ we denote by τR the τ-dilation of R, that is, the cube with center X_R and with sidelength $\tau\, \ell(R)$.

- Given a cube $Q \subset \mathbb{R}^n$ we set $X_Q = (x_Q, 4\, \ell(Q))$ and $A_Q = (x_Q, \ell(Q))$.

- A weight w is a non-negative locally integrable function. A weight induces a Borel measure as follows: for any measurable set E we write $w(E) := \int_E w(x)\, dx$.

- Given a weight w and $1 < p < \infty$ we say that $w \in RH_p$ if there exists a constant C_p such that for every Q

$$
\left(\fint_Q w(x)^p\, dx \right)^{\frac{1}{p}} \le C_p \fint_Q w(x)\, dx.
$$

Given a cube Q_0, if the previous condition holds for any cube $Q \subset Q_0$ we write $w \in RH_p(Q_0)$.

- Let A_∞ be the set of Muckenhoupt weights in $\mathbb{R}^n$. That is, given ω a non-negative Borel measure on $\mathbb{R}^n$ we say that $\omega \in A_\infty$ if there exist $0 < \alpha, \beta < 1$ such that for every cube Q and for every measurable set $E \subset Q$ we have

$$\frac{|E|}{|Q|} < \alpha \quad \Longrightarrow \quad \frac{\omega(E)}{\omega(Q)} < \beta.$$

It is easy to see that this yields that ω is doubling —one estimates $\omega(\lambda Q \setminus Q)/\omega(\lambda Q)$ for λ sufficiently close to 1 and then iterates. This condition implies that ω is absolutely continuous with respect to the Lebesgue measure and that its Radon-Nikodym derivative $k = d\omega/dx$ (which is a weight) satisfies $k \in RH_p$, see Proposition 2.2 or [GR, Chapter 4] for details. Indeed one can alternatively define A_∞ as the class of non-negative Borel measures absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivatives in $\cup_q RH_q$. Also, A_∞ can be defined in terms of (2.2) or the corresponding conditions in Proposition 2.2.

3.2. **Introduction.** We work with real symmetric second order elliptic operators: $Lf(X) = -\operatorname{div}(A(X)\nabla f(X))$, $X \in \mathbb{R}_+^{n+1}$, with $A(X) = (a_{i,j}(X))_{1 \leq i,j \leq n+1}$ being a real, symmetric $(n+1) \times (n+1)$ matrix such that $a_{i,j} \in L^\infty(\mathbb{R}_+^{n+1})$ for $1 \leq i,j \leq n+1$, and A is uniformly elliptic, that is, there exists $0 < \lambda \leq 1$ such that

$$\lambda |\xi|^2 \leq A(X)\xi \cdot \xi \leq \lambda^{-1} |\xi|^2,$$

for all $\xi \in \mathbb{R}^{n+1}$ and almost every $X \in \mathbb{R}_+^{n+1}$.

Some of the material below is taken from [Ken, Chapter 1], the reader might find convenient to have this reference handy.

The solutions of the Dirichlet problem are represented by the harmonic measure. Namely, there exists a family of regular Borel probability measures $\{\omega_L^X\}_{X \in \mathbb{R}_+^{n+1}}$ in $\mathbb{R}^n$ such that for every $f \in C_0(\mathbb{R}^n)$, the function

$$u(X) = \int_{\mathbb{R}^n} f(y) \, d\omega_L^X(y)$$

is a classical solution of the Dirichlet problem

$$\begin{cases} Lu = 0 \text{ in } \mathbb{R}_+^{n+1} \\ \\ u\big|_{\mathbb{R}^n} = f \end{cases} \tag{3.1}$$

This family $\{\omega_L^X\}_{X \in \mathbb{R}_+^{n+1}}$ is called the L-harmonic measure. Sometimes, we will drop the subindex L. For a fixed $X_0 \in \mathbb{R}_+^{n+1}$ we let $\omega = \omega^{X_0}$ and abusing notation ω is called the harmonic measure.

If $\omega_L^X \ll dx$, we write the Poisson kernel as k_L^X, that is, $k_L^X = d\omega_L^X/dx$ is the Radon-Nikodym derivative of ω_L^X with respect to dx. Again for a fixed $X_0 \in \mathbb{R}_+^{n+1}$ we let $k = k_L^{X_0}$ and k is called the Poisson kernel (notice that for every $X \in \mathbb{R}_+^{n+1}$, ω^X and ω are mutually absolutely continuous).

We recall the fundamental relationship between solvability of the Dirichlet problem with L^p data, and higher integrability of the Poisson kernel, essentially as stated in [Ken, Theorem 1.7.3].

Theorem 3.1. *Given an operator L as above and $1 < p < \infty$, the following statements are equivalent:*

(a) *If $u \in C_0(\mathbb{R}_+^{n+1})$ is a classical solution of the Dirichlet problem (3.1) with data $f \in C_0(\mathbb{R}^n)$ then*

$$\|u^*\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^{p'}(\mathbb{R}^n)}, \tag{3.2}$$

where $u^*(x) = \sup_{Y \in \Gamma_\eta(x)} |u(Y)|$ with $\Gamma_\eta(x) = \{Y \in \mathbb{R}^{n+1}_+ : |x - y|_\infty < \eta \varrho(Y)\}$, $\eta > 0$.

(b) *$\omega \in RH_p$; by this we mean that $\omega \ll dx$ and for each cube $Q \subset \mathbb{R}^n$, we have that the Poisson kernel satisfies $k^{X_Q} \in RH_p(Q)$, uniformly in Q.[‡‡] That is, there exists a uniform constant C_0 such that for all $Q \subset \mathbb{R}^n$,*

$$\left(\fint_{Q'} k^{X_Q}(y)^p \, dy \right)^{1/p} \le C_0 \fint_{Q'} k^{X_Q}(y) \, dy, \qquad \forall Q' \subset Q. \tag{3.3}$$

(c) *$\omega \ll dx$, and there is a uniform constant C_0 such that for every Q in $\mathbb{R}^n$, we have the scale invariant L^p estimate*

$$\int_Q k^{X_Q}(y)^p \, dy \le C_0 |Q|^{1-p}. \tag{3.4}$$

When (a) occurs we say that $(D)_{p'}$ is solvable for L or that L is solvable in $L^{p'}$. In such case, for every $f \in L^{p'}(\mathbb{R}^n)$ there exists a unique u such that $Lu = 0$ in $\mathbb{R}^{n+1}_+$, (3.2) holds and u converges non-tangentially to f a.e..

Given two operators L_0 and L as above with associated matrices A_0 and A, we define their disagreement as

$$a(X) := \sup_{|X-Y|_\infty < \varrho(X)/2} |\mathcal{E}(Y)|, \qquad \mathcal{E}(Y) = A(Y) - A_0(Y).$$

3.3. **Main application.** In this section, to illustrate the use of Theorem 2.6, we present an alternative proof of a well known result of [FKP].

Theorem 3.2 ([FKP]). *Let L_0 and L be two operators as above with a being their disagreement, and let ω_0, ω denote their respective harmonic measures. Assume that*

$$\sup_{Q \in \mathbb{R}^n} \frac{1}{|Q|} \int_{R_Q} \frac{a(X)^2}{\varrho(X)} \, dX < \infty. \tag{3.5}$$

Then, we have that $\omega_0 \in A_\infty$ implies $\omega \in A_\infty$. More precisely, if L_0 is solvable in some $L^{p'}$, $1 < p' < \infty$, there exists $1 < q' < \infty$ such that L is solvable in $L^{q'}$.

We prove this result by using the extrapolation of Carleson measures Theorem 2.6. We take $d\mu(X) = \frac{a(X)^2}{\varrho(X)} \, dX$, that is, $d\mu(x,t) = a(x,t)^2 \frac{dt}{t} \, dx$ and (3.5) gives $\mu \in \mathcal{C}$. Therefore, to show that the harmonic measure $\omega \in A_\infty$, it suffices to fix Q and a family $\mathcal{F}$ such that (2.3) holds and show that $\mathcal{P}_\mathcal{F} \omega$ satisfies the A_∞ condition in (2.4). We will introduce some intermediate operators that allow us to pass from L_0 to L. Since the smallness in (2.3) is guaranteed above the discrete sawtooth region, we first introduce L_1 such that the disagreement with L_0 lives in that region (this is done in the first step). Once we have the solvability of L_1 we will be changing this operator in subsequent steps and in the end we will end up with L.

Let us call the reader's attention to the fact that in any given step we work with L_i and L_{i+1} in such a way that L_i is the "known" and L_{i+1} is the "unknown" in the sense that we have some nice properties for L_i and we want to infer them to L_{i+1}. For any of these operators L_i we write ω_i for the harmonic measure and, where it exists, k_i for the Poisson kernel.

