

UNIFORM RECTIFIABILITY AND HARMONIC MEASURE IV: AHLFORS REGULARITY PLUS POISSON KERNELS IN L^p IMPLIES UNIFORM RECTIFIABILITY.

STEVE HOFMANN AND JOSÉ MARÍA MARTELL

ABSTRACT. Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors-David regular set of dimension n . We show that the weak- A_∞ property of harmonic measure, for the open set $\Omega := \mathbb{R}^{n+1} \setminus E$, implies uniform rectifiability of E .

This is a preliminary version of our work on this topic, as presented by the first author at the Workshop on Harmonic Analysis and PDE held at ICMAT in Madrid, in January 2015. The final published version will be jointly authored with K. Nyström and P. Le, and in addition to the present results, will treat also the analogous theory for the p -Laplacian.

CONTENTS

1. Introduction	1
2. Preliminaries	5
3. Proof of Theorem 1.1, step 1: preliminary arguments	11
4. Proof of Theorem 1.1, step 2: the Lewis-Vogel argument	14
5. WHSA implies UR: Proof of Proposition 1.7	23
6. Proof of Corollary 1.3	28
References	29

1. INTRODUCTION

In this note, we present a quantitative, scale invariant result of free boundary type. Somewhat more precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (not necessarily connected) satisfying an interior Corkscrew condition, whose boundary is n -dimensional Ahlfors-David regular (ADR). Given these background hypotheses,

Date: May 24, 2015.

2000 Mathematics Subject Classification. 31B05, 31B25, 35J08, 42B25, 42B37, 28A75, 28A78.

Key words and phrases. Harmonic measure, Poisson kernel, uniform rectifiability, Carleson measures.

The first author was supported by NSF grant DMS-1361701. The second author was supported by ICMAT Severo Ochoa project SEV-2011-0087. He also acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC agreement no. 615112 HAPDEGMT..

we show that if ω , the harmonic measure for Ω , is absolutely continuous with respect to surface measure σ , and if the Poisson kernel $k = d\omega/d\sigma$ verifies an appropriate scale invariant higher integrability estimate (in particular, if ω belongs to weak- A_∞ with respect to σ), then $\partial\Omega$ is uniformly rectifiable, in the sense of [DS1, DS2]. Here $\sigma := H^n|_{\partial\Omega}$ is, as usual, the restriction to $\partial\Omega$ of n -dimensional Hausdorff measure (the other notation and terminology used here will be defined in the sequel). In particular, our background hypotheses hold in the case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR (hence closed) set of co-dimension 1: in that case, it is well known that the Corkscrew condition is verified automatically in Ω , i.e., in every ball $B = B(x, r)$ centered on $\partial\Omega$, there is some component of $\Omega \cap B$ that contains a point Y with $\text{dist}(Y, \partial\Omega) \approx r$.

In previous work with I. Uriarte-Tuero [HMU], the authors had proved such a result under the additional hypothesis that Ω is a connected domain, satisfying an interior Harnack Chain condition. In hindsight, under that extra assumption, one obtains the stronger conclusion that in fact, $\Omega_{ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ also satisfies a Corkscrew condition, and hence that Ω is an NTA domain in the sense of [JK]; see [AHMNT] for the details. The new advance in the present paper, then, is the removal of any connectivity hypothesis; in particular, we avoid the Harnack Chain condition.

Before discussing further historical background, let us now state our main result. To this end, given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a Euclidean ball $B = B(x, r) \subset \mathbb{R}^{n+1}$, centered on $\partial\Omega$, we let $\Delta = \Delta(x, r) := B \cap \partial\Omega$ denote the corresponding surface ball, and for a constant $C > 0$, we set $C\Delta := \Delta(x, Cr)$. For $X \in \Omega$, let ω^X be harmonic measure for Ω , with pole at X . As mentioned above, all other terminology and notation will be defined below.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set, whose boundary is Ahlfors-David regular of dimension n . Suppose that there is a constant $C_0 \geq 1$, and an exponent $p > 1$, such that for every ball surface ball $\Delta = \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam } \partial\Omega$, there exists $Y_\Delta \in B(x, C_0 r)$ with $\text{dist}(Y_\Delta, \partial\Omega) \geq C_0^{-1} r$, satisfying*

- (a) **Bourgain's estimate:** $\omega^{Y_\Delta}(\Delta) \geq C_0^{-1}$.
- (b) **Scale-invariant higher integrability:** $\omega^{Y_\Delta} \ll \sigma$ in $C_1\Delta$ and $k^{Y_\Delta} = d\omega^{Y_\Delta}/d\sigma$ satisfies

$$(1.2) \quad \int_{C_1\Delta} k^{Y_\Delta}(y)^p d\sigma(y) \leq C_0 \sigma(C_1\Delta)^{1-p},$$

where C_1 is a large enough constant depending only on n and the ADR constant of $\partial\Omega$. Then $\partial\Omega$ is uniformly rectifiable and moreover the “UR character” (see Definition 2.4) depends only on n , the ADR constants, p and C_0 .

Remark. As mentioned above, the background hypotheses hold in the special case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an n -dimensional ADR set E , and in that setting, condition (a) is automatically verified. Indeed, by a result of Bourgain [Bo] (see Lemma 2.25 below), the Ahlfors-David regularity of the boundary implies that there is always a point Y_Δ as above, and a sufficiently large C_0 , such that estimate (a) holds; in fact, ω^Y satisfies (a) for every $Y \in \Omega \cap B(x, c_1 r)$, for c_1 small enough

depending only on dimension and the ADR constants, and as we have noted, the ADR property ensures that some such Y satisfies $\text{dist}(Y, \partial\Omega) \approx r$. Thus, the theorem will hold in this setting, if (b) holds for this Y .

The observations in the preceding remark will allow us to deduce, as an easy corollary, the following variant of Theorem 1.1 (we shall give the short proof of the corollary in Section 6).

Corollary 1.3. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors-David regular set of dimension n , and let $\Omega := \setminus E$. Suppose that for every ball $B(x, r)$, $x \in E$, $0 < r < \text{diam } E$, and for all $Y \in \Omega \setminus B(x, 2r)$, harmonic measure $\omega^Y \in \text{weak-}A_\infty(\Delta(x, r))$, that is, there is a constant $C_0 \geq 1$ and an exponent $p > 1$, each of which is uniform with respect to x, r and Y , such that $\omega^Y \ll \sigma$ in $\Delta(x, r)$, and $k^Y = d\omega^Y/d\sigma$ satisfies*

$$(1.4) \quad \left(\int_{\Delta'} k^Y(z)^p d\sigma(z) \right)^{\frac{1}{p}} \leq C_0 \int_{2\Delta'} k^Y(z) d\sigma(z),$$

for every $\Delta' = B' \cap E$ with $2B' \subset B(x, r)$. Then E is uniformly rectifiable and moreover the “UR character” (see Definition 2.4) depends only on n , the ADR constant of E , p and C_0 .

Combining Theorem 1.1 with the results in [BH], we obtain as an immediate consequence a “big pieces” characterization of uniformly rectifiable sets of co-dimension 1, in terms of harmonic measure. Here and in the sequel, given an ADR set E , Q will denote a “dyadic cube” on E in the sense of [DS1, DS2] and [Ch], and $\mathbb{D}(E)$ will denote the collection of all such cubes; see Lemma 2.5 below.

Theorem 1.5. *Let $E \subset \mathbb{R}^{n+1}$ be an n -dimensional ADR set. Let $\Omega := \mathbb{R}^{n+1} \setminus E$. Then E is uniformly rectifiable if and only if it has “big pieces of good harmonic measure estimates” in the following sense: for each $Q \in \mathbb{D}(E)$ there exists an open set $\widetilde{\Omega} = \widetilde{\Omega}_Q$ with the following properties, with uniform control of the various implicit constants:*

- $\partial\widetilde{\Omega}$ is ADR;
- the interior Corkscrew condition holds in $\widetilde{\Omega}$;
- $\partial\widetilde{\Omega}$ has a “big pieces” overlap with $\partial\Omega$, in the sense that

$$(1.6) \quad \sigma(Q \cap \partial\widetilde{\Omega}) \gtrsim \sigma(Q);$$

- for each surface ball $\Delta = \Delta(x, r) := B(x, r) \cap \partial\widetilde{\Omega}$, with $x \in \partial\widetilde{\Omega}$ and $r \in (0, \text{diam}(\widetilde{\Omega}))$, there is an interior corkscrew point $X_\Delta \in \Omega_0$, such that $\omega^{X_\Delta} := \omega_{\widetilde{\Omega}}^{X_\Delta}$, the harmonic measure for $\widetilde{\Omega}$ with pole at X_Δ , satisfies $\omega^{X_\Delta}(\Delta) \gtrsim 1$, and belongs to $\text{weak-}A_\infty(\Delta)$.

The “only if” direction is proved in [BH], and in fact the open sets $\widetilde{\Omega}$ constructed there even satisfy a 2-sided Corkscrew condition, and moreover, $\widetilde{\Omega} \subset \Omega$, with $\text{diam}(\widetilde{\Omega}) \approx \text{diam}(Q)$. To obtain the converse direction, we simply observe that by Theorem 1.1, the subdomains $\widetilde{\Omega}$ have uniformly rectifiable boundaries, with uniform control of the “UR” character of each $\partial\widetilde{\Omega}$, and thus E is uniformly rectifiable, by [DS2].

Let us now discuss some related earlier results. Our approach in the present paper owes a great deal to prior work of Lewis and Vogel [LV], who proved a version of Theorem 1.1 under the stronger hypothesis that ω itself is an Ahlfors-David regular measure, and thus the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. With this assumption, they were able to show that $\partial\Omega$ satisfies the so-called “Weak Exterior Convexity” (WEC) condition, which characterizes uniform rectifiability [DS2]. To weaken the hypotheses on ω , as we have done here, requires two further considerations. The first is quite natural in this context: a stopping time argument, in the spirit of the proofs of the Kato square root conjecture [HMc], [HLMc], [AHLMcT] (and of local Tb theorems [Ch], [AHMTT], [H]), by means of which we extract ample dyadic sawtooth regimes on which averages of harmonic measure are bounded and accretive (see Lemma 3.4 below). This will allow us to use the arguments of [LV] within these good sawtooth regions. The second new consideration is necessitated by the fact that in our setting, the doubling property may fail for harmonic measure. In the absence of doubling, we are unable to obtain the WEC condition directly. Nonetheless, we shall be able to follow very closely the arguments of [LV] up to a point, to obtain a condition on $\partial\Omega$ that we have called the “Weak Half Space Approximation” (WHS) property. Indeed, extracting the essence of the [LV] argument, while dispensing with the doubling property, one realizes that the WHS is precisely what one obtains. To fix ideas, and for the sake of self-containment, we shall summarize this fact, and present the argument of [LV] here as Lemma 4.16. Of course, in the proof of Lemma 4.16, we shall follow [LV] quite closely. Finally then, having obtained that $\partial\Omega$ satisfies the WHS property, we are then left with showing that WHS implies uniform rectifiability:

Proposition 1.7. *An n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable if and only if it satisfies the WHS property.*

We shall give the definition of WHS in Section 2, and the proof of the proposition in Section 5. While the WHS condition, per se, is new, even in this last step we shall make use of a modified version of part of the argument in [LV].

We note that in [LV], the authors treated also the case that the “ p -harmonic measure” (i.e., the Riesz measure associated to a non-negative p -harmonic function vanishing on a surface ball) was ADR, for all $1 < p < \infty$. Of course, the case $p = 2$ corresponds to the classical Laplacian. In a forthcoming joint paper with K. Nyström and P. Le, we plan to extend the results of the present paper to the case of the p -Laplacian, with $1 < p < \infty$.

To provide some additional context, we mention that our results here may be viewed as a “large constant” analogue of a result of Kenig and Toro [KT], which states that in the presence of a Reifenberg flatness condition and Ahlfors-David regularity, $\log k \in VMO$ implies that the unit normal ν to the boundary belongs to VMO , where k is the Poisson kernel with pole at some fixed point. Moreover, under the same background hypotheses, the condition that $\nu \in VMO$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace, thus $\log k \in VMO \implies \text{vanishing UR}$, given sufficient Reifenberg flatness. On the other hand, our large constant version “almost” says “ $\log k \in BMO \implies UR$ ”. Indeed, it is well known that the A_∞ condition (i.e., weak- A_∞ plus the doubling property) implies

that $\log k \in BMO$, while if $\log k \in BMO$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [KT] may be viewed as an “endpoint” version of the free boundary results of [AC] and [Je], which say, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal ν (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

The paper is organized as follows. In Section 2, we state several definitions and basic lemmas. In Section 3, we begin the proof of Theorem 1.1 with some preliminary arguments, and in Section 4, we complete the proof, modulo Proposition 1.7, following the arguments of [LV]. In Section 5, we give the proof of Proposition 1.7, i.e., the proof of the fact that the WHSA condition implies uniform rectifiability. Finally, in Section 6, we give the (very short) proof of Corollary 1.3.

