Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces.

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Abstract

Let μ be a non-negative Borel measure on $\mathbb{R}^d$. Fix a real number n, $0 < n \leq d$, and assume that μ is "n-dimensional" in the following sense: the measure of a cube is smaller than the length of its side raised to the n-th power. Calderón-Zygmund operators, Hardy and BMO spaces, and some other topics in Harmonic Analysis have been successfully handled in this setting recently, although the measure may be non-doubling. The aim of this paper is to study two-weight norm inequalities for radial fractional maximal functions associated to such μ. Namely, we characterize those pairs of weights for which these maximal operators satisfy strong and weak type inequalities. Sawyer and radial Muckenhoupt type conditions are respectively the solutions for these problems. Furthermore, if we strengthen Muckenhoupt conditions by adding a "power-bump" to the right-hand side weight or even by introducing certain Orlicz norm, strong type inequalities can be achieved. As a consequence, two-weight norm inequalities for fractional integrals associated to μ are obtained. Finally, for the particular case of the Hardy-Littlewood radial maximal function, we show how, in contrast with the classical situation, radial Muckenhoupt weights may fail to satisfy a reverse Hölder's inequality and also strong type inequalities do not necessarily hold for them.

1 Introduction.

Let μ be a non-negative "n-dimensional" Borel measure on $\mathbb{R}^d$, that is, a measure satisfying

$$\mu(Q) \leq \ell(Q)^n$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $\ell(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \leq d$. Throughout this

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paper, the only cubes we shall consider will be those with sides parallel to the coordinate axes and we shall always denote the side length as above. Besides, for $r > 0$, $r\,Q$ will mean the cube with the same centre as Q and with $\ell(r\,Q) = r\,\ell(Q)$. Moreover, $Q(x,r)$ will be the cube centered at x with side length r.

In the case of the Lebesgue measure, the translation invariance and the good behaviour with respect to dilations are strong tools for the development of Harmonic Analysis. A natural extension is obtained with the concept of space of homogeneous type, which is a quasi-metric space endowed with a doubling measure. The doubling condition says that for every ball B, the measure of the 2-dilated ball, $2\,B$ having the same center and double radius, is controlled by the measure of B. In other words, the measure behaves well under dilations. Nowadays, it seems that this doubling condition can be removed and many results of Harmonic Analysis are still true without it. X. Tolsa in [To1], [To2] managed to handle the Cauchy integral operator associated to a "1-dimensional" measure in $\mathbb{C}$ which might be non-doubling. He characterized those measures for which this operator is bounded in L^2 and he also treated the existence of principal values. On the other hand, F. Nazarov, S. Treil and A. Volberg in [NTV1], [NTV2], introduce the non-homogeneous spaces which are metric spaces with an "n-dimensional" measure. They deal with Calderón-Zygmund operators obtaining a $T(1)$ theorem and the expected weak and strong type inequalities. Again, there is no need of any doubling condition. After these works, the new field of non-homogeneous Harmonic Analysis has experienced a great development. For questions about H^1 and BMO, the reader is referred to [MMNO], [To3], [To4]. Vector-valued inequalities, their relation with weights and the existence of principal values in weighted spaces are treated in [GM1], [GM2]. Finally, in [OP], the authors study weighted norm inequalities for the centered Hardy-Littlewood maximal function and Muckenhoupt weights.

In this line, the aim of the present work is to consider two-weight norm inequalities for the following maximal operators: given $0 \le \alpha < n$, define the radial fractional maximal functions

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\ell(Q)^{n-\alpha}} \int_Q |f(y)|\,d\mu(y).$$

When $\alpha = 0$, we are considering the Hardy-Littlewood radial maximal function $\mathcal{M}_0 = \mathcal{M}$. We shall investigate for which pairs of weights $\mathcal{M}_\alpha$ satisfies a strong or a weak type inequality. A weight w will be a locally integrable function which is positive almost everywhere (with respect to the measure μ). For any measurable set E we shall write $w(E) = \int_E w\,d\mu$ and $L^p(w) = L^p(w\,d\mu)$ for $0 < p < \infty$. If $1 \le p \le \infty$, then, as usual, p' will be the exponent conjugate to p, that is, the one satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

The plan of the paper is the following: Section 2 contains some immediate estimates for $\mathcal{M}_\alpha$. In Section 3, we characterize those pairs of weights for which this maximal operator satisfies a strong type inequality (see Theorem 3.1). Sawyer type conditions are obtained by means of a discretization method based on ideas of [Saw] and the recent work [Cru]. Let us mention that variants of this basic technique will be used all throughout the paper. We devote Section 4 to show that radial Muckenhoupt classes $\mathcal{A}^\alpha_{p,q}$ (see Definition 4.1) are the precise classes giving the two-weight weak type inequalities. The main drawback of Sawyer type conditions is that they involve the operator itself.

It would be better if one could prove strong estimates by using some modification of the Muckenhoupt classes. We are interested in strengthening $\mathcal{A}^{\alpha}_{p,q}$ to obtain sufficient conditions for the strong type inequalities. In Theorem 5.1, we prove that this can be done by introducing a "power-bump" on the right-hand side weight. Namely, if $1 < p < q < \infty$, $0 \le \alpha < n$ and the pair of weights (u, v) satisfies

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v(x)^{r\,(1-p')}\,d\mu(x)\Big)^{\frac{1}{r\,p'}} \le C, \tag{1}$$

for some $r > 1$ and for every cube Q, then $\mathcal{M}_\alpha$ is bounded from $L^p(v)$ to $L^q(u)$. The proof of this result relies on the ideas we use to prove Theorem 3.1. In fact, we can go further because these techniques allow us even to relax condition (1) and still obtain the same estimate for $\mathcal{M}_\alpha$. This can be achieved by replacing the term in v by certain Orlicz norm localized in the cube Q (see Subsection 5.2 for the details). In Section 6, we study some estimates for fractional integrals. For $0 < \alpha < n$ and $f \in L^\infty(\mu)$ a boundedly supported function, we define the fractional integral of order α as