[‡‡]In [Ken], condition (b) is stated in slightly different form, involving a global reverse Hölder estimate for harmonic measure with one fixed pole; it is well known that the present version of (b), as well as (c), are also equivalent to condition (a).

3.4. **Auxiliary results.** We summarize some well known results for divergence form elliptic equations that we will use in the sequel. The reader is referred to [Ken, Chapter 1] and the references therein for full details (see also [HM].)

Theorem 3.3. *There exists a unique function $G = G_L : \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} \longrightarrow \mathbb{R} \cup \{+\infty\}$, $G \geq 0$, such that $G(X, Y) = G(Y, X)$ for each $X, Y \in \mathbb{R}_+^{n+1}$, $G(\cdot, Y) \in W_1^2(\mathbb{R}_+^{n+1} \setminus R(Y, r)) \cap \dot{W}_{1,0}^1(\mathbb{R}_+^{n+1})$ for each $Y \in \mathbb{R}_+^{n+1}$ and $r > 0$, and $LG(\cdot, Y) = -\delta_Y$ for each $Y \in \mathbb{R}_+^{n+1}$.*

Lemma 3.4 (Caccioppoli)**.** *Let $Q \subset \mathbb{R}^n$ and let R be a cube in $\mathbb{R}_+^{n+1}$ such that $\overline{\tau R} \subset R_Q$ with $\tau > 1$. If $Lu = 0$ in R_Q, then*

$$\fint_R |\nabla u(Y)|^2 \, dY \leq C_{\lambda, n, \tau} \, \ell(R)^{-2} \fint_{\tau R} u(Y)^2 \, dY. \tag{3.6}$$

Lemma 3.5 (Doubling)**.** *There exists $C = C(\lambda, n)$ such that for every cube $Q \in \mathbb{R}^n$*

$$\omega^X(2Q) \leq C \, \omega^X(Q).$$

Lemma 3.6 (Caffarelli-Fabes-Mortola-Salsa)**.** *There exists a constant $C = C_{n,\lambda} < \infty$ such that for every cube Q, we have*

$$\omega^X(Q) \geq 1/C, \qquad \forall X \in 4Q \times [\ell(Q), 5\ell(Q)]. \tag{3.7}$$

Moreover, given $X, Y \in \mathbb{R}_+^{n+1}$ such that $|X - Y|_\infty > 2\varrho(Y)$ we have

$$G(X, Y) \approx \frac{\omega^X(Q(y, \varrho(Y)))}{\varrho(Y)^{n-1}}, \tag{3.8}$$

where the implicit constants depend only on dimension and ellipticity.

Lemma 3.7. *Given $Q \subset \mathbb{R}^n$, let L_1 and L_2 be elliptic operators such that $L_1 \equiv L_2$ in R_Q. If the corresponding harmonic measures ω_1, ω_2 are absolutely continuous with respect to the Lebesgue measure (we write k_1 and k_2 for the Poisson kernels), then*

$$k_1^{X_Q}(y) \approx k_2^{X_Q}(y), \qquad for \ a.e. \ y \in \tfrac{1}{2}Q.$$

Lemma 3.8. *Let $Q \subset Q_0$ and set $X_0 = (x_{Q_0}, 4\ell(Q_0))$, $X_Q = (x_Q, 4\ell(Q))$ where x_{Q_0} and x_Q are respectively the centers of Q_0 and Q. If $\omega \ll dx$ then*

$$k^{X_Q}(y) \approx \frac{k^{X_0}(y)}{\omega^{X_0}(Q)}, \qquad for \ a.e. \ y \in Q. \tag{3.9}$$

For an elliptic operator L, given u such that $Lu = 0$ in $\mathbb{R}_+^{n+1}$, we define the square function

$$S_\eta u(x) = \Big(\iint_{\Gamma_\eta(x)} |\nabla u(x, t)|^2 \, t^{1-n} \, dt \Big)^{\frac{1}{2}},$$

where

$$\Gamma_\eta(x) := \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \eta \, t\}$$

is the cone with vertex x and aperture η. We then have the following:

Theorem 3.9 (Dahlberg-Jerison-Kenig [DJK][*])**.**   *Suppose that for some $p' \in (1, \infty)$, $(D)_{p'}$ is solvable for L. Then, if u is a solution of the Dirichlet problem with data $f \in L^{p'}(\mathbb{R}^n)$, we have, for all $\eta > 0$,*

$$\|S_\eta u\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)},$$

where the implicit constant depends on dimension, ellipticity, η, and on the constants in the L^p estimates for the Poisson kernel of L.

---

[*]In fact, the theorem in [DJK] is somewhat more general than the result stated here, but we do not requiere the full version.

4. Proof of Theorem 3.2

We want to apply Theorem 2.6 with the Carleson measure $d\mu(X) = \frac{a(X)^2}{\varrho(X)} \, dX$. Given $\delta > 0$ to be chosen, we fix Q_0 and a family of pairwise disjoint subcubes $\mathcal{F} = \{Q_k\}_k \in \mathcal{D}(Q_0)$ such that

$$\sup_{Q \in \mathcal{D}(Q_0)} \frac{\mu(R_Q \cap \Omega_{\mathcal{F}})}{|Q|} \le \delta. \tag{4.1}$$

Set $X_0 = (x_0, 4\, \ell(Q_0))$ with x_0 being the center of Q_0.

As L_0 is solvable in some space $L^{p'}$ then $\omega_{L_0}^{X_0} = \omega_0^{X_0} \in RH_p(Q_0)$. This means that $\omega_0^{X_0} \ll dx$ and $k_0^{X_0} \in RH_p(Q_0)$. Without loss of generality we can assume that $1 < p < 2$ (as $RH_{p_1} \subset RH_{p_2}$ for $p_2 < p_1$). As $\omega_L^{X_0}$ is doubling, it suffices to work with $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}_{\mathcal{F}}'$, thus our goal is to show that $\mathcal{P}_{\mathcal{F}} \, \omega_L^{X_0}$ satisfies (2.4), with uniform constants. Notice that for a Borel set E, from the definition we have

$$\mathcal{P}_{\mathcal{F}} \, \omega_L^{X_0}(E) = \int_{\mathbb{R}^n} \mathcal{P}_{\mathcal{F}}(\chi_E)(x) d\omega_L^{X_0}(x) = u(X_0),$$

where u is a solution of the Dirichlet problem with data $\mathcal{P}_{\mathcal{F}}(\chi_E)$.

4.1. **An overview of the proof.** The proof that we present here runs parallel to that in [HM]. Indeed, Steps 0, 1 and 2 remain the same and therefore we only give the main ideas. Steps 3 and 4 need to be changed according to the $A_\infty^{\mathrm{dyadic}}$ condition that is contained in Theorem 2.6.

Step 0. We first make a reduction that allows us to use qualitative properties of the unknown harmonic measure. Indeed, we replace L by L_γ with $\gamma > 0$, which eventually goes to 0, so that L_γ coincides with L on a γ-strip along the boundary. This allows us to use qualitative properties of the corresponding harmonic measures. In particular, $\omega_{L_\gamma} \ll dx$ and also $\omega_{L_\gamma} \in RH_p$. Of course in the latter the constant will depend very badly on γ, but we will use this only in a qualitative way. Taking this reduction into account we can assume without loss of generality that all the harmonic measures below are absolutely continuous with respect to the Lebesgue measure and also that the Poisson kernels satisfy (qualitatively) RH_p. In our estimates the constants will not depend on γ.

Step 1. We define a new operator L_1 that agrees with L_0 everywhere except for the discrete sawtooth domain on which the new operator L_1 becomes L. That is, $L_1 = L$ in $\Omega_0 := R_{Q_0} \cap \Omega_{\mathcal{F}} = R_{Q_0} \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k})$ and $L_1 = L_0$ otherwise (see Figure 2). This means that the disagreement between L_0 and L_1 lives in Ω_0 and the harmonic measure μ restricted to Ω_0 is small at all the scales (see (4.1)).

