A SUFFICIENT GEOMETRIC CRITERION FOR QUANTITATIVE ABSOLUTE CONTINUITY OF HARMONIC MEASURE

STEVE HOFMANN AND JOSÉ MARÍA MARTELL

ABSTRACT. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set, not necessarily connected, with an n-dimensional uniformly rectifiable boundary. We show that harmonic measure for Ω is weak-A_∞ with respect to surface measure on $\partial\Omega$, provided that Ω satisfies a certain weak version of a local John condition.

CONTENTS

1. INTRODUCTION

A classical result of F. and M. Riesz [RR] states that for a simply connected domain Ω in the complex plane, rectifiability of $\partial\Omega$ implies that harmonic measure for Ω is absolutely continuous with respect to arclength measure on the boundary. A quantitative version of this theorem was later proved by Lavrentiev [Lav]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones

Date: December 11, 2017.

2000 *Mathematics Subject Classification.* 31B05, 35J08, 35J25, 42B20, 42B25, 42B37.

Key words and phrases. Harmonic measure, Poisson kernel, uniform rectifiability. Carleson measures.

The first author was supported by NSF grants DMS-1361701 and DMS-1664047. The second author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the "Severo Ochoa" Programme for Centres of Excellence in R&D (SEV-2015- 0554). He also acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC agreement no. 615112 HAPDEGMT. In addition, both authors were supported by NSF Grant DMS-1440140 while in residence at the MSRI in Berkeley, California, during Spring semester 2017.

[BJ] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion. They also present a counter-example to show that the result of [RR] may fail in the absence of some connectivity hypothesis (e.g., simple connectedness).

In dimensions greater than 2, a fundamental result of Dahlberg [Dah] establishes a quantitative version of absolute continuity, namely that harmonic measure belongs to the class A_∞ in an appropriate local sense (see Definitions 1.19 and 1.23 below), with respect to surface measure on the boundary of a Lipschitz domain.

The result of Dahlberg was extended to the class of Chord-arc domains (see Definition 1.13) by David and Jerison [DJ], and independently by Semmes [Sem]. The Chord-arc hypothesis was weakened to that of a two-sided Corkscrew condition (Definition 1.10) by Bennewitz and Lewis [BL], who then drew the conclusion that harmonic measure is weak-A_∞ (in an appropriate local sense, see Definitions 1.19 and 1.23) with respect to surface measure on the boundary; the latter condition is similar to the A_∞ condition, but without the doubling property, and is the best conclusion that can be obtained under the weakened geometric conditions considered in [BL]. We note that weak-A_∞ is still a quantitative, scale invariant version of absolute continuity.

While the present paper was in preparation, we learned of some very interesting recent work of J. Azzam [Azz], who has given a geometric characterization of the A_∞ property of harmonic measure with respect to surface measure for domains with Ahlfors-David regular (ADR) boundary (see Definition 1.6). This work is related to our own, so let us describe it in a bit more detail. Specifically, Azzam shows that for a domain Ω with ADR boundary, harmonic measure is in A_∞ with respect to surface measure, if and only if 1) $\partial\Omega$ is uniformly rectifiable (this is a quantitative, scale-invariant version of rectifiability, see Definition 1.8 and the ensuing comments), and 2) Ω is semi-uniform in the sense of Aikawa and Hirata [AH]. The semi-uniform condition is a connectivity condition which states that for some uniform constant M, every pair of points $X \in \Omega$ and $y \in \partial\Omega$ may be connected by a rectifiable curve $\gamma = \gamma(y, X)$, with $\gamma \setminus \{y\} \subset \Omega$, with length $\ell(\gamma) \leq M|X - y|$ and which satisfies the "cigar path" condition

$$\min\{\ell(\gamma(y, Z)), \ell(\gamma(Z, X))\} \leq M \operatorname{dist}(Z, \partial\Omega), \quad \forall Z \in \gamma. \tag{1.1}$$

Semi-uniformity is a weak version of the well known uniform condition, whose definition is similar, except that it applies to all pairs of points $X, Y \in \Omega$. For example, the unit disk centered at the origin, with the slit $-1/2 \leq x \leq 1/2, y = 0$ removed, is semi-uniform, but not uniform. It was shown in [AH] that for a domain satisfying a John condition and the Capacity Density Condition (in particular, for a domain with an ADR boundary), semi-uniformity characterizes the doubling property of harmonic measure. The method of [Azz] is, broadly speaking, related to that of [DJ], and of [BL]. In [DJ], the authors show that a Chord-arc domain Ω may be approximated in a "Big Pieces" sense (see [DJ] or [BL] for a precise statement) by Lipschitz subdomains $\Omega' \subset \Omega$; this fact allows one to reduce matters to the result of Dahlberg via the maximum principle (a method which, to the present authors' knowledge, first appears in [JK] in the context of BMO_1 domains). The same strategy, i.e., Big Piece approximation by Lipschitz subdomains, is employed in [BL]. Similarly, in [Azz], matters are reduced to the result of [DJ], by showing that for a domain Ω with an ADR boundary, Ω is semi-uniform with a uniformly rectifiable boundary if and only if it has "Big Pieces" of Chord-arc subdomains (see [Azz] for a precise statement of the latter condition). As mentioned above, the converse direction is also treated in [Azz]. In that case, given an interior Corkscrew condition (which holds automatically in the presence of the doubling property of harmonic measure), and provided that $\partial\Omega$ is ADR, the A_∞ (or even weak-A_∞) property of harmonic measure was already known to imply uniform rectifiability of the boundary [HM3] (although the published version appears in [HLMN]; see also [MT] for an alternative proof, and a somewhat more general result); as in [AH], semi-uniformity follows from the doubling property, although in [Azz], the author manages to show this while dispensing with the John domain background assumption (given a harmlessly strengthened version of the doubling property).

In light of the example of [BJ], it is an interesting open question to try to determine the minimal connectivity assumption, which, in conjunction with uniform rectifiability of the boundary, yields quantitative absolute continuity of harmonic measure with respect to surface measure. In the present work, we impose a significantly milder connectivity hypothesis than semi-uniformity, and we then show that harmonic measure ω satisfies a weak-A_∞ condition with respect to surface measure σ on the boundary, provided that $\partial\Omega$ is uniformly rectifiable. The weak-A_∞ conclusion is best possible in this generality: indeed, the stronger conclusion that $\omega \in A_\infty(\sigma)$, which entails doubling of ω, necessarily requires semi-uniformity, as Azzam has shown.

Let us now describe our connectivity hypothesis, which says, roughly speaking, that from each point $X \in \Omega$, there is local non-tangential access to an ample portion of a surface ball at a scale on the order of $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. Let us make this a bit more precise. A "carrot path" (aka non-tangential path) joining a point $X \in \Omega$, and a point $y \in \partial\Omega$, is a connected rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, such that for some $\lambda \in (0, 1)$ and for all $Z \in \gamma$,

$$\lambda\, \ell(\gamma(y, Z)) \leq \delta(Z)\,. \tag{1.2}$$

For $X \in \Omega$, and $R \geq 2$, set

$$\Delta_X = \Delta_X^R := B(X, R\delta(X)) \cap \partial\Omega\,.$$

We assume that every point $X \in \Omega$ may be joined by a carrot path to each y in a "Big Piece" of Δ_X, i.e., to each y in a Borel subset $F \subset \Delta_X$, with $\sigma(F) \geq \theta\sigma(\Delta_X)$, where σ denotes surface measure on $\partial\Omega$, and where the parameters $R \geq 2$, $\lambda \in (0, 1)$, and $\theta \in (0, 1]$ are uniformly controlled. We refer to this condition as a "weak local John condition", although "weak local semi-uniformity" would probably be equally appropriate. See Definitions 1.14, 1.16 and 1.18 for more details. We remark that a strong version of the local John condition (i.e., with $\theta = 1$) has appeared in [HMT], in connection with boundary Poincaré inequalities for non-smooth domains.

We observe that the weak local John condition is strictly weaker than semi-uniformity: for example, the unit disk centered a the origin, with either the cross $\{-1/2 \leq x \leq 1/2, y = 0\} \cup \{-1/2 \leq y \leq 1/2, x = 0\}$ removed, or with the slit $0 \leq x \leq 1, y = 0$ removed, satisfies the weak local John condition.

The main result of this paper is the following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$ be an open set, not necessarily connected, with a uniformly rectifiable (UR) boundary. If Ω satisfies the weak local John condition, then harmonic measure ω is locally in weak-A_∞ (see Definition 1.23) with respect to surface measure σ on $\partial\Omega$.*

We expect that the converse holds, assuming in addition that Ω satisfies an interior Corkscrew condition, and we hope to treat this direction in a future paper. Of course, as noted above, the weak-A_∞ condition already yields uniform rectifiability of the boundary [HM3] (also [HLMN] and [MT]), so it remains to show that weak-A_∞ implies the weak local John condition. As noted above, the stronger assumption that $\omega \in A_\infty$, since it entails doubling, yields the stronger conclusion that Ω is semi-uniform, by [Azz]. We include in Section 4 of the present paper an easy direct proof of the fact that doubling of harmonic measure implies a strong local John condition (i.e., with $\theta = 1$).

As is well known, quantitative absolute continuity, more precisely that $\omega \in$ weak-$A_\infty(\sigma)$ in the sense of Definition 1.23, is equivalent to an L^p solvability result for the Dirichlet problem. We therefore have the following.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$ be an open set, not necessarily connected, with a uniformly rectifiable boundary. Suppose in addition that Ω satisfies the weak local John condition. Then the L^p Dirichlet problem for Ω is solvable in L^p, for some $p < \infty$, i.e., given continuous data g defined on $\partial\Omega$, for the harmonic measure solution u to the Dirichlet problem with data g, we have for some $p < \infty$ that*

$$\|N_* u\|_{L^p(\partial\Omega)} \leq C \|g\|_{L^p(\partial\Omega)}, \tag{1.5}$$

where $N_* u$ is a suitable version of the non-tangential maximal function of u.

We refer the reader to, e.g., [HLe, Section 4] for details.

1.1. Further notation and definitions.

- Unless otherwise stated, we use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.
- Ω will always denote an open set in $\mathbb{R}^{n+1}$, not necessarily connected unless otherwise specified.
- We use the notation $\gamma(X, Y)$ to denote a rectifiable path with endpoints X and Y, and its arc-length will be denoted $\ell(\gamma(X, Y))$. Given such a path, if $Z \in \gamma(X, Y)$, we use the notation $\gamma(Z, Y)$ to denote the portion of the original path with endpoints Z and Y.

- Given an open set $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in Ω (or more generally in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- We let e_j, $j = 1, 2, \dots, n+1$, denote the standard unit basis vectors in $\mathbb{R}^{n+1}$.
- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \Omega$. A *surface ball* is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$.
- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.
- Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.
- Given an open set $\Omega \subset \mathbb{R}^{n+1}$, for $X \in \Omega$, we set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.
- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\lfloor_{\partial\Omega}$ denote the surface measure on $\partial\Omega$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A.
- Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $⨍_A f d\mu := \mu(A)^{-1} \int_A f d\mu$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$. Given an ADR set $E \subset \mathbb{R}^{n+1}$, we use Q (or sometimes P) to denote a dyadic "cube" on E. The latter exist (cf. [DS1], [Chr]), and enjoy certain properties which we enumerate in Lemma 1.26 below.

Definition 1.6. (**ADR**) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \leq \sigma(\Delta(x, r)) \leq C\, r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{1.7}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x, r) := E \cap B(x, r)$ is the *surface ball* of radius r, and as above, $\sigma := H^n\lfloor_E$ is the "surface measure" on E.

Definition 1.8. (**UR**) (aka *uniformly rectifiable*). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants c_1 and C_1, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than C_1, such that

$$H^n\Big(E \cap B(x, r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \geq c_1 r^n .$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [DS1]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NTV] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions); cf. [DS1, DS2].

Definition 1.9. (**"UR character"**). Given a UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants (c_1, C_1) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 1.10. (**Corkscrew condition**). Following [JK], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *Corkscrew point* relative to Δ. We note that we may allow $r < C \operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c. In order to emphasize that $B(X_\Delta, cr) \subset \Omega$, we shall sometimes refer to this property as the *interior Corkscrew condition*.

Definition 1.11. (**Harnack Chains, and the Harnack Chain condition** [JK]). Given two points $X, X' \in \Omega$, and a pair of numbers $M, N \geq 1$, an (M, N)-*Harnack Chain connecting X to X'*, is a chain of open balls $B_1, \dots, B_N \subset \Omega$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $M^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq M \operatorname{diam}(B_k)$. We say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant M such that for any two points $X, X' \in \Omega$, there is an (M, N)-Harnack Chain connecting them, with N depending only on the ratio $|X - X'|/ (\min(\delta(X), \delta(X')))$.

Definition 1.12. (**NTA**). Again following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA (*Non-tangentially accessible*) if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\rm ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the Corkscrew condition.

Definition 1.13. (**CAD**). We say that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a CAD (*Chord-arc domain*), if it is NTA, and if $\partial\Omega$ is ADR.

Definition 1.14. (**Carrot path**). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Given a point $X \in \Omega$, and a point $y \in \partial\Omega$, we say that a connected rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, is a *carrot path* (more precisely, a *λ-carrot path*) connecting y to X, if $\gamma \setminus \{y\} \subset \Omega$, and if for some $\lambda \in (0, 1)$ and for all $Z \in \gamma$,

$$\lambda\, \ell(\gamma(y, Z)) \leq \delta(Z)\,. \tag{1.15}$$

With a slight abuse of terminology, we shall sometimes refer to such a path as a *λ-carrot path in* Ω, although of course the endpoint y lies on $\partial\Omega$.

A carrot path is sometimes referred to as a non-tangential path.

Definition 1.16. (**(θ, λ, R)-weak local John point**). Let $X \in \Omega$, and for constants $\theta \in (0, 1]$, $\lambda \in (0, 1)$, and $R \geq 2$, set

$$\Delta_X = \Delta_X^R := B(X, R\delta(X)) \cap \partial\Omega \,.$$

We say that a point $X \in \Omega$ is a (θ, λ, R)-*weak local John point* if there is a Borel set $F \subset \Delta_X^R$, with $\sigma(F) \geq \theta\sigma(\Delta_X^R)$, such that for every $y \in F$, there is a λ-carrot path connecting y to X.

Thus, a weak local John point is non-tangentially connected to an ample portion of the boundary, locally. We observe that one can always choose R smaller, for possibly different values of θ and λ, by moving from X to a point X' on a line segment joining X to the boundary.

Remark 1.17. We observe that it is a slight abuse of notation to write Δ_X, since the latter is not centered on $\partial\Omega$, and thus it is not a true surface ball; on the other hand, there are true surface balls, $\Delta'_X := \Delta(\hat{x}, (R-1)\delta(X))$ and $\Delta''_X := \Delta(\hat{x}, (R+1)\delta(X))$, centered at a "touching point" $\hat{x} \in \partial\Omega$ with $\delta(X) = |X - \hat{x}|$, which, respectively, are contained in, and contain, Δ_X.

Definition 1.18. (**Weak local John condition**). We say that Ω satisfies a *weak local John condition* if there are constants $\lambda \in (0, 1)$, $\theta \in (0, 1]$, and $R \geq 2$, such that every $X \in \Omega$ is a (θ, λ, R)-weak local John point.

Definition 1.19. (A_∞, weak-A_∞, and weak-RH_q). Given an ADR set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and s such that for each surface ball $\Delta = B \cap E$, with $B \subseteq B_0$, we have

$$\mu(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(\Delta)\,, \qquad \text{for every Borel set } A \subset \Delta\,. \tag{1.20}$$

Similarly, we say that $\mu \in$ weak-$A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$, with $2B \subseteq B_0$,

$$\mu(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(2\Delta)\,, \qquad \text{for every Borel set } A \subset \Delta\,. \tag{1.21}$$

We recall that, as is well known, the condition $\mu \in$ weak-$A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0, and that for some $q > 1$, the Radon-Nikodym derivative $k := d\mu/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_\Delta k^q d\sigma\right)^{1/q} \lesssim \fint_{2\Delta} k\, d\sigma \approx \frac{\mu(2\Delta)}{\sigma(\Delta)}\,, \quad \forall\, \Delta = B \cap E,\ \text{with } 2B \subseteq B_0\,. \tag{1.22}$$

We shall refer to the inequality in (1.22) as an "RH_q" estimate, and we shall say that $k \in RH_q(\Delta_0)$ if k satisfies (1.22).