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\,d\mu(y).$$

If we introduce in (1) another "power-bump" on the left-hand term, that is, if

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)^r\,d\mu(x)\Big)^{\frac{1}{r\,q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v(x)^{r\,(1-p')}\,d\mu(x)\Big)^{\frac{1}{r\,p'}} \le C, \tag{2}$$

holds for some $r > 1$ and for every Q, then I_α is bounded from $L^p(v)$ to $L^q(u)$ for $1 < p < q < \infty$. The proof is a consequence of the corresponding result for $\mathcal{M}_\alpha$ and some version of the inequality proved in [Wel] in the doubling case which relates fractional integrals and radial fractional maximal functions (see Theorems 6.4 and 6.5). Finally, Section 7 pays special attention to the Hardy-Littlewood radial maximal function $\mathcal{M}$. We are only interested in the case $p = q$. Proposition 7.1 collects some easy properties of Muckenhoupt and Sawyer classes. We also study what happens when the weight is the same in both sides of the inequalities, that is, when $u = v$. It is well known that in the classical setting —$\mathbb{R}^d$, with the Lebesgue measure— the Muckenhoupt condition admits certain self-improvement. For $1 \le p < \infty$, denote by A_p the classical Muckenhoupt weight class and denote the classical Hardy-Littlewood maximal function by M. Observe that the radial classes $\mathcal{A}_p$ (see Section 7) become A_p when μ is the Lebesgue measure (in that case $M = \mathcal{M}$). As we can see in [GR, Chapter IV], some of the most relevant properties of A_p-weights are the following:

(a) If $w \in A_p$, $1 < p < \infty$, then w satisfies a reverse Hölder's inequality.

(b) If $w \in A_p$, $1 < p < \infty$, then $w^{1+\varepsilon} \in A_p$ for some $\varepsilon > 0$.

(c) If $w \in A_p$, $1 < p < \infty$, then $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$.

(d) If $w \in A_p$, $1 < p < \infty$, then M is bounded in $L^p(w\,dx)$.

So, it is natural to wonder whether these properties hold in this new context. As we shall see in Theorem 7.2, $\mathcal{A}_p$-weights may fail to satisfy (a), (b), (c) or (d).

To finish this section, let us say something about the development of these topics in the classical setting. The classes of weights for which the Hardy-Littlewood maximal function is bounded on $L^p(w\,dx)$ were found by Muckenhoupt in [Muc] and further systematized in the paper [CF]. For a complete account of this topic see [GR, Chapter IV]. E.T. Sawyer in [Saw] characterizes those pairs of weights for which the Hardy-Littlewood or the fractional maximal functions satisfy a two-weight strong type inequality. A new proof of this result is given in [Cru], where one can find many references about the evolution of this subject. On the other hand, sharp two-weight estimates for the Hardy-Littlewood operator are obtained in [Pe1]. Concerning the classical version of condition (1) and the analogues for the scale of Orlicz spaces (Theorem 5.3), it is worth mentioning that in [Pe2] their sufficiency is proved for the boundedness of the fractional maximal function and fractional integrals. In [SW], it is shown how the doubling version of (2) implies two-weight inequalities for fractional integrals. For spaces of homogeneous type see [PW] and for some recent unweighted estimates for fractional integrals on this non-homogeneous setting see [GG].

2 Basic facts.

Let us define the centered maximal functions:

$$\mathcal{M}^c_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{Q(x,r)} |f(y)|\, d\mu(y).$$

Then, it is easy to see that

$$\mathcal{M}^c_\alpha f(x) \leq \mathcal{M}_\alpha f(x) \leq 2^{n-\alpha}\, \mathcal{M}^c_\alpha f(x). \tag{3}$$

We can obtain some inequalities for $\mathcal{M}_\alpha$ acting on Lebesgue spaces. The proofs are classical and we include them for the reader's convenience.

Proposition 2.1 *For $0 \leq \alpha < n$, the maximal operator $\mathcal{M}_\alpha$ satisfies:*

$$\begin{aligned} \mu\{x \in \mathbb{R}^d : \mathcal{M}_\alpha f(x) > \lambda\} &\leq C\Big(\frac{1}{\lambda}\int_{\mathbb{R}^d} |f(x)|\, d\mu(x)\Big)^{\frac{n}{n-\alpha}}, \\ \|\mathcal{M}_\alpha f\|_{L^\infty(\mu)} &\leq \|f\|_{L^{\frac{n}{\alpha}}(\mu)}, \end{aligned}$$

where, in the case $\alpha = 0$, we have to write ∞ instead of $\frac{n}{\alpha}$. Consequently, if $0 < \alpha < n$, $\mathcal{M}_\alpha : L^p(\mu) \longrightarrow L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For $\alpha = 0$, $\mathcal{M}$ is of strong type (p,p), $1 < p \leq \infty$.

Proof. Take $f \in L^1(\mu)$ and define $E_\lambda = \{x \in \mathbb{R}^d : \mathcal{M}_\alpha f(x) > \lambda\}$. If $x \in E_\lambda$, by (3) there exists $r_x > 0$ such that

$$\frac{1}{r_x^{n-\alpha}} \int_{Q(x,r_x)} |f(y)|\, d\mu(y) > 2^{\alpha-n}\, \lambda.$$

In particular, $r_x \leq (2^{n-\alpha}\,\lambda^{-1}\,\|f\|_{L^1(\mu)})^{\frac{1}{n-\alpha}}$. So, Vitali covering lemma provides a pairwise disjoint collection of cubes $\{Q(x_j, r_j)\}_j$, with $x_j \in E_\lambda$ and $r_j = r_{x_j}$, for which

$$E_\lambda \subset \bigcup_{x\in E_\lambda} Q(x, r_x) \subset \bigcup_j Q(x_j, 3\,r_j).$$

Thus,

$$\begin{aligned}\mu(E_\lambda) &\leq \sum_j \mu(Q(x_j, 3\,r_j)) \leq 3^n \sum_j r_j^n \leq 3^n \sum_j \Big(\frac{1}{2^{\alpha-n}\,\lambda} \int_{Q(x_j,r_j)} |f(y)|\,d\mu(y)\Big)^{\frac{n}{n-\alpha}} \\ &\leq 6^n \Big(\sum_j \frac{1}{\lambda} \int_{Q(x_j,r_j)} |f(y)|\,d\mu(y)\Big)^{\frac{n}{n-\alpha}} \leq 6^n \Big(\frac{1}{\lambda} \int_{\mathbb{R}^d} |f(y)|\,d\mu(y)\Big)^{\frac{n}{n-\alpha}},\end{aligned}$$

where we have used that $\frac{n}{n-\alpha} \geq 1$ and that the cubes $Q(x_j, r_j)$ are pairwise disjoint. This ends the proof of part (i). For (ii), when $\alpha = 0$ the inequality is trivial since $\mathcal{M}f(x) \leq \|f\|_{L^\infty(\mu)}$. In the other case, $0 < \alpha < n$, consider $Q \ni x$. By using Hölder's inequality with exponent $\frac{n}{\alpha} > 1$, it follows that

$$\frac{1}{\ell(Q)^{n-\alpha}} \int_Q |f(y)|\,d\mu(y) \leq \frac{1}{\ell(Q)^{n-\alpha}}\,\|f\|_{L^{\frac{n}{\alpha}}(\mu)}\,\mu(Q)^{1/(\frac{n}{\alpha})'} \leq \|f\|_{L^{\frac{n}{\alpha}}(\mu)}.$$