2. PRELIMINARIES

Definition 2.1. (ADR) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n , is ADR if it is closed, and if there is some uniform constant C such that

$$(2.2) \quad \frac{1}{C} r^n \leq \sigma(\Delta(x, r)) \leq C r^n, \quad \forall r \in (0, \text{diam}(E)), x \in E,$$

where $\text{diam}(E)$ may be infinite. Here, $\Delta(x, r) := E \cap B(x, r)$ is the “surface ball” of radius r , and $\sigma := H^n|_E$ is the “surface measure” on E , where H^n denotes n -dimensional Hausdorff measure.

Definition 2.3. (UR) (aka *uniformly rectifiable*). An n -dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains “Big Pieces of Lipschitz Images” of \mathbb{R}^n (“BPLI”). This means that there are positive constants θ and M_0 , such that for each $x \in E$ and each $r \in (0, \text{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0 , such that

$$H^n(E \cap B(x, r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})) \geq \theta r^n.$$

We recall that n -dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of \mathbb{R}^n ; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all “sufficiently nice” singular integrals are L^2 -bounded [DS1]. In fact, for n -dimensional ADR sets in \mathbb{R}^{n+1} , the L^2 boundedness of certain special singular integral operators (the “Riesz Transforms”), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NToV] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett’s “4-corners Cantor set” in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions); see [DS1, DS2], and in particular Theorem 2.12 below. In this paper, we shall also present a new characterization of UR sets of co-dimension 1 (see Proposition 1.7 below), which will be very useful in the proof of Theorem 1.1.

Definition 2.4. (“UR character”). Given a UR set $E \subset \mathbb{R}^{n+1}$, its “UR character” is just the pair of constants (θ, M_0) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Lemma 2.5. (Existence and properties of the “dyadic grid”) [DS1, DS2], [Ch]. Suppose that $E \subset \mathbb{R}^{n+1}$ is closed n -dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$ and $C_* < \infty$, depending only on dimension and the ADR constants, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets (“cubes”)

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where \mathfrak{I}_k denotes some (possibly finite) index set depending on k , satisfying

- (i) $E = \cup_j Q_j^k$ for each $k \in \mathbb{Z}$
- (ii) If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.
- (iii) For each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.
- (iv) $\text{diam}(Q_j^k) \leq C_* 2^{-k}$.
- (v) Each Q_j^k contains some “surface ball” $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.
- (vi) $H^n(\{x \in Q_j^k : \text{dist}(x, E \setminus Q_j^k) \leq \varrho 2^{-k}\}) \leq C_* \varrho^\gamma H^n(Q_j^k)$, for all k, j and for all $\varrho \in (0, a_0)$.

Let us make a few remarks are concerning this lemma, and discuss some related notation and terminology.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Ch], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property (2.2), the result already appears in [DS1, DS2].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \text{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k , i.e.,

$$\mathbb{D} := \cup_k \mathbb{D}_k,$$

where, if $\text{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \text{diam}(E)$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \text{diam}(Q)$ and

$$(2.6) \quad \Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr),$$

for some uniform constant C . We shall denote this ball and surface ball by

$$(2.7) \quad B_Q := B(x_Q, r), \quad \Delta_Q := \Delta(x_Q, r),$$

and we shall refer to the point x_Q as the “center” of Q .

- Given a dyadic cube $Q \in \mathbb{D}$, we define its “ κ -dilate” by

$$(2.8) \quad \kappa Q := E \cap B(x_Q, \kappa \operatorname{diam}(Q)).$$

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the “length” of Q . Clearly, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the “dyadic generation” to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

Definition 2.9. (“ ε -local BAUP”) Given $\varepsilon > 0$, we shall say that $Q \in \mathbb{D}(E)$ satisfies the ε -local BAUP condition if there is a family \mathcal{P} of hyperplanes (depending on Q) such that every point in $10Q$ is at a distance at most $\varepsilon\ell(Q)$ from $\cup_{P \in \mathcal{P}} P$, and every point in $(\cup_{P \in \mathcal{P}} P) \cap B(x_Q, 10 \operatorname{diam}(Q))$ is at a distance at most $\varepsilon\ell(Q)$ from E .

Definition 2.10. (BAUP). We shall say that an n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the condition of *Bilateral Approximation by Unions of Planes* (“BAUP”), if for some $\varepsilon_0 > 0$, and for every positive $\varepsilon < \varepsilon_0$, there is a constant $C_0 = C_0(\varepsilon)$ such that the set \mathcal{B} of bad cubes in $\mathbb{D}(E)$, for which the ε -local BAUP condition fails, satisfies the packing condition

$$(2.11) \quad \sum_{Q' \subset Q, Q' \in \mathcal{B}} \sigma(Q') \leq C_0 \sigma(Q), \quad \forall Q \in \mathbb{D}(E).$$

For future reference, we recall the following result of David and Semmes [DS2].

Theorem 2.12 ([DS2, Theorem I.2.18, p. 36]). *Let $E \subset \mathbb{R}^{n+1}$ be an n -dimensional ADR set. Then, E is uniformly rectifiable if and only if it satisfies BAUP.*

We remark that the definition of BAUP in [DS2] is slightly different in superficial appearance, but it is not hard to verify that the dyadic version stated here is equivalent to the condition in [DS2]. We note that we shall not need the full strength of this equivalence here, but only the fact that our version of BAUP implies the version in [DS2], and hence implies UR.

We shall also require a new characterization of UR sets of co-dimension 1, which is related to the BAUP and its variants. For a sufficiently large constant K_0 to be chosen (see Lemma 3.14 below), we set

$$(2.13) \quad B_Q^* := B(x_Q, K_0^2 \ell(Q)), \quad \Delta_Q^* := B_Q^* \cap E.$$

Given a small positive number ε , which we shall typically assume to be much smaller than K_0^{-6} , we also set

$$(2.14) \quad B_Q^{**} = B_Q^{**}(\varepsilon) := B(x_Q, \varepsilon^{-2} \ell(Q)), \quad B_Q^{***} = B_Q^{***}(\varepsilon) := B(x_Q, \varepsilon^{-5} \ell(Q)).$$

Definition 2.15. (“ ε -local WHSA”) Given $\varepsilon > 0$, we shall say that $Q \in \mathbb{D}(E)$ satisfies the ε -local WHSA condition (or more precisely, the “ ε -local WHSA with parameter K_0 ”) if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number K_0 satisfying

- (1) $\operatorname{dist}(Z, E) \leq \varepsilon\ell(Q)$, for every $Z \in P \cap B_Q^{**}(\varepsilon)$.
- (2) $\operatorname{dist}(Q, P) \leq K_0^{3/2} \ell(Q)$.
- (3) $H \cap B_Q^{**}(\varepsilon) \cap E = \emptyset$.

Let us note that in particular, part (2) of the previous definition says that the hyperplane P has an “ample” intersection with the ball $B_Q^{**}(\varepsilon)$. Indeed,

$$(2.16) \quad \text{dist}(x_Q, P) \lesssim K_0^{\frac{3}{2}} \ell(Q) \ll \varepsilon^{-2} \ell(Q).$$

Definition 2.17. (“WHSA”) We shall say that an n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the *Weak Half-Space Approximation* property (“WHSA”) if for some pair of positive constants ε_0 and K_0 , and for every positive $\varepsilon < \varepsilon_0$, there is a constant $C_1 = C_1(\varepsilon)$ such that the set \mathcal{B} of bad cubes in $\mathbb{D}(E)$, for which the ε -local WHSA condition with parameter K_0 fails, satisfies the packing condition

$$(2.18) \quad \sum_{Q \subset Q_0, Q \in \mathcal{B}} \sigma(Q) \leq C_1 \sigma(Q_0), \quad \forall Q_0 \in \mathbb{D}(E).$$

Next, we develop some further notation and terminology. Let \mathcal{W} be a fixed collection of closed Whitney cubes for an open set Ω with ADR boundary $E = \partial\Omega$, and given $Q \in \mathbb{D}(E)$, for the same constant K_0 as in (2.13), we set

$$(2.19) \quad \mathcal{W}_Q := \{I \in \mathcal{W} : K_0^{-1} \ell(Q) \leq \ell(I) \leq K_0 \ell(Q), \text{ and } \text{dist}(I, Q) \leq K_0 \ell(Q)\}.$$

We fix a small, positive parameter τ , to be chosen momentarily, and given $I \in \mathcal{W}$, we let

$$(2.20) \quad I^* = I^*(\tau) := (1 + \tau)I$$

denote the corresponding “fattened” Whitney cube. We now choose τ sufficiently small that the cubes I^* will retain the usual properties of Whitney cubes, in particular that

$$\text{diam}(I) \approx \text{diam}(I^*) \approx \text{dist}(I^*, E) \approx \text{dist}(I, E).$$

We then define Whitney regions with respect to Q by setting

$$(2.21) \quad U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*.$$

We observe that these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon K_0 and dimension. We enumerate the components of U_Q as $\{U_Q^i\}_i$.

Moreover, we enlarge the Whitney regions as follows.

Definition 2.22. For $\varepsilon > 0$, and given $Q \in \mathbb{D}(E)$, we write $X \approx_{\varepsilon, Q} Y$ if X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3 \ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3} \ell(Q)$. Given a sufficiently small parameter $\varepsilon > 0$, we then set

$$(2.23) \quad \widetilde{U}_Q^i := \{X \in \mathbb{R}^{n+1} \setminus E : X \approx_{\varepsilon, Q} Y, \text{ for some } Y \in U_Q^i\}.$$

Remark 2.24. Since \widetilde{U}_Q^i is an enlarged version of U_Q , it may be that there are some $i \neq j$ for which \widetilde{U}_Q^i meets \widetilde{U}_Q^j . This overlap will be harmless.

Lemma 2.25 (Bourgain [Bo]). *Suppose that $\partial\Omega$ is n -dimensional ADR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, depending only on n and ADR, such that for every $x \in \partial\Omega$, and every $r \in (0, \text{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$(2.26) \quad \omega^Y(\Delta(x, r)) \geq 1/C > 0.$$

We refer the reader to [Bo, Lemma 1] for the proof. We note for future reference that in particular, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$, and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$(2.27) \quad \omega^X(\Delta_X) \geq 1/C.$$

Indeed, the latter bound follows immediately from (2.26), and the fact that we can form a Harnack Chain connecting X to a point Y that lies on the line segment from X to \hat{x} , and satisfies $|Y - \hat{x}| = c\delta(X)$.

As a consequence of Lemma 2.25, we have the following:

Corollary 2.28 ([HMT]). *Let $\partial\Omega$ be n -dimensional ADR. Suppose that $u \geq 0$ is harmonic in $\Omega \cap B(x, 2r)$, and vanishes continuously on the surface ball $\Delta(x, 2r) = B(x, 2r) \cap \partial\Omega$, with $x \in \partial\Omega$, and $0 < r < \text{diam } \partial\Omega$. Then for some $\alpha > 0$,*

$$(2.29) \quad u(Y) \leq C \left(\frac{\delta(Y)}{r} \right)^\alpha \frac{1}{|B(x, 2r)|} \iint_{B(x, 2r) \cap \Omega} u, \quad \forall Y \in B(x, r) \cap \Omega,$$

where the constants C and α depend only on dimension and the ADR constants for $\partial\Omega$.

Lemma 2.30 ([HMT]). *Let Ω be an open set with n -dimensional ADR boundary. There are positive, finite constants C , depending only on dimension, Λ and c_θ , depending on dimension, Λ , and $\theta \in (0, 1)$, such that the Green function satisfies*

$$(2.31) \quad G(X, Y) \leq C |X - Y|^{1-n};$$

$$(2.32) \quad c_\theta |X - Y|^{1-n} \leq G(X, Y), \quad \text{if } |X - Y| \leq \theta \delta(X), \quad \theta \in (0, 1);$$

$$(2.33) \quad G(X, \cdot) \in C(\overline{\Omega} \setminus \{X\}) \quad \text{and} \quad G(X, \cdot)|_{\partial\Omega} \equiv 0, \quad \forall X \in \Omega;$$

$$(2.34) \quad G(X, Y) \geq 0, \quad \forall X, Y \in \Omega, \quad X \neq Y;$$

$$(2.35) \quad G(X, Y) = G(Y, X), \quad \forall X, Y \in \Omega, \quad X \neq Y;$$

and for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$(2.36) \quad \int_{\partial\Omega} \Phi d\omega^X - \Phi(X) = - \iint_{\Omega} \nabla_Y G(Y, X) \cdot \nabla \Phi(Y) dY, \quad \text{for a.e. } X \in \Omega.$$

Next we present a version of one of the estimates obtained by Caffarelli-Fabes-Mortola-Salsa in [CFMS], which remains true even in the absence of connectivity:

Lemma 2.37 (CFMS). *Suppose that $\partial\Omega$ is n -dimensional ADR. For every $Y \in \Omega$ and $X \in \Omega$ such that $|X - Y| \geq \delta(Y)/2$ we have*

$$(2.38) \quad \frac{G(Y, X)}{\delta(Y)} \leq C \frac{\omega^X(\Delta_Y)}{\sigma(\Delta_Y)},$$

where $\Delta_Y = B(\hat{y}, 10\delta(Y))$ with $\hat{y} \in \partial\Omega$ such that $|Y - \hat{y}| = \delta(Y)$.