Definition 1.23. (**Local A_∞ and local weak-A_∞**). We say that harmonic measure ω is locally in A_∞ (resp., locally in weak-A_∞) on $\partial\Omega$, if there are uniform positive constants C and s such that for every ball $B = B(x,r)$ centered on $\partial\Omega$, with radius $r < \operatorname{diam}(\partial\Omega)/4$, and associated surface ball $\Delta = B \cap \partial\Omega$,

$$\omega^X(A) \le C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \omega^X(\Delta), \qquad \forall\, X \in \Omega\setminus 4B,\ \forall \text{ Borel } A \subset \Delta, \tag{1.24}$$

or, respectively, that

$$\omega^X(A) \le C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \omega^X(2\Delta), \qquad \forall\, X \in \Omega\setminus 4B,\ \forall \text{ Borel } A \subset \Delta; \tag{1.25}$$

equivalently, if for every ball B and surface ball $\Delta = B \cap \partial\Omega$ as above, and for each point $X \in \Omega\setminus 4B$, $\omega^X \in A_\infty(\Delta)$ (resp., $\omega^X \in$ weak-$A_\infty(\Delta)$) with uniformly controlled A_∞ (resp., weak-A_∞) constants.

Lemma 1.26. (**Existence and properties of the "dyadic grid"**) [DS1, DS2], [Chr]. *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-dimensional ADR set. Then there exist constants $a_0 > 0$, $s > 0$ and $C_1 < \infty$, depending only on n and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(i) *$E = \cup_j Q_j^k$ for each $k \in \mathbb{Z}$.*

(ii) *If $m \ge k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.*

(iii) *For each (j,k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.*

(iv) $\operatorname{diam}(Q_j^k) \le C_1 2^{-k}$.

(v) *Each Q_j^k contains some "surface ball" $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.*

(vi) *$H^n(\{x \in Q_j^k : \operatorname{dist}(x, E\setminus Q_j^k) \le \vartheta\, 2^{-k}\}) \le C_1\, \vartheta^s\, H^n(Q_j^k)$, for all k, j and for all $\vartheta \in (0, a_0)$.*

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Chr], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0,1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property (1.7), the result already appears in [DS1, DS2]. Some predecessors of this construction have appeared in [D1] and [D2].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

  $$\mathbb{D} := \cup_k \mathbb{D}_k,$$

  where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$ such that $r_Q \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r_Q) \subset Q \subset \Delta(x_Q, Cr_Q), \tag{1.27}$$

for some uniform constant C. We shall denote this ball and surface ball by

$$B_Q := B(x_Q, r_Q)\,, \qquad \Delta_Q := \Delta(x_Q, r_Q), \tag{1.28}$$

and we shall refer to the point x_Q as the "center" of Q.

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the dyadic generation to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- For a pair of cubes $Q', Q \in \mathbb{D}$, if Q' is a dyadic child of Q, i.e., if $Q' \subset Q$, and $\ell(Q) = 2\ell(Q')$, then we write $Q' \lhd Q$.

With the dyadic cubes in hand, we may now define the notion of a Corkscrew point relative to a cube Q.

Definition 1.29. (**Corkscrew point relative to** Q). Let Ω satisfy the Corkscrew condition (Definition 1.10), suppose that $\partial\Omega$ is ADR, and let $Q \in \mathbb{D}(\partial\Omega)$. A *Corkscrew point relative to* Q is simply a Corkscrew point relative to the surface ball Δ_Q defined (1.27)-(1.28).

Definition 1.30. (**Coherency**). [DS2]. Let $E \subset \mathbb{R}^{n+1}$ be an ADR set. Let $\mathbf{S} \subset \mathbb{D}(E)$. We say that $\mathbf{S}$ is *coherent* if the following conditions hold:

(a) $\mathbf{S}$ contains a unique maximal element $Q(\mathbf{S})$ which contains all other elements of $\mathbf{S}$ as subsets.

(b) If Q belongs to $\mathbf{S}$, and if $Q \subset \widetilde{Q} \subset Q(\mathbf{S})$, then $\widetilde{Q} \in \mathbf{S}$.

(c) Given a cube $Q \in \mathbf{S}$, either all of its children belong to $\mathbf{S}$, or none of them do.

We say that $\mathbf{S}$ is *semi-coherent* if conditions (a) and (b) hold.

2. Preliminaries

We begin by recalling a bilateral version of the David-Semmes "Corona decomposition" of a UR set. We refer the reader to [HMM] for the proof.

Lemma 2.1. ([HMM, Lemma 2.2]) *Let $E \subset \mathbb{R}^{n+1}$ be a UR set of dimension n. Then given any positive constants $\eta \ll 1$ and $K \gg 1$, there is a disjoint decomposition $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, satisfying the following properties.*

(1) *The "Good" collection $\mathcal{G}$ is further subdivided into disjoint stopping time regimes, such that each such regime $\mathbf{S}$ is coherent (Definition 1.30).*

(2) *The "Bad" cubes, as well as the maximal cubes $Q(\mathbf{S})$, $\mathbf{S} \subset \mathcal{G}$, satisfy a Carleson packing condition:*

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') \;+\; \sum_{\mathbf{S} \subset \mathcal{G}: Q(\mathbf{S}) \subset Q} \sigma(Q(\mathbf{S})) \;\leq\; C_{\eta,K}\, \sigma(Q)\,, \quad \forall Q \in \mathbb{D}(E)\,.$$

(3) *For each* $\mathbf{S} \subset \mathcal{G}$*, there is a Lipschitz graph* $\Gamma_{\mathbf{S}}$*, with Lipschitz constant at most* η*, such that, for every* $Q \in \mathbf{S}$*,*

$$\sup_{x\in\Delta_Q^*} \operatorname{dist}(x,\Gamma_{\mathbf{S}}) + \sup_{y\in B_Q^*\cap\Gamma_{\mathbf{S}}} \operatorname{dist}(y,E) < \eta\,\ell(Q)\,, \tag{2.2}$$

where $B_Q^* := B(x_Q, K\ell(Q))$ *and* $\Delta_Q^* := B_Q^* \cap E$*, and* x_Q *is the "center" of* Q *as in* (1.27)-(1.28).

We mention that David and Semmes, in [DS1], had previously proved a unilateral version of Lemma 2.1, in which the bilateral estimate (2.2) is replaced by the unilateral bound

$$\sup_{x\in\Delta_Q^*} \operatorname{dist}(x,\Gamma_{\mathbf{S}}) < \eta\,\ell(Q)\,, \qquad \forall\, Q\in\mathbf{S}\,. \tag{2.3}$$

Next, we make a standard Whitney decomposition of $\Omega_E := \mathbb{R}^{n+1}\setminus E$, for a given UR set E (in particular, Ω_E is open, since UR sets are closed by definition). Let $\mathcal{W} = \mathcal{W}(\Omega_E)$ denote a collection of (closed) dyadic Whitney cubes of Ω_E, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω_E, which satisfy

$$4\operatorname{diam}(I) \le \operatorname{dist}(4I,\partial\Omega) \le \operatorname{dist}(I,\partial\Omega) \le 40\operatorname{diam}(I)\,, \qquad \forall\, I\in\mathcal{W} \tag{2.4}$$

(just dyadically divide the standard Whitney cubes, as constructed in [Ste, Chapter VI], into cubes with side length 1/8 as large) and also

$$(1/4)\operatorname{diam}(I_1) \le \operatorname{diam}(I_2) \le 4\operatorname{diam}(I_1)\,,$$

whenever I_1 and I_2 touch.

We fix a small parameter $\tau_0 > 0$, so that for any $I \in \mathcal{W}$, and any $\tau \in (0,\tau_0]$, the concentric dilate

$$I^*(\tau) := (1+\tau)I \tag{2.5}$$

still satisfies the Whitney property

$$\operatorname{diam} I \approx \operatorname{diam} I^*(\tau) \approx \operatorname{dist}\left(I^*(\tau),E\right) \approx \operatorname{dist}(I,E)\,, \quad 0<\tau\le\tau_0\,. \tag{2.6}$$

Moreover, for $\tau \le \tau_0$ small enough, and for any $I, J \in \mathcal{W}$, we have that $I^*(\tau)$ meets $J^*(\tau)$ if and only if I and J have a boundary point in common, and that, if $I \neq J$, then $I^*(\tau)$ misses $(3/4)J$.

Pick two parameters $\eta \ll 1$ and $K \gg 1$ (eventually, we shall take $K = \eta^{-3/4}$). For $Q \in \mathbb{D}(E)$, define

$$\mathcal{W}_Q^0 := \left\{I\in\mathcal{W} : \eta^{1/4}\ell(Q) \le \ell(I) \le K^{1/2}\ell(Q),\ \operatorname{dist}(I,Q) \le K^{1/2}\ell(Q)\right\}. \tag{2.7}$$

Remark 2.8. We note that $\mathcal{W}_Q^0$ is non-empty, provided that we choose η small enough, and K large enough, depending only on dimension and ADR, since the ADR condition implies that Ω_E satisfies a Corkscrew condition. In the sequel, we shall always assume that η and K have been so chosen.

Next, we recall a construction in [HMM, Section 3], leading up to and including in particular [HMM, Lemma 3.24]. We summarize this construction as follows.

Lemma 2.9. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set, and let $\Omega_E := \mathbb{R}^{n+1} \setminus E$. Given positive constants $\eta \ll 1$ and $K \gg 1$, as in (2.7) and Remark 2.8, let $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, be the corresponding bilateral Corona decomposition of Lemma 2.1. Then for each $\mathbf{S} \subset \mathcal{G}$, and for each $Q \in \mathbf{S}$, the collection $\mathcal{W}_Q^0$ in (2.7) has an augmentation $\mathcal{W}_Q^* \subset \mathcal{W}$ satisfying the following properties.*

(1) *$\mathcal{W}_Q^0 \subset \mathcal{W}_Q^* = \mathcal{W}_Q^{*,+} \cup \mathcal{W}_Q^{*,-}$, where (after a suitable rotation of coordinates) each $I \in \mathcal{W}_Q^{*,+}$ lies above the Lipschitz graph $\Gamma_{\mathbf{S}}$ of Lemma 2.1, each $I \in \mathcal{W}_Q^{*,-}$ lies below $\Gamma_{\mathbf{S}}$. Moreover, if Q' is a child of Q, also belonging to $\mathbf{S}$, then $\mathcal{W}_Q^{*,+}$ (resp. $\mathcal{W}_Q^{*,-}$) belongs to the same connected component of Ω_E as does $\mathcal{W}_{Q'}^{*,+}$ (resp. $\mathcal{W}_{Q'}^{*,-}$) and $\mathcal{W}_{Q'}^{*,+} \cap \mathcal{W}_Q^{*,+} \neq \emptyset$ (resp., $\mathcal{W}_{Q'}^{*,-} \cap \mathcal{W}_Q^{*,-} \neq \emptyset$).*

(2) *There are uniform constants c and C such that*

$$\begin{aligned} c\eta^{1/2}\ell(Q) \le \ell(I) \le CK^{1/2}\ell(Q)\,, &\quad \forall I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I,Q) \le CK^{1/2}\ell(Q)\,, &\quad \forall I \in \mathcal{W}_Q^*, \\ c\eta^{1/2}\ell(Q) \le \operatorname{dist}(I^*(\tau),\Gamma_{\mathbf{S}})\,, &\quad \forall I \in \mathcal{W}_Q^*\,, \quad \forall \tau \in (0,\tau_0]\,. \end{aligned} \tag{2.10}$$

Moreover, given $\tau \in (0,\tau_0]$, set

$$U_Q^\pm = U_{Q,\tau}^\pm := \bigcup_{I \in \mathcal{W}_Q^{*,\pm}} \operatorname{int}(I^*(\tau))\,, \qquad U_Q := U_Q^+ \cup U_Q^-\,, \tag{2.11}$$

and given $\mathbf{S}'$, a semi-coherent subregime of $\mathbf{S}$, define

$$\Omega_{\mathbf{S}'}^\pm = \Omega_{\mathbf{S}'}^\pm(\tau) := \bigcup_{Q \in \mathbf{S}'} U_Q^\pm\,. \tag{2.12}$$

Then each of $\Omega_{\mathbf{S}'}^\pm$ is a CAD, with Chord-arc constants depending only on n, τ, η, K, and the ADR/UR constants for $\partial\Omega$.

Remark 2.13. In particular, for each $\mathbf{S} \subset \mathcal{G}$, if Q' and Q belong to $\mathbf{S}$, and if Q' is a dyadic child of Q, then $U_{Q'}^+ \cup U_Q^+$ is Harnack Chain connected, and every pair of points $X, Y \in U_{Q'}^+ \cup U_Q^+$ may be connected by a Harnack Chain in Ω_E of length at most $C = C(n, \tau, \eta, K, \text{ADR/UR})$. The same is true for $U_{Q'}^- \cup U_Q^-$.

Remark 2.14. Let $0 < \tau \le \tau_0/2$. Given any $\mathbf{S} \subset \mathcal{G}$, and any semi-coherent subregime $\mathbf{S}' \subset \mathbf{S}$, define $\Omega_{\mathbf{S}'}^\pm = \Omega_{\mathbf{S}'}^\pm(\tau)$ as in (2.12), and similarly set $\widehat{\Omega}_{\mathbf{S}'}^\pm = \Omega_{\mathbf{S}'}^\pm(2\tau)$. Then by construction, for any $X \in \overline{\Omega_{\mathbf{S}'}^\pm}$,

$$\operatorname{dist}(X,E) \approx \operatorname{dist}(X, \partial\widehat{\Omega}_{\mathbf{S}'}^\pm)\,,$$

where of course the implicit constants depend on τ.

As in [HMM], it will be useful for us to extend the definition of the Whitney region U_Q to the case that $Q \in \mathcal{B}$, the "bad" collection of Lemma 2.1. Let $\mathcal{W}_Q^*$ be the augmentation of $\mathcal{W}_Q^0$ as constructed in Lemma 2.9, and set

$$\mathcal{W}_Q := \begin{cases} \mathcal{W}_Q^*\,, & Q \in \mathcal{G}, \\ \mathcal{W}_Q^0\,, & Q \in \mathcal{B} \end{cases}\,. \tag{2.15}$$

For $Q \in \mathcal{G}$ we shall henceforth simply write $\mathcal{W}_Q^{\pm}$ in place of $\mathcal{W}_Q^{*,\pm}$. For arbitrary $Q \in \mathbb{D}(E)$, we may then define

$$U_Q = U_{Q,\tau} := \bigcup_{I \in \mathcal{W}_Q} \operatorname{int}(I^*(\tau)) \,. \tag{2.16}$$

Let us note that for $Q \in \mathcal{G}$, the latter definition agrees with that in (2.11).

For future reference, we introduce dyadic sawtooth regions as follows. Set

$$\mathbb{D}_Q := \{Q' \in \mathbb{D}(E) : Q' \subset Q\} \,, \tag{2.17}$$

and given $k \geq 1$,

$$\mathbb{D}_Q^k := \left\{Q' \in \mathbb{D}(E) : Q' \subset Q,\ \ell(Q') = 2^{-k}\,\ell(Q)\right\} \,, \tag{2.18}$$

Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the *global discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} \,, \tag{2.19}$$

i.e., $\mathbb{D}_{\mathcal{F}}$ is the collection of all $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$. We may allow $\mathcal{F}$ to be empty, in which case $\mathbb{D}_{\mathcal{F}} = \mathbb{D}$. Given some fixed cube Q, the *local discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q. \tag{2.20}$$

Note that with this convention, $\mathbb{D}_Q = \mathbb{D}_{\emptyset,Q}$ (i.e., if one takes $\mathcal{F} = \emptyset$ in (2.20)).