We recall that $k_0^{X_0} \in RH_p(Q_0)$, and in particular we have

$$\int_{Q_0} k_0^{X_0}(y)^p \, dy \le C_0 |Q_0|^{1-p}. \tag{4.2}$$

Our immediate goal in Step 1 is to show that (4.2) remains true (with a different but uniform constant, independent of Q_0), when $k_0^{X_0}$ is replaced by $k_1^{X_0}$, the Poisson kernel for the operator L_1 defined above. To do that, we proceed by duality and fix a smooth function $g \ge 0$ supported on Q_0, such that $\|g\|_{L^{p'}(Q_0)} = 1$. Let u_0 and u_1 be the corresponding solutions to the Dirichlet problems for L_0 and L_1 with boundary data g. As the disagreement between L_0 and L_1 gives rise to a Carleson measure that it small at all scales by (4.1), it can be proved that u_1 is a small perturbation of u_0. To be more precise, we show the following:

$$|u_1(X_0) - u_0(X_0)| \lesssim \delta^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0)}. \tag{4.3}$$
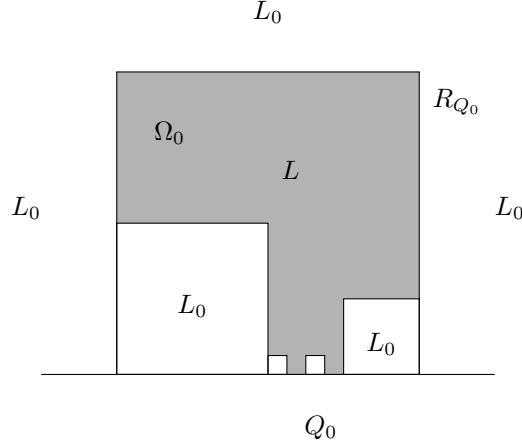
FIGURE 2. Definition of L_1

Since $k_0^{X_0}$ satisfies (4.2), we may therefore obtain (4.2) for $k_1^{X_0}$ by taking a supremum over all g as above, and then hiding the error in (4.3) for δ small enough (here we use the qualitative estimate $\|k_1^{X_0}\|_{L^p(Q_0)} < \infty$, see Step 0.)

In order to carry out Step 2, we need to extend (4.2) and obtain a reverse Hölder estimate on every dyadic subcube of Q_0. The key fact that will allow us to do so is that, in (4.1), the sup is taken with respect to all such cubes. The idea of the proof is to repeat the previous argument for a fixed $Q \in \mathcal{D}(Q_0)$ to obtain the analogue of (4.2) on Q, for the Poisson kernel associated to L_1, which is now defined with respect to

$$\Omega_Q := R_Q \cap \Omega_{\mathcal{F}} = R_Q \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k}).$$

The definition of the operator L_1 will depend on Q, but we will address this issue by use of the comparison principle. Eventually we show the following:

Conclusion (Step 1). *There exists $1 < r < \infty$ such that for every $Q \in \mathcal{D}(Q_0)$,*

$$\left(\fint_Q k_1^{X_0}(x)^r \, dx \right)^{\frac{1}{r}} \leq C \fint_Q k_1^{X_0}(x) \, dx. \tag{4.4}$$
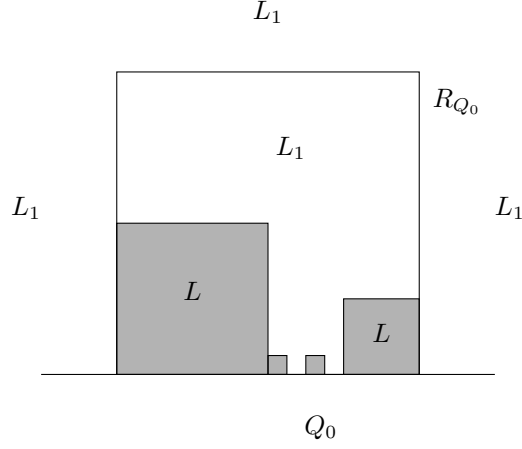
That is, $\omega_1^{X_0} \in A_\infty^{\mathrm{dyadic}}(Q_0)$. Hence we deduce that the same is true for $\mathcal{P}_{\mathcal{F}} \omega_1^{X_Q}$, by the following lemma.

Lemma 4.1. *Suppose that $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$, for some fixed cube Q_0, and suppose that $\mathcal{F} = \{Q_k\} \subset \mathcal{D}(Q_0)$ is a non-overlapping family. Then also $\mathcal{P}_{\mathcal{F}} \omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$.*

Step 2. We define the operator L_2 such that the disagreement with L_1 lives inside the Carleson boxes corresponding to the family $\mathcal{F}$. That is, set $L_2 = L$ in $R_{Q_0} \setminus \Omega_{\mathcal{F}} = \cup_{Q_k \in \mathcal{F}} R_{Q_k}$ and $L_2 = L_1$ otherwise (see Figure 3). We write $\omega_1 = \omega_{L_1}^{X_0}$ and $\omega_2 = \omega_{L_2}^{X_0}$ for the corresponding harmonic measures for L_1 and L_2 in $\mathbb{R}_+^{n+1}$ with fixed pole at $X_0 = (x_0, 4\,\ell(Q_0))$. We also let $\nu_1 = \nu_1^{X_0}$ and $\nu_2 = \nu_2^{X_0}$ denote the harmonic measures of L_1 and L_2 with pole at X_0, with respect to the domain $\Omega_{\mathcal{F}} = \mathbb{R}_+^{n+1} \setminus \cup_{Q_k \in \mathcal{F}} R_{Q_k}$. We notice that $L_1 = L_2$ in $\Omega_{\mathcal{F}}$ and therefore $\nu_1 = \nu_2$.

We apply the sawtooth lemma for projections (see Lemma 4.3 below) to both L_1 and L_2 and then we obtain that for all $Q \subset \mathcal{D}(Q_0)$ and $F \subset Q$

$$\left(\frac{\mathcal{P}_{\mathcal{F}} \omega_i(F)}{\mathcal{P}_{\mathcal{F}} \omega_i(Q)} \right)^{\theta_i} \lesssim \frac{\mathcal{P}_{\mathcal{F}} \bar{\nu}_i(F)}{\mathcal{P}_{\mathcal{F}} \bar{\nu}_i(Q)} \lesssim \frac{\mathcal{P}_{\mathcal{F}} \omega_i(F)}{\mathcal{P}_{\mathcal{F}} \omega_i(Q)}, \qquad i = 1, 2;$$
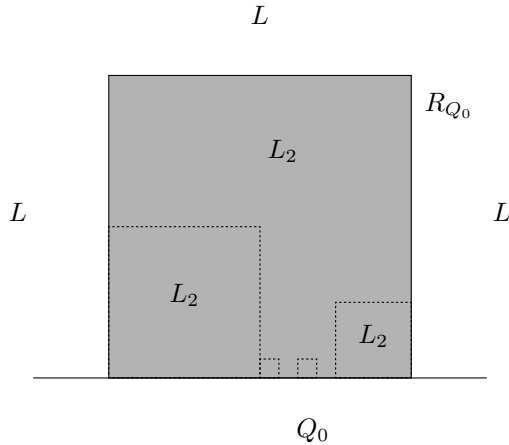
FIGURE 3. Definition of L_2

that is, $\mathcal{P}_{\mathcal{F}} \omega_i \in A_\infty^{\mathrm{dyadic}}(\mathcal{P}_{\mathcal{F}} \bar{\nu}_i, Q_0)$, for $i = 1, 2$ —here we use that $\mathcal{P}_{\mathcal{F}} \omega_i$ and $\mathcal{P}_{\mathcal{F}} \bar{\nu}_i$ are dyadically doubling, see [HM]. As observed above, $\nu_1 = \nu_2$ and therefore (4.13) implies that $\mathcal{P}_{\mathcal{F}} \bar{\nu}_1 = \mathcal{P}_{\mathcal{F}} \bar{\nu}_2$. Since $A_\infty^{\mathrm{dyadic}}(Q_0, \cdot)$ defines an equivalence relationship, and since we showed in Step 1 that $\mathcal{P}_{\mathcal{F}} \omega_1 \in A_\infty^{\mathrm{dyadic}}(Q_0)$ (with respect to Lebesgue measure), therefore we conclude also that $\mathcal{P}_{\mathcal{F}} \omega_2 \in A_\infty^{\mathrm{dyadic}}(Q_0)$:

Conclusion (Step 2). *There exist* $\theta, \theta' > 0$ *such that*

$$\left(\frac{|F|}{|Q|}\right)^\theta \lesssim \frac{\mathcal{P}_{\mathcal{F}} \omega_2^{X_0}(F)}{\mathcal{P}_{\mathcal{F}} \omega_2^{X_0}(Q)} \lesssim \left(\frac{|F|}{|Q|}\right)^{\theta'}, \qquad Q \in \mathcal{D}(Q_0), \quad F \subset Q.$$

Step 3. It remains to change the operator outside R_{Q_0}. Thus, we define $L_3 = L_2$ in R_{Q_0} and $L_3 = L$ otherwise (see Figure 4). Let us observe that $L_3 = L$ in $\mathbb{R}^{n+1}_+$.