For future use, we note that as a consequence of (2.38), it follows directly that for every $Q \in \mathbb{D}(\partial\Omega)$, if $Y \in C_0 B_Q$ with $\delta(Y) \gtrsim C'_0 \ell(Q)$, then there exists $C = C(n, \text{ADR}, C_0, C'_0)$ such that

$$(2.39) \quad \frac{G(Y, X)}{\ell(Q)} \lesssim \frac{\omega^X(CQ)}{\sigma(CQ)} \lesssim \int_Q \mathcal{M}(k^X 1_{CQ}) d\sigma, \quad \forall X \notin C B_Q,$$

where \mathcal{M} is the usual Hardy-Littlewood maximal operator on $\partial\Omega$, and k^X is the Poisson kernel for Ω with pole at X .

Proof of Lemma 2.37. We follow the well-known argument of [CFMS] (see also [Ke, Lemma 1.3.3]). Fix $Y \in \Omega$ and write $B^Y = \overline{B(Y, \delta(Y)/2)}$. Consider the open set $\widehat{\Omega} = \Omega \setminus B^Y$ for which clearly $\partial\widehat{\Omega} = \partial\Omega \cup \partial B^Y$. Set

$$u(X) := G(Y, X)/\delta(Y), \quad v(X) := \omega^X(\Delta_Y)/\sigma(\Delta_Y),$$

for every $X \in \widehat{\Omega}$. Note that both u and v are non-negative harmonic functions in $\widehat{\Omega}$. If $X \in \partial\Omega$ then $u(X) = 0 \leq v(X)$. Take now $X \in \partial B^Y$ so that $u(X) \lesssim \delta(Y)^{-n}$ by (2.31). On the other hand, if we fix $X_0 \in \partial B^Y$ with X_0 on the line segment that joints Y and \hat{y} , then $2\Delta_{X_0} = \Delta_Y$, so that $v(X_0) \gtrsim \delta(Y)^{-n}$, by (2.27). By Harnack's inequality, we then obtain $v(X) \gtrsim \delta(Y)^{-n}$, for all $X \in \partial B^Y$. Thus, $u \lesssim v$ in $\partial\widehat{\Omega}$ and by the maximum principle this immediately extends to $\widehat{\Omega}$ as desired. \square

Lemma 2.40. *Suppose that $\partial\Omega$ is n -dimensional ADR. Let $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, and set $\Delta = B \cap \partial\Omega$. There exists $\kappa_0 > 2$ depending only in n and the ADR constant of $\partial\Omega$ such that for $X \in \Omega \setminus \kappa_0 B$ we have*

$$(2.41) \quad \sup_{\frac{1}{2}B} G(\cdot, X) \lesssim \frac{1}{|B|} \iint_B G(Y, X) dY \leq C r \frac{\omega^X(C\Delta)}{\sigma(C\Delta)}.$$

where C depends in n and the ADR constant of $\partial\Omega$.

Proof. Extending $G(\cdot, X)$ to be 0 outside of Ω , we obtain a sub-harmonic function in B . The first inequality in (2.41) follows immediately. To prove the second inequality, we set $\Sigma_B = \{I \in \mathcal{W} : I \cap B \neq \emptyset\}$ and note that if $I \in \Sigma_B$ then

$$\ell(I) \approx \text{dist}(I, \partial\Omega) \leq \text{dist}(I, x) \leq r.$$

In particular we can find κ_0 depending only in the implicit constants in the previous estimate so that $d(X, 4I) \geq 4r$ for every $I \in \Sigma_B$. Let $Q_I \in \mathbb{D}$ be so that $\ell(Q_I) = \ell(I)$ and $\text{dist}(I, \partial\Omega) = \text{dist}(I, Q_I)$. Note that then $\ell(Q_I) \lesssim r$ and $Y(I)$, the center of I , satisfies $Y(I) \in CB_{Q_I}$ and $\delta(Y(I)) \approx \ell(I) \approx \ell(Q_I)$. Hence we can invoke (2.39) (taking κ_0 larger if needed) and obtain that for every $Y \in I$,

$$G(Y, X) \approx G(Y(I), X) \lesssim \ell(I) \frac{\omega^X(CQ_I)}{\sigma(CQ_I)},$$

where the first estimate uses Harnack's inequality in $2I \subset \Omega$. Hence,

$$\begin{aligned} \iint_B G(Y, X) dY &\leq \sum_{I \in \Sigma_B} \iint_I G(Y, X) dY \lesssim \sum_{I \in \Sigma_B} \ell(I)^2 \omega^X(CQ_I) \\ &\leq \sum_{k: 2^{-k} \lesssim r} 2^{-2k} \sum_{I \in \Sigma_B: \ell(I) = 2^{-k}} \omega^X(CQ_I) \lesssim r^2 \omega^X(C'\Delta). \end{aligned}$$

where in the last inequality we have used that the cubes Q_I have uniformly bounded overlap whenever $\ell(I) = 2^{-k}$ and they are all contained in $C'\Delta$. This and the ADR property readily yields the desired estimate. \square

3. PROOF OF THEOREM 1.1, STEP 1: PRELIMINARY ARGUMENTS

Let us introduce some notation. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. Recall (2.7) and take $B_{Q_0} = B(x_{Q_0}, r_{Q_0})$ with $r_{Q_0} \approx \ell(Q_0)$ so that $\Delta_{Q_0} = B_{Q_0} \cap \partial\Omega \subset Q_0$. Let X_0 be the associated “corkscrew” point given in the statement of Theorem 1.1. In particular, for C_1 large enough (so that $2Q_0 \subset C_1\Delta_{Q_0}$) we have that $X_0 \in C_0 B_{Q_0}$ with $\delta(X_0) \geq C_0^{-1} r_{Q_0} \approx \ell(Q_0)$ for which $\omega^{X_0} \ll \sigma$ in $2Q_0$ and moreover

$$(3.1) \quad \omega^{X_0}(Q_0) \geq C_0^{-1}, \quad \int_{2Q_0} k^{X_0}(y)^p d\sigma \leq C_0 \sigma(2Q_0)^{1-p}$$

Set $\omega := C_0 \sigma(Q_0) \omega^{X_0}$, and let $k := d\omega/d\sigma$ be the corresponding normalized Poisson kernel. We then have

$$(3.2) \quad 1 \leq \frac{\omega(Q_0)}{\sigma(Q_0)} \leq \frac{\omega(\partial\Omega)}{\sigma(Q_0)} \leq C_0$$

and

$$(3.3) \quad \left(\int_{2Q_0} k(y)^p d\sigma(y) \right)^{\frac{1}{p}} \leq C_0^{1+\frac{1}{p}}.$$

Given a family $\mathcal{F} = \{Q_j\}$ of disjoint sub-cubes of Q_0 , we set

$$\mathbb{D}_{\mathcal{F}, Q_0} := \{Q \subset Q_0 : Q \text{ is not contained in any } Q_j \in \mathcal{F}\},$$

and for any $Q \in \mathbb{D}(\partial\Omega)$, we set

$$\mathbb{D}_Q := \{Q' \subset Q\}.$$

As above, let \mathcal{M} denote the usual Hardy-Littlewood maximal operator on $\partial\Omega$.

Lemma 3.4. *Under the assumptions of Theorem 1.1 and following the previous notation, there is a pairwise disjoint family $\mathcal{F} = \{Q_j\}_j \setminus \{Q_0\}$, such that*

$$(3.5) \quad \sigma(Q_0 \setminus (\cup_j Q_j)) \geq c_0 \sigma(Q_0)$$

where $0 < c_0 \leq 1$ depends only on the implicit constants in the hypotheses of Theorem 1.1, and

$$(3.6) \quad \frac{1}{2} \leq \frac{\omega(Q)}{\sigma(Q)} \leq \int_Q \mathcal{M}(k1_{2Q_0}) d\sigma \leq C, \quad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0},$$

where $C > 1$ depends only on the implicit constants in the hypotheses.

Proof. The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [HMc], [HLMc], [AHLMcT], and in local Tb theorems. We begin by noting that

$$(3.7) \quad \begin{aligned} \int_{Q_0} \mathcal{M}(k1_{2Q_0}) d\sigma &\leq \left(\int_{Q_0} (\mathcal{M}(k1_{2Q_0}))^p d\sigma \right)^{1/p} \\ &\leq C_1 \left(\int_{2Q_0} k^p d\sigma \right)^{1/p} \leq C_1 C_0^{1+\frac{1}{p}} =: C_2, \end{aligned}$$

where we have used the $L^p(\sigma)$ boundedness of the Hardy-Littlewood maximal function and (3.3). We now let $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ be the collection of sub-cubes that are maximal with respect to the property that either

$$(3.8) \quad \frac{\omega(Q_j)}{\sigma(Q_j)} < \frac{1}{2},$$

and/or

$$(3.9) \quad \int_{Q_j} \mathcal{M}(k1_{2Q_0}) d\sigma > C_2 K,$$

where $K \geq 1$ is a sufficiently large number to be chosen momentarily. Note that $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ by (3.2) and (3.7). We shall say that Q_j is of “type I” if (3.8) holds, and Q_j is of “type II” if (3.9) holds but (3.8) does not. Set $A := Q_0 \setminus (\cup_j Q_j)$, and $F := \cup_{Q_j \text{ type II}} Q_j$. Then by (3.2),

$$(3.10) \quad \sigma(Q_0) \leq \omega(Q_0) = \sum_{Q_j \text{ type I}} \omega(Q_j) + \omega(F) + \omega(A).$$

By definition of the type I cubes,

$$(3.11) \quad \sum_{Q_j \text{ type I}} \omega(Q_j) \leq \frac{1}{2} \sum_j \sigma(Q_j) \leq \frac{1}{2} \sigma(Q_0).$$

To handle the remaining terms, observe that

$$(3.12) \quad \begin{aligned} \sigma(F) &= \sum_{Q_j \text{ type II}} \sigma(Q_j) \leq \frac{1}{C_2 K} \sum_j \int_{Q_j} \mathcal{M}(k1_{2Q_0}) d\sigma \\ &\leq \frac{1}{C_2 K} \int_{Q_0} \mathcal{M}(k1_{2Q_0}) d\sigma \leq \frac{1}{K} \sigma(Q_0), \end{aligned}$$

by the definition of the type II cubes and (3.7). Combining (3.3) and (3.12), we find that

$$(3.13) \quad \omega(F) = \int_F k d\sigma \leq \left(\int_{2Q_0} k^p d\sigma \right)^{\frac{1}{p}} \sigma(F)^{\frac{1}{p'}} \leq C_0^{1+\frac{1}{p}} K^{-1/p'} \sigma(2Q_0) \leq \frac{1}{4} \sigma(Q_0),$$

by choice of K large enough. By (3.11) and (3.13), we may hide the two small terms on the left hand side of (3.10), and then use (3.3), to obtain

$$\sigma(Q_0) \leq 4 \omega(A) = 4 \int_A k d\sigma \lesssim \left(\int_{2Q_0} k^p d\sigma \right)^{\frac{1}{p}} \sigma(A)^{\frac{1}{p'}} \lesssim \sigma(A)^{1/p'} \sigma(Q_0)^{\frac{1}{p}}.$$

Estimate (3.5) now follows readily. Moreover, (3.6) holds, by the maximality of the cubes Q_j , and our choice of K . \square

We recall that the ball B_Q^* and surface ball Δ_Q^* are defined in (2.13).

Lemma 3.14. *Under the notation of Lemma 3.4, if the constant K_0 in (2.19) is chosen sufficiently large, for each $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ with $\ell(Q) \leq K_0^{-1} \ell(Q_0)$ there exists*

$Y_Q \in U_Q$ with $\delta(Y_Q) \leq |Y_Q - x_Q| \lesssim \ell(Q)$ (where the implicit constant is independent of K_0) such that

$$(3.15) \quad \frac{\omega^{X_0}(Q)}{\sigma(Q)} \leq C |\nabla G(X_0, Y_Q)|,$$

where C depends on K_0 and the implicit constants in the hypotheses of Theorem 1.1.

Remark 3.16. Our choice of K_0 in the previous result will guarantee that $\delta(X_0) \approx \ell(Q_0) \geq K_0^{-1/2} \ell(Q_0)$. Note also that the point Y_Q is an effective corkscrew relative to Q since $\delta(Y_Q) \gtrsim K_0^{-1} \ell(Q)$ (as $Y \in U_Q$) and also $|Y_Q - x_Q| \lesssim \ell(Q)$ (with constant independent of K_0). Abusing the notation we will say that Y_Q is a corkscrew point relative to Q and we observe that the corresponding constant depends on K_0 .