Finally, we conclude this section with a well-known consequence of the ADR property of $\partial\Omega$.

Lemma 2.21. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with ADR boundary. Let $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is non-negative and harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \overline{\Omega}$, and that $u \equiv 0$ on $\partial\Omega \cap B(x, 2r)$. Then, there exist constants $\alpha \in (0, 1)$, and $C > 0$, depending only on n and the ADR constant, such that*

$$u(Z) \leq C\left(\frac{\delta(Z)}{r}\right)^{\alpha} \max_{B(x,2r)\cap\Omega} u. \tag{2.22}$$

We omit the proof, which is a consequence of the fact that an open set with ADR boundary satisfies the Capacity Density Condition. This is standard, but see, e.g., [HKM, Theorem 6.38], or [HLMN, Remark 3.26, Lemma 3.27, Lemma 3.31].

3. Proof of Theorem 1.3

In the proof of Theorem 1.3, we shall employ a two-parameter induction argument, which is a refinement of the method of "extrapolation" of Carleson measures. The latter is a bootstrapping scheme for lifting the Carleson measure constant, developed by J. L. Lewis [LM], and based on the corona construction of Carleson [Car] and Carleson and Garnett [CG] (see also [HLw], [AHLT], [AHMTT], [HM1], [HM2],[HMM]).

3.1. **Step 1: the set-up.** To set the stage for the induction procedure, let us begin by making some preliminary reductions. First, by the method of [BL], more precisely, from the combination of [BL, Lemma 2.2] and its proof, and [BL, Lemma 3.1], it suffices to show that there are positive constants ε and c such that for each $X \in \Omega$ with $\delta(X) \lesssim \operatorname{diam}(\partial\Omega)$, if $\Delta_X = \Delta_X^R = B(X, R\delta(X)) \cap \partial\Omega$, for some fixed $R \geq 2$ as in Definition 1.16, and if A is a Borel subset of Δ_X, then

$$\sigma(A) \geq (1-\varepsilon)\sigma(\Delta_X) \implies \omega^X(A) \geq c\,. \tag{3.1}$$

It will be convenient to work with a certain dyadic version of (3.1). To this end, let $X \in \Omega$, and let $\hat{x} \in \partial\Omega$ be a touching point for X, i.e., $|X - \hat{x}| = \delta(X)$. Choose X_1 on the line segment joining X to $\hat{x}$, with $\delta(X_1) = \delta(X)/2$. Then $\Delta_{X_1} = B(X_1, R\delta(X)/2) \cap \partial\Omega$. Note that $B(X_1, R\delta(X)/2) \subset B(X, R\delta(X))$, and furthermore,

$$\operatorname{dist}\Big(B(X_1, R\delta(X)/2), \partial B(X, R\delta(X)\Big) > \frac{R-1}{2}\delta(X) \geq \frac{1}{2}\delta(X).$$

We may therefore cover Δ_{X_1} by a disjoint collection $\{Q_i\}_{i=1}^N \subset \mathbb{D}(\partial\Omega)$, of equal length $\ell(Q_i) \approx \delta(X)$, such that each $Q_i \subset \Delta_X$, and such that the implicit constants depend only on n and ADR, and thus the cardinality N of the collection depends on n, ADR, and R. With $E = \partial\Omega$, we make the Whitney decomposition of the set $\Omega_E = \mathbb{R}^{n+1}\setminus E$ as in Section 2 (thus, $\Omega \subset \Omega_E$). Moreover, for sufficiently small η and sufficiently large K in (2.7), we then have that $X \in U_{Q_i}$ for each $i = 1, 2, \dots, N$. By hypothesis, there are constants $\theta_0 \in (0,1]$, $\lambda_0 \in (0,1)$, and $R \geq 2$ as above, such that every $X \in \Omega$ is a (θ_0, λ_0, R)-weak local John point (Definition 1.16). In particular, this is true for X_1, hence there is a Borel set $F \subset \Delta_{X_1}$, with $\sigma(F) \geq \theta_0\sigma(\Delta_{X_1})$, such that every $y \in F$ may be connected to X_1 via a λ_0-carrot path. By ADR, $\sigma(\Delta_{X_1}) \approx \sum_{i=1}^N \sigma(Q_i)$ and thus by pigeon-holing, there is at least one $Q_i =: Q$ such that $\sigma(F \cap Q) \geq \theta_1\sigma(Q)$, with θ_1 depending only on θ_0, n and ADR. Moreover, the λ_0-carrot path connecting each $y \in F$ to X_1 may be extended to a λ_1-carrot path connecting y to X, where λ_1 depends only on λ_0.

We have thus reduced matters to the following dyadic scenario. Let $Q \in \mathbb{D}(\partial\Omega)$, and let $U_Q = U_{Q,\tau}$ be the associated Whitney region as in (2.16), with $\tau \leq \tau_0/2$ fixed, and suppose that U_Q meets Ω (recall that by construction $U_Q \subset \Omega_E$, with $E = \partial\Omega$). For $X \in U_Q \cap \Omega$, and for a constant $\lambda \in (0,1)$, let

$$F_{car}(X, Q) = F_{car}(X, Q, \lambda) \tag{3.2}$$

denote the set of $y \in Q$ which may be joined to X by a λ-carrot path $\gamma(y, X)$, and for $\theta \in (0,1]$, set

$$T_Q = T_Q(\theta, \lambda) := \{X \in U_Q \cap \Omega :\ \sigma(F_{car}(X, Q, \lambda)) \geq \theta\sigma(Q)\}\,. \tag{3.3}$$

Our goal is to prove that, given $\lambda \in (0,1)$ and $\theta \in (0,1]$, there are positive constants ε and c, depending on θ, λ, and the allowable parameters, such that for each $Q \in \mathbb{D}(\partial\Omega)$, if A is a Borel subset of Q, then

$$\sigma(A) \geq (1-\varepsilon)\sigma(Q) \implies \omega^X(A) \geq c\,, \qquad \forall\, X \in T_Q(\theta, \lambda)\,. \tag{3.4}$$

Remark 3.5. For some $Q \in \mathbb{D}(\partial\Omega)$, it may be that T_Q is empty. On the other hand, by the preceding discussion, each $X \in \Omega$ belongs to $T_Q(\theta_1, \lambda_1)$ for suitable Q, θ_1 and λ_1, so that (3.4) (with $\theta = \theta_1, \lambda = \lambda_1$) implies (3.1), for a slightly different choice of sufficiently small ε; more precisely, the left hand inequality in (3.1), with

ε replaced by ε/C, implies the left hand inequality in (3.4), with $A \cap Q$ in place of A, where Q is the particular Q_i selected in the previous paragraph.

The rest of this section is therefore devoted to proving (3.4) (when it is not vacuous). To this end, we let $\lambda \in (0,1)$ (by Remark 3.5, any fixed $\lambda \leq \lambda_1$ will suffice). We also fix positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, and for these values of η and K, we make the bilateral Corona decomposition of Lemma 2.1, so that $\mathbb{D}(\partial\Omega) = \mathcal{G} \cup \mathcal{B}$. We also construct the Whitney collections $\mathcal{W}_Q^0$ in (2.7), and $\mathcal{W}_Q^*$ of Lemma 2.9 for this same choice of η and K.

Given a cube $Q \in \mathbb{D}(\partial\Omega)$, we set

$$\mathbb{D}_*(Q) := \{Q' \subset Q : \ \ell(Q)/4 \leq \ell(Q') \leq \ell(Q),\ Q'\}\,. \tag{3.6}$$

Thus, $\mathbb{D}_*(Q)$ consists of the cube Q itself, along with its dyadic children and grandchildren. Let

$$\mathcal{M} := \{Q(\mathbf{S})\}_{\mathbf{S}}$$

denote the collection of cubes which are the maximal elements of the stopping time regimes in $\mathcal{G}$. We define

$$\alpha_Q := \begin{cases} \sigma(Q)\,, & \text{if } (\mathcal{M} \cup \mathcal{B}) \cap \mathbb{D}_*(Q) \neq \emptyset, \\ 0\,, & \text{otherwise.} \end{cases} \tag{3.7}$$

Given any collection $\mathbb{D}' \subset \mathbb{D}(\partial\Omega)$, we set

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q. \tag{3.8}$$

Then $\mathfrak{m}$ is a discrete Carleson measure, i.e., recalling that $\mathbb{D}_Q$ is the discrete Carleson region relative to Q defined in (2.17), we claim that there is a uniform constant C such that

$$\mathfrak{m}(\mathbb{D}_Q) \;=\; \sum_{Q' \subset Q} \alpha_{Q'} \;\leq\; C\sigma(Q)\,, \qquad \forall\, Q \in \mathbb{D}(\partial\Omega)\,. \tag{3.9}$$

Indeed, note that for any $Q' \in \mathbb{D}_Q$, there are at most 3 cubes Q such that $Q' \in \mathbb{D}_*(Q)$ (namely, Q' itself, its dyadic parent, and its dyadic grandparent), and that by ADR, $\sigma(Q) \approx \sigma(Q')$, if $Q' \in \mathbb{D}_*(Q)$. Thus, given any $Q_0 \in \mathbb{D}(\partial\Omega)$,

$$\mathfrak{m}(\mathbb{D}_{Q_0}) \;=\; \sum_{Q \subset Q_0} \alpha_Q \;\leq\; \sum_{Q' \in \mathcal{M} \cup \mathcal{B}} \; \sum_{Q \subset Q_0 :\, Q' \in \mathbb{D}_*(Q)} \sigma(Q)$$

$$\lesssim \sum_{Q' \in \mathcal{M} \cup \mathcal{B} :\, Q' \subset Q_0} \sigma(Q') \;\leq\; C\sigma(Q_0)\,,$$

by Lemma 2.1 (2). Here, and throughout the remainder of this section, a generic constant C, and implicit constants, are allowed to depend upon the choice of the parameters η and K that we have fixed, along with the usual allowable parameters.

With (3.9) in hand, we therefore have

$$M_0 := \sup_{Q \in \mathbb{D}(E)} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} \leq C < \infty\,. \tag{3.10}$$

As mentioned above, our proof will be based on a two parameter induction scheme. Given $\lambda \in (0, \lambda_1]$ fixed as above, we recall that the set $F_{car}(X, Q, \lambda)$ is defined in (3.2). The induction hypothesis, which we formulate for any $a \geq 0$, and any $\theta \in (0,1]$ is as follows:

$\boxed{H[a,\theta]}$

There exist $c_a = c_a(\theta), \varepsilon_a = \varepsilon_a(\theta) \in (0,1)$ such that for any given $Q \in \mathbb{D}(\partial\Omega)$, if

$$\mathfrak{m}(\mathbb{D}_Q) \leq a\sigma(Q), \tag{3.11}$$

if there is a subset $V_Q \subset U_Q \cap \Omega$ for which

$$\sigma\left(\bigcup_{X \in V_Q} F_{car}(X,Q,\lambda)\right) \geq \theta\sigma(Q), \tag{3.12}$$

and if $A \subset Q$, then

$$\sigma(A) \geq (1-\varepsilon_a)\sigma(Q) \implies \frac{1}{|\widehat{U}_Q|}\iint_{\widehat{U}_Q} \omega^Y(A)\,dY \geq c_a\,, \tag{3.13}$$

In the last expression

$$\widehat{U}_Q = \widehat{U}_Q(\theta,\lambda,V_Q) := \bigcup_{i:U_Q^i \cap V_Q \neq \emptyset} U_Q^i\,, \tag{3.14}$$

where each U_Q^i is a connected component of U_Q, and where the union runs over those U_Q^i that meet V_Q (thus, each such $U_Q^i \subset \Omega$, by construction).

Let us briefly sketch the strategy of the proof. We first fix $\theta = 1$, and by induction on a, establish $H[M_0, 1]$. We then show that there is a fixed $\zeta \in (0,1)$ such that $H[M_0,\theta]$ implies $H[M_0,\zeta\theta]$, for every $\theta \in (0,1]$. Iterating, we then obtain $H[M_0,\theta_1]$ for any $\theta_1 \in (0,1]$. Now, by (3.10), we have (3.11) with $a = M_0$, for every $Q \in \mathbb{D}(\partial\Omega)$. Thus, $H[M_0,\theta_1]$ may be applied in every cube Q such that $T_Q(\theta_1,\lambda)$ (see (3.3)) is non-empty, with $V_Q = \{X\}$, for any $X \in T_Q(\theta_1,\lambda)$. For $\lambda \leq \lambda_1$, and an appropriate choice of θ_1, by Remark 3.5, we obtain (3.1), and thus that Theorem 1.3 holds, as desired.

We begin with some preliminary observations. In what follows we have fixed $\lambda \in (0,\lambda_1]$ and two positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, for which the bilateral Corona decomposition of $\mathbb{D}(\partial\Omega)$ in Lemma 2.1 is applied. We now fix $k_0 \in \mathbb{N}$, $k_0 \geq 4$, such that

$$2^{-k_0} \leq \frac{\eta}{K} < 2^{-k_0+1}\,. \tag{3.15}$$

Lemma 3.16. *Let $Q \in \mathbb{D}(\partial\Omega)$, and suppose that $Q' \subset Q$, with $\ell(Q') \leq 2^{-k_0}\ell(Q)$. Suppose that there are points $X \in U_Q \cap \Omega$ and $y \in Q'$, that are connected by a λ-carrot path $\gamma = \gamma(y,X)$ in Ω. Then γ meets $U_{Q'} \cap \Omega$.*

Proof. By construction (see (2.7), Lemma 2.9, (2.15) and (2.16)), $X \in U_Q$ implies that

$$\eta^{1/2}\ell(Q) \lesssim \delta(X) \lesssim K^{1/2}\ell(Q)\,.$$

Since $2^{-k_0} \ll \eta$, and $\ell(Q') \leq 2^{-k_0}\ell(Q)$, we then have that $X \in \Omega \setminus B(y, 2\ell(Q'))$. Thus, $\gamma(y,X)$ meets $B(y,2\ell(Q')) \setminus B(y,\ell(Q'))$, say at a point Z. Since $\gamma(y,X)$ is a

λ-carrot path, and since we have previously specified that $\eta \ll \lambda^4$,

$$\delta(Z) \geq \lambda\ell(\gamma(y,Z)) \geq \lambda|y-Z| \geq \lambda\ell(Q') \gg \eta^{1/4}\ell(Q').$$

On the other hand

$$\delta(Z) \leq \operatorname{dist}(Z,Q') \leq |Z-y| \leq 2\ell(Q') \ll K^{1/2}\ell(Q').$$

In particular then, the Whitney box I containing Z must belong to $\mathcal{W}^0_{Q'}$ (see (2.7)), so $Z \in U_{Q'}$. Note that $Z \in \Omega$ since $\gamma \subset \Omega$. $\square$

We shall also require the following. We recall that by Lemma 2.9, for $Q \in \mathbf{S}$, the Whitney region U_Q has the splitting $U_Q = U_Q^+ \cup U_Q^-$, with U_Q^+ (resp. U_Q^-) lying above (resp., below) the Lipschitz graph $\Gamma_\mathbf{S}$ of Lemma 2.1.

