HARMONIC MEASURE AND QUANTITATIVE CONNECTIVITY: GEOMETRIC CHARACTERIZATION OF THE L^p-SOLVABILITY OF THE DIRICHLET PROBLEM. PART I

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Abstract. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with an n-dimensional uniformly rectifiable boundary. We show that $\partial\Omega$ may be approximated in a "Big Pieces" sense by boundaries of chord-arc subdomains of Ω, and hence that harmonic measure for Ω is weak-A_∞ with respect to surface measure on $\partial\Omega$, provided that Ω satisfies a certain weak version of a local John condition. Under the further assumption that Ω satisfies an interior Corkscrew condition, and combined with our previous work, and with recent work of Azzam, Mourgoglou and Tolsa, this yields a geometric characterization of domains whose harmonic measure is quantitatively absolutely continuous with respect to surface measure and hence a characterization of the fact that the associated L^p-Dirichlet problem is solvable for some finite p.

Contents

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1. Introduction

A classical result of F. and M. Riesz [RR] states that for a simply connected domain Ω in the complex plane, rectifiability of $\partial\Omega$ implies that harmonic measure for Ω is absolutely continuous with respect to arclength measure on the boundary. A quantitative version of this theorem was later proved by Lavrentiev [Lav]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones [BJ] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion. They also present a counter-example to show that the result of [RR] may fail in the absence of some connectivity hypothesis (e.g., simple connectedness).

In dimensions greater than 2, a fundamental result of Dahlberg [Dah] establishes a quantitative version of absolute continuity, namely that harmonic measure belongs to the class A_∞ in an appropriate local sense (see Definitions 1.22 and 1.26 below), with respect to surface measure on the boundary of a Lipschitz domain.

The result of Dahlberg was extended to the class of Chord-arc domains (see Definition 1.14) by David and Jerison [DJ], and independently by Semmes [Sem]. The Chord-arc hypothesis was weakened to that of a two-sided Corkscrew condition (Definition 1.11) by Bennewitz and Lewis [BL], who then drew the conclusion that harmonic measure is weak-A_∞ (in an appropriate local sense, see Definitions 1.22 and 1.26) with respect to surface measure on the boundary; the latter condition is similar to the A_∞ condition, but without the doubling property, and is the best conclusion that can be obtained under the weakened geometric conditions considered in [BL]. We note that weak-A_∞ is still a quantitative, scale invariant version of absolute continuity.

More recently, J. Azzam [Azz], has given a geometric characterization of the A_∞ property of harmonic measure with respect to surface measure for domains with Ahlfors-David regular (ADR) boundary (see Definition 1.7). This work is related to our own, so let us describe it in a bit more detail. Specifically, Azzam shows that for a domain Ω with ADR boundary, harmonic measure is in A_∞ with respect to surface measure, if and only if 1) $\partial\Omega$ is uniformly rectifiable (UR)[1], and 2) Ω is semi-uniform in the sense of Aikawa and Hirata [AH]. The semi-uniform condition is a connectivity condition which states that for some uniform constant M, every pair of points $X \in \Omega$ and $y \in \partial\Omega$ may be connected by a rectifiable curve $\gamma = \gamma(y, X)$, with $\gamma \setminus \{y\} \subset \Omega$, with length $\ell(\gamma) \leq M|X - y|$, and which satisfies the "cigar path" condition

$$\min\{\ell(\gamma(y,Z)), \ell(\gamma(Z,X))\} \leq M \operatorname{dist}(Z, \partial\Omega), \qquad \forall Z \in \gamma. \tag{1.1}$$

Semi-uniformity is a weak version of the well known uniform condition, whose definition is similar, except that it applies to all pairs of points $X, Y \in \Omega$. For example, the unit disk centered at the origin, with the slit $-1/2 \leq x \leq 1/2, y = 0$ removed, is semi-uniform, but not uniform. It was shown in [AH] that for a domain satisfying a John condition and the Capacity Density Condition (in particular, for a domain with an ADR boundary), semi-uniformity characterizes the doubling

[1] This is a quantitative, scale-invariant version of rectifiability, see Definition 1.9 and the ensuing comments.

property of harmonic measure. The method of [Azz] is, broadly speaking, related to that of [DJ], and of [BL]. In [DJ], the authors show that a Chord-arc domain Ω may be approximated in a "Big Pieces" sense (see [DJ] or [BL] for a precise statement; also cf. Definition 1.19 below) by Lipschitz subdomains $\Omega' \subset \Omega$; this fact allows one to reduce matters to the result of Dahlberg via the maximum principle (a method which, to the present authors' knowledge, first appears in [JK] in the context of BMO_1 domains). The same strategy, i.e., Big Piece approximation by Lipschitz subdomains, is employed in [BL]. Similarly, in [Azz], matters are reduced to the result of [DJ], by showing that for a domain Ω with an ADR boundary, Ω is semi-uniform with a uniformly rectifiable boundary if and only if it has "Very Big Pieces" of Chord-arc subdomains (see [Azz] for a precise statement of the latter condition). As mentioned above, the converse direction is also treated in [Azz]. In that case, given an interior Corkscrew condition (which holds automatically in the presence of the doubling property of harmonic measure), and provided that $\partial\Omega$ is ADR, the A_∞ (or even weak-A_∞) property of harmonic measure was already known to imply uniform rectifiability of the boundary [HM3] (although the published version appears in [HLMN]; see also [MT] for an alternative proof, and a somewhat more general result); as in [AH], semi-uniformity follows from the doubling property, although in [Azz], the author manages to show this while dispensing with the John domain background assumption (given a harmlessly strengthened version of the doubling property).

In light of the example of [BJ], it has been an important open problem to determine the minimal connectivity assumption, which, in conjunction with uniform rectifiability of the boundary, yields quantitative absolute continuity of harmonic measure with respect to surface measure. We observe that in [Azz], the connectivity condition (semi-uniformity), is tied to the doubling property of harmonic measure, and not to absolute continuity. In the present work, we impose a significantly milder connectivity hypothesis than semi-uniformity, and we then show that $\partial\Omega$ may be approximated in a big pieces sense by boundaries of chord-arc subdomains, and hence that harmonic measure ω satisfies a weak-A_∞ condition with respect to surface measure σ on the boundary, provided that $\partial\Omega$ is uniformly rectifiable. The weak-A_∞ conclusion is best possible in this generality: indeed, the stronger conclusion that $\omega \in A_\infty(\sigma)$, which entails doubling of ω, necessarily requires semi-uniformity, as Azzam has shown. One may then combine our results here with our previous work [HM3], and with recent work of Azzam, Mourgoglou and Tolsa [AMT2], to obtain a geometric characterization of quantitative absolute continuity of harmonic measure (see Theorem 1.6 below).

Let us now describe our connectivity hypothesis, which says, roughly speaking, that from each point $X \in \Omega$, there is local non-tangential access to an ample portion of a surface ball at a scale on the order of $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. Let us make this a bit more precise. A "carrot path" (aka non-tangential path) joining a point $X \in \Omega$, and a point $y \in \partial\Omega$, is a connected rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, such that for some $\lambda \in (0, 1)$ and for all $Z \in \gamma$,

$$\lambda\, \ell(\gamma(y, Z)) \leq \delta(Z)\,. \tag{1.2}$$

For $X \in \Omega$, and $R \geq 2$, set

$$\Delta_X = \Delta_X^R := B(X, R\delta(X)) \cap \partial\Omega .$$

We assume that every point $X \in \Omega$ may be joined by a carrot path to each y in a "Big Piece" of Δ_X, i.e., to each y in a Borel subset $F \subset \Delta_X$, with $\sigma(F) \geq \theta\sigma(\Delta_X)$, where σ denotes surface measure on $\partial\Omega$, and where the parameters $R \geq 2$, $\lambda \in (0,1)$, and $\theta \in (0,1]$ are uniformly controlled. We refer to this condition as a "weak local John condition", although "weak local semi-uniformity" would be equally appropriate. See Definitions 1.15, 1.17 and 1.19 for more details. We remark that a strong version of the local John condition (i.e., with $\theta = 1$) has appeared in [HMT], in connection with boundary Poincaré inequalities for non-smooth domains.

We observe that the weak local John condition is strictly weaker than semi-uniformity: for example, the unit disk centered a the origin, with either the cross $\{-1/2 \leq x \leq 1/2, y = 0\} \cup \{-1/2 \leq y \leq 1/2, x = 0\}$ removed, or with the slit $0 \leq x \leq 1, y = 0$ removed, satisfies the weak local John condition, although semi-uniformity fails in each case.

The main result in the present work is the following. The terminology used here will be defined in the sequel.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with an Ahlfors-David regular (ADR) boundary. Then the following are equivalent:*

(i) *$\partial\Omega$ is uniformly rectifiable (UR; see Definition 1.9 below), and Ω satisfies the weak local John condition.*
(ii) *Ω satisfies an Interior Big Pieces of Chord-Arc Domains (IBPCAD) condition (see Definition 1.20 below).*

Only the direction (i) implies (ii) is non-trivial. For the converse, the fact that IBPCAD implies the weak local John condition is immediate from the definitions. Moreover, the boundary of a chord-arc domain is UR, and an ADR set with big pieces of UR is also UR (see [DS2]).

We note that condition (ii) (hence also condition (i)) implies that harmonic measure is locally in weak-A_∞ (see Definition 1.26) with respect to surface measure: this fact is well known, and follows from the maximum principle and the result of [DJ] and [Sem] for chord-arc domains, and the method of [BL].

Moreover, for an open set with ADR boundary, the weak-A_∞ property implies (and in fact is equivalent to) solvability of the Dirichlet problem for some $p < \infty$; we refer the reader to, e.g., [HLe, Section 4] for details. We therefore have the following.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with a uniformly rectifiable boundary. Suppose in addition that Ω satisfies the weak local John condition. Then the L^p Dirichlet problem for Ω is solvable in L^p, for some $p < \infty$, i.e., given continuous data g defined on $\partial\Omega$, for the harmonic measure solution u to the Dirichlet problem with data g, we have for some $p < \infty$ that*

$$\|N_* u\|_{L^p(\partial\Omega)} \leq C \, \|g\|_{L^p(\partial\Omega)} , \tag{1.5}$$

where $N_* u$ is a suitable version of the non-tangential maximal function of u.

Combining the previous results with certain other recent works (to be discussed momentarily), one obtains the following geometric characterization of quantitative absolute continuity of harmonic measure, and of the L^p solvability of the Dirichlet problem.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying an interior Corkscrew condition (see Definition 1.11 below), and suppose that $\partial\Omega$ is Ahlfors-David regular (ADR). Then the following are equivalent:*

(1) *$\partial\Omega$ is Uniformly Rectifiable (UR) and Ω satisfies a weak local John condition.*
(2) *Ω satisfies an Interior Big Pieces of Chord-Arc Domains (IBPCAD) condition.*
(3) *Harmonic measure ω is locally in weak-A_∞ (see Definition 1.26 below) with respect to surface measure σ on $\partial\Omega$.*
(4) *The L^p Dirichlet problem is solvable in the sense of Corollary 1.4, for some $p < \infty$.*

Some comments are in order. In the present paper we shall prove that (1) implies (2) (this is the content of Theorem 1.3), and we note that the interior Corkscrew condition is not needed for this particular implication (nor for (2) implies (3) if and only if (4)). Rather, it is crucial for the converse direction (3) implies (1). As noted above, the fact that (2) implies (3) follows by a well-known argument using the maximum principle and the result of [DJ] and [Sem] for chord-arc domains, along with the criterion for weak-A_∞ obtained in [BL]; the equivalence of (3) and (4) is well known. The implication (3) implies (1) has two parts: that weak-A_∞ implies UR is the main result of [HM3]; an alternative proof, with a more general result, appears in [MT], and see also [HLMN] for the final published version of the results of [HM3], along with an extension to the p-harmonic setting. The remaining implication, that weak-A_∞ implies weak local John, is a very recent result of Azzam, Mourgoglou and Tolsa [AMT2]. We remark that in a preliminary version of this work [HM4], we have given a direct proof that (1) implies (3), using an approach similar to that of the present paper. We also mention that our background hypotheses (upper and lower ADR, and interior Corkscrew) are in the nature of best possible: one may construct a counter-example in the absence of any one of them, for at least one direction of this chain of implications, as we shall discuss in Appendix A.

1.1. Further notation and definitions.

- Unless otherwise stated, we use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Ω will always denote an open set in $\mathbb{R}^{n+1}$, not necessarily connected unless otherwise specified.
- We use the notation $\gamma(X, Y)$ to denote a rectifiable path with endpoints X and Y, and its arc-length will be denoted $\ell(\gamma(X, Y))$. Given such a path, if $Z \in \gamma(X, Y)$, we use the notation $\gamma(Z, Y)$ to denote the portion of the original path with endpoints Z and Y.
- Given an open set $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in Ω (or more generally in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- We let e_j, $j = 1, 2, \dots, n + 1$, denote the standard unit basis vectors in $\mathbb{R}^{n+1}$.
- The open $(n + 1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \Omega$. A *surface ball* is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$.
- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.
- Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.
- Given an open set $\Omega \subset \mathbb{R}^{n+1}$, for $X \in \Omega$, we set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.
- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\lfloor_{\partial\Omega}$ denote the surface measure on $\partial\Omega$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A.
- Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $⨍_A f d\mu := \mu(A)^{-1} \int_A f d\mu$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$. Given an ADR set $E \subset \mathbb{R}^{n+1}$, we use Q (or sometimes P) to denote a dyadic "cube" on E. The latter exist (cf. [DS1], [Chr]), and enjoy certain properties which we enumerate in Lemma 1.29 below.

Definition 1.7. (**ADR**) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \le \sigma(\Delta(x, r)) \le C\, r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{1.8}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x, r) := E \cap B(x, r)$ is the *surface ball* of radius r, and as above, $\sigma := H^n\lfloor_E$ is the "surface measure" on E.

Definition 1.9. (**UR**) (aka *uniformly rectifiable*). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "Big Pieces of Lipschitz

Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants c_1 and C_1, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than C_1, such that

$$H^n\Big(E \cap B(x,r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \geq c_1 r^n .$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [DS1]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NTV] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions); cf. [DS1, DS2].

Definition 1.10. (**"UR character"**). Given a UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants (c_1, C_1) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 1.11. (**Corkscrew condition**). Following [JK], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *Corkscrew point* relative to Δ. We note that we may allow $r < C \operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c. In order to emphasize that $B(X_\Delta, cr) \subset \Omega$, we shall sometimes refer to this property as the *interior Corkscrew condition*.

Definition 1.12. (**Harnack Chains, and the Harnack Chain condition** [JK]). Given two points $X, X' \in \Omega$, and a pair of numbers $M, N \geq 1$, an (M,N)-*Harnack Chain connecting X to X'*, is a chain of open balls $B_1, \dots, B_N \subset \Omega$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $M^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq M \operatorname{diam}(B_k)$. We say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant M such that for any two points $X, X' \in \Omega$, there is an (M,N)-Harnack Chain connecting them, with N depending only on the ratio $|X - X'|/\left(\min\left(\delta(X), \delta(X')\right)\right)$.

Definition 1.13. (**NTA**). Again following [JK], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA (*Non-tangentially accessible*) if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the Corkscrew condition.

Definition 1.14. (**CAD**). We say that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a CAD (*Chord-arc domain*), if it is NTA, and if $\partial\Omega$ is ADR.

Definition 1.15. (**Carrot path**). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Given a point $X \in \Omega$, and a point $y \in \partial\Omega$, we say that a connected rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, is a *carrot path* (more precisely, a λ*-carrot path*) connecting y to X, if $\gamma \setminus \{y\} \subset \Omega$, and if for some $\lambda \in (0,1)$ and for all $Z \in \gamma$,

$$\lambda\, \ell(\gamma(y,Z)) \leq \delta(Z)\,. \tag{1.16}$$

With a slight abuse of terminology, we shall sometimes refer to such a path as a λ*-carrot path in* Ω, although of course the endpoint y lies on $\partial\Omega$.

A carrot path is sometimes referred to as a non-tangential path.

Definition 1.17. (**(θ, λ, R)-weak local John point**). Let $X \in \Omega$, and for constants $\theta \in (0,1]$, $\lambda \in (0,1)$, and $R \geq 2$, set

$$\Delta_X = \Delta_X^R := B(X, R\delta(X)) \cap \partial\Omega\,.$$

We say that a point $X \in \Omega$ is a (θ, λ, R)-*weak local John point* if there is a Borel set $F \subset \Delta_X^R$, with $\sigma(F) \geq \theta\sigma(\Delta_X^R)$, such that for every $y \in F$, there is a λ-carrot path connecting y to X.

Thus, a weak local John point is non-tangentially connected to an ample portion of the boundary, locally. We observe that one can always choose R smaller, for possibly different values of θ and λ, by moving from X to a point X' on a line segment joining X to the boundary.