Proof. Fix $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ with $\ell(Q) \leq K_0^{-1} \ell(Q_0)$ and K_0 large enough to be chosen. Recall that $\delta(X_0) \approx \ell(Q_0)$ and therefore if K_0 is large enough we may assume that $\delta(X_0) \approx \ell(Q_0) \geq K_0^{-1/2} \ell(Q_0)$. Recall (2.6) and write $\hat{B}_Q = B(x_Q, \hat{r}_Q)$, $\hat{\Delta}_Q = \hat{B}_Q \cap E$ so that $\hat{r}_Q \approx \ell(Q)$ and $Q \subset \frac{1}{2} \hat{\Delta}_Q$. Let $0 \leq \phi_Q \in C_0^\infty(\hat{B}_Q)$ so that $\phi_Q \equiv 1$ in $\frac{1}{2} \hat{B}_Q$ and $\|\nabla \phi_Q\| \lesssim \ell(Q)^{-1}$. Note that

$$K_0^{1/2} \ell(Q) \leq K_0^{-1/2} \ell(Q_0) \lesssim \delta(X_0) \leq |X_0 - x_Q|$$

which implies that $X_0 \notin 4 \hat{B}_Q$ provided K_0 is large enough. We can next use (2.36) (if needed we can slightly move X_0 so that the equality holds at X_0 and then use Harnack's inequality when needed to move back to X_0 , details are left to the interested reader). Then

$$(3.17) \quad \begin{aligned} \ell(Q) \omega^{X_0}(Q) &\leq \ell(Q) \int_{\partial\Omega} \phi_Q d\omega^{X_Q} \lesssim \iint_{\hat{B}_Q \cap \Omega} |\nabla G(X_0, Y)| dY \\ &\leq \iint_{\hat{B}_Q \cap U_Q} |\nabla G(X_0, Y)| dY + \iint_{(\hat{B}_Q \cap \Omega) \setminus U_Q} |\nabla G(X_0, Y)| dY =: I + II. \end{aligned}$$

Notice that by construction $(\hat{B}_Q \cap \Omega) \setminus U_Q \subset \{Y \in \hat{B}_Q : \delta(Y) \lesssim K_0^{-1} \ell(Q)\}$. Writing $\Sigma_Q := \{I \in \mathcal{W} : I \cap \hat{B}_Q \neq \emptyset, \ell(I) \lesssim K_0^{-1} \ell(Q)\}$ and using interior estimates (note that $4I \subset \Omega$) we have

$$II \leq \sum_{I \in \Sigma_Q} \iint_I |\nabla G(X_0, Y)| dY \lesssim \sum_{I \in \Sigma_Q} \ell(I)^n G(X_0, Y_I)$$

where Y_I is the center of I . We next use Corollary 2.28 and Lemma 2.40 (we may need to take K_0 larger) observe that

$$(3.18) \quad \begin{aligned} G(X_0, Y_I) &\lesssim \left(\frac{\ell(I)}{\ell(Q)} \right)^\alpha \frac{1}{|2 \hat{B}_Q|} \iint_{2 \hat{B}_Q \cap \Omega} G(X_0, Y) dY \lesssim \left(\frac{\ell(I)}{\ell(Q)} \right)^\alpha \ell(Q) \frac{\omega^{X_0}(CQ)}{\sigma(CQ)} \\ &\leq \sigma(Q_0)^{-1} \left(\frac{\ell(I)}{\ell(Q)} \right)^\alpha \ell(Q) \int_Q \mathcal{M}(k1_{2Q_0}) d\sigma \lesssim \sigma(Q_0)^{-1} \left(\frac{\ell(I)}{\ell(Q)} \right)^\alpha \ell(Q), \end{aligned}$$

where we recall that $\omega = C_0 \sigma(Q_0) \omega^{X_0}$, we have used that if K_0 is large enough and $Q \in \mathbb{D}_{Q_0}$ with $\ell(Q) \leq K_0^{-1} \ell(Q_0)$ then $CQ \subset 2Q_0$, and we have also invoked

(3.6) (recall that $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$). As before let $Q_I \in \mathbb{D}$ be so that $\ell(Q_I) = \ell(I)$ and $\text{dist}(I, E) = \text{dist}(I, Q_I)$ and observe that since I meets \hat{B}_Q then $Q_I \subset C Q$ for some uniform constant. Let us observe that from the Properties of the Whitney cubes, for every k , the family of cubes $\{Q_I\}_{\ell(I)=2^{-k}}$ has bounded overlap uniformly in k . Hence we can use (3.18), (3.6), and the definition of ω to obtain

$$\begin{aligned} II &\lesssim \sigma(Q_0)^{-1} \sum_{I \in \Sigma_Q} \ell(I)^n \left(\frac{\ell(I)}{\ell(Q)} \right)^\alpha \ell(Q) \\ &\lesssim \sigma(Q_0)^{-1} \ell(Q)^{1-\alpha} \sum_{k: 2^{-k} \lesssim K_0^{-1} \ell(Q)} 2^{-k\alpha} \sum_{I \in \Sigma_Q: \ell(I)=2^{-k}} \sigma(Q_I) \\ &\lesssim \sigma(Q_0)^{-1} K_0^{-\alpha} \ell(Q) \sigma(C Q) \\ &\leq \frac{1}{2} \ell(Q) \omega^{X_0}(Q), \end{aligned}$$

provided K_0 is large enough. We can then plug this estimate in (3.17) and hide it to obtain that

$$\begin{aligned} \ell(Q) \omega^{X_0}(Q) &\leq I \leq \iint_{\hat{B}_Q \cap U_Q} |\nabla G(X_0, Y)| dY = \sum_i \iint_{\hat{B}_Q \cap U_Q^i} |\nabla G(X_0, Y)| dY \\ &\lesssim \ell(Q)^{n+1} \max_i \sup_{Y \in \hat{B}_Q \cap U_Q^{i_0}} |\nabla G(X_0, Y)| \lesssim \ell(Q) \sigma(Q) \max_i \sup_{Y \in \hat{B}_Q \cap U_Q^{i_0}} |\nabla G(X_0, Y)|, \end{aligned}$$

where we recall that number of components of U_Q is uniformly bounded. This clearly implies that we can find $Y_Q \in \hat{B}_Q \cap U_Q^i$ for some i such that $\omega^{X_0}(Q)/\sigma(Q) \lesssim |\nabla G(X_0, Y_Q)|$. To complete the proof we simply observe that $\delta(Y_Q) \leq |Y_Q - x_Q| \leq \hat{r}_Q \lesssim \ell(Q)$. \square

4. PROOF OF THEOREM 1.1, STEP 2: THE LEWIS-VOGEL ARGUMENT

We recall that $B_Q^{***}(\varepsilon) = B(x_Q, \varepsilon^{-5} \ell(Q))$, as in (2.14). Set $\Delta_Q^{***}(\varepsilon) := E \cap B_Q^{***}(\varepsilon)$.

Our proof here is a refinement/extension of the arguments in [LV], who, as mentioned in the introduction, treated the special case that the Poisson kernel $k \approx 1$. Our goal in this section is to show that $E = \partial\Omega$ satisfies WHSA, and hence is UR, by Proposition 1.7. Turning to the details, we fix $Q_0 \in \mathbb{D}(\partial\Omega)$, and we set

$$u(Y) := C_0 \sigma(Q_0) G(X_0, Y),$$

where X_0 is the “corkscrew” associated with Δ_{Q_0} (see the beginning of Section 3) and C_0 is the constant in (3.1). As above, for the same constant C_0 , we set

$$\omega := C \sigma(Q_0) \omega^{X_0},$$

and we recall that by (3.2),

$$(4.1) \quad \frac{\omega(Q_0)}{\sigma(Q_0)} \approx 1.$$

Let $\mathcal{F} = \{Q_j\}_j$ be the family of maximal stopping time cubes constructed in Lemma 3.4. Combining (3.15) and (3.6), we see that

$$(4.2) \quad |\nabla u(Y_Q)| \gtrsim 1, \quad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0}^* := \{Q \in \mathbb{D}_{\mathcal{F}, Q_0} : \ell(Q) \leq K_0^{-1} \ell(Q_0)\},$$

where $Y_Q \in U_Q$ is the point constructed in Lemma 3.14. We recall that the Whitney region U_Q has a uniformly bounded number of connected components, which we have enumerated as $\{U_Q^i\}_i$. We now fix the particular i such that $Y_Q \in U_Q^i \subset \widetilde{U}_Q^i$, where the latter is the enlarged Whitney region constructed in Definition 2.22.

For a suitably small ε_0 , say $\varepsilon_0 \ll K_0^{-6}$, we fix an arbitrary positive $\varepsilon < \varepsilon_0$, and we fix also a large positive number M to be chosen. For each point $Y \in \Omega$, we set

$$(4.3) \quad B_Y := \overline{B(Y, (1 - \varepsilon^{2M/\alpha})\delta(Y))}, \quad \widetilde{B}_Y := \overline{B(Y, \delta(Y))},$$

where $\alpha > 0$ is the DG/N exponent at the boundary (see Corollary 2.28).

For $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, we consider three cases.

Case 0: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) > \varepsilon^{10} \ell(Q_0)$.

Case 1: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) \leq \varepsilon^{10} \ell(Q_0)$ (in particular $Q \in \mathbb{D}_{\mathcal{F}, Q_0}^*$; see (4.2)), and

$$(4.4) \quad \sup_{X, Y \in \widetilde{U}_Q^i} \sup_{Z_1 \in B_Y, Z_2 \in B_X} |\nabla u(Z_1) - \nabla u(Z_2)| > \varepsilon^{2M}.$$

Case 2: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) \leq \varepsilon^{10} \ell(Q_0)$ (in particular $Q \in \mathbb{D}_{\mathcal{F}, Q_0}^*$), and

$$(4.5) \quad \sup_{X, Y \in \widetilde{U}_Q^i} \sup_{Z_1 \in B_Y, Z_2 \in B_X} |\nabla u(Z_1) - \nabla u(Z_2)| \leq \varepsilon^{2M}.$$

We trivially see that the cubes in Case 0 satisfy a packing condition:

$$(4.6) \quad \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 0 holds}}} \sigma(Q) \leq \sum_{Q \in \mathbb{D}_{Q_0}, \ell(Q) > \varepsilon^{10} \ell(Q_0)} \sigma(Q) \lesssim (\log \varepsilon^{-1}) \sigma(Q).$$

Before proceeding further, let us first note that if $\ell(Q) \leq \varepsilon^{10} \ell(Q_0)$, then by (4.2), (2.39) (which we may apply with $X = X_0$, since $\ell(Q) \ll \ell(Q_0)$), and (3.6),

$$(4.7) \quad 1 \lesssim |\nabla u(Y_Q)| \lesssim \frac{u(Y_Q)}{\delta(Y_Q)} \lesssim 1.$$

Next, we treat Case 1, and for these cubes, we shall also obtain a packing condition. We now augment \widetilde{U}_Q^i as follows. Set

$$\mathcal{W}_Q^{i,*} := \left\{ I \in \mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y \in \left(\bigcup_{X \in \widetilde{U}_Q^i} B_X \right) \right\}$$

(and define $\mathcal{W}_Q^{j,*}$ analogously for all other \widetilde{U}_Q^j), and set

$$U_Q^{i,*} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{**}, \quad U_Q^* := \bigcup_j U_Q^{j,*}$$

where $I^{**} = (1 + 2\tau)I$ is a suitably fattened Whitney cube, with τ fixed as above. By construction,

$$\widetilde{U}_Q^i \subset \bigcup_{X \in \widetilde{U}_Q^i} B_X \subset \bigcup_{Y \in \bigcup_{X \in \widetilde{U}_Q^i} B_X} B_Y \subset U_Q^{i,*},$$

and for all $Y \in U_Q^{i,*}$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on ε). Moreover, also by construction, there is a Harnack path connecting any pair of points in $U_Q^{i,*}$ (depending again on ε), and furthermore, for every $I \in \mathcal{W}_Q^{i,*}$ (or for that matter for every $I \in \mathcal{W}_Q^{j,*}$, $j \neq i$),

$$\varepsilon^s \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3} \ell(Q), \quad \text{dist}(I, Q) \lesssim \varepsilon^{-4} \ell(Q),$$

where $0 < s = s(M, \alpha)$. Thus, by Harnack's inequality and (4.7),

$$(4.8) \quad C^{-1} \delta(Y) \leq u(Y) \leq C \delta(Y), \quad \forall Y \in U_Q^{i,*},$$

with $C = C(K_0, \varepsilon, M)$, where we have used that u is a solution in Ω away from the pole at X_0 , and that X_0 is far from U_Q^* , since $\ell(Q) \ll \ell(Q_0)$. Moreover, for future reference, we note that the upper bound for u holds in all of U_Q^* , i.e.,

$$(4.9) \quad u(Y) \leq C \delta(Y), \quad \forall Y \in U_Q^*,$$

by (2.39) and (3.6), where again $C = C(K_0, \varepsilon, M)$. Choosing Z_1, Z_2 as in (4.4), and then using the mean value property of harmonic functions, we find that

$$\varepsilon^{2M} \leq C_\varepsilon (\ell(Q))^{-(n+1)} \iint_{B_{Z_1} \cup B_{Z_2}} |\nabla u(Y) - \vec{\beta}| dY,$$

where $\vec{\beta}$ is a constant vector at our disposal. In turn, by Poincaré's inequality (see, e.g., [HM2, Section 4] in this context), we obtain that

$$\sigma(Q) \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 \delta(Y) dY \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 u(Y) dY,$$

where the implicit constants depend on ε , and in the last step we have used (4.8). Consequently,

$$(4.10) \quad \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \ell(Q) \leq \varepsilon^{10} \ell(Q_0)}} \iint_{U_Q^*} |\nabla^2 u(Y)|^2 u(Y) dY \\ \lesssim \iint_{\Omega_{\mathcal{F}, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY,$$

where

$$(4.11) \quad \Omega_{\mathcal{F}, Q_0}^* := \text{int} \left(\bigcup_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \ell(Q) \leq \varepsilon^{10} \ell(Q_0)}} U_Q^* \right),$$

and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps.