Taking the supremum over all cubes Q which contain x, $\mathcal{M}_\alpha f(x) \leq \|f\|_{L^{\frac{n}{\alpha}}(\mu)}$. The other estimates are obtained by means of Marcinkiewicz interpolation theorem between (i) and (ii). □

3 Strong type inequalities.

Sawyer [Saw] obtained necessary and sufficient conditions on a pair of weights in order to have two-weight strong inequalities for the Hardy-Littlewood and the corresponding fractional maximal functions. In the recent work [Cru], a new proof of this result is given. We generalize this characterization for the radial operators we are concerned with. We prove that Sawyer type conditions are the suitable ones for this problem.

Theorem 3.1 *Consider p, q with $1 < p \leq q < \infty$; α, $0 \leq \alpha < n$, and a pair of weights (u, v). Then, the following statements are equivalent:*

(i) *For every cube Q,*

$$\Big(\int_Q (\mathcal{M}_\alpha(v^{1-p'}\,\chi_Q)(x))^q\,u(x)\,d\mu(x)\Big)^{\frac{1}{q}} \leq C\,\Big(\int_Q v(x)^{1-p'}\,d\mu(x)\Big)^{\frac{1}{p}} < \infty.$$

(ii) *For every $f \in L^p(v)$,*

$$\Big(\int_{\mathbb{R}^d} (\mathcal{M}_\alpha f(x))^q\,u(x)\,d\mu(x)\Big)^{\frac{1}{q}} \leq C\,\Big(\int_{\mathbb{R}^d} |f(x)|^p\,v(x)\,d\mu(x)\Big)^{\frac{1}{p}}.$$

In order to prove this result, we need the following lemma.

Lemma 3.2 *Consider $0 \le \alpha < n$ and $f \ge 0$ a locally integrable function. If for some cube Q and for some $t > 0$ we have*

$$\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(y)\, d\mu(y) > t,$$

then there exists a dyadic cube P such that $Q \subset 3\,P$ and

$$\frac{1}{\ell(P)^{n-\alpha}} \int_P f(y)\, d\mu(y) > 2^{\alpha-n-d}\, t.$$

Proof. Take $k \in \mathbb{Z}$ such that $2^{k-1} \le \ell(Q) < 2^k$. Then, there exist $P_1, \dots, P_N$, $1 \le N \le 2^d$, dyadic cubes of the generation 2^k which intersect Q. Since $\ell(P_j) = 2^k > \ell(Q)$, then $Q \subset 3\,P_j$, for every j. Besides, for at least one of them, say P, the following condition holds

$$\int_P f(y)\, d\mu(y) > \frac{t\,\ell(Q)^{n-\alpha}}{2^d}.$$

Indeed, if it were not true,

$$\int_Q f(y)\, d\mu(y) \le \sum_{j=1}^{N} \int_{P_j} f(y)\, d\mu(y) \le \sum_{j=1}^{N} \frac{t\,\ell(Q)^{n-\alpha}}{2^d} \le t\,\ell(Q)^{n-\alpha},$$

contradicting our hypothesis. Then, it is clear that

$$\frac{1}{\ell(P)^{n-\alpha}} \int_P f(y)\, d\mu(y) > \frac{t\,\ell(Q)^{n-\alpha}}{2^d\,\ell(P)^{n-\alpha}} \ge 2^{\alpha-n-d}\, t.$$

□

Proof of Theorem 3.1. Write $\sigma(x) = v(x)^{1-p'}$. The fact that (ii) implies (i) is obtained by taking $f = \sigma\,\chi_Q$ in (ii). The other implication is proved as follows. First, without loss of generality, we can assume that $f \in L^p(v)$ is a non-negative bounded function with compact support. This guarantees that $\mathcal{M}_\alpha f$ is finite μ-almost everywhere. Decompose $\mathbb{R}^d$ in the following way

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} \Omega_k, \qquad \text{with} \qquad \Omega_k = \{x \in \mathbb{R}^d : 2^k < \mathcal{M}_\alpha f(x) \le 2^{k+1}\}.$$

Then, for every k and every $x \in \Omega_k$ there is a cube Q_x^k containing x, such that

$$\frac{1}{\ell(Q_x^k)^{n-\alpha}} \int_{Q_x^k} f(y)\, d\mu(y) > 2^k.$$

Thus, Lemma 3.2 provides a dyadic cube P_x^k with $Q_x^k \subset 3\,P_x^k$ and

$$\frac{1}{\ell(P_x^k)^{n-\alpha}} \int_{P_x^k} f(y)\, d\mu(y) > 2^{\alpha-n-d}\, 2^k. \tag{4}$$

This estimate says that for every fixed k, the dyadic cubes P_x^k have bounded size. Then, there is a subcollection of maximal dyadic cubes (and so disjoint) $\{P_j^k\}_j$ in such a way that every Q_x^k is contained in $3\,P_j^k$ for some j. As a consequence, $\Omega_k \subset \bigcup_j 3\,P_j^k$. Next, decompose Ω_k by using the sets:

$$E_1^k = 3\,P_1^k \bigcap \Omega_k,\, E_2^k = \left(3\,P_2^k \setminus 3\,P_1^k\right) \bigcap \Omega_k, \dots, E_j^k = \left(3\,P_j^k \setminus \bigcup_{r=1}^{j-1} 3\,P_r^k\right) \bigcap \Omega_k, \dots.$$