Remark 1.18. We observe that it is a slight abuse of notation to write Δ_X, since the latter is not centered on $\partial\Omega$, and thus it is not a true surface ball; on the other hand, there are true surface balls, $\Delta'_X := \Delta(\hat{x}, (R-1)\delta(X))$ and $\Delta''_X := \Delta(\hat{x}, (R+1)\delta(X))$, centered at a "touching point" $\hat{x} \in \partial\Omega$ with $\delta(X) = |X - \hat{x}|$, which, respectively, are contained in, and contain, Δ_X.

Definition 1.19. (**Weak local John condition**). We say that Ω satisfies a *weak local John condition* if there are constants $\lambda \in (0,1)$, $\theta \in (0,1]$, and $R \geq 2$, such that every $X \in \Omega$ is a (θ, λ, R)-weak local John point.

Definition 1.20. (**IBPCAD**). We say that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ has *Interior Big Pieces of Chord-Arc Domains* (IBPCAD) if there exist positive constants η and C, and $R \geq 2$, such that for every $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, there is a chord-arc domain $\Omega_X \subset \Omega$ satisfying

- $X \in \Omega_X$.
- $\operatorname{dist}(X, \partial\Omega_X) \geq \eta\delta(X)$.
- $\operatorname{diam}(\Omega_X) \leq C\delta(X)$.
- $\sigma(\partial\Omega_X \cap \Delta_X^R) \geq \eta\,\sigma(\Delta_X^R) \approx_R \eta\,\delta(X)^n$.
- The chord-arc constants of the domains Ω_X are uniform in X.

Remark 1.21. In the presence of an interior Corkscrew condition, Definition 1.20 is easily seen to be equivalent to the following more standard "Big Pieces" condition: there is a constant $\eta > 0$ (perhaps slightly different to that in Definition 1.20), such that for each surface ball $\Delta := \Delta(x,r) = B(x,r) \cap \partial\Omega$, $x \in \partial\Omega$ and $r < \operatorname{diam}(\partial\Omega)$, there is a chord-arc domain Ω_Δ satisfying

- $\Omega_\Delta \subset B(x,r) \cap \Omega$.
- Ω_Δ contains a Corkscrew point X_Δ, with $\operatorname{dist}(X_\Delta, \partial\Omega_\Delta) \geq \eta r$.
- $\sigma(\partial\Omega_\Delta \cap \Delta) \geq \eta\,\sigma(\Delta) \approx \eta r^n$.

- The chord-arc constants of the domains Ω_Δ are uniform in Δ.

Definition 1.22. (A_∞, weak-A_∞, and weak-RH_q). Given an ADR set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$ centered at E, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and s such that for each surface ball $\Delta = B \cap E$ centered on E, with $B \subseteq B_0$, we have

$$\mu(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(\Delta)\,, \qquad \text{for every Borel set } A \subset \Delta\,. \tag{1.23}$$

Similarly, we say that $\mu \in$ weak-$A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$ centered on E, with $2B \subseteq B_0$,

$$\mu(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(2\Delta)\,, \qquad \text{for every Borel set } A \subset \Delta\,. \tag{1.24}$$

We recall that, as is well known, the condition $\mu \in$ weak-$A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0, and that for some $q > 1$, the Radon-Nikodym derivative $k := d\mu/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_\Delta k^q d\sigma\right)^{1/q} \lesssim \fint_{2\Delta} k\, d\sigma \approx \frac{\mu(2\Delta)}{\sigma(\Delta)}\,, \quad \forall\, \Delta = B \cap E,\ \text{with } 2B \subseteq B_0\,, \tag{1.25}$$

with B centered on E. We shall refer to the inequality in (1.25) as an "RH_q" estimate, and we shall say that $k \in RH_q(\Delta_0)$ if k satisfies (1.25).

Definition 1.26. (**Local A_∞ and local weak-A_∞**). We say that harmonic measure ω is locally in A_∞ (resp., locally in weak-A_∞) on $\partial\Omega$, if there are uniform positive constants C and s such that for every ball $B = B(x, r)$ centered on $\partial\Omega$, with radius $r < \operatorname{diam}(\partial\Omega)/4$, and associated surface ball $\Delta = B \cap \partial\Omega$,

$$\omega^X(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \omega^X(\Delta)\,, \qquad \forall\, X \in \Omega \setminus 4B\,,\ \forall \text{ Borel } A \subset \Delta\,, \tag{1.27}$$

or, respectively, that

$$\omega^X(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \omega^X(2\Delta)\,, \qquad \forall\, X \in \Omega \setminus 4B\,,\ \forall \text{ Borel } A \subset \Delta\,; \tag{1.28}$$

equivalently, if for every ball B and surface ball $\Delta = B \cap \partial\Omega$ as above, and for each point $X \in \Omega \setminus 4B$, $\omega^X \in A_\infty(\Delta)$ (resp., $\omega^X \in$ weak-$A_\infty(\Delta)$) with uniformly controlled A_∞ (resp., weak-A_∞) constants.

Lemma 1.29. (**Existence and properties of the "dyadic grid"**) [DS1, DS2], [Chr]. *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-dimensional ADR set. Then there exist constants $a_0 > 0$, $s > 0$ and $C_1 < \infty$, depending only on n and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(*i*) $E = \cup_j Q_j^k$ *for each* $k \in \mathbb{Z}$.

(*ii*) *If* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(*iii*) *For each* (j, k) *and each* $m < k$, *there is a unique* i *such that* $Q_j^k \subset Q_i^m$.

(iv) $\operatorname{diam}(Q_j^k) \le C_1 2^{-k}$.

(v) *Each* Q_j^k *contains some "surface ball"* $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.

(vi) $H^n(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \le \vartheta\, 2^{-k}\}) \le C_1\, \vartheta^s\, H^n(Q_j^k)$, *for all* k, j *and for all* $\vartheta \in (0, a_0)$.

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Chr], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0,1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property (1.8), the result already appears in [DS1, DS2]. Some predecessors of this construction have appeared in [Da1] and [Da2].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D} := \cup_k \mathbb{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$ such that $r_Q \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r_Q) \subset Q \subset \Delta(x_Q, C r_Q), \tag{1.30}$$

for some uniform constant C. We shall denote this ball and surface ball by

$$B_Q := B(x_Q, r_Q)\,, \qquad \Delta_Q := \Delta(x_Q, r_Q), \tag{1.31}$$

and we shall refer to the point x_Q as the "center" of Q.

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the dyadic generation to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- For a pair of cubes $Q', Q \in \mathbb{D}$, if Q' is a dyadic child of Q, i.e., if $Q' \subset Q$, and $\ell(Q) = 2\ell(Q')$, then we write $Q' \lhd Q$.

With the dyadic cubes in hand, we may now define the notion of a Corkscrew point relative to a cube Q.

Definition 1.32. (**Corkscrew point relative to** Q). Let Ω satisfy the Corkscrew condition (Definition 1.11), suppose that $\partial\Omega$ is ADR, and let $Q \in \mathbb{D}(\partial\Omega)$. A *Corkscrew point relative to* Q is simply a Corkscrew point relative to the surface ball Δ_Q defined (1.30)-(1.31).

Definition 1.33. (**Coherency and Semi-coherency**). [DS2]. Let $E \subset \mathbb{R}^{n+1}$ be an ADR set. Let $\mathbf{S} \subset \mathbb{D}(E)$. We say that $\mathbf{S}$ is *coherent* if the following conditions hold:

(a) $\mathbf{S}$ contains a unique maximal element $Q(\mathbf{S})$ which contains all other elements of $\mathbf{S}$ as subsets.

(b) If Q belongs to $\mathbf{S}$, and if $Q \subset \widetilde{Q} \subset Q(\mathbf{S})$, then $\widetilde{Q} \in \mathbf{S}$.

(c) Given a cube $Q \in \mathbf{S}$, either all of its children belong to $\mathbf{S}$, or none of them do.

We say that $\mathbf{S}$ is *semi-coherent* if conditions (a) and (b) hold.

2. Preliminaries

We begin by recalling a bilateral version of the David-Semmes "Corona decomposition" of a UR set. We refer the reader to [HMM] for the proof.

Lemma 2.1. ([HMM, Lemma 2.2]) *Let $E \subset \mathbb{R}^{n+1}$ be a UR set of dimension n. Then given any positive constants $\eta \ll 1$ and $K \gg 1$, there is a disjoint decomposition $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, satisfying the following properties.*

(1) *The "Good" collection $\mathcal{G}$ is further subdivided into disjoint stopping time regimes, such that each such regime $\mathbf{S}$ is coherent (Definition 1.33).*

(2) *The "Bad" cubes, as well as the maximal cubes $Q(\mathbf{S})$, $\mathbf{S} \subset \mathcal{G}$, satisfy a Carleson packing condition:*

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') \; + \sum_{\mathbf{S} \subset \mathcal{G}: Q(\mathbf{S}) \subset Q} \sigma(Q(\mathbf{S})) \;\leq\; C_{\eta,K}\, \sigma(Q)\,, \quad \forall Q \in \mathbb{D}(E)\,.$$

(3) *For each $\mathbf{S} \subset \mathcal{G}$, there is a Lipschitz graph $\Gamma_{\mathbf{S}}$, with Lipschitz constant at most η, such that, for every $Q \in \mathbf{S}$,*

$$\sup_{x \in \Delta_Q^*} \operatorname{dist}(x, \Gamma_{\mathbf{S}}) \;+ \sup_{y \in B_Q^* \cap \Gamma_{\mathbf{S}}} \operatorname{dist}(y, E) < \eta\, \ell(Q)\,, \tag{2.2}$$

where $B_Q^* := B(x_Q, K\ell(Q))$ and $\Delta_Q^* := B_Q^* \cap E$, and x_Q is the "center" of Q as in (1.30)-(1.31).

We mention that David and Semmes, in [DS1], had previously proved a unilateral version of Lemma 2.1, in which the bilateral estimate (2.2) is replaced by the unilateral bound

$$\sup_{x \in \Delta_Q^*} \operatorname{dist}(x, \Gamma_{\mathbf{S}}) \;<\; \eta\, \ell(Q)\,, \qquad \forall\, Q \in \mathbf{S}\,. \tag{2.3}$$

Next, we make a standard Whitney decomposition of $\Omega_E := \mathbb{R}^{n+1} \setminus E$, for a given UR set E (in particular, Ω_E is open, since UR sets are closed by definition). Let $\mathcal{W} = \mathcal{W}(\Omega_E)$ denote a collection of (closed) dyadic Whitney cubes of Ω_E, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω_E, which satisfy

$$4\operatorname{diam}(I) \leq \operatorname{dist}(4I, \partial\Omega) \leq \operatorname{dist}(I, \partial\Omega) \leq 40\operatorname{diam}(I)\,, \qquad \forall\, I \in \mathcal{W} \tag{2.4}$$

(just dyadically divide the standard Whitney cubes, as constructed in [Ste, Chapter VI], into cubes with side length 1/8 as large) and also

$$(1/4)\operatorname{diam}(I_1) \leq \operatorname{diam}(I_2) \leq 4\operatorname{diam}(I_1)\,,$$

whenever I_1 and I_2 touch.

We fix a small parameter $\tau_0 > 0$, so that for any $I \in \mathcal{W}$, and any $\tau \in (0, \tau_0]$, the concentric dilate

$$I^*(\tau) := (1 + \tau)I \tag{2.5}$$

still satisfies the Whitney property

$$\operatorname{diam} I \approx \operatorname{diam} I^*(\tau) \approx \operatorname{dist}(I^*(\tau), E) \approx \operatorname{dist}(I, E), \quad 0 < \tau \le \tau_0 . \tag{2.6}$$

Moreover, for $\tau \le \tau_0$ small enough, and for any $I, J \in \mathcal{W}$, we have that $I^*(\tau)$ meets $J^*(\tau)$ if and only if I and J have a boundary point in common, and that, if $I \ne J$, then $I^*(\tau)$ misses $(3/4)J$.

Pick two parameters $\eta \ll 1$ and $K \gg 1$ (eventually, we shall take $K = \eta^{-3/4}$). For $Q \in \mathbb{D}(E)$, define

$$\mathcal{W}_Q^0 := \left\{ I \in \mathcal{W} : \eta^{1/4}\ell(Q) \le \ell(I) \le K^{1/2}\ell(Q), \ \operatorname{dist}(I, Q) \le K^{1/2}\ell(Q) \right\}. \tag{2.7}$$

Remark 2.8. We note that $\mathcal{W}_Q^0$ is non-empty, provided that we choose η small enough, and K large enough, depending only on dimension and ADR, since the ADR condition implies that Ω_E satisfies a Corkscrew condition. In the sequel, we shall always assume that η and K have been so chosen.

Next, we recall a construction in [HMM, Section 3], leading up to and including in particular [HMM, Lemma 3.24]. We summarize this construction as follows.

Lemma 2.9. *Let $E \subset \mathbb{R}^{n+1}$ be n-dimensional UR, and set $\Omega_E := \mathbb{R}^{n+1} \setminus E$. Given positive constants $\eta \ll 1$ and $K \gg 1$, as in (2.7) and Remark 2.8, let $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, be the corresponding bilateral Corona decomposition of Lemma 2.1. Then for each $\mathbf{S} \subset \mathcal{G}$, and for each $Q \in \mathbf{S}$, the collection $\mathcal{W}_Q^0$ in (2.7) has an augmentation $\mathcal{W}_Q^* \subset \mathcal{W}$ satisfying the following properties.*

1. *$\mathcal{W}_Q^0 \subset \mathcal{W}_Q^* = \mathcal{W}_Q^{*,+} \cup \mathcal{W}_Q^{*,-}$, where (after a suitable rotation of coordinates) each $I \in \mathcal{W}_Q^{*,+}$ lies above the Lipschitz graph $\Gamma_{\mathbf{S}}$ of Lemma 2.1, each $I \in \mathcal{W}_Q^{*,-}$ lies below $\Gamma_{\mathbf{S}}$. Moreover, if Q' is a child of Q, also belonging to $\mathbf{S}$, then $\mathcal{W}_Q^{*,+}$ (resp. $\mathcal{W}_Q^{*,-}$) belongs to the same connected component of Ω_E as does $\mathcal{W}_{Q'}^{*,+}$ (resp. $\mathcal{W}_{Q'}^{*,-}$) and $\mathcal{W}_{Q'}^{*,+} \cap \mathcal{W}_Q^{*,+} \ne \emptyset$ (resp., $\mathcal{W}_{Q'}^{*,-} \cap \mathcal{W}_Q^{*,-} \ne \emptyset$).*
2. *There are uniform constants c and C such that*

$$\begin{gathered} c\eta^{1/2}\ell(Q) \le \ell(I) \le CK^{1/2}\ell(Q), \quad \forall I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I, Q) \le CK^{1/2}\ell(Q), \quad \forall I \in \mathcal{W}_Q^*, \\ c\eta^{1/2}\ell(Q) \le \operatorname{dist}(I^*(\tau), \Gamma_{\mathbf{S}}), \quad \forall I \in \mathcal{W}_Q^*, \quad \forall \tau \in (0, \tau_0]. \end{gathered} \tag{2.10}$$

Moreover, given $\tau \in (0, \tau_0]$, set

$$U_Q^{\pm} = U_{Q,\tau}^{\pm} := \bigcup_{I \in \mathcal{W}_Q^{*,\pm}} \operatorname{int}(I^*(\tau)), \qquad U_Q := U_Q^+ \cup U_Q^-, \tag{2.11}$$

and given $\mathbf{S}'$, a semi-coherent subregime of $\mathbf{S}$, define

$$\Omega_{\mathbf{S}'}^{\pm} = \Omega_{\mathbf{S}'}^{\pm}(\tau) := \bigcup_{Q \in \mathbf{S}'} U_Q^{\pm}. \tag{2.12}$$

Then each of $\Omega^\pm_{\mathbf{S}'}$ is a CAD, with Chord-arc constants depending only on n, τ, η, K, and the ADR/UR constants for $\partial\Omega$.

Remark 2.13. In particular, for each $\mathbf{S} \subset \mathcal{G}$, if Q' and Q belong to $\mathbf{S}$, and if Q' is a dyadic child of Q, then $U^+_{Q'} \cup U^+_Q$ is Harnack Chain connected, and every pair of points $X, Y \in U^+_{Q'} \cup U^+_Q$ may be connected by a Harnack Chain in Ω_E of length at most $C = C(n, \tau, \eta, K, \text{ADR/UR})$. The same is true for $U^-_{Q'} \cup U^-_Q$.

Remark 2.14. Let $0 < \tau \le \tau_0/2$. Given any $\mathbf{S} \subset \mathcal{G}$, and any semi-coherent subregime $\mathbf{S}' \subset \mathbf{S}$, define $\Omega^\pm_{\mathbf{S}'} = \Omega^\pm_{\mathbf{S}'}(\tau)$ as in (2.12), and similarly set $\widehat{\Omega}^\pm_{\mathbf{S}'} = \Omega^\pm_{\mathbf{S}'}(2\tau)$. Then by construction, for any $X \in \overline{\Omega^\pm_{\mathbf{S}'}}$,

$$\operatorname{dist}(X, E) \approx \operatorname{dist}(X, \partial\widehat{\Omega}^\pm_{\mathbf{S}'}) ,$$

where of course the implicit constants depend on τ.