Take an arbitrary $N > 1/\varepsilon$ (eventually $N \rightarrow \infty$), and augment \mathcal{F} by adding to it all subcubes $Q \subset Q_0$ with $\ell(Q) \leq 2^{-N} \ell(Q_0)$. Let $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ denote the collection of maximal cubes of this augmented family. Thus, $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ iff $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ and $\ell(Q) > 2^{-N} \ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N, Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'}, Q_0}$ if $N \leq N'$ and therefore $\Omega_{\mathcal{F}_N, Q_0}^* \subset$

$\Omega_{\mathcal{F}_{N'}, Q_0}^*$ (where $\Omega_{\mathcal{F}_N, Q_0}^*$ is defined as in (4.11) with \mathcal{F}_N replacing \mathcal{F}). By monotone convergence and (4.10), we have that

$$(4.12) \quad \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \limsup_{N \rightarrow \infty} \iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY.$$

It therefore suffices to establish bounds for the latter integral that are uniform in N , with N large.

Let us then fix $N > 1/\varepsilon$. Since $\Omega_{\mathcal{F}_N, Q_0}^*$ is a finite union of fattened Whitney boxes, we may now integrate by parts, using the identity $2|\nabla \partial_k u|^2 = \operatorname{div} \nabla (\partial_k u)^2$ for harmonic functions, to obtain that

$$(4.13) \quad \iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY \lesssim \int_{\partial \Omega_{\mathcal{F}_N, Q_0}^*} (|\nabla^2 u| |\nabla u| u + |\nabla u|^3) dH^n \leq C_\varepsilon H^n(\partial \Omega_{\mathcal{F}_N, Q_0}^*)$$

where in the second inequality we have used the standard estimate

$$\delta(Y) |\nabla^2 u(Y)| \lesssim |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)},$$

along with (4.9). We observe that $\Omega_{\mathcal{F}_N, Q_0}^*$ is a sawtooth domain in the sense of [HMM], or to be more precise, it is a union of a bounded number, depending on ε , of such sawtooths, the maximal cube for each of which is a sub-cube of Q_0 with length on the order of $\varepsilon^{10} \ell(Q_0)$. Thus, by [HMM, Appendix A], $\partial \Omega_{\mathcal{F}_N, Q_0}^*$ is ADR, uniformly in N , and therefore

$$H^n(\partial \Omega_{\mathcal{F}_N, Q_0}^*) \leq C_\varepsilon \left(\operatorname{diam}(\partial \Omega_{\mathcal{F}_N, Q_0}^*) \right)^n \leq C_\varepsilon \sigma(Q_0).$$

Combining the latter estimate with (4.12) and (4.13), we obtain

$$(4.14) \quad \frac{1}{\sigma(Q_0)} \sum_{Q \in \mathbb{D}_{\mathcal{F}, Q_0} : \text{Case 1 holds}} \sigma(Q) \leq C(\varepsilon, K_0, M, \eta).$$

Now we turn to Case 2. We claim that for every Q as in Case 2, the ε -local WHSA property (see Definition 2.15) holds, provided that M is taken large enough. Momentarily taking this claim for granted, we may complete the proof of Theorem 1.1 as follows. Given the claim, it follows that the cubes $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, which belong to the bad collection \mathcal{B} of cubes in $\mathbb{D}(\partial \Omega)$ for which the ε -local WHSA condition fails, must be as in Case 0 or Case 1, and therefore, by (4.6) and (4.14), satisfy the packing estimate

$$(4.15) \quad \sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0).$$

For each $Q_0 \in \mathbb{D}(\partial \Omega)$, there is a family $\mathcal{F} \subset \mathbb{D}_{Q_0}$ for which (4.15), and also the ‘‘ampleness’’ condition (3.5), hold uniformly. We may therefore invoke a well known lemma of John-Nirenberg type to deduce that (2.18) holds for all $\varepsilon \in (0, \varepsilon_0)$, and therefore to conclude that $\partial \Omega$ satisfies the WHSA condition (Definition 2.17), and hence is UR, by Proposition 1.7.

Thus, it remains to show that every Q as in Case 2 satisfies the ε -local WHSA property. In fact, we shall prove the following. Given $\varepsilon > 0$, we set

$$B_Q^{\text{big}} = B_Q^{\text{big}}(\varepsilon) := B(x_Q, \varepsilon^{-8}\ell(Q)), \quad \Delta_Q^{\text{big}} := B_Q^{\text{big}} \cap \partial\Omega.$$

Lemma 4.16. *Fix $\varepsilon \in (0, K_0^{-6})$. Suppose that $u \geq 0$ is harmonic in $\Omega_Q := \Omega \cap B_Q^{\text{big}}$, $u \in C(\overline{\Omega_Q})$, $u \equiv 0$ on Δ_Q^{big} . Suppose also that for some i , there exists a point $Y_Q \in U_Q^i$ such that*

$$(4.17) \quad |\nabla u(Y_Q)| \approx 1,$$

and furthermore, that

$$(4.18) \quad \sup_{B_Q^{***}} u \lesssim \varepsilon^{-5}\ell(Q);$$

$$(4.19) \quad \sup_{X, Y \in \tilde{U}_Q^i} \sup_{Z_1 \in B_Y, Z_2 \in B_X} |\nabla u(Z_1) - \nabla u(Z_2)| \leq \varepsilon^{2M}.$$

Then Q satisfies the ε -local WHSA, provided that M is large enough, depending only on dimension and on the implicit constants in the stated hypotheses.

Let us recall that in the scenario of Case 2, $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, $\ell(Q) \leq \varepsilon^{10}\ell(Q_0)$, and (4.5) holds. Then (4.17) holds by virtue of (4.7), while (4.18) holds by Lemma 2.40 applied with $B = 2B_Q^{***}$, and (3.6). Moreover, (4.19) is merely a restatement of (4.5). Thus, the hypotheses of Lemma 4.16 are all verified for the Case 2 cubes, so modulo the proof of Proposition 1.7, it remains only to prove the lemma.

Proof of Lemma 4.16. Our approach here will follow that of [LV] very closely, but with some modifications owing to the fact that in contrast to the situation in [LV], our solution u need not be Lipschitz, and our harmonic measures need not be doubling (it is the latter obstacle that has forced us to introduce the WHSA condition, rather than to work with the “Weak Exterior Convexity” condition used in [LV]). In fact, Lemma 4.16 is essentially a distillation of the main argument of the corresponding part of [LV], but with the doubling hypothesis removed.

For convenience, in the proof of the lemma, we shall use the notational convention that implicit and generic constants are allowed to depend upon K_0 , but not on ε or M . Dependence on the latter will be noted explicitly.

We begin with the following. We remind the reader that the balls B_Y and \tilde{B}_Y are defined in (4.3).

Lemma 4.20. *Let $Y \in U_Q^i$, $X \in \tilde{U}_Q^i$. Suppose first that $w \in \partial\tilde{B}_Y \cap \partial\Omega$, and let W be the radial projection of w onto ∂B_Y . Then*

$$(4.21) \quad u(W) \lesssim \varepsilon^{2M-5}\delta(Y).$$

If $w \in \partial\tilde{B}_X \cap \partial\Omega$, and W now is the radial projection of w onto ∂B_X , then

$$(4.22) \quad u(W) \lesssim \varepsilon^{2M-5}\ell(Q).$$

Proof. Since $K_0^{-1}\ell(Q) \lesssim \delta(Y) \lesssim K_0\ell(Q)$ for $Y \in U_Q^i$, it is enough to prove (4.22).

To prove (4.22), we first note that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q),$$

by definition of B_X, \widetilde{B}_X and the fact that by construction of \widetilde{U}_Q^i ,

$$(4.23) \quad \varepsilon^3\ell(Q) \lesssim \delta(X) \lesssim \varepsilon^{-3}\ell(Q), \quad \forall X \in \widetilde{U}_Q^i.$$

In addition,

$$(4.24) \quad \text{diam}(\widetilde{U}_Q^i) \lesssim \varepsilon^{-4}\ell(Q),$$

again by construction of \widetilde{U}_Q^i . Consequently, $W \in (1/2)B_Q^{***} = B(x_Q, (1/2)\varepsilon^{-5}\ell(Q))$, so by Corollary 2.28 and (4.18),

$$u(W) \lesssim \left(\frac{\varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q)}{\varepsilon^{-5}\ell(Q)} \right)^\alpha \frac{1}{|B_Q^{***}|} \iint_{B_Q^{***}} u \lesssim \varepsilon^{2M+2\alpha-5}\ell(Q) \leq \varepsilon^{2M-5}\ell(Q).$$

□

Claim 4.25. Let $Y \in U_Q^i$. For all $W \in B_Y$ (see (4.3)),

$$(4.26) \quad |u(W) - u(Y) - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M}\delta(Y).$$

Proof of Claim 4.25. Let $W \in B_Y$. Then

$$u(W) - u(Y) = \nabla u(\widetilde{W}) \cdot (W - Y),$$

for some $\widetilde{W} \in B_Y$. We may then invoke (4.19), with $X = Y$, $Z_1 = \widetilde{W}$, and $Z_2 = Y$, to obtain (4.26). □

Claim 4.27. Let $Y \in U_Q^i$. Suppose that $w \in \partial\widetilde{B}_Y \cap \partial\Omega$. Then

$$(4.28) \quad |u(Y) - \nabla u(Y) \cdot (Y - w)| = |u(w) - u(Y) - \nabla u(Y) \cdot (w - Y)| \lesssim \varepsilon^{2M-5}\delta(Y).$$

Proof of Claim 4.27. Given $w \in \widetilde{B}_Y \cap \partial\Omega$, let W be the radial projection of w onto ∂B_Y , so that $|W - w| = \varepsilon^{2M/\alpha}\delta(Y)$. Since $u(w) = 0$, by (4.21) we have

$$|u(W) - u(w)| = u(W) \lesssim \varepsilon^{2M-5}\delta(Y).$$

Since (4.26) holds for W , we obtain (4.28). □

To simplify notation, let us now set $Y := Y_Q$, the point in U_Q^i satisfying (4.17). By (4.17) and (4.19), for $\varepsilon < 1/2$, and M chosen large enough, we have that

$$(4.29) \quad |\nabla u(Z)| \approx 1, \quad \forall Z \in \widetilde{U}_Q^i.$$

By translation and rotation, we may assume that $0 \in \widetilde{B}_Y \cap \partial\Omega$, and that $Y = \delta(Y)e_{n+1}$, where as usual $e_{n+1} := (0, \dots, 0, 1)$.

Claim 4.30. We claim that

$$(4.31) \quad \left| \langle \nabla u(Y), e_{n+1} \rangle - |\nabla u(Y)| \right| \lesssim \varepsilon^{2M-5}.$$

Proof of Claim 4.30. We apply (4.28), with $w = 0$, to obtain

$$|u(Y) - \nabla u(Y) \cdot Y| \lesssim \varepsilon^{2M-5} \delta(Y).$$

Combining the latter bound with (4.26), we find that

$$(4.32) \quad |u(W) - \nabla u(Y) \cdot W| = |u(W) - \nabla u(Y) \cdot Y - \nabla u(Y) \cdot (W - Y)| \\ \lesssim \varepsilon^{2M-5} \delta(Y), \quad \forall W \in B_Y.$$

Fix $W \in \partial B_Y$ so that

$$\nabla u(Y) \cdot \frac{W - Y}{|W - Y|} = -|\nabla u(Y)|.$$

Since $|W - Y| = (1 - \varepsilon^{2M/\alpha})\delta(Y)$, and since $u \geq 0$, we have

$$(4.33) \quad 0 \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} + \frac{u(W)}{\delta(Y)} \\ \leq \frac{1}{\delta(Y)} \left(-\nabla u(Y) \cdot \frac{(W - Y)}{1 - \varepsilon^{2M/\alpha}} - \nabla u(Y) \cdot Y + u(W) \right) \\ \lesssim (\varepsilon^{2M-5} + \varepsilon^{2M/\alpha}) \approx \varepsilon^{2M-5},$$

by (4.32) and (4.17). \square

Claim 4.34. Suppose that $M > 5$. Then

$$(4.35) \quad \left| |\nabla u(Y)|e_{n+1} - \nabla u(Y) \right| \lesssim \varepsilon^{M-3}.$$

