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THE WEAK-A_∞ PROPERTY OF HARMONIC AND p-HARMONIC MEASURES IMPLIES UNIFORM RECTIFIABILITY.

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ABSTRACT. Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors-David regular set of dimension n. We show that the weak-A_∞ property of harmonic measure, for the open set $\Omega := \mathbb{R}^{n+1} \setminus E$, implies uniform rectifiability of E. More generally, we establish a similar result for the Riesz measure, p-harmonic measure, associated to the p-Laplace operator, $1 < p < \infty$.

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1. Introduction

In this paper we prove quantitative, scale invariant results of free boundary type, for harmonic measure and, more generally, for p-harmonic measure. More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (not necessarily connected nor bounded) satisfying an interior Corkscrew condition, whose boundary is n-dimensional Ahlfors-David regular (ADR) (see Definition 2.1). Given these background hypotheses we prove, see Theorem 1.1 and Corollary 1.5 below, that if ω, the harmonic measure for Ω, is absolutely continuous with respect to σ, and if the Poisson kernel $k = d\omega/d\sigma$ verifies an appropriate scale invariant higher integrability estimate (in particular, if ω belongs to weak-A_∞ with respect to σ), then $\partial\Omega$ is uniformly rectifiable in the sense of [DS1, DS2]. In particular, our background hypotheses hold in the case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set of co-dimension 1, as in that case it is well known that the Corkscrew condition is verified automatically in Ω, i.e., in every ball $B = B(x, r)$ centered on E, there is some component of $\Omega \cap B$ that contains a point Y with $\operatorname{dist}(Y, E) \approx r$. Furthermore, our argument is general enough to allow us to establish a non-linear version of Theorem 1.1, see Theorem 1.12 below, involving the p-Laplace operator, p-harmonic functions and p-harmonic measure.

To briefly outline previous work, in [HMU] the first and third authors, together with I. Uriarte-Tuero, proved the same result (cf. Theorem 1.1 and Corollary 1.5) under the additional strong hypothesis that Ω is a connected domain, satisfying an interior Harnack Chain condition. In hindsight, under that extra assumption, one obtains the stronger conclusion that the exterior domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ in fact also satisfies a Corkscrew condition, and hence that Ω is an NTA domain in the sense of [JK], see [AHMNT] for the details. Compared to [HMU] the main new advances in the present paper are two. First, the removal of any connectivity hypothesis, in particular, we avoid the Harnack Chain condition. Second, we are able to establish a version of our results also in the non-linear case $1 < p < \infty$. Our main results, Theorem 1.1, Corollary 1.5 and Theorem 1.12, are new even in the linear case $p = 2$.

Our approach is decidedly influenced by prior work of Lewis and Vogel [LV1], [LV2]. In particular, in [LV2] the authors proved a version of Theorem 1.12, and Theorem 1.1, under the stronger hypothesis that p-harmonic measure μ itself is an Ahlfors-David regular measure which in the linear case $p = 2$ implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. However, to weaken the hypotheses on ω and μ, as we have done here, requires further considerations that we discuss below in Subsection 1.2.

To provide some additional context, we mention that out results here may be viewed as "large constant" analogues of results of Kenig and Toro [KT], in the linear case $p = 2$, and of J. Lewis and the fourth named author of the present paper [LN], in the general p-harmonic case $1 < p < \infty$. These authors show that in the presence of a Reifenberg flatness condition and Ahlfors-David regularity, $\log k \in VMO$ implies that the unit normal ν to the boundary belongs to VMO, where k is the Poisson kernel with pole at some fixed point (resp., the density of p-harmonic Riesz measure associated to a particular ball $B(x, r)$). Moreover, under the same background hypotheses, the condition that $\nu \in VMO$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace, thus $\log k \in VMO \implies$ *vanishing UR*, given sufficient Reifenberg flatness. On the other hand, our large constant version "almost" says "$\log k \in BMO \implies UR$". Indeed, it is well known that the A_∞ condition, i.e., weak-A_∞ plus the doubling property, implies that $\log k \in BMO$, while if $\log k \in BMO$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [KT] may be viewed as an "endpoint" version of the free boundary results of [AC] and [Je], which say, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal ν (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

1.1. **Statement of main results.** Given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a Euclidean ball $B = B(x,r) \subset \mathbb{R}^{n+1}$, centered on $\partial\Omega$, we let $\Delta = \Delta(x,r) := B \cap \partial\Omega$ denote the corresponding surface ball. For $X \in \Omega$, let ω^X be harmonic measure for Ω, with pole at X. As mentioned above, all other terminology and notation will be defined below.

Concerning the Laplace operator and harmonic measure we prove the following results.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set, whose boundary is Ahlfors-David regular of dimension n (see Definition 2.1). Suppose that there are positive constants C_0 and c_0, and an exponent $q > 1$, such that for every surface ball $\Delta = \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there exists $X_\Delta \in B(x,r) \cap \Omega$, with $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$, satisfying*

($\star$) ***Scale-invariant higher integrability:*** *$\omega^{X_\Delta} \ll \sigma$ in 2Δ, and $k^{X_\Delta} := \frac{d\omega^{X_\Delta}}{d\sigma}$ satisfies*

$$\int_{2\Delta} k^{X_\Delta}(y)^q \, d\sigma(y) \leq C_0 \, \sigma(\Delta)^{1-q} . \tag{1.2}$$

Then $\partial\Omega$ is uniformly rectifiable and moreover the "UR character" (see Definition 2.4) depends only on n, the ADR constants, q, c_0, and C_0.

We note that the point X_Δ in Theorem 1.1 is a "Corkscrew point" for Ω, relative to Δ. An open set Ω for which there is such a point relative to every surface ball $\Delta(x,r)$, $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, with a uniform constant c_0, is said to satisfy the "Corkscrew condition" (see Definition 2.5 below).

Remark 1.3. We note that, in lieu of absolute continuity and ($\star$), only the following apparently weaker condition is actually used in the proof of Theorem 1.1.

($\star\star$) **Local non-degeneracy:** there exist uniform constants $\eta, \beta > 0$, such that if $A \subset \Delta$ is Borel measurable, then

$$\sigma(A) \geq (1-\eta)\,\sigma(\Delta) \quad \implies \quad \omega^{X_\Delta}(A) \geq \beta\, \omega^{X_\Delta}(\Delta).\text{[1]} \tag{1.4}$$

Here, $\Delta = \Delta(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $X_\Delta \in B(x,r/2) \cap \Omega$, with $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r/2$.[2] We observe that there turns out to be some flexibility in the

[1] This formulation is adapted from [MT]; see the discussion in Subsection 1.4 below.

[2] For aesthetic reasons, and for convenience in the sequel, in contrast to condition ($\star$), we prefer to state condition ($\star\star$) in terms of Δ rather than 2Δ, and with $X_\Delta \in B(x,r/2)$ rather than $B(x,r)$.

choice of X_Δ (see the discussion at the beginning of Section 4), and consequently it is not hard to see that ($\star$) implies ($\star\star$); see Lemma 4.3.

We also have the following easy corollary of Theorem 1.1 (we shall give the short proof of the corollary in Section 5.4).

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set, satisfying the Corkscrew condition, whose boundary is Ahlfors-David regular of dimension n. Suppose further that for every ball $B = B(x,r)$, $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and for all $Y \in \Omega \setminus B(x, 2r)$, harmonic measure $\omega^Y \in$ weak-$A_\infty(\Delta(x,r))$, i.e., there is a constant $C_0 \geq 1$, and an exponent $q > 1$, each of which is uniform with respect to x, r and Y, such that $\omega^Y \ll \sigma$ in $\Delta(x,r)$, and $k^Y = d\omega^Y/d\sigma$ satisfies*

$$\left(\fint_{\Delta'} k^Y(z)^q \, d\sigma(z)\right)^{\frac{1}{q}} \leq C_0 \fint_{2\Delta'} k^Y(z) \, d\sigma(z), \tag{1.6}$$

for every surface ball centered on the boundary $\Delta' = B' \cap \partial\Omega$ with $2B' \subset B(x,r)$. Then $\partial\Omega$ is uniformly rectifiable and moreover the "UR character", see Definition 2.4, depends only on n, the ADR constant of $\partial\Omega$, q, C_0, and the Corkscrew constant.

Remark 1.7. As mentioned above, the Corkscrew condition is automatically satisfied in the case that E is an n-dimensional ADR set (hence closed, see Definition 2.1 below), and $\Omega = \mathbb{R}^{n+1} \setminus E$ is its complement, with the Corkscrew constant for Ω depending only on n and the ADR constant of E. Thus, in particular, Corollary 1.5 applies in that setting, so in the presence of the weak reverse Hölder condition (1.6), we deduce that E is uniformly rectifiable.

Combining Theorem 1.1 with the results in [BH], we obtain as an immediate consequence a "big pieces" characterization of uniformly rectifiable sets of co-dimension 1, in terms of harmonic measure. Here and in the sequel, given an ADR set E, Q will denote a "dyadic cube" on E in the sense of [DS1, DS2] and [Ch], and $\mathbb{D}(E)$ will denote the collection of all such cubes, see Lemma 2.6 below.

Theorem 1.8. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an n-dimensional ADR set. Let $\Omega := \mathbb{R}^{n+1} \setminus E$. Then E is uniformly rectifiable if and only if it has "big pieces of good harmonic measure estimates" in the following sense: for each $Q \in \mathbb{D}(E)$ there exists an open set $\widetilde{\Omega} = \widetilde{\Omega}_Q$ with the following properties, with uniform control of the various implicit constants:*

- *$\partial\widetilde{\Omega}$ is ADR;*
- *the interior Corkscrew condition holds in $\widetilde{\Omega}$;*
- *$\partial\widetilde{\Omega}$ has a "big pieces" overlap with E, in the sense that $\sigma(Q \cap \partial\widetilde{\Omega}) \gtrsim \sigma(Q)$;*
- *for each surface ball $\Delta = \Delta(x,r) := B(x,r) \cap \partial\widetilde{\Omega}$, with $x \in \partial\widetilde{\Omega}$ and $r \in (0, \operatorname{diam}(\widetilde{\Omega}))$; there is an interior corkscrew point $X_\Delta \in \widetilde{\Omega}$, such that $\omega_{\widetilde{\Omega}}^{X_\Delta}$, the harmonic measure for $\widetilde{\Omega}$ with pole at X_Δ, satisfies $\omega_{\widetilde{\Omega}}^{X_\Delta}(\Delta) \gtrsim 1$, and belongs to weak-$A_\infty(\Delta)$.*

The "only if" direction is proved in [BH], and the open sets $\widetilde{\Omega}$ constructed in [BH] even satisfy a 2-sided Corkscrew condition, and moreover, $\widetilde{\Omega} \subset \Omega$, with $\operatorname{diam}(\widetilde{\Omega}) \approx \operatorname{diam}(Q)$. To obtain the converse direction, we simply observe that by Theorem 1.1, the subdomains $\widetilde{\Omega}$ have uniformly rectifiable boundaries, with uniform control of the "UR character" of each $\partial\widetilde{\Omega}$, and thus, by [DS2], E is uniformly rectifiable.

To formulate our main result in the non-linear setting we first need to introduce some notation. If $O \subset \mathbb{R}^{n+1}$ is an open set and $1 \leq p \leq \infty$, then by $W^{1,p}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_{n+1}})$, both of which are q th power integrable on O. Let $\|f\|_{1,p} = \|f\|_p + \big\| |\nabla f| \big\|_p$ be the norm in $W^{1,p}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue p norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,p}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,p}(O)$. We let $W^{1,p}_{\rm loc}(O)$ be the set of all functions u such that $u\Theta \in W_0^{1,p}(O)$ whenever $\Theta \in C_0^\infty(O)$.

Given an open set O, and $1 < p < \infty$, we say that u is p-harmonic in O provided $u \in W^{1,p}_{\rm loc}(O)$ and

$$\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \Theta \, dX = 0, \qquad \forall\, \Theta \in C_0^\infty(O)\,. \tag{1.9}$$

Observe that if u is smooth and $\nabla u \neq 0$ in O, then

$$\nabla \cdot (|\nabla u|^{p-2}\, \nabla u) \equiv 0 \quad \text{in} \quad O, \tag{1.10}$$

and u is a classical solution in O to the p-Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with n-dimensional ADR boundary. Let $p \in (1, \infty)$. Given $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$, let u be a non-negative p-harmonic function in $\Omega \cap B(x,r)$ which vanishes continuously on $\Delta(x,r) := B(x,r) \cap \partial\Omega$. Extend u to all of $B(x,r)$ by putting $u \equiv 0$ on $B(x,r) \setminus \overline{\Omega}$. Then there exists (see [HKM, Chapter 21] and Lemma 3.43 below), a unique non-negative finite Borel measure μ on $\mathbb{R}^{n+1}$, with support contained in $\Delta(x,r)$, such that

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dX = \int_{\partial\Omega} \phi \, d\mu \,, \forall\, \phi \in C_0^\infty(B(x,r)). \tag{1.11}$$

We refer to μ as the p-harmonic measure associated to u. In the case $p = 2$, and if u is the Green function for Ω with pole at $X \in \Omega$, then the measure μ coincides with harmonic measure at X, $\omega = \omega^X$.

Concerning the p-Laplace operator, p-harmonic functions and p-harmonic measure we prove the following theorem.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set, whose boundary is Ahlfors-David regular of dimension n. Let p, $1 < p < \infty$, be given. Let C be a sufficiently large constant (to be specified), depending only on n and the ADR constant, and Suppose that there exist $q > 1$, and a positive constant C_0, for which the following holds: for each $x \in \partial\Omega$ and each $0 < r < \operatorname{diam}(\partial\Omega)$, there is a non-trivial,* *non-negative p-harmonic function $u = u_{x,r}$ in $\Omega \cap B(x, Cr)$, and corresponding p-harmonic measure $\mu = \mu_{x,r}$, such that $\mu \ll \sigma$ in $\Delta(x, Cr)$, and such that $k := d\mu/d\sigma$ satisfies*

$$\left(\fint_{\Delta(x,Cr)} k(y)^q \, d\sigma(y)\right)^{1/q} \leq C_0 \frac{\mu(\Delta(x,r))}{\sigma(\Delta(x,r))}. \tag{1.13}$$

Then $\partial\Omega$ is uniformly rectifiable, and moreover the "UR character", see Definition 2.4, depends only on n, the ADR constant, p, q and C_0.

Some remarks are in order concerning the hypotheses of Theorem 1.12. Let us observe that, in particular, Ahlfors-David regularity and (1.13) imply that

$$\mu(\Delta(x, Cr)) \leq C_1 \, \mu(\Delta(x,r)), \tag{1.14}$$

with $C_1 \approx C_0$. In the linear case, the latter estimate will follow automatically, with $\mu = \omega^Y$, for some $Y \in B(x,r)$ such that $\operatorname{dist}(Y, E) \approx r$, and with C_1 depending only on n and the ADR constant, by Bourgain's Lemma 3.1 below, even though ω^Y need not be a doubling measure (i.e., (1.14) says nothing about points other than x nor about scales other than r). In the non-linear case, it seems that we must impose condition (1.14) by hypothesis. We also observe that (1.13) holds in particular if $\mu \in$ weak-$A_\infty(\Delta(x, 2Cr))$ and satisfies (1.14) (with radius $2C$ in place of C). Of course, (1.14) holds trivially if μ is a doubling measure, but we do not assume doubling.

Remark 1.15. We note that, as in Remark 1.3, the proof of Theorem 1.12 will in fact use, in lieu of absolute continuity and (1.13), only the apparently weaker condition that there exist uniform constants $\eta, \beta \in (0,1)$ such that for all $\Delta = \Delta(x,r)$, and for all Borel sets $A \subset \Delta$,

$$\sigma(A) \geq (1-\eta)\,\sigma(\Delta) \quad \Longrightarrow \quad \mu(A) \geq \beta\,\mu(\Delta). \tag{1.16}$$

1.2. **Brief outline of the proofs of the main results.** As mentioned, the approach in the present paper is strongly influenced by prior work due to Lewis and Vogel [LV1], [LV2], who in [LV2] proved a version of Theorem 1.12, and Theorem 1.1, under the stronger hypothesis that p-harmonic measure μ itself is an Ahlfors-David regular measure. In the linear case $p = 2$, this implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. Assuming that p-harmonic measure μ is an Ahlfors-David regular measure, Lewis and Vogel were able to show that E satisfies the so-called Weak Exterior Convexity (WEC) condition, which characterizes uniform rectifiability [DS2]. To weaken the hypotheses on ω and μ, as we have done here, requires two further considerations. The first is quite natural in this context: a stopping time argument, in the spirit of the proofs of the Kato square root conjecture [HMc], [HLMc], [AHLMcT] (and of local Tb theorems [Ch], [AHMTT], [Ho]), by means of which we extract ample dyadic sawtooth regimes on which averages of harmonic measure and p-harmonic measure are bounded and accretive, see Lemma 4.12 below. This allows us to use the arguments of [LV2] within these good sawtooth regions. The second new consideration is necessitated by the fact that in our setting, the doubling property may fail for harmonic and p-harmonic measure. In the absence of doubling, we are unable to obtain the WEC condition directly. Nonetheless, we are able to follow the arguments of [LV2] very closely up to a point, to obtain a condition on $\partial\Omega$ which we call the "Weak Half Space Approximation" (WHSA) property (see Definition 2.19). Indeed, extracting the essence of the [LV2] argument, while dispensing with the doubling property, one realizes that the WHSA is precisely what one obtains. In the sequel, we present the argument of [LV2] as Lemma 5.10. Finally, having obtained that $\partial\Omega$ satisfies the WHSA property, we are able prove the following proposition stating that WHSA implies uniform rectifiability.

Proposition 1.17. *An n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable if and only if it satisfies the WHSA property.*

While the WHSA condition, per se, is new, our proof of Proposition 1.17 is based on a modified version of part of the argument in [LV2].

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we state several definitions, including definitions of ADR, UR, and dyadic grids, and introduce further notions and notation. In Section 3, we state, and either prove, or give references for, the PDE estimates needed in the proofs of our main results. In Section 4, we begin the (simultaneous) proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. In Section 5, following [LV1], [LV2], we complete the proofs of Theorem 1.1 and Theorem 1.12, modulo Proposition 1.17. At the end of Section 5 we also give the (very short) proof of Corollary 1.5. In Section 6, we give the proof of Proposition 1.17, i.e., the proof of the fact that the WHSA condition implies uniform rectifiability.

1.4. **Discussion of recent related work.** We note that some related work has recently appeared, or been carried out, while this manuscript was in preparation. In the setting of uniform domains with lower ADR boundary with locally finite n-dimensional Hausdorff measure Mourgoglou [Mo] has shown that rectifiability of the boundary implies absolute continuity of surface measure with respect to harmonic measure (for the Laplacian). Akman, Badger and the first and third authors of the present paper [ABHM], in the setting of uniform domains with ADR boundary, have characterized the rectifiability of the boundary in terms of the absolute continuity of harmonic measure and some elliptic measures and surface measure or in terms of some qualitative A_∞ condition. Also, Azzam, Mourgoglou and Tolsa [AMT] have obtained that absolute continuity of harmonic measure with respect to surface measure on a H^n-finite piece of the boundary implies that harmonic measure is rectifiable in that piece. The setting is very general as they only assume a "porosity" (i.e. Corkscrew) condition in the complement of $\partial\Omega$. In [HMMTV], Mayboroda, Tolsa, Volberg and the first and third authors of the present paper , the same result is proved removing the porosity assumption. Both [AMT] and the follow-up version [HMMTV] (which will be combined in the forthcoming paper [AHMMMTV]) rely on recent deep results of [NToV1], [NToV2], concerning connections between rectifiability and the behavior of Riesz transforms.