FIGURE 4. Definition of L_3

We want to show that (2.4) holds with $\mathcal{P}_{\mathcal{F}}$ in place of $\mathcal{P}'_{\mathcal{F}}$, that is, we want to obtain

$$|E_\beta| := \left|\left\{x \in Q_0 : \mathcal{P}_{\mathcal{F}} k_3^{X_0}(x) \leq \beta \left(\mathcal{P}_{\mathcal{F}} \omega_3^{X_0}\right)_{Q_0}\right\}\right| \leq \alpha |Q_0|. \tag{4.5}$$

We fix $0 < \alpha < 1$ and let $\beta > 0$ to be chosen. Let us observe that we can disregard the trivial case $\mathcal{F} = \{Q_0\}$ since we have $\mathcal{P}_\mathcal{F} k_3^{X_0}(x)/(\mathcal{P}_\mathcal{F} \omega_3^{X_0})_{Q_0} = \chi_{Q_0}(x)$. Therefore the lefthand side of (4.5) vanishes for $0 < \beta < 1$ and the desired estimate follows at once.

Once we have disregarded this trivial change we take $j \geq 2$ large enough such that $2^{-j+1} < 1 - (1 - \alpha/2)^{1/n}$. We set $\tilde{Q}_0 = (1 - 2^{-j+1}) Q_0$ and observe that $Q_0 \setminus \tilde{Q}_0 = \cup_\Lambda Q$ where $\Lambda \subset \mathcal{D}(Q_0)$ and $\ell(Q) = 2^{-j} \ell(Q_0)$ for every $Q \in \Lambda$. Notice that Λ consists of all dyadic cubes in $\mathcal{D}(Q_0)$ with sidelength $2^{-j} \ell(Q_0)$ which are adjacent to the boundary of Q_0. The choice of j yields that $|Q_0 \setminus \tilde{Q}_0| < \alpha/2 \, |Q_0|$. On the other hand, we claim that by the comparison principle

$$\mathcal{P}_\mathcal{F} k_2^{X_0}(x) \leq C_\alpha \mathcal{P}_\mathcal{F} k_3^{X_0}(x), \qquad \text{a.e. } x \in \tilde{Q}_0. \tag{4.6}$$

That the harmonic measure is a probability implies $(\mathcal{P}_\mathcal{F} \omega_3^{X_0})_{Q_0} = \omega_3^{X_0}(Q_0)/|Q_0| \leq 1/|Q_0|$. Then we obtain

$$|E_\beta| \leq |Q_0 \setminus \tilde{Q}_0| + |E_\beta \cap \tilde{Q}_0| < \frac{\alpha}{2} |Q_0| + \left| \left\{ x \in Q_0 : \mathcal{P}_\mathcal{F} k_2^{X_0}(x) \leq \beta \, C_\alpha/|Q_0| \right\} \right|$$

$$=: \frac{\alpha}{2} |Q_0| + |F|$$

Next, we use the conclusion of Step 2 and also that $\mathcal{P}_\mathcal{F} \omega_2^{X_0}(Q_0) = \omega_2^{X_0}(Q_0) \gtrsim 1$ to obtain

$$\left(\frac{|F|}{|Q_0|} \right)^\theta \leq C \frac{\mathcal{P}_\mathcal{F} \omega_2^{X_0}(F)}{\mathcal{P}_\mathcal{F} \omega_2^{X_0}(Q_0)} \leq C \int_F \mathcal{P}_\mathcal{F} k_2^{X_0}(x) \, dx \leq C \, \beta \, C_\alpha \frac{|F|}{|Q_0|} \leq C_0 \, \beta \, C_\alpha < \left(\frac{\alpha}{2} \right)^\theta$$

provided we pick β so that $0 < \beta < (\alpha/2)^\theta (C_0 \, C_\alpha)^{-1}$. This allows to obtain the desired estimate (4.5).

Let us summarize what we have obtained so far (we recall that $L_3 \equiv L$):

Conclusion (Step 3). *There exists $\delta > 0$ for which the following statement holds: given $0 < \alpha < 1$, there is $\beta > 0$ such that for every $Q_0 \subset \mathbb{R}^n$, if $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q_0)$ is a pairwise disjoint collection of dyadic subcubes of Q_0 satisfying $\|\mu_\mathcal{F}\|_{\mathcal{C}(Q_0)} \leq \delta$, then*

$$\left| \left\{ x \in Q_0 : \mathcal{P}_\mathcal{F} k_L^{X_0}(x) \leq \beta \, (\mathcal{P}_\mathcal{F} \omega_L^{X_0})_{Q_0} \right\} \right| \leq \alpha \, |Q_0|.$$

Step 4. In order to use the extrapolation result we need to be able to fix the pole relative to a given cube Q_0, and obtain the last estimate for any dyadic subcubes of Q_0. Fixed $Q \in \mathcal{D}(Q_0)$ and $\mathcal{F} \subset Q$ as before, we use the conclusion of Step 3 and then pass from the pole X_Q to X_0 by means of Lemma 3.8. Thus, we may apply the extrapolation result Theorem 2.6 and conclude that $\omega^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ uniformly in Q_0:

Proposition 4.2. *There exists $\delta > 0$ for which the following statement holds: given $0 < \alpha < 1$, there is $\beta > 0$ such that for every $Q_0 \subset \mathbb{R}^n$ and for all $Q \in \mathcal{D}(Q_0)$, if $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ is a pairwise disjoint collection of dyadic subcubes of Q satisfying $\|\mu_\mathcal{F}\|_{\mathcal{C}(Q)} \leq \delta$, then*

$$\left| \left\{ x \in Q : \mathcal{P}_\mathcal{F} k_L^{X_0}(x) \leq \beta \, (\mathcal{P}_\mathcal{F} \omega_L^{X_0})_Q \right\} \right| \leq \alpha \, |Q|.$$

Consequently, $\omega^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ uniformly in Q_0. In particular, there exist $1 < q < \infty$ and a uniform constant C_0 such that we have the following reverse Hölder inequalities for all $Q_0 \subset \mathbb{R}^n$,

$$\left(\fint_{Q_0} k_L^{X_{Q_0}}(y)^q \, dy \right)^{\frac{1}{q}} \leq C_0 \fint_{Q_0} k_L^{X_{Q_0}}(y) \, dy \approx \frac{1}{|Q_0|} \tag{4.7}$$

From this result, we see that (4.7) and Theorem 3.1 yield as desired that L is solvable in $L^{q'}$ and then the proof of Theorem 3.2 is completed.

R_{Q_0}

Q_0
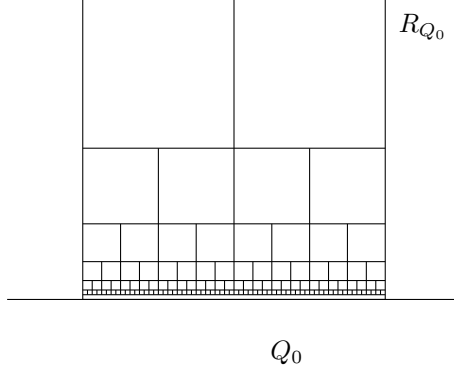
FIGURE 5. Whitney decomposition of R_{Q_0}

4.2. **Some details of the proof.** In this section we present some of the details needed to carry out the previous scheme of the proof. As mentioned above, Steps 0, 1, 2 are taken from [HM], therefore we only sketch the argument. Steps 3 and 4 need to be adapted from [HM] since the A_∞^{dyadic} condition used there is not the one in the present extrapolation result.

Step 0. We define $A_\gamma(x,t) = A(x,t)$ for $t > \gamma$ and $A_\gamma(x,t) = A_0(x,t)$ for $0 \le t \le \gamma$. In the following steps we work with L_γ in place of L. We note that the ellipticity constants of A_γ are controlled by those of A and A_0, uniformly in γ. Also, $|A_0(X) - A_\gamma(X)| \le |A_0(X) - A(X)|$ and thus the Carleson condition is controlled independently of γ. Notice that $L_\gamma = L_0$ in the strip $\{(x,t) : 0 \le t < \gamma\}$ and then in every step, by the comparison principle, we can use that all the harmonic measures are in RH_p (that is, they are absolutely continuous with respect to dx and the Poisson kernels are in RH_p: the constants will depend on γ but in our arguments we will only use this qualitatively and not quantitatively). In particular in Step 1 we have *a priori* that $\omega_1^{X_0} \ll dx$ and that $k_1^{X_0} \in L^p(Q_0)$ (this depends on γ, but we only use this in a qualitatively way). Therefore, we can carry out the whole argument and in the end we shall establish the reverse Hölder inequality (4.7) below for k_{L_γ} with q and C_0 independent of γ. One may then pass to the limit as follows: by [Ken, p. 41] for any smooth function φ we have $\langle \varphi, \omega_{L_\gamma}^{X_0} \rangle \longrightarrow \langle \varphi, \omega_L^{X_0} \rangle$ as $\gamma \to 0^+$. For any cube Q_0, and for every smooth function φ in $L^{q'}(Q_0)$ with $\|\varphi\|_{L^{q'}(Q_0)} = 1$ we have

$$|\langle \varphi, \omega_L^{X_0} \rangle| = \lim_{\gamma \to 0^+} |\langle \varphi, \omega_{L_\gamma}^{X_0} \rangle| \le \sup_{\gamma > 0} \|k_{L_\gamma}^{X_0}\|_{L^q(Q_0)} \|\varphi\|_{L^{q'}(Q_0)} \le C_0 |Q_0|^{-1/q'}.$$

Thus, $\Lambda_{\omega_L^{X_0}}(\varphi) := \langle \varphi, \omega_L^{X_0} \rangle$ is a functional in $(L^{q'}(Q))^*$. Therefore $\omega_L^{X_0} \ll dx$ in Q_0 and $k_L^{X_0}$ verifies (4.7). This in turn implies as desired that L is solvable in $L^{q'}$ by Theorem 3.1.