As in [HMM], it will be useful for us to extend the definition of the Whitney region U_Q to the case that $Q \in \mathcal{B}$, the "bad" collection of Lemma 2.1. Let $\mathcal{W}^*_Q$ be the augmentation of $\mathcal{W}^0_Q$ as constructed in Lemma 2.9, and set

$$\mathcal{W}_Q := \begin{cases} \mathcal{W}^*_Q, & Q \in \mathcal{G}, \\ \mathcal{W}^0_Q, & Q \in \mathcal{B} \end{cases} . \tag{2.15}$$

For $Q \in \mathcal{G}$ we shall henceforth simply write $\mathcal{W}^\pm_Q$ in place of $\mathcal{W}^{*,\pm}_Q$. For arbitrary $Q \in \mathbb{D}(E)$, good or bad, we may then define

$$U_Q = U_{Q,\tau} := \bigcup_{I \in \mathcal{W}_Q} \operatorname{int}(I^*(\tau)) . \tag{2.16}$$

Let us note that for $Q \in \mathcal{G}$, the latter definition agrees with that in (2.11). Note that by construction

$$U_Q \subset \{Y \in \Omega : \operatorname{dist}(Y, \partial\Omega) > c\eta^{1/2}\ell(Q)\} \cap B(x_Q, CK^{1/2}\ell(Q)), \tag{2.17}$$

for some uniform constants $C \ge 1$ and $0 < c < 1$ (see (2.4), (2.7), and (2.10)). In particular, for every $Q \in \mathbb{D}$ if follows that

$$\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \subset B(x_Q, K\ell(Q)) =: B^*_Q. \tag{2.18}$$

For future reference, we introduce dyadic sawtooth regions as follows. Set

$$\mathbb{D}_Q := \{Q' \in \mathbb{D}(E) : Q' \subset Q\} , \tag{2.19}$$

and given $k \ge 1$,

$$\mathbb{D}^k_Q := \left\{Q' \in \mathbb{D}(E) : Q' \subset Q,\ \ell(Q') = 2^{-k}\,\ell(Q)\right\} , \tag{2.20}$$

Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the *global discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} , \tag{2.21}$$

i.e., $\mathbb{D}_{\mathcal{F}}$ is the collection of all $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$. We may allow $\mathcal{F}$ to be empty, in which case $\mathbb{D}_{\mathcal{F}} = \mathbb{D}$. Given some fixed cube Q, the *local discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q. \tag{2.22}$$

Note that with this convention, $\mathbb{D}_Q = \mathbb{D}_{\emptyset,Q}$ (i.e., if one takes $\mathcal{F} = \emptyset$ in (2.22)).

3. Proof of Theorem 1.3

In the proof of Theorem 1.3, we shall employ a two-parameter induction argument, which is a refinement of the method of "extrapolation" of Carleson measures. The latter is a bootstrapping scheme for lifting the Carleson measure constant, developed by J. L. Lewis [LM], and based on the corona construction of Carleson [Car] and Carleson and Garnett [CG] (see also [HLw], [AHLT], [AHMTT], [HM1], [HM2],[HMM]).

3.1. **Step 1: the set-up.** To set the stage for the induction procedure, let us begin by making a preliminary reduction. It will be convenient to work with a certain dyadic version of Definition 1.20. To this end, let $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, and set $\Delta_X = \Delta_X^R = B(X, R\delta(X)) \cap \partial\Omega$, for some fixed $R \geq 2$ as in Definition 1.17. Let $\hat{x} \in \partial\Omega$ be a touching point for X, i.e., $|X - \hat{x}| = \delta(X)$. Choose X_1 on the line segment joining X to $\hat{x}$, with $\delta(X_1) = \delta(X)/2$, and set $\Delta_{X_1} = B(X_1, R\delta(X)/2) \cap \partial\Omega$. Note that $B(X_1, R\delta(X)/2) \subset B(X, R\delta(X))$, and furthermore,

$$\operatorname{dist}\Big(B(X_1, R\delta(X)/2), \partial B(X, R\delta(X)\Big) > \frac{R-1}{2}\delta(X) \geq \frac{1}{2}\delta(X).$$

We may therefore cover Δ_{X_1} by a disjoint collection $\{Q_i\}_{i=1}^N \subset \mathbb{D}(\partial\Omega)$, of equal length $\ell(Q_i) \approx \delta(X)$, such that each $Q_i \subset \Delta_X$, and such that the implicit constants depend only on n and ADR, and thus the cardinality N of the collection depends on n, ADR, and R. With $E = \partial\Omega$, we make the Whitney decomposition of the set $\Omega_E = \mathbb{R}^{n+1} \setminus E$ as in Section 2 (thus, $\Omega \subset \Omega_E$). Moreover, for sufficiently small η and sufficiently large K in (2.7), we then have that $X \in U_{Q_i}$ for each $i = 1, 2, \dots, N$. By hypothesis, there are constants $\theta_0 \in (0, 1]$, $\lambda_0 \in (0, 1)$, and $R \geq 2$ as above, such that every $Z \in \Omega$ is a (θ_0, λ_0, R)-weak local John point (Definition 1.17). In particular, this is true for X_1, hence there is a Borel set $F \subset \Delta_{X_1}$, with $\sigma(F) \geq \theta_0 \sigma(\Delta_{X_1})$, such that every $y \in F$ may be connected to X_1 via a λ_0-carrot path. By ADR, $\sigma(\Delta_{X_1}) \approx \sum_{i=1}^N \sigma(Q_i)$ and thus by pigeon-holing, there is at least one $Q_i =: Q$ such that $\sigma(F \cap Q) \geq \theta_1 \sigma(Q)$, with θ_1 depending only on θ_0, n and ADR. Moreover, the λ_0-carrot path connecting each $y \in F$ to X_1 may be extended to a λ_1-carrot path connecting y to X, where λ_1 depends only on λ_0.

We have thus reduced matters to the following dyadic scenario: let $Q \in \mathbb{D}(\partial\Omega)$, let $U_Q = U_{Q,\tau}$ be the associated Whitney region as in (2.16), with $\tau \leq \tau_0/2$ fixed, and suppose that U_Q meets Ω (recall that by construction $U_Q \subset \Omega_E = \mathbb{R}^{n+1} \setminus E$, with $E = \partial\Omega$). For $X \in U_Q \cap \Omega$, and for a constant $\lambda \in (0, 1)$, let

$$F_{car}(X, Q) = F_{car}(X, Q, \lambda) \tag{3.1}$$

denote the set of $y \in Q$ which may be joined to X by a λ-carrot path $\gamma(y, X)$, and for $\theta \in (0, 1]$, set

$$T_Q = T_Q(\theta, \lambda) := \{X \in U_Q \cap \Omega : \sigma(F_{car}(X, Q, \lambda)) \geq \theta\sigma(Q)\}. \tag{3.2}$$

Remark 3.3. Our goal is to prove that, given $\lambda \in (0, 1)$ and $\theta \in (0, 1]$, there are positive constants η and C, depending on θ, λ, and the allowable parameters, such that for each $Q \in \mathbb{D}(\partial\Omega)$, and for each $X \in T_Q(\theta, \lambda)$, there is a chord-arc domain Ω_X, constructed as a union $\cup_k I_k^*$ of fattened Whitney boxes I_k^*, with uniformly controlled chord-arc constants, such that

$$U_Q^i \subset \Omega_X \subset \Omega \cap B(X, C\delta(X)),$$

where U_Q^i is the particular connected component of U_Q containing X, and

$$\sigma(\partial\Omega_X \cap Q) \geq \eta\sigma(Q). \tag{3.4}$$

For some $Q \in \mathbb{D}(\partial\Omega)$, it may be that T_Q is empty. On the other hand, by the preceding discussion, each $X \in \Omega$ belongs to $T_Q(\theta_1, \lambda_1)$ for suitable Q, θ_1 and λ_1, so that (3.4) (with $\theta = \theta_1, \lambda = \lambda_1$) implies

$$\sigma(\partial\Omega_X \cap \Delta_X) \geq \eta_1\sigma(\Delta_X),$$

with $\eta_1 \approx \eta$, where Q is the particular Q_i selected in the previous paragraph. Moreover, since $X \in T_Q \subset U_Q$, we can modify Ω_X if necessary, by adjoining to it one or more fattened Whitney boxes I^* with $\ell(I) \approx \ell(Q)$, to ensure that for the modified Ω_X, we have in addition that $\operatorname{dist}(X, \partial\Omega_X) \gtrsim \ell(Q) \approx \delta(X)$, and therefore Ω_X verifies all the conditions in Definition 1.20.

The rest of this section is therefore devoted to proving that there exists, for a given Q and for each $X \in T_Q(\theta, \lambda)$, a chord-arc domain Ω_X satisfying the stated properties (when the set $T_Q(\theta, \lambda)$ is not vacuous). To this end, we let $\lambda \in (0, 1)$ (by Remark 3.3, any fixed $\lambda \leq \lambda_1$ will suffice). We also fix positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, and for these values of η and K, we make the bilateral Corona decomposition of Lemma 2.1, so that $\mathbb{D}(\partial\Omega) = \mathcal{G} \cup \mathcal{B}$. We also construct the Whitney collections $\mathcal{W}_Q^0$ in (2.7), and $\mathcal{W}_Q^*$ of Lemma 2.9 for this same choice of η and K.

Given a cube $Q \in \mathbb{D}(\partial\Omega)$, we set

$$\mathbb{D}_*(Q) := \{Q' \subset Q : \ell(Q)/4 \leq \ell(Q') \leq \ell(Q)\}. \tag{3.5}$$

Thus, $\mathbb{D}_*(Q)$ consists of the cube Q itself, along with its dyadic children and grandchildren. Let

$$\mathcal{M} := \{Q(\mathbf{S})\}_{\mathbf{S}}$$

denote the collection of cubes which are the maximal elements of the stopping time regimes in $\mathcal{G}$. We define

$$\alpha_Q := \begin{cases} \sigma(Q), & \text{if } (\mathcal{M} \cup \mathcal{B}) \cap \mathbb{D}_*(Q) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Given any collection $\mathbb{D}' \subset \mathbb{D}(\partial\Omega)$, we set

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q. \tag{3.7}$$

Then $\mathfrak{m}$ is a discrete Carleson measure, i.e., recalling that $\mathbb{D}_Q$ is the discrete Carleson region relative to Q defined in (2.19), we claim that there is a uniform constant C such that

$$\mathfrak{m}(\mathbb{D}_Q) = \sum_{Q' \subset Q} \alpha_{Q'} \leq C\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(\partial\Omega). \tag{3.8}$$

Indeed, note that for any $Q' \in \mathbb{D}_Q$, there are at most 3 cubes Q such that $Q' \in \mathbb{D}_*(Q)$ (namely, Q' itself, its dyadic parent, and its dyadic grandparent), and that by ADR, $\sigma(Q) \approx \sigma(Q')$, if $Q' \in \mathbb{D}_*(Q)$. Thus, given any $Q_0 \in \mathbb{D}(\partial\Omega)$,

$$\begin{aligned} \mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \leq \sum_{Q' \in \mathcal{M}\cup\mathcal{B}} \ \sum_{Q \subset Q_0:\, Q' \in \mathbb{D}_*(Q)} \sigma(Q) \\ \lesssim \sum_{Q' \in \mathcal{M}\cup\mathcal{B}:\, Q' \subset Q_0} \sigma(Q') \leq C\sigma(Q_0), \end{aligned}$$

by Lemma 2.1 (2). Here, and throughout the remainder of this section, a generic constant C, and implicit constants, are allowed to depend upon the choice of the parameters η and K that we have fixed, along with the usual allowable parameters.

With (3.8) in hand, we therefore have

$$M_0 := \sup_{Q \in \mathbb{D}(E)} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} \leq C < \infty. \tag{3.9}$$

As mentioned above, our proof will be based on a two parameter induction scheme. Given $\lambda \in (0, \lambda_1]$ fixed as above, we recall that the set $F_{car}(X, Q, \lambda)$ is defined in (3.1). The induction hypothesis, which we formulate for any $a \geq 0$, and any $\theta \in (0, 1]$ is as follows:

$\boxed{H[a,\theta]}$

There is a positive constant $c_a = c_a(\theta) < 1$ such that for any given $Q \in \mathbb{D}(\partial\Omega)$, if

$$\mathfrak{m}(\mathbb{D}_Q) \leq a\sigma(Q), \tag{3.10}$$

and if there is a subset $V_Q \subset U_Q \cap \Omega$ for which

$$\sigma\Bigg(\bigcup_{X \in V_Q} F_{car}(X, Q, \lambda)\Bigg) \geq \theta\sigma(Q), \tag{3.11}$$

then there is a subset $V_Q^* \subset V_Q$, such that for each connected component U_Q^i of U_Q which meets V_Q^*, there is a chord-arc domain Ω_Q^i which is the interior of the union of a collection of fattened Whitney cubes I^*, and whose chord-arc constants depend only on dimension, λ, a, θ, and the ADR constants for Ω. Moreover, $U_Q^i \subset \Omega_Q^i \subset B_Q^* \cap \Omega = B(x_Q, K\ell(Q)) \cap \Omega$, and $\sum_i \sigma(\partial\Omega_Q^i \cap Q) \geq c_a\sigma(Q)$, where the sum runs over those i such that U_Q^i meets V_Q^*.

Let us briefly sketch the strategy of the proof. We first fix $\theta = 1$, and by induction on a, establish $H[M_0, 1]$. We then show that there is a fixed $\zeta \in (0, 1)$ such that $H[M_0, \theta]$ implies $H[M_0, \zeta\theta]$, for every $\theta \in (0, 1]$. Iterating, we then obtain

$H[M_0, \theta_1]$ for any $\theta_1 \in (0, 1]$. Now, by (3.9), we have (3.10) with $a = M_0$, for every $Q \in \mathbb{D}(\partial\Omega)$. Thus, $H[M_0, \theta_1]$ may be applied in every cube Q such that $T_Q(\theta_1, \lambda)$ (see (3.2)) is non-empty, with $V_Q = \{X\}$, for any $X \in T_Q(\theta_1, \lambda)$. For $\lambda \leq \lambda_1$, and an appropriate choice of θ_1, by Remark 3.3, we obtain the existence of a chord-arc domain Ω_X verifying the conditions of Definition 1.20, and thus that Theorem 1.3 holds, as desired.

We begin with some preliminary observations. In what follows we have fixed $\lambda \in (0, \lambda_1]$ and two positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, for which the bilateral Corona decomposition of $\mathbb{D}(\partial\Omega)$ in Lemma 2.1 is applied. We now fix $k_0 \in \mathbb{N}$, $k_0 \geq 4$, such that

$$2^{-k_0} \leq \frac{\eta}{K} < 2^{-k_0+1}\,. \tag{3.12}$$

Lemma 3.13. *Let $Q \in \mathbb{D}(\partial\Omega)$, and suppose that $Q' \subset Q$, with $\ell(Q') \leq 2^{-k_0}\ell(Q)$. Suppose that there are points $X \in U_Q \cap \Omega$ and $y \in Q'$, that are connected by a λ-carrot path $\gamma = \gamma(y, X)$ in Ω. Then γ meets $U_{Q'} \cap \Omega$.*

Proof. By construction (see (2.7), Lemma 2.9, (2.15) and (2.16)), $X \in U_Q$ implies that

$$\eta^{1/2}\ell(Q) \lesssim \delta(X) \lesssim K^{1/2}\ell(Q)\,.$$

Since $2^{-k_0} \ll \eta$, and $\ell(Q') \leq 2^{-k_0}\ell(Q)$, we then have that $X \in \Omega \setminus B(y, 2\ell(Q'))$, so $\gamma(y, X)$ meets $B(y, 2\ell(Q')) \setminus B(y, \ell(Q'))$, say at a point Z. Since $\gamma(y, X)$ is a λ-carrot path, and since we have previously specified that $\eta \ll \lambda^4$,

$$\delta(Z) \geq \lambda\ell(\gamma(y, Z)) \geq \lambda|y - Z| \geq \lambda\ell(Q') \gg \eta^{1/4}\ell(Q')\,.$$

On the other hand

$$\delta(Z) \leq \operatorname{dist}(Z, Q') \leq |Z - y| \leq 2\ell(Q') \ll K^{1/2}\ell(Q')\,.$$

In particular then, the Whitney box I containing Z must belong to $\mathcal{W}^0_{Q'}$ (see (2.7)), so $Z \in U_{Q'}$. Note that $Z \in \Omega$ since $\gamma \subset \Omega$. □

We shall also require the following. We recall that by Lemma 2.9, for $Q \in \mathbf{S}$, the Whitney region U_Q has the splitting $U_Q = U_Q^+ \cup U_Q^-$, with U_Q^+ (resp. U_Q^-) lying above (resp., below) the Lipschitz graph $\Gamma_{\mathbf{S}}$ of Lemma 2.1.