Finally, we discuss two closely related papers treating the case $p = 2$. First, we mention that a preliminary version of our results, treating only the linear harmonic case (i.e., Theorem 1.1 of the present paper) under hypothesis ($\star$), appeared earlier in the unpublished preprint [HM2]. The result of [HM2], again in the case $p = 2$, was then essentially reproved, by a different method, in the work of Mourgoglou and Tolsa [MT], but assuming condition ($\star\star$) in place of ($\star$). While the present paper was in preparation, we learned of the work in [MT], and we realized that our arguments (and those of [HM2]), almost unchanged, also allow ($\star$) to be replaced by ($\star\star$) or its p-harmonic equivalent. The current version of this manuscript incorporates this observation.[3] Let us mention also that the approach in [MT] is based on showing that ($\star\star$) for harmonic measure implies L^2 boundedness of the Riesz transforms, and thus it is a quantitative version of the method of [AHMMMTV]. An interesting feature of the proof in [MT], is that it works even without the lower bound in the Ahlfors-David condition; in that case, one may deduce rectifiability, as opposed to uniform rectifiability, of the underlying measure on $\partial\Omega$. On the other hand, it seems difficult to generalize the approach of [MT] to the p-Laplace setting, since it is based on Riesz transforms, which are tied to the linear harmonic case.

2. ADR, UR, and dyadic grids

Definition 2.1. (**ADR**) (aka *Ahlfors-David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed, and if there is some uniform constant C such that

$$C^{-1}\, r^n \le \sigma(\Delta(x,r)) \le C\, r^n, \qquad \forall\, r \in (0, \operatorname{diam}(E)), \quad x \in E, \tag{2.2}$$

where diam(E) may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the "surface ball" of radius r, and $\sigma := H^n|_E$ is the "surface measure" on E, where H^n denotes n-dimensional Hausdorff measure.

Definition 2.3. (**UR**) (aka *uniformly rectifiable*). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants θ and M_0, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$H^n\Big(E \cap B(x,r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \ge \theta\, r^n .$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [DS1]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [MMV] for the case $n = 1$, and [NToV1] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [DS2, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions);

[3]We thank the authors of [MT] for making their preprint available to us, while our manuscript was in preparation.

see [DS1, DS2], and in particular Theorem 2.14 below. In this paper, we shall also present a new characterization of UR sets of co-dimension 1 (see Proposition 1.17 below), which will be very useful in the proof of Theorem 1.1.

Definition 2.4. (**UR character**). Given a UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants (θ, M_0) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 2.5. (**Corkscrew condition**). Following [JK], we say that an opent set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "Corkscrew condition" if for some uniform constant $c_0 > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a point $X_\Delta \in B(x, r) \cap \Omega$ such that $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$. The point $X_\Delta \subset \Omega$ is called a "Corkscrew point" relative to Δ.

Lemma 2.6. *(***Existence and properties of the "dyadic grid"***)* [DS1, DS2], [Ch]. *Suppose that $E \subset \mathbb{R}^{n+1}$ is closed n-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$ and $C_* < \infty$, depending only on dimension and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(*i*) $E = \cup_j Q_j^k$ *for each $k \in \mathbb{Z}$.*

(*ii*) *If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.*

(*iii*) *For each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.*

(*iv*) $\operatorname{diam}(Q_j^k) \leq C_* 2^{-k}$.

(*v*) *Each Q_j^k contains some "surface ball" $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.*

(*vi*) $H^n(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \varrho\, 2^{-k}\}) \leq C_*\, \varrho^\gamma\, H^n(Q_j^k)$, *for all k, j and for all $\varrho \in (0, a_0)$.*

Let us make a few remarks are concerning this lemma, and discuss some related notation and terminology.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Ch], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property (2.2), the result already appears in [DS1, DS2].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D} := \cup_k \mathbb{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr), \tag{2.7}$$

for some uniform constant C. We shall denote this ball and surface ball by

$$B_Q := B(x_Q, r)\,, \qquad \Delta_Q := \Delta(x_Q, r), \tag{2.8}$$

and we shall refer to the point x_Q as the "center" of Q.

- Given a dyadic cube $Q \in \mathbb{D}$, we define its "κ-dilate" by

$$\kappa Q := E \cap B\left(x_Q, \kappa \operatorname{diam}(Q)\right). \tag{2.9}$$

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Clearly, $\ell(Q) \approx \operatorname{diam}(Q)$.

- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

- For any $Q \in \mathbb{D}(E)$, we set $\mathbb{D}_Q := \{Q' \in \mathbb{D} : \; Q' \subset Q\}$.

- Given $Q_0 \in \mathbb{D}(E)$ and a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$ of pairwise disjoint cubes, we set

$$\mathbb{D}_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{Q_0} : \; Q \text{ is not contained in any } Q_j \in \mathcal{F}\} = \mathbb{D}_{Q_0} \setminus \Big(\bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} \Big). \tag{2.10}$$

Definition 2.11. (**ε-local BAUP**) Given $\varepsilon > 0$, we shall say that $Q \in \mathbb{D}(E)$ satisfies the *ε-local BAUP* condition if there is a family $\mathcal{P}$ of hyperplanes (depending on Q) such that every point in $10Q$ is at a distance at most $\varepsilon\ell(Q)$ from $\cup_{P \in \mathcal{P}} P$, and every point in $(\cup_{P \in \mathcal{P}} P) \cap B(x_Q, 10 \operatorname{diam}(Q))$ is at a distance at most $\varepsilon\ell(Q)$ from E.

Definition 2.12. (**BAUP**). We shall say that an n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the condition of *Bilateral Approximation by Unions of Planes* ("BAUP"), if for some $\varepsilon_0 > 0$, and for every positive $\varepsilon < \varepsilon_0$, there is a constant C_ε such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the ε-local BAUP condition fails, satisfies the packing condition

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') \leq C_\varepsilon\, \sigma(Q)\,, \qquad \forall\, Q \in \mathbb{D}(E)\,. \tag{2.13}$$

For future reference, we recall the following result of David and Semmes [DS2], see [DS2, Theorem I.2.18, p. 36].

Theorem 2.14. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ADR set. Then, E is uniformly rectifiable if and only if it satisfies BAUP.*

We remark that the definition of BAUP in [DS2] is slightly different in superficial appearance, but it is not hard to verify that the dyadic version stated here is equivalent to the condition in [DS2]. We note that we shall not need the full strength of this equivalence here, but only the fact that our version of BAUP implies the version in [DS2], and hence implies UR.

We shall also require a new characterization of UR sets of co-dimension 1, which is related to the BAUP and its variants. For a sufficiently large constant K_0 to be chosen (see Lemma 4.24 below), we set

$$B_Q^* := B(x_Q, K_0^2 \ell(Q)), \qquad \Delta_Q^* := B_Q^* \cap E. \tag{2.15}$$

Given a small positive number ε, which we shall typically assume to be much smaller than K_0^{-6}, we also set

$$B_Q^{**} = B_Q^{**}(\varepsilon) := B(x_Q, \varepsilon^{-2}\ell(Q)), \quad B_Q^{***} = B_Q^{***}(\varepsilon) := B(x_Q, \varepsilon^{-5}\ell(Q)). \tag{2.16}$$

Definition 2.17. (**ε-local WHSA**) Given $\varepsilon > 0$, we shall say that $Q \in \mathbb{D}(E)$ satisfies the *ε-local WHSA* condition (or more precisely, the "ε-local WHSA with parameter K_0") if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number K_0 satisfying

(1) $\operatorname{dist}(Z, E) \le \varepsilon\ell(Q)$, for every $Z \in P \cap B_Q^{**}(\varepsilon)$.
(2) $\operatorname{dist}(Q, P) \le K_0^{3/2}\ell(Q)$.
(3) $H \cap B_Q^{**}(\varepsilon) \cap E = \emptyset$.

Note that part (2) of the previous definition says that the hyperplane P has an "ample" intersection with the ball $B_Q^{**}(\varepsilon)$. Indeed,

$$\operatorname{dist}(x_Q, P) \lesssim K_0^{\frac{3}{2}}\,\ell(Q) \ll \varepsilon^{-2}\ell(Q). \tag{2.18}$$

Definition 2.19. (**WHSA**) We shall say that an n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the *Weak Half-Space Approximation* property ("WHSA") if for some pair of positive constants ε_0 and K_0, and for every positive $\varepsilon < \varepsilon_0$, there is a constant C_ε such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the ε-local WHSA condition with parameter K_0 fails, satisfies the packing condition

$$\sum_{Q\subset Q_0,\, Q\in\mathcal{B}} \sigma(Q) \le C_\varepsilon\, \sigma(Q_0), \qquad \forall\, Q_0 \in \mathbb{D}(E). \tag{2.20}$$

Next, we develop some further notation and terminology. Given a closed set E, we set $\delta_E(Y) := \operatorname{dist}(Y, E)$, and we shall simply write $\delta(Y)$ when the set has been fixed.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a covering of Ω with non-overlapping interiors, and which satisfy

$$4\operatorname{diam}(I) \le \operatorname{dist}(4I, \partial\Omega) \le \operatorname{dist}(I, \partial\Omega) \le 40\operatorname{diam}(I) \tag{2.21}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2), \text{ whenever } I_1 \text{ and } I_2 \text{ touch.} \tag{2.22}$$

Assuming that $E = \partial\Omega$ is ADR and given $Q \in \mathbb{D}(E)$, for the same constant K_0 as in (2.15), we set

$$\mathcal{W}_Q := \left\{ I \in \mathcal{W} :\ K_0^{-1}\ell(Q) \le \ell(I) \le K_0\,\ell(Q), \text{ and } \operatorname{dist}(I, Q) \le K_0\,\ell(Q) \right\}. \tag{2.23}$$

We fix a small, positive parameter τ, to be chosen momentarily, and given $I \in \mathcal{W}$, we let

$$I^* = I^*(\tau) := (1+\tau)I \tag{2.24}$$

denote the corresponding "fattened" Whitney cube. We now choose τ sufficiently small that the cubes I^* will retain the usual properties of Whitney cubes, in particular that

$$\operatorname{diam}(I) \approx \operatorname{diam}(I^*) \approx \operatorname{dist}(I^*, E) \approx \operatorname{dist}(I, E)\,.$$

We then define Whitney regions with respect to Q by setting

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*\,. \tag{2.25}$$

We observe that these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon K_0 and dimension. We enumerate the components of U_Q as $\{U_Q^i\}_i$.

Moreover, we enlarge the Whitney regions as follows.

Definition 2.26. For $\varepsilon > 0$, and given $Q \in \mathbb{D}(E)$, we write $X \approx_{\varepsilon,Q} Y$ if X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3 \ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3}\ell(Q)$. Given a sufficiently small parameter $\varepsilon > 0$, we then set

$$\widetilde{U}_Q^i := \left\{X \in \mathbb{R}^{n+1} \setminus E : \, X \approx_{\varepsilon,Q} Y\,, \text{ for some } Y \in U_Q^i\right\}\,. \tag{2.27}$$

Remark 2.28. Since $\widetilde{U}_Q^i$ is an enlarged version of U_Q, it may be that there are some $i \neq j$ for which $\widetilde{U}_Q^i$ meets $\widetilde{U}_Q^j$. This overlap will be harmless.

3. PDE estimates

In this section we recall several estimates for harmonic measure and harmonic functions, and also for p-harmonic measure and p-harmonic functions. Although some of the PDE results in the harmonic case $p = 2$ can be subsumed into the general p-harmonic theory, we choose to present some aspects of the harmonic theory separately, in part for the convenience of those readers who are more familiar with the case $p = 2$, and in part because the presence of the Green function is unique to that case.

3.1. **PDE estimates: the harmonic case.**

Next, we recall several facts concerning harmonic measure and Green's functions. Let Ω be an open set, not necessarily connected, and set $\delta(X) = \delta_{\partial\Omega}(X) = \operatorname{dist}(X, \partial\Omega)$.

Lemma 3.1 (Bourgain [Bo])**.** *Suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0,1)$ and $C \in (1,\infty)$, depending only on n and ADR, such that for every $x \in \partial\Omega$, and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$, then*

$$\omega^Y(\Delta(x,r)) \geq 1/C > 0\,. \tag{3.2}$$

We refer the reader to [Bo, Lemma 1] for the proof. We note for future reference that in particular, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$, and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$\omega^X(\Delta_X) \geq 1/C\,. \tag{3.3}$$

Indeed, the latter bound follows immediately from (3.2), and the fact that we can form a Harnack Chain connecting X to a point Y that lies on the line segment from X to $\hat{x}$, and satisfies $|Y-\hat{x}| = c\delta(X)$.

A proof of the next lemma may be found, e.g., in [HMT]. We note that, in particular, the ADR hypothesis implies that $\partial\Omega$ is Wiener regular at every point (see Lemma 3.27 below).

Lemma 3.4. *Let Ω be an open set with n-dimensional ADR boundary. There are positive, finite constants C, depending only on dimension and c_θ, depending on dimension, and $\theta \in (0,1)$, such that the Green function satisfies*

$$G(X,Y) \le C\,|X-Y|^{1-n} \tag{3.5}$$

$$c_\theta\,|X-Y|^{1-n} \le G(X,Y)\,, \quad \text{if } |X-Y| \le \theta\,\delta(X)\,,\ \theta\in(0,1)\,; \tag{3.6}$$

$$G(X,\cdot) \in C(\overline{\Omega}\setminus\{X\}) \qquad \textit{and} \qquad G(X,\cdot)\big|_{\partial\Omega} \equiv 0\,, \qquad \forall X\in\Omega; \tag{3.7}$$

$$G(X,Y) \ge 0\,, \qquad \forall X,Y\in\Omega\,,\ X\neq Y; \tag{3.8}$$

$$G(X,Y) = G(Y,X)\,, \qquad \forall X,Y\in\Omega\,,\ X\neq Y; \tag{3.9}$$

and for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \Phi\, d\omega^X - \Phi(X) = -\iint_\Omega \nabla_Y G(Y,X)\cdot\nabla\Phi(Y)\,dY, \qquad \forall\, X\in\Omega. \tag{3.10}$$

Next we present a version of one of the estimates obtained by Caffarelli-Fabes-Mortola-Salsa in [CFMS], which remains true even in the absence of connectivity:

Lemma 3.11 ("CFMS" estimates)**.** *Suppose that $\partial\Omega$ is n-dimensional ADR. For every $Y\in\Omega$ and $X\in\Omega$ such that $|X-Y|\ge\delta(Y)/2$ we have*

$$\frac{G(Y,X)}{\delta(Y)} \le C\,\frac{\omega^X(\Delta_Y)}{\sigma(\Delta_Y)}, \tag{3.12}$$

where $\Delta_Y = B(\hat{y}, 10\delta(Y))\cap E$, with $\hat{y}\in\partial\Omega$ such that $|Y-\hat{y}| = \delta(Y)$.

For future use, we note that as a consequence of (3.12), it follows directly that for every $Q\in\mathbb{D}(\partial\Omega)$, if $Y\in B(x_Q, C\ell(Q))$, with $\delta(Y)\ge c\ell(Q)$, then there exists $\kappa = \kappa(C,c)$ such that

$$\frac{G(Y,X)}{\ell(Q)} \lesssim \frac{\omega^X(\kappa Q)}{\sigma(Q)} \lesssim \kappa^n \left(\fint_Q \left(\mathcal{M}\omega^X\right)^{1/2} d\sigma\right)^2, \qquad \forall\, X\notin B(x_Q,\kappa\ell(Q))\,, \tag{3.13}$$

where κQ is defined in (2.9), and $\mathcal{M}$ is the usual Hardy-Littlewood maximal operator on $\partial\Omega$.

Proof of Lemma 3.11. We follow the well-known argument of [CFMS] (see also [Ke, Lemma 1.3.3]). Fix $Y\in\Omega$ and write $B^Y = \overline{B(Y,\delta(Y)/2)}$. Consider the open set $\widehat{\Omega} = \Omega\setminus B^Y$ for which clearly $\partial\widehat{\Omega} = \partial\Omega\cup\partial B^Y$. Set

$$u(X) := G(Y,X)/\delta(Y)\,, \qquad v(X) := \omega^X(\Delta_Y)/\sigma(\Delta_Y)\,,$$

for every $X\in\widehat{\Omega}$. Note that both u and v are non-negative harmonic functions in $\widehat{\Omega}$. If $X\in\partial\Omega$ then $u(X) = 0 \le v(X)$. Take now $X\in\partial B^Y$ so that $u(X)\lesssim\delta(Y)^{-n}$

by (3.5). On the other hand, if we fix $X_0 \in \partial B^Y$ with X_0 on the line segment that joints Y and $\hat{y}$, then $2\Delta_{X_0} = \Delta_Y$, so that $v(X_0) \gtrsim \delta(Y)^{-n}$, by (3.3). By Harnack's inequality, we then obtain $v(X) \gtrsim \delta(Y)^{-n}$, for all $X \in \partial B^Y$. Thus, $u \lesssim v$ in $\partial\widehat{\Omega}$ and by the maximum principle this immediately extends to $\widehat{\Omega}$ as desired. □

Lemma 3.14. *Let $\partial\Omega$ be n-dimensional ADR. Let $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and set $\Delta = B \cap \partial\Omega$. There exist constants $\kappa_0 > 2$, $C > 1$, and $M_1 > 1$, depending only on n and the ADR constant of $\partial\Omega$, such that for $X \in \Omega \setminus B(x, \kappa_0 r)$, we have*

$$\sup_{\frac{1}{2}B} G(\cdot, X) \lesssim \frac{1}{|B|} \iint_B G(Y,X)\,dY \le C\,r\,\frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}. \tag{3.15}$$

Moreover, for each $\gamma \in (0,1]$

$$\frac{1}{|B|} \iint_{B\cap\{Y:\,\delta(Y)<\gamma r\}} G(Y,X)\,dY \le C\,\gamma^2 r\,\frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}. \tag{3.16}$$

where C depends on n and the ADR constant of $\partial\Omega$.

We note that in the previous estimates it is implicitly understood that $G(\cdot, X)$ is extended to be 0 outside of Ω.