Then, we can write

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} \Omega_k = \bigcup_{j,k} E_j^k$$

and these sets are pairwise disjoint. Fix a large integer $K > 0$, which will go to infinity later, and let $\Lambda_K = \{(j,k) \in \mathbb{N}\times\mathbb{Z} : |k| \le K\}$. By using that $E_j^k \subset \Omega_k$ and that the cubes P_j^k verify (4), we obtain

$$\begin{aligned}
\mathcal{I}_K &= \int_{\bigcup_{k=-K}^K \Omega_k} (\mathcal{M}_\alpha f(x))^q\, u(x)\, d\mu(x) = \sum_{(j,k)\in\Lambda_K} \int_{E_j^k} (\mathcal{M}_\alpha f(x))^q\, u(x)\, d\mu(x) \\
&\le \sum_{(j,k)\in\Lambda_K} u(E_j^k)\, (2^{k+1})^q \le C \sum_{(j,k)\in\Lambda_K} u(E_j^k) \left(\frac{1}{\ell(P_j^k)^{n-\alpha}} \int_{P_j^k} f(y)\, d\mu(y)\right)^q \\
&= C \sum_{(j,k)\in\Lambda_K} u(E_j^k) \left(\frac{1}{\ell(3\,P_j^k)^{n-\alpha}} \int_{3\,P_j^k} \sigma(y)\, d\mu(y)\right)^q \left(\frac{\int_{P_j^k} (f\,\sigma^{-1})(y)\,\sigma(y)\, d\mu(y)}{\int_{3\,P_j^k} \sigma(y)\, d\mu(y)}\right)^q \\
&= C \int_{\mathcal{Y}} T_K(f\,\sigma^{-1})^q\, d\nu,
\end{aligned}$$

where $\mathcal{Y} = \mathbb{N}\times\mathbb{Z}$, ν is the measure in $\mathcal{Y}$ given by

$$\nu(j,k) = u(E_j^k) \left(\frac{1}{\ell(3\,P_j^k)^{n-\alpha}} \int_{3\,P_j^k} \sigma(y)\, d\mu(y)\right)^q,$$

and, for every measurable function h, the operator T_K is defined by the expression

$$T_K h(j,k) = \frac{\int_{P_j^k} h(y)\,\sigma(y)\, d\mu(y)}{\int_{3\,P_j^k} \sigma(y)\, d\mu(y)}\, \chi_{\Lambda_K}(j,k).$$

In this way, if we prove that $T_K : L^p(\mathbb{R}^d, \sigma) \longrightarrow L^q(\mathcal{Y}, \nu)$ is bounded independently of K, we shall obtain

$$\mathcal{I}_K \le C \int_{\mathcal{Y}} T_K(f\,\sigma^{-1})^q\, d\nu \le C \left(\int_{\mathbb{R}^d} (f\,\sigma^{-1})^p\,\sigma\, d\mu\right)^{\frac{q}{p}} = C\left(\int_{\mathbb{R}^d} f^p\, v\, d\mu\right)^{\frac{q}{p}}.$$

The uniformity in K of this estimate and the monotone convergence theorem will lead to the desired inequality. So, it remains to see that T_K is uniformly bounded from $L^p(\mathbb{R}^d, \sigma)$ to $L^q(\mathcal{Y}, \nu)$. It is clear, that $T_K : L^\infty(\mathbb{R}^d, \sigma) \longrightarrow L^\infty(\mathcal{Y}, \nu)$ with constant less or equal than 1. Marcinkiewicz interpolation theorem says that it is enough to prove the uniform boundedness of the operators T_K from $L^1(\mathbb{R}^d,\sigma)$ to $L^{\frac{q}{p},\infty}(\mathcal{Y},\nu)$ or, what is the same,

$$\nu\{(j,k)\in\mathcal{Y}: T_K h(j,k)>\lambda\}\le C\Big(\frac{1}{\lambda}\int_{\mathbb{R}^d}|h(x)|\,\sigma(x)\,d\mu(x)\Big)^{\frac{q}{p}},\qquad \text{for every } \lambda>0.$$

For this, let us fix $h\ge 0$ bounded with compact support and put

$$F_\lambda=\{(j,k)\in\mathcal{Y}: T_K h(j,k)>\lambda\}=\{(j,k)\in\Lambda_K: T_K h(j,k)>\lambda\}.$$

Since $E_j^k\subset 3\,P_j^k$, we observe

$$\begin{aligned}
\nu(F_\lambda) &= \sum_{(j,k)\in F_\lambda} u(E_j^k)\left(\frac{1}{\ell(3\,P_j^k)^{n-\alpha}}\int_{3\,P_j^k}\sigma(y)\,d\mu(y)\right)^q\\
&= \sum_{(j,k)\in F_\lambda}\int_{E_j^k}\left(\frac{1}{\ell(3\,P_j^k)^{n-\alpha}}\int_{3\,P_j^k}\sigma(y)\,d\mu(y)\right)^q u(x)\,d\mu(x)\\
&\le \sum_{(j,k)\in F_\lambda}\int_{E_j^k}(\mathcal{M}_\alpha(\sigma\,\chi_{3\,P_j^k})(x))^q\,u(x)\,d\mu(x).
\end{aligned}$$

The dyadic cubes of the collection $\{P_j^k:(j,k)\in F_\lambda\}$ have bounded size, just because if $(j,k)\in F_\lambda$, then $|k|\le K$ and for every k the cubes $\{P_j^k\}_j$ do have bounded size. This allows us to extract a maximal subcollection $\{P_i\}_i$, in such a way that for every $(j,k)\in F_\lambda$, $P_j^k\subset P_i$ for some i. The pairwise disjointness of the sets E_j^k and the fact that $E_j^k\subset 3\,P_j^k$, lead to

$$\begin{aligned}
\nu(F_\lambda) &\le \sum_i\sum_{P_j^k\subset P_i}\int_{E_j^k}(\mathcal{M}_\alpha(\sigma\,\chi_{3\,P_j^k})(x))^q\,u(x)\,d\mu(x)\\
&\le \sum_i\int_{3\,P_i}(\mathcal{M}_\alpha(\sigma\,\chi_{3\,P_i})(x))^q\,u(x)\,d\mu(x)\\
&\le C\sum_i\Big(\int_{3\,P_i}\sigma(x)\,d\mu(x)\Big)^{\frac{q}{p}}.
\end{aligned}$$