Proof of Claim 4.34. Note that

$$\left| |\nabla u(Y)|e_{n+1} - \langle \nabla u(Y), e_{n+1} \rangle e_{n+1} \right| \lesssim \varepsilon^{2M-5},$$

by Claim 4.30. Therefore, it is enough to consider $\nabla_{\parallel} u := \nabla u - \langle \nabla u, e_{n+1} \rangle e_{n+1}$. Observe that

$$|\nabla_{\parallel} u(Y)|^2 = |\nabla u(Y)|^2 - (\langle \nabla u(Y), e_{n+1} \rangle)^2 \\ = (|\nabla u(Y)| - \langle \nabla u(Y), e_{n+1} \rangle)(|\nabla u(Y)| + \langle \nabla u(Y), e_{n+1} \rangle) \lesssim \varepsilon^{2M-5},$$

by (4.31) and (4.17). \square

Now for $Y = \delta(Y)e_{n+1} \in U_Q^i$ fixed as above, we consider another point $X \in \widetilde{U}_Q^i$. We form a polygonal path in \widetilde{U}_Q^i , joining Y to X , with vertices

$$Y_0 := Y, Y_1, Y_2, \dots, Y_N := X,$$

such that $Y_{k+1} \in B_{Y_k} \cap B(Y_k, \ell(Q))$, $0 \leq k \leq N-1$, and such that the distance between consecutive vertices is comparable to $\ell(Q)$, and therefore the total length of the path is on the order of $N\ell(Q)$. Let us note that we may take $N \lesssim \varepsilon^{-4}$, by (4.24). We further note that by (4.19) and (4.35),

$$(4.36) \quad \left| \nabla u(W) - |\nabla u(Y)|e_{n+1} \right| \\ \leq |\nabla u(W) - \nabla u(Y)| + \left| \nabla u(Y) - |\nabla u(Y)|e_{n+1} \right| \\ \lesssim \varepsilon^{2M} + \varepsilon^{M-3} \lesssim \varepsilon^{M-3}, \quad \forall W \in B_Z, \forall Z \in \widetilde{U}_Q^i.$$

Claim 4.37. Assume $M > 7$. Then for each $k = 1, 2, \dots, N$,

$$(4.38) \quad |u(Y_k) - |\nabla u(Y)|\langle Y_k, e_{n+1} \rangle| \lesssim k \varepsilon^{M-3} \ell(Q).$$

Moreover,

$$(4.39) \quad |u(W) - |\nabla u(Y)|W_{n+1}| \lesssim \varepsilon^{M-7} \ell(Q), \quad \forall W \in B_X, \forall X \in \widetilde{U}_Q^i.$$

Proof of Claim 4.37. By (4.32) and (4.35), we have

$$(4.40) \quad \begin{aligned} |u(W) - |\nabla u(Y)|W_{n+1}| &\lesssim |u(W) - \nabla u(Y) \cdot W| + |(|\nabla u(Y) - |\nabla u(Y)|e_{n+1}|) \cdot W| \\ &\lesssim \varepsilon^{2M-5} \delta(Y) + \varepsilon^{M-3} |W| \lesssim \varepsilon^{M-3} \ell(Q), \quad \forall W \in B_Y, \end{aligned}$$

since $\delta(Z) \approx \ell(Q)$, for all $Z \in U_Q^i$ (so in particular, for $Z = Y$), and since $|W| \leq 2\delta(Y) \lesssim \ell(Q)$, for all $W \in B_Y$. Thus, (4.38) holds with $k = 1$, since $Y_1 \in B_Y$, by construction.

Now suppose that (4.38) holds for all $1 \leq i \leq k$, with $k \leq N$. Let $W \in B_{Y_k}$, so that W may be joined to Y_k by a line segment of length less than $\delta(Y_k) \lesssim \varepsilon^{-3} \ell(Q)$ (the latter bound holds by (4.23)). We note also that if $k \leq N - 1$, and if $W = Y_{k+1}$, then this line segment has length at most $\ell(Q)$, by construction. Then

$$\begin{aligned} &|u(W) - |\nabla u(Y)|W_{n+1}| \\ &\leq |u(W) - u(Y_k) + |\nabla u(Y)|\langle Y_k - W, e_{n+1} \rangle| + |u(Y_k) - |\nabla u(Y)|\langle Y_k, e_{n+1} \rangle| \\ &= |(W - Y_k) \cdot \nabla u(W_1) + |\nabla u(Y)|\langle Y_k - W, e_{n+1} \rangle| + O(k \varepsilon^{M-3} \ell(Q)), \end{aligned}$$

where W_1 is an appropriate point on the line segment joining W and Y_k , and where we have used that Y_k satisfies (4.38). By (4.36), applied to W_1 , we find in turn that

$$(4.41) \quad |u(W) - |\nabla u(Y)|W_{n+1}| \lesssim \varepsilon^{M-3} |W - Y_k| + k \varepsilon^{M-3} \ell(Q),$$

which, by our previous observations, is bounded by $C(k+1)\varepsilon^{M-3} \ell(Q)$, if $W = Y_{k+1}$, or by $(\varepsilon^{M-6} + k \varepsilon^{M-3})\ell(Q)$, in general. In the former case, we find that (4.38) holds for all $k = 1, 2, \dots, N$, and in the latter case, taking $k = N \lesssim \varepsilon^{-4}$, we obtain (4.39). \square

Claim 4.42. Let $X \in \widetilde{U}_Q^i$, and let $w \in \partial\Omega \cap \partial\widetilde{B}_X$. Then

$$(4.43) \quad |\nabla u(Y)|w_{n+1}| \lesssim \varepsilon^{M/2} \ell(Q).$$

Proof of Claim 4.42. Let W be the radial projection of w onto ∂B_X , so that

$$(4.44) \quad |W - w| = \varepsilon^{2M/\alpha} \delta(X) \lesssim \varepsilon^{(2M/\alpha)-3} \ell(Q),$$

by (4.23). We write

$$\begin{aligned} |\nabla u(Y)|w_{n+1}| &\leq |\nabla u(Y)|W_{n+1}| + |u(W) - |\nabla u(Y)|W_{n+1}| + u(W) \\ &=: I + II + u(W). \end{aligned}$$

Note that $I \lesssim \varepsilon^{(2M/\alpha)-3} \ell(Q)$, by (4.44) and (4.17) (recall that $Y = Y_Q$), and that $II \lesssim \varepsilon^{M-7} \ell(Q)$, by (4.39). Furthermore, $u(W) \lesssim \varepsilon^{2M-5} \ell(Q)$, by (4.22). For M chosen large enough, we obtain (4.43). \square

We note that since we have fixed $Y = Y_Q$, it then follows from (4.43) and (4.17) that

$$(4.45) \quad |w_{n+1}| \lesssim \varepsilon^{M/2} \ell(Q), \quad \forall w \in \partial\Omega \cap \partial\widetilde{B}_X, \quad \forall X \in \widetilde{U}_Q^i.$$

Recall now that x_Q denotes the “center” of Q (see (2.6)-(2.7)). Set

$$(4.46) \quad O := B(x_Q, 2\varepsilon^{-2}\ell(Q)) \cap \{W : W_{n+1} > \varepsilon^2\ell(Q)\}.$$

Claim 4.47. *For every point $X \in O$, we have $X \approx_{\varepsilon, Q} Y$ (see Definition 2.22). Thus, in particular, $O \subset \widetilde{U}_Q^i$.*

Proof of Claim 4.47. Let $X \in O$. We need to show that X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3}\ell(Q)$ (for convenience, we shall refer to such balls as “admissible”). We first observe that if $X = te_{n+1}$, with $\varepsilon^3\ell(Q) \leq t \leq \varepsilon^{-3}\ell(Q)$, then by an iteration argument using (4.45) (with M chosen large enough), we may join X to Y by at most $C \log(1/\varepsilon)$ admissible balls. The point $(2\varepsilon)^{-3}\ell(Q)e_{n+1}$ may then be joined to any point of the form $(X', (2\varepsilon)^{-3}\ell(Q))$ by a chain of at most C admissible balls, whenever $X' \in \mathbb{R}^n$ with $|X'| \leq \varepsilon^{-3}\ell(Q)$. In turn, the latter point may then be joined to $(X', \varepsilon^3\ell(Q))$. \square

We note that Claim 4.47 implies that

$$(4.48) \quad \partial\Omega \cap O = \emptyset.$$

Indeed, $O \subset \widetilde{U}_Q^i \subset \Omega$.

Let P_0 denote the hyperplane

$$P_0 := \{Z : Z_{n+1} = 0\}.$$

Claim 4.49. *If $Z \in P_0$, with $|Z - x_Q| \leq \varepsilon^{-2}\ell(Q)$, then*

$$(4.50) \quad \delta(Z) = \text{dist}(Z, \partial\Omega) \leq 16\varepsilon^2\ell(Q).$$

Proof of Claim 4.49. Observe that $B(Z, 2\varepsilon^2\ell(Q))$ meets O . Then by Claim 4.47, there is a point $X \in \widetilde{U}_Q^i \cap B(Z, 2\varepsilon^2\ell(Q))$. Suppose now that (4.50) is false, so that $\delta(X) \geq 14\varepsilon^2\ell(Q)$. Then $B(Z, 4\varepsilon^2\ell(Q)) \subset B_X$, so by (4.39), we have

$$(4.51) \quad |u(W) - |\nabla u(Y)|W_{n+1}| \leq C\varepsilon^{M-7}\ell(Q), \quad \forall W \in B(Z, 4\varepsilon^2\ell(Q)).$$

In particular, since $Z_{n+1} = 0$, we may choose W such that $W_{n+1} = -\varepsilon^2\ell(Q)$, to obtain that

$$|\nabla u(Y)|\varepsilon^2\ell(Q) \leq C\varepsilon^{M-7}\ell(Q),$$

since $u \geq 0$. But for $\varepsilon < 1/2$, and M large enough, this is a contradiction, by (4.17) (recall that we have fixed $Y = Y_Q$). \square

It now follows by Definition 2.15 that Q satisfies the ε -local WHSA condition, with

$$P = P(Q) := \{Z : Z_{n+1} = \varepsilon^2\ell(Q)\}, \quad H = H(Q) := \{Z : Z_{n+1} > \varepsilon^2\ell(Q)\}.$$

This concludes the proof of Lemma 4.16, and therefore also that of Theorem 1.1, modulo Proposition 1.7. \square

5. WHSA IMPLIES UR: PROOF OF PROPOSITION 1.7

We suppose that E satisfies the WHSA property. Given a positive $\varepsilon < \varepsilon_0 \ll K_0^{-6}$, we let \mathcal{B}_0 denote the collection of bad cubes for which ε -local WHSA fails. By Definition 2.17, \mathcal{B}_0 satisfies the Carleson packing condition (2.18). We now introduce a variant of the packing measure for \mathcal{B}_0 . We recall that $B_Q^* = B(x_Q, K_0^2 \ell(Q))$, and given $Q \in \mathbb{D}(E)$, we set

$$(5.1) \quad \mathbb{D}_\varepsilon(Q) := \{Q' \in \mathbb{D}(E) : \varepsilon^{3/2} \ell(Q) \leq \ell(Q') \leq \ell(Q), Q' \text{ meets } B_Q^*\}.$$

Set

$$(5.2) \quad \alpha_Q := \begin{cases} \sigma(Q), & \text{if } \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and define

$$(5.3) \quad \mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q, \quad \mathbb{D}' \subset \mathbb{D}(E).$$

Then \mathfrak{m} is a discrete Carleson measure, with

$$(5.4) \quad \mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \leq C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E).$$

Indeed, note that for any Q' , the cardinality of $\{Q : Q' \in \mathbb{D}_\varepsilon(Q)\}$, is uniformly bounded, depending on n, ε and ADR , and that $\sigma(Q) \leq C_\varepsilon \sigma(Q')$, if $Q' \in \mathbb{D}_\varepsilon(Q)$. Then given any $Q_0 \in \mathbb{D}(E)$,

$$\begin{aligned} \mathfrak{m}(\mathbb{D}_{Q_0}) &= \sum_{Q \subset Q_0 : \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset} \sigma(Q) \leq \sum_{Q' \in \mathcal{B}_0} \sum_{Q \subset Q_0 : Q' \in \mathbb{D}_\varepsilon(Q)} \sigma(Q) \\ &\leq C_\varepsilon \sum_{Q' \in \mathcal{B}_0 : Q' \subset 2B_{Q_0}^*} \sigma(Q') \leq C_\varepsilon \sigma(Q_0), \end{aligned}$$

by (2.18) and ADR .

To prove Proposition 1.7, we are required to show that the collection \mathcal{B} of bad cubes for which the $\sqrt{\varepsilon}$ -local BAUP condition fails, satisfies a packing condition. That is, we shall establish the discrete Carleson measure estimate

$$(5.5) \quad \tilde{\mathfrak{m}}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0 : Q \in \mathcal{B}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E).$$

To this end, by (5.4), it suffices to show that if $Q \in \mathcal{B}$, then $\alpha_Q \neq 0$ (and thus $\alpha_Q = \sigma(Q)$, by definition). In fact, we shall prove the contrapositive statement.