Lemma 3.14. *Let $Q' \subset Q$, and suppose that Q' and Q both belong to $\mathcal{G}$, and moreover that both Q' and Q belong to the same stopping time regime* **S**. *Suppose that $y \in Q'$ and $X \in U_Q \cap \Omega$ are connected via a λ-carrot path $\gamma(y, X)$ in Ω, and assume that there is a point $Z \in \gamma(y, X) \cap U_{Q'} \cap \Omega$ (by Lemma 3.13 we know that such a Z exists provided $\ell(Q') \leq 2^{-k_0}\ell(Q)$). Then $X \in U_Q^+$ if and only if $Z \in U_{Q'}^+$ (thus, $X \in U_Q^-$ if and only if $Z \in U_{Q'}^-$).*

Proof. We suppose for the sake of contradiction that, e.g., $X \in U_Q^+$, and that $Z \in U_{Q'}^-$. Thus, in traveling from y to Z and then to X along the path $\gamma(y, X)$, one must cross the Lipschitz graph $\Gamma_{\mathbf{S}}$ at least once between Z and X. Let Y_1 be the first point on $\gamma(y, X) \cap \Gamma_{\mathbf{S}}$ that one encounters *after* Z, when traveling toward X. By Lemma 2.9,

$$K^{1/2}\ell(Q) \gtrsim \delta(X) \geq \lambda\ell(\gamma(y, X)) \gg K^{-1/4}\ell(\gamma(y, X))\,,$$

where we recall that we have fixed $K \gg \lambda^{-4}$. Consequently, $\ell(\gamma(y,X)) \ll K^{3/4}\ell(Q)$, so in particular, $\gamma(y,X) \subset B_Q^* := B(x_Q, K\ell(Q))$, as in Lemma 2.1. On the other hand, $Y_1 \notin B_{Q'}^*$. Indeed, $Y_1 \in \Gamma_S$, so if $Y_1 \in B_{Q'}^*$, then by (2.2), $\delta(Y_1) \leq \eta\ell(Q')$. However,

$$\delta(Y_1) \geq \lambda\ell(\gamma(y,Y_1)) \geq \lambda\ell(\gamma(y,Z)) \geq \lambda|y-Z| \geq \lambda\,\mathrm{dist}(Z,Q') \gtrsim \lambda\eta^{1/2}\ell(Q')\,,$$

where in the last step we have used Lemma 2.9. This contradicts our choice of $\eta \ll \lambda^4$.

We now form a chain of consecutive dyadic cubes $\{P_i\} \subset \mathbb{D}_Q$, connecting Q' to Q, i.e.,

$$Q' = P_0 \lhd P_1 \lhd P_2 \lhd \cdots \lhd P_M \lhd P_{M+1} = Q\,,$$

where the introduced notation $P_i \lhd P_{i+1}$ means that P_i is the dyadic child of P_{i+1}, that is, $P_i \subset P_{i+1}$ and $\ell(P_{i+1}) = 2\ell(P_i)$. Let $P := P_{i_0}$, $1 \leq i_0 \leq M+1$, be the smallest of the cubes P_i such that $Y_1 \in B_{P_i}^*$. Setting $P' := P_{i_0-1}$, we then have that $Y_1 \in B_P^*$, and $Y_1 \notin B_{P'}^*$. By the coherency of $\mathbf{S}$, it follows that $P \in \mathbf{S}$, so by (2.2),

$$\delta(Y_1) \leq \eta\ell(P)\,. \tag{3.15}$$

On the other hand,

$$\mathrm{dist}(Y_1, P') \gtrsim K\ell(P') \approx K\ell(P)\,,$$

and therefore, since $y \in Q' \subset P'$,

$$\delta(Y_1) \geq \lambda\ell(\gamma(y,Y_1)) \geq \lambda|y-Y_1| \geq \lambda\,\mathrm{dist}(Y_1,P') \gtrsim \lambda K\ell(P)\,. \tag{3.16}$$

Combining (3.15) and (3.16), we see that $\lambda \lesssim \eta/K$, which contradicts that we have fixed $\eta \ll \lambda^4$, and $K \gg \lambda^{-4}$. □

Lemma 3.17. *Fix $\lambda \in (0,1)$. Given $Q \in \mathbb{D}(\partial\Omega)$ and a non-empty set $V_Q \subset U_Q \cap \Omega$, such that each $X \in V_Q$ may be connected by a λ-carrot path to some $y \in Q$, set*

$$F_Q := \bigcup_{X \in V_Q} F_{car}(X,Q,\lambda)\,, \tag{3.18}$$

where we recall that $F_{car}(X,Q,\lambda)$ is the set of $y \in Q$ that are connected via a λ-carrot path to X (see (3.1)). Let $Q' \subset Q$ be such that $\ell(Q') \leq 2^{-k_0}\ell(Q)$ and $F_Q \cap Q' \neq \emptyset$. Then, there exists a non-empty set $V_{Q'} \subset U_{Q'} \cap \Omega$ such that if we define $F_{Q'}$ as in (3.18) with Q' replacing Q, then $F_Q \cap Q' \subset F_{Q'}$. Moreover, for every $Y \in V_{Q'}$, there exist $X \in V_Q$, $y \in Q'$ (indeed $y \in F_Q \cap Q'$) and a λ-carrot path $\gamma = \gamma(y,X)$ such that $Y \in \gamma$.

Proof. For every $y \in F_Q \cap Q'$, by definition of F_Q, there exist $X \in V_Q$ and a λ-carrot path $\gamma = \gamma(y,X)$. By Lemma 3.13, there is $Y = Y(y) \in \gamma \cap U_{Q'} \cap \Omega$ (there can be more than one Y, but we just pick one). Note that the sub-path $\gamma(y,Y) \subset \gamma(y,X)$ is also a λ-carrot path, for the same constant λ. All the conclusions in the lemma follow easily from the construction by letting $V_{Q'} = \bigcup_{y \in F_Q \cap Q'} Y(y)$. □

Remark 3.19. It follows easily from the previous proof that under the same assumptions, if one further assumes that $\ell(Q') < 2^{-k_0}\,\ell(Q)$, we can then repeat the argument with both Q' and $(Q')^*$ (the dyadic parent of Q') to obtain respectively $V_{Q'}$ and $V_{(Q')^*}$. Moreover, this can be done in such a way that every point in $V_{Q'}$ (resp. $V_{(Q')^*}$) belongs to a λ-carrot path which also meets $V_{(Q')^*}$ (resp. $V_{Q'}$), connecting U_Q and Q'.

Given a family $\mathcal{F} := \{Q_j\} \subset \mathbb{D}(\partial\Omega)$ of pairwise disjoint cubes, we recall that the "discrete sawtooth" $\mathbb{D}_{\mathcal{F}}$ is the collection of all cubes in $\mathbb{D}(\partial\Omega)$ that are not contained in any $Q_j \in \mathcal{F}$ (see (2.21)), and we define the restriction of $\mathfrak{m}$ (cf. (3.6), (3.7)) to the sawtooth $\mathbb{D}_{\mathcal{F}}$ by

$$\mathfrak{m}_{\mathcal{F}}(\mathbb{D}') := \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q \in \mathbb{D}' \setminus (\cup_{\mathcal{F}} \mathbb{D}_{Q_j})} \alpha_Q. \tag{3.20}$$

We then set

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} := \sup_{Q' \subset Q} \frac{\mathfrak{m}_{\mathcal{F}}(\mathbb{D}_{Q'})}{\sigma(Q')}.$$

Let us note that we may allow $\mathcal{F}$ to be empty, in which case $\mathbb{D}_{\mathcal{F}} = \mathbb{D}$ and $\mathfrak{m}_{\mathcal{F}}$ is simply $\mathfrak{m}$. We note that the following claims remain true when $\mathcal{F}$ is empty, with some straightforward changes that are left to the interested reader.

Claim 3.21. *Given $Q \in \mathbb{D}(\partial\Omega)$, and a family $\mathcal{F} = \mathcal{F}_Q := \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$ of pairwise disjoint sub-cubes of Q, if $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \le 1/2$, then each $Q' \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$, each $Q_j \in \mathcal{F}$, and every dyadic child Q_j' of any $Q_j \in \mathcal{F}$, belong to the good collection $\mathcal{G}$, and moreover, every such cube belongs to the* **same** *stopping time regime* $\mathbf{S}$. *In particular,* $\mathbf{S}' := \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ *is a semi-coherent subregime of* $\mathbf{S}$*, and so is* $\mathbf{S}'' := (\mathbb{D}_{\mathcal{F}} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathbb{D}_Q$*, where $\mathcal{F}'$ denotes the collection of all dyadic children of cubes in $\mathcal{F}$.*

Indeed, if any $Q' \in \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ were in $\mathcal{M} \cup \mathcal{B}$ (recall that $\mathcal{M} := \{Q(\mathbf{S})\}_{\mathbf{S}}$ is the collection of cubes which are the maximal elements of the stopping time regimes in $\mathcal{G}$), then by construction $\alpha_{Q'} = \sigma(Q')$ for that cube (see (3.6)), so by definition of $\mathfrak{m}$ and $\mathfrak{m}_{\mathcal{F}}$, we would have

$$1 = \frac{\sigma(Q')}{\sigma(Q')} \le \frac{\mathfrak{m}_{\mathcal{F}}(\mathbb{D}_{Q'})}{\sigma(Q')} \le \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \le \frac{1}{2},$$

a contradiction. Similarly, if some $Q_j \in \mathcal{F}$ (respectively, $Q_j' \in \mathcal{F}'$) were in $\mathcal{M} \cup \mathcal{B}$, then its dyadic parent (respectively, dyadic grandparent) Q_j^* would belong to $\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$, and by definition $\alpha_{Q_j^*} = \sigma(Q_j^*)$, so again we reach a contradiction. Consequently, $\mathcal{F} \cup \mathcal{F}' \cup (\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q)$ does not meet $\mathcal{M} \cup \mathcal{B}$, and the claim follows.

For future reference, we now prove the following. Recall that for $Q \in \mathcal{G}$, U_Q has precisely two connected components $U_Q^\pm$ in $\mathbb{R}^{n+1} \setminus \partial\Omega$.

Lemma 3.22. *Let $Q \in \mathbb{D}(\partial\Omega)$, let k_1 be such that $2^{k_1} > 2^{k_0} \gg 100K$, see (3.12), and suppose that there is a family $\mathcal{F} = \mathcal{F}_Q := \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$ of pairwise disjoint sub-cubes of Q, with $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \le 1/2$ (hence by Claim 3.21, there is some* $\mathbf{S}$ *with* $\mathbf{S} \supset (\mathbb{D}_{\mathcal{F}} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathbb{D}_Q$*), and a non-empty subcollection $\mathcal{F}^* \subset \mathcal{F}$, such that:*

(i) *$\ell(Q_j) \le 2^{-k_1}\ell(Q)$, for each cube $Q_j \in \mathcal{F}^*$;*

(ii) *the collection of balls $\{\kappa B_{Q_j}^* := B(x_{Q_j}, \kappa K \ell(Q_j)) : Q_j \in \mathcal{F}^*\}$ is pairwise disjoint, where $\kappa \gg K^4$ is a sufficiently large positive constant; and*

(iii) *$\mathcal{F}^*$ has a disjoint decomposition $\mathcal{F}^* = \mathcal{F}_+^* \cup \mathcal{F}_-^*$, where for each $Q_j \in \mathcal{F}_\pm^*$, there is a chord-arc subdomain $\Omega_{Q_j}^\pm \subset \Omega$, consisting of a union of fattened Whitney cubes I^*, with $U_{Q_j}^\pm \subset \Omega_{Q_j}^\pm \subset B_{Q_j}^* := B(x_{Q_j}, K\ell(Q_j))$, and with uniform control of the chord-arc constants.*

Define a semi-coherent subregime $\mathbf{S}^* \subset \mathbf{S}$ *by*

$$\mathbf{S}^* = \left\{ Q' \in \mathbb{D}_Q : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}^* \right\},$$

and for each choice of $\pm$ *for which* $\mathcal{F}^*_\pm$ *is non-empty, set*

$$\Omega_Q^\pm := \Omega^\pm_{\mathbf{S}^*} \bigcup \left(\bigcup_{Q_j \in \mathcal{F}^*_\pm} \Omega^\pm_{Q_j} \right) \tag{3.23}$$

Then for κ *large enough, depending only on allowable parameters,* $\Omega_Q^\pm$ *is a chord-arc domain, with chord arc constants depending only on the uniformly controlled chord-arc constants of* $\Omega^\pm_{Q_j}$ *and on the other allowable parameters. Moreover,* $\Omega_Q^\pm \subset B_Q^* \cap \Omega = B(x_Q, K\ell(Q)) \cap \Omega$*, and* $\Omega_Q^\pm$ *is a union of fattened Whitney cubes.*

Remark 3.24. Note that we define $\Omega_Q^\pm$ if and only if $\mathcal{F}^*_\pm$ is non-empty. It may be that one of $\mathcal{F}^*_+, \mathcal{F}^*_-$ is empty, but $\mathcal{F}^*_+$ and $\mathcal{F}^*_-$ cannot both be empty, since $\mathcal{F}^*$ is non-empty by assumption.

Proof of Lemma 3.22. Without loss of generality we may assume that $\Omega_{Q_j}\pm$ is not contained in $\Omega^\pm_{\mathbf{S}^*}$ for all $Q_j \in \mathcal{F}^*$ (otherwise we can drop those cubes from $\mathcal{F}^*$). On the other hand, we notice that $\Omega^\pm_Q$ is a union of (open) fattened Whitney cubes (assuming that it is non-empty): each $\Omega^\pm_{Q_j}$ has this property by assumption, as does $\Omega^\pm_{\mathbf{S}^*}$ by construction.

We next observe that if Ω_Q^+ (resp. Ω_Q^-) is non-empty, then it is contained in Ω. Indeed, by construction, Ω_Q^+ is non-empty if and only if $\mathcal{F}^*_+$ is non-empty. In turn, $\mathcal{F}^*_+$ is non-empty if and only if there is some $Q_j \in \mathcal{F}^*$ such that $U^+_{Q_j} \subset \Omega^+_{Q_j} \subset \Omega$, and moreover, the latter is true for every $Q_j \in \mathcal{F}^*_+$, by definition. But each such Q_j belongs to $\mathbf{S}^*$, hence $U^+_{Q_j} \subset \Omega^+_{\mathbf{S}^*}$, again by construction (see (2.12)). Thus, $\Omega^+_{\mathbf{S}^*}$ meets Ω, and since $\Omega^+_{\mathbf{S}^*} \subset \mathbb{R}^{n+1} \setminus \partial\Omega$, therefore $\Omega^+_{\mathbf{S}^*} \subset \Omega$. Combining these observations, we see that $\Omega_Q^+ \subset \Omega$. Of course, the same reasoning applies to Ω_Q^-, provided it is non-empty.

In addition, since $\mathbf{S}^* \subset \mathbf{S}$, and since $K \gg K^{1/2}$, by Lemma 2.9 we have $\Omega^\pm_{\mathbf{S}^*} \subset B_Q^* = B(x_Q, K\ell(Q))$. Furthermore, $\Omega^\pm_{Q_j} \subset B^*_{Q_j} := B(x_{Q_j}, K\ell(Q_j))$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q) \leq (100K)^{-1}\ell(Q)$, we obtain

$$\operatorname{dist}(\Omega^\pm_{Q_j}, Q) + \operatorname{diam}(\Omega^\pm_{Q_j}) \leq 3K\ell(Q_j) \leq 3K2^{-k_1}\ell(Q) \ll \ell(Q).$$

Thus, in particular, $\Omega^\pm_{Q_j} \subset B_Q^*$, and therefore also $\Omega_Q^\pm \subset B_Q^*$.

It therefore remains to establish the chord-arc properties. It is straightforward to prove the interior Corkscrew condition and the upper ADR bound, and we omit the details. Thus, we must verify the Harnack Chain condition, the lower ADR bound, and the exterior Corkscrew condition.

Harnack Chains. Suppose, without loss of generality, that Ω_Q^+ is non-empty, and let $X, Y \in \Omega_Q^+$, with $|X - Y| = R$. If X and Y both lie in $\Omega^+_{\mathbf{S}^*}$, or in the same $\Omega^+_{Q_j}$, then we can connect X and Y by a suitable Harnack path, since each of these domains is chord-arc. Thus, we may suppose either that 1) $X \in \Omega^+_{\mathbf{S}^*}$ and Y lies in some $\Omega^+_{Q_j}$, or that 2) X and Y lie in two distinct $\Omega^+_{Q_{j_1}}$ and $\Omega^+_{Q_{j_2}}$. We may reduce the latter case to the former case: by the separation property (ii) in Lemma 3.22, we must have $R \gtrsim \kappa \max\big(\operatorname{diam}(\Omega^+_{Q_{j_1}}), \operatorname{diam}(\Omega^+_{Q_{j_2}})\big)$, so given case 1), we can connect $X \in \Omega^+_{Q_{j_1}}$ to the center Z_1 of some $I_1^* \subset U^+_{Q_1}$, and $Y \in \Omega^+_{Q_{j_2}}$ to the center Z_2 of some $I_2 \subset U^+_{Q_2}$, where $Q_1, Q_2 \in \mathbf{S}^*$, with $Q_{j_i} \subset Q_i \subset Q$, and $\ell(Q_i) \approx R$, $i = 1, 2$. Finally, we can connect Z_1 and Z_2 using that $\Omega^+_{\mathbf{S}^*}$ is chord-arc.