Step 1. We recall that L_1 is defined as $L_1 = L$ in Ω_0 and $L_1 = L_0$ otherwise (see Figure 2). That is, L_1 is the divergence form elliptic operator with associated matrix $A_1 = A$ in Ω_0 and $A_1 = A_0$ otherwise. We set $\mathcal{E}_1(Y) = A_1(Y) - A_0(Y) = \mathcal{E}(Y) \chi_{\Omega_0}(Y)$. In what follows we write $\omega_0 = \omega_{L_0}$, $\omega_1 = \omega_{L_1}$, $G_1 = G_{L_1}$.

We perform a Whitney decomposition of R_{Q_0} with respect to the distance to the boundary $\mathbb{R}^n$: $R_{Q_0} = \cup_{Q \in \mathcal{D}(Q_0)^*} U_Q$ where $\mathcal{D}(Q_0)^* = \mathcal{D}(Q_0) \setminus \{Q_0\}$, for every
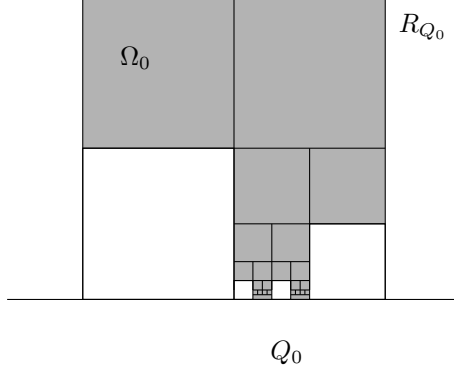
FIGURE 6. Whitney decomposition of Ω_0

cube Q we write $U_Q = Q \times [\ell(Q), 2\,\ell(Q))$ (see Figure 5) and it follows that the sets U_Q are pairwise disjoint. Let us observe that $\Omega_0 = R_{Q_0} \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k}) = \cup_{Q \in \mathcal{F}_1} U_Q$ where $\mathcal{F}_1 = \mathcal{D}(Q_0)^* \setminus \bigcup_{Q_k \in \mathcal{F}} \mathcal{D}(Q_k)^*$, see Figure 6.

We show (4.3), the argument is taken from [HM] and some details are skipped. As in [FKP], we have

$$F_1(X_0) := |u_1(X_0) - u_0(X_0)| = \Big| \int_{\mathbb{R}_+^{n+1}} \nabla_Y G_1(X_0, Y)\, \mathcal{E}_1(Y)\, \nabla u_0(Y)\, dY \Big|$$

$$\leq \int_{\Omega_0} |\nabla_Y G_1(X_0, Y)|\, |\mathcal{E}(Y)|\, |\nabla u_0(Y)|\, dY$$

$$\leq \sum_{Q \in \mathcal{F}_1} \sup_{U_Q} |\mathcal{E}| \Big(\int_{U_Q} |\nabla_Y G_1(X_0, Y)|^2\, dY \Big)^{\frac{1}{2}} \Big(\int_{U_Q} |\nabla u_0(Y)|^2\, dY \Big)^{\frac{1}{2}}.$$

As X_0 is away from R_{Q_0} we have that $G_1(X_0, \cdot)$ is a non-negative solution of L_1 in $R_{2\,Q_0}$ we can apply Caccioppoli's inequality (Lemma 3.4) to this function. Also, we use (3.8) and we conclude that

$$\int_{U_Q} |\nabla_Y G_1(X_0, Y)|^2\, dY \lesssim \ell(Q)^{-2} \int_{2\,U_Q} G_1(X_0, Y)^2\, dY$$

$$\lesssim \Big(\frac{\omega_1^{X_0}(Q)}{|Q|} \Big)^2 |2\,U_Q| \approx \Big(\frac{\omega_1^{X_0}(Q)}{|Q|} \Big)^{2-p} \int_{\frac{1}{4}\,U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y) \Big)^p\, dy\, ds,$$

where $P_s^{Q_0}$ is the dyadic averaging operator defined as follows:

$$P_s^{Q_0} f(y) := \sum_{Q \in \mathcal{D}(Q_0)^*} \Big(\fint_Q f(z)\, dz \Big)\, \chi_{U_Q}(y, s).$$

Note that in the sum there is at most one non-zero term since the sets U_Q are a disjoint partition of R_{Q_0}. Next we use that $\sup_{U_Q} |\mathcal{E}| \leq a(Y)$ for every $Y \in \frac{1}{4}\,U_Q$, by a routine geometric argument that we leave to the reader, and we obtain

$$F_1(X_0) \lesssim \Big(\sum_{Q \in \mathcal{F}_1} \int_{\frac{1}{4}\,U_Q} \Big(P_s^{Q_0} k_1^{X_0}(y) \Big)^p \frac{a(y, s)^2}{s}\, dy\, ds \Big)^{\frac{1}{2}}$$

$$\times \Big(\sum_{Q \in \mathcal{F}_1} \Big(\frac{\omega_1^{X_0}(Q)}{|Q|} \Big)^{2-p} \int_{U_Q} |\nabla u_0(y, s)|^2\, s\, dy\, ds \Big)^{\frac{1}{2}} =: I \cdot II.$$

We estimate each factor in turn. For I, we define

$$d\tilde{\mu}(y,s) = \chi_{\Omega_0}(y,s)\, d\mu(y,s) = \chi_{\Omega_0}(y,s)\, a(y,s)^2 \frac{dy\,ds}{s}$$

so by the dyadic Carleson Embedding [HM, Lemma 3.12] and (4.1) we have

$$I^2 \leq \int_{R_{Q_0}} \left(P_s^{Q_0} k_1^{X_0}(y) \right)^p d\tilde{\mu}(y,s) \lesssim \|\tilde{\mu}\|_{\mathcal{C}(Q_0)} \int_{Q_0} k_1^{X_0}(y)^p\, dy \lesssim \delta^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0)}^{\frac{p}{2}}.$$

We now estimate II. For a sufficiently large $\eta > 0$ we have

$$
\begin{aligned}
II^2 &= \sum_{Q \in \mathcal{F}_1} \frac{1}{|Q|} \int_Q \left[\left(\frac{\omega_1^{X_0}(Q)}{|Q|} \right)^{2-p} \int_{U_Q} |\nabla u_0(y,s)|^2\, s\, dy\, ds \right] dx \\
&\lesssim \sum_{Q \in \mathcal{F}_1} \iint_{U_Q} \left(M(k_1^{X_0} \chi_{Q_0})(x) \right)^{2-p} \int_{|x-y|<\eta s} |\nabla u_0(y,s)|^2\, s^{1-n}\, dy\, ds\, dx \\
&\leq \|S_\eta u_0\|_{L^{p'}}^2 \|M(k_1^{X_0} \chi_{Q_0})\|_{L^p}^{2-p} \\
&\lesssim \|k_1^{X_0}\|_{L^p(Q_0)}^{2-p},
\end{aligned}
$$

where we have used that $1 < p < 2$ and Theorem 3.9 (and the fact that $(D)_{p'}$ is solvable for L_0). Collecting our estimates for I and II we conclude as desired (4.3):

$$F_1(X_0) = |u_1(X_0) - u_0(X_0)| \lesssim \delta^{\frac{1}{2}} \|k_1^{X_0}\|_{L^p(Q_0)}.$$

Since $k_0^{X_0}$ satisfies (4.2), we may therefore obtain (4.2) for $k_1^{X_0}$ by taking a supremum over all g as above, and then hiding the error in (4.3) for δ small enough (here we use the qualitative estimate $\|k_1^{X_0}\|_{L^p(Q_0)} < \infty$, see Step 0.)