Proof. Extending $G(\cdot, X)$ to be 0 outside of Ω, we obtain a sub-harmonic function in B. The first inequality in (3.15) follows immediately. The second inequality in (3.15) is just the special case $\gamma = 1$ of (3.16), so it suffices to prove the latter. Set $\Sigma_\gamma = \{I \in \mathcal{W} : I \cap B \neq \emptyset, \operatorname{dist}(I, \partial\Omega) < \gamma r\}$, and note that if $I \in \Sigma_\gamma$ then by (2.21)

$$40^{-1}\operatorname{dist}(I,\partial\Omega) \le \operatorname{diam}(I) \le \operatorname{dist}(I,\partial\Omega) < \gamma r \le r, \qquad \operatorname{dist}(I,x) \le r.$$

In particular, $I \subset B(x, 2r)$. Moreover, we can find κ_0 depending only dimension so that $d(X, 4I) \ge 4r$ for every $I \in \Sigma_\gamma$ and $X \in \Omega \setminus B(x, \kappa_0 r)$. Let $Q_I \in \mathbb{D}$ be so that $\ell(Q_I) = \ell(I)$ and $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, Q_I)$. Then $\ell(Q_I) \le \gamma r$, and $Y(I)$, the center of I, satisfies $Y(I) \in B(x_{Q_I}, C\ell(Q_I))$, and $\delta(Y(I)) \approx \ell(I) = \ell(Q_I)$. Hence we can invoke (3.13) (taking κ_0 larger if needed) and obtain that for every $Y \in I$,

$$G(Y,X) \approx G(Y(I),X) \lesssim \ell(I)\,\frac{\omega^X(\kappa Q_I)}{\sigma(Q_I)},$$

where the first estimate uses Harnack's inequality in $2I \subset \Omega$. Hence,

$$\iint_{B\cap\{Y:\,\delta(Y)<\gamma r\}} G(Y,X)\,dY \le \sum_{I\in\Sigma_\gamma} \iint_I G(Y,X)\,dY \lesssim \sum_{I\in\Sigma_\gamma} \ell(I)^2\,\omega^X(\kappa Q_I)$$
$$\le \sum_{k:2^{-k}\lesssim\gamma r} 2^{-2k} \sum_{I\in\Sigma_\gamma:\ell(I)=2^{-k}} \omega^X(\kappa Q_I) \lesssim (\gamma r)^2\,\omega^X(\Delta(x, M_1 r)),$$

where the last step we have used that for each fixed k, the cubes κQ_I with $\ell(I) = 2^{-k}$ have uniformly bounded overlaps, and are all contained in $\Delta(x, M_1 r)$, for M_1 chosen large enough. Dividing by $|B| \approx r^{n+1}$, and using the ADR property, we obtain the desired estimate. □

3.2. **PDE estimates: the p-harmonic case.** We now recall several fundamental estimates for p-harmonic functions and p-harmonic measure, some of which generalize certain of the preceding estimates that we have stated in the harmonic case. We ask the reader to forgive a moderate amount of redundancy. Given a closed set E, as above we set $\delta(Y) := \mathrm{dist}(Y, E)$.

Lemma 3.17. *Let p, $1 < p < \infty$, be given. Let u be a positive p-harmonic function in $B(X, 2r)$. Then*

$$\left(\frac{1}{|B(X, r/2)|}\iint_{B(X,r/2)} |\nabla u|^p\, dy\right)^{\frac{1}{p}} \leq \frac{C}{r} \max_{B(X,r)} u, \tag{3.18}$$

$$\max_{B(X,r)} u \leq C \min_{B(X,r)} u. \tag{3.19}$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $Y, Y' \in B(X, r)$, then

$$|u(Y) - u(Y')| \leq C\left(\frac{|Y - Y'|}{r}\right)^{\alpha} \max_{B(X,2r)} u. \tag{3.20}$$

Proof. (3.18) is a standard energy estimate. (3.19) is the well known Harnack inequality for positive solutions to the p-Laplace operator. (3.20) is a well known interior Hölder continuity estimate for solutions to equations of p-Laplace type. We refer to [Se] for these results. □

Definition 3.21. Let $O \subset \mathbb{R}^{n+1}$ be open and let K be a compact subset of O. Given p, $1 < p < \infty$, we let

$$\mathrm{Cap}_p(K, O) = \inf\left\{\iint_O |\nabla\phi|^p\, dY : \ \phi \in C_0^\infty(O),\ \phi \geq 1 \text{ in } K\right\}.$$

$\mathrm{Cap}_p(K, O)$ is referred to as the p-capacity of K relative to O. The p-capacity of an arbitrary set $E \subset O$ is defined by

$$\mathrm{Cap}_p(E, O) = \inf_{\substack{E\subset G\subset O\\ G \text{ open}}} \ \sup_{\substack{K\subset G\\ K \text{ compact}}} \mathrm{Cap}_p(K, O). \tag{3.22}$$

Definition 3.23. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \mathrm{diam}(E)$. Given p, $1 < p < \infty$, we say that $E \cap B(x, 4r)$ is p-thick if for every $x \in E \cap B(x, 4r)$ there exists $r_x > 0$ such that

$$\int_0^{r_x} \left[\frac{\mathrm{Cap}_p(E \cap B(x,\rho), B(x, 2\rho))}{\mathrm{Cap}_p(B(x,\rho), B(x, 2\rho))}\right]^{\frac{1}{p-1}} \frac{d\rho}{\rho} = \infty$$

We note that this definition is just the Wiener criterion in the p-harmonic case. As it can be seen in [HKM, Chapter 6] p-thickness implies that all points on $E \cap B(x, 4r)$ are regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$.

Definition 3.24. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \mathrm{diam}(E)$. Given p, $1 < p < \infty$, and $\eta > 0$ we say that $E \cap B(x, 4r)$ is uniformly p-thick with constant η if

$$\frac{\mathrm{Cap}_p(E \cap B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))}{\mathrm{Cap}_p(B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))} \geq \eta, \tag{3.25}$$

whenever $\hat{x} \in E \cap B(x, 4r)$ and $B(\hat{x}, 2\hat{r}) \subset B(x, 4r)$.

Remark 3.26. In the case $p = 2$, the condition defined in Definition 3.24 is sometimes called the Capacity Density Condition (CDC), see for instance [Ai]. Note that uniform p-thickness is a strong quantitative version of the p-thickness defined above and hence of the Wiener regularity for the Laplace and the p-Laplace operator.

Lemma 3.27. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors-David regular of dimension n. Let p, $1 < p < \infty$, be given.Then $E \cap B(x, 4r)$ is uniformly p-thick for some constant η, depending only on p, n, and the ADR constant, whenever $x \in E$, $0 < r < \frac{1}{4} \operatorname{diam} E$.*

Proof. We first observe that since the ADR condition is scale-invariant we may translate and rescale to prove (3.25) only for $\hat{x} = 0$ and $\hat{r} = 1$ (we would also need to rescale E but abusing the notation we call it again E). Write then $B = B(0, 1)$ and observe that, for every $1 < p < \infty$, [HKM, Example 2.12] gives

$$\operatorname{Cap}_p(B, 2B) = C(n, p). \tag{3.28}$$

The desired bound from below follows at once if $p > n + 1$ from the estimate in [HKM, Example 2.12]:

$$\operatorname{Cap}_p(E \cap B, 2B) \geq \operatorname{Cap}_p(\{0\}, 2B) = C(n, p)'.$$

Let us now consider the case $1 < p \leq n + 1$. Write $K = E \cap \overline{\frac{1}{2} B}$. Combining [HKM, Theorem 2.38], [AH, Theorem 2.2.7] and [AH, Theorem 4.5.2] we have that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \widetilde{\operatorname{Cap}}_p(K) \gtrsim \sup_{\mu} \left(\frac{\mu(K)}{\|W_p(\mu)\|_{L^1(\mu)}^{\frac{1}{p'}}} \right)^p. \tag{3.29}$$

In the previous expression the implicit constants depend only on p and n; $\widetilde{\operatorname{Cap}}_p$ stands for the inhomogeneous p-capacity, that is,

$$\widetilde{\operatorname{Cap}}_p(K) = \inf \left\{ \iint_{\mathbb{R}^{n+1}} (|\phi|^p + |\nabla \phi|^p) \, dY : \ \phi \in C_0^\infty(\mathbb{R}), \ \phi \geq 1 \text{ in } K \right\};$$

the sup runs over all Radon positive measures supported on K; and

$$W_p(\mu)(y) := \int_0^1 \left(\frac{\mu(B(y, t))}{t^{n+1-p}} \right)^{p'-1} \frac{dt}{t}, \qquad x \in \operatorname{supp} \mu.$$

We choose $\mu = H^n|_K$ and observe that, if $y \in \operatorname{supp} \mu \subset K \subset E$ and $0 < t < 1$ then $\mu(B(y, t)) = \sigma(B(y, t) \cap B(0, 1/2) \lesssim t^n$ by ADR. This easily gives $W_p(\mu)(y) \lesssim 1$ for every $y \in \operatorname{supp} \mu$ and by ADR

$$\int_K W_p(\mu)(y) \, d\mu(y) \leq \mu(K) \leq \sigma(B) \lesssim 1.$$

We can now use (3.29) and ADR again to conclude that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \mu(K) \geq \sigma(B(0, 1/2))^p \gtrsim 1$$

Combining this with (3.28) we readily obtain (3.25). □

Lemma 3.30. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors-David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in E$ and let $0 < r < \operatorname{diam}(E)$. Then, given $f \in W^{1,p}(B(x,4r))$ there exists a unique p-harmonic function $u \in W^{1,p}(B(x,4r)\setminus E)$ such that $u - f \in W_0^{1,p}(B(x,4r) \setminus E)$. Furthermore, let $u, v \in W_{\mathrm{loc}}^{1,p}(B(x,4r) \setminus E)$ be a p-superharmonic function and a p-subharmonic function in Ω, respectively. If $\inf\{u-v, 0\} \in W_0^{1,p}(B(x,4r)\setminus E)$, then $u \geq v$ a.e in $B(x,4r)\setminus E$. Finally, every point $\hat{x} \in E\cap B(x,4r)$ is regular for the continuous Dirichlet problem for $\nabla\cdot(|\nabla u|^{p-2}\nabla u) = 0$.*

Proof. The first part of the lemma is a standard maximum principle. The fact that every point $\hat{x} \in E \cap B(x,4r)$ is regular in the continuous Dirichlet problem for $\nabla\cdot(|\nabla u|^{p-2}\nabla u) = 0$ follows from the fact that Lemma 3.27 implies that $E\cap B(x,4r)$ is uniformly p-thick for every $1 < p < \infty$ and hence we can invoke [HKM, Chapter 6]. □

Lemma 3.31. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with Ahlfors-David regular of dimension n boundary. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is non-negative and p-harmonic in $B(x,4r)\cap\Omega$, continuous on $B(x,4r) \cap \overline{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x,4r)$. Then, extending u to be 0 in $B(x,4r) \setminus \overline{\Omega}$ there holds*

$$\left(\frac{1}{|B(x,r/2)|}\iint_{B(x,r/2)} |\nabla u|^p\,dy\right)^{\frac{1}{p}} \leq \frac{C}{r}\left(\frac{1}{|B(x,r)|}\iint_{B(x,r)} u^{p-1}\right)^{\frac{1}{p-1}}. \tag{3.32}$$

Furthermore, there exists $\alpha \in (0,1)$, depending only on p, n and the ADR constant, such that if $Y, Y' \in B(x,r)$,

$$|u(Y) - u(Y')| \leq C\left(\frac{|Y-Y'|}{r}\right)^{\alpha} \max_{B(x,2r)} u. \tag{3.33}$$

Proof. Since u, extended as above to all of $B(x,4r)$, is a non-negative p-subsolution in $B(x,4r)$, (3.32) is just a standard energy or Caccioppoli estimate plus a standard interior estimate. Thus, we only prove (3.33). Since $E \cap B(x,4r)$ is uniformly p-thick as seen in Lemma 3.27, we can invoke [HKM, Theorem 6.38] to obtain that there exist $C \geq 1$ and $\alpha = \alpha \in (0,1)$, depending only on n, p, and the ADR constant, such that

$$\max_{B(x,\rho)} u \leq C\left(\frac{\rho}{r}\right)^{\alpha} \max_{B(x,r)} u, \qquad \text{whenever } 0 < \rho \leq r. \tag{3.34}$$

This, the triangle inequality and elementary arguments give (3.33). □

Lemma 3.35. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with Ahlfors-David regular of dimension n boundary. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is non-negative and p-harmonic in $B(x,4r)\cap\Omega$, continuous on $B(x,4r) \cap \overline{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x,4r)$. Then, extending u to be 0 in $B(x,4r) \setminus \overline{\Omega}$, there exists $\alpha > 0$, such that*

$$u(Y) \leq C\left(\frac{\delta(Y)}{r}\right)^{\alpha}\left(\frac{1}{|B(x,2r)|}\iint_{B(x,2r)} u^{p-1}(Z)\,dZ\right)^{\frac{1}{p-1}}, \tag{3.36}$$

for all $Y \in B(x,r)$, where the constants C and α depend only on n, p, and the ADR constant of $\partial\Omega$.

Proof. Follows from Lemma 3.31 and standard estimates for p-subsolutions. Let us note that in the linear case (i.e, $p = 2$) one can give an alternative proof based on Bourgain's Lemma 3.1 and an iteration argument (see [HMT] for details). □

Lemma 3.37. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with Ahlfors-David regular of dimension n boundary. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is non-negative and p-harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \overline{\Omega}$, $u = 0$ on $\partial\Omega \cap B(x, 4r)$, and assume that u is extended to be 0 in $B(x, 4r) \setminus \overline{\Omega}$. Then u has a representative in $W^{1,p}(B(x, 4r))$ with Hölder continuous partial derivatives in $B(x, 4r) \setminus \partial\Omega$. Furthermore, there exists $\beta \in (0, 1]$, such that if $Y, Y' \in B(X, \hat{r}/2)$, with $B(X, 4\hat{r}) \subset B(x, 4r) \setminus \partial\Omega$, then*

$$
(3.38)\qquad |\nabla u(Y) - \nabla u(Y')| \lesssim \left(\frac{|Y - Y'|}{\hat{r}}\right)^{\beta} \max_{B(X,\hat{r})} |\nabla u| \lesssim \frac{1}{\hat{r}} \left(\frac{|Y - Y'|}{\hat{r}}\right)^{\beta} \max_{B(X,2\hat{r})} u\,,
$$

where β and the implicit constants depend only on p and n. Furthermore, if

$$
(3.39)\qquad \frac{u(Y)}{\delta(Y))} \approx |\nabla u(Y)|\,, \qquad Y \in B(X, 3\hat{r})\,,
$$

then u has continuous second derivatives in $B(X, 3\hat{r})$, and there exists $C \geq 1$, depending only on n, p and the implicit constants in (3.39), such that

$$
(3.40)\qquad \max_{B(X,\frac{\hat{r}}{2})} |\nabla^2 u| \leq C \left(\frac{1}{|B(X, \hat{r})|} \iint_{B(X,\hat{r})} |\nabla^2 u(Y)|^2\, dY\right)^{\frac{1}{2}} \leq C^2\, \frac{u(X)}{\delta(X)^2}.
$$

Proof. For (3.38) we refer, for example, to [To]. (3.40) is a consequence of (3.38), (3.39) and Schauder type estimates, see [GT]. For a more detailed proof of (3.40), see [LV1, Lemma 2.4 (d)] for example. □

Remark 3.41. We note that the second inequality in (3.38) and (3.19) give

$$
(3.42)\qquad |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)}\,, \qquad Y \in B(x, 2r) \setminus \partial\Omega.
$$

Lemma 3.43. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors-David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that u is non-negative and p-harmonic in $B(x, 4r) \cap \Omega$, vanishing continuously on $B(x, 4r) \cap \Omega$ (hence u is continuous in $B(x, 4r)$ after being extended by 0 in $B(x, 4r) \setminus \overline{\Omega}$). There exists a unique finite positive Borel measure μ on $\mathbb{R}^{n+1}$, with support in $\partial\Omega \cap B(x, 4r)$, such that*

$$
(3.44)\qquad -\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla\phi\, dY = \int \phi\, d\mu
$$

whenever $\phi \in C_0^\infty(B(x, 4r))$. Furthermore, there exists $C < \infty$, depending only on p, n and the ADR constant, such that

$$
(3.45)\qquad \left(\frac{\max_{B(x,r)} u}{r}\right)^{p-1} \leq C\, \frac{\mu(\Delta(x, 2r))}{\sigma(\Delta(x, 2r))}.
$$

Note that (3.45) is the p-harmonic analogue of Lemma 3.11.

Proof. For the proof of (3.44), see [HKM, Chapter 21]. Using Lemma 3.27 and Lemma 3.31, (3.45) follows directly from [KZ, Lemma 3.1], see also [EL]. □

The following lemma generalizes Lemma 3.14 to the case $1 < p < \infty$.

Lemma 3.46. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors-David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that u and μ are as in Lemma 3.43. Then there exist constants C and M_1, depending only on n and the ADR constant, such that if $B(y, M_1 s) \subset B(x, 2r)$ with $y \in \partial\Omega$, then*

$$\max_{B(y,s/2)} u^{p-1} \lesssim \frac{1}{|B(y,s)|} \iint_{B(y,s)} u^{p-1}(Z)\, dZ \leq C\, s^{p-1}\, \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y,s))}\,.$$

Moreover, for all $\gamma \in (0,1]$,

$$\frac{1}{|B(y,s)|} \iint_{B(y,s)\cap\{Y:\,\delta(Y)\leq\gamma s\}} u^{p-1}(Z)\, dZ \leq C\,\gamma^p\, s^{p-1}\, \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y,s))}\,.$$

We note that in the previous estimates it is implicitly understood that u is extended to be 0 on $B(x,4r) \setminus \overline{\Omega}$

Proof. Using (3.45) the proof of Lemma 3.46 is the same *mutatis mutandi* as that of Lemma 3.14. We omit further details. □

4. Proof of Theorem 1.1 and Theorem 1.12: preliminary arguments

We start the proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. We first show that (1.2) implies (1.4). To this end, we claim that, without loss of generality, we may suppose that for a surface ball $\Delta = \Delta(x,r)$, the point X_Δ in the statement of Theorem 1.1 satisfies (3.2), i.e., there is some $c_1 = c_1(n, ADR) > 0$ such that

$$\omega^{X_\Delta}(\Delta) \geq c_1. \tag{4.1}$$

The only price to be paid is that the constants c_0, C_0 may now be slightly different (depending only on n and ADR), and that (1.2) will now hold with Δ in place of 2Δ, i.e., for the (possibly) new point X_Δ, we shall have

$$\int_\Delta k^{X_\Delta}(y)^q\, d\sigma(y) \leq C_0\, \sigma(\Delta)^{1-q}\,. \tag{4.2}$$

Indeed, set $\Delta' := \Delta(x, r/2)$, and let $X' := X_{\Delta'} \in B(x, r/2) \cap \Omega$ be the point such that (1.2) holds for Δ'. Fix $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$. Suppose first that $\delta(X') \leq r/4$ in which case $\Delta(\hat{x}, r/4) \subset \Delta$. Thus, if in addition $\delta(X') < cr/4$, where $c \in (0,1)$ is the constant in Lemma 3.1, then we set $X_\Delta := X'$, and (4.1) holds by Lemma 3.1. On the other hand, if $cr/4 \leq \delta(X_\Delta) \leq r/4$, we select X_Δ along the line segment joining X' to $\hat{x}$, such that $\delta(X_\Delta) = |X_\Delta - \hat{x}| = cr/8$, and (4.1) holds exactly as before. Moreover, (4.2) holds for this new X_Δ, in the first case, immediately by (1.2) applied to $X' = X_{\Delta'}$, and in the second case, by moving from X' to X_Δ via Harnack's inequality (which may be used within the touching ball $B(X', \delta(X'))$.) Let us finally consider the case $\delta(X') > r/4$. Then we can use Harnack within the ball $B(X', r/4)$, to pass to a point X'', on the line segment joining X' to x such that $|X' - X''| = r/8$, and consequently $\delta(X'') \leq |X'' - x| < 3r/8$ (since $X' \in B(x, r/2)$). Hence (1.2) holds (with different constant) for Δ' with X'' in place of $X_{\Delta'}$. Take now $\hat{x} \in \partial\Omega$ such that $\delta(X'') = |X'' - \hat{x}|$ and note that $\Delta(\hat{x}, r/4) \subset \Delta$. We can now repeat the previous argument with X'' in place of X'. Details are left to the interested reader.