Hence, we need only construct a suitable Harnack Chain in Case 1). We note that by assumption and construction, $U^+_{Q_j} \subset \Omega^+_{\mathbf{S}^*} \cap \Omega^+_{Q_j}$.

Suppose first that

$$|X - Y| = R \le c'\ell(Q_j)\,, \tag{3.25}$$

where $c' \le 1$ is a sufficiently small positive constant to be chosen. Since $Y \in \Omega^+_{Q_j} \subset B^*_{Q_j}$, we then have that $X \in 2B^*_{Q_j}$, so by the construction of $\Omega^+_{\mathbf{S}^*}$ and the separation property (ii), it follows that $\delta(X) \ge c\ell(Q_j)$, where c is a uniform constant depending only on the allowable parameters (in particular, this fact is true for all $X \in \Omega^+_{\mathbf{S}^*} \cap 2B^*_{Q_j}$, so it does not depend on the choice of $c' < 1$). Now choosing $c' \le c/2$ (eventually, it may be even smaller), we find that $\delta(Y) \ge (c/2)\ell(Q_j)$. Moreover, $Y \in \Omega^+_{Q_j} \subset B^*_{Q_j}$ implies that $\delta(Y) \le K\ell(Q_j)$. Also, since $X \in 2B^*_{Q_j}$ we have that $\delta(X) \le 2K\ell(Q_j)$. Since $\Omega^+_{Q_j}$ and $\Omega^+_{\mathbf{S}^*}$ are each the interior of a union of fattened Whitney cubes, it follows that there are Whitney cubes I and J, with $X \in I^*$, $Y \in J^*$, and

$$\ell(I) \approx \ell(J) \approx \ell(Q_j)\,,$$

where the implicit constants depend on K. For c' small enough in (3.25), depending on the implicit constants in the last display, and on the parameter τ in (2.5), this can happen only if I^* and J^* overlap (recall that we have fixed τ small enough that I^* and J^* overlap if and only if I and J have a boundary point in common), in which case we may trivially connect X and Y by a suitable Harnack Chain.

On the other hand, suppose that

$$|X - Y| = R \ge c'\ell(Q_j)\,.$$

Let $Z \in U^+_{Q_j} \subset \Omega^+_{\mathbf{S}^*} \cap \Omega^+_{Q_j}$, with $\operatorname{dist}(Z, \partial\Omega^+_Q) \gtrsim \ell(Q_j)$ (we may find such a Z, since $U^+_{Q_j}$ is a union of fattened Whitney cubes, all of length $\ell(I^*) \approx \ell(Q_j)$; just take Z to be the center of such an I^*). We may then construct an appropriate Harnack Chain from Y to X by connecting Y to Z via a Harnack Chain in the chord-arc domain $\Omega^+_{Q_j}$, and Z to X via a Harnack Chain in the chord-arc domain $\Omega^+_{\mathbf{S}^*}$.

Lower ADR and Exterior Corkscrews. We will establish these two properties essentially simultaneously. Again suppose that, e.g., Ω^+_Q is non-empty. Let $x \in \partial\Omega^+_Q$, and consider $B(x, r)$, with $r < \operatorname{diam}\Omega^+_Q \approx_K \ell(Q)$. Our main goal at this stage is to prove the following:

$$\Big|B(x, r) \setminus \overline{\Omega^+_Q}\Big| \ge c r^{n+1}\,, \tag{3.26}$$

with c a uniform positive constant depending only upon allowable parameters (including κ). Indeed, momentarily taking this estimate for granted, we may combine (3.26) with the interior Corkscrew condition to deduce the lower ADR bound via the relative isoperimetric inequality [EG, p. 190]. In turn, with both the lower and upper ADR bounds in hand, (3.26) implies the existence of exterior Corkscrews (see, e.g., [HM2, Lemma 5.7]).

Thus, it is enough to prove (3.26). We consider the following cases.

Case 1: $B(x, r/2)$ does not meet $\partial\Omega^+_{Q_j}$ for any $Q_j \in \mathcal{F}^*_+$. In this case, the exterior Corkscrew for $\Omega^+_{\mathbf{S}^*}$ associated with $B(x, r/2)$ easily implies (3.26).

Case 2: $B(x, r/2)$ meets $\partial\Omega^+_{Q_j}$ for at least one $Q_j \in \mathcal{F}^*_+$, and $r \leq \kappa^{1/2}\ell(Q_{j_0})$, where Q_{j_0} is chosen to have the largest length $\ell(Q_{j_0})$ among those Q_j such that $\partial\Omega^+_{Q_j}$ meets $B(x, r/2)$. We now further split the present case into subcases.

Subcase 2a: $B(x, r/2)$ meets $\partial\Omega^+_{Q_{j_0}}$ at a point Z with $\delta(Z) \leq (M\kappa^{1/2})^{-1}\ell(Q_{j_0})$, where M is a large number to be chosen. Then $B(Z, (M\kappa^{1/2})^{-1}r) \subset B(x, r)$, for M large enough. In addition, we claim that $B(Z, (M\kappa^{1/2})^{-1}r)$ misses $\Omega^+_{\mathbf{S}^*} \cup (\cup_{j\neq j_0}\Omega^+_{Q_j})$. The fact that $B(Z, (M\kappa^{1/2})^{-1}r)$ misses every other $\Omega^+_{Q_j}$, $j \neq j_0$, follows immediately from the restriction $r \leq \kappa^{1/2}\ell(Q_{j_0})$, and the separation property (ii). To see that $B(Z, (M\kappa^{1/2})^{-1}r)$ misses $\Omega^+_{\mathbf{S}^*}$, note that if $|Z - Y| < (M\kappa^{1/2})^{-1}r$, then

$$\delta(Y) \leq \delta(Z) + (M\kappa^{1/2})^{-1}r \leq \left((M\kappa^{1/2})^{-1} + M^{-1}\right)\ell(Q_{j_0}) \ll \ell(Q_{j_0}),$$

for M large. On the other hand,

$$\delta(Y) \gtrsim \ell(Q_{j_0}), \qquad \forall\, Y \in \Omega^+_{\mathbf{S}^*} \cap B(Z, \kappa^{1/2}\ell(Q_{j_0})),$$

by the construction of $\Omega^+_{\mathbf{S}^*}$ and the separation property (ii). Thus, the claim follows, for a sufficiently large (fixed) choice of M. Since $B(Z, (M\kappa^{1/2})^{-1}r)$ misses $\Omega^+_{\mathbf{S}^*}$ and all other $\Omega^+_{Q_j}$, we inherit an exterior Corkscrew point in the present case (depending on M and κ) from the chord-arc domain $\Omega^+_{Q_{j_0}}$. Again (3.26) follows.

Subcase 2b: $\delta(Z) \geq (M\kappa^{1/2})^{-1}\ell(Q_{j_0})$, for every $Z \in B(x, r/2) \cap \partial\Omega^+_{Q_{j_0}}$ (hence $\delta(Z) \approx_{\kappa,K} \ell(Q_{j_0})$, since $\Omega^+_{Q_{j_0}} \subset B^*_{Q_{j_0}}$). We claim that consequently, $x \in \partial I^*$, for some I with $\ell(I) \approx \ell(Q_{j_0}) \gtrsim r$, such that $\operatorname{int} I^* \subset \Omega^+_Q$. To see this, observe that it is clear if $x \in \partial\Omega^+_{Q_{j_0}}$ (just take $Z = x$). Otherwise, by the separation property (ii), the remaining possibility in the present scenario is that $x \in \partial U^+_{Q'} \cap \partial\Omega^+_{\mathbf{S}^*}$, for some $Q' \in \mathbf{S}^*$ with $Q_{j_0} \subset Q'$, in which case $\delta(x) \approx \ell(Q') \geq \ell(Q_{j_0})$. Since also $\delta(x) \leq |x - Z| + \delta(Z) \lesssim_{\kappa,K} \ell(Q_{j_0})$, for any $Z \in B(x, r/2) \cap \partial\Omega^+_{Q_{j_0}}$, the claim follows.

On the other hand, since $x \in \partial\Omega^+_Q$, there is a $J \in \mathcal{W}$ with $\ell(J) \approx \ell(Q_{j_0})$, such that J^* is not contained in Ω^+_Q. We then have an exterior Corkscrew point in $J^* \cap B(x, r)$, and (3.26) follows in this case.

Case 3: $B(x, r/2)$ meets $\partial\Omega^+_{Q_j}$ for at least one $Q_j \in \mathcal{F}^*_+$, and $r > \kappa^{1/2}\ell(Q_{j_0})$, where as above Q_{j_0} has the largest length $\ell(Q_{j_0})$ among those Q_j such that $\partial\Omega^+_{Q_j}$ meets $B(x, r/2)$. In particular then, $r \gg 2K\ell(Q_{j_0}) = \operatorname{diam}(B^*_{Q_{j_0}}) \geq \operatorname{diam}(\Omega^+_{Q_{j_0}})$, since we assume $\kappa \gg K^4$.

We next claim that $B(x, r/4)$ contains some $x_1 \in \partial\Omega^+_{\mathbf{S}^*} \cap \partial\Omega^+_Q$. This is clear if $x \in \partial\Omega^+_{\mathbf{S}^*}$ by taking $x_1 = x$. Otherwise, $x \in \partial\Omega^+_{Q_j}$ for some $Q_j \in \mathcal{F}^*$. Note that

$U^{\pm}_{Q_j} \subset B(x_{Q_j}, K\ell(Q_j)) \subset B(x, 2K\ell(Q_j))$. Also, $U^{\pm}_{Q_j} \subset \Omega^{\pm}_{\mathbf{S}^*}$, by construction. On the other hand we note that if $Z \in U^{\pm}_Q$ we have by (2.17)

$$|Z - x_{Q_j}| \geq \delta(Z) \gtrsim \eta^{1/2}\ell(Q) \geq \eta^{1/2}2^{k_1}\ell(Q_j) \gg K\ell(Q_j)$$

by our choice of k_1. By this fact, and the definition of $\Omega_{\mathbf{S}^*}$, we have

$$U^{\pm}_Q \subset \Omega^{\pm}_{\mathbf{S}^*} \setminus B(x, 3K\ell(Q_j)).$$

Using then that $\Omega^{\pm}_{\mathbf{S}^*}$ is connected, we see that a path within $\Omega^{\pm}_{\mathbf{S}^*}$ joining $U^{\pm}_{Q_j}$ with $U^{\pm}_Q$ must meet $\partial B(x, 3K\ell(Q_j))$. Hence we can find $Y^{\pm} \in \Omega^{\pm}_{\mathbf{S}^*} \cap \partial B(x, 3K\ell(Q_j))$. By Lemma 2.9, $\Omega^{+}_{\mathbf{S}^*}$ and $\Omega^{-}_{\mathbf{S}^*}$ are disjoint (they live respectively above and below the graph $\Gamma_{\mathbf{S}}$), so a path joining Y^+ and Y^- within $\partial B(x, 3K\ell(Q_j))$ meets some $x_1 \in \partial\Omega^{+}_{\mathbf{S}^*} \cap \partial B(x, 3K\ell(Q_j))$. On the other hand, $x_1 \notin \overline{\Omega^{+}_{Q_j}}$, since $\overline{\Omega^{+}_{Q_j}} \subset \overline{B^*_{Q_j}} \subset B(x, 3K\ell(Q_j))$. Furthermore, $x_1 \in \partial B(x, 3K\ell(Q_j)) \subset \kappa B^*_{Q_j}$, so by assumption (ii), we necessarily have that $x_1 \notin \overline{\Omega^{+}_{Q_k}}$ for $k \neq j$. Thus, $x_1 \in \partial\Omega^{+}_{Q}$, and moreover, since $B(x, r/2)$ meets $\partial\Omega^{+}_{Q_j}$ (at x) we have $\ell(Q_j) \leq \ell(Q_{j_0})$. Therefore, x_1 is the claimed point, since in the current case $3K\ell(Q_j) \leq 3K\ell(Q_{j_0}) \ll r$.

With the point x_1 in hand, we note that

$$B(x_1, r/4) \subset B(x, r/2) \quad \text{and} \quad B(x_1, r/2) \subset B(x, r). \tag{3.27}$$

By the exterior Corkscrew condition for $\Omega^{+}_{\mathbf{S}^*}$,

$$\left|B(x_1, r/4) \setminus \overline{\Omega^{+}_{\mathbf{S}^*}}\right| \geq c_1 r^{n+1}, \tag{3.28}$$

for some constant c_1 depending only on n and the ADR/UR constants for $\partial\Omega$, by Lemma 2.9. Also, for each $\Omega^{+}_{Q_j}$ whose boundary meets $B(x_1, r/4) \setminus \overline{\Omega^{+}_{\mathbf{S}^*}}$ (and thus meets $B(x, r/2)$),

$$\kappa^{1/4}\operatorname{diam}(B^*_{Q_j}) \leq \kappa^{1/4}\operatorname{diam}(B^*_{Q_{j_0}}) \leq 2K\kappa^{1/4}\ell(Q_{j_0}) \leq \frac{2Kr}{\kappa^{1/4}} \ll r, \tag{3.29}$$

in the present scenario. Consequently, $\kappa^{1/4}B^*_{Q_j} \subset B(x_1, r/2)$, for all such Q_j.

We now make the following claim.

Claim 1:

$$\left|B(x_1, r/2) \setminus \overline{\Omega^{+}_{Q}}\right| \geq c_2 r^{n+1}, \tag{3.30}$$

for some $c_2 > 0$ depending only on allowable parameters.

Observe that by the second containment in (3.27), we obtain (3.26) as an immediate consequence of (3.30), and thus the proof will be complete once we have established Claim 1.

To prove the claim, we suppose first that

$$\sum \left|B^*_{Q_j} \setminus \overline{\Omega^{+}_{\mathbf{S}^*}}\right| \leq \frac{c_1}{2} r^{n+1}, \tag{3.31}$$

where the sum runs over those j such that $\overline{B^*_{Q_j}}$ meets $B(x_1, r/4) \setminus \overline{\Omega^{+}_{\mathbf{S}^*}}$, and c_1 is the constant in (3.28). In that case, (3.30) holds with $c_2 = c_1/2$ (and even with

$B(x_1, r/4)$), by definition of Ω_Q^+ (see (3.23)), and the fact that $\Omega_{Q_j} \subset B^*_{Q_j}$. On the other hand, if (3.31) fails, then summing over the same subset of indices j, we have

$$CK \sum \ell(Q_j)^{n+1} \geq \sum \left|B^*_{Q_j} \setminus \overline{\Omega^+_{\mathbf{S}^*}}\right| \geq \frac{c_1}{2}\, r^{n+1} \tag{3.32}$$

We now make a second claim:

Claim 2: For j appearing in the previous sum, we have

$$\left|\left(\kappa^{1/4} B^*_{Q_j} \setminus B^*_{Q_j}\right) \setminus \overline{\Omega^+_{\mathbf{S}^*}}\right| \geq c\, \ell(Q_j)^{n+1}\,, \tag{3.33}$$

for some uniform $c > 0$.

Taking the latter claim for granted momentarily, we insert estimate (3.33) into (3.32) and sum, to obtain

$$\sum \left|\left(\kappa^{1/4} B^*_{Q_j} \setminus B^*_{Q_j}\right) \setminus \overline{\Omega^+_{\mathbf{S}^*}}\right| \gtrsim r^{n+1}\,. \tag{3.34}$$

By the separation property (ii), the balls $\kappa^{1/4} B^*_{Q_j}$ are pairwise disjoint, and by assumption $\Omega^+_{Q_j} \subset B^*_{Q_j}$. Thus, for any given j_1, $\kappa^{1/4} B^*_{Q_{j_1}} \setminus \overline{B^*_{Q_{j_1}}}$ misses $\cup_j \overline{\Omega^+_{Q_j}}$. Moreover, as noted above (see (3.29) and the ensuing comment), $\kappa^{1/4} B^*_{Q_j} \subset B(x_1, r/2)$ for each j under consideration in (3.31)-(3.34). Claim 1 now follows.