Self-improvement of Step 1. We fix $Q \in \mathcal{D}(Q_0)$ and set $X_Q = (x_Q, 4\,\ell(Q))$ where x_Q is the center of Q. Let us define a new operator $L_1^Q = L$ in $\Omega_Q = R_Q \cap \Omega_{\mathcal{F}} = R_Q \setminus (\cup_{Q_k \in \mathcal{F}} R_{Q_k})$ and $L_1^Q = L_0$ otherwise in $\mathbb{R}_+^{n+1}$, and let $k_{L_1^Q}^{X_Q}$ denote the Poisson kernel for L_1^Q with pole at X_Q. We claim that

$$\int_Q k_{L_1^Q}^{X_Q}(x)^p dx \lesssim |Q|^{1-p}, \tag{4.8}$$

where the constant is independent of Q. Indeed, if $Q \subset Q_k$ for some $Q_k \in \mathcal{F}$ then we obtain that $\Omega_Q = \emptyset$ and $L_1^Q \equiv L_0$ in $\mathbb{R}_+^{n+1}$. In that case, (4.8) holds by hypothesis. Otherwise, since trivially $\|\mu\|_{\mathcal{C}(Q)} \leq \|\mu\|_{\mathcal{C}(Q_0)}$ for every $Q \in \mathcal{D}(Q_0)$, we have that the analogue of (4.1) obviously holds on Q, for the same family $\mathcal{F}$ (or to be more precise, for the family $\mathcal{F}_Q$ defined as the family of cubes in $\mathcal{F}$ that meet Q). Consequently, if Q is not contained in any $Q_k \in \mathcal{F}$, then we may simply repeat the previous argument with respect to Q, and we obtain (4.8) exactly as before. This proves the claim.

Now by (3.7) and (4.8) we obtain

$$\left(\fint_Q k_{L_1^Q}^{X_Q}(x)^p\, dx \right)^{\frac{1}{p}} \lesssim \fint_Q k_{L_1^Q}^{X_Q}(x)\, dx. \tag{4.9}$$

Next, we want to pass from $k_{L_1^Q}^{X_Q}$ to $k_{L_1}^{X_Q}$. Notice that $L_1 \equiv L_1^Q$ in R_Q, therefore Lemma 3.7 yields that $k_1^{X_Q}(y) = k_{L_1}^{X_Q}(y) \approx k_{L_1^Q}^{X_Q}(y)$, for a.e. $y \in \frac{1}{2}Q$. The latter fact, (4.9) and the doubling property imply that

$$\left(\fint_{\frac{1}{2}Q} k_1^{X_Q}(x)^p\, dx \right)^{\frac{1}{p}} \lesssim \left(\fint_Q k_{L_1^Q}^{X_Q}(x)^p\, dx \right)^{\frac{1}{p}} \lesssim \fint_Q k_{L_1^Q}^{X_Q}(x)\, dx \lesssim \fint_{\frac{1}{2}Q} k_1^{X_Q}(x)\, dx. \tag{4.10}$$

Consequently, by Lemma 3.8 we have

$$\left(\fint_{\frac{1}{2}Q} k_1^{X_0}(x)^p \, dx\right)^{\frac{1}{p}} \lesssim \fint_{\frac{1}{2}Q} k_1^{X_0}(x) \, dx, \qquad \forall Q \in \mathcal{D}(Q_0). \tag{4.11}$$

Then [HM, Lemma B.7] yields as desired (4.4) and therefore we have obtained the conclusion of Step 1 stated above.

Proof of Lemma 4.1. That $\mathcal{P}_\mathcal{F}\omega$ is dyadically doubling follows from [HM] and the fact that so is ω. As $\omega \in A_\infty^{\text{dyadic}}(Q_0)$, we have $k = d\omega/dx \in RH_q^{\text{dyadic}}(Q_0)$ for some $1 < q < \infty$. It is trivial to see that $\mathcal{P}_\mathcal{F}\omega \ll dx$ and that $d(\mathcal{P}_\mathcal{F}\omega)/dx = \mathcal{P}_\mathcal{F}k$. We show that $\mathcal{P}_\mathcal{F}k \in RH_q^{\text{dyadic}}(Q_0)$. Let $Q \in \mathcal{D}(Q_0)$. If $Q \subset Q_k$ for some $Q_k \in \mathcal{F}$ then $\mathcal{P}_\mathcal{F}k(x) \equiv w_{Q_k}$ for every $x \in Q_k$, thus we trivially obtain the desired estimate

$$\left(\fint_Q \mathcal{P}_\mathcal{F}k(x)^q \, dx\right)^{\frac{1}{q}} = w_{Q_k} = \fint_Q \mathcal{P}_\mathcal{F}k(x) \, dx.$$

Otherwise, Q is not contained in any Q_k and it follows that if $Q \cap Q_k \neq \emptyset$ then $Q_k \subsetneq Q$. It is straightforward to show that $\mathcal{P}_\mathcal{F}k(x) = \mathcal{P}_\mathcal{F}(k\,\chi_Q)(x)$ for every $x \in Q$. Then, we obtain as desired

$$\fint_Q \mathcal{P}_\mathcal{F}k(x)^q \, dx = \fint_Q \mathcal{P}_\mathcal{F}(k\,\chi_Q)(x)^q \, dx \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} \mathcal{P}_\mathcal{F}(k\,\chi_Q)(x)^q \, dx \leq \fint_Q k(x)^q \, dx$$
$$\lesssim \left(\fint_Q k(x) \, dx\right)^q = \left(\fint_Q \mathcal{P}_\mathcal{F}k(x) \, dx\right)^q.$$

Gathering the two cases we conclude that $\mathcal{P}_\mathcal{F}k \in RH_q^{\text{dyadic}}(Q_0)$ and this leads to $\mathcal{P}_\mathcal{F}k \in A_\infty^{\text{dyadic}}(Q_0)$ by Proposition 2.2. $\qquad\square$

Step 2. To complete this step we just need to state the following sawtooth lemma for projections:

Lemma 4.3 (Discrete sawtooth lemma for projections, [HM]). *Let Q_0 be a fixed cube in $\mathbb{R}^n$, let $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q_0)$ be a family of pairwise disjoint dyadic cubes and let $\mathcal{P}_\mathcal{F}$ be the corresponding projection operator. Set $\Omega_\mathcal{F} = \mathbb{R}_+^{n+1}\setminus(\cup_{Q_k \in \mathcal{F}} R_{Q_k})$. We write $\omega = \omega^{X_0}$ and $\nu = \nu^{X_0}$ for the harmonic measures of L with fixed pole at $X_0 = (x_{Q_0}, 4\,\ell(Q_0))$ with respect to the domains $\mathbb{R}_+^{n+1}$ and $\Omega_\mathcal{F}$. Let $\bar\nu = \bar\nu^{X_0}$ be the measure defined by*

$$\bar\nu(F) = \nu(F\setminus(\cup_{Q_k \in \mathcal{F}} R_{Q_k})) + \sum_{Q_k \in \mathcal{F}} \frac{\omega(F \cap Q_k)}{\omega(Q_k)}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F}), \qquad F \subset Q_0. \tag{4.12}$$

We observe that $\mathcal{P}_\mathcal{F}\bar\nu$ depends only on ν and not on ω since

$$\mathcal{P}_\mathcal{F}\bar\nu(F) = \nu(F\setminus(\cup_{Q_k \in \mathcal{F}} R_{Q_k})) + \sum_{Q_k \in \mathcal{F}} \frac{|F \cap Q_k|}{|Q_k|}\,\nu(\overline{R_{Q_k}} \cap \partial\Omega_\mathcal{F}), \qquad F \subset Q_0. \tag{4.13}$$

Then, there exists $\theta > 0$ such that for all $Q \in \mathcal{D}(Q_0)$ and $F \subset Q$, we have

$$\left(\frac{\mathcal{P}_\mathcal{F}\omega(F)}{\mathcal{P}_\mathcal{F}\omega(Q)}\right)^\theta \lesssim \frac{\mathcal{P}_\mathcal{F}\bar\nu(F)}{\mathcal{P}_\mathcal{F}\bar\nu(Q)} \lesssim \frac{\mathcal{P}_\mathcal{F}\omega(F)}{\mathcal{P}_\mathcal{F}\omega(Q)}. \tag{4.14}$$

Step 3. We show (4.6). Notice that $L_2 \equiv L_3$ in R_{Q_0}, then, as in Lemma 3.7, by the comparison principle we have that $k_2^{X_0}(y) \approx k_3^{X_0}(y)$ for a.e. $y \in \tilde{Q}_0$ where the constants depend on j and hence on α. This implies that for a.e. $x \in \tilde{Q}_0$ we obtain

$$\mathcal{P}_\mathcal{F}k_2^{X_0}(x) \leq C_\alpha\, k_3^{X_0}(x)\,\chi_{\mathbb{R}^n\setminus(\cup_{Q_k \in \mathcal{F}} Q_k)}(x) + \sum_{Q_k \in \mathcal{F}} \frac{\omega_2^{X_0}(Q_k)}{|Q_k|}\,\chi_{Q_k}(x).$$

Note that the sum can be restricted to those cubes in $\mathcal{F}$ that meet $\tilde{Q}_0$. Therefore we pick such a cube Q_k and show that $\omega_2^{X_0}(Q_k) \leq C_\alpha \omega_3^{X_0}(Q_k)$ which in turn implies (4.6).