Claim 5.6. *Suppose then that $\alpha_Q = 0$. Then $\sqrt{\varepsilon}$ -local BAUP condition holds for Q .*

Proof of Claim 5.6. We first note that since $\alpha_Q = 0$, then by definition of α_Q ,

$$(5.7) \quad \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) = \emptyset.$$

Thus, the ε -local WWSA condition (Definition 2.15) holds for every $Q' \in \mathbb{D}_\varepsilon(Q)$ (in particular, for Q itself). By rotation and translation, we may suppose that the hyperplane $P = P(Q)$ in Definition 2.15 is

$$P = \{Z \in \mathbb{R}^{n+1} : Z_{n+1} = 0\},$$

and that the half-space $H = H(Q)$ is the upper half-space $\mathbb{R}_+^{n+1} = \{Z : Z_{n+1} > 0\}$. We recall that by Definition 2.15, P and H satisfy

$$(5.8) \quad \text{dist}(Z, E) \leq \varepsilon \ell(Q), \quad \forall Z \in P \cap B_Q^{**}(\varepsilon).$$

$$(5.9) \quad \text{dist}(P, Q) \leq K_0^{3/2} \ell(Q),$$

and

$$(5.10) \quad H \cap B_Q^{**}(\varepsilon) \cap E = \emptyset.$$

The proof will now follow a similar construction in [LV], which was used to establish the Weak Exterior Convexity condition. By (5.10), there are two cases.

Case 1: $10Q \subset \{Z : -\sqrt{\varepsilon} \ell(Q) \leq Z_{n+1} \leq 0\}$. In this case, the $\sqrt{\varepsilon}$ -local BAUP condition holds trivially for Q , with $\mathcal{P} = \{P\}$.

Case 2. There is a point $x \in 10Q$ such that $x_{n+1} < -\sqrt{\varepsilon} \ell(Q)$. In this case, we choose $Q' \ni x$, with $\varepsilon^{3/4} \ell(Q) \leq \ell(Q') < 2\varepsilon^{3/4} \ell(Q)$. Thus,

$$(5.11) \quad Q' \subset \{Z : Z_{n+1} \leq -\frac{1}{2} \sqrt{\varepsilon} \ell(Q)\}.$$

Moreover, $Q' \in \mathbb{D}_\varepsilon(Q)$, so by (5.7), $Q' \notin \mathcal{B}_0$, i.e., Q' satisfies the ε -local WWSA. Let $P' = P(Q')$, and $H' = H(Q')$ denote the hyperplane and half-space corresponding to Q' in Definition 2.15, so that

$$(5.12) \quad \text{dist}(Z, E) \leq \varepsilon \ell(Q') \leq 2\varepsilon^{7/4} \ell(Q), \quad \forall Z \in P' \cap B_{Q'}^{**}(\varepsilon),$$

$$(5.13) \quad \text{dist}(P', Q') \leq K_0^{3/2} \ell(Q') \approx K_0^{3/2} \varepsilon^{3/4} \ell(Q) \ll \varepsilon^{1/2} \ell(Q)$$

(where the last inequality holds since $\varepsilon \ll K_0^{-6}$), and

$$(5.14) \quad H' \cap B_{Q'}^{**}(\varepsilon) \cap E = \emptyset,$$

where we recall that $B_{Q'}^{**}(\varepsilon) := B(x_{Q'}, \varepsilon^{-2} \ell(Q'))$ (see (2.14)). We note that

$$(5.15) \quad B_Q^* \subset \widetilde{B}_Q(\varepsilon) := B(x_Q, \varepsilon^{-1} \ell(Q)) \subset B_{Q'}^{**}(\varepsilon) \cap B_Q^{**}(\varepsilon),$$

by construction, since $\varepsilon \ll K_0^{-6}$. Let ν' denote the unit normal vector to P' , pointing into H' . Then ν' points “downward”, i.e., $\nu' \cdot e_{n+1} < 0$, otherwise $H' \cap \widetilde{B}_Q(\varepsilon)$ would meet E , by (5.8), (5.11), and (5.13). Moreover, by (5.10), (5.12), and the definition of H ,

$$(5.16) \quad P' \cap \widetilde{B}_Q(\varepsilon) \cap \{Z : Z_{n+1} > 2\varepsilon^{7/4} \ell(Q)\} = \emptyset.$$

Consequently,

Claim 5.17. *The angle θ between ν' and $-e_{n+1}$ satisfies $\theta \approx \sin \theta \lesssim \varepsilon$.*

Indeed, since Q' meets $10Q$, (5.9) and (5.13) imply that $\text{dist}(P, P') \lesssim K_0^{3/2} \ell(Q)$, and that the latter estimate is attained near Q . By (5.16) and a trigonometric argument, one then obtains Claim 5.17 (more precisely, one obtains $\theta \lesssim K_0^{3/2} \varepsilon$, but in this section, we continue to use the notational convention that implicit constants may depend upon K_0 , but K_0 is fixed, and $\varepsilon \ll K_0^{-6}$). The interested reader could probably supply the remaining details of the argument that we have just sketched, but for the sake of completeness, we shall give the full proof at the end of this section.

We therefore take Claim 5.17 for granted, and proceed with the argument. We note first that every point in $(P \cup P') \cap B_Q^*$ is at a distance at most $\varepsilon \ell(Q)$ from E , by (5.8), (5.12) and (5.15). To complete the proof of Claim 5.6, it therefore remains only to verify the following. As with the previous claim, we shall provide a condensed proof immediately, and present a more detailed argument at the end of the section.

Claim 5.18. *Every point in $10Q$ lies within $\sqrt{\varepsilon} \ell(Q)$ of a point in $P \cup P'$.*

Suppose not. We could then repeat the previous argument, to construct a cube Q'' , a hyperplane P'' , a unit vector v'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'' , with the same properties as Q' , P' , v' and H' . In particular, we have the respective analogues of (5.13), (5.11), and (5.14), namely

$$(5.19) \quad \text{dist}(P'', Q'') \leq K_0^{3/2} \ell(Q') \approx K_0^{3/2} \varepsilon^{3/4} \ell(Q) \ll \varepsilon^{1/2} \ell(Q),$$

$$(5.20) \quad \text{dist}(Q'', P') \geq \frac{1}{2} \sqrt{\varepsilon} \ell(Q), \quad \text{and} \quad Q'' \cap H' = \emptyset,$$

and

$$(5.21) \quad H'' \cap B_{Q''}^{**}(\varepsilon) \cap E = \emptyset,$$

In addition, as in (5.15), we also have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$. On the other hand, the angle between v' and v'' is very small. Thus, combining (5.12), (5.19) and (5.20), we see that $H'' \cap B_Q^*$ captures points in E , which contradicts (5.21).

Claim (5.6) therefore holds (in fact, with a union of at most 2 planes), and thus we obtain the conclusion of Proposition 1.7. \square

We now provide detailed proofs of Claims 5.17 and 5.18.

Proof of Claim 5.17. By (5.13) we can pick $x' \in Q'$, $y' \in P'$ such that $|y' - x'| \ll \varepsilon^{1/2} \ell(Q)$ and therefore $y' \in 11Q$. Also, from (5.9) and (5.10) we can find $\bar{x} \in Q$ such that $-K_0^{3/2} \ell(Q) < \bar{x}_{n+1} \leq 0$. This and (5.11) yield

$$(5.22) \quad -2K_0^{3/2} \ell(Q) < y'_{n+1} < -\frac{1}{4} \sqrt{\varepsilon} \ell(Q)$$

Write π to denote the orthogonal projection onto P . Let $Z \in P$ (i.e., $Z_{n+1} = 0$) be such that $|Z - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Then, $Z \in B(x_Q, 3K_0^{3/2} \ell(Q)) \subset B_Q^*$. Hence $Z \in P \cap B_Q^{**}(\varepsilon)$ and by (5.8), $\text{dist}(Z, E) \leq \varepsilon \ell(Q)$. Then there exists $x_Z \in E$ with $|Z - x_Z| \leq \varepsilon \ell(Q)$ which in turn implies that $|(x_Z)_{n+1}| \leq \varepsilon \ell(Q)$. Note that $x_Z \in B(x_Q, 4K_0^{3/2} \ell(Q)) \subset B_Q^*$ and by (5.15) $x_Z \in E \cap B_Q^{**}(\varepsilon) \cap B_{Q'}^{**}(\varepsilon)$. This, (5.10) and

(5.14) imply that $x_Z \notin H \cup H'$. Hence, $(x_Z)_{n+1} \leq 0$ and $(x_Z - y') \cdot \nu' \leq 0$, since $y' \in P'$ and ν' denote the unit normal vector to P' pointing into H' . Observe that by (5.22)

$$(5.23) \quad \frac{1}{8} \sqrt{\varepsilon} \ell(Q) < -\varepsilon \ell(Q) + \frac{1}{4} \sqrt{\varepsilon} \ell(Q) < (x_Z - y')_{n+1} < 2 K_0^{3/2} \ell(Q)$$

and

$$(5.24) \quad (x_Z - y')_{n+1} \nu'_{n+1} \leq -\pi(x_Z - y') \cdot \pi(\nu') \\ \leq |x_Z - z| - \pi(Z - y') \cdot \pi(\nu') \leq \varepsilon \ell(Q) - \pi(Z - y') \cdot \pi(\nu')$$

and that,

We prove that $\nu'_{n+1} < -\frac{1}{8} < 0$ considering two cases:

Case 1: $|\pi(\nu')| \geq \frac{1}{2}$.

We pick

$$Z_1 = \pi(y') + K_0^{3/2} \ell(Q) \frac{\pi(\nu')}{|\pi(\nu')|}.$$

By construction $Z_1 \in P$ and $|Z_1 - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Hence we can use (5.24)

$$(x_{Z_1} - y')_{n+1} \nu'_{n+1} \leq \varepsilon \ell(Q) - \pi(Z_1 - y') \cdot \pi(\nu') \\ = \varepsilon \ell(Q) - K_0^{3/2} \ell(Q) |\pi(\nu')| \leq -\frac{1}{4} K_0^{3/2} \ell(Q).$$

This together with (5.23) give that $\nu'_{n+1} < -1/8 < 0$.

Case 2: $|\pi(\nu')| < \frac{1}{2}$.

This case is much simpler. Note first that $|\nu'_{n+1}|^2 = 1 - |\pi(\nu')|^2 > 3/4$ and thus either $\nu'_{n+1} < -\sqrt{3}/2$ or $\nu'_{n+1} > \sqrt{3}/2$. We see that the second scenario leads to a contradiction. Assume then that $\nu'_{n+1} > \sqrt{3}/2$. We take $Z_2 = \pi(y') \in P$ which clearly satisfies and $|Z_2 - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Again (5.24) and (5.23) are applicable

$$\frac{1}{8} \sqrt{\varepsilon} \ell(Q) \frac{\sqrt{3}}{2} < (x_Z - y')_{n+1} \nu'_{n+1} \leq \varepsilon \ell(Q) \ll \sqrt{\varepsilon} \ell(Q),$$

and we get a contradiction. Hence necessarily $\nu'_{n+1} \leq -\sqrt{3}/2 < -1/8 < 0$.

Once we know that $\nu'_{n+1} < -1/8 < 0$ we estimate θ the angle between ν' and $-e_{n+1}$. Note first $\cos \theta = -\nu'_{n+1} > 1/8$. If $\cos \theta = 1$ (which occurs if $\nu' = -e_{n+1}$) then $\theta = \sin \theta = 0$ and the proof is complete. Assume then that $\cos \theta \neq 1$ in which case $1/8 < -\nu'_{n+1} < 1$ and hence $|\pi(\nu')| \neq 0$. Pick

$$Z_3 = y' + \frac{\ell(Q)}{2\varepsilon} \hat{\nu}', \quad \hat{\nu}' = \frac{e_{n+1} - \nu'_{n+1} \nu'}{|\pi(\nu')|}.$$

Note that $\hat{\nu}' \cdot \nu' = 0$ and then $Z_3 \in P'$ since $y' \in P'$. Also $|\hat{\nu}'| = 1$ and therefore $|Z_3 - y'| = \ell(Q)/(2\varepsilon)$. This in turn gives that $Z_3 \in \widetilde{B}_Q(\varepsilon)$. We have then obtained that $Z_3 \in P' \cap \widetilde{B}_Q(\varepsilon)$ and hence $(Z_3)_{n+1} \leq 2\varepsilon^{7/4} \ell(Q)$ by (5.16). This and (5.23) easily give

$$4 K_0^{3/2} \ell(Q) \geq 2\varepsilon^{7/4} \ell(Q) \geq (Z_3)_{n+1} = y'_{n+1} + \frac{\ell(Q)}{2\varepsilon} \frac{1 - (\nu'_{n+1})^2}{|\pi(\nu')|}$$

$$= y'_{n+1} + \frac{\ell(Q)}{2\varepsilon} |\pi(v')| \geq -2K_0^{3/2} \ell(Q) + \frac{\ell(Q)}{2\varepsilon} |\pi(v')|.$$

This readily yields $|\sin \theta| = |\pi(v')| \leq 8K_0^{3/2} \varepsilon$ and the proof is then complete. \square

Proof of Claim 5.18. We seek to show that every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$. Suppose not and we will get a contradiction.