We turn to the proof of Claim 2. There are two cases: if $\frac{1}{2}\kappa^{1/4} B^*_{Q_j} \subset \mathbb{R}^{n+1} \setminus \overline{\Omega^+_{\mathbf{S}^*}}$, then (3.33) is trivial, since $\kappa \gg 1$. Otherwise, $\frac{1}{2}\kappa^{1/4} B^*_{Q_j}$ contains a point $z \in \partial\Omega^+_{\mathbf{S}^*}$. In the latter case, by the exterior Corkscrew condition for $\Omega^+_{\mathbf{S}^*}$,

$$\left|B(z, 2^{-1}\kappa^{1/4} K\ell(Q_j)) \setminus \overline{\Omega^+_{\mathbf{S}^*}}\right| \gtrsim \kappa^{(n+1)/4} (K\ell(Q_j))^{n+1} \gg |B^*_{Q_j}|\,,$$

since $\kappa \gg 1$. On the other hand, $B(z, 2^{-1}\kappa^{1/4} K\ell(Q_j)) \subset \kappa^{1/4} B^*_{Q_j}$, and (3.33) follows. $\square$

3.2. Step 2: Proof of $H[M_0, 1]$.

We shall deduce $H[M_0, 1]$ from the following pair of claims.

Claim 3.35. *$H[0, \theta]$ holds for every $\theta \in (0, 1]$.*

Proof of Claim 3.35. If $a = 0$ in (3.10), then $\|\mathfrak{m}\|_{\mathcal{C}(Q)} = 0$, whence it follows by Claim 3.21, with $\mathcal{F} = \emptyset$, that there is a stopping time regime $\mathbf{S} \subset \mathcal{G}$, with $\mathbb{D}_Q \subset \mathbf{S}$. Hence $\mathbf{S}' := \mathbb{D}_Q$ is a coherent subregime of $\mathbf{S}$, so by Lemma 2.9, each of $\Omega^\pm_{\mathbf{S}'}$ is a CAD, containing $U_Q^\pm$, respectively, with $\Omega^\pm_{\mathbf{S}'} \subset B^*_Q$ by (2.18). Moreover, by [HMM, Proposition A.14]

$$Q \subset \partial\Omega^\pm_{\mathbf{S}'} \cap \partial\Omega\,,$$

so that $\sigma(Q) \leq \sigma(\partial\Omega^\pm_{\mathbf{S}'} \cap \partial\Omega)$. Thus, $H[0, \theta]$ holds trivially. $\square$

Claim 3.36. *There is a uniform constant $b > 0$ such that $H[a, 1] \implies H[a + b, 1]$, for all $a \in [0, M_0)$.*

Combining Claims 3.35 and 3.36, we find that $H[M_0, 1]$ holds.

To prove Claim 3.36, we shall require the following.

Lemma 3.37 ([HM2, Lemma 7.2]). *Suppose that E is an n-dimensional ADR set, and let $\mathfrak{m}$ be a discrete Carleson measure, as in (3.7)-(3.9) above. Fix $Q \in \mathbb{D}(E)$. Let $a \geq 0$ and $b > 0$, and suppose that $\mathfrak{m}(\mathbb{D}_Q) \leq (a+b)\,\sigma(Q)$. Then there is a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q$ of pairwise disjoint cubes, and a constant C depending only on n and the ADR constant such that*

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq Cb, \tag{3.38}$$

$$\sigma\Big(\bigcup_{\mathcal{F}_{bad}} Q_j\Big) \leq \frac{a+b}{a+2b}\,\sigma(Q)\,, \tag{3.39}$$

where $\mathcal{F}_{bad} := \{Q_j \in \mathcal{F} : \mathfrak{m}(\mathbb{D}_{Q_j} \setminus \{Q_j\}) > a\sigma(Q_j)\}$.

We refer the reader to [HM2, Lemma 7.2] for the proof. We remark that the lemma is stated in [HM2] in the case that E is the boundary of a connected domain, but the proof actually requires only that E have a dyadic cube structure, and that σ be a non-negative, dyadically doubling Borel measure on E. In our case, we shall of course apply the lemma with $E = \partial\Omega$, where Ω is open, but not necessarily connected.

Proof of Claim 3.36. We assume that $H[a, 1]$ holds, for some $a \in [0, M_0)$. Set $b = 1/(2C)$, where C is the constant in (3.38). Consider a cube $Q \in \mathbb{D}(\partial\Omega)$ with $\mathfrak{m}(\mathbb{D}_Q) \leq (a+b)\sigma(Q)$. Suppose that there is a set $V_Q \subset U_Q \cap \Omega$ such that (3.11) holds with $\theta = 1$. We fix $k_1 > k_0$ (see (3.12)) large enough so that $2^{k_1} > 100K$.

Case 1: There exists $Q' \in \mathbb{D}_Q^{k_1}$ (see (2.20)) with $\mathfrak{m}(\mathbb{D}_{Q'}) \leq a\sigma(Q')$.

In the present scenario $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (3.11) and (3.18)), which implies $\sigma(F_Q \cap Q') = \sigma(Q')$. We apply Lemma 3.17 to obtain $V_{Q'} \subset U_{Q'} \cap \Omega$ and the corresponding $F_{Q'}$ which satisfies $\sigma(F_{Q'}) = \sigma(Q')$. That is, (3.11) holds for Q', with $\theta = 1$. Consequently, we may apply the induction hypothesis to Q', to find $V^*_{Q'} \subset V_{Q'}$, such that for each $U^i_{Q'}$ meeting $V^*_{Q'}$, there is a chord-arc domain $\Omega^i_{Q'} \supset U^i_{Q'}$ formed by a union of fattened Whitney cubes with $\Omega^i_{Q'} \subset B(x'_Q, K\ell(Q')) \cap \Omega$, and

$$\sum_{i:U^i_{Q'} \text{ meets } V^*_{Q'}} \sigma(\partial\Omega^i_{Q'} \cap Q') \geq c_a \sigma(Q')\,. \tag{3.40}$$

By Lemma 3.17, and since $k_1 > k_0$, each $Y \in V^*_{Q'}$ lies on a λ-carrot path connecting some $y \in Q'$ to some $X \in V_Q$; let V^{**}_Q denote the set of all such X, and let $\mathbf{U}^{**}_Q$ (respectively, $\mathbf{U}^*_{Q'}$) denote the collection of connected components of U_Q (resp., of $U_{Q'}$) which meet V^{**}_Q (resp., $V^*_{Q'}$). By construction, each component $U^i_{Q'} \in \mathbf{U}^*_{Q'}$ may be joined to some corresponding component in $\mathbf{U}^{**}_Q$, via one of the carrot paths. After possible renumbering, we designate this component as U^i_Q, we let X_i, Y_i denote the points in $V^{**}_Q \cap U^i_Q$ and in $V^*_Q \cap U^i_{Q'}$, respectively, that are joined by this carrot path, and we let γ_i be the portion of the carrot path joining X_i to Y_i (if there is more than one such path or component, we just pick one). We also let $V^*_Q = \{X_i\}_i$ be the collection of all of the selected points X_i. We let $\mathcal{W}_i$ be the collection of Whitney cubes meeting γ_i, and we then define

$$\Omega_Q^i := \Omega_{Q'}^i \bigcup \operatorname{int}\Bigg(\bigcup_{I\in\mathcal{W}_i} I^*\Bigg) \bigcup U_Q^i \,.$$

By the definition of a λ-carrot path, since $\ell(Q') \approx_{k_1} \ell(Q)$, and since $\Omega_{Q'}^i$ is a CAD, one may readily verify that Ω_Q^i is also a CAD consisting of a union $\cup_k I_k^*$ of fattened Whitney cubes I_k^*. We omit the details. Moreover, by construction,

$$\partial\Omega_Q^i \cap Q \supset \partial\Omega_{Q'}^i \cap Q',$$

so that the analogue of (3.40) holds with Q' replaced by Q, and with c_a replaced by $c_{k_1} c_a$.

It remains to verify that $\Omega_Q^i \subset B_Q^* = B(x_Q, K\ell(Q))$. By the induction hypothesis, and our choice of k_1, since $\ell(Q') = 2^{-k_1}\ell(Q)$ we have

$$\Omega_{Q'}^i \subset B_{Q'}^* \cap \Omega = B(x_{Q'}, K\ell(Q')) \cap \Omega \subset B_Q^* \cap \Omega.$$

Moreover, $U_Q \subset B_Q^*$, by (2.18). We therefore need only to consider I^* with $I \in \mathcal{W}_i$. For such an I, by definition there is a point $Z_i \in I \cap \gamma_i$ and $y_i \in Q'$, so that $Z_i \in \gamma(y_i, X_i)$ and thus,

$$\delta(Z_i) \le |Z_i - y_i| \le \ell(y_i, Z_i) \le \ell(y_i, X_i) \le \lambda^{-1}\delta(X_i) \le \lambda^{-1}|X_i - x_Q| \le \lambda^{-1} C K^{1/2}\ell(Q)\,,$$

where in the last inequality we have used (2.17) and the fact that $X_i \in U_Q$. Hence, for every $Z \in I^*$ by (2.4)

$$|Z - x_Q| \le \operatorname{diam}(2I) + |Z_i - y_i| + |y_i - x_Q| \le C|Z_i - y_i| + \operatorname{diam}(Q) < K\ell(Q),$$

by our choice of the parameters K and λ.

We then obtain the conclusion of $H[a+b, 1]$ in the present case.

Case 2: $\mathfrak{m}(\mathbb{D}_{Q'}) > a\sigma(Q')$ for every $Q' \in \mathbb{D}_Q^{k_1}$.

In this case, we apply Lemma 3.37 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q$ such that (3.38) and (3.39) hold. In particular, by our choice of $b = 1/(2C)$,

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \le 1/2\,, \tag{3.41}$$

so that the conclusions of Claim 3.21 hold.

We set

$$F_0 := Q \setminus \Big(\bigcup_{\mathcal{F}} Q_j\Big), \tag{3.42}$$

define

$$\mathcal{F}_{good} := \mathcal{F} \setminus \mathcal{F}_{bad} = \{Q_j \in \mathcal{F} : \ \mathfrak{m}(\mathbb{D}_{Q_j} \setminus \{Q_j\}) \le \ a\sigma(Q_j)\}\,, \tag{3.43}$$

and let

$$G_0 := \bigcup_{\mathcal{F}_{good}} Q_j\,.$$

Then by (3.39)

$$\sigma(F_0 \cup G_0) \ \ge \ \rho\sigma(Q)\,, \tag{3.44}$$

where $\rho \in (0,1)$ is defined by

$$\frac{a+b}{a+2b} \le \frac{M_0+b}{M_0+2b} =: 1-\rho \in (0,1)\,. \tag{3.45}$$

We claim that

$$\ell(Q_j) \le 2^{-k_1}\,\ell(Q), \qquad \forall\, Q_j \in \mathcal{F}_{good}. \tag{3.46}$$

Indeed, if this were not true for some Q_j, then by definition of $\mathcal{F}_{good}$ and pigeon-holing there will be $Q_j' \in \mathbb{D}_{Q_j}$ with $\ell(Q_j') = 2^{-k_1}\,\ell(Q)$ such that $\mathfrak{m}(\mathbb{D}_{Q_j'}) \le a\,\sigma(Q_j')$. This contradicts the assumptions of the current case.

Note also that $Q \notin \mathcal{F}_{good}$ by (3.46) and $Q \notin \mathcal{F}_{bad}$ by (3.39), hence $\mathcal{F} \subset \mathbb{D}_Q \setminus \{Q\}$. By (3.41) and Claim 3.21, there is some stopping time regime $\mathbf{S} \subset \mathcal{G}$ so that $\mathbf{S}'' = (\mathbb{D}_{\mathcal{F}} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathbb{D}_Q$ is a semi-coherent subregime of $\mathbf{S}$, where $\mathcal{F}'$ denotes the collection of all dyadic children of cubes in $\mathcal{F}$.

Case 2a: $\sigma(F_0) \ge \frac{1}{2}\rho\sigma(Q)$.

In this case, Q has an ample overlap with the boundary of a chord-arc domain with controlled chord-arc constants. Indeed, let $\mathbf{S}' = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$ which, by (3.41) and Claim 3.21, is a semi-coherent subregime of some $\mathbf{S} \subset \mathcal{G}$. Hence, by Lemma 2.9, each of $\Omega^{\pm}_{\mathbf{S}'}$ is a CAD with constants depending on the allowable parameters, formed by the union of fattened Whitney boxes, which satisfies $\Omega^{\pm}_{\mathbf{S}'} \subset B^*_Q \cap \Omega$ (see (2.11), (2.12), and (2.18)). Moreover, by [HMM, Proposition A.14] and [HM2, Proposition 6.3] and our current assumptions,

$$\sigma(Q \cap \partial\Omega^{\pm}_{\mathbf{S}'}) = \sigma(F_0) \ge \frac{\rho}{2}\sigma(Q)\,.$$

Recall that in establishing $H[a+b,1]$, we assume that there is a set $V_Q \subset U_Q \cap \Omega$ for which (3.11) holds with $\theta = 1$. Pick then $X \in V_Q$ and set $V^*_Q := \{X\} \subset V_Q$. Note that since $U_Q = U_Q^+ \cup U_Q^-$ it follows that X belongs to either $U_Q^+ \cap \Omega$ or $U_Q^- \cap \Omega$. For the sake of specificity assume that $X \in U_Q^+ \cap \Omega$ hence, in particular, $U_Q^+ \subset \Omega^+_{\mathbf{S}'} \subset \Omega$. Note also that U_Q^+ is the only component of U_Q meeting V^*_Q. All these together give at once that the conclusion of $H[a+b,1]$ holds in the present case.

Case 2b: $\sigma(F_0) < \frac{1}{2}\rho\sigma(Q)$.

In this case by (3.44)

$$\sigma(G_0) \;\ge\; \frac{\rho}{2}\,\sigma(Q)\,. \tag{3.47}$$

In addition, by the definition of $\mathcal{F}_{good}$ (3.43), and pigeon-holing, every $Q_j \in \mathcal{F}_{good}$ has a dyadic child Q_j' (there could be more children satisfying this, but we just pick one) so that

$$\mathfrak{m}(\mathbb{D}_{Q_j'}) \le a\sigma(Q_j')\,. \tag{3.48}$$

Under the present assumptions $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (3.11) and (3.18)), hence $\sigma(F_Q \cap Q_j') = \sigma(Q_j')$. We apply Lemma 3.17 (recall (3.46)) to obtain $V_{Q_j'} \subset U_{Q_j'} \cap \Omega$ and $F_{Q_j'}$ which satisfies $\sigma(F_{Q_j'}) = \sigma(Q_j')$. That is, (3.11) holds for Q_j', with $\theta = 1$. Consequently, recalling that $Q_j' \in \mathbf{S} \subset \mathcal{G}$ (see Claim 3.21), and applying the induction hypothesis to Q_j', we find $V^*_{Q_j'} \subset V_{Q_j'}$, such that for each $U^{\pm}_{Q_j'}$ meeting

$V^*_{Q'_j}$, there is a chord-arc domain $\Omega^\pm_{Q'_j} \supset U^\pm_{Q'_j}$ formed by a union of fattened Whitney cubes with $\Omega^\pm_{Q'_j} \subset B^*_{Q'_j} \cap \Omega$. Moreover, since in particular, the cubes in $\mathcal{F}$ along with all of their children belong to the same stopping time regime $\mathbf{S}$ (see Claim 3.21), the connected component $U^\pm_{Q_j}$ overlaps with the corresponding component $U^\pm_{Q'_j}$ for its child, so we may augment $\Omega^\pm_{Q'_j}$ by adjoining to it the appropriate component $U^\pm_{Q_j}$, to form a chord arc domain

$$\Omega^\pm_{Q_j} := \Omega^\pm_{Q'_j} \cup U^\pm_{Q_j}\,. \tag{3.49}$$

Moreover, since $K \gg 1$, and since $Q'_j \subset Q_j$, we have that $B^*_{Q'_j} \subset B^*_{Q_j}$, hence $\Omega^\pm_{Q_j} \subset B^*_{Q_j}$ by construction.

By a covering lemma argument, for a sufficiently large constant $\kappa \gg K^4$, we may extract a subcollection $\mathcal{F}^*_{good} \subset \mathcal{F}_{good}$ so that $\{\kappa B^*_{Q_j}\}_{Q_j \in \mathcal{F}^*_{good}}$ is a pairwise disjoint family, and

$$\bigcup_{Q_j \in \mathcal{F}_{good}} Q_j \subset \bigcup_{Q_j \in \mathcal{F}^*_{good}} 5\kappa B^*_{Q_j}.$$

In particular, by (3.47),

$$\sum_{Q_j \in \mathcal{F}^*_{good}} \sigma(Q_j) \gtrsim_{\kappa,K} \sum_{Q_j \in \mathcal{F}_{good}} \sigma(Q_j) = \sigma(G_0) \gtrsim \rho\sigma(Q), \tag{3.50}$$

where the implicit constants depend on ADR, K, and the dilation factor κ.

By the induction hypothesis, and by construction (3.49) and ADR,

$$\sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q'_j) \gtrsim \sigma(Q_j)\,, \tag{3.51}$$

where Ω_{Q_j} is equal either to $\Omega^+_{Q_j}$ or to $\Omega^-_{Q_j}$ (if (3.51) holds for both choices, we arbitrarily set $\Omega_{Q_j} = \Omega^+_{Q_j}$).