Similarly, if (1.4) holds for $\Delta = \Delta(x,r)$, with $X_\Delta \in B(x,r/2)\cap\Omega$, then again without loss of generality we may suppose that (4.1) holds, for possibly a new $X_\Delta \in B(x,r)\cap\Omega$. Indeed if we let $X' \in B(x,r/2)\cap\Omega$ be the original point X_Δ for which (1.4) holds, we may then follow the argument in the previous paragraph, *mutatis mutandi*. We choose $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$ and suppose first that $\delta(X') \le r/4$ so that $\Delta(\hat{x}, r/4) \subset \Delta$. Considering the same two cases as before we pick X_Δ and in either case (4.1) holds by Lemma 3.1 applied to the surface ball $\Delta(\hat{x}, r/4)$. Note that in the second case (1.4) continues to hold for X_Δ, with a different but still uniform β, by the use of Harnack's inequality within the touching ball $B(X', \delta(X'))$, to move from X' to X_Δ. When $r/4 < \delta(X')$ we choose X'' as before, and by Harnack's inequality, (1.4) holds with X'' in place of X', for a different but still uniform β. Again, if we let $\hat{x} \in \partial\Omega$, with $\delta(X'') = |X'' - \hat{x}|$ then $\Delta(\hat{x}, r/4) \subset \Delta$, and we may now repeat the previous argument with X'' in place of X'.

We are now ready to show that (1.2) implies (1.4).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-dimensional ADR boundary, and let $\Delta = \Delta(x,r)$ be a surface ball on $\partial\Omega$. Let μ be a measure on $\partial\Omega$, such that $\mu|_\Delta \ll \sigma$, and that for some $q > 1$, and for some $\Lambda < \infty$, that*

$$\fint_\Delta k^q \, d\sigma \le \Lambda\,, \tag{4.4}$$

where $k := d\mu/d\sigma$ on Δ. Suppose also that

$$\frac{\mu(\Delta)}{\sigma(\Delta)} \ge 1. \tag{4.5}$$

Then there are constants $\eta, \beta \in (0,1)$, depending only on n, q, Λ and ADR, such that for any Borel set $A \subset \Delta$

$$\sigma(A) \ge (1-\eta)\,\sigma(\Delta) \implies \mu(A) \ge \beta\,\mu(\Delta). \tag{4.6}$$

Remark 4.7. Let k be a normalized version of harmonic measure: $k = c_1^{-1}\sigma(\Delta)\,k^{X_\Delta}$, with X_Δ a point for which (4.1) and (4.2) hold. Then clearly (4.4) and (4.5) hold for k, and the conclusion (4.6) is just a reformulation of (1.4). We note that in the sequel, we shall actually use only (4.6)/(1.4), rather than condition (4.4)/(4.2). Thus, Theorem 1.1 could just as well have been stated with condition ($\star\star$) (see Remark 1.3) in place of ($\star$).

Proof of Lemma 4.3. Set $F := \Delta\setminus A$, so $\sigma(F) \le \eta\,\sigma(\Delta)$. Then

$$\begin{aligned}\mu(F) = \int_F k\,d\sigma &\le \sigma(F)^{1/q'}\left(\int_\Delta k^q d\sigma\right)^{1/q} \\ &\le \Lambda^{1/q}\,\sigma(F)^{1/q'}\,\sigma(\Delta)^{1/q} \le \Lambda^{1/q}\,\eta^{1/q'}\sigma(\Delta) \le \Lambda^{1/q}\,\eta^{1/q'}\mu(\Delta)\,,\end{aligned}$$

where in the last step we have used (4.5). Thus,

$$\mu(A) \ge \left(1 - \Lambda^{1/q}\,\eta^{1/q'}\right)\mu(\Delta) \ge \frac{1}{2}\,\mu(\Delta)\,,$$

for η small enough. This completes the proof. $\square$

Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. As in (2.8), we set $B_{Q_0} = B(x_{Q_0}, r_0)$, with $r_0 := r_{Q_0} \approx \ell(Q_0)$, so that $\Delta_{Q_0} = B_{Q_0} \cap \partial\Omega \subset Q_0$.

Proceeding first in the setting of Theorem 1.1, let $X_0 := X_{\Delta_{Q_0}}$ be the the point relative to $\Delta = \Delta_{Q_0}$ such that (4.1) and (4.2) hold. Note that (4.1) trivially implies that

$$\omega^{X_0}(Q_0) \geq c_1 \,.$$

With the pole X_0 fixed, we define the normalized harmonic measure and the normalized Green's function, respectively, by

$$\mu := \frac{1}{c_1}\,\sigma(Q_0)\,\omega^{X_0}\,, \qquad u(Y) := \frac{1}{c_1}\,\sigma(Q_0)\,G(X_0, Y)\,. \tag{4.8}$$

Then under this normalization, setting $\|\mu\| = \mu(\partial\Omega)$, we have

$$1 \leq \frac{\mu(Q_0)}{\sigma(Q_0)} \leq \frac{\|\mu\|}{\sigma(Q_0)} \leq C_1\,, \tag{4.9}$$

with $C_1 = 1/c_1$. Furthermore, we may apply Lemma 4.3 (using (4.1) and with $\Lambda \approx C_0/c_1$) to obtain (4.6) for μ, with $\Delta = \Delta_{Q_0}$. In turn, the latter bound, in conjunction with (4.1) and ADR, clearly implies an analogous estimate for Q_0, namely that there are constants that we again call $\eta, \beta \in (0, 1)$ such that for any Borel set $A \subset Q_0$,

$$\sigma(A) \geq (1 - \eta)\,\sigma(Q_0) \quad \Longrightarrow \quad \mu(A) \geq \beta\,\mu(Q_0)\,. \tag{4.10}$$

Here, of course, we may have different values of the parameters η and β, but these have the same dependence as the original values, so for convenience we maintain the same notation.

In the p-harmonic case, proceeding under the setup of Theorem 1.12, we let u, μ be the p-harmonic function and its associated p-harmonic measure, corresponding to the point $x = x_{Q_0}$ and the radius $r = Cr_0 := Cr_{Q_0}$, satisfying the hypotheses of Theorem 1.12, where we choose the constant C depending only on n and ADR, such that $Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ (thus in particular, μ is defined on Q_0). Since we assume that u is non-trivial and non-negative, we can apply Lemma 3.43 in $B(x_{Q_0}, Cr_0)$ and use (1.14) to conclude that $\mu(\Delta_{Q_0}) > 0$. We can therefore normalize u and μ (abusing the notation we call the normalizations u and μ) so that $\mu(\Delta_{Q_0})/\sigma(Q_0) = 1$, and since $\Delta_{Q_0} \subset Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ by (1.14), we also have $\mu(\Delta(x_{Q_0}, Cr_0))/\sigma(\Delta(x_{Q_0}, Cr_0)) \approx \mu(Q_0)/\sigma(Q_0) \approx 1$. Set $k := d\mu/d\sigma$. As above, by (1.13) and (1.14), we may then use Lemma 4.3 to see that again μ satisfies (4.9), now with $\|\mu\| := \mu(\Delta(x_{Q_0}, Cr_0))$, and (4.10). The constants C_1, η and β depend on C, n, the ADR constant, C_0, and q.

Remark 4.11. Under the assumptions of Theorems 1.1 and 1.12 and throughout this section and Section 6, for $Q_0 \in \mathbb{D}(E)$ fixed, u, μ will continue to denote the normalized Green function and harmonic measure or the normalized non-negative p-harmonic solution and p-harmonic Riesz measure, as defined above. In particular, (4.9) and (4.10) hold for all $1 < p < \infty$.

As above, let $\mathcal{M}$ denote the usual Hardy-Littlewood maximal operator on $\partial\Omega$ and recall the definition of $\mathbb{D}_{\mathcal{F},Q_0}$ in (2.10).

Lemma 4.12. *Let $Q_0 \in \mathbb{D}$, and suppose that μ satisfies (4.9) and (4.10). Then there is a pairwise disjoint family $\mathcal{F} = \{Q_j\}_{j\geq 1} \subset \mathbb{D}_{Q_0}$, such that*

$$\sigma\Big(Q_0 \setminus \big(\cup_j Q_j\big)\Big) \geq \frac{1}{C}\,\sigma(Q_0) \tag{4.13}$$

and

$$\frac{\beta}{2} \leq \frac{\mu(Q)}{\sigma(Q)} \leq \left(\fint_Q (\mathcal{M}\mu)^{1/2}\, d\sigma\right)^2 \leq C\,, \qquad \forall\, Q \in \mathbb{D}_{\mathcal{F},Q_0}\,, \tag{4.14}$$

where $C > 1$ depends only on η, β, C_1, n and ADR.

Proof. The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [HMc],[HLMc], [AHLMcT], and in local Tb theorems. We begin by noting that

$$\|\mathcal{M}\mu\|_{L^{1,\infty}(\sigma)} := \sup_{\lambda>0} \lambda\, \sigma\{\mathcal{M}\mu > \lambda\} \lesssim \|\mu\| \lesssim \sigma(Q_0)\,, \tag{4.15}$$

by the Hardy-Littlewood Theorem and (4.9). Consequently, by Kolmogorov's criterion,

$$\fint_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \leq C = C(n, ADR, C_1)\,. \tag{4.16}$$

We now perform a stopping time argument to extract a family $\mathcal{F} = \{Q_j\}$ of dyadic sub-cubes of Q_0 that are maximal with respect to the property that either

$$\frac{\mu(Q_j)}{\sigma(Q_j)} < \frac{\beta}{2}\,, \tag{4.17}$$

and/or

$$\fint_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma > K\,, \tag{4.18}$$

where $K \geq 1$ is a sufficiently large number to be chosen momentarily. Note that $Q_0 \notin \mathcal{F}$, by (4.9) and (4.16). We shall say that Q_j is of "type I" if (4.17) holds, and Q_j is of "type II" if (4.18) holds but (4.17) does not. Set $A := Q_0 \setminus (\cup_j Q_j)$, and $F := \cup_{Q_j \,\text{type II}}\, Q_j$. Then by (4.9),

$$\sigma(Q_0) \leq \mu(Q_0) = \sum_{Q_j \,\text{type I}} \mu(Q_j) + \mu(F) + \mu(A)\,. \tag{4.19}$$

By definition of the type I cubes,

$$\sum_{Q_j \,\text{type I}} \mu(Q_j) \leq \frac{\beta}{2} \sum_j \sigma(Q_j) \leq \frac{\beta}{2}\, \sigma(Q_0)\,. \tag{4.20}$$

To handle the remaining terms, observe that

$$\begin{aligned} \sigma(F) = \sum_{Q_j \,\text{type II}} \sigma(Q_j) &\leq \frac{1}{K} \sum_j \int_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma \\ &\leq \frac{1}{K} \int_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \leq \eta\, \sigma(Q_0)\,, \end{aligned} \tag{4.21}$$

by the definition of the type II cubes, (4.16), and the choice of $K = C\eta^{-1}$. By (4.10) and complementation, we therefore find that

$$\mu(F) \leq (1-\beta)\,\mu(Q_0)\,. \tag{4.22}$$

Next, if $x \in A$, then every $Q \in \mathbb{D}_{Q_0}$ that contains x, must satisfy the opposite inequality to (4.18), and therefore, by Lebesgue's differentiation theorem,

$$\mathcal{M}\mu(x) \leq K^2\,, \qquad \text{for} \quad \text{a.e. } x \in A\,.$$

Thus $\mu|_A \ll \sigma$, with $d\mu|_A/d\sigma \leq K^2$, and thus,

$$\mu(A) \leq K^2\sigma(A)\,.$$

Combining the latter estimate with (4.19), (4.20), and (4.22), we obtain

$$\beta\,\mu(Q_0) \leq \frac{\beta}{2}\,\sigma(Q_0) + K^2\sigma(A)\,.$$

Using (4.9), we then find that

$$\beta\,\sigma(Q_0) \leq \beta\,\mu(Q_0) \leq \frac{\beta}{2}\,\sigma(Q_0) + K^2\sigma(A)\,.$$

The conclusion of the lemma now follows readily. $\square$

For future reference, let us note an easy consequence of the last inequality in (4.14) and the ADR property: for all $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, and for any constant $b > 1$, we have

$$\mu\left(\Delta(x_Q,\, b\,\mathrm{diam}(Q))\right) \lesssim b^n\sigma(Q)\left(\fint_Q (\mathcal{M}\mu)^{1/2}\,d\sigma\right)^2 \lesssim b^n\sigma(Q)\,. \tag{4.23}$$

We recall that the ball B_Q^* and surface ball Δ_Q^* are defined in (2.15).

Lemma 4.24. *Let u, μ, be as in Remark 4.11. If the constant K_0 in (2.15) and (2.23) is chosen sufficiently large, then for each $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ with $\ell(Q) \leq K_0^{-1}\,\ell(Q_0)$, there exists $Y_Q \in U_Q$ with $\delta(Y_Q) \leq |Y_Q - x_Q| \lesssim \ell(Q)$, where the implicit constant is independent of K_0, such that*

$$\frac{\mu(Q)}{\sigma(Q)} \leq C\,|\nabla u(Y_Q)|^{p-1}, \tag{4.25}$$

where C depends on K_0 and the implicit constants in the hypotheses of Theorems 1.1 and 1.12.

Remark 4.26. Recalling the construction at the beginning of Section 4, and the fact that we have defined $X_0 := X_{\Delta_{Q_0}}$, we see that $\ell(Q_0) \approx \delta(X_0) \geq K_0^{-1/2}\,\ell(Q_0)$, for K_0 chosen large enough. We note further that the point Y_Q whose existence is guaranteed by Lemma 4.24, is essentially a Corkscrew point relative to Q. Indeed, $\delta(Y_Q) \gtrsim K_0^{-1}\ell(Q)$ (since $Y \in U_Q$), and also $|Y_Q - x_Q| \lesssim \ell(Q)$ (with constant independent of K_0). With a slight abuse of terminology, we shall refer to Y_Q as a Corkscrew point relative to Q, with corkscrew constant depending on K_0.

Proof of Lemma 4.24. Fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \leq K_0^{-1}\,\ell(Q_0)$, where, as in Remark 4.26, we have chosen K_0 large enough that $\ell(Q_0) \approx \delta(X_0) \geq K_0^{-1/2}\,\ell(Q_0)$. Recall (2.7), (2.8) and set $\hat{B}_Q = B(x_Q, \hat{r}_Q)$, $\hat{\Delta}_Q = \hat{B}_Q \cap \partial\Omega$, with $\hat{r}_Q \approx \ell(Q)$ and

$Q \subset \hat{\Delta}_Q$. Let $0 \le \phi_Q \in C_0^\infty(2\hat{B}_Q)$, such that $\phi_Q \equiv 1$ in $\hat{B}_Q$ and $\|\nabla\phi_Q\| \lesssim \ell(Q)^{-1}$. Note that

$$K_0^{1/2}\,\ell(Q) \le K_0^{-1/2}\,\ell(Q_0) \le \delta(X_0) \le |X_0 - x_Q|\,,$$

which implies that $X_0 \notin 4\hat{B}_Q$ provided K_0 is large enough. Thus, by (3.10) in the linear case, or (3.44) in general,

$$\begin{aligned}
(4.27)\qquad \ell(Q)\,\mu(Q) \le \ell(Q)\int_{\partial\Omega}\phi_Q\,d\mu &\lesssim \iint_{\hat{B}_Q\cap\Omega}|\nabla u(Y)|^{p-1}\,dY\\
&\le \iint_{\hat{B}_Q\cap U_Q}|\nabla u(Y)|^{p-1}\,dY + \iint_{(\hat{B}_Q\cap\Omega)\setminus U_Q}|\nabla u(Y)|^{p-1}\,dY\\
&=: \mathcal{I} + \mathcal{II}.
\end{aligned}$$

Notice that by construction $(\hat{B}_Q \cap \Omega) \setminus U_Q \subset \{Y \in \hat{B}_Q : \delta(Y) \le CK_0^{-1}\,\ell(Q)\}$. We may therefore cover the latter region by a family of ball $\{B_k\}_k$, centered on $\partial\Omega$, of radius $CK_0^{-1}\ell(Q)$, such that their doubles $\{2B_k\}$ have bounded overlaps, and satisfy

$$\bigcup_k 2B_k \subset \{Y \in 2\hat{B}_Q : \delta(Y) \le 2CK_0^{-1}\,\ell(Q)\} =: \Sigma(K_0).$$

By the boundary Cacciopoli estimate in Lemma 3.31, plus Hölder's inequality, we obtain

$$\begin{aligned}
\mathcal{II} &\le \sum_k \iint_{B_k}|\nabla u(Y)|^{p-1}dY \lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1}\sum_k\iint_{2B_k}|u(Y)|^{p-1}dY\\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1}\iint_{\Sigma(K_0)}|u(Y)|^{p-1}dY\\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} K_0^{-p}\,\ell(Q)^p\,\mu(\Delta(x_Q, 2M_1\hat{r}_Q))\\
&\lesssim K_0^{-1}\ell(Q)\,\sigma(Q) \le \frac{1}{2}\,\ell(Q)\,\mu(Q)\,,
\end{aligned}$$

where in the last three steps we have used, (3.16) (when $p = 2$) or Lemma 3.46 ($1 < p < \infty$), (4.23), and finally the choice of K_0 large enough. We can then hide this term on the left hand side of (4.27), so that

$$\begin{aligned}
\ell(Q)\,\mu(Q) \lesssim \mathcal{I} &= \iint_{\hat{B}_Q\cap U_Q}|\nabla u(Y)|^{p-1}\,dY = \sum_i\iint_{\hat{B}_Q\cap U_Q^i}|\nabla u(Y)|^{p-1}\,dY\\
&\lesssim \ell(Q)^{n+1}\max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i}|\nabla u(Y)|^{p-1}\\
&\approx \ell(Q)\,\sigma(Q)\max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i}|\nabla u(Y)|^{p-1},
\end{aligned}$$

and we recall that $\{U_Q^i\}_i$ is an enumeration of the connected components of U_Q, and that the number of these components is uniformly bounded. Thus, for some i, there is a point $Y_Q \in \hat{B}_Q \cap U_Q^i$, such that $\mu(Q)/\sigma(Q) \lesssim |\nabla u(Y_Q)|^{p-1}$. To complete the proof we simply observe that $\delta(Y_Q) \le |Y_Q - x_Q| \le \hat{r}_Q \lesssim \ell(Q)$, by construction. $\square$

5. Proof of Theorem 1.1, Corollary 1.5 and Theorem 1.12

In this section we complete the proofs of Theorem 1.1 and Theorem 1.12 by proving that $E := \partial\Omega$ satisfies WHSA, and hence, by Proposition 1.17, E is UR. The proof of Corollary 1.5 follows almost immediately from Theorem 1.1 and we supply the proof at the end of the section. Our approach to the proofs of Theorems 1.1 and 1.12 is a refinement/extension of the arguments in [LV2], who, as mentioned in the introduction, treated the special case that $k \approx 1$.