Note that it has been precisely in the last inequality, where we have used the condition (i) assumed on the weights. Since the cubes P_i were extracted from the collection $\{P_j^k:(j,k)\in F_\lambda\}$, for every i, there exists an index $(j,k)\in F_\lambda$ with $P_i=P_j^k$. In this case, $T_K h(j,k)>\lambda$, because $(j,k)\in F_\lambda$, and we have

$$\int_{3\,P_i}\sigma(x)\,d\mu(x)=\int_{3\,P_j^k}\sigma(x)\,d\mu(x)<\frac{1}{\lambda}\int_{P_j^k}h(x)\,\sigma(x)\,d\mu(x)=\frac{1}{\lambda}\int_{P_i}h(x)\,\sigma(x)\,d\mu(x).$$

By using that $\frac{q}{p}\ge 1$ and that the cubes P_i are maximal and so pairwise disjoint, we obtain

$$\begin{aligned}
\nu(F_\lambda) &\le C\sum_i\Big(\frac{1}{\lambda}\int_{P_i}h(x)\,\sigma(x)\,d\mu(x)\Big)^{\frac{q}{p}}\le C\Big(\sum_i\frac{1}{\lambda}\int_{P_i}h(x)\,\sigma(x)\,d\mu(x)\Big)^{\frac{q}{p}}\\
&\le C\Big(\frac{1}{\lambda}\int_{\mathbb{R}^d}h(x)\,\sigma(x)\,d\mu(x)\Big)^{\frac{q}{p}},
\end{aligned}$$

where the constant C does not depend on K as we wanted. $\square$

4 Weak type inequalities.

Throughout this section we are concerned with the problem of finding for which pairs of weights (u, v) does the maximal operator $\mathcal{M}_\alpha$ satisfy a weak type inequality.

Definition 4.1 *Let $1 \le p \le q < \infty$ and $0 \le \alpha < n$. We shall say that the pair of weights $(u, v) \in \mathcal{A}^\alpha_{p,q}$, if for every cube Q*

(i) $\displaystyle \frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x) \Big)^{\frac{1}{q}} \Big(\int_Q v(x)^{1-p'}\, d\mu(x) \Big)^{\frac{1}{p'}} \le C$, *when $1 < p < \infty$.*

(ii) $\displaystyle \frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x) \Big)^{\frac{1}{q}} \le C\, v(x)$ *for μ-almost every $x \in Q$, when $p = 1$.*

Observe that we are implicitly assuming that $u, v^{1-p'} \in L^1_{\mathrm{loc}}(\mu)$ and so $u < \infty$, $v > 0$ μ-a.e.. The next result provides an equivalent definition for the class of weights $\mathcal{A}^\alpha_{1,q}$.

Lemma 4.2 *Take $1 \le q \le \frac{n}{n-\alpha}$. Then, the pair of weights $(u, v) \in \mathcal{A}^\alpha_{1,q}$, if and only if, for μ-almost every $x \in \mathbb{R}^d$, $(\mathcal{M}_\beta u(x))^{\frac{1}{q}} \le C\, v(x)$ with $\beta = n - (n-\alpha)\, q$. Indeed, both conditions hold with the same constant.*

In order to prove this lemma, we shall need the following remark: let $\mathcal{Q}$ be the countable set consisting of those cubes with center in $\mathbb{Q}^d$ and radius a positive rational number. An easy continuity argument shows that in the definition of $\mathcal{M}_\alpha$ it is enough to consider just the cubes in $\mathcal{Q}$.

Proof. Let $(u, v) \in \mathcal{A}^\alpha_{1,q}$ with constant C_0. Define

$$N(Q) = \Big\{ x \in Q : \frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(y)\, d\mu(y) \Big)^{\frac{1}{q}} > C_0\, v(x) \Big\} \quad \text{and} \quad N = \bigcup_{Q \in \mathcal{Q}} N(Q).$$

Then, by definition $\mu(N(Q)) = 0$ and, since $\mathcal{Q}$ is countable, $\mu(N) = 0$. Let us set

$$F = \{ y \in \mathbb{R}^d : (\mathcal{M}_\beta u(y))^{\frac{1}{q}} > C_0\, v(y) \}.$$

Use that $\frac{n-\beta}{q} = n - \alpha$ and the remark above to prove that $F \subset N$. In this way, $\mu(F) = 0$, or equivalently,

$$(\mathcal{M}_\beta u(x))^{\frac{1}{q}} \le C_0\, v(x), \qquad \text{for } \mu\text{-almost every } x \in \mathbb{R}^d.$$

Finally, the other implication is trivial. $\square$

Now, we can prove a result characterizing those pairs of weights for which $\mathcal{M}_\alpha$ satisfies weak type inequalities.

Theorem 4.3 *Given p, q with $1 \leq p \leq q < \infty$; α, $0 \leq \alpha < n$, and a pair of weights (u, v), the following statements are equivalent:*

(i) $(u, v) \in \mathcal{A}^{\alpha}_{p,q}$.

(ii) $\mathcal{M}_\alpha : L^p(v) \longrightarrow L^{q,\infty}(u)$, *that is, for every* $\lambda > 0$

$$u\{x \in \mathbb{R}^d : \mathcal{M}_\alpha f(x) > \lambda\} \leq \frac{C}{\lambda^q} \Big(\int_{\mathbb{R}^d} |f(x)|^p \, v(x) \, d\mu(x) \Big)^{\frac{q}{p}}.$$

(iii) For every $f \geq 0$ *and every cube* Q,

$$\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x) \, d\mu(x) \Big)^q u(Q) \leq C \Big(\int_Q f(x)^p \, v(x) \, d\mu(x) \Big)^{\frac{q}{p}}.$$

Proof. The proof follows the ideas that, for the classical setting, can be found in [GR, Chapter IV]. We shall follow the scheme:

$$(ii) \Longrightarrow (iii) \Longrightarrow (i) \Longrightarrow (iii) \Longrightarrow (ii).$$

$\boxed{(ii) \Longrightarrow (iii)}$ Take $f \geq 0$ and a cube Q such that

$$\widetilde{f}_{\alpha,Q} = \frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x) \, d\mu(x) > 0.$$

If $0 < \lambda < \widetilde{f}_{\alpha,Q}$ and $x \in Q$, we have

$$\lambda < \widetilde{f}_{\alpha,Q} = \frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x) \, \chi_Q(x) \, d\mu(x) \leq \mathcal{M}_\alpha(f \, \chi_Q)(x)$$

and so $Q \subset \{y \in \mathbb{R}^d : \mathcal{M}_\alpha(f \, \chi_Q)(y) > \lambda\}$. By using *(ii)*,

$$u(Q) \leq u\{y \in \mathbb{R}^d : \mathcal{M}_\alpha(f \, \chi_Q)(y) > \lambda\} \leq \frac{C}{\lambda^q} \Big(\int_Q f(x)^p \, v(x) \, d\mu(x) \Big)^{\frac{q}{p}},$$

that is,

$$\lambda^q \, u(Q) \leq C \Big(\int_Q f(x)^p \, v(x) \, d\mu(x) \Big)^{\frac{q}{p}}, \quad \text{for } 0 < \lambda < \widetilde{f}_{\alpha,Q}.$$

Then,

$$(\widetilde{f}_{\alpha,Q})^q \, u(Q) \leq C \Big(\int_Q f(x)^p \, v(x) \, d\mu(x) \Big)^{\frac{q}{p}},$$

which is exactly *(iii)*.