Case 1: $Q_k \subset \tilde{Q}_0$. As before $\omega_2^{X_0}(Q_k) \leq C_\alpha \, \omega_3^{X_0}(Q_k)$.

Case 2: $Q_k \not\subset \tilde{Q}_0$. As $Q_k \cap \tilde{Q}_0 \neq \emptyset$, it is not difficult to show that there exists $\bar{Q}_k$ a dyadic "child" of Q_k such that $\bar{Q}_k \subset \tilde{Q}_0$. Given this, since $\omega_2^{X_0}$ is doubling we have

$$\omega_2^{X_0}(Q_k) \leq C \, \omega_2^{X_0}(\bar{Q}_k) \leq C_\alpha \, \omega_3^{X_0}(\bar{Q}_k) \geq C_\alpha \, \omega_3^{X_0}(Q_k).$$

Step 4. We only need to give the proof of Proposition 4.2.

Proof of Proposition 4.2. Take an arbitrary $\alpha \in (0,1)$ and let $\beta, \delta > 0$ be given by the conclusion of Step 3. We fix $Q_0 \subset \mathbb{R}^n$ and $Q \in \mathcal{D}(Q_0)$. Let $\mathcal{F} = \{Q_k\}_k \subset \mathcal{D}(Q)$ be such that $\|\mu_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \delta$. Then, we use Lemma 3.8 and for a.e. $x \in Q$ we obtain

$$\mathcal{P}_{\mathcal{F}} k_L^{X_Q}(x) \approx \frac{k_L^{X_{Q_0}}(x)}{\omega_L^{X_{Q_0}}(Q)} \, \chi_{\mathbb{R}^n \setminus (\cup_{Q_k \in \mathcal{F}} Q_k)}(x) + \sum_{Q_k \in \mathcal{F}} \frac{\omega_L^{X_{Q_0}}(Q_k)}{|Q_k| \, \omega_L^{X_{Q_0}}(Q)} \, \chi_{Q_k}(x)$$

$$= \frac{\mathcal{P}_{\mathcal{F}} k_L^{X_{Q_0}}(x)}{\mathcal{P}_{\mathcal{F}} \omega_L^{X_{Q_0}}(Q)},$$

where we have used that $\mathcal{P}_{\mathcal{F}} \omega_L^{X_{Q_0}}(Q) = \omega_L^{X_{Q_0}}(Q)$. This and (3.7) imply

$$\frac{\mathcal{P}_{\mathcal{F}} k_L^{X_{Q_0}}(x)}{(\mathcal{P}_{\mathcal{F}} \omega_L^{X_{Q_0}})_Q} \geq C \, \mathcal{P}_{\mathcal{F}} k_L^{X_Q}(x) \, |Q| \geq C \, \frac{\mathcal{P}_{\mathcal{F}} k_L^{X_Q}(x)}{\omega_L^{X_Q}(Q)/|Q|} = C_0 \, \frac{\mathcal{P}_{\mathcal{F}} k_L^{X_Q}(x)}{(\mathcal{P}_{\mathcal{F}} \omega_L^{X_Q})_Q}.$$

We apply this estimate and the conclusion of Step 3 with Q in place of Q_0 to conclude that

$$\left| \left\{ x \in Q : \mathcal{P}_{\mathcal{F}} k_L^{X_{Q_0}}(x) \leq C_0 \, \beta \, (\mathcal{P}_{\mathcal{F}} \omega_L^{X_{Q_0}})_Q \right\} \right|$$

$$\leq \left| \left\{ x \in Q : \mathcal{P}_{\mathcal{F}} k_L^{X_Q}(x) \leq \beta \, (\mathcal{P}_{\mathcal{F}} \omega_L^{X_Q})_Q \right\} \right| \leq \alpha \, |Q|.$$

Next, by the extrapolation of Carleson measures Theorem 2.6, there exist $0 < \alpha_0 < 1$ and $\beta_0 > 0$ such that for every $Q \in \mathcal{D}(Q_0)$,

$$\left| \left\{ x \in Q : k_L^{X_{Q_0}}(x) \leq \beta_0 \, (\mathcal{P}_{\mathcal{F}} \omega_L^{X_{Q_0}})_Q \right\} \right| \leq \alpha_0 \, |Q|.$$

This fact plus Proposition 2.2 imply the existence of $q = q_L$ and a uniform constant C_1 such that for all $Q \in \mathcal{D}(Q_0)$,

$$\left(\fint_Q k_L^{X_{Q_0}}(y)^q \, dy \right)^{\frac{1}{q}} \leq C_1 \fint_Q k_L^{X_{Q_0}}(y) \, dy.$$

If we specify this estimate to $Q = Q_0$ we obtain as desired (4.7). We notice that the previous estimate and the fact that $\omega_L^{X_{Q_0}}$ is doubling imply $k_L^{X_{Q_0}} \in RH_q(Q_0)$. $\square$

5. Proof of Propositions 2.2 and 2.4

The proofs that we present here follow the classical ideas in [CF] (see also [GR], [Gra], [Per]).

5.1. **Proof of Proposition 2.2.** We show that $(b) \implies (a) \implies (c) \implies (d) \implies (e) \implies (b)$.

$\boxed{(b) \implies (a)}$ We pick $0 < \alpha < 1$ such that $C_0 \, \alpha^\theta < 1$, and $C_0 \, \alpha^\theta < \beta < 1$. Then (b) easily implies $\omega \preceq \nu$.

$\boxed{(a) \implies (c)}$ We first show that $\omega \ll \nu$. We remind the reader at this point that our dyadic cubes are "1/2-open", i.e., they are Cartesian products of intervals closed at the left-hand endpoint, and open on the right. We note that any open set $G \subset \mathbb{R}^n$ may be realized as the *disjoint* union of a countable collection of such cubes.

Let α, β be the constants in the condition $\omega \preceq \nu$. Suppose that ω is not absolutely continuous with respect to ν, that is, there exists $E \subset Q_0$ such that $\nu(E) = 0$ and $\omega(E) > 0$. If $Q_0 \subsetneq \mathbb{R}^n$ we extend the measure ω to $\mathbb{R}^n$ so that is identically zero outside Q_0 (abusing notation, we call the new measure ω). Since ω is a regular measure there exists an open set $G \supset E$ such that $\omega(G) < \beta^{-1}\omega(E)$. As noted above, we can cover G by a pairwise disjoint collection of cubes $\{Q_j\}_j$, belonging to the dyadic grid induced by Q_0. If $Q_0 \subset Q_{j_0}$ for some j_0 then $0 = \nu(E) < \alpha\,\nu(Q_0)$ implies, by $\omega \preceq \nu$,

$$\omega(E) < \beta\,\omega(Q_0) \leq \beta\,\omega(Q_{j_0}) \leq \beta\,\omega(G)\,,$$

and we obtain a contradiction. Thus, Q_0 is not contained in any of the cubes Q_j. Therefore, if $E \cap Q_j \neq \emptyset$ then $Q_j \subsetneq Q_0$ and thus $Q_j \in \mathcal{D}(Q_0)$. Using $\omega \preceq \nu$, we have that $0 = \nu(E \cap Q_j) < \alpha\,\nu(Q_j)$ yields $\omega(E \cap Q_j) < \beta\,\omega(Q_j)$. We sum on j and conclude that

$$\omega(E) = \sum_{j:Q_j \cap E \neq \emptyset} \omega(E \cap Q_j) \leq \beta \sum_j \omega(Q_j) = \beta\,\omega(G),$$

which leads us again to a contradiction. Therefore, we have shown that $\omega \ll \nu$.

Next, we take $F = \{x \in Q : k_\omega(x) \leq (1 - \beta) \fint_Q k_\omega \, d\nu\}$. Then,

$$\omega(F) = \int_F k_\omega(x)\,d\nu(x) \leq (1 - \beta) \left(\fint_Q k_\omega \, d\nu \right) \nu(F) \leq (1 - \beta)\,\omega(Q).$$

which implies that $\omega(Q \setminus F)/\omega(Q) \geq \beta$. We apply (a) to $E = Q \setminus F$ and then $\nu(E)/\nu(Q) \geq \alpha$. Passing to the complement we readily obtain $\nu(F) \leq (1 - \alpha)\,\nu(Q)$.