Combining (3.51) with (3.50), we obtain

$$\sum_{Q_j \in \mathcal{F}^*_{good}} \sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q)\,. \tag{3.52}$$

We now assign each $Q_j \in \mathcal{F}^*_{good}$ either to $\mathcal{F}^*_+$ or to $\mathcal{F}^*_-$, depending on whether we chose Ω_{Q_j} satisfying (3.51) to be $\Omega^+_{Q_j}$, or $\Omega^-_{Q_j}$. We note that at least one of the sub-collections $\mathcal{F}^*_\pm$ is non-empty, since for each j, there was at least one choice of "+' or "-" such that (3.51) holds for the corresponding choice of Ω_{Q_j}. Moreover, the two collections are disjoint, since we have arbitrarily designated $\Omega_{Q_j} = \Omega^+_{Q_j}$ in the case that there were two choices for a particular Q_j.

To proceed, as in Lemma 3.22 we set

$$\mathbf{S}^* = \left\{Q' \in \mathbb{D}_Q : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}^*_{good}\right\}$$

which is semi-coherent by construction. For $\mathcal{F}^*_\pm$ non-empty, we now define

$$\Omega^\pm_Q = \Omega^\pm_{\mathbf{S}^*} \bigcup \Big(\bigcup_{Q_j \in \mathcal{F}^*_\pm} \Omega_{Q_j} \Big). \tag{3.53}$$

Observe that by the induction hypothesis, and our construction (see (3.49) and the ensuing comment), for an appropriate choice of $\pm$, $U^{\pm}_{Q_j} \subset \Omega_{Q_j} \subset B^*_{Q_j}$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q)$, by (3.52) and Lemma 3.22, with $\mathcal{F}^* = \mathcal{F}^*_{good}$, each (non-empty) choice of $\Omega^{\pm}_Q$ defines a chord-arc domain with the requisite properties.

Thus, we have proved Claim 3.36 and therefore, as noted above, it follows that $H[M_0, 1]$ holds. □

3.3. **Step 3: bootstrapping θ.** In this last step, we shall prove that there is a uniform constant $\zeta \in (0,1)$ such that for each $\theta \in (0,1]$, $H[M_0,\theta] \implies H[M_0,\zeta\theta]$. Since we have already established $H[M_0,1]$, we then conclude that $H[M_0,\theta_1]$ holds for any given $\theta_1 \in (0,1]$. As noted above, it then follows that Theorem 1.3 holds, as desired.

In turn, it will be enough to verify the following.

Claim 3.54. *There is a uniform constant $\beta \in (0,1)$ such that for every $a \in [0, M_0)$, $\theta \in (0,1]$, $\vartheta \in (0,1)$, and $b = 1/(2C)$ as in Step 2/Proof of Claim 3.36, if $H[M_0,\theta]$ holds, then*

$$H[a,(1-\vartheta)\theta] \implies H[a+b,(1-\vartheta\beta)\theta]\,.$$

Let us momentarily take Claim 3.54 for granted. Recall that by Claim 3.35, $H[0,\theta]$ holds for all $\theta \in (0,1]$. In particular, given $\theta \in (0,1]$ fixed, for which $H[M_0,\theta]$ holds, we have that $H[0,\theta/2]$ holds. Combining the latter fact with Claim 3.54, and iterating, we obtain that $H[kb,(1-2^{-1}\beta^k)\theta]$ holds. We eventually reach $H[M_0,(1-2^{-1}\beta^\nu)\theta]$, with $\nu \approx M_0/b$. The conclusion of Step 3 now follows, with $\zeta := 1-2^{-1}\beta^\nu$.

Proof of Claim 3.54. The proof will be a refinement of that of Claim 3.36. We are given some $\theta \in (0,1]$ such that $H[M_0,\theta]$ holds, and we assume that $H[a,(1-\vartheta)\theta]$ holds, for some $a \in [0,M_0)$ and $\vartheta \in (0,1)$. Set $b = 1/(2C)$, where as before C is the constant in (3.38). Consider a cube $Q \in \mathbb{D}(\partial\Omega)$ with $\mathfrak{m}(\mathbb{D}_Q) \leq (a+b)\sigma(Q)$. Suppose that there is a set $V_Q \subset U_Q \cap \Omega$ such that (3.11) holds with θ replaced by $(1-\vartheta\beta)\theta$, for some $\beta \in (0,1)$ to be determined. Our goal is to show that for a sufficiently small, but uniform choice of β, we may deduce the conclusion of the induction hypothesis, with C_{a+b}, c_{a+b} in place of C_a, c_a.

By assumption, and recalling the definition of F_Q in (3.18), we have that (3.11) holds with constant $(1-\vartheta\beta)\theta$, i.e.,

$$\sigma(F_Q) \geq (1-\vartheta\beta)\theta\sigma(Q)\,. \tag{3.55}$$

As in Step 2, we fix $k_1 > k_0$ (see (3.12)) large enough so that $2^{k_1} > 100K$. There are two principal cases. The first is as follows.

Case 1: There exists $Q' \in \mathbb{D}^{k_1}_Q$ (see (2.20)) with $\mathfrak{m}(\mathbb{D}_{Q'}) \leq a\sigma(Q')$.

We split Case 1 into two subcases.

Case 1a: $\sigma(F_Q \cap Q') \geq (1-\vartheta)\theta\sigma(Q')$.

In this case, we follow the Case 1 argument for $\theta = 1$ in Subsection 3.2 *mutatis mutandis*, so we merely sketch the proof. By Lemma 3.17, we may construct $V_{Q'}$ and $F_{Q'}$ so that $F_Q \cap Q' = F_{Q'}$ and hence $\sigma(F_{Q'}) \geq (1-\vartheta)\theta\sigma(Q')$. We may then apply the induction hypothesis $H[a,(1-\vartheta)\theta]$ in Q', and then proceed exactly as in Case 1 of Step 2 to construct a subset $V_Q^* \subset V_Q$ and a family of chord-arc domains Ω_Q^i satisfying the various desired properties, and such that

$$\sum_{i:U_Q^i \text{ meets } V_Q^*} \sigma(\partial\Omega_Q^i \cap Q) \geq c_a \sigma(Q') \gtrsim_{k_1} c_a \sigma(Q)\,.$$

The conclusion of $H[a+b,(1-\vartheta\beta)\theta]$ then holds in the present scenario.

Case 1b: $\sigma(F_Q \cap Q') < (1-\vartheta)\theta\sigma(Q')$.

By (3.55)

$$(1-\vartheta\beta)\theta\sigma(Q) \;\leq\; \sigma(F_Q) \;=\; \sigma(F_Q \cap Q') + \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,.$$

In the scenario of Case 1b, this leads to

$$\begin{aligned}(1-\vartheta\beta)\theta\sigma(Q') + (1-\vartheta\beta)\theta \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(Q'') &= (1-\vartheta\beta)\theta\sigma(Q) \\ &\leq (1-\vartheta)\theta\sigma(Q') + \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,,\end{aligned}$$

that is,

$$(1-\beta)\vartheta\theta\sigma(Q') + (1-\vartheta\beta)\theta \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(Q'') \;\leq\; \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,. \tag{3.56}$$

Note that we have the dyadic doubling estimate

$$\sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(Q'') \leq \sigma(Q) \leq M_1 \sigma(Q')\,,$$

where $M_1 = M_1(k_1, n, ADR)$. Combining this estimate with (3.56), we obtain

$$\left[(1-\beta)\frac{\vartheta}{M_1} + (1-\vartheta\beta)\right]\theta \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(Q'') \leq \sum_{Q'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}} \sigma(F_Q \cap Q'')\,.$$

We now choose $\beta \leq 1/(M_1+1)$, so that $(1-\beta)/M_1 \geq \beta$, and therefore the expression in square brackets is at least 1. Consequently, by pigeon-holing, there exists a particular $Q_0'' \in \mathbb{D}_Q^{k_1} \setminus \{Q'\}$ such that

$$\theta\sigma(Q_0'') \leq \sigma(F_Q \cap Q_0'')\,. \tag{3.57}$$

By Lemma 3.17, we can find $V_{Q_0''}$ such that $F_Q \cap Q_0'' = F_{Q_0''}$, where the latter is defined as in (3.18), with Q_0'' in place of Q. By assumption, $H[M_0,\theta]$ holds, so combining (3.57) with the fact that (3.10) holds with $a = M_0$ for every $Q \in \mathbb{D}(\partial\Omega)$, we find that there exists a subset $V_{Q_0''}^* \subset V_{Q_0''}$, along with a family of chord-arc domains $\{\Omega_{Q_0''}^i\}_i$ enjoying all the appropriate properties relative to Q_0''. Using that $\ell(Q_0'') \approx_{k_1} \ell(Q)$, we may now proceed exactly as in Case 1a above, and also Case 1 in Step 2, to construct V_Q^* and $\{\Omega_Q^i\}_i$ such that the conclusion of $H[a+b,(1-\vartheta\beta)\theta]$ holds in the present case also.

Case 2: $\mathfrak{m}(\mathbb{D}_{Q'}) > a\sigma(Q')$ for every $Q' \in \mathbb{D}_Q^{k_1}$.

In this case, we apply Lemma 3.37 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_Q$ such that (3.38) and (3.39) hold. In particular, by our choice of $b = 1/(2C)$, $\|\mathfrak{m}_{\mathcal{F}}\|_{C(Q)} \leq 1/2$.

Recall that F_Q is defined in (3.18), and satisfies (3.55).

We define $F_0 = Q \setminus (\bigcup_{\mathcal{F}} Q_j)$ as in (3.42), and $\mathcal{F}_{good} := \mathcal{F} \setminus \mathcal{F}_{bad}$ as in (3.43). Let $G_0 := \bigcup_{\mathcal{F}_{good}} Q_j$. Then as above (see (3.44)),

$$\sigma(F_0 \cup G_0) \geq \rho\sigma(Q)\,, \tag{3.58}$$

where again $\rho = \rho(M_0, b) \in (0,1)$ is defined as in (3.45). Just as in Case 2 for $\theta = 1$ in Subsection 3.2, we have that

$$\ell(Q_j) \leq 2^{-k_1}\,\ell(Q), \quad \forall\, Q_j \in \mathcal{F}_{good}, \qquad \text{and} \qquad \mathcal{F} \subset \mathbb{D}_Q \setminus \{Q\} \tag{3.59}$$

(see (3.46)). Hence, the conclusions of Claim 3.21 hold.

We first observe that if $\sigma(F_0) \geq \varepsilon\sigma(Q)$, for some $\varepsilon > 0$ to be chosen (depending on allowable parameters), then the desired conclusion holds. Indeed, in this case, we may proceed exactly as in the analogous scenario in Case 2a of Step 2: the promised chord-arc domain is again simply one of $\Omega_S^{\pm}$, since at least one of these contains a point in V_Q and hence in particular is a subdomain of Ω. The constant c_{a+b} in our conclusion will depend on ε, but in the end this will be harmless, since ε will be chosen to depend only on allowable parameters.

We may therefore suppose that

$$\sigma(F_0) < \varepsilon\sigma(Q)\,. \tag{3.60}$$

Next, we refine the decomposition $\mathcal{F} = \mathcal{F}_{good} \cup \mathcal{F}_{bad}$. With ρ as in (3.45) and (3.58), we choose $\beta < \rho/4$. Set

$$\mathcal{F}_{good}^{(1)} := \left\{Q_j \in \mathcal{F}_{good} : \sigma(F_Q \cap Q_j) \geq (1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j)\right\}\,,$$

and define $\mathcal{F}_{good}^{(2)} := \mathcal{F}_{good} \setminus \mathcal{F}_{good}^{(1)}$. Let

$$\mathcal{F}_{bad}^{(1)} := \left\{Q_j \in \mathcal{F}_{bad} : \sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)\right\}\,,$$

and define $\mathcal{F}_{bad}^{(2)} := \mathcal{F}_{bad} \setminus \mathcal{F}_{bad}^{(1)}$.

We split the remaining part of Case 2 into two subcases. The first of these will be easy, based on our previous arguments.

Case 2a: There is $Q_j \in \mathcal{F}_{bad}^{(1)}$ such that $\ell(Q_j) > 2^{-k_1}\,\ell(Q)$.

By definition of $\mathcal{F}_{bad}^{(1)}$, $\sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)$. By pigeon-holing, Q_j has a descendant Q' with $\ell(Q') = 2^{-k_1}\ell(Q)$, such that $\sigma(F_Q \cap Q') \geq \theta\sigma(Q')$. We may then apply $H[M_0, \theta]$ in Q', and proceed exactly as we did in Case 1b above with the cube Q_0'', which enjoyed precisely the same properties as does our current Q'. Thus, we draw the desired conclusion in the present case.

The main case is the following.

Case 2b: Every $Q_j \in \mathcal{F}_{bad}^{(1)}$ satisfies $\ell(Q_j) \leq 2^{-k_1}\,\ell(Q)$.

Observe that by definition,

$$\sigma(F_Q \cap Q_j) \le (1 - 4\vartheta\beta\rho^{-1})\theta\sigma(Q_j), \qquad \forall\, Q_j \in \mathcal{F}^{(2)}_{good}, \tag{3.61}$$

and also

$$\sigma(F_Q \cap Q_j) \le \theta\sigma(Q_j), \qquad \forall\, Q_j \in \mathcal{F}^{(2)}_{bad}, \tag{3.62}$$

Set $\mathcal{F}_* := \mathcal{F} \setminus \mathcal{F}^{(2)}_{good}$. For future reference, we shall derive a certain ampleness estimate for the cubes in $\mathcal{F}_*$.

By (3.55),

$$\begin{aligned}(1-\vartheta\beta)\theta\sigma(Q) \le\ \sigma(F_Q)\ &\le\ \sigma(F_0) + \sum_{\mathcal{F}_*}\sigma(Q_j) + \sum_{\mathcal{F}^{(2)}_{good}}\sigma(F_Q\cap Q_j)\\ &\le\ \varepsilon\sigma(Q) + \sum_{\mathcal{F}_*}\sigma(Q_j) + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q),\end{aligned} \tag{3.63}$$

where in the last step have used (3.60) and (3.61). Observe that

$$(1-\vartheta\beta)\theta = \left(4\rho^{-1} - 1\right)\vartheta\beta\theta + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta. \tag{3.64}$$

Using (3.63) and (3.64), for $\varepsilon \ll (4\rho^{-1} - 1)\vartheta\beta\theta$, we obtain

$$2^{-1}\left(4\rho^{-1} - 1\right)\vartheta\beta\theta\sigma(Q) \le \sum_{\mathcal{F}_*}\sigma(Q_j)$$

and thus

$$\sigma(Q) \le\ C(\vartheta,\rho,\beta,\theta)\sum_{\mathcal{F}_*}\sigma(Q_j). \tag{3.65}$$

We now make the following claim.

Claim 3.66. *For ε chosen sufficiently small,*

$$\max\left(\sum_{\mathcal{F}^{(1)}_{good}}\sigma(Q_j),\ \sum_{\mathcal{F}^{(1)}_{bad}}\sigma(Q_j)\right) \ge \varepsilon\sigma(Q).$$

Proof of Claim 3.66. If $\sum_{\mathcal{F}^{(1)}_{good}}\sigma(Q_j) \ge \varepsilon\sigma(Q)$, then we are done. Therefore, suppose that

$$\sum_{\mathcal{F}^{(1)}_{good}}\sigma(Q_j) < \varepsilon\sigma(Q). \tag{3.67}$$

We have made the decomposition

$$\mathcal{F} = \mathcal{F}^{(1)}_{good} \cup \mathcal{F}^{(2)}_{good} \cup \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(2)}_{bad}. \tag{3.68}$$

Consequently

$$\sigma(F_Q)\ \le \sum_{\mathcal{F}^{(2)}_{good}}\sigma(F_Q\cap Q_j) + \sum_{\mathcal{F}_{bad}}\sigma(F_Q\cap Q_j) + O\left(\varepsilon\sigma(Q)\right),$$

where we have used (3.60), and (3.67) to estimate the contributions of F_0, and of $\mathcal{F}^{(1)}_{good}$, respectively. This, (3.55), (3.61), and (3.62) yield

$$(1-\vartheta\beta)\theta\left(\sum_{\mathcal{F}^{(2)}_{good}}\sigma(Q_j)+\sum_{\mathcal{F}^{(2)}_{bad}}\sigma(Q_j)\right)\leq(1-\vartheta\beta)\theta\sigma(Q)\leq\sigma(F_Q)$$
$$\leq\left(1-4\vartheta\beta\rho^{-1}\right)\theta\sum_{\mathcal{F}^{(2)}_{good}}\sigma(Q_j)+\sum_{\mathcal{F}^{(1)}_{bad}}\sigma(Q_j)+\theta\sum_{\mathcal{F}^{(2)}_{bad}}\sigma(Q_j)+O\left(\varepsilon\sigma(Q)\right).$$

In turn, applying (3.64) in the latter estimate, and rearranging terms, we obtain

$$(4\rho^{-1}-1)\vartheta\beta\theta\sum_{\mathcal{F}^{(2)}_{good}}\sigma(Q_j)-\vartheta\beta\theta\sum_{\mathcal{F}^{(2)}_{bad}}\sigma(Q_j)\leq\sum_{\mathcal{F}^{(1)}_{bad}}\sigma(Q_j)+O\left(\varepsilon\sigma(Q)\right). \tag{3.69}$$

Recalling that $G_0=\cup_{\mathcal{F}_{good}}Q_j$, and that $\mathcal{F}_{good}=\mathcal{F}^{(1)}_{good}\cup\mathcal{F}^{(2)}_{good}$, we further note that by (3.58), choosing $\varepsilon\ll\rho$, and using (3.60) and (3.67), we find in particular that

$$\sum_{\mathcal{F}^{(2)}_{good}}\sigma(Q_j)\geq\frac{\rho}{2}\sigma(Q). \tag{3.70}$$

Applying (3.70) and the trivial estimate $\sum_{\mathcal{F}^{(2)}_{bad}}\sigma(Q_j)\leq\sigma(Q)$ in (3.69), we then have

$$\vartheta\beta\theta\left[1-\frac{\rho}{2}\right]\sigma(Q)=\left[(4\rho^{-1}-1)\vartheta\beta\theta\frac{\rho}{2}-\vartheta\beta\theta\right]\sigma(Q)$$
$$\leq(4\rho^{-1}-1)\vartheta\beta\theta\sum_{\mathcal{F}^{(2)}_{good}}\sigma(Q_j)-\vartheta\beta\theta\sum_{\mathcal{F}^{(2)}_{nbad}}\sigma(Q_j)\leq\sum_{\mathcal{F}^{(1)}_{bad}}\sigma(Q_j)+O\left(\varepsilon\sigma(Q)\right).$$

Since $\rho<1$, we conclude, for $\varepsilon\leq(4C)^{-1}\vartheta\beta\theta$, that

$$\frac{1}{4}\vartheta\beta\theta\,\sigma(Q)\leq\sum_{\mathcal{F}^{(1)}_{bad}}\sigma(Q_j),$$

and Claim 3.66 follows. □

With Claim 3.66 in hand, let us return to the proof of Case 2b of Claim 3.54.