Lemma 3.17. *Let $Q' \subset Q$, and suppose that Q' and Q both belong to $\mathcal{G}$, and moreover that both Q' and Q belong to the same stopping time regime* $\mathbf{S}$. *Suppose that $y \in Q'$ and $X \in U_Q \cap \Omega$ are connected via a λ-carrot path $\gamma(y,X)$ in Ω, and assume that there is a point $Z \in \gamma(y,X) \cap U_{Q'} \cap \Omega$ (by Lemma 3.16 we know that such a Z exists provided $\ell(Q') \leq 2^{-k_0}\ell(Q)$). Then $X \in U_Q^+$ if and only if $Z \in U_{Q'}^+$ (thus, $X \in U_Q^-$ if and only if $Z \in U_{Q'}^-$).*

Proof of Lemma 3.17. We suppose for the sake of contradiction that, e.g., $X \in U_Q^+$, and that $Z \in U_{Q'}^-$. Thus, in traveling from y to Z and then to X along the path $\gamma(y,X)$, one must cross the Lipschitz graph $\Gamma_\mathbf{S}$ at least once between Z and X. Let Y_1 be the first point on $\gamma(y,X) \cap \Gamma_\mathbf{S}$ that one encounters *after* Z, when traveling toward X. By Lemma 2.9,

$$K^{1/2}\ell(Q) \gtrsim \delta(X) \geq \lambda\ell(\gamma(y,X)) \gg K^{-1/4}\ell(\gamma(y,X)),$$

where we recall that we have fixed $K \gg \lambda^{-4}$. Consequently, $\ell(\gamma(y,X)) \ll K^{3/4}\ell(Q)$, so in particular, $\gamma(y,X) \subset B_Q^* := B(x_Q, K\ell(Q))$, as in Lemma 2.1. On the other hand, $Y_1 \notin B_{Q'}^*$. Indeed, $Y_1 \in \Gamma_S$, so if $Y_1 \in B_{Q'}^*$, then by (2.2), $\delta(Y_1) \leq \eta\ell(Q')$. However,

$$\delta(Y_1) \geq \lambda\ell(\gamma(y,Y_1)) \geq \lambda\ell(\gamma(y,Z)) \geq \lambda|y-Z| \geq \lambda \operatorname{dist}(Z,Q') \gtrsim \lambda\eta^{1/2}\ell(Q'),$$

where in the last step we have used Lemma 2.9. This contradicts our choice of $\eta \ll \lambda^4$.

We now form a chain of consecutive dyadic cubes $\{P_i\} \subset \mathbb{D}_Q$, connecting Q' to Q, i.e.,

$$Q' = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_M \subset P_{M+1} = Q, \qquad \ell(P_{i+1}) = 2\ell(P_i),$$

and let $P := P_{i_0}$, $1 \leq i_0 \leq M+1$, be the smallest of the cubes P_i such that $Y_1 \in B_{P_i}^*$. Setting $P' := P_{i_0-1}$, we then have that $Y_1 \in B_P^*$, and $Y_1 \notin B_{P'}^*$. By the coherency of $\mathbf{S}$, $P \in \mathbf{S}$, so by (2.2),

$$\delta(Y_1) \leq \eta\ell(P). \tag{3.18}$$

On the other hand,

$$\operatorname{dist}(Y_1,P') \gtrsim K\ell(P') \approx K\ell(P),$$

and therefore, since $y \in Q' \subset P'$,

$$\delta(Y_1) \geq \lambda\ell(\gamma(y,Y_1)) \geq \lambda|y-Y_1| \geq \lambda \operatorname{dist}(Y_1,P') \gtrsim \lambda K\ell(P). \tag{3.19}$$

Combining (3.18) and (3.19), we see that $\lambda \lesssim \eta/K$, which contradicts that we have fixed $\eta \ll \lambda^4$, and $K \gg \lambda^{-4}$. □

Lemma 3.20. *Fix $\lambda \in (0,1)$. Given $Q \in \mathbb{D}(\partial\Omega)$ and a non-empty set $V_Q \subset U_Q \cap \Omega$, let*

$$F_Q := \bigcup_{X \in V_Q} F_{car}(X, Q, \lambda)\,, \tag{3.21}$$

where we recall that $F_{car}(X, Q, \lambda)$ is the set of $y \in Q$ that are connected via a λ-carrot path to X (see (3.2)). Let $Q' \subset Q$ be such that $\ell(Q') \leq 2^{-k_0}\ell(Q)$ and $F_Q \cap Q' \neq \emptyset$. Then, there exists a non-empty set $V_{Q'} \subset U_{Q'} \cap \Omega$ such that if we define $F_{Q'}$ as in (3.21) with Q' replacing Q then $F_Q \cap Q' \subset F_{Q'}$. Moreover, for every $Y \in V_{Q'}$, there exist $X \in V_Q$, $y \in Q'$ and a λ-carrot path $\gamma = \gamma(y, X)$ such that $Y \in \gamma$.

Proof of Lemma 3.20. For every $y \in F_Q \cap Q'$, by definition of F_Q, there exist $X \in V_Q$ and a λ-carrot path $\gamma = \gamma(y, X)$. By Lemma 3.16, there is $Y = Y(y) \in \gamma \cap U_{Q'} \cap \Omega$ (there can be more than one Y, but we just pick one). Note that the sub-path $\gamma(y, Y) \subset \gamma(y, X)$ is also a λ-carrot path, for the same constant λ. All the conclusions in the lemma follow easily from the construction by letting $V_{Q'} = \bigcup_{y \in F_Q \cap Q'} Y(y)$. □

Remark 3.22. It follows easily from the previous proof that under the same assumptions, if one further assumes that $\ell(Q') < 2^{-k_0}\,\ell(Q)$, we can then repeat the argument with both Q' and $(Q')^*$ (the dyadic parent of Q') to obtain respectively $V_{Q'}$ and $V_{(Q')^*}$. Moreover, this can be done in such a way that every point in $V_{Q'}$ (resp. $V_{(Q')^*}$) belongs to a λ-carrot path which also meets $V_{(Q')^*}$ (resp. $V_{Q'}$), connecting U_Q and Q'.

Given a family $\mathcal{F} := \{Q_j\} \subset \mathbb{D}(\partial\Omega)$ of pairwise disjoint cubes, we recall that the "discrete sawtooth" $\mathbb{D}_{\mathcal{F}}$ is the collection of all cubes in $\mathbb{D}(\partial\Omega)$ that are not contained in any $Q_j \in \mathcal{F}$ (see (2.19)), and we define the restriction of $\mathfrak{m}$ to the sawtooth $\mathbb{D}_{\mathcal{F}}$ by

$$\mathfrak{m}_{\mathcal{F}}(\mathbb{D}') := \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q \in \mathbb{D}' \setminus (\cup_{\mathcal{F}} \mathbb{D}_{Q_j})} \alpha_Q. \tag{3.23}$$

We then set

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} := \sup_{Q' \subset Q} \frac{\mathfrak{m}_{\mathcal{F}}(\mathbb{D}_{Q'})}{\sigma(Q')}.$$

Let us note that we may allow $\mathcal{F}$ to be empty, in which case $\mathbb{D}_{\mathcal{F}} = \mathbb{D}$ and $\mathfrak{m}_{\mathcal{F}}$ is simply $\mathfrak{m}$. We note that the following claims remain true when $\mathcal{F}$ is empty, with some straightforward changes that are left to the interested reader.

Claim 3.24. *Given $Q \in \mathbb{D}(\partial\Omega)$, and a family $\mathcal{F} = \mathcal{F}_Q := \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$ of pairwise disjoint sub-cubes of Q, if $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq 1/2$, then each $Q' \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$, each $Q_j \in \mathcal{F}$, and every dyadic child Q'_j of any $Q_j \in \mathcal{F}$, belong to the good collection $\mathcal{G}$, and moreover, every such cube belongs to the* **same** *stopping time regime* **S**. *In particular,* $\mathbf{S}' := \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ *is a semi-coherent sub-regime of* **S**, *and so is* $\mathbf{S}'' := (\mathbb{D}_{\mathcal{F}} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathbb{D}_Q$, *where $\mathcal{F}'$ denotes the collection of all dyadic children of cubes in $\mathcal{F}$.*

Indeed, if any $Q' \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ were in $\mathcal{M} \cup \mathcal{B}$ (recall that $\mathcal{M} := \{Q(\mathbf{S})\}_{\mathbf{S}}$ is the collection of cubes which are the maximal elements of the stopping time regimes in $\mathcal{G}$), then by construction $\alpha_{Q'} = \sigma(Q')$ for that cube (see (3.7)), so by definition of $\mathfrak{m}$ and $\mathfrak{m}_{\mathcal{F}}$, we would have

$$1 = \frac{\sigma(Q')}{\sigma(Q')} \leq \frac{\mathfrak{m}_{\mathcal{F}}(\mathbb{D}_{Q'})}{\sigma(Q')} \leq \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \frac{1}{2}\,,$$

a contradiction. Similarly, if some $Q_j \in \mathcal{F}$ (respectively, $Q'_j \in \mathcal{F}'$) were in $\mathcal{M} \cup \mathcal{B}$, then its dyadic parent (respectively, dyadic grandparent) Q_j^* would belong to $\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$, and by definition $\alpha_{Q_j^*} = \sigma(Q_j^*)$, so again we reach a contradiction. Consequently, $\mathcal{F} \cup \mathcal{F}' \cup (\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q)$ does not meet $\mathcal{M} \cup \mathcal{B}$, and the claim follows.

Lemma 3.25. *Let $Q \subset \mathbb{D}(\partial\Omega)$, and suppose that $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$ is a family of pairwise disjoint sub-cubes of Q, such that $\mathbf{S}' := \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ is a semi-coherent sub-regime of some $\mathbf{S} \subset \mathcal{G}$, and such that also $\mathcal{F} \subset \mathbf{S}$. Suppose that U_Q^+ meets Ω (and thus, $\Omega_{\mathbf{S}'}^+ \subset \Omega$). Then given any $Q_j \in \mathcal{F}$, there is a point $Z_1 \in \partial\Omega_{\mathbf{S}'}^+$ such that*

$$\operatorname{dist}(Z_1, Q_j) \approx \delta(Z_1) \approx \ell(Q_j)\,,$$

where the implicit constants may depend on η, and moreover,

$$|Z_1 - x_{Q_j}| \leq \ell(Q_j)\,. \tag{3.26}$$

Proof of Lemma 3.25. We first observe that by rotation we may assume that U_Q^+ (and hence $\Omega_{\mathbf{S}'}^+$) lies above the Lipschitz graph $\Gamma_{\mathbf{S}}$. Let Q_j^* denote the dyadic parent of Q_j. Observe that by hypothesis, $Q_j^* \in \mathbf{S}'$. Let x_{Q_j} be the "center" of Q_j, as in (1.27)-(1.28), and by translation we may suppose that $x_{Q_j} = 0$. Let $\gamma_0 := \{te_{n+1} : t \geq 0\}$ be the upward vertical ray emanating from the origin, and set $Y_1 := \ell(Q_j)e_{n+1}$, $Y_2 := 5\eta\ell(Q_j)e_{n+1}$. Let $\Gamma_{\mathbf{S}}$ be the Lipschitz graph of Lemma 2.1, and note that by (2.2), along with the fact that $\Gamma_{\mathbf{S}}$ has Lipschitz constant at most η,

$$\operatorname{dist}(Y_1, Q_j^*) \approx \delta(Y_1) \approx \ell(Q_j^*)\,, \tag{3.27}$$

and therefore (2.7) yields

$$Y_1 \in U_{Q_j^*}^+ \subset \Omega_{\mathbf{S}'}^+\,.$$

Moreover, by (2.2), $Y_2 \in \Omega$, with

$$\delta(Y_2) \gtrsim \eta\ell(Q_j)\,,$$

and in general

$$\eta\ell(Q_j) \lesssim \delta(Z) \leq \operatorname{dist}(Z, Q_j) \leq |Z - 0| \leq \ell(Q_j)\,, \tag{3.28}$$

for every point $Z \in \gamma_0$ between Y_1 and Y_2. On the other hand, we claim that $Y_2 \notin \Omega_{\mathbf{S}'}^+$. Indeed, note that

$$\delta(Y_2) \leq |Y_2 - 0| = 5\eta\ell(Q_j)\,. \tag{3.29}$$

Suppose by way of contradiction that $Y_2 \in \Omega_{\mathbf{S}'}^+$. Then by definition, $Y_2 \in U_{Q'}^+$, for some cube $Q' \in \mathbf{S}'$, and therefore

$$\delta(Y_2) \gtrsim \eta^{1/2}\ell(Q')\,,$$

by (2.10) and the definition of $U^+_{Q'}$. Combining the last two inequalities, we find that

$$\ell(Q') \lesssim \eta^{1/2}\ell(Q_j) \ll \ell(Q_j). \tag{3.30}$$

Thus, $Q' \cap Q_j = \emptyset$, by the semi-coherency of $\mathbf{S}'$. Therefore, by (1.27)-(1.28), and the fact that we have fixed $\eta \le K^{-4/3}$,

$$\operatorname{dist}(Q', 0) \gtrsim \ell(Q_j) \gtrsim \eta^{-1/2}\ell(Q') \gtrsim K^{2/3}\ell(Q') \gg K^{1/2}\ell(Q')\,.$$

On the other hand, by (3.29) and (3.30),

$$\begin{aligned}\operatorname{dist}(Q', 0) &\le \operatorname{dist}(Q', Y_2) + \operatorname{diam}(Q') + |Y_2 - 0| \\ &\le\ \operatorname{dist}(Q', Y_2) + C\eta^{1/2}\ell(Q_j) + 5\eta\ell(Q_j)\ \le\ \operatorname{dist}(Q', Y_2) + C\eta^{1/2}\operatorname{dist}(Q', 0)\end{aligned}$$

Consequently,

$$\operatorname{dist}(Q', Y_2) \gg K^{1/2}\ell(Q')$$

so $Y_2 \notin U^+_{Q'}$, by (2.10). This proves the claim.

Now travel downward along γ_0 from Y_1 toward Y_2, and let Z_1 be the first point on $\partial\Omega^+_{\mathbf{S}'}$. By (3.28), Z_1 enjoys the desired properties. □

For future reference, we now state and prove the following.

Lemma 3.31. *Fix $\lambda, \epsilon_0, \rho_0, c_0 \in (0,1)$. Given $Q \subset \mathbb{D}(\partial\Omega)$ and a non-empty set $V_Q \subset U_Q \cap \Omega$, let F_Q be defined as in (3.21). Suppose that $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$ is a family of pairwise disjoint sub-cubes of Q, such that $\mathbf{S}' := \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ is a semi-coherent sub-regime of some $\mathbf{S} \subset \mathcal{G}$, and such that also $\mathcal{F} \subset \mathbf{S}$.*

Let $\mathcal{F}' \subset \mathcal{F}$ be such that

$$\sigma\Big(\bigcup_{\mathcal{F}'} Q_j\Big) \ge \rho_0\, \sigma(Q). \tag{3.32}$$

Assume also that for every $Q_j \in \mathcal{F}'$ there is a set $V_{Q_j} \subset U_{Q_j} \cap \Omega$ such that every point in V_{Q_j} belongs to some λ-carrot path connecting V_Q with Q_j. Defining $\widehat{U}_Q$ and $\widehat{U}_{Q_j}$, for $Q_j \in \mathcal{F}'$, as in (3.14), if $A \subset Q$ is a Borel set such that

$$\frac{1}{|\widehat{U}_{Q_j}|}\iint_{\widehat{U}_{Q_j}} \omega^Y(A \cap Q_j)\, dY \ge c_0, \qquad \forall\, Q_j \in \mathcal{F}', \tag{3.33}$$

then

$$\frac{1}{|\widehat{U}_Q|}\iint_{\widehat{U}_Q} \omega^Y(A)\, dY \ge c\,, \tag{3.34}$$

where c depends on $\lambda, \epsilon_0, \rho_0, c_0$ as well as n, τ, η, K and the ADR/UR constants of Ω.