We fix $Q_0 \in \mathbb{D}(E)$, and we let u and μ be as in Remark 4.11. We recall that by (4.9),

$$\frac{\mu(Q_0)}{\sigma(Q_0)} \approx 1\,. \tag{5.1}$$

Let $\mathcal{F} = \{Q_j\}_j$ be the family of maximal stopping time cubes constructed in Lemma 4.12. Combining (4.25) and (4.14), we see that

$$|\nabla u(Y_Q)| \gtrsim 1\,, \qquad \forall\, Q \in \mathbb{D}^*_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{\mathcal{F},Q_0} :\ \ell(Q) \le K_0^{-1}\,\ell(Q_0)\}\,, \tag{5.2}$$

where $Y_Q \in U_Q$ is the point constructed in Lemma 4.24. We recall that the Whitney region U_Q has a uniformly bounded number of connected components, which we have enumerated as $\{U^i_Q\}_i$. We now fix the particular i such that $Y_Q \in U^i_Q \subset \widetilde{U}^i_Q$, where the latter is the enlarged Whitney region constructed in Definition 2.26.

For a suitably small ε_0, say $\varepsilon_0 \ll K_0^{-6}$, we fix an arbitrary positive $\varepsilon < \varepsilon_0$, and we fix also a large positive number M to be chosen. For each point $Y \in \Omega$, we set

$$B_Y := \overline{B(Y,(1-\varepsilon^{2M/\alpha})\delta(Y))}\,, \qquad \widetilde{B}_Y := \overline{B(Y,\delta(Y))}\,, \tag{5.3}$$

where $0 < \alpha < 1$ is the exponent appearing in Lemma 3.35. For $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, we consider three cases.

Case 0: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) > \varepsilon^{10}\,\ell(Q_0)$.

Case 1: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\,\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}^i_Q}\ \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| > \varepsilon^{2M}\,. \tag{5.4}$$

Case 2: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\,\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}^i_Q}\ \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| \le \varepsilon^{2M}\,. \tag{5.5}$$

We trivially see that the cubes in Case 0 satisfy a packing condition:

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 0 holds}}} \sigma(Q) \le \sum_{Q\in\mathbb{D}_{Q_0},\,\ell(Q)>\varepsilon^{10}\,\ell(Q_0)} \sigma(Q) \lesssim (\log\varepsilon^{-1})\,\sigma(Q_0). \tag{5.6}$$

Note that in Case 1 and Case 2 we have $Q \in \mathbb{D}^*_{\mathcal{F},Q_0}$, see (5.2). Furthermore, if $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$, then by (5.2), (3.42), and either (3.13) (which we apply in the case $p = 2$, with $X = X_0$, since $\ell(Q) \ll \ell(Q_0)$) or (3.45) (for general p, $1 < p < \infty$), and (4.14), we have

$$1 \lesssim |\nabla u(Y_Q)| \lesssim \frac{u(Y_Q)}{\delta(Y_Q)} \lesssim 1\,. \tag{5.7}$$

Regarding Case 1 we shall obtain the following packing condition:

Lemma 5.8. *Under the previous assumptions, the following packing condition holds:*

$$\frac{1}{\sigma(Q_0)} \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \leq C(\varepsilon, K_0, M, \eta)\,, \tag{5.9}$$

On the other hand, we shall see that the cubes in Case 2 satisfy the ε-local WHSA property. Given $\varepsilon > 0$, we recall that $B_Q^{***}(\varepsilon) = B(x_Q, \varepsilon^{-5}\ell(Q))$, see (2.16). We also introduce

$$B_Q^{\text{big}} = B_Q^{\text{big}}(\varepsilon) := B\Big(x_Q, \varepsilon^{-8}\ell(Q)\Big)\,, \qquad \Delta_Q^{\text{big}} := B_Q^{\text{big}} \cap E.$$

Lemma 5.10. *Fix $\varepsilon \in (0, K_0^{-6})$, and let $1 < p < \infty$. Suppose that u is non-negative and p-harmonic in $\Omega_Q := \Omega \cap B_Q^{\text{big}}$, $u \in C(\overline{\Omega_Q})$, $u \equiv 0$ on Δ_Q^{big}. Suppose also that for some i, there exists a point $Y_Q \in U_Q^i$ such that*

$$|\nabla u(Y_Q)| \approx 1\,, \tag{5.11}$$

and furthermore, that

$$\sup_{B_Q^{***}} u \lesssim \varepsilon^{-5}\ell(Q)\,, \tag{5.12}$$

and

$$\sup_{X,Y \in \widetilde{U}_Q^i} \ \sup_{Z_1 \in B_Y,\, Z_2 \in B_X} |\nabla u(Z_1) - \nabla u(Z_2)| \leq 2\varepsilon^{2M}\,. \tag{5.13}$$

Then Q satisfies the ε-local WHSA, provided that M is large enough, depending only on dimension and on the implicit constants in the stated hypotheses.

Assuming these results momentarily we can complete the proof of Theorem 1.1 and Theorem 1.12 as follows. First we see that we can apply Lemma 5.10 to the cubes in Case 2. Indeed, let Q be a cube such that $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, $\ell(Q) \leq \varepsilon^{10}\,\ell(Q_0)$, and (5.5) holds. Hence (5.11) follows by virtue of (5.7), while (5.12) holds by Lemma 3.14 applied with $B = 2B_Q^{***}$ (or Lemma 3.46, with $B(y, s) = 2B_Q^{***}$), and (4.23). Moreover, (5.13) follows trivially from (5.5). Thus, the hypotheses of Lemma 5.10 are all verified and hence Q satisfies the ε-local WHSA condition. In particular, the cubes $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, which belong to the bad collection $\mathcal{B}$ of cubes in $\mathbb{D}(E)$ for which the ε-local WHSA condition fails, must be as in Case 0 or Case 1. By (5.6) and (5.9) these cubes satisfy the packing estimate

$$\sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F},Q_0}} \sigma(Q) \leq C_\varepsilon\, \sigma(Q_0)\,. \tag{5.14}$$

For each $Q_0 \in \mathbb{D}(E)$, there is a family $\mathcal{F} \subset \mathbb{D}_{Q_0}$ for which (5.14), and also the "ampleness" condition (4.13), hold uniformly. We may therefore invoke a well known lemma of John-Nirenberg type to deduce that (2.20) holds for all $\varepsilon \in (0, \varepsilon_0)$, and therefore to conclude that E satisfies the WHSA condition, Definition 2.19. Hence E is UR by Proposition 1.17.

The rest of the section is devoted to the proof Lemmas 5.8 and Lemma 5.10. We shall first prove Lemma 5.8 in the relatively simpler linear case $p = 2$, see subsection 5.1. The proof of Lemma 5.8 in the general case $1 < p < \infty$ is a bit more delicate and given in subsection 5.2. Lemma 5.10 is proved in subsection 5.3. Finally, the proof of Corollary 1.5 is given in subsection 5.4.

Before passing to the subsections we first introduce some additional notation to be used in the sequel. We augment $\widetilde{U}^i_Q$ as follows. Set

$$\mathcal{W}^{i,*}_Q := \left\{ I \in \mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y \in \left(\cup_{X \in \widetilde{U}^i_Q} B_X \right) \right\} \tag{5.15}$$

(and define $\mathcal{W}^{j,*}_Q$ analogously for all other $\widetilde{U}^j_Q$), and set

$$U^{i,*}_Q := \bigcup_{I \in \mathcal{W}^{i,*}_Q} I^{**}, \qquad U^*_Q := \bigcup_j U^{j,*}_Q \tag{5.16}$$

where $I^{**} = (1 + 2\tau)I$ is a suitably fattened Whitney cube, with τ fixed as above. By construction,

$$\widetilde{U}^i_Q \subset \bigcup_{X \in \widetilde{U}^i_Q} B_X \subset \bigcup_{Y \in \cup_{X \in \widetilde{U}^i_Q} B_X} B_Y \subset U^{i,*}_Q ,$$

and for all $Y \in U^{i,*}_Q$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on ε). Moreover, also by construction, there is a Harnack path connecting any pair of points in $U^{i,*}_Q$ (depending again on ε), and furthermore, for every $I \in \mathcal{W}^{i,*}_Q$ (or for that matter for every $I \in \mathcal{W}^{j,*}_Q$, $j \neq i$),

$$\varepsilon^s \, \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3} \, \ell(Q), \qquad \operatorname{dist}(I, Q) \lesssim \varepsilon^{-4} \, \ell(Q) ,$$

where $0 < s = s(M, \alpha)$. Thus, by Harnack's inequality and (5.7),

$$C^{-1} \delta(Y) \le u(Y) \le C \delta(Y) , \qquad \forall \, Y \in U^{i,*}_Q , \tag{5.17}$$

with $C = C(K_0, \varepsilon, M)$. Moreover, for future reference, we note that the upper bound for u holds in all of U^*_Q, i.e.,

$$u(Y) \le C \delta(Y) , \qquad \forall \, Y \in U^*_Q , \tag{5.18}$$

by (3.12) (resp. (3.45)) and (4.14), where again $C = C(K_0, \varepsilon, M)$.

5.1. **Proof of Lemma 5.8 in the linear case ($p = 2$).** We here complete the proof of estimate (5.9) in the relatively simpler linear case $p = 2$. To start the proof of (5.9), we fix $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ so that Case 1 holds. We see that if we choose Z as in (5.4), and use the mean value property of harmonic functions, then

$$\varepsilon^{2M} \le C_\varepsilon \, (\ell(Q))^{-(n+1)} \iint_{B_Z \cup B_{Y_Q}} |\nabla u(Y) - \vec{\beta}| dY ,$$

where $\vec{\beta}$ is a constant vector at our disposal. By Poincaré's inequality, see, e.g., [HM1, Section 4] in this context, we obtain that

$$\sigma(Q) \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 \delta(Y)\, dY \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 u(Y)\, dY\,,$$

where the implicit constants depend on ε, and in the last step we have used (5.17). Consequently,

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \ell(Q)\le\varepsilon^{10}\ell(Q_0)}} \iint_{U_Q^*} |\nabla^2 u(Y)|^2 u(Y)\, dY \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} |\nabla^2 u(Y)|^2 u(Y)\, dY, \tag{5.19}$$

where

$$\Omega^*_{\mathcal{F},Q_0} := \operatorname{int}\Big(\bigcup_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \ell(Q)\le\varepsilon^{10}\ell(Q_0)}} U_Q^* \Big), \tag{5.20}$$

and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps.

Take an arbitrary $N > 1/\varepsilon$ (eventually $N \to \infty$), and augment $\mathcal{F}$ by adding to it all subcubes $Q \subset Q_0$ with $\ell(Q) \le 2^{-N}\,\ell(Q_0)$. Let $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ denote the collection of maximal cubes of this augmented family. Thus, $Q \in \mathbb{D}_{\mathcal{F}_N,Q_0}$ iff $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and $\ell(Q) > 2^{-N}\,\ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N,Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'},Q_0}$ if $N \le N'$ and therefore $\Omega^*_{\mathcal{F}_N,Q_0} \subset \Omega^*_{\mathcal{F}_{N'},Q_0}$ (where $\Omega^*_{\mathcal{F}_N,Q_0}$ is defined as in (5.20) with $\mathcal{F}_N$ replacing $\mathcal{F}$). By monotone convergence and (5.19), we have that

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \limsup_{N\to\infty} \iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\, dY. \tag{5.21}$$

It therefore suffices to establish bounds for the latter integral that are uniform in N, with N large.

Let us then fix $N > 1/\varepsilon$. Since $\Omega^*_{\mathcal{F}_N,Q_0}$ is a finite union of fattened Whitney boxes, we may now integrate by parts, using the identity $2|\nabla\partial_k u|^2 = \operatorname{div}\nabla(\partial_k u)^2$ for harmonic functions, to obtain that

$$\iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\, dY \lesssim \int_{\partial\Omega^*_{\mathcal{F}_N,Q_0}} \Big(|\nabla^2 u|\,|\nabla u|\, u + |\nabla u|^3\Big)\, dH^n \le C_\varepsilon\, H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}), \tag{5.22}$$

where in the second inequality we have used the standard estimates

$$\delta(Y)|\nabla^2 u(Y)|,\ |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)}\,,$$

along with (5.18). We observe that $\Omega^*_{\mathcal{F}_N,Q_0}$ is a sawtooth domain in the sense of [HMM], or to be more precise, it is a union of a bounded number, depending on ε, of such sawtooths, one for each maximal sub-cube of Q_0 with length on the order of $\varepsilon^{10}\ell(Q_0)$. By [HMM, Appendix A] each of the previous sawtooth domains is ADR uniformly in N. Hence, its union is upper ADR uniformly in N with constant depending on the number of sawtooth domains in the union, which ultimately depends on ε. Therefore

$$H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}) \le C_\varepsilon \left(\operatorname{diam}(\partial\Omega^*_{\mathcal{F}_N,Q_0})\right)^n \le C_\varepsilon\, \sigma(Q_0)\,.$$

Combining the latter estimate with (5.21) and (5.22), we obtain (5.9), as desired, in the case $p = 2$.

5.2. **Proof of Lemma 5.8 in the general case** $(1 < p < \infty)$**.** We here prove (5.9) for general p, $1 < p < \infty$, by proceeding along the lines of the proof of Lemma 2.5 in [LV1]. We fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1 holds and hence (5.4) holds. Let us recall that we have verified estimates (5.7), (5.17), and (5.18) for all p, $1 < p < \infty$.

Recall that if $X \in \widetilde{U}^i_Q$, then by definition X can be connected to some $\tilde{Y} \in U^i_Q$, and then to $Y_Q \in U^i_Q$, by a chain of at most $C\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$. Note that using the triangle inequality and the definition of $\widetilde{U}^i_Q$, we may suppose that $Y_{k+1} \in B(Y_k, 3\delta(Y_k)/4) \subset B_{Y_k}$, otherwise we increase the chain by introducing some intermediate points and the new chain will have essentially the same length. Fix now Q, a cube in Case 1, and by (5.4) we can pick $X \in \widetilde{U}^i_Q$ so that

$$\sup_{Y\in B_X} |\nabla u(Y) - \nabla u(Y_Q)| > \varepsilon^{2M}\,.$$

As observed before we can form a Harnack chain connecting X and Y_Q so that $Y_1 = Y_Q$ and $Y_l = X$ and $l \le C\varepsilon^{-1}$. Then, the previous expression can be written as

$$\sup_{Y\in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_1)| > \varepsilon^{2M}\,. \tag{5.23}$$

Obviously we may assume that

$$\sup_{Y\in B_{Y_j}} |\nabla u(Y) - \nabla u(Y_1)| \le \varepsilon^{2M}\,, \tag{5.24}$$

whenever $1 < j \le l-1$, and $l > 1$, since otherwise we shorten the chain (and work with the first Y_j for which (5.23) holds). This and the fact that $Y_{j+1} \in B_{Y_j}$ for every $1 \le j \le l-1$ imply that

$$|\nabla u(Y_j)| \ge |\nabla u(Y_1)| - \varepsilon^{2M}, \quad \text{for } 1 \le j \le l. \tag{5.25}$$

Furthermore, using the triangle inequality

$$\varepsilon^{2M} \le \sup_{Y\in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| + \sum_{j=1}^{l-1} |\nabla u(Y_{j+1}) - \nabla u(Y_j)|. \tag{5.26}$$

Hence, using this and the fact that $l \lesssim \varepsilon^{-1}$ we have that either

$$\begin{aligned} &(i) \quad \sup_{Y\in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| \ge \varepsilon^{2M+2}, \text{ or} \\ &(ii) \quad |\nabla u(Y_{j+1}) - \nabla u(Y_j)| \ge \varepsilon^{2M+2}, \text{ for some } 1 \le j \le l-1. \end{aligned} \tag{5.27}$$

By (5.18) and (3.42) we have

$$|\nabla u(Y)| \le C_\varepsilon\,, \qquad \forall\, Y \in U^*_Q. \tag{5.28}$$

In scenario (i) of (5.27) we take Y, a point where the sup is attained. This choice, (5.28) and the first inequality in (3.38), imply that $|Y - Y_l| \approx_\varepsilon \ell(Q)$. We then construct $\Gamma_0(Q)$ a (possibly rotated) rectangle as follows. The base and the top are two n-dimensional cubes of side length $c_\varepsilon\,\ell(Q)$, with c_ε chosen sufficiently small, centered respectively at the points Y and Y_l, and lying in the two parallel hyperplanes passing through the points Y and Y_l being perpendicular to the vector joining these two points. Note that for this rectangle, all side lengths are of the order of $\ell(Q)$ with implicit constants possibly depending on ε. In scenario (ii) of (5.27) we do the same construction with Y_{j+1} and Y_j in place of Y and Y_l and define $\Gamma_0(Q)$ which will verify the same properties. Note that in either case, (5.28) and the first inequality in (3.38) give with the property that

$$|\nabla u(Y) - \nabla u(W)| \geq \varepsilon^{2M+4} \tag{5.29}$$

whenever W, Y are in the base and top of the parallelepiped, respectively. By construction, at least the top, which we denote by $t(Q)$, is centered on Y_j, for some $1 \leq j \leq l$. We observe that by (5.25) and (5.7), since $Y_1 := Y_Q$, and since ε is very small, we have for each Y_j, $1 \leq j \leq l$,

$$|\nabla u(Y_j)| \geq a\,, \tag{5.30}$$

for some uniform constant a independent of ε, and therefore by (3.38), we also have

$$|\nabla u(Y)| \geq a/2\,, \qquad \forall\, Y \in t(Q)\,, \tag{5.31}$$

provided that we take c_ε small enough, since $\operatorname{diam}(t(Q)) \approx c_\varepsilon\,\ell(Q)$. Moving downward, that is, from top to base, through $\Gamma_0(Q)$, along slices parallel to $t(Q)$, we stop the first time that we reach a slice $b(Q)$ which contains a point Z with $|\nabla u(Z)| \leq a/4$. If there is such a slice, we form a new rectangle $\Gamma(Q)$ with base $b(Q)$ and top $t(Q)$; otherwise, we set $\Gamma(Q) := \Gamma_0(Q)$, and let $b(Q)$ denote the base in this case as well. In either case, $\operatorname{dist}(b(Q), t(Q)) \approx \ell(Q)$, with implicit constants possibly depending on ε, by (3.38) and (5.31). Note that by construction, and the continuity of ∇u,

$$|\nabla u(Y)| \geq a/4\,, \qquad \forall\, Y \in \Gamma(Q)\,, \tag{5.32}$$

and that $|\Gamma(Q)| \approx \ell(Q)^{n+1}$, again with implicit constants that may depend on ε. Moreover, if $\Gamma(Q) = \Gamma_0(Q)$, then (5.29) holds for all $W \in b(Q)$ and $Y \in t(Q)$. Otherwise, if $\Gamma(Q)$ is strictly contained in $\Gamma_0(Q)$, then, since $\operatorname{diam}(b(Q)) \approx c_\varepsilon\,\ell(Q)$ with c_ε small, and since by construction $b(Q)$ contains a point Z with $|\nabla u(Z)| = a/4$, it follows that $|\nabla u(W)| \leq 3a/8$, for all $W \in b(Q)$, by (3.38). Hence, in either situation, since $a/8 \gg \varepsilon^{2M+4}$, we have

$$|\nabla u(Y) - \nabla u(W)| \geq \varepsilon^{2M+4}\,, \qquad \forall\, W \in b(Q),\ Y \in t(Q)\,. \tag{5.33}$$