$\boxed{(iii) \Longrightarrow (i)}$ Let $f \geq 0$. For any $S \subset Q$, apply (iii) to $f\,\chi_S$ to obtain

$$\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_S f(x)\, d\mu(x)\Big)^q u(Q) \leq C \Big(\int_S f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}}. \tag{5}$$

If we take $f \equiv 1$, this turns out to be

$$\Big(\frac{\mu(S)}{\ell(Q)^{n-\alpha}}\Big)^q u(Q) \leq C\, v(S)^{\frac{q}{p}}. \tag{6}$$

As in [GR, pag. 388], from this inequality it follows that $v > 0$ μ-almost everywhere, unless $u = 0$ μ-almost everywhere; and that $u \in L^1_{\text{loc}}(\mu)$, unless $v = \infty$ μ-almost everywhere. Once we have observed these properties about u and v, which allow us to discard the trivial cases, we are going to show that $(u, v) \in \mathcal{A}^\alpha_{p,q}$. First, we do it for $1 < p < \infty$. Take f such that $f(x) = f(x)^p\, v(x)$, that is, $f(x) = v(x)^{1-p'}$. Since *a priori* we do not know f to be locally integrable, we fix Q and define

$$S_j = \Big\{x \in Q : v(x) > \frac{1}{j}\Big\}, \qquad \text{for } j = 1, 2, \dots.$$

Then f is bounded in every S_j and $\int_{S_j} v^{1-p'}\, d\mu < \infty$. Use (5) with $S = S_j$ and the function f chosen before to get

$$\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_{S_j} v(x)^{1-p'}\, d\mu(x)\Big)^q u(Q) \leq C \Big(\int_{S_j} v(x)^{1-p'}\, d\mu(x)\Big)^{\frac{q}{p}}.$$

Each integral is finite and consequently:

$$\frac{1}{\ell(Q)^{(n-\alpha)\,q}} \Big(\int_{S_j} v(x)^{1-p'}\, d\mu(x)\Big)^{q-\frac{q}{p}} u(Q) \leq C,$$

which can be rewritten as

$$\frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x)\Big)^{\frac{1}{q}} \Big(\int_{S_j} v(x)^{1-p'}\, d\mu(x)\Big)^{\frac{1}{p'}} \leq C.$$

Moreover, we have $S_1 \subset S_2 \subset \dots$ and $\bigcup_j S_j = \{x \in Q : v(x) > 0\}$. Taking limits as $j \to \infty$ we get

$$\frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x)\Big)^{\frac{1}{q}} \Big(\int_{\{x\in Q: v(x)>0\}} v(x)^{1-p'}\, d\mu(x)\Big)^{\frac{1}{p'}} \leq C$$

and since $v > 0$ μ-almost everywhere, $(u, v) \in \mathcal{A}^\alpha_{p,q}$.

For the case $p = 1$, note that (6) can be written in the following manner:

$$\frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x)\Big)^{\frac{1}{q}} \leq C\, \frac{v(S)}{\mu(S)}, \qquad \text{for every } Q \text{ and every } S \subset Q \text{ with } \mu(S) > 0.$$

Fix Q and consider

$$a > \operatorname*{ess\,inf}_{Q} v = \inf \big\{ t > 0 : \mu\{x \in Q : v(x) < t\} > 0 \big\}.$$

Take $S_a = \{x \in Q : v(x) < a\} \subset Q$. Then, $\mu(S_a) > 0$ and

$$\frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x) \Big)^{\frac{1}{q}} \leq \frac{C}{\mu(S_a)} \int_{S_a} v(x)\, d\mu(x) \leq \frac{C}{\mu(S_a)}\, a\, \mu(S_a) = C\, a.$$

Since this happens for every $a > \operatorname{ess\,inf}_Q v$, we get

$$\frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x) \Big)^{\frac{1}{q}} \leq C \operatorname*{ess\,inf}_{Q} v \leq C\, v(x) \qquad \text{for } \mu\text{-almost every } x \in Q$$

and $(u, v) \in \mathcal{A}^{\alpha}_{1,q}$.

$\boxed{(i) \Longrightarrow (iii)}$ First, we do the case $p = 1$. For $f \geq 0$ and for every Q we observe

$$\begin{aligned}\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x)\, d\mu(x) \Big)^q u(Q) &= \Big(\int_Q f(x) \frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(y)\, d\mu(y) \Big)^{\frac{1}{q}} d\mu(x) \Big)^q \\ &\leq C \Big(\int_Q f(x)\, v(x)\, d\mu(x) \Big)^q,\end{aligned}$$

where in the last inequality we have used that $(u, v) \in \mathcal{A}^{\alpha}_{1,q}$. On the other hand, when $1 < p < \infty$, by means of Hölder's inequality we obtain

$$\begin{aligned}\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x)\, d\mu(x) \Big)^q &= \frac{1}{\ell(Q)^{(n-\alpha)\, q}} \Big(\int_Q f(x)\, v(x)^{\frac{1}{p}}\, v(x)^{-\frac{1}{p}}\, d\mu(x) \Big)^q \\ &\leq \frac{1}{\ell(Q)^{(n-\alpha)\, q}} \Big(\int_Q f(x)^p\, v(x)\, d\mu(x) \Big)^{\frac{q}{p}} \Big(\int_Q v(x)^{1-p'}\, d\mu(x) \Big)^{\frac{q}{p'}}.\end{aligned}$$