Proof. Let us consider a particular $Q_j \in \mathcal{F}'$. Since $Q_j \in \mathbf{S}$, by (3.33) there is at least one choice of $\pm$ (or possibly both) such that V_{Q_j} meets $U^+_{Q_j}$ (resp., $U^-_{Q_j}$), and such that (3.33) holds (with a slightly different constant) with $\widehat{U}_{Q_j}$ replaced by $U^+_{Q_j}$ (resp., by $U^-_{Q_j}$). Consider the case that, for example,

$$\frac{1}{|U^+_{Q_j}|}\iint_{U^+_{Q_j}} \omega^Y(A\cap Q_j)\,dY \gtrsim c_0\,, \tag{3.35}$$

and V_{Q_j} meets $U^+_{Q_j}$, say at a point Z_j. By assumption and Lemma 3.17 applied with $Q' = Q_j$, we find that $Z_j \in \gamma(y, X^j)$, for some $y \in Q_j$ and $X^j \in U^+_Q \cap V_Q$, where $\gamma(y, X^j)$ is a λ-carrot path in Ω. By Harnack's inequality, Remark 2.13, and (3.35)

$$\omega^Y(A\cap Q_j) \gtrsim c_0\,, \qquad \forall\, Y \in U^+_{Q^*_j}\,, \tag{3.36}$$

where Q^*_j is the dyadic parent of Q_j

We now write $\mathcal{F}' = \mathcal{F}^+ \cup \mathcal{F}^-$, where $Q_j \in \mathcal{F}^+$ if (3.35) and hence also (3.36) hold, and $Q_j \in \mathcal{F}^-$, if the analogous estimates hold with $U^-_{Q_j}, U^-_{Q^*_j}$ in place of $U^+_{Q_j}, U^+_{Q^*_j}$. Of course, it is possible that these estimates may hold for both choices of $\pm$ for some j, so that $\mathcal{F}^+$ and $\mathcal{F}^-$ need not be disjoint, but this is harmless for our purposes. Setting $G := \bigcup_{\mathcal{F}'} Q_j$ and $G^\pm := \bigcup_{\mathcal{F}^\pm} Q_j$, we see that (3.32) continues to hold, with constant $c\,\rho_0$, for at least one choice of $G^\pm$ in place of G, say without loss of generality that

$$\sigma(G^+) \,\geq\, c\rho_0\sigma(Q)\,. \tag{3.37}$$

We recall that $B^*_{Q_j} := B(x_{Q_j}, K\ell(Q_j))$. By a covering lemma argument, we may extract a sub-collection of $\mathcal{F}^+$, call it $\mathcal{F}^*$, such that $\{B^*_{Q_j} : Q_j \in \mathcal{F}^*\}$

is pairwise disjoint, and such that $G^* := \bigcup_{\mathcal{F}^*} Q_j$ satisfies

$$\sum_{\mathcal{F}^*}\sigma(Q_j) = \sigma(G^*) \,\gtrsim\, \rho_0\sigma(Q)\,. \tag{3.38}$$

By assumption, $\mathbf{S}' := \mathbb{D}_\mathcal{F} \cap \mathbb{D}_Q$ is a semi-coherent sub-regime of $\mathbf{S}$, so $\Omega^+_{\mathbf{S}'}$ is a chord-arc domain, by Lemma 2.9. We let $\widehat{\Omega}^+_{\mathbf{S}'}$ be the analogous chord-arc domain, constructed with dilation parameter $2\tau \leq \tau_0$ in place of τ in (2.5) and (2.11).

Given $Q_j \in \mathcal{F}^*$, let $Z_1 = Z_{1,j} \in \partial\Omega^+_{\mathbf{S}'}$ be the point constructed in Lemma 3.25, relative to Q_j. Then $\operatorname{dist}(Z_{1,j}, Q_j) \approx \delta(Z_{1,j}) \approx \ell(Q_j)$, and by Remark 2.14, we therefore also have

$$\operatorname{dist}\left(Z_{1,j}, \partial\widehat{\Omega}^+_{\mathbf{S}'}\right) \approx \ell(Q_j)\,,$$

Thus, by (3.36), and Harnack's inequality applied within the chord arc domain $\widehat{\Omega}^+_{\mathbf{S}'}$, we find that

$$\omega^Y(A) \geq \omega^Y(A\cap Q_j) \gtrsim c_0\,, \qquad \forall\, Y \in B(Z_{1,j}, c\ell(Q_j)) =: B_{1,j} \tag{3.39}$$

for a sufficiently small but uniform constant $c > 0$.

By (3.26), $B_{1,j} \subset B^*_{Q_j}$. Consequently, the balls in the collection $\{B_{1,j} : Q'_j \in \mathcal{F}^*\}$ are pairwise disjoint. Therefore, by (3.38) and the fact that $\partial\Omega^+_{\mathbf{S}'}$ is ADR and has diameter of the order of $\ell(Q)$ by construction, setting $\sigma_\star := H^n\lfloor_{\partial\Omega^+_{\mathbf{S}'}}$, we find that

$$\rho_0\sigma_\star\left(\partial\Omega^+_{\mathbf{S}'}\right) \approx \rho_0\sigma(Q) \lesssim \sum_{\mathcal{F}^*}\sigma(Q_j) \approx \sum_{\mathcal{F}^*}\sigma_\star\left(B_{1,j}\cap\partial\Omega^+_{\mathbf{S}'}\right) = \sigma_\star(A_\star)\,,$$

where $A_\star := \bigcup_{\mathcal{F}^*}\left(B_{1,j}\cap\partial\Omega^+_{\mathbf{S}'}\right)$. Thus, by the result of [DJ], letting $\omega_\star$ denote harmonic measure for $\Omega^+_{\mathbf{S}'}$, we find that

$$\omega^X_\star(A_\star) \geq c_\star\,, \qquad \forall X\in\Omega^+_{\mathbf{S}'} \text{ with } \operatorname{dist}\left(X,\partial\Omega^+_{\mathbf{S}'}\right)\geq \eta\ell(Q)\,, \tag{3.40}$$

where $c_\star$ depends on ρ_0,η and the chord-arc constants for $\Omega^+_{\mathbf{S}'}$. In particular, we note that (3.40) holds for the center X_I of every $I\in\mathcal{W}^{*,+}_Q$. We now fix some $I\in\mathcal{W}^{*,+}_Q$, and observe that by the Markov property

$$\omega^{X_I}(A)=\int_{\partial\Omega^+_{\mathbf{S}'}}\omega^Y(A)\,d\omega^{X_I}_\star(Y)\geq\int_{A_\star}\omega^Y(A)\,d\omega^{X_I}_\star(Y)\gtrsim c_0\,\omega^{X_I}_\star(A_\star)\gtrsim c_0c_\star\,,$$

where in the last two steps we have used first the definition of $A_\star$ and (3.39), and then (3.40). By Harnack's inequality, we find that $\omega^X(A)\geq c$ for every $X\in U^+_Q$. Moreover, U^+_Q meets V_Q (at X^j), for each $Q_j\in\mathcal{F}^+$, and $\mathcal{F}^+\neq\emptyset$ by (3.37), hence $U^+_Q\subset\widehat{U}_Q$. We therefore obtain (3.34). This completes the proof. □

3.2. **Step 2: Proof of $H[M_0,1]$.** We shall deduce $H[M_0,1]$ from the following pair of claims.

Claim 3.41. *$H[0,\theta]$ holds for every $\theta\in(0,1]$.*

Proof of Claim 3.41. If $a=0$ in (3.11), then $\|\mathfrak{m}\|_{\mathcal{C}(Q)}=0$, whence it follows by Claim 3.24, with $\mathcal{F}=\emptyset$, that there is a stopping time regime $\mathbf{S}\subset\mathcal{G}$, with $\mathbb{D}_Q\subset\mathbf{S}$. Hence $\mathbf{S}':=\mathbb{D}_Q$ is a coherent sub-regime of $\mathbf{S}$, so by Lemma 2.9, each of $\Omega^\pm_{\mathbf{S}'}$ is a CAD. Moreover, by [HMM, Proposition A.14]

$$Q\subset\partial\Omega^\pm_{\mathbf{S}'}\cap\partial\Omega\subset\Delta^*_Q=\partial\Omega\cap B(x_Q,K\ell(Q))\,,$$

and in fact Q coincides with the Lipschitz graph $\Gamma_{\mathbf{S}}$, by Lemma 2.1. By the chord arc property, for $\lambda\in(0,1)$ chosen small enough, depending only on the NTA constants of $\Omega^\pm_{\mathbf{S}'}$, for any fixed $X\in U^\pm_Q$, there is a λ-carrot path joining X to every $y\in Q$. To verify $H[0,\theta]$, we may assume that U_Q meets Ω. Thus, at least one of $U^\pm_Q\cap\Omega$ is non-empty, for the sake of specificity, say U^+_Q meets Ω (hence $\Omega^+_{\mathbf{S}'}\subset\Omega$). We fix $X_0=X^+_0\in U^+_Q$. Let $\omega^{X_0}_\star$ denote harmonic measure for $\Omega^+_{\mathbf{S}'}$. By the result of [DJ], $\omega^{X_0}_\star\in A_\infty(\partial\Omega^+_{\mathbf{S}'})$ with respect to surface measure on $\partial\Omega^+_{\mathbf{S}'}$, thus for every $\varepsilon\in(0,1]$, there is a constant $c_\varepsilon\in(0,1)$ such that

$$\sigma(A)\geq(1-\varepsilon)\sigma(Q)\implies\omega^{X_0}_\star(A)\geq c_\varepsilon\,.$$

Since $\Omega^+_{\mathbf{S}'}\subset\Omega$, we have $\omega^{X_0}(A)\geq c_\varepsilon$, by the maximum principle. The conclusion of (3.13) then follows by Harnack's inequality. □

Claim 3.42. *There is a uniform constant $b>0$ such that $H[a,1]\implies H[a+b,1]$, for all $a\in[0,M_0)$.*

Combining Claims 3.41 and 3.42, we find that $H[M_0,1]$ holds.

To prove Claim 3.42, we shall require the following.

Lemma 3.43 ([HM2, Lemma 7.2])**.** *Suppose that E is an n-dimensional ADR set, and let $\mathfrak{m}$ be a discrete Carleson measure, as in (3.8)-(3.10) above. Fix $Q\in\mathbb{D}(E)$. Let $a\geq0$ and $b>0$, and suppose that $\mathfrak{m}(\mathbb{D}_Q)\leq(a+b)\,\sigma(Q)$. Then there is a family*

$\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q$ of pairwise disjoint cubes, and a constant C depending only on n and the ADR constant such that

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq Cb, \tag{3.44}$$

$$\sigma\Big(\bigcup_{\mathcal{F}_{bad}} Q_j\Big) \leq \frac{a+b}{a+2b}\,\sigma(Q)\,, \tag{3.45}$$

where $\mathcal{F}_{bad} := \{Q_j \in \mathcal{F} : \mathfrak{m}(\mathbb{D}_{Q_j} \setminus \{Q_j\}) > a\sigma(Q_j)\}$.

We refer the reader to [HM2, Lemma 7.2] for the proof. We remark that the lemma is stated in [HM2] in the case that E is the boundary of a connected domain, but the proof actually requires only that E have a dyadic cube structure, and that σ be a non-negative, dyadically doubling Borel measure on E. In our case, we shall of course apply the lemma with $E = \partial\Omega$, where Ω is open, but not necessarily connected.

Proof of Claim 3.42. We assume that $H[a, 1]$ holds, for some $a \in [0, M_0)$. Set $b = 1/(2C)$, where C is the constant in (3.44). Consider a cube $Q \in \mathbb{D}(\partial\Omega)$ with $\mathfrak{m}(\mathbb{D}_Q) \leq (a+b)\sigma(Q)$. Suppose that there is a set $V_Q \subset U_Q \cap \Omega$ such that (3.12) holds with $\theta = 1$, and let A be a Borel subset of Q, with

$$\sigma(A) \geq (1-\varepsilon)\sigma(Q)\,, \tag{3.46}$$

where $\varepsilon > 0$ is a sufficiently small number to be chosen. There are two cases.

Case 1: There exists $Q' \in \mathbb{D}_Q^{k_0}$ (see (2.18) and (3.15)) with $\mathfrak{m}(\mathbb{D}_{Q'}) \leq a\sigma(Q')$.

In this case, choosing $\varepsilon > 0$ small enough, depending on n, k_0, ADR, and ε_a, we have that

$$\sigma(A \cap Q') \geq (1-\varepsilon_a)\sigma(Q')\,. \tag{3.47}$$

Moreover, in the present scenario $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (3.12) and (3.21)), which implies $\sigma(F_Q \cap Q') = \sigma(Q')$. We apply Lemma 3.20 to obtain $V_{Q'} \subset U_{Q'} \cap \Omega$ and the corresponding $F_{Q'}$ which satisfies $\sigma(F_{Q'}) = \sigma(Q')$. That is, (3.12) holds for Q', with $\theta = 1$. Consequently, we may apply the induction hypothesis to Q', to obtain that

$$|\widehat{U}_{Q'}|^{-1} \iint_{\widehat{U}_{Q'}} \omega^Y(A \cap Q')\, dY \geq c_a\,, \tag{3.48}$$

with $\widehat{U}_{Q'}$ defined as in (3.14). Since $U_{Q'}$ has at most a bounded number of connected components $U^i_{Q'}$ (depending on η, K and n), there is at least one particular $U^i_{Q'}$ meeting $V_{Q'}$ such that

$$|U^i_{Q'}|^{-1} \iint_{U^i_{Q'}} \omega^Y(A \cap Q')\, dY \gtrsim c_a\,,$$

and thus by Harnack's inequality, $\omega^Z(A \cap Q') \gtrsim c_a$, where $Z \in U^i_{Q'} \cap V_{Q'}$. By Lemma 3.20, Z lies on a λ-carrot path $\gamma(y, X)$, for some $y \in Q'$, and some $X \in$

$V_Q \subset U_Q$. This, the fact that $\ell(Q') \approx \ell(Q)$ (depending on k_0, hence on η and K), and Harnack's inequality yield $\omega^X(A) \geq \omega^X(A \cap Q') \gtrsim c_a$, and furthermore

$$|\widehat{U}_Q|^{-1} \iint_{\widehat{U}_Q} \omega^Y(A)\, dY \geq c c_a\,. \tag{3.49}$$

Taking $c_{a+b} = c c_a$, we obtain the conclusion of $H[a+b, 1]$ in the present case.

Case 2: $\mathfrak{m}(\mathbb{D}_{Q'}) > a\sigma(Q')$ for every $Q' \in \mathbb{D}_Q^{k_0}$.

In this case, we apply Lemma 3.43 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q$ such that (3.44) and (3.45) hold. In particular, by our choice of $b = 1/(2C)$,

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq 1/2\,, \tag{3.50}$$

so that the conclusions of Claim 3.24 hold.

We set

$$F_0 := Q \setminus \Big(\bigcup_{\mathcal{F}} Q_j\Big), \tag{3.51}$$

define

$$\mathcal{F}_{good} := \mathcal{F} \setminus \mathcal{F}_{bad} = \{Q_j \in \mathcal{F} : \mathfrak{m}(\mathbb{D}_{Q_j} \setminus \{Q_j\}) \leq a\sigma(Q_j)\}\,, \tag{3.52}$$

and let

$$G_0 := \bigcup_{\mathcal{F}_{good}} Q_j\,.$$

Then by (3.45)

$$\sigma(F_0 \cup G_0) \geq \rho\sigma(Q)\,, \tag{3.53}$$

where $\rho \in (0,1)$ is defined by

$$\frac{a+b}{a+2b} \leq \frac{M_0+b}{M_0+2b} =: 1-\rho \in (0,1)\,. \tag{3.54}$$

We claim that

$$\ell(Q_j) \leq 2^{-k_0}\,\ell(Q), \qquad \forall\, Q_j \in \mathcal{F}_{good}. \tag{3.55}$$

Indeed, if this were not true for some Q_j, then by definition of $\mathcal{F}_{good}$ and pigeon-holing there will be $Q_j' \in \mathbb{D}_{Q_j}$ with $\ell(Q_j') = 2^{-k_0}\,\ell(Q)$ such that $\mathfrak{m}(\mathbb{D}_{Q_j'}) \leq a\,\sigma(Q_j')$. This contradicts the assumptions of the current case.

Note also that $Q \notin \mathcal{F}_{good}$ by (3.55) and $Q \notin \mathcal{F}_{bad}$ by (3.45), hence $\mathcal{F} \subset \mathbb{D}_Q \setminus \{Q\}$.

Recall that we are given a Borel set $A \subset Q$, satisfying (3.46). Assuming that $\varepsilon < \rho/2$, we then define the "extra good" collection

$$\mathcal{F}_{eg} := \left\{Q_j \in \mathcal{F}_{good} : \sigma(A \cap Q_j) \geq \left(1 - \frac{2\varepsilon}{\rho}\right)\sigma(Q_j)\right\}\,, \tag{3.56}$$

and set $G_1 := \bigcup_{\mathcal{F}_{eg}} Q_j$

Claim 3.57. $\sigma(F_0 \cup G_1) \geq \frac{\rho}{2}\,\sigma(Q)$.