Assume then that there is $x' \in 10Q$ with $\text{dist}(x', P \cup P') > \varepsilon \ell(Q)$. In particular Then $x'_{n+1} < -\sqrt{\varepsilon}\ell(Q)$ and we repeat the previous argument, to construct a cube Q'' , a hyperplane P'' , a unit vector v'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'' , with the same properties as Q' , P' , v' and H' . In particular, we have the respective analogues of (5.13) and (5.14), namely

$$(5.25) \quad \text{dist}(P'', Q'') \leq K_0^{3/2} \ell(Q') \approx K_0^{3/2} \varepsilon^{3/4} \ell(Q) \ll \varepsilon^{1/2} \ell(Q),$$

and

$$(5.26) \quad H'' \cap B_{Q''}^{**}(\varepsilon) \cap E = \emptyset,$$

Also,

$$\varepsilon \ell(Q) \leq \text{dist}(x', P') \leq \text{diam}(Q'') + \text{dist}(Q'', P') \leq \frac{1}{2} \varepsilon \ell(Q) + \text{dist}(Q'', P'),$$

and, by (5.14),

$$(5.27) \quad \text{dist}(Q'', P') \geq \frac{1}{2} \sqrt{\varepsilon} \ell(Q), \quad \text{and} \quad Q'' \cap H' = \emptyset,$$

In addition, as in (5.15), we also have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$.

By (5.25) there is $y'' \in Q''$ and $z'' \in P''$ such that $|y'' - z''| \ll \varepsilon^{1/2} \ell(Q)$. By (5.26) $y'' \notin H'$. Write π' to denote the orthogonal projection onto P' and note that (5.27) give $\text{dist}(y'', P') = |y'' - \pi'(y'')| \geq \frac{1}{2} \sqrt{\varepsilon} \ell(Q)$. Note also that

$$|y'' - \pi'(y'')| = \text{dist}(y'', P') \leq |y'' - x'| + |x' - x| + \text{diam}(Q') + \text{dist}(Q', P') \leq 11 \text{diam}(Q)$$

and

$$|\pi'(y'') - x_Q| \leq |\pi'(y'') - y''| + |y'' - x'| + |x' - x_Q| < 22 \text{diam}(Q) < K^2 \ell(Q).$$

Hence $\pi'(y'') \in B_Q^* \subset \widetilde{B}_Q(\varepsilon)$ and since $\pi'(y'') \in P'$ we have that (5.12) give that there is $\tilde{y} \in E$ with $|\pi'(y'') - \tilde{y}| \leq 2\varepsilon^{7/4} \ell(Q)$. Then $\tilde{y} \in 23Q \subset B_Q^* \cap E$ and $|\tilde{y} - z''| < 12 \text{diam}(Q)$. To complete our proof we just need to show that $\tilde{y} \in H''$ which gets into contradiction with (5.26).

Let us then proof that $\tilde{y} \in H''$. Write v'' to denote the unit normal vector to P'' , pointing into H'' and let us claim that

$$(5.28) \quad |v' - v''| \leq 16 \sqrt{2} K_0^{2/3} \varepsilon.$$

Assuming this momentarily and recalling that $y'' \notin H'$ we obtain. Then,

$$\begin{aligned} \frac{1}{2} \sqrt{\varepsilon} \ell(Q) &\leq |y'' - \pi'(y'')| = (\pi(y'') - y'') \cdot v' \\ &\leq |\pi(y'') - \tilde{y}| + |\tilde{y} - z''| |v' - v''| + (\tilde{y} - z'') \cdot v'' + |z'' - y''| \\ &< \frac{1}{4} \sqrt{\varepsilon} \ell(Q) + (\tilde{y} - z'') \cdot v''. \end{aligned}$$

This immediately gives that $(\tilde{y} - z'') \cdot v'' > \frac{1}{4} \sqrt{\varepsilon} \ell(Q) > 0$ and hence $y'' \in H''$ as desired.

To complete the proof we obtain (5.28). Note first that if $|\alpha| < \pi/4$ then

$$1 - \cos \alpha = 1 - \sqrt{1 - \sin^2 \alpha} \leq \sin^2 \alpha.$$

In particular we can apply this to θ (resp. θ') which is the angle between v' (resp. v'') and $-e_{n+1}$ since $|\sin \theta|, |\sin \theta'| \leq 8 K_0^{3/2} \varepsilon$:

$$\sqrt{1 - \cos \theta} + \sqrt{1 - \cos \theta'} \leq 16 K_0^{3/2} \varepsilon$$

Using the trivial formula

$$|a - b|^2 = 2(1 - ab), \quad \forall, a, b \in \mathbb{R}^{n+1}, |a| = |b| = 1.$$

we conclude that

$$\begin{aligned} |v' - v''| &\leq |v' - (-e_{n+1})| + |(-e_{n+1}) - v''| = \sqrt{2(1 + v' e_{n+1})} + \sqrt{2(1 + v'' e_{n+1})} \\ &= \sqrt{2(1 - \cos \theta)} + \sqrt{2(1 - \cos \theta')} \leq 16 \sqrt{2} K_0^{3/2} \varepsilon. \end{aligned}$$

This proves (5.28) and this completes the proof. \square

6. PROOF OF COROLLARY 1.3

This result will follow almost immediately from Theorem 1.1. Let $B = B(x, r)$ and $\Delta = B \cap E$, with $x \in E$ and $0 < r < \text{diam}(E)$. Let c be the constant in Lemma 2.25. By the ADR property of E , there is a point $Y_\Delta \in B' := B(x, cr)$, which is a Corkscrew point relative to the surface ball $\Delta' := B' \cap E$, and therefore also a Corkscrew point relative to Δ , albeit with slightly different Corkscrew constants now depending also on c , that is, there is c_1 such that $\delta(Y_\Delta) \geq c_1 r$. Note that Y_Δ satisfies the conditions in Theorem 1.1 and in order to apply that result we need to check the validity of (a) and (b). That (a) holds is an immediate consequence of Lemma 2.25. Let us now prove (b). Given C_1 a large enough constant, write $\hat{\Delta} = C_1 \Delta$. Cover $\hat{\Delta}$ by a collection of surface balls $\{\Delta_i\}_{i=1}^N$ with $\Delta_i = B_i \cap E = B(x_i, c_1 r/4)$, $x_i \in \Delta$ and where N is uniformly bounded. By construction $Y_\Delta \in \Omega \setminus 4B_i$ and by hypothesis $\omega^{Y_\Delta} \in \text{weak-}A_\infty(2\Delta_i)$. Hence $\omega^{Y_\Delta} \ll \sigma$ in $2\Delta_i$ and (1.4) holds with Y_Δ in place of Y and with $\Delta' = \Delta_i$. Then, we clearly have that $\omega^{Y_\Delta} \ll \sigma$ in $\hat{\Delta}$ and if we write $k^{Y_\Delta} = d\omega^{Y_\Delta}/d\sigma$ we obtain

$$\begin{aligned} \int_{\hat{\Delta}} k^{Y_\Delta}(z)^p d\sigma(z) &\leq \sum_{i=1}^N \int_{\Delta_i} k^{Y_\Delta}(z)^p d\sigma(z) \lesssim \sum_{i=1}^N \sigma(\Delta_i) \left(\int_{2\Delta_i} k^{Y_\Delta}(z) d\sigma(z) \right)^p \\ &\lesssim \sum_{i=1}^N \sigma(2\Delta_i)^{1-p} \omega^{Y_\Delta}(2\Delta_i) \lesssim \sigma(\hat{\Delta})^{1-p}, \end{aligned}$$

where in the last estimate we have used that $\omega(E)^{Y_\Delta} \lesssim 1$, the ADR property and that N is uniformly bounded. This gives (b) in Theorem 1.1, which in turn can be applied to obtain that E is UR as desired. \square

REFERENCES

- [AC] H. Alt and L. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* **325** (1981), 105-144. [5](#)
- [AHLMcT] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, The solution of the Kato Square Root Problem for Second Order Elliptic operators on \mathbb{R}^n , *Annals of Math.* **156** (2002), 633-654. [4](#), [11](#)
- [AHLT] P. Auscher, S. Hofmann, J.L. Lewis and P. Tchamitchian, Extrapolation of Carleson measures and the analyticity of Kato's square-root operators, *Acta Math.* **187** (2001), no. 2, 161-190.
- [AHMTT] P. Auscher, S. Hofmann, C. Muscalu, T. Tao and C. Thiele, Carleson measures, trees, extrapolation, and $T(b)$ theorems, *Publ. Mat.* **46** (2002), no. 2, 257-325. [4](#)
- [AHMNT] J. Azzam, S. Hofmann, J. M. Martell, K. Nyström, T. Toro, A new characterization of chord-arc domains, to appear, *Journ. Euro. Math. Soc.* [2](#)
- [BH] S. Bortz and S. Hofmann, Harmonic measure and approximation of uniformly rectifiable sets, preprint. [3](#)
- [Bo] J. Bourgain, On the Hausdorff dimension of harmonic measure in higher dimensions, *Invent. Math.* **87** (1987), 477-483. [2](#), [8](#), [9](#)
- [CFMS] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ. Math. J.* **30** (1981), no. 4, 621-640. [9](#), [10](#)
- [Car] L. Carleson, Interpolation by bounded analytic functions and the corona problem, *Ann. of Math. (2)* **76** (1962), 547-559.
- [CG] L. Carleson and J. Garnett, Interpolating sequences and separation properties, *J. Analyse Math.* **28** (1975), 273-299.
- [Ch] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.*, **LX/LXI** (1990), 601-628. [3](#), [4](#), [6](#)
- [DS1] G. David and S. Semmes, Singular integrals and rectifiable sets in \mathbb{R}^n : Beyond Lipschitz graphs, *Asterisque* **193** (1991). [2](#), [3](#), [5](#), [6](#)
- [DS2] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Monographs and Surveys **38**, AMS 1993. [2](#), [3](#), [4](#), [5](#), [6](#), [7](#)
- [H] S. Hofmann, Local Tb Theorems and applications in PDE, *Proceedings of the ICM Madrid*, Vol. **II**, pp. 1375-1392, European Math. Soc., 2006. [4](#)
- [HLMc] S. Hofmann, M. Lacey and A. McIntosh, The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds, *Annals of Math.* **156** (2002), pp 623-631. [4](#), [11](#)
- [HL] S. Hofmann and J.L. Lewis, The Dirichlet problem for parabolic operators with singular drift terms, *Mem. Amer. Math. Soc.* **151** (2001), no. 719.
- [HM1] S. Hofmann and J.M. Martell, A_∞ estimates via extrapolation of Carleson measures and applications to divergence form elliptic operators, *Trans. Amer. Math. Soc.* **364** (2012), no. 1, 65-101
- [HM2] S. Hofmann and J.M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p , to appear, *Ann. Sci. École Norm. Sup.* **47** (2014), no. 3, 577-654. [16](#)
- [HMM] S. Hofmann, J.-M. Martell and S. Mayboroda, Uniform Rectifiability, Carleson measure estimates, and approximation of harmonic functions, preprint. [17](#)
- [HMU] S. Hofmann, J. M. Martell and I. Uriarte-Tuero, Uniform Rectifiability and Harmonic Measure II: Poisson kernels in L^p imply uniform rectifiability, *Duke Math. J.* **163** (2014), 1601-1654. [2](#)
- [HMT] S. Hofmann, J.M. Martell and T. Toro, General divergence form elliptic operators on domains with ADR boundaries, and on 1-sided NTA domains, in progress. [9](#)
- [HMc] S. Hofmann and A. McIntosh, The solution of the Kato problem in two dimensions, Proceedings of the Conference on Harmonic Analysis and PDE held in El Escorial, Spain in July 2000, *Publ. Mat.* Vol. extra, 2002 pp. 143-160. [4](#), [11](#)

- [HMMM] S. Hofmann, D. Mitrea, M. Mitrea, A. Morris, L^p -Square Function Estimates on Spaces of Homogeneous Type and on Uniformly Rectifiable Sets, preprint. [6](#)
- [Je] D. Jerison, Regularity of the Poisson kernel and free boundary problems, *Colloquium Mathematicum* LX/LXI (1990) 547-567. [5](#)
- [JK] D. Jerison and C. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), no. 1, 80–147. [2](#)
- [Ke] C.E. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, *CBMS Regional Conference Series in Mathematics*, **83**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994. [10](#)
- [KT] C. Kenig and T. Toro, Poisson kernel characterizations of Reifenberg flat chord arc domains, *Ann. Sci. École Norm. Sup. (4)* **36** (2003), no. 3, 323–401. [4](#), [5](#)
- [LM] J. Lewis and M. Murray, The method of layer potentials for the heat equation in time-varying domains, *Mem. Amer. Math. Soc.* **114** (1995), no. 545.
- [LV] J. L. Lewis and A. Vogel, Symmetry theorems and uniform rectifiability, *Boundary Value Problems* Vol. 2007 (2007), article ID 030190, 59 pages. [4](#), [5](#), [14](#), [18](#), [24](#)
- [MMV] P. Mattila, M. Melnikov and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, *Ann. of Math. (2)* **144** (1996), no. 1, 127–136. [5](#)
- [NToV] F. Nazarov, X. Tolsa, and A. Volberg, On the uniform rectifiability of ad-regular measures with bounded Riesz transform operator: The case of codimension 1, *Acta Math.*, to appear. [5](#)

STEVE HOFMANN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

Email address: hofmanns@missouri.edu

JOSÉ MARÍA MARTELL, INSTITUTO DE CIENCIAS MATEMÁTICAS CSIC-UAM-UC3M-UCM, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, C/ NICOLÁS CABRERA, 13-15, E-28049 MADRID, SPAIN

Email address: chema.martell@icmat.es