Thus, since $(u, v) \in \mathcal{A}^{\alpha}_{p,q}$,

$$\begin{aligned}&\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x)\, d\mu(x) \Big)^q u(Q) \\ &\leq \frac{1}{\ell(Q)^{(n-\alpha)\, q}} \Big(\int_Q f(x)^p\, v(x)\, d\mu(x) \Big)^{\frac{q}{p}} \Big(\int_Q v(x)^{1-p'}\, d\mu(x) \Big)^{\frac{q}{p'}} \int_Q u(x)\, d\mu(x) \\ &= \Big(\int_Q f(x)^p\, v(x)\, d\mu(x) \Big)^{\frac{q}{p}} \Big\{ \frac{1}{\ell(Q)^{n-\alpha}} \Big(\int_Q u(x)\, d\mu(x) \Big)^{\frac{1}{q}} \Big(\int_Q v(x)^{1-p'}\, d\mu(x) \Big)^{\frac{1}{p'}} \Big\}^q \\ &\leq C \Big(\int_Q f(x)^p\, v(x)\, d\mu(x) \Big)^{\frac{q}{p}},\end{aligned}$$

and we have proved (iii).

$\boxed{(iii) \Longrightarrow (ii)}$ Observe that it is enough to obtain the desired inequality for $f \in L^p(v)$ with $f \geq 0$. On the other hand, if $f \in L^p_{\text{loc}}(v)$ and $u(Q) > 0$,

$$\Big(\frac{1}{\ell(Q)^{n-\alpha}} \int_Q f(x)\, d\mu(x)\Big)^q u(Q) \leq C \Big(\int_Q f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}} < \infty,$$

and thereby $f \in L^1_{\text{loc}}(\mu)$. As a consequence, we can assume that $f \in L^1(\mu)$, since by defining $f_k = f\,\chi_{Q(0,k)}$ then $f_k \nearrow f$ as $k \to \infty$ and if we get (ii) for each f_k with a constant independent of k, taking limits in k yields (ii) for f. Taking into account all these remarks, we shall prove the desired inequality for $f \geq 0$, $f \in L^p(v) \bigcap L^1(\mu)$. Define

$$E_\lambda = \{x \in \mathbb{R}^d : \mathcal{M}_\alpha f(x) > \lambda\}.$$

If $x \in E_\lambda$, by (3), there will exist $r_x > 0$ with

$$\frac{1}{r_x^{n-\alpha}} \int_{Q(x,r_x)} f(y)\, d\mu(y) > 2^{\alpha-n}\, \lambda.$$

In particular, $r_x \leq (2^{n-\alpha}\, \lambda^{-1}\, \|f\|_{L^1(\mu)})^{\frac{1}{n-\alpha}}$. By Vitali covering lemma, there exists a subcollection of pairwise disjoint cubes $\{Q(x_j, r_j)\}_j$, with $x_j \in E_\lambda$ and $r_j = r_{x_j}$, for which

$$E_\lambda \subset \bigcup_{x \in E_\lambda} Q(x, r_x) \subset \bigcup_j Q(x_j, 3\, r_j).$$

Recall that (iii) led to (5). By this estimate with $Q = Q(x_j, 3\, r_j)$ and $S = Q(x_j, r_j) \subset Q$ it is proved that

$$\begin{aligned}
u(E_\lambda) &\leq \sum_j u(Q(x_j, 3\, r_j)) \\
&\leq C \sum_j \Big(\frac{1}{\ell(Q(x_j, 3\, r_j))^{n-\alpha}} \int_{Q(x_j,r_j)} f(x)\, d\mu(x)\Big)^{-q} \Big(\int_{Q(x_j,r_j)} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}} \\
&= 3^{(n-\alpha)\, q}\, C \sum_j \Big(\frac{1}{r_j^{n-\alpha}} \int_{Q(x_j,r_j)} f(x)\, d\mu(x)\Big)^{-q} \Big(\int_{Q(x_j,r_j)} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}} \\
&\leq \frac{6^{(n-\alpha)\, q}\, C}{\lambda^q} \sum_j \Big(\int_{Q(x_j,r_j)} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}} \\
&\leq \frac{C}{\lambda^q} \Big(\sum_j \int_{Q(x_j,r_j)} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}} \leq \frac{C}{\lambda^q} \Big(\int_{\mathbb{R}^d} f(x)^p\, v(x)\, d\mu(x)\Big)^{\frac{q}{p}},
\end{aligned}$$

where we have taken into account that $\frac{q}{p} \geq 1$ and the fact that the cubes $Q(x_j, r_j)$ are pairwise disjoint. $\square$

5 Sufficient conditions for strong type inequalities.

In this section, we would like to impose certain conditions on the weights which provide strong type inequalities for $\mathcal{M}_\alpha$. We would like to replace (i) in Theorem 3.1 by conditions which do not involve the operator but geometrical properties on the weights.

5.1 Adding a "power-bump" to v.

In order to establish strong type inequalities, first we shall add a power r greater than 1 to the weight v in the following way: write condition $\mathcal{A}_{p,q}^{\alpha}$ as

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v(x)^{1-p'}\,d\mu(x)\Big)^{\frac{1}{p'}}\leq C.$$

Replacing the term in v by

$$\Big(\frac{1}{\ell(Q)^n}\int_Q v(x)^{r\,(1-p')}\,d\mu(x)\Big)^{\frac{1}{r\,p'}},\qquad r>1,$$

we obtain a stronger condition —called "power-bump" condition— which will allows us to get strong type inequalities. In the classical setting this was done by C. Pérez (see [Pe2]).

Theorem 5.1 *Let p,q with $1<p<q<\infty$ and α with $0\leq\alpha<n$. Let (u,v) be a pair of weights for which there exists $r>1$ such that, for every cube Q,*

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v(x)^{r\,(1-p')}\,d\mu(x)\Big)^{\frac{1}{r\,p'}}\leq C. \tag{7}$$

Then, for every $f\in L^p(v)$ it follows that

$$\Big(\int_{\mathbb{R}^d}(\mathcal{M}_\alpha f(x))^q\,u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\leq C\Big(\int_{\mathbb{R}^d}|f(x)|^p\,v(x)\,d\mu(x)\Big)^{\frac{1}{p}}.$$

Note that by putting $r=1$ in (7), we have $\mathcal{A}_{p,q}^{\alpha}$. In addition, if a pair of weights (u,v) satisfies this condition with $r>1$, then $(u,v)\in\mathcal{A}_{p,q}^{\alpha}$, or what is equivalent, $\mathcal{M}_\alpha : L^p(v)\longrightarrow L^{q,\infty}(u)$. This result says that, by assuming (7), which is stronger than $\mathcal{A}_{p,q}^{\alpha}$, the operator $\mathcal{M}_\alpha$ turns out to be of strong type.