Indeed, define the "bad good" collection

$$\mathcal{F}_{bg} := \left\{ Q_j \in \mathcal{F}_{good} : \, \sigma(A \cap Q_j) < \left(1 - \frac{2\varepsilon}{\rho}\right) \sigma(Q_j) \right\}.$$

Then, since $\mathcal{F}$ is a pairwise disjoint collection,

$$\frac{2\varepsilon}{\rho} \sum_{\mathcal{F}_{bg}} \sigma(Q_j) \le \sum_{\mathcal{F}_{bg}} \sigma(Q_j \setminus A) \le \sigma(Q \setminus A) \le \varepsilon \sigma(Q),$$

by (3.46). Consequently, $\sum_{\mathcal{F}_{bg}} \sigma(Q_j) \le \frac{\rho}{2} \sigma(Q)$, and the claim holds by (3.53).

If F_0 (see (3.51)) satisfies $\sigma(F_0) \ge (\rho/4)\sigma(Q)$, then Q has an ample overlap with the boundary of a chord-arc domain with controlled chord-arc constants; indeed, by (3.50) and Claim 3.24, $\mathbf{S}' = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ is a semi-coherent sub-regime of some $\mathbf{S} \subset \mathcal{G}$, and up to a set of σ-measure 0 (see [HMM, Proposition A.14] and [HM2, Proposition 6.3]),

$$Q \cap \partial\Omega^{\pm}_{\mathbf{S}'} = F_0\,,$$

where by Lemma 2.9, each of $\Omega^{\pm}_{\mathbf{S}'}$ is a CAD. Recall that in establishing $H[a+b, 1]$, we assume that there is a set $V_Q \subset U_Q \cap \Omega$ for which (3.12) holds with $\theta = 1$; in particular, at least one of $U_Q^{\pm}$ meets Ω, and without loss of generality we may suppose that this is true for U_Q^{+}; thus, $\Omega^{+}_{\mathbf{S}'} \subset \Omega$. Then, if $\varepsilon \le \rho/8$, it follows that $\sigma(A \cap F_0) \ge (\rho/8)\,\sigma(Q)$ and by [DJ], the maximum principle, and Harnack's inequality, $\omega^X(A) \ge c$, for every $X \in U_Q^{+}$. Consequently, (3.13) follows in this case, with $a + b$ in place of a.

We may therefore suppose that $\sigma(F_0) \le (\rho/4)\sigma(Q)$, so by Claim 3.57,

$$\sigma(G_1) \ge \frac{\rho}{4}\,\sigma(Q)\,. \tag{3.58}$$

In addition, by the definition of $\mathcal{F}_{good}$ (3.52), and pigeon-holing, every $Q_j \in \mathcal{F}_{eg}$ has a dyadic child Q_j' (there could be more children satisfying this, but we just pick one) satisfying

$$\mathfrak{m}(\mathbb{D}_{Q_j'}) \le a\sigma(Q_j')\,. \tag{3.59}$$

Moreover, by definition of $\mathcal{F}_{eg}$ (see (3.56)) and the ADR property, choosing ε sufficiently small, we find that for this same child $Q_j' \subset Q_j \in \mathcal{F}_{eg}$,

$$\sigma(A \cap Q_j') \ge (1 - \varepsilon_a)\sigma(Q_j')\,. \tag{3.60}$$

We observe that under the present assumptions $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (3.12) and (3.21)), hence $\sigma(F_Q \cap Q_j') = \sigma(Q_j')$. We apply Lemma 3.20 to obtain $V_{Q_j'} \subset U_{Q_j'} \cap \Omega$ and $F_{Q_j'}$ which satisfies $\sigma(F_{Q_j'}) = \sigma(Q_j')$. That is, (3.12) holds for Q_j', with $\theta = 1$. Consequently, by the induction hypothesis $H[a, 1]$,

$$\frac{1}{|\widehat{U}_{Q_j'}|} \iint_{\widehat{U}_{Q_j'}} \omega^Y(A \cap Q_j')\, dY \ge c_a\,, \tag{3.61}$$

for every $Q_j \in \mathcal{F}_{eg}$. Let us consider a particular $Q_j \in \mathcal{F}_{eg}$. We note that $Q_j' \in \mathbf{S}$, by Claim 3.24 and (3.50)). Thus, there is at least one choice of $\pm$ (or possibly both) such that $V_{Q_j'}$ meets $U^{+}_{Q_j'}$ (resp., $U^{-}_{Q_j'}$), and such that (3.61) holds (with a slightly different constant) with $\widehat{U}_{Q'_j}$ replaced by $U^+_{Q'_j}$ (resp., by $U^-_{Q'_j}$). For example, suppose that

$$\frac{1}{|U^+_{Q'_j}|}\iint_{U^+_{Q'_j}} \omega^Y(A\cap Q'_j)\,dY \gtrsim c_a\,, \tag{3.62}$$

and that $V_{Q'_j}$ meets $U^+_{Q'_j}$, say at a point Z'. By Lemmas 3.20 and Lemma 3.17 applied with $Q' = Q'_j$, we find that $Z' \in \gamma(y,X)$, for some $y\in Q'_j$ and $X\in U^+_{Q}$, where $\gamma(y,X)$ is a λ-carrot path in Ω. By Remark 3.22 (with $Q' = Q'_j$ and hence $(Q')^* = Q_j$) and Lemma 3.17 we can also construct $V_{Q_j}\subset U_{Q_j}$ and find $Z\in V_{Q_j}\cap U_{Q^+_j}\cap\gamma(y,X)$. By Harnack's inequality, Remark 2.13, and (3.62) it follows that (3.62) holds for Q_j in place of Q'_j with a slightly different constant and hence

$$\frac{1}{|\widehat{U}_{Q_j}|}\iint_{\widehat{U}_{Q_j}} \omega^Y(A\cap Q_j)\,dY \gtrsim c_a\,. \tag{3.63}$$

This, (3.58), and the properties of the constructed V_{Q_j}'s allow us to apply Lemma 3.31 with $\mathcal{F}' = \mathcal{F}_{eg}$ to conclude (3.34) and hence that (3.13) holds with $a+b$ in place of a. Thus, we have proved Claim 3.42, and therefore, as noted above, it follows that $H[M_0,1]$ holds. □

3.3. **Step 3: bootstrapping θ.** In this last step, we shall prove that there is a uniform constant $\zeta\in(0,1)$ such that for each $\theta\in(0,1]$, $H[M_0,\theta] \implies H[M_0,\zeta\theta]$. Since we have already established $H[M_0,1]$, we then conclude that $H[M_0,\theta_1]$ holds for any given $\theta_1\in(0,1]$. As noted above, it then follows that Theorem 1.3 holds, as desired.

In turn, it will be enough to verify the following.

Claim 3.64. *There is a uniform constant $\beta\in(0,1)$ such that for every $a\in[0,M_0)$, $\theta\in(0,1]$, $\vartheta\in(0,1)$, and $b = 1/(2C)$ as in Step 2/Proof of Claim 3.42, if $H[M_0,\theta]$ holds, then*

$$H[a,(1-\vartheta)\theta] \implies H[a+b,(1-\vartheta\beta)\theta]\,.$$

Let us momentarily take Claim 3.64 for granted. Recall that by Claim 3.41, $H[0,\theta]$ holds for all $\theta\in(0,1]$. In particular, given $\theta\in(0,1]$ fixed, for which $H[M_0,\theta]$ holds, we have that $H[0,\theta/2]$ holds. Combining the latter fact with Claim 3.64, and iterating, we obtain that $H[kb,(1-2^{-1}\beta^k)\theta]$ holds. We eventually reach $H[M_0,(1-2^{-1}\beta^\nu)\theta]$, with $\nu\approx M_0/b$. The conclusion of Step 3 now follows, with $\zeta := 1-2^{-1}\beta^\nu$.

Proof of Claim 3.64. The proof will be a refinement of that of Claim 3.42. We are given some $\theta\in(0,1]$ such that $H[M_0,\theta]$ holds, and we assume that $H[a,(1-\vartheta)\theta]$ holds, for some $a\in[0,M_0)$ and $\vartheta\in(0,1)$. Set $b = 1/(2C)$, where as before C is the constant in (3.44). Consider a cube $Q\in\mathbb{D}(\partial\Omega)$ with $\mathfrak{m}(\mathbb{D}_Q)\le(a+b)\sigma(Q)$. Suppose that there is a set $V_Q\subset U_Q\cap\Omega$ such that (3.12) holds with θ replaced by $(1-\vartheta\beta)\theta$, for some $\beta\in(0,1)$ to be determined, and let A be a Borel subset of Q, with

$$\sigma(A)\ge(1-\varepsilon)\sigma(Q)\,, \tag{3.65}$$

where $\varepsilon > 0$ is a sufficiently small number to be chosen, eventually depending only on a, θ, ϑ, and the various allowable parameters. Our goal is to show that for a sufficiently small, but uniform choice of β, we may conclude that (3.13) holds, with $\varepsilon_{a+b}, c_{a+b}$ in place of ε_a, c_a. There are two principal cases.

Case 1: There exists $Q' \in \mathbb{D}_Q^{k_0}$ (cf. (2.18)) with $\mathfrak{m}(\mathbb{D}_{Q'}) \leq a\sigma(Q')$.

In this case, choosing $\varepsilon > 0$ small enough, depending on n, k_0, ADR, and ε_a, we have that

$$\sigma(A \cap Q') \geq (1 - \varepsilon_a)\sigma(Q')\,. \tag{3.66}$$

By assumption, and recalling the definition of F_Q in (3.21), (3.12) holds with constant $(1 - \vartheta\beta)\theta$, i.e.,

$$\sigma(F_Q) \geq (1 - \vartheta\beta)\theta\sigma(Q)\,. \tag{3.67}$$

We split Case 1 into two subcases.

Case 1a: $\sigma(F_Q \cap Q') \geq (1 - \vartheta)\theta\sigma(Q')$.

In this case, we follow the Case 1 argument for $\theta = 1$ in Subsection 3.2 *mutatis mutandi*, so we merely sketch the proof. By Lemma 3.20, we may construct $V_{Q'}$ and $F_{Q'}$ so that $F_Q \cap Q' \subset F_{Q'}$ and hence $\sigma(F_{Q'}) \geq (1-\vartheta)\theta\sigma(Q')$. This and (3.66) allow us to apply the induction hypothesis $H[a, (1-\vartheta)\theta]$ in Q' and to obtain (3.48). We then use Harnack's inequality and the λ-carrot property to conclude that (3.49) holds, as desired.

Case 1b: $\sigma(F_Q \cap Q') < (1 - \vartheta)\theta\sigma(Q')$.

Set $\varepsilon_0 := \varepsilon_{M_0}(\theta)$, i.e., the value of ε_a corresponding to $H[a, \theta]$, with $a = M_0$. Choosing $\varepsilon > 0$ small enough in (3.65), depending on n, k_0, ADR, and ε_0, we then have the following extension of (3.66), valid for every $Q'' \in \mathbb{D}_Q^{k_0}$:

$$\sigma(A \cap Q'') \geq (1 - \varepsilon_0)\sigma(Q'')\,. \tag{3.68}$$

By (3.67)

$$(1 - \vartheta\beta)\theta\sigma(Q) \;\leq\; \sigma(F_Q) \;=\; \sigma(F_Q \cap Q') + \sum_{Q'' \in \mathbb{D}_Q^{k_0} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,.$$

In the scenario of Case 1b, this leads to

$$\begin{aligned}(1 - \vartheta\beta)\theta\sigma(Q') + (1 - \vartheta\beta)\theta \sum_{Q'' \in \mathbb{D}_Q^{k_0} \setminus \{Q'\}} \sigma(Q'') &= (1 - \vartheta\beta)\theta\sigma(Q)\\ &\leq\; (1 - \vartheta)\theta\sigma(Q') + \sum_{Q'' \in \mathbb{D}_Q^{k_0} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,,\end{aligned}$$

that is,

$$(1 - \beta)\vartheta\theta\sigma(Q') + (1 - \vartheta\beta)\theta \sum_{Q'' \in \mathbb{D}_Q^{k_0} \setminus \{Q'\}} \sigma(Q'') \;\leq\; \sum_{Q'' \in \mathbb{D}_Q^{k_0} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,. \tag{3.69}$$

Note that we have the dyadic doubling estimate

$$\sum_{Q''\in\mathbb{D}_Q^{k_0}\setminus\{Q'\}} \sigma(Q'') \le \sigma(Q) \le M_1\sigma(Q')\,,$$

where $M_1 = M_1(k_0, n, ADR)$. Combining this estimate with (3.69), we obtain

$$\left[(1-\beta)\frac{\vartheta}{M_1} + (1-\vartheta\beta)\right]\theta \sum_{Q''\in\mathbb{D}_Q^{k_0}\setminus\{Q'\}} \sigma(Q'') \le \sum_{Q''\in\mathbb{D}_Q^{k_0}\setminus\{Q'\}} \sigma(F_Q\cap Q'')\,.$$

We now choose $\beta \le 1/(M_1+1)$, so that $(1-\beta)/M_1 \ge \beta$, and therefore the expression in square brackets is at least 1. Consequently, by pigeon-holing, there exists a particular $Q_0'' \in \mathbb{D}_Q^{k_0}\setminus\{Q'\}$ such that

$$\theta\sigma(Q_0'') \le \sigma(F_Q\cap Q_0'')\,. \tag{3.70}$$

By Lemma 3.20, we can find $V_{Q_0''}$ such that $F_Q\cap Q_0'' \subset F_{Q_0''}$, where the latter is defined as in (3.21), with Q_0'' in place of Q. By assumption, $H[M_0,\theta]$ holds, so combining (3.68) (which holds for Q_0'') and (3.70), along with the fact that (3.11) holds with $a = M_0$ for every $Q\in\mathbb{D}(\partial\Omega)$, we find that

$$\frac{1}{|\widehat{U}_{Q_0''}|}\iint_{\widehat{U}_{Q_0''}} \omega^Y(A\cap Q_0'')\,dY \ge c_0 := c_{M_0}(\theta)\,.$$

Using a now familiar argument, we observe that by Lemma 3.20, points in $\widehat{U}_{Q_0''}$ are connected to points in $\widehat{U}_Q$ by a Harnack chain of length at most $C = C(\lambda, k_0, \eta, K)$, thus

$$|\widehat{U}_Q|^{-1}\iint_{\widehat{U}_Q} \omega^Y(A)\,dY \ge cc_0\,,$$

as desired.

Case 2: $\mathfrak{m}(\mathbb{D}_{Q'}) > a\sigma(Q')$ for every $Q'\in\mathbb{D}_Q^{k_0}$.

In this case, we apply Lemma 3.43 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\}\subset\mathbb{D}_Q$ such that (3.44) and (3.45) hold. In particular, by our choice of $b = 1/(2C)$, $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \le 1/2$.

Recall that we are given a Borel set $A\subset Q$, satisfying (3.65), for a sufficiently small choice of ε depending only on $a, \theta, \vartheta, \beta$, and other allowable parameters. Recall also that F_Q is defined in (3.21), and satisfies (3.67).

We define $F_0 = Q\setminus(\bigcup_{\mathcal{F}} Q_j)$ as in (3.51), and $\mathcal{F}_{good} := \mathcal{F}\setminus\mathcal{F}_{bad}$ as in (3.52). Let $G_0 := \bigcup_{\mathcal{F}_{good}} Q_j$. Then as above (see (3.53)),

$$\sigma(F_0\cup G_0) \ge \rho\sigma(Q)\,, \tag{3.71}$$

where again $\rho = \rho(M_0, b)\in(0,1)$ is defined as in (3.54). Much as in Case 2 for $\theta = 1$ in Subsection 3.2, one can show that

$$\ell(Q_j) \le 2^{-k_0}\,\ell(Q), \quad \forall\, Q_j\in\mathcal{F}_{good}, \qquad \text{and} \qquad \mathcal{F}\subset\mathbb{D}_Q\setminus\{Q\} \tag{3.72}$$

Hence, the conclusions of Claim 3.24 hold.