We begin by noting that by definition of $\mathcal{F}^{(1)}_{bad}$, and Lemma 3.17, we can apply $H[M_0,\theta]$ to any $Q_j\in\mathcal{F}^{(1)}_{bad}$, hence for each such Q_j there is a family of chord-arc domains $\{\Omega^i_{Q_j}\}_i$ satisfying the desired properties.

Now consider $Q_j\in\mathcal{F}^{(1)}_{good}$. Since $\mathcal{F}^{(1)}_{good}\subset\mathcal{F}_{good}$, by pigeon-holing Q_j has a dyadic child Q'_j satisfying

$$\mathfrak{m}(\mathbb{D}_{Q'_j})\leq a\sigma(Q'_j), \tag{3.71}$$

(there may be more than one such child, but we just pick one). Our immediate goal is to find a child Q''_j of Q_j, which may or may not equal Q'_j, for which we may construct a family of chord-arc domains $\{\Omega^i_{Q''_j}\}_i$ satisfying the desired properties. To this end, we assume first that Q'_j satisfies

$$\sigma(F_Q \cap Q'_j) \geq (1-\vartheta)\theta\sigma(Q'_j)\,. \tag{3.72}$$

In this case, we set $Q''_j := Q'_j$, and using Lemma 3.17, by the induction hypothesis $H[a,(1-\vartheta)\theta]$, we obtain the desired family of chord-arc domains.

We therefore consider the case

$$\sigma(F_Q \cap Q'_j) < (1-\vartheta)\theta\sigma(Q'_j)\,. \tag{3.73}$$

In this case, we shall select $Q''_j \neq Q'_j$. Recall that we use the notation $Q'' \lhd Q$ to mean that Q'' is a dyadic child of Q. Set

$$\mathcal{F}''_j := \left\{Q''_j \lhd Q_j :\, Q''_j \neq Q'_j\right\}.$$

Note that we have the dyadic doubling estimate

$$\sum_{Q''_j \in \mathcal{F}''_j} \sigma(Q''_j) \leq \sigma(Q_j) \leq M_1\sigma(Q'_j)\,, \tag{3.74}$$

where $M_1 = M_1(n, ADR)$. We also note that

$$(1-4\vartheta\beta\rho^{-1})\theta = (1-4\beta\rho^{-1})\vartheta\theta + (1-\vartheta)\theta\,. \tag{3.75}$$

By definition of $\mathcal{F}^{(1)}_{good}$,

$$(1-4\vartheta\beta\rho^{-1})\theta\sigma(Q_j) \;\leq\; \sigma(F_Q\cap Q_j) \;=\; \sigma(F_Q\cap Q'_j) + \sum_{Q''_j\in\mathcal{F}''_j}\sigma(F_Q\cap Q''_j)\,.$$

By (3.73), it follows that

$$\begin{aligned}(1-4\vartheta\beta\rho^{-1})\theta\sigma(Q'_j) + (1-4\vartheta\beta\rho^{-1})\theta\sum_{Q''_j\in\mathcal{F}''_j}\sigma(Q''_j) &= (1-4\vartheta\beta\rho^{-1})\theta\sigma(Q_j)\\ &\leq\; (1-\vartheta)\theta\sigma(Q'_j) + \sum_{Q''_j\in\mathcal{F}''_j}\sigma(F_Q\cap Q''_j)\,.\end{aligned}$$

In turn, using (3.75), we obtain

$$(1-4\beta\rho^{-1})\vartheta\theta\sigma(Q'_j) + (1-4\vartheta\beta\rho^{-1})\theta\sum_{Q''_j\in\mathcal{F}''_j}\sigma(Q''_j) \;\leq\; \sum_{Q''_j\in\mathcal{F}''_j}\sigma(F_Q\cap Q''_j)\,.$$

By the dyadic doubling estimate (3.74), this leads to

$$\Big[(1-4\beta\rho^{-1})\vartheta M_1^{-1} + (1-4\vartheta\beta\rho^{-1})\Big]\theta\sum_{Q''_j\in\mathcal{F}''_j}\sigma(Q''_j) \;\leq\; \sum_{Q''_j\in\mathcal{F}''_j}\sigma(F_Q\cap Q''_j)\,.$$

Choosing $\beta \leq \rho/(4(M_1+1))$, we find that the expression in square brackets is at least 1, and therefore, by pigeon holing, we can pick $Q''_j \in \mathcal{F}''_j$ satisfying

$$\sigma(F_Q\cap Q''_j) \geq \theta\sigma(Q''_j)\,. \tag{3.76}$$

Hence, using Lemma 3.17, we see that the induction hypothesis $H[M_0,\theta]$ holds for $Q''_j \in \mathcal{F}''_j$, and once again we obtain the desired family of chord-arc domains.

Recall that we have constructed our packing measure $\mathfrak{m}$ in such a way that each $Q_j \in \mathcal{F}$, as well as all of its children, along with the cubes in $\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q$, belong to the same stopping time regime $\mathbf{S}$; see Claim 3.21. This means in particular that for each such Q_j, the Whitney region U_{Q_j} has exactly two components $U^{\pm}_{Q_j} \subset \Omega^{\pm}_{\mathbf{S}}$, and the analogous statement is true for each child of Q_j. This fact has the following consequences:

Remark 3.77. For each $Q_j \in \mathcal{F}^{(1)}_{bad}$, and for the selected child Q''_j of each $Q_j \in \mathcal{F}^{(1)}_{good}$, the conclusion of the induction hypothesis produces at most two chord-arc domains $\Omega^{\pm}_{Q_j} \supset U^{\pm}_{Q_j}$ (resp. $\Omega^{\pm}_{Q''_j} \supset U^{\pm}_{Q''_j}$), which we enumerate as $\Omega^i_{Q_j}$ (resp. $\Omega^i_{Q''_j}$), $i = 1, 2$, with $i = 1$ corresponding "+", and $i = 2$ corresponding to "-", respectively.

Remark 3.78. For each $Q_j \in \mathcal{F}^{(1)}_{good}$, the connected component $U^{\pm}_{Q_j}$ overlaps with the corresponding component $U^{\pm}_{Q''_j}$ for its child, so we may augment $\Omega^i_{Q''_j}$ by adjoining to it the appropriate component $U^{\pm}_{Q_j}$, to form a chord arc domain

$$\Omega^i_{Q_j} := \Omega^i_{Q''_j} \cup U^i_{Q_j}\,.$$

By the induction hypothesis, for each $Q_j \in \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}$ (and by ADR, in the case of $\mathcal{F}^{(1)}_{good}$), the chord-arc domains $\Omega^i_{Q_j}$ that we have constructed satisfy

$$\sum_i \sigma(Q_j \cap \partial\Omega^i_{Q_j}) \gtrsim \sigma(Q_j)\,,$$

where the sum has either one or two terms, and where the implicit constant depends either on M_0 and θ, or on a and $(1-\vartheta)\theta$, depending on which part of the induction hypothesis we have used. In particular, for each such Q_j, there is at least one choice of index i such that $\Omega^i_{Q_j} =: \Omega_{Q_j}$ satisfies

$$\sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q_j) \tag{3.79}$$

(if the latter is true for both choices $i = 1, 2$, we arbitrarily choose $i = 1$, which we recall corresponds to "+"). Combining the latter bound with Claim 3.66, and recalling that ε has now been fixed depending only on allowable parameters, we see that

$$\sum_{Q_j \in \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}} \sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q)$$

For $Q_j \in \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}$, as above set $B^*_{Q_j} := B(x_{Q_j}, K\ell(Q_j))$. By a covering lemma argument, we may extract a subfamily $\mathcal{F}^* \subset \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}$ such that $\{\kappa B^*_{Q_j}\}_{Q_j \in \mathcal{F}^*}$ is pairwise disjoint, where again $\kappa \gg K^4$ is a large dilation factor, and such that

$$\sum_{Q_j \in \mathcal{F}^*} \sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim_\kappa \sigma(Q) \tag{3.80}$$

Let us now build (at most two) chord-arc domains Ω^i_Q satisfying the desired properties.

Recall that for each $Q_j \in \mathcal{F}^*$, we defined the corresponding chord-arc domain $\Omega_{Q_j} := \Omega^i_{Q_j}$, where the choice of index i (if there was a choice), was made so that (3.79) holds. We then assign each $Q_j \in \mathcal{F}^*$ either to $\mathcal{F}^*_+$ or to $\mathcal{F}^*_-$, depending on whether we chose Ω_{Q_j} satisfying (3.79) to be $\Omega^1_{Q_j} = \Omega^+_{Q_j}$, or $\Omega^2_{Q_j} = \Omega^-_{Q_j}$. We note that at least one of the sub-collections $\mathcal{F}_{\pm}^*$ is non-empty, since for each j, there was at least one choice of index i such that (3.79) holds with $\Omega_{Q_j} := \Omega^i_{Q_j}$. Moreover, the two collections are disjoint, since we have arbitrarily designated $\Omega_{Q_j} = \Omega^1_{Q_j}$ (corresponding to "+") in the case that there were two choices for a particular Q_j.

We further note that if $Q_j \in \mathcal{F}_{\pm}^*$, then $\Omega_{Q_j} = \Omega^{\pm}_{Q_j} \supset U^{\pm}_{Q_j}$.

We are now in position to apply Lemma 3.22. Set

$$\mathbf{S}^* = \left\{Q' \in \mathbb{D}_Q : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}^*\right\},$$

which is a semi-coherent subregime of $\mathbf{S}$, with maximal cube Q. Without loss of generality, we may suppose that $\mathcal{F}_+^*$ is non-empty, and we then define

$$\Omega_Q^+ := \Omega_{\mathbf{S}^*}^+ \bigcup \left(\bigcup_{Q_j \in \mathcal{F}_+^*} \Omega_{Q_j} \right),$$

and similarly with "+" replaced by "-", provided that $\mathcal{F}_-^*$ is also non-empty. Observe that by the induction hypothesis, and our construction (see Remarks 3.77-3.78 and Lemma 2.9), for an appropriate choice of $\pm$, $U^{\pm}_{Q_j} \subset \Omega_{Q_j} \subset B^*_{Q_j}$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q)$, by (3.80) and Lemma 3.22, each (non-empty) choice defines a chord-arc domain with the requisite properties. □

Appendix A. Some counter-examples

We shall discuss some counter-examples which show that our background hypotheses in Theorem 1.6 (namely, ADR and interior Corkscrew condition) are in some sense in the nature of best possible. In the first two examples, Ω is a domain satisfying an interior Corkscrew condition, such that $\partial\Omega$ satisfies exactly one (but not both) of the upper or the lower ADR bounds, and for which harmonic measure ω fails to be weak-A_∞ with respect to surface measure σ on $\partial\Omega$. In this setting, in which full ADR fails, there is no established notion of uniform rectifiability, but in each case, the domain will enjoy some substitute property which would imply uniform rectifiability of the boundary in the presence of full ADR.

In the last example, we construct an open set Ω with ADR boundary, and for which $\omega \in$ weak-A_∞ with respect to surface measure, but for which the interior Corkscrew condition fails, and $\partial\Omega$ is not uniformly rectifiable.

Failure of the upper ADR bound. In [AMT1], the authors construct an example of a Reifenberg flat domain $\Omega \subset \mathbb{R}^{n+1}$ for which surface measure $\sigma = H^n\lfloor_{\partial\Omega}$ is locally finite on $\partial\Omega$, but for which the upper ADR bound

$$\sigma(\Delta(x, r) \leq Cr^n \tag{A.1}$$

fails, and for which harmonic measure ω is not absolutely continuous with respect to σ. Note that the hypothesis of Reifenberg flatness implies in particular that Ω and $\Omega_{ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ are both NTA domains, hence both enjoy the Corkscrew condition, so by the relative isoperimetric inequality, the lower ADR bound

$$\sigma(\Delta(x, r) \geq cr^n \tag{A.2}$$

holds. Thus, it is the failure of (A.1) which causes the failure of absolute continuity: in the presence of (A.1), the results of [DJ] apply, and one has that $\omega \in A_\infty(\sigma)$, and that $\partial\Omega$ satisfies a "big pieces of Lipschitz graphs" condition (see [DJ] for a precise statement), and hence is uniformly rectifiable. We note that by a result of Badger [Bad], a version of the Lipschitz approximation result of [DJ] still holds for NTA domains with locally finite surface measure, even in the absence of the upper ADR condition.

Failure of the lower ADR bound. In [ABHM, Example 5.5], the authors give an example of a domain satisfying the interior Corkscrew condition, whose boundary is rectifiable (indeed, it is contained in a countable union of hyperplanes), and satisfies the upper ADR condition (A.1), but not the lower ADR condition (A.2), but for which surface measure σ fails to be absolutely continuous with respect to harmonic measure, and in fact, for which the non-degeneracy condition

$$A \subset \Delta_X := B(X, 10\delta(X)) \cap \partial\Omega, \quad \sigma(A) \geq (1-\eta)\sigma(\Delta_X) \implies \omega^X(A) \geq c\,, \tag{A.3}$$

fails to hold uniformly for $X \in \Omega$, for any fixed positive η and c, and therefore ω cannot be weak-A_∞ with respect to σ. We note that in the presence of the full ADR condition, if $\partial\Omega$ were contained in a countable union of hyperplanes (as it is in the example), then in particular it would satisfy the "BAUP" condition of [DS2], and thus would be uniformly rectifiable [DS2, Theorem I.2.18, p. 36].

Failure of the interior Corkscrew condition. The example is based on the construction of Garnett's 4-corners Cantor set $C \subset \mathbb{R}^2$ (see, e.g., [DS2, Chapter 1]). Let I_0 be a unit square positioned with lower left corner at the origin in the plane, and in general for each $k = 0, 1, 2, ...$, we let I_k be the unit square positioned with lower left corner at the point $(2k, 0)$ on the x-axis. Set $\Omega_0 := I_0$. Let Ω_1 be the first stage of the 4-corners construction, i.e., a union of four squares of side length $1/4$, positioned in the corners of the unit square I_1, and similarly, for each k, let Ω_k be the k-th stage of the 4-corners construction, positioned inside I_k. Note that $\operatorname{dist}(\Omega_k, \Omega_{k+1}) = 1$ for every k. Set $\Omega := \cup_k \Omega_k$. It is easy to check that $\partial\Omega$ is ADR, and that the non-degeneracy condition (A.3) holds in Ω for some uniform positive η and c, and thus by the criterion of [BL], $\omega \in$ weak-$A_\infty(\sigma)$. On the other hand, the interior Corkscrew condition clearly fails to hold in Ω (it holds only for decreasingly small scales as k increases), and certainly $\partial\Omega$ cannot be uniformly rectifiable: indeed, if it were, then $\partial\Omega_k$ would be UR, with uniform constants, for each k, and this would imply that C itself was UR, whereas in fact, as is well known, it is totally non-rectifiable. One can produce a similar set in 3 dimensions by simply taking the cylinder $\Omega' = \Omega \times [0, 1]$. Details are left to the interested reader.

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