We let $\gamma = a/8$ and set

$$F_\gamma(|\nabla u|) := \max(|\nabla u|^2 - \gamma^2, 0)\,.$$

Then by (5.32) we see that

$$F_\gamma(|\nabla u|) \geq a^2/64\,, \qquad \forall\, Y \in \Gamma(Q)\,. \tag{5.34}$$

Furthermore, by (5.33), fundamental theorem of calculus, (5.17), (5.32) and (5.34), we have,

$$\ell(Q)^n \lesssim \iint_{\Gamma(Q)} u\,|\nabla^2 u|^2\,dX \lesssim \iint_{\Gamma(Q)} u\,F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}\,|\nabla^2 u|^2\,dY\,,$$

where the implicit constants depend on ε. In particular, since $\Gamma(Q) \subset U_Q^{i,*} \subset U_Q^*$, by ADR we obtain

$$\sigma(Q) \lesssim \iint_{U_Q^*} uF_\gamma(|\nabla u|)|\nabla u|^{p-2}|\nabla^2 u|^2\,dY\,,$$

where the implicit constants still depend on ε, and this estimate holds for all cubes $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1 holds. Hence,

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} u\,F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}\,|\nabla^2 u|^2\,dY\,, \tag{5.35}$$

where $\Omega^*_{\mathcal{F},Q_0}$ was defined in (5.20) and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps. To prove (5.9) in the general case $1 < p < \infty$ it therefore suffices to establish the local square function bound

$$\iint_{\Omega^*_{\mathcal{F},Q_0}} u\,F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}\,|\nabla^2 u|^2\,dY \lesssim \sigma(Q_0)\,, \tag{5.36}$$

where, as we recall, u is a non-negative p-harmonic function in the open set $\Omega_0 := \Omega \cap B(x_{Q_0}, Cr_{Q_0})$, vanishing on $\Delta(x_{Q_0}, Cr_{Q_0})$.

To start the proof of (5.36), for each $Q \in \mathbb{D}(E)$, we define a further fattening of U_Q^* as follows. Set

$$U_Q^{i,**} := \bigcup_{I\in\mathcal{W}_Q^{i,*}} I^{***}\,, \qquad U_Q^{**} := \bigcup_i U_Q^{i,**},$$
$$U_Q^{i,***} := \bigcup_{I\in\mathcal{W}_Q^{i,*}} I^{****}\,, \qquad U_Q^{***} := \bigcup_i U_Q^{i,***},$$

where $I^{***} = (1+3\tau)I$, and $I^{****} = (1+4\tau)I$ are fattened Whitney regions, for some fixed small τ as above, see (5.15)-(5.16). Notice that $I^{**} \subset I^{***} \subset I^{****}$. We observe that the fattened Whitney regions U_Q^{***} have bounded overlaps, say

$$\sum_{Q\in\mathbb{D}(E)} 1_{U_Q^{***}}(Y) \le M_0\,, \tag{5.37}$$

where $M_0 < \infty$ is a uniform constant depending on K_0, ε, τ and n. Next, let $\{\eta_Q\}_Q$ be a partition of unity adapted to U_Q^{**}. That is

(1) $\sum_Q \eta_Q(Y) \equiv 1$ whenever $Y \in \Omega$.
(2) $\operatorname{supp} \eta_Q \subset U_Q^{**}$.
(3) $\eta_Q \in C_0^\infty(\mathbb{R}^{n+1})$, with $0 \le \eta_Q \le 1$, $\eta_Q \ge c$ on U_Q^* and $|\nabla \eta_Q| \le C\ell(Q)^{-1}$.

Set

$$\mathbb{D}_{\mathcal{F},Q_0,\varepsilon} := \left\{Q \in \mathbb{D}_{\mathcal{F},Q_0} : \ \ell(Q) \le \varepsilon^{10}\ell(Q_0)\right\},$$

and recall, see (5.20), that

$$\Omega^*_{\mathcal{F},Q_0} := \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0,\varepsilon}} U_Q^* \Big).$$

Given a large number $N \gg \varepsilon^{-10}$, set

$$\Lambda = \Lambda(N) = \left\{Q \in \mathbb{D}(E) : \ U_Q^{**} \cap \Omega^*_{\mathcal{F},Q_0} \ne \emptyset \ \text{ and } \ \ell(Q) \ge N^{-1}\ell(Q_0)\right\}.$$

Eventually, we shall let $N \to \infty$. Let

$$I_1(N) := \sum_{Q\in\Lambda(N)} \iint u\, F_\gamma(|\nabla u|)\Big(\sum_{i,j=1}^{n+1} u_{y_iy_j}^2\Big)\eta_Q\, dY$$

and note, by positivity of u, the properties of η_Q, that we then have

$$\iint_{\Omega^*_{\mathcal{F},Q_0}} u\, F_\gamma(|\nabla u|)\, |\nabla^2 u|^2\, dY \lesssim \lim_{N\to\infty} I_1(N).$$

We now fix N and we intend to perform integration by parts and in this argument we will exploit that $|\nabla u|^2$ is a subsolution to a certain linear PDE defined based on u. To describe this in detail, let $Q \in \Lambda(N)$ be such that $F_\gamma(|\nabla u(Y)|) \ne 0$ for some $Y \in U_Q^{**}$. Then $|\nabla u(Y)| \ge \gamma$ and there exists $C = C(\gamma) \ge 1$, such that

$$C^{-1} \le |\nabla u(X)| \lesssim 1 \text{ whenever } X \in B(Y, \delta(Y)/C), \tag{5.38}$$

and where the upper bound follows from (5.18) and the lower bound uses also (3.38). Let $\zeta = \nabla u \cdot \xi$, for some $\xi \in \mathbb{R}^{n+1}$. Then ζ satisfies, at $X \in B(Y, \delta(Y)/C)$, the partial differential equation

$$L\zeta = \nabla\cdot\left[(p-2)\,|\nabla u|^{p-4}\,(\nabla u\cdot\nabla\zeta)\,\nabla u + |\nabla u|^{p-2}\,\nabla\zeta\right] = 0 \tag{5.39}$$

as is seen by a straightforward calculation from differentiating the p-Laplace partial differential equation for u with respect to ξ. Note that (5.39) can be written in the form

$$L\zeta = \sum_{i,j=1}^{n+1} \frac{\partial}{\partial y_i}\left[\, b_{ij}(\cdot)\, \zeta_{y_j}(\cdot)\right] = 0, \tag{5.40}$$

where,

$$b_{ij}(Y) = |\nabla u|^{p-4}\left[(p-2)\, u_{y_i}\, u_{y_j} + \delta_{ij}\, |\nabla u|^2\right](Y), \qquad 1 \le i, j \le n+1, \tag{5.41}$$

and δ_{ij} is the Kronecker δ. Clearly we also have

$$Lu(Y) = (p-1)\,\nabla\cdot\left[\,|\nabla u|^{p-2}\,\nabla u\,\right](Y) = 0. \tag{5.42}$$

In particular, u, and $(\nabla u\cdot\xi)$ for each $\xi \in \mathbb{R}^{n+1}$ all satisfy the divergence form partial differential equation (5.40).

It is easy to see that $(b_{ij})_{ij}$ satisfies the following degenerate ellipticity condition: for every $\xi \in \mathbb{R}^{n+1}$ one has

$$\sum_{i,j=1}^{n+1} b_{ij}\,\xi_i\,\xi_j = (p-2)\,|\nabla u|^{p-4}\sum_{i,j=1}^{n+1} u_i\,u_j\,\xi_i\,\xi_j + |\nabla u|^{p-2}\sum_{i,j=1}^{n+1}\delta_{ij}\,\xi_i\,\xi_j$$
$$= (p-2)\,|\nabla u|^{p-4}\,(\nabla u\cdot\xi)^2 + |\nabla u|^{p-2}\,|\xi|^2 \geq \min\{1,p-1\}\,|\nabla u|^{p-2}\,|\xi|^2, \tag{5.43}$$

where the last inequality is immediate when $p \geq 2$ and uses the Cauchy-Schwarz inequality when $1 < p < 2$. Hence, $|\nabla u|^2$ is a subsolution to the PDE defined in (5.40), (5.41) as it is seen from the calculation

$$L(|\nabla u|^2) = 2\sum_{i,j,k=1}^{n+1} b_{ij}\,u_{y_iy_k}\,u_{y_jy_k} \gtrsim |\nabla u|^{p-2}\Big(\sum_{i,j=1}^{n+1} u^2_{y_iy_j}\Big). \tag{5.44}$$

Now, using (5.44) and that (5.38) holds for every Y such that $F_\gamma(|\nabla u(Y)|) \neq 0$ we see that $I_1(N) \lesssim J_1(N)$ where

$$J_1(N) := \sum_{Q\in\Lambda(N)} \iint u\,F_\gamma(|\nabla u|)L(|\nabla u|^2)\,\eta_Q\,dY\,.$$

Hence it suffices to establish bounds for the integral $J_1 := J_1(N)$ that are uniform in N, with N large. In the following we let $v = F_\gamma(|\nabla u|)$ and we note that $\nabla v = \nabla(|\nabla u|^2)$ whenever $v > 0$. Using this and integration by parts we see that

$$J_1 = -J_2 - J_3 - J_4,$$

where

$$J_2 = \sum_{Q\in\Lambda(N)} \iint v\sum_{i,j=1}^{n+1} b_{ij}u_{y_i}v_{y_j}\,\eta_Q\,dY,$$

$$J_3 = \sum_{Q\in\Lambda(N)} \iint u\sum_{i,j=1}^{n+1} b_{ij}v_{y_i}v_{y_j}\,\eta_Q\,dY,$$

$$J_4 = \sum_{Q\in\Lambda(N)} \iint uv\sum_{i,j=1}^{n+1} b_{ij}v_{y_j}\,(\eta_Q)_{y_i}\,dY.$$

We will estimate J_4 first. Set $\Lambda_1 = \Lambda_{11}\cup\Lambda_{12}$, where

$$\Lambda_{11} := \left\{Q\in\Lambda:\ U_Q^{**}\text{ meets }\Omega\setminus\Omega_{\mathcal{F},Q_0}\right\},$$

and

$$\Lambda_{12} := \left\{Q\in\Lambda:\ U_Q^{**}\text{ meets }U_{Q'}^{**}\text{ such that }\ell(Q') < N^{-1}\ell(Q_0)\right\}.$$

From the definition of η_Q, we obtain

$$|J_4| \lesssim \sum_{Q\in\Lambda_{11}} \iint u\,v\sum_{i,j=1}^{n+1} |u_{ij}|\,|u_i|\,|(\eta_Q)_j|\,dY + \sum_{Q\in\Lambda_{11}} \iint u\,v\sum_{i,j=1}^{n+1} |u_{ij}|\,|u_i|\,|(\eta_Q)_j|\,dY$$
$$=: J_{51} + J_{52}.$$

Notice that, equivalently, Λ_{11} is the subcollection of $Q\in\Lambda_1$ such that U_Q^{**} meets $\partial\Omega^*_{\mathcal{F},Q_0}$. We first estimate J_{51}. Note that by (3.38), (5.18) and Harnack's inequality,

$$\delta(Y)|\nabla u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q)\,, \tag{5.45}$$

whenever $Y \in U_Q^{***}$. Furthermore, if $v \neq 0$ for some $Y \in U_Q^{***}$, then using (5.38) and (3.40), we also have

$$(\delta(Y))^2 |\nabla^2 u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q)\,. \tag{5.46}$$

In particular, $u|\nabla \eta_Q| \lesssim 1$, by the construction of η_Q, $|\nabla u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$, and $\delta(Y)|\nabla^2 u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$ and $v \neq 0$. Thus,

$$J_{51} \lesssim \sum_{Q\in\Lambda_{11}} \ell(Q)^n \lesssim \sum_{Q\in\Lambda_{11}} H^n(U_Q^{***} \cap \partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sum_{Q\in\Lambda_{11}} H^n(\partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sigma(Q_0)$$

where we have used that $\partial\Omega^*_{\mathcal{F},Q_0}$ is ADR, see [HMM], and the bounded overlap property (5.37). To estimate J_{52} we observe that for each $Q \in \Lambda_{12}$, $\ell(Q) \approx N^{-1}\ell(Q_0)$, by properties of Whitney regions. Hence, by a slightly simpler version of the argument used for the estimate of J_{51} we obtain

$$J_{52} \lesssim \sum_{Q\in\Lambda_{12}} \sigma(Q) \lesssim \sigma(Q_0).$$

Therefore, $|J_4| \lesssim J_{51} + J_{52} \lesssim \sigma(Q_0)$.

To handle J_2 we use that u is a solution to (5.40). Indeed, by integration by parts, using the identity $2vv_{y_j} = (v^2)_{y_j}$ we see that

$$2\,J_2 = \sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} (v^2)_{y_j}\, \eta_Q\, dY = -\sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v^2\, (\eta_Q)_{y_j}\, dY,$$

and by the same argument as in the estimate of J_4 we obtain $|J_2| \lesssim \sigma(Q_0)$.

To conclude we collect the estimates for J_2 and J_4, and use use that J_3 is non-negative by (5.43) to obtain $J_1(N) \lesssim \sigma(Q_0)$, with constants independent of N. The proof of (5.9) in the general case $1 < p < \infty$ is then complete.

5.3. **Proof of Lemma 5.10.** To prove Lemma 5.10, we will follow the corresponding argument in [LV2] closely, but with some modifications due to the fact that in contrast to the situation in [LV2], our solution u need not be Lipschitz up to the boundary, and our harmonic/p-harmonic measures need not be doubling. It is the latter obstacle that has forced us to introduce the WHSA condition, rather than to work with the Weak Exterior Convexity condition used in [LV2]. Lemma 5.10 is essentially a distillation of the main argument of the corresponding part of [LV2], but with the doubling hypothesis removed.

In the remainder of this section, we will, for convenience, use the notational convention that implicit and generic constants are allowed to depend upon K_0, but not on ε or M. Dependence on the latter will be stated explicitly. We first prove the following lemma and we recall that the balls B_Y and $\widetilde{B}_Y$ are defined in (5.3).

Lemma 5.47. *Let $Y \in U_Q^i$, $X \in \widetilde{U}_Q^i$. Suppose first that $w \in \partial\widetilde{B}_Y \cap E$, and let W be the radial projection of w onto ∂B_Y. Then*

$$u(W) \lesssim \varepsilon^{2M-5}\delta(Y)\,. \tag{5.48}$$

If $w \in \partial\widetilde{B}_X \cap E$, and W now is the radial projection of w onto ∂B_X, then

$$u(W) \lesssim \varepsilon^{2M-5}\ell(Q)\,. \tag{5.49}$$

Proof. Since $K_0^{-1}\ell(Q) \lesssim \delta(Y) \lesssim K_0\,\ell(Q)$ for $Y \in U_Q^i$, it is enough to prove (5.49). To prove (5.49), we first note that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q)\,,$$

by definition of $B_X, \widetilde{B}_X$ and the fact that by construction of $\widetilde{U}_Q^i$,

$$\varepsilon^3\ell(Q) \lesssim \delta(X) \lesssim \varepsilon^{-3}\ell(Q)\,, \quad \forall\, X \in \widetilde{U}_Q^i\,. \tag{5.50}$$

In addition, again by construction of $\widetilde{U}_Q^i$,

$$\operatorname{diam}(\widetilde{U}_Q^i) \lesssim \varepsilon^{-4}\ell(Q)\,. \tag{5.51}$$

Consequently, $W \in \frac{1}{2}B_Q^{***} = B(x_Q, \frac{1}{2}\varepsilon^{-5}\ell(Q))$, so by Lemma 3.35 and (5.12),

$$u(W) \lesssim \left(\frac{\varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q)}{\varepsilon^{-5}\ell(Q)}\right)^{\alpha} \frac{1}{|B_Q^{***}|}\iint_{B_Q^{***}} u \lesssim \varepsilon^{2M+2\alpha-5}\ell(Q) \le \varepsilon^{2M-5}\ell(Q)\,.$$

□

Claim 5.52. *Let $Y \in U_Q^i$. For all $W \in B_Y$,*

$$|u(W) - u(Y) - \nabla u(Y)\cdot(W - Y)| \lesssim \varepsilon^{2M}\delta(Y)\,. \tag{5.53}$$

Proof of Claim 5.52. Let $W \in B_Y$. Then for some $\widetilde{W} \in B_Y$,

$$u(W) - u(Y) = \nabla u(\widetilde{W})\cdot(W - Y)\,.$$

We may then invoke (5.13), with $X = Y$, $Z_1 = \widetilde{W}$, and $Z_2 = Y$, to obtain (5.53). □

Claim 5.54. *Let $Y \in U_Q^i$. Suppose that $w \in \partial\widetilde{B}_Y \cap E$. Then*

$$|u(Y) - \nabla u(Y)\cdot(Y - w)| = |u(w) - u(Y) - \nabla u(Y)\cdot(w - Y)| \lesssim \varepsilon^{2M-5}\delta(Y)\,. \tag{5.55}$$

Proof of Claim 5.54. Given $w \in \partial\widetilde{B}_Y \cap E$, let W be the radial projection of w onto ∂B_Y, so that $|W - w| = \varepsilon^{2M/\alpha}\delta(Y)$. Since $u(w) = 0$, by (5.48) we have

$$|u(W) - u(w)| = u(W) \lesssim \varepsilon^{2M-5}\delta(Y).$$

Since (5.53) holds for W, we obtain (5.55) by (5.11) and (5.13). □

To simplify notation, we now set $Y := Y_Q$, the point in U_Q^i satisfying (5.11). By (5.11) and (5.13), for $\varepsilon < 1/2$, and M chosen large enough, we have that

$$|\nabla u(Z)| \approx 1\,, \qquad \forall\, Z \in \widetilde{U}_Q^i\,. \tag{5.56}$$

By translation and rotation, we assume that $0 \in \partial\widetilde{B}_Y \cap E$, and that $Y = \delta(Y)e_{n+1}$, where as usual $e_{n+1} := (0, \dots, 0, 1)$.