With $\varepsilon > 0$ as in (3.65), we observe first that if $\sigma(F_0) \ge \sqrt{\varepsilon}\sigma(Q)$, then (3.49) holds, as desired. Indeed, as in the analogous scenario in Subsection 3.2, Q has an ample overlap with the boundary of a chord-arc domain with controlled chord-arc constants. More precisely, by (3.50) and Claim 3.24, $\mathbf{S}' = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ is a semi-coherent sub-regime of some $\mathbf{S}$, and up to a set of σ-measure 0 (see [HMM, Proposition A.14] and [HM2, Proposition 6.3]),

$$Q \cap \partial\Omega^{\pm}_{\mathbf{S}'} = F_0 \,,$$

where by Lemma 2.9, each of $\Omega^{\pm}_{\mathbf{S}'}$ is a CAD. Recall that our goal is to establish $H[a+b,(1-\vartheta\beta)\theta]$, in which case we assumed that there is a set $V_Q \subset U_Q \cap \Omega$ for which (3.12) holds with $(1-\vartheta\beta)\theta$ in place of θ. In particular, at least one of $U^{\pm}_Q$ meets Ω, and without loss of generality we may suppose that this is true for U^{+}_Q; thus, $\Omega^{+}_{\mathbf{S}'} \subset \Omega$. Then, if $\varepsilon < 1/4$ one has that $\sigma(A \cap F_0) \geq (\sqrt{\varepsilon}/2)\sigma(Q)$ and by [DJ], the maximum principle, and Harnack's inequality, $\omega^X(A) \geq c_\varepsilon$, for every $X \in U^{+}_Q$, and (3.49) follows.

We may therefore suppose that

$$\sigma(F_0) < \sqrt{\varepsilon}\sigma(Q)\,. \tag{3.73}$$

Next, we refine the decomposition $\mathcal{F} = \mathcal{F}_{good} \cup \mathcal{F}_{bad}$. Modifying slightly our previous construction, we define the "extra good" collection

$$\mathcal{F}_{eg} := \left\{Q_j \in \mathcal{F}_{good} : \sigma(A \cap Q_j) \geq \left(1 - \sqrt{\varepsilon}\right)\sigma(Q_j)\right\}\,,$$

and let $\mathcal{F}_{bg} := \mathcal{F}_{good} \setminus \mathcal{F}_{eg}$ be the "bad good" collection. We similarly decompose $\mathcal{F}_{bad}$. Let

$$\mathcal{F}_{nsb} := \left\{Q_j \in \mathcal{F}_{bad} : \sigma(A \cap Q_j) \geq \left(1 - \sqrt{\varepsilon}\right)\sigma(Q_j)\right\}$$

be the "not so bad" collection, and let $\mathcal{F}_{eb} := \mathcal{F}_{bad} \setminus \mathcal{F}_{nsb}$ be the "extra bad" collection. Note that by definition,

$$\sigma(Q_j \setminus A) > \sqrt{\varepsilon}\sigma(Q_j)\,, \qquad \forall\, Q_j \in \mathcal{F}_{bg} \cup \mathcal{F}_{eb}\,,$$

and therefore

$$\sum_{\mathcal{F}_{bg}\cup\mathcal{F}_{eb}} \sigma(Q_j) \leq \varepsilon^{-1/2} \sum_{\mathcal{F}_{bg}\cup\mathcal{F}_{eb}} \sigma(Q_j \setminus A) \leq \varepsilon^{-1/2}\sigma(Q \setminus A) \leq \sqrt{\varepsilon}\sigma(Q)\,, \tag{3.74}$$

by (3.65).

We now further refine $\mathcal{F}_{eg}$ and $\mathcal{F}_{nsb}$ as follows. With ρ as in(3.54) and (3.71), we choose $\beta < \rho/4$. Set

$$\mathcal{F}^{(1)}_{eg} := \left\{Q_j \in \mathcal{F}_{eg} : \sigma(F_Q \cap Q_j) \geq (1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j)\right\}\,,$$

and define $\mathcal{F}^{(2)}_{eg} := \mathcal{F}_{eg} \setminus \mathcal{F}^{(1)}_{eg}$. Let

$$\mathcal{F}^{(1)}_{nsb} := \left\{Q_j \in \mathcal{F}_{nsb} : \sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)\right\}\,,$$

and define $\mathcal{F}^{(2)}_{nsb} := \mathcal{F}_{nsb} \setminus \mathcal{F}^{(1)}_{nsb}$.

We split Case 2 into two subcases.

Case 2a: There is $Q_j \in \mathcal{F}^{(1)}_{nsb}$ such that $\ell(Q_j) > 2^{-k_0}\,\ell(Q)$.

By definition of $\mathcal{F}^{(1)}_{nsb}$, $\sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)$. By pigeon-holing, Q_j has a descendent Q' with $\ell(Q') = 2^{-k_0}\ell(Q)$, such that $\sigma(F_Q \cap Q') \geq \theta\sigma(Q')$. Also, by (3.65), for ε small enough, $\sigma(A \cap Q') \geq (1 - \varepsilon_{M_0}(\theta))\,\sigma(Q')$. Using Lemma 3.20 we can find $V_{Q'}$ and F'_Q so that $\sigma(F_{Q'}) \geq \sigma(F_Q \cap Q') \geq \theta\sigma(Q')$. Hence, we may apply $H[M_0, \theta]$ in Q', to obtain that

$$\frac{1}{|\widehat{U}_{Q'}|} \iint_{\widehat{U}_{Q'}} \omega^Y(A \cap Q')\, dY \geq c_0 := c_{M_0}(\theta)\,.$$

As before, by Lemma 3.20, points in $\widehat{U}_{Q'}$ are connected to points in $\widehat{U}_Q$ by a Harnack chain of length at most $C = C(\lambda, k_0, \eta, K)$, thus

$$|\widehat{U}_Q|^{-1} \iint_{\widehat{U}_Q} \omega^Y(A)\, dY \geq c c_0\,,$$

as desired.

Case 2b: Every $Q_j \in \mathcal{F}^{(1)}_{nsb}$ satisfies $\ell(Q_j) \leq 2^{-k_0}\,\ell(Q)$.

Observe that by definition,

$$\sigma(F_Q \cap Q_j) \leq (1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j)\,, \qquad \forall\, Q_j \in \mathcal{F}^{(2)}_{eg}\,, \tag{3.75}$$

and also

$$\sigma(F_Q \cap Q_j) \leq \theta\sigma(Q_j)\,, \qquad \forall\, Q_j \in \mathcal{F}^{(2)}_{nsb}\,, \tag{3.76}$$

Set $\mathcal{F}_* := \mathcal{F} \setminus \mathcal{F}^{(2)}_{eg}$. For future reference, we shall derive a certain ampleness estimate for the cubes in $\mathcal{F}_*$.

By (3.67),

$$\begin{aligned}(1 - \vartheta\beta)\theta\sigma(Q) \leq\ \sigma(F_Q)\ &\leq\ \sigma(F_0) + \sum_{\mathcal{F}_*} \sigma(Q_j) + \sum_{\mathcal{F}^{(2)}_{eg}} \sigma(F_Q \cap Q_j)\\ &\leq\ \sqrt{\varepsilon}\sigma(Q) + \sum_{\mathcal{F}_*} \sigma(Q_j) + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q)\,,\end{aligned} \tag{3.77}$$

where in the last step have used (3.73) and (3.75). Observe that

$$(1 - \vartheta\beta)\theta = \left(4\rho^{-1} - 1\right)\vartheta\beta\theta + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\,. \tag{3.78}$$

Using (3.77) and (3.78), for $\sqrt{\varepsilon} \ll (4\rho^{-1} - 1)\vartheta\beta\theta$, we obtain

$$2^{-1}\left(4\rho^{-1} - 1\right)\vartheta\beta\theta\sigma(Q) \leq \sum_{\mathcal{F}_*} \sigma(Q_j)$$

and thus

$$\sigma(Q) \leq\ C(\vartheta, \rho, \beta, \theta) \sum_{\mathcal{F}_*} \sigma(Q_j)\,. \tag{3.79}$$

We now make the following claim.

Claim 3.80. *For ε chosen sufficiently small,*

$$\max\left(\sum_{\mathcal{F}^{(1)}_{eg}} \sigma(Q_j)\,, \sum_{\mathcal{F}^{(1)}_{nsb}} \sigma(Q_j)\right) \geq\ \sqrt{\varepsilon}\sigma(Q)\,.$$

Proof of Claim 3.80. If $\sum_{\mathcal{F}_{eg}^{(1)}} \sigma(Q_j) \geq \sqrt{\varepsilon}\sigma(Q)$, then we are done. Therefore, suppose that

$$\sum_{\mathcal{F}_{eg}^{(1)}} \sigma(Q_j) < \sqrt{\varepsilon}\sigma(Q)\,. \tag{3.81}$$

We have made the decomposition

$$\mathcal{F} = \mathcal{F}_{eg}^{(1)} \cup \mathcal{F}_{eg}^{(2)} \cup \mathcal{F}_{bg} \cup \mathcal{F}_{nsb} \cup \mathcal{F}_{eb}\,, \tag{3.82}$$

where also $\mathcal{F}_{nsb} = \mathcal{F}_{nsb}^{(1)} \cup \mathcal{F}_{nsb}^{(2)}$.

Consequently

$$\sigma(F_Q) \leq \sum_{\mathcal{F}_{eg}^{(2)}} \sigma(F_Q \cap Q_j) + \sum_{\mathcal{F}_{nsb}} \sigma(F_Q \cap Q_j) + O\Big(\sqrt{\varepsilon}\sigma(Q)\Big)\,,$$

where we have used (3.73), (3.74), and (3.81) to estimate the contributions of F_0, $\mathcal{F}_{bg}$ and $\mathcal{F}_{eb}$, and $\mathcal{F}_{eg}^{(1)}$, respectively. This, (3.67), (3.75), and (3.76) yield

$$\begin{aligned}(1-\vartheta\beta)\theta\left(\sum_{\mathcal{F}_{eg}^{(2)}} \sigma(Q_j) + \sum_{\mathcal{F}_{nsb}^{(2)}} \sigma(Q_j)\right) &\leq (1-\vartheta\beta)\theta\sigma(Q) \leq \sigma(F_Q) \\ &\leq \Big(1-4\vartheta\beta\rho^{-1}\Big)\theta \sum_{\mathcal{F}_{eg}^{(2)}} \sigma(Q_j) + \sum_{\mathcal{F}_{nsb}^{(1)}} \sigma(Q_j) + \theta \sum_{\mathcal{F}_{nsb}^{(2)}} \sigma(Q_j) + O\Big(\sqrt{\varepsilon}\sigma(Q)\Big)\,.\end{aligned}$$

In turn, applying (3.78) in the latter estimate, and rearranging terms, we obtain

$$(4\rho^{-1}-1)\vartheta\beta\theta \sum_{\mathcal{F}_{eg}^{(2)}} \sigma(Q_j) - \vartheta\beta\theta \sum_{\mathcal{F}_{nsb}^{(2)}} \sigma(Q_j) \leq \sum_{\mathcal{F}_{nsb}^{(1)}} \sigma(Q_j) + O\Big(\sqrt{\varepsilon}\sigma(Q)\Big)\,. \tag{3.83}$$

Recalling that $G_0 = \cup_{\mathcal{F}_{good}} Q_j$, and that $\mathcal{F}_{good} = \mathcal{F}_{bg} \cup \mathcal{F}_{eg}^{(1)} \cup \mathcal{F}_{eg}^{(2)}$, we further note that by (3.71), choosing $\varepsilon \ll \rho^2$, and using (3.73), (3.74), and (3.81), we find in particular that

$$\sum_{\mathcal{F}_{eg}^{(2)}} \sigma(Q_j) \geq \frac{\rho}{2}\sigma(Q). \tag{3.84}$$

Applying (3.84) and the trivial estimate $\sum_{\mathcal{F}_{nsb}^{(2)}} \sigma(Q_j) \leq \sigma(Q)$ in (3.83), we then have

$$\begin{aligned}\vartheta\beta\theta\left[1-\frac{\rho}{2}\right]\sigma(Q) &= \left[(4\rho^{-1}-1)\vartheta\beta\theta\frac{\rho}{2} - \vartheta\beta\theta\right]\sigma(Q) \\ &\leq (4\rho^{-1}-1)\vartheta\beta\theta \sum_{\mathcal{F}_{eg}^{(2)}} \sigma(Q_j) - \vartheta\beta\theta \sum_{\mathcal{F}_{nsb}^{(2)}} \sigma(Q_j) \leq \sum_{\mathcal{F}_{nsb}^{(1)}} \sigma(Q_j) + O\Big(\sqrt{\varepsilon}\sigma(Q)\Big)\,.\end{aligned}$$

Since $\rho < 1$, we conclude, for $\sqrt{\varepsilon} \leq (4C)^{-1}\vartheta\beta\theta$, that

$$\frac{1}{4}\vartheta\beta\theta\,\sigma(Q) \leq \sum_{\mathcal{F}_{nsb}^{(1)}} \sigma(Q_j)\,,$$

and Claim 3.80 follows. □

With Claim 3.80 in hand, let us return to the proof of Case 2 of Claim 3.64.

Claim 3.85. *Choosing ε small enough, every $Q_j \in \mathcal{F}_{nsb}^{(1)}$ satisfies*

$$\frac{1}{|\widehat{U}_{Q_j}|} \iint_{\widehat{U}_{Q_j}} \omega^Y(A \cap Q_j)\, dY \geq c_0\,, \tag{3.86}$$

with $c_0 = c_{M_0}(\theta)$. Here $\widehat{U}_{Q_j}$ is defined as in (3.14) with V_{Q_j} being the set constructed in Lemma 3.20 so that $F_Q \cap Q_j \subset F_{Q_j}$.

Proof of Claim 3.87. Fix $Q_j \in \mathcal{F}_{nsb}^{(1)}$. Then $\sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)$, by definition of $\mathcal{F}_{nsb}^{(1)}$, and for ε small enough, $\sigma(A \cap Q_j) \geq (1 - \varepsilon_{M_0}(\theta))\sigma(Q_j)$, by definition of $\mathcal{F}_{nsb}$. In the scenario of Case 2b, $\ell(Q_j) \leq 2^{-k_0}\,\ell(Q)$. Thus, we can use Lemma 3.20 to construct the corresponding V_{Q_j} so that $\sigma(F_{Q_j}) \geq \sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)$. We may therefore apply $H[M_0, \theta]$ to Q_j, to obtain that (3.86) holds with $c_0 = c_{M_0}(\theta)$. □

Claim 3.87. *Choosing β and ε small enough, (3.86) holds for every $Q_j \in \mathcal{F}_{eg}^{(1)}$ with c_0 being either of the order of $c_a((1-\vartheta)\theta)$ or $c_{M_0}(\theta)$. Again, $\widehat{U}_{Q_j}$ is defined as in (3.14) with V_{Q_j} being the set constructed in Lemma 3.20 and Remark 3.22.*

Proof of Claim 3.87. Fix $Q_j \in \mathcal{F}_{eg}^{(1)}$. In particular, $Q_j \in \mathcal{F}_{good}$, so by pigeon-holing, Q_j has a child Q_j' satisfying (3.59). In addition, $\ell(Q_j) \leq 2^{-k_0}\ell(Q)$ by (3.72). Moreover, for ε chosen small enough, Q_j' satisfies (3.60), by definition of $\mathcal{F}_{eg}$.

Our immediate goal is to find a child Q_j'' of Q_j, which may or may not equal Q_j', for which we have the estimate

$$\frac{1}{|\widehat{U}_{Q_j''}|} \iint_{\widehat{U}_{Q_j''}} \omega^Y(A \cap Q_j'')\, dY \geq c_0\,, \tag{3.88}$$

with $c_0 = c_a((1-\vartheta)\theta)$. To this end, we assume first that Q_j' satisfies

$$\sigma(F_Q \cap Q_j') \geq (1-\vartheta)\theta\sigma(Q_j')\,. \tag{3.89}$$

In this case, we set $Q_j'' := Q_j'$ and use (3.89) and Lemma 3.20 to find $V_{Q_j''}$ such that $\sigma(F_{Q_j''}) \geq (1-\vartheta)\theta\sigma(Q_j'')$. By the induction hypothesis $H[a, (1-\vartheta)\theta]$, applied in $Q_j'' = Q_j'$, we obtain (3.88).