$\boxed{(c) \Longrightarrow (d)}$ Given $Q \in \mathcal{D}(Q_0)$ and $\lambda > \fint_Q k_\omega \, d\nu$ we use the Calderón-Zygmund decomposition with respect to the dyadic doubling measure ν to obtain that there exists a family of maximal, therefore disjoint, cubes $\{Q_j\}_j \subset \mathcal{D}(Q)$ such that

$$\{x \in Q : M_{\nu,Q}^d k_\omega(x) > \lambda\} = \bigcup_j Q_j, \qquad \lambda < \fint_{Q_j} k_\omega(x)\,d\nu(x) \leq C_\nu\,\lambda,$$

here $M_{\nu,Q}^d$ is the dyadic maximal operator with respect to the measure ν and in the sup the cubes are in $\mathcal{D}(Q)$. We apply (c) to each Q_j to conclude that

$$\nu\{x \in Q_j : k_\omega(x) > \beta\,\lambda\} \geq \nu\{x \in Q_j : k_\omega(x) > \beta \fint_{Q_j} k_\omega \, d\nu\} \geq (1 - \alpha)\,\nu(Q_j).$$

Then the desired estimate follows easily:

$$\omega\{x \in Q : k_\omega(x) > \lambda\} \leq \omega\{x \in Q : M_{\nu,Q}^d k_\omega(x) > \lambda\} = \sum_j \omega(Q_j) \leq C_\nu\,\lambda \sum_j \nu(Q_j)$$

$$\leq \frac{C_\nu}{1 - \alpha}\,\lambda \sum_j \nu\{x \in Q_j : k_\omega(x) > \beta\,\lambda\} \leq \frac{C_\nu}{1 - \alpha}\,\lambda\,\nu\{x \in Q : k_\omega(x) > \beta\,\lambda\}.$$

$\boxed{(d) \Longrightarrow (e)}$ We take $N > c_Q := \fint_Q k_\omega \, d\nu$ and write $k_{\omega,N} = \min\{k_w, N\}$. We observe that

$$\fint_Q k_{\omega,N}(x)^{1+\delta}\,d\nu(x) \leq \frac{1}{\nu(Q)} \int_Q k_{\omega,N}(x)^\delta\,d\omega(x)$$

$$= \frac{\delta}{\nu(Q)} \int_0^N \lambda^\delta\,\omega\{x \in Q : k_\omega(x) > \lambda\} \frac{d\lambda}{\lambda}$$

$$= \frac{\delta}{\nu(Q)} \int_0^{c_Q} \cdots + \frac{\delta}{|Q|} \int_{c_Q}^N \cdots = I + II.$$

The estimate for I is trivial:

$$I \leq \frac{\delta}{\nu(Q)}\,\omega(Q) \int_0^{c_Q} \lambda^\delta \frac{d\lambda}{\lambda} = c_Q^{1+\delta}.$$

For II we first observe that in (d) we can assume that $0 < \beta \leq 1$ (otherwise we make the right hand side bigger replacing β by 1). Then, using (d) we obtain

$$
\begin{aligned}
II &\leq \frac{\delta\, C_0}{\nu(Q)} \int_{c_Q}^{N} \lambda^{\delta+1}\, \nu\{x \in Q : k_\omega(x) > \beta\, \lambda\}\, \frac{d\lambda}{\lambda} \\
&\leq \frac{\delta\, C_0}{\nu(Q)\, \beta^{1+\delta}} \int_0^N \lambda^{\delta+1}\, \nu\{x \in Q : k_\omega(x) > \lambda\}\, \frac{d\lambda}{\lambda} \\
&= \frac{\delta\, C_0}{(\delta+1)\, \beta^{\delta+1}} \fint_Q k_{\omega,N}(x)^{1+\delta}\, d\nu(x).
\end{aligned}
$$

We next pick $\delta > 1$ small enough so that the constant in front of the integral is smaller than $1/2$. Then, we have

$$
\fint_Q k_{\omega,N}(x)^{1+\delta}\, d\nu(x) \leq c_Q^{1+\delta} + \frac{1}{2} \fint_Q k_{\omega,N}(x)^{1+\delta}\, d\nu(x)
$$

and we can hide the last term into the left hand side (this term is finite since $k_{\omega,N} \leq N$). Thus the desired estimate follows at once by the monotonous convergence theorem.

$\boxed{(e) \Longrightarrow (b)}$ Using Hölder's inequality we obtain

$$
\frac{\omega(E)}{\nu(Q)} = \fint_Q \chi_E\, k_\omega\, d\nu \leq \left(\frac{\nu(E)}{\nu(Q)} \right)^{\frac{1}{(1+\delta)'}} \left(\fint_Q k_\omega^{1+\delta}\, d\nu \right)^{\frac{1}{1+\delta}} \leq C_2 \left(\frac{\nu(E)}{\nu(Q)} \right)^{\frac{1}{(1+\delta)'}} \frac{\omega(Q)}{\nu(Q)},
$$

and the desired estimate follows at once.

5.2. **Proof of Proposition 2.4.** For (i) it suffices to show that $\omega \preceq \nu$ implies $\nu \preceq \omega$. Let $\alpha, \beta \in (0,1)$ be the constants in the condition $\omega \preceq \nu$. Let $\alpha' = 1 - \beta$ and $1 - \alpha < \beta' < 1$. If $E \subset Q \in \mathcal{D}(Q_0)$ with $\omega(E)/\omega(Q) < \alpha'$ then $\omega(Q \setminus E)/\omega(Q) > 1 - \alpha' = \beta$. By $\omega \preceq \nu$ it follows that $\nu(Q \setminus E)/\nu(Q) \geq \alpha$ which in turn implies $\nu(E)/\nu(Q) \leq 1 - \alpha < \beta'$, and this shows $\nu \preceq \omega$.

To prove (ii) we first observe that $\preceq$ is clearly reflexive (i.e., $\nu \preceq \nu$) and we have just proved that it is also symmetric (i.e., $\omega \preceq \nu$ implies $\nu \preceq \omega$). To show the transitivity we use (b) in Proposition 2.2. If ω, ν, μ are non-negative regular Borel measures dyadically doubling such that $\omega \preceq \nu$ and $\nu \preceq \mu$, we have

$$
\frac{\omega(E)}{\omega(Q)} \leq C_0 \left(\frac{\nu(E)}{\nu(Q)} \right)^{\theta} \leq C_0\, C_0' \left(\frac{\mu(E)}{\mu(Q)} \right)^{\theta \cdot \theta'},
$$

where in the first (resp. second) inequality we have used (b) in Proposition 2.2 applied to $\omega \preceq \nu$ (resp. $\nu \preceq \mu$) —notice that ν and μ are dyadically doubling. Then, using again Proposition 2.2 it follows as desired that $\omega \preceq \mu$ (here we use that μ is dyadically doubling).

REFERENCES

[AHLT]   P. Auscher, S. Hofmann, J.L. Lewis and P. Tchamitchian, *Extrapolation of Carleson measures and the analyticity of Kato's square-root operators*, Acta Math. **187** (2001), no. 2, 161–190.

[AHMTT] P. Auscher, S. Hofmann, C. Muscalu, T. Tao and C. Thiele, *Carleson measures, trees, extrapolation, and T(b) theorems*, Publ. Mat. **46** (2002), no. 2, 257–325.

[Car]   L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547–559.

[CF]   R.R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.

[DJK]   B.E. Dahlberg, D.S. Jerison and C.E. Kenig, *Area integral estimates for elliptic differential operators with nonsmooth coefficients*, Ark. Mat. **22** (1984), no. 1, 97–108.

[DS]   G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs*, Asterisque **193** (1991).

[FKP]   R.A. Fefferman, C.E. Kenig and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. of Math. (2) **134** (1991), no. 1, 65–124.

[GR]   J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, **116**. North-Holland Publishing Co., Amsterdam, 1985.

[Gra]   L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.

[HL]   S. Hofmann and J.L. Lewis, *The Dirichlet problem for parabolic operators with singular drift terms*, Mem. Amer. Math. Soc. **151** (2001), no. 719

[HM]   S. Hofmann and J.M. Martell, A_∞ *estimates via extrapolation of Carleson measures and applications to divergence form elliptic operators*, Preprint 2009. `http://www.uam.es/chema.martell`

[Ken]   C.E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, **83**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

[LM]   J. Lewis and M. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Mem. Amer. Math. Soc. **114** (1995), no. 545.

[Per]   C. Pérez, Personal communication.

STEVE HOFMANN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
  E-mail address: `hofmanns@missouri.edu`

JOSÉ MARÍA MARTELL, INSTITUTO DE CIENCIAS MATEMÁTICAS CSIC-UAM-UC3M-UCM, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, C/ SERRANO 121, E-28006 MADRID, SPAIN
  E-mail address: `chema.martell@uam.es`