Claim 5.57. *We claim that*

$$\Big|\nabla u(Y)\cdot e_{n+1} - |\nabla u(Y)|\Big| \lesssim \varepsilon^{2M-5}\,. \tag{5.58}$$

Proof of Claim 5.57. We apply (5.55), with $w = 0$, to obtain

$$|u(Y) - \nabla u(Y) \cdot Y| \lesssim \varepsilon^{2M-5}\delta(Y).$$

Combining the latter bound with (5.53), we find that

$$(5.59) \quad |u(W) - \nabla u(Y) \cdot W| = |u(W) - \nabla u(Y) \cdot Y - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M-5}\delta(Y)\,, \quad \forall\, W \in B_Y\,.$$

Fix $W \in \partial B_Y$ so that $\nabla u(Y) \cdot \frac{W-Y}{|W-Y|} = -|\nabla u(Y)|$. Since $|W - Y| = (1 - \varepsilon^{2M/\alpha})\delta(Y)$, and since $u \geq 0$, we have

$$(5.60) \quad \begin{aligned} 0 \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} &\leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} + \frac{u(W)}{\delta(Y)} \\ &\leq \frac{1}{\delta(Y)}\left(-\nabla u(Y) \cdot \frac{(W-Y)}{1-\varepsilon^{2M/\alpha}} - \nabla u(Y) \cdot Y + u(W)\right) \\ &\lesssim \left(\varepsilon^{2M-5} + \varepsilon^{2M/\alpha}\right) \approx \varepsilon^{2M-5}\,, \end{aligned}$$

by (5.59) and (5.11). □

Claim 5.61. *Suppose that $M > 5$. Then*

$$(5.62) \quad \Big|\,|\nabla u(Y)|e_{n+1} - \nabla u(Y)\Big| \lesssim \varepsilon^{M-3}\,.$$

Proof of Claim 5.61. By Claim 5.57,

$$\Big|\,|\nabla u(Y)|e_{n+1} - (\nabla u(Y) \cdot e_{n+1})e_{n+1}\Big| \lesssim \varepsilon^{2M-5}\,.$$

Therefore, it is enough to consider $\nabla_{||}u := \nabla u - (\nabla u(Y) \cdot e_{n+1})e_{n+1}$. Observe that

$$\begin{aligned} |\nabla_{||}u(Y)|^2 &= |\nabla u(Y)|^2 - (\nabla u(Y) \cdot e_{n+1})^2 \\ &= (|\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1})\,(|\nabla u(Y)| + \nabla u(Y) \cdot e_{n+1}) \lesssim \varepsilon^{2M-5}\,, \end{aligned}$$

by (5.58) and (5.11). □

Now for $Y = \delta(Y)e_{n+1} \in U^i_Q$ fixed as above, we consider another point $X \in \widetilde{U}^i_Q$. By definition of $\widetilde{U}^i_Q$, there is a polygonal path in $\widetilde{U}^i_Q$, joining Y to X, with vertices

$$Y_0 := Y, Y_1, Y_2, \dots, Y_N := X\,, \qquad N \lesssim \varepsilon^{-4}\,,$$

such that $Y_{k+1} \in B_{Y_k} \cap B(Y_k, \ell(Q))$, $0 \leq k \leq N-1$, and such that the distance between consecutive vertices is at most $C\ell(Q)$. Indeed, by definition of $\widetilde{U}^i_Q$, we may connect Y to X by a polygonal path connecting the centers of at most ε^{-1} balls, such that the distance between consecutive vertices is between $\varepsilon^3\,\ell(Q)/2$ and $\varepsilon^{-3}\,\ell(Q)/2$. If any such distance is greater than $\ell(Q)$, we take at most $C\varepsilon^{-3}$ intermediate vertices with distances on the order of $\ell(Q)$. The total length of the path is thus on the order of $N\ell(Q)$ with $N \lesssim \varepsilon^{-4}$. Furthermore, by (5.13) and (5.62),

$$(5.63) \quad \begin{aligned} &\Big|\nabla u(W) - |\nabla u(Y)|e_{n+1}\Big| \\ &\qquad \leq |\nabla u(W) - \nabla u(Y)| + \Big|\nabla u(Y) - |\nabla u(Y)|e_{n+1}\Big| \\ &\qquad\qquad \lesssim \varepsilon^{2M} + \varepsilon^{M-3} \lesssim \varepsilon^{M-3}\,, \quad \forall\, W \in B_Z\,,\ \forall Z \in \widetilde{U}^i_Q\,. \end{aligned}$$

Claim 5.64. *Assume $M > 7$. Then for each $k = 1, 2, \dots, N$,*

$$\big|u(Y_k) - |\nabla u(Y)| Y_k \cdot e_{n+1}\big| \lesssim k\, \varepsilon^{M-3} \ell(Q)\,. \tag{5.65}$$

Moreover,

$$\big|u(W) - |\nabla u(Y)| W_{n+1}\big| \lesssim \varepsilon^{M-7} \ell(Q)\,, \quad \forall\, W \in B_X\,, \forall\, X \in \widetilde{U}^i_Q\,. \tag{5.66}$$

Proof of Claim 5.64. By (5.59) and (5.62), we have

$$\begin{aligned}\big|u(W) - |\nabla u(Y)| W_{n+1}\big| &\lesssim |u(W) - \nabla u(Y) \cdot W| \\ &\quad + \Big|\Big(\nabla u(Y) - |\nabla u(Y)| e_{n+1}\Big) \cdot W\Big| \\ &\lesssim \varepsilon^{2M-5} \delta(Y) + \varepsilon^{M-3} |W| \lesssim \varepsilon^{M-3} \ell(Q)\,, \quad \forall\, W \in B_Y\,,\end{aligned} \tag{5.67}$$

since $\delta(Z) \approx \ell(Q)$, for all $Z \in U^i_Q$ (so in particular, for $Z = Y$), and since $|W| \le 2\delta(Y) \lesssim \ell(Q)$, for all $W \in B_Y$. Thus, (5.65) holds with $k = 1$, since $Y_1 \in B_Y$, by construction. Now suppose that (5.65) holds for all $1 \le i \le k$, with $k \le N$. Let $W \in B_{Y_k}$, so that W may be joined to Y_k by a line segment of length less than $\delta(Y_k) \lesssim \varepsilon^{-3} \ell(Q)$ (the latter bound holds by (5.50)). We note also that if $k \le N - 1$, and if $W = Y_{k+1}$, then this line segment has length at most $\ell(Q)$, by construction. Then

$$\begin{aligned}&\big|u(W) - |\nabla u(Y)| W_{n+1}\big| \\ &\quad \le |u(W) - u(Y_k) + |\nabla u(Y)| (Y_k - W) \cdot e_{n+1}\big| + \big|u(Y_k) - |\nabla u(Y)| Y_k \cdot e_{n+1}\big| \\ &\qquad = \big|(W - Y_k) \cdot \nabla u(W_1) + |\nabla u(Y)| (Y_k - W) \cdot e_{n+1}\big| + O\Big(k\, \varepsilon^{M-3} \ell(Q)\Big)\,,\end{aligned}$$

where W_1 is an appropriate point on the line segment joining W and Y_k, and where we have used that Y_k satisfies (5.65). By (5.63), applied to W_1, we find in turn that

$$\big|u(W) - |\nabla u(Y)| W_{n+1}\big| \lesssim \varepsilon^{M-3}\, |W - Y_k| + k\, \varepsilon^{M-3} \ell(Q)\,, \tag{5.68}$$

which, by our previous observations, is bounded by $C(k+1)\varepsilon^{M-3}\ell(Q)$, if $W = Y_{k+1}$, or by $(\varepsilon^{M-6} + k\, \varepsilon^{M-3})\ell(Q)$, in general. In the former case, we find that (5.65) holds for all $k = 1, 2, \dots, N$, and in the latter case, taking $k = N \lesssim \varepsilon^{-4}$, we obtain (5.66). □

Claim 5.69. *Let $X \in \widetilde{U}^i_Q$, and let $w \in E \cap \partial \widetilde{B}_X$. Then*

$$|\nabla u(Y)|\, |w_{n+1}| \lesssim \varepsilon^{M/2} \ell(Q)\,. \tag{5.70}$$

Proof of Claim 5.69. Let W be the radial projection of w onto ∂B_X, so that

$$|W - w| = \varepsilon^{2M/\alpha} \delta(X) \lesssim \varepsilon^{(2M/\alpha)-3} \ell(Q)\,, \tag{5.71}$$

by (5.50). We write

$$\begin{aligned}|\nabla u(Y)|\, |w_{n+1}| \le |\nabla u(Y)|\, |W - w| + \big|u(W) - |\nabla u(Y)| W_{n+1}\big| + u(W) \\ =: I + II + u(W).\end{aligned}$$

Note that $I \lesssim \varepsilon^{(2M/\alpha)-3} \ell(Q)$, by (5.71) and (5.11) (recall that $Y = Y_Q$), and that $II \lesssim \varepsilon^{M-7} \ell(Q)$, by (5.66). Furthermore, $u(W) \lesssim \varepsilon^{2M-5} \ell(Q)$, by (5.49). For M chosen large enough, we obtain (5.70). □

We note that since we have fixed $Y = Y_Q$, it then follows from (5.70) and (5.11) that

$$|w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q), \quad \forall\, w \in E \cap \partial\widetilde{B}_X, \quad \forall X \in \widetilde{U}^i_Q. \tag{5.72}$$

Recall that x_Q denotes the "center" of Q (see (2.7)-(2.8)). Set

$$O := B\left(x_Q, 2\varepsilon^{-2}\ell(Q)\right) \cap \left\{W : W_{n+1} > \varepsilon^2\ell(Q)\right\}. \tag{5.73}$$

Claim 5.74. *For every point $X \in O$, we have $X \approx_{\varepsilon,Q} Y$ (see Definition 2.26). Thus, in particular, $O \subset \widetilde{U}^i_Q$.*

Proof of Claim 5.74. Let $X \in O$. We need to show that X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$ (for convenience, we shall refer to such balls as "admissible"). We first observe that if $X = te_{n+1}$, with $\varepsilon^3\ell(Q) \le t \le \varepsilon^{-3}\ell(Q)$, then by an iteration argument using (5.72) (with M chosen large enough), we may join X to Y by at most $C\log(1/\varepsilon)$ admissible balls. The point $(2\varepsilon)^{-3}\ell(Q)e_{n+1}$ may then be joined to any point of the form $(X', (2\varepsilon)^{-3}\ell(Q))$ by a chain of at most C admissible balls, whenever $X' \in \mathbb{R}^n$ with $|X'| \le \varepsilon^{-3}\ell(Q)$. In turn, the latter point may then be joined to $(X', \varepsilon^3\ell(Q))$ by at most $C\log(1/\varepsilon)$ admissible balls. □

We note that Claim 5.74 implies that

$$E \cap O = \emptyset. \tag{5.75}$$

Indeed, $O \subset \widetilde{U}^i_Q \subset \Omega$. Let P_0 denote the hyperplane

$$P_0 := \{Z : Z_{n+1} = 0\}.$$

Claim 5.76. *If $Z \in P_0$, with $|Z - x_Q| \le \frac{3}{2}\varepsilon^{-2}\ell(Q)$, then*

$$\delta(Z) = \operatorname{dist}(Z, E) \le 16\varepsilon^2\ell(Q). \tag{5.77}$$

Proof of Claim 5.76. Observe that $B(Z, 2\varepsilon^2\ell(Q))$ meets O. Then by Claim 5.74, there is a point $X \in \widetilde{U}^i_Q \cap B(Z, 2\varepsilon^2\ell(Q))$. Suppose now that (5.77) is false, which in particular implies so that $\delta(X) \ge 14\varepsilon^2\ell(Q)$. Then $B(Z, 4\varepsilon^2\ell(Q)) \subset B_X$, so by (5.66), we have

$$|u(W) - |\nabla u(Y)| W_{n+1}| \le C\,\varepsilon^{M-7}\ell(Q), \quad \forall\, W \in B(Z, 4\varepsilon^2\ell(Q)). \tag{5.78}$$

In particular, since $Z_{n+1} = 0$, we may choose W such that $W_{n+1} = -\varepsilon^2\ell(Q)$, to obtain that

$$|\nabla u(Y)|\,\varepsilon^2\ell(Q) \le C\varepsilon^{M-7}\ell(Q),$$

since $u \ge 0$. But for $\varepsilon < 1/2$, and M large enough, this is a contradiction, by (5.11) (recall that we have fixed $Y = Y_Q$). □

It now follows by Definition 2.17 that Q satisfies the ε-local WHSA condition, with

$$P = P(Q) := \{Z : Z_{n+1} = \varepsilon^2\ell(Q)\}, \quad H = H(Q) := \{Z : Z_{n+1} > \varepsilon^2\ell(Q)\}.$$

This concludes the proof of Lemma 5.10.

5.4. **Proof of Corollary 1.5.** Corollary 1.5 follows almost immediately from Theorem 1.1. Let $B = B(x, r)$ and $\Delta = B \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Let c be the constant in Lemma 3.1. By hypothesis, there is a point $X_\Delta \in B \cap \Omega$ which is a corkscrew point relative to Δ, that is, there is a uniform constant $c_0 > 0$ such that $\delta(X_\Delta) \geq c_0 r$. Thus, to apply Theorem 1.1, it remains only to verify hypothesis ($\star$). For a sufficiently large constant C_1, set $\Delta^{fat} = \Delta(x, C_1 r)$. Cover Δ^{fat} by a collection of surface balls $\{\Delta_i\}_{i=1}^N$ with $\Delta_i = B_i \cap \partial\Omega$, and $B_i := B(x_i, c_0 r/4)$, where $x_i \in \Delta^{fat}$ and where N is uniformly bounded, depending only on n, c_0, C_1 and ADR. By construction, $X_\Delta \in \Omega \setminus 4B_i$, so by hypothesis, $\omega^{X_\Delta} \in$ weak-$A_\infty(2\Delta_i)$. Hence, $\omega^{X_\Delta} \ll \sigma$ in $2\Delta_i$, and (1.6) holds with $Y = X_\Delta$, and with $\Delta' = \Delta_i$. Consequently, $\omega^{X_\Delta} \ll \sigma$ in Δ^{fat}, and if we write $k^{X_\Delta} = d\omega^{X_\Delta}/d\sigma$ we obtain

$$\int_{\Delta^{fat}} k^{X_\Delta}(z)^q \, d\sigma(z) \leq \sum_{i=1}^N \int_{\Delta_i} k^{X_\Delta}(z)^q \, d\sigma(z) \lesssim \sum_{i=1}^N \sigma(\Delta_i) \left(\fint_{2\Delta_i} k^{X_\Delta}(z) \, d\sigma(z) \right)^q$$
$$\lesssim \sum_{i=1}^N \sigma(2\Delta_i)^{1-q} \, \omega^{X_\Delta}(2\Delta_i) \lesssim \sigma(\Delta^{fat})^{1-q},$$

where in the last estimate we have used the ADR property, the uniform boundedness of N, and the fact that $\omega^{X_\Delta}(2\Delta_i) \leq 1$. By Theorem 1.1, it then follows that $\partial\Omega$ is UR as desired. □

6. Proof of Proposition 1.17

We here prove Proposition 1.17. We first observe that if E is UR then it satisfies the so-called "bilateral weak geometric lemma (BWGL)" (see [DS1, Theorem I.2.4, p. 32]). In turn, in [DS1, Section II.2.1, pp. 97], one can find a dyadic formulation of the BWGL as follows. Given ε small enough and $k > 1$ large to be chosen, $\mathbb{D}(E)$ can be split in two collections, one of "bad cubes" and another of "good cubes", so that the "bad cubes" satisfy a packing condition and each "good cube" Q verifies the following: there is a hyperplane $P = P(Q)$ such that $\operatorname{dist}(Z, E) \leq \varepsilon \, \ell(Q)$ for every $Z \in P \cap B(x_Q, k \, \ell(Q))$, and $\operatorname{dist}(Z, P) \leq \varepsilon \, \ell(Q)$ for every $Z \in B(x_Q, k \, \ell(Q)) \cap E$. In turn, this implies that $B(x_Q, k \, \ell(Q)) \cap E$ is sandwiched between to planes parallel to P at distance $\varepsilon \, \ell(Q)$. Hence, at that scale, we have a half-space (indeed we have two) free of E and clearly the $2\,\varepsilon$-local WHSA holds provided K is taken of the order of ε^{-2} or larger. Further details are left to the interested reader. Thus we obtain the easy implication UR $\implies$ WHSA.

The main part of the proof is to establish the opposite implication. To this end, we assume that E satisfies the WHSA property and we shall show that E is UR. Given a positive $\varepsilon < \varepsilon_0 \ll K_0^{-6}$, we let $\mathcal{B}_0$ denote the collection of bad cubes for which ε-local WHSA fails. By Definition 2.19, $\mathcal{B}_0$ satisfies the Carleson packing condition (2.20). We now introduce a variant of the packing measure for $\mathcal{B}_0$. We recall that $B_Q^* = B(x_Q, K_0^2 \ell(Q))$, and given $Q \in \mathbb{D}(E)$, we set

$$\mathbb{D}_\varepsilon(Q) := \left\{ Q' \in \mathbb{D}(E) : \, \varepsilon^{3/2} \ell(Q) \leq \ell(Q') \leq \ell(Q), \; Q' \text{ meets } B_Q^* \right\}. \tag{6.1}$$

Set

$$\alpha_Q := \begin{cases} \sigma(Q), & \text{if } \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{6.2}$$

and define

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q, \qquad \mathbb{D}' \subset \mathbb{D}(E). \tag{6.3}$$

Then $\mathfrak{m}$ is a discrete Carleson measure, with

$$\mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \leq C_\varepsilon \, \sigma(Q_0), \qquad Q_0 \in \mathbb{D}(E). \tag{6.4}$$

Indeed, note that for any Q', the cardinality of $\{Q : Q' \in \mathbb{D}_\varepsilon(Q)\}$, is uniformly bounded, depending on n, ε and ADR, and that $\sigma(Q) \leq C_\varepsilon \, \sigma(Q')$, if $Q' \in \mathbb{D}_\varepsilon(Q)$. Then given any $Q_0 \in \mathbb{D}(E)$,

$$\begin{aligned} \mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0 \colon \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset} \sigma(Q) &\leq \sum_{Q' \in \mathcal{B}_0} \sum_{Q \subset Q_0 \colon Q' \in \mathbb{D}_\varepsilon(Q)} \sigma(Q) \\ &\leq C_\varepsilon \sum_{Q' \in \mathcal{B}_0 \colon Q' \subset 2B^*_{Q_0}} \sigma(Q') \leq C_\varepsilon \, \sigma(Q_0), \end{aligned}$$

by (2.20) and ADR.