Let us next consider the case

$$\sigma(F_Q \cap Q_j') < (1-\vartheta)\theta\sigma(Q_j')\,. \tag{3.90}$$

In this case, we shall select $Q_j'' \neq Q_j'$. Recall that we use the notation $Q'' \lhd Q$ to mean that Q'' is a dyadic child of Q. Set

$$\mathcal{F}_j'' := \left\{Q_j'' \lhd Q_j :\, Q_j'' \neq Q_j'\right\}.$$

Note that we have the dyadic doubling estimate

$$\sum_{Q_j'' \in \mathcal{F}_j''} \sigma(Q_j'') \leq \sigma(Q_j) \leq M_1\sigma(Q_j')\,, \tag{3.91}$$

where $M_1 = M_1(n, ADR)$. We also note that

$$(1 - 4\vartheta\beta\rho^{-1})\theta = (1 - 4\beta\rho^{-1})\vartheta\theta + (1 - \vartheta)\theta . \tag{3.92}$$

By definition of $\mathcal{F}_{eg}^{(1)}$,

$$(1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j) \le \sigma(F_Q \cap Q_j) = \sigma(F_Q \cap Q_j') + \sum_{Q_j''\in\mathcal{F}_j''} \sigma(F_Q \cap Q_j'') .$$

By (3.90), it follows that

$$\begin{aligned}(1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j') + (1 - 4\vartheta\beta\rho^{-1})\theta \sum_{Q_j''\in\mathcal{F}_j''} \sigma(Q_j'') &= (1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j) \\ &\le (1 - \vartheta)\theta\sigma(Q_j') + \sum_{Q_j''\in\mathcal{F}_j''} \sigma(F_Q \cap Q_j'') .\end{aligned}$$

In turn, using (3.92), we obtain

$$(1 - 4\beta\rho^{-1})\vartheta\theta\sigma(Q_j') + (1 - 4\vartheta\beta\rho^{-1})\theta \sum_{Q_j''\in\mathcal{F}_j''} \sigma(Q_j'') \le \sum_{Q_j''\in\mathcal{F}_j''} \sigma(F_Q \cap Q_j'') .$$

By the dyadic doubling estimate (3.91), this leads to

$$\Big[(1 - 4\beta\rho^{-1})\vartheta M_1^{-1} + (1 - 4\vartheta\beta\rho^{-1})\Big]\theta \sum_{Q_j''\in\mathcal{F}_j''} \sigma(Q_j'') \le \sum_{Q_j''\in\mathcal{F}_j''} \sigma(F_Q \cap Q_j'') .$$

Choosing $\beta \le \rho/(4(M_1 + 1))$, we find that the expression in square brackets is at least 1, and therefore, by pigeon holing, we can pick $Q_j'' \in \mathcal{F}_j''$ satisfying

$$\sigma(F_Q \cap Q_j'') \ge \theta\sigma(Q_j'') . \tag{3.93}$$

Moreover, for ε sufficiently small, by definition of $\mathcal{F}_{eg}$ (recall that $Q_j \in \mathcal{F}_{eg}^{(1)}$),

$$\sigma(A \cap Q_j'') \ge (1 - \varepsilon_{M_0}(\theta))\sigma(Q_j'') .$$

Hence, using (3.93) and Lemma 3.20, we see that the induction hypothesis $H[M_0, \theta]$ holds for $Q_j'' \in \mathcal{F}_j''$, and hence (3.88) follows with $c_0 = c_{M_0}(\theta)$.

Thus, (3.88) holds in both cases, and therefore $V_{Q_j''}$ must meet $U_{Q_j''}$, say at a point Z''. Let $U^i_{Q_j''}$ be the component of $U_{Q_j''}$ containing Z'', and note that (3.88) still holds with $U^i_{Q_j''}$ in place of $\widehat{U}_{Q_j''}$, possibly with a slightly different constant. By Harnack's inequality it follows that

$$\omega^{Z''}(A \cap Q_j'') \gtrsim c_0 . \tag{3.94}$$

By Lemma 3.20 we find that $Z'' \in \gamma(y, X)$, for some $y \in Q_j''$ and $X \in U_Q$, where $\gamma(y, X)$ is a λ-carrot path in Ω. By Remark 3.22 (with $Q' = Q_j''$ and hence $(Q')^* = Q_j$) we can also construct $V_{Q_j} \subset U_{Q_j}$ and find $Z \in V_{Q_j} \cap U_{Q_j} \cap \gamma(y, X)$. Since Z and Z'' lie in the same λ-carrot path and $\ell(Q_j) = 2\ell(Q_j'')$ we can use Harnack's inequality to see that (3.94) holds for Z and hence, by Harnack's inequality again for the component of U_{Q_j} containing Z. This easily yields (3.86) as desired. $\square$

To complete the proof of Claim 3.64 we write $\mathcal{F}' = \mathcal{F}_{nsb}^{(1)} \cup \mathcal{F}_{eg}^{(1)} \subset \mathcal{F}$ and note that by Claim 3.80

$$\sum_{Q_j \in \mathcal{F}'} \sigma(Q_j) = \sum_{\mathcal{F}_{nsb}^{(1)}} \sigma(Q_j) + \sum_{Q_j \in \mathcal{F}_{eg}} \sigma(Q_j) \geq \sqrt{\varepsilon}\sigma(Q)\,.$$

Also, for every $Q_j \in \mathcal{F}'$, Claims (3.85) and (3.87) give

$$\frac{1}{|\widehat{U}_{Q_j}|} \iint_{\widehat{U}_{Q_j}} \omega^Y(A \cap Q_j)\, dY \geq c_0\,,$$

where $\widehat{U}_{Q_j}$ is defined as in (3.14) and V_{Q_j} is the set constructed in Claim (3.85) or (3.87), which in turn comes from Lemma 3.20 and/or Remark 3.22. We now have all the ingredients to apply Lemma 3.31 and eventually obtain (3.34). Thus, (3.13) holds with $a + b$ in place of a. This finishes the proof of of Claim 3.64 and consequently that of Theorem 1.3. □

4. Doubling implies strong local John

In this section, we give a direct proof of a fact which already follows from stronger results of [Azz], namely that doubling of harmonic measure implies that a strong version of the local John condition holds (i.e., Definition 1.18, with $\theta = 1$). To this end, we recall the following fact from [AH].

Lemma 4.1. [AH, Lemma 2]. *Let $G(X, Y)$ denote the Green function for $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, a domain with ADR boundary. Suppose that $\delta(X) = r$, and that $G(X, Y) \geq A_0 r^{1-n}$, for some positive constant A_0. Then there is a constant $A = A(A_0, ADR, n)$ and a rectifiable path $\gamma \subset \Omega$ joining X to Y, with $\ell(\gamma) \leq Ar$, and $\delta(Z) \geq r/A$, for all $Z \in \gamma$.*

We remark that the full strength of the ADR hypothesis is not needed in Lemma 4.1: in [AH], the authors assume only the capacity density condition.

We shall define the doubling property essentially as in [Azz]: harmonic measure is doubling if there is a constant $A \geq 2$, and a function $C : (0, \infty) \to (1, \infty)$ such that, for any ball $B = B(x_B, r_B)$ centered on $\partial\Omega$ with radius $r_B < \operatorname{diam}(\partial\Omega)$, and corresponding surface ball $\Delta = B \cap \partial\Omega$, and for any $\alpha > 0$,

$$\omega^X(2\Delta) \leq C(\alpha)\omega^X(\Delta)\,,$$

for all X such that $\operatorname{dist}(X, A\Delta) \geq \alpha r_B$.

Let $X \in \Omega$, set $r := \delta(X)$, and define

$$\Delta_X := \Delta_X^{10} = B(X, 10r) \cap \partial\Omega\,,$$

and let $y \in \Delta_X$. Our goal is to show that there is a λ-carrot path joining X to y, with λ depending only on n, ADR, and the doubling constant. For a sufficiently large number M to be chosen, and for each $k = 1, 2, 3, \ldots$, set

$$B_k := B(y, M^{-k}r)\,, \qquad \Delta_k := B_k \cap \partial\Omega\,.$$

By a result of [Bou], there is a constant $c_0 = c_0(n, ADR)$ such that $\omega^X(\Delta_X) \geq c_0$. By doubling,

$$\omega^X(\Delta_2) \geq c_1 c_0\,, \tag{4.2}$$

with $c_1 = c_1(M)$. We then have a weak version of the well-known "CFMS" estimate, namely that for M large enough, there is a point $X_1 \in 2B_2 \cap \Omega$, with $\delta(X_1) \geq 2M^{-3}r$, such that

$$G(X, X_1) \geq c_M r^{1-n} . \tag{4.3}$$

The latter estimate may be proved by a modification of the proof of the classical CFMS estimates, see, e.g., [Ken, Lemma 1.3.5], using (4.2), Lemma 2.21, and the reverse CFMS estimate. We omit the details.

By (4.3) and Lemma 4.1, there is a path γ joining X to X_1, of length at most $C_M r$, with $\delta(Z) \geq r/C_M$ for all $Z \in \gamma$. Moreover, invoking the result of [Bou] again, we have that for M large enough, $\omega^{X_1}(\Delta_1) \geq c_0$, and then by doubling,

$$\omega^{X_1}(\Delta_3) \geq c_1 c_0 .$$

We may then construct a path to y by iterating.

We have the following immediate corollary.

Corollary 4.4. *Suppose that harmonic measure is doubling, and that $\partial\Omega$ is UR. Then ω satisfies the local A_∞ condition (Definition 1.23).*

Proof. We have just shown, as a consequence of the doubling property, that Ω satisfies a strong version of the local John condition, i.e., Definition 1.18, with $\theta = 1$. Thus, by Theorem 1.3, harmonic measure is in local weak-A_∞, and by the doubling hypothesis, the latter condition improves to local A_∞. □

References

[AH] H. Aikawa and K. Hirata, Doubling conditions for harmonic measure in John domains, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 2, 429-445. 2, 3, 33

[AHLT] P. Auscher, S. Hofmann, J.L. Lewis and P. Tchamitchian, Extrapolation of Carleson measures and the analyticity of Kato's square-root operators, *Acta Math.* **187** (2001), no. 2, 161–190. 12

[AHMTT] P. Auscher, S. Hofmann, C. Muscalu, T. Tao and C. Thiele, Carleson measures, trees, extrapolation, and $T(b)$ theorems, *Publ. Mat.* **46** (2002), no. 2, 257–325. 12

[Azz] J. Azzam, Semi-uniform domains and a characterization of the A_∞ property for harmonic measure, preprint, arXiv:1711.03088. 2, 3, 4, 33

[BL] B. Bennewitz and J.L. Lewis, On weak reverse Hölder inequalities for nondoubling harmonic measures, *Complex Var. Theory Appl.* **49** (2004), no. 7–9, 571-582. 2, 3, 13

[BJ] C. Bishop and P. Jones, Harmonic measure and arclength, *Ann. of Math. (2)* **132** (1990), 511–547. 2, 3

[Bou] J. Bourgain, On the Hausdorff dimension of harmonic measure in higher dimensions, *Invent. Math.* **87** (1987), 477–483. 33, 34

[Car] L. Carleson, Interpolation by bounded analytic functions and the corona problem, *Ann. of Math. (2)* **76** (1962), 547–559. 12

[CG] L. Carleson and J. Garnett, Interpolating sequences and separation properties, *J. Analyse Math.* **28** (1975), 273–299. 12

[Chr] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.*, **LX/LXI** (1990), 601–628. 5, 8

[Dah] B. Dahlberg, On estimates for harmonic measure, *Arch. Rat. Mech. Analysis* **65** (1977), 272–288 2

[D1] G. David, Morceaux de graphes lipschitziens et intégrales singulières sur une surface. (French) [Pieces of Lipschitz graphs and singular integrals on a surface] *Rev. Mat. Iberoamericana* **4** (1988), no. 1, 73-114. 8

[D2] G. David, Wavelets and singular integrals on curves and surfaces. *Lecture Notes in Mathematics,* **1465**. Springer-Verlag, Berlin, 1991. 8

[DJ] G. David and D. Jerison, Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals, *Indiana Univ. Math. J.* **39** (1990), no. 3, 831–845. 2, 3, 21, 24, 28

[DS1] G. David and S. Semmes, Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs, *Asterisque* **193** (1991). 5, 6, 8, 10

[DS2] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Monographs and Surveys **38**, AMS 1993. 6, 8, 9

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover (Rev. ed.), 2006. 12

[HLe] S. Hofmann and P. Le, BMO solvability and absolute continuity of harmonic measure, *J. Geom. Anal*, to appear. 4

[HLMN] S. Hofmann, P. Le, J. M. Martell, and K. Nyström, The weak-A_∞ property of harmonic and p-harmonic measures implies uniform rectifiability, *Anal. PDE.* **10** (2017), no. 3, 513–558. 3, 4, 12

[HLw] S. Hofmann and J.L. Lewis, The Dirichlet problem for parabolic operators with singular drift terms, *Mem. Amer. Math. Soc.* **151** (2001), no. 719. 12

[HM1] S. Hofmann and J.M. Martell, A_∞ estimates via extrapolation of Carleson measures and applications to divergence form elliptic operators, *Trans. Amer. Math. Soc.* **364** (2012), no. 1, 65–101 12

[HM2] S. Hofmann and J.M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p, Ann. Sci. École Norm. Sup. **47** (2014), no. 3, 577–654. 12, 21, 22, 24, 28

[HM3] S. Hofmann and J.M. Martell, Uniform Rectifiability and harmonic measure IV: Ahlfors regularity plus Poisson kernels in L^p implies uniform rectifiability, preprint, *arXiv:1505.06499*. 3, 4

[HMM] S. Hofmann, J.M. Martell, and S. Mayboroda, Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions, *Duke Math. J.* 165 (2016), no. 12, 2331–2389. 9, 10, 11, 12, 21, 24, 28

[HMMM] S. Hofmann, D. Mitrea, M. Mitrea, and A.J. Morris. L^p-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets. *Mem. Amer. Math. Soc.* **245** (2017), no. 1159. 8

[HMT] S. Hofmann, M. Mitrea and M. Taylor, Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains, *Int. Math. Res. Not. IMRN* **2010** (2010), 2567-2865. 3

[JK] D. Jerison and C. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), no. 1, 80–147. 2, 6

[Ken] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994 34

[Lav] M. Lavrentiev, Boundary problems in the theory of univalent functions (Russian), *Math Sb.* **43** (1936), 815-846; *AMS Transl. Series 2* **32** (1963), 1–35. 1

[LM] J. Lewis and M. Murray, The method of layer potentials for the heat equation in time-varying domains, *Mem. Amer. Math. Soc.* **114** (1995), no. 545. 12

[MMV] P. Mattila, M. Melnikov and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, *Ann. of Math. (2)* **144** (1996), no. 1, 127–136. 6

[MT] M. Mourgoglou and X. Tolsa, Harmonic measure and Riesz transform in uniform and general domains, *J. Reine Angew. Math.*, to appear. 3, 4

[NTV] F. Nazarov, X. Tolsa, and A. Volberg, On the uniform rectifiability of ad-regular measures with bounded Riesz transform operator: The case of codimension 1, *Acta Math.*, to appear. 6

[RR] F. and M. Riesz, Über die randwerte einer analtischen funktion, *Compte Rendues du Quatrième Congrès des Mathématiciens Scandinaves*, Stockholm 1916, Almqvists and Wilksels, Uppsala, 1920. 1, 2

[Sem] S. Semmes, A criterion for the boundedness of singular integrals on on hypersurfaces, *Trans. Amer. Math. Soc.* **311** (1989), 501-513. 2

[Ste] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princteon University Press, Princeton, NJ, 1970. 10

Steve Hofmann, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: `hofmanns@missouri.edu`

José María Martell, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13-15, E-28049 Madrid, Spain

E-mail address: `chema.martell@icmat.es`