To prove Proposition 1.17, we are required to show that the collection $\mathcal{B}$ of bad cubes for which the $\sqrt{\varepsilon}$-local BAUP condition fails, satisfies a packing condition. That is, we shall establish the discrete Carleson measure estimate

$$\widetilde{\mathfrak{m}}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0 \colon Q \in \mathcal{B}} \sigma(Q) \leq C_\varepsilon \, \sigma(Q_0), \qquad Q_0 \in \mathbb{D}(E). \tag{6.5}$$

To this end, by (6.4), it suffices to show that if $Q \in \mathcal{B}$, then $\alpha_Q \neq 0$ (and thus $\alpha_Q = \sigma(Q)$, by definition). In fact, we shall prove the contrapositive statement.

Claim 6.6. *Suppose then that $\alpha_Q = 0$. Then $\sqrt{\varepsilon}$-local BAUP condition holds for Q.*

Proof of Claim 6.6. We first note that since $\alpha_Q = 0$, then by definition of α_Q,

$$\mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) = \emptyset. \tag{6.7}$$

Thus, the ε-local WHSA condition (Definition 2.17) holds for every $Q' \in \mathbb{D}_\varepsilon(Q)$ (in particular, for Q itself). By rotation and translation, we may suppose that the hyperplane $P = P(Q)$ in Definition 2.17 is

$$P = \left\{ Z \in \mathbb{R}^{n+1} : Z_{n+1} = 0 \right\},$$

and that the half-space $H = H(Q)$ is the upper half-space $\mathbb{R}^{n+1}_+ = \{Z : Z_{n+1} > 0\}$. We recall that by Definition 2.17, P and H satisfy

$$\operatorname{dist}(Z, E) \leq \varepsilon \ell(Q), \qquad \forall\, Z \in P \cap B^{**}_Q(\varepsilon). \tag{6.8}$$

$$\operatorname{dist}(P, Q) \leq K_0^{3/2} \ell(Q), \tag{6.9}$$

and

$$H \cap B^{**}_Q(\varepsilon) \cap E = \emptyset\,. \tag{6.10}$$

The proof will now follow by a construction similar to the construction in [LV2]. In [LV2] the authors used the construction to establish the Weak Exterior Convexity condition. By (6.10), there are two cases.

Case 1: $10Q \subset \{Z : -\sqrt{\varepsilon}\ell(Q) \leq Z_{n+1} \leq 0\}$. In this case, the $\sqrt{\varepsilon}$-local BAUP condition holds trivially for Q, with $\mathcal{P} = \{P\}$.

Case 2. There is a point $x \in 10Q$ such that $x_{n+1} < -\sqrt{\varepsilon}\ell(Q)$. In this case, we choose $Q' \ni x$, with $\varepsilon^{3/4}\ell(Q) \leq \ell(Q') < 2\varepsilon^{3/4}\ell(Q)$. Thus,

$$Q' \subset \{Z : Z_{n+1} \leq -\frac{1}{2}\sqrt{\varepsilon}\ell(Q)\}\,. \tag{6.11}$$

Moreover, $Q' \in \mathbb{D}_\varepsilon(Q)$, so by (6.7), $Q' \notin \mathcal{B}_0$, i.e., Q' satisfies the ε-local WHSA. Let $P' = P(Q')$, and $H' = H(Q')$ denote the hyperplane and half-space corresponding to Q' in Definition 2.17, so that

$$\operatorname{dist}(Z, E) \leq \varepsilon\ell(Q') \leq 2\varepsilon^{7/4}\ell(Q)\,, \quad \forall\, Z \in P' \cap B^{**}_{Q'}(\varepsilon)\,, \tag{6.12}$$

$$\operatorname{dist}(P', Q') \leq K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q) \tag{6.13}$$

(where the last inequality holds since $\varepsilon \ll K_0^{-6}$), and

$$H' \cap B^{**}_{Q'}(\varepsilon) \cap E = \emptyset\,, \tag{6.14}$$

where we recall that $B^{**}_{Q'}(\varepsilon) := B\left(x_{Q'}, \varepsilon^{-2}\ell(Q')\right)$ (see (2.16)). We note that

$$B^*_Q \subset \widetilde{B}_Q(\varepsilon) := B\left(x_Q, \varepsilon^{-1}\ell(Q)\right) \subset B^{**}_{Q'}(\varepsilon) \cap B^{**}_Q(\varepsilon)\,, \tag{6.15}$$

by construction, since $\varepsilon \ll K_0^{-6}$. Let ν' denote the unit normal vector to P', pointing into H'. Note that by (6.10), (6.12), and the definition of H,

$$P' \cap \widetilde{B}_Q(\varepsilon) \cap \{Z : Z_{n+1} > 2\varepsilon^{7/4}\,\ell(Q)\} = \emptyset\,. \tag{6.16}$$

Moreover, ν' points "downward", i.e., $\nu' \cdot e_{n+1} < 0$, otherwise $H' \cap \widetilde{B}_Q(\varepsilon)$ would meet E, by (6.8), (6.11), and (6.13). More precisely, we have the following.

Claim 6.17. *The angle θ between ν' and $-e_{n+1}$ satisfies* $0 \leq \theta \approx \sin\theta \lesssim \varepsilon$.

Indeed, since Q' meets $10Q$, (6.9) and (6.13) imply that $\operatorname{dist}(P, P') \lesssim K_0^{3/2}\ell(Q)$, and that the latter estimate is attained near Q. By (6.16) and a trigonometric argument, one then obtains Claim 6.17 (more precisely, one obtains $\theta \lesssim K_0^{3/2}\varepsilon$, but in this section, we continue to use the notational convention that implicit constants may depend upon K_0, but K_0 is fixed, and $\varepsilon \ll K_0^{-6}$). The interested reader could probably supply the remaining details of the argument that we have just sketched, but for the sake of completeness, we shall give the full proof at the end of this section.

We therefore take Claim 6.17 for granted, and proceed with the argument. We note first that every point in $(P \cup P') \cap B^*_Q$ is at a distance at most $\varepsilon\ell(Q)$ from E, by (6.8), (6.12) and (6.15). To complete the proof of Claim 6.6, it therefore remains only to verify the following. As with the previous claim, we shall provide a condensed proof immediately, and present a more detailed argument at the end of the section.

Claim 6.18. *Every point in* $10Q$ *lies within* $\sqrt{\varepsilon}\ell(Q)$ *of a point in* $P \cup P'$.

Suppose not. We could then repeat the previous argument, to construct a cube Q'', a hyperplane P'', a unit vector ν'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'', with the same properties as Q', P', ν' and H'. In particular, we have the respective analogues of (6.13) and (6.14), namely

$$\operatorname{dist}(P'', Q'') \leq K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q)\,, \tag{6.19}$$

and

$$H'' \cap B^{**}_{Q''}(\varepsilon) \cap E = \emptyset\,, \tag{6.20}$$

Also, we have the analogue of (6.11), with Q'', P' in place of Q', P, thus

$$\operatorname{dist}(Q'', P') \geq \frac{1}{2}\sqrt{\varepsilon}\ell(Q)\,, \quad \text{and} \quad Q'' \cap H' = \emptyset\,, \tag{6.21}$$

In addition, as in (6.15), we also have $B^*_Q \subset B^{**}_{Q''}(\varepsilon)$. On the other hand, the angle between ν' and ν'' is very small. Thus, combining (6.12), (6.19) and (6.21), we see that $H'' \cap B^*_Q$ captures points in E, which contradicts (6.20).

Claim 6.6 therefore holds (in fact, with a union of at most 2 planes), and thus we obtain the conclusion of Proposition 1.17. □

We now provide detailed proofs of Claims 6.17 and 6.18.

Proof of Claim 6.17. By (6.13) we can pick $x' \in Q'$, $y' \in P'$ such that $|y' - x'| \ll \varepsilon^{1/2}\,\ell(Q)$ and therefore $y' \in 11\,Q$. Also, from (6.9) and (6.10) we can find $\bar{x} \in Q$ such that $-K_0^{3/2}\,\ell(Q) < \bar{x}_{n+1} \leq 0$. This and (6.11) yield

$$-\,2\,K_0^{3/2}\,\ell(Q) < y'_{n+1} < -\frac{1}{4}\,\sqrt{\varepsilon}\,\ell(Q). \tag{6.22}$$

Let π denote the orthogonal projection onto P. Let $Z \in P$ (i.e., $Z_{n+1} = 0$) be such that $|Z - \pi(y')| \leq K_0^{3/2}\,\ell(Q)$. Then, $Z \in B(x_Q, 4\,K_0^{3/2}\,\ell(Q)) \subset B^*_Q$. Hence $Z \in P \cap B^{**}_Q(\varepsilon)$ and by (6.8), $\operatorname{dist}(Z, E) \leq \varepsilon\,\ell(Q)$. Then there exists $x_Z \in E$ with $|Z - x_Z| \leq \varepsilon\,\ell(Q)$ which in turn implies that $|(x_Z)_{n+1}| \leq \varepsilon\,\ell(Q)$. Note that $x_Z \in B(x_Q, 5\,K_0^{3/2}\,\ell(Q)) \subset B^*_Q$ and by (6.15), $x_Z \in E \cap B^{**}_Q(\varepsilon) \cap B^{**}_{Q'}(\varepsilon)$. This, (6.10) and (6.14) imply that $x_Z \notin H \cup H'$. Hence, $(x_Z)_{n+1} \leq 0$ and $(x_Z - y') \cdot \nu' \leq 0$, since $y' \in P'$ and ν' denote the unit normal vector to P' pointing into H'. Using (6.22) we observe that

$$\frac{1}{8}\,\sqrt{\varepsilon}\,\ell(Q) < -\varepsilon\,\ell(Q) + \frac{1}{4}\,\sqrt{\varepsilon}\,\ell(Q) < (x_Z - y')_{n+1} < 2\,K_0^{3/2}\,\ell(Q) \tag{6.23}$$

and that

$$\begin{aligned}(x_Z - y')_{n+1}\,\nu'_{n+1} &\leq -\pi(x_Z - y') \cdot \pi(\nu') \\ &\leq |x_Z - z| - \pi(Z - y') \cdot \pi(\nu') \leq \varepsilon\,\ell(Q) - \pi(Z - y') \cdot \pi(\nu').\end{aligned} \tag{6.24}$$

We shall prove that $\nu'_{n+1} < -\frac{1}{8} < 0$ by considering two cases:

Case 1: $|\pi(\nu')| \geq \frac{1}{2}$. We pick

$$Z_1 = \pi(y') + K_0^{3/2}\,\ell(Q)\,\frac{\pi(\nu')}{|\pi(\nu')|}.$$

By construction $Z_1 \in P$ and $|Z_1 - \pi(y')| \leq K_0^{3/2}\,\ell(Q)$. Hence we can use (6.24) with Z_1

$$\begin{aligned}(x_{Z_1} - y')_{n+1}\,\nu'_{n+1} &\leq \varepsilon\,\ell(Q) - \pi(Z_1 - y')\cdot\pi(\nu') \\ &= \varepsilon\,\ell(Q) - K_0^{3/2}\,\ell(Q)\,|\pi(\nu')| \leq -\frac{1}{4}\,K_0^{3/2}\,\ell(Q).\end{aligned}$$

This together with (6.23) give that $\nu'_{n+1} < -1/8 < 0$.

Case 2: $|\pi(\nu')| < \frac{1}{2}$. This case is much simpler. Note first that $|\nu'_{n+1}|^2 = 1-|\pi(\nu')|^2 > 3/4$ and thus either $\nu'_{n+1} < -\sqrt{3}/2$ or $\nu'_{n+1} > \sqrt{3}/2$. We see that the second scenario leads to a contradiction. Assume then that $\nu'_{n+1} > \sqrt{3}/2$. We take $Z_2 = \pi(y') \in P$ which clearly satisfies and $|Z_2 - \pi(y')| \leq K_0^{3/2}\,\ell(Q)$. Again (6.24) and (6.23) are applicable with Z_2

$$\frac{1}{8}\,\sqrt{\varepsilon}\,\ell(Q)\,\frac{\sqrt{3}}{2} < (x_{Z_2} - y')_{n+1}\,\nu'_{n+1} \leq \varepsilon\,\ell(Q) \ll \sqrt{\varepsilon}\,\ell(Q),$$

and we get a contradiction. Hence necessarily $\nu'_{n+1} \leq -\sqrt{3}/2 < -1/8 < 0$.

Having proved that $\nu'_{n+1} < -1/8 < 0$ we estimate θ, the angle between ν' and $-e_{n+1}$. Note first $\cos\theta = -\nu'_{n+1} > 1/8$. If $\cos\theta = 1$ (which occurs if $\nu' = -e_{n+1}$) then $\theta = \sin\theta = 0$ and the proof is complete. Assume then that $\cos\theta \neq 1$ in which case $1/8 < -\nu'_{n+1} < 1$ and hence $|\pi(\nu')| \neq 0$. Pick

$$Z_3 = y' + \frac{\ell(Q)}{2\,\varepsilon}\,\hat{\nu}', \qquad\qquad \hat{\nu}' = \frac{e_{n+1} - \nu'_{n+1}\,\nu'}{|\pi(\nu')|}.$$

Then $\hat{\nu}'\cdot\nu' = 0$ and hence $Z_3 \in P'$ as $y' \in P'$. Also $|\hat{\nu}'| = 1$ and therefore $|Z_3 - y'| = \ell(Q)/(2\,\varepsilon)$. This in turn gives that $Z_3 \in \widetilde{B}_Q(\varepsilon)$. We have obtained that $Z_3 \in P' \cap \widetilde{B}_Q(\varepsilon)$, and hence $(Z_3)_{n+1} \leq 2\varepsilon^{7/4}\,\ell(Q)$ by (6.16). This and (6.23) applied to Z_3 easily give

$$\begin{aligned}4\,K_0^{3/2}\,\ell(Q) \geq 2\varepsilon^{7/4}\,\ell(Q) \geq (Z_3)_{n+1} &= y'_{n+1} + \frac{\ell(Q)}{2\,\varepsilon}\,\frac{1-(\nu'_{n+1})^2}{|\pi(\nu')|} \\ &= y'_{n+1} + \frac{\ell(Q)}{2\,\varepsilon}\,|\pi(\nu')| \geq -2\,K_0^{3/2}\,\ell(Q) + \frac{\ell(Q)}{2\,\varepsilon}\,|\pi(\nu')|.\end{aligned}$$

This readily yields $|\sin\theta| = |\pi(\nu')| \leq 8\,K_0^{3/2}\,\varepsilon$ and the proof is complete. □

Proof of Claim 6.18. We want to prove that every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P\cup P'$. We will argue by contradiction and hence we assume that there exists $x' \in 10Q$ with $\operatorname{dist}(x', P\cup P') > \sqrt{\varepsilon}\,\ell(Q)$. In particular, $x'_{n+1} < -\sqrt{\varepsilon}\,\ell(Q)$ and as observed above, we may repeat the previous argument, to construct a cube Q'', a hyperplane P'', a unit vector ν'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'', with the same properties as Q', P', ν' and H', namely (6.19), (6.21) and (6.20). Also,

$$\sqrt{\varepsilon}\,\ell(Q) \le \operatorname{dist}(x', P') \le \operatorname{diam}(Q'') + \operatorname{dist}(Q'', P') \le \frac{1}{2}\sqrt{\varepsilon}\,\ell(Q) + \operatorname{dist}(Q'', P'),$$

and, in addition, as in (6.15), we have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$.

By (6.19) there is $y'' \in Q''$ and $z'' \in P''$ such that $|y'' - z''| \ll \varepsilon^{1/2}\,\ell(Q)$. By (6.20) $y'' \notin H'$. Write π' to denote the orthogonal projection onto P' and note that (6.21) give $\operatorname{dist}(y'', P') = |y'' - \pi'(y'')| \ge \frac{1}{2}\sqrt{\varepsilon}\ell(Q)$. Note also that

$$\begin{aligned} |y'' - \pi'(y'')| &= \operatorname{dist}(y'', P') \\ &\le |y'' - x'| + |x' - x| + \operatorname{diam}(Q') + \operatorname{dist}(Q', P') \le 11 \operatorname{diam}(Q) \end{aligned}$$

and that

$$|\pi'(y'') - x_Q| \le |\pi'(y'') - y''| + |y'' - x'| + |x' - x_Q| < 22 \operatorname{diam}(Q) < K_0^2\,\ell(Q).$$

Hence $\pi'(y'') \in B_Q^* \subset \widetilde{B}_Q(\varepsilon)$ and since $\pi'(y'') \in P'$ we have that (6.12) gives that there is $\tilde{y} \in E$ with $|\pi'(y'') - \tilde{y}| \le 2\,\varepsilon^{7/4}\,\ell(Q)$. Then $\tilde{y} \in 23Q \subset B_Q^* \cap E$ and $|\tilde{y} - z''| < 12 \operatorname{diam}(Q)$. To complete our proof we just need to show that $\tilde{y} \in H''$ which contradicts (6.20).

We now prove that $\tilde{y} \in H''$. Write ν'' to denote the unit normal vector to P'', pointing into H'' and let us momentarily assume that

$$|\nu' - \nu''| \le 16\sqrt{2}\,K_0^{2/3}\,\varepsilon. \tag{6.25}$$

We then obtain, recalling that $y'' \notin H'$, that

$$\begin{aligned} \frac{1}{2}\sqrt{\varepsilon}\,\ell(Q) &\le |y'' - \pi'(y'')| = (\pi'(y'') - y'') \cdot \nu' \\ &\le |\pi'(y'') - \tilde{y}| + |\tilde{y} - z''|\,|\nu' - \nu''| + (\tilde{y} - z'') \cdot \nu'' + |z'' - y''| \\ &< \frac{1}{4}\sqrt{\varepsilon}\,\ell(Q) + (\tilde{y} - z'') \cdot \nu''. \end{aligned}$$

This immediately gives that $(\tilde{y} - z'') \cdot \nu'' > \frac{1}{4}\sqrt{\varepsilon}\,\ell(Q) > 0$ and hence $\tilde{y} \in H''$ as desired. Hence to complete the proof we have to prove (6.25). To start the proof, we first note that if $|\alpha| < \pi/4$, then

$$1 - \cos\alpha = 1 - \sqrt{1 - \sin^2\alpha} \le \sin^2\alpha.$$

In particular, we can apply this to θ (resp. θ'), which is the angle between ν' (resp. ν'') and $-e_{n+1}$, and as we shows that $|\sin\theta|, |\sin\theta'| \le 8\,K_0^{3/2}\,\varepsilon$, we see that

$$\sqrt{1 - \cos\theta} + \sqrt{1 - \cos\theta'} \le 16\,K_0^{3/2}\,\varepsilon$$

Using the trivial formula

$$|a - b|^2 = 2(1 - a\dot{b}), \qquad \forall\, a, b \in \mathbb{R}^{n+1}, \quad |a| = |b| = 1.$$

we conclude that

$$\begin{aligned} |\nu' - \nu''| &\le |\nu' - (-e_{n+1})| + |(-e_{n+1}) - \nu''| \\ &= \sqrt{2\,(1 + \nu'\,e_{n+1})} + \sqrt{2\,(1 + \nu''\,e_{n+1})} \\ &= \sqrt{2\,(1 - \cos\theta)} + \sqrt{2\,(1 - \cos\theta')} \le 16\sqrt{2}\,K_0^{3/2}\,\varepsilon. \end{aligned}$$

This proves (6.25) and hence the proof of Claim 6.18 is complete. □

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