Proof. Initially, we are going to follow, with a slight modification, the steps of the proof of Theorem 3.1. Assume that $0\leq f\in L^p(v)$ is bounded with compact support. Put $\sigma(x)=v(x)^{1-p'}$. Take $a>2^n$ and consider the disjoint partition of $\mathbb{R}^d$ given by the sets

$$\mathbb{R}^d=\bigcup_{k\in\mathbb{Z}}\Omega_k,\qquad\text{with}\qquad\Omega_k=\{x\in\mathbb{R}^d : a^k<\mathcal{M}_\alpha f(x)\leq a^{k+1}\}.$$

Just like we did before, for every k there exists a collection of pairwise disjoint dyadic cubes $\{P_j^k\}_{j\in\mathbb{N}}$ such that, $\Omega_k\subset\bigcup_j 3\,P_j^k$ and

$$\frac{1}{\ell(P_j^k)^{n-\alpha}}\int_{P_j^k}f(y)\,d\mu(y)>2^{\alpha-n-d}\,a^k. \tag{8}$$

These P_j^k are indeed the maximal cubes for which (8) holds. Thus, writing $\widetilde{P}_j^k$ for the dyadic cube with $\ell(\widetilde{P}_j^k)=2\,\ell(P_j^k)$ and $P_j^k\subset\widetilde{P}_j^k$ (the so-called "father" of P_j^k),

$$\frac{1}{\ell(\widetilde{P}_j^k)^{n-\alpha}}\int_{\widetilde{P}_j^k}f(y)\,d\mu(y)\leq 2^{\alpha-n-d}\,a^k.$$

and hence,

$$\frac{1}{\ell(P_j^k)^{n-\alpha}} \int_{P_j^k} f(y)\, d\mu(y) \le 2^n \frac{1}{\ell(\widetilde{P}_j^k)^{n-\alpha}} \int_{\widetilde{P}_j^k} f(y)\, d\mu(y) \le 2^{\alpha-d}\, a^k. \tag{9}$$

Use that $a > 1$, (8) and the maximality of the cubes $\{P_j^k\}_j$ to obtain that for every i there exists $j = j(i,k)$ in such a way that $P_i^{k+1} \subset P_j^k$. If these cubes were equal, by (9)

$$2^{\alpha-n-d}\, a^{k+1} < \frac{1}{\ell(P_i^{k+1})^{n-\alpha}} \int_{P_i^{k+1}} f(y)\, d\mu(y) = \frac{1}{\ell(P_j^k)^{n-\alpha}} \int_{P_j^k} f(y)\, d\mu(y) \le 2^{\alpha-d}\, a^k,$$

and we would have $a < 2^n$, contradicting the previous assumption. This proves that $P_i^{k+1} \subsetneq P_j^k$. Thereby, there are no repeated cubes in the collection $\{P_j^k : j \in \mathbb{N}, k \in \mathbb{Z}\}$, since for a fixed k the cubes $\{P_j^k\}_{j\in\mathbb{N}}$ are pairwise disjoint and they are strictly nested if we let k change. Once we have done this observation, we follow again the scheme of the proof of Theorem 3.1. In the same way, we define the pairwise disjoint sets E_j^k. Let $K > 0$ be a large integer and $\Lambda_K = \{(j,k) \in \mathbb{N} \times \mathbb{Z} : |k| \le K\}$. Then,

$$\mathcal{I}_K \le C \sum_{(j,k)\in\Lambda_K} u(E_j^k) \left(\frac{1}{\ell(P_j^k)^{n-\alpha}} \int_{P_j^k} f(y)\, d\mu(y) \right)^q = C \int_{\mathcal{Y}} T_K(f\,\sigma^{-1})^q\, d\nu, \tag{10}$$

where in this case $\mathcal{Y} = \mathbb{N} \times \mathbb{Z}$, ν is a measure in $\mathcal{Y}$ given by

$$\nu(j,k) = \ell(P_j^k)^{q\,\alpha}\, u(E_j^k) \left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma(y)^r\, d\mu(y) \right)^{\frac{q}{r}},$$

and the operator T_K is defined as

$$T_K h(j,k) = \frac{\frac{1}{\ell(P_j^k)^n} \int_{P_j^k} h(y)\,\sigma(y)\, d\mu(y)}{\left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma(y)^r\, d\mu(y) \right)^{\frac{1}{r}}} \chi_{\Lambda_K}(j,k)$$

for any measurable function h. We want to show that $T_K : L^p(\mathbb{R}^d, \sigma) \longrightarrow L^q(\mathcal{Y}, \nu)$ independently of K. Note that

$$|T_K h(j,k)| \le \frac{\left(\frac{1}{\ell(P_j^k)^n} \int_{P_j^k} \sigma(y)^r\, d\mu(y) \right)^{\frac{1}{r}}}{\left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma(y)^r\, d\mu(y) \right)^{\frac{1}{r}}} \|h\|_{L^\infty(\sigma)} = 3^{\frac{n}{r}}\, \|h\|_{L^\infty(\sigma)}.$$

and therefore $T_K : L^\infty(\mathbb{R}^d, \sigma) \longrightarrow L^\infty(\mathcal{Y}, \nu)$ uniformly on K. Then, by Marcinkiewicz interpolation theorem, it is enough to prove that $T_K : L^1(\mathbb{R}^d, \sigma) \longrightarrow L^{\frac{q}{p},\infty}(\mathcal{Y}, \nu)$. Let us fix $h \ge 0$ a bounded function with compact support and define

$$F_\lambda = \{(j,k) \in \mathcal{Y} : T_K h(j,k) > \lambda\} = \{(j,k) \in \Lambda_K : T_K h(j,k) > \lambda\}.$$

We know that $E_j^k \subset 3\,P_j^k$ and so

$$\nu(F_\lambda) \le \frac{1}{3^{q\,\alpha}} \sum_{(j,k)\in F_\lambda} \ell(3\,P_j^k)^{q\,\alpha}\, u(3\,P_j^k) \left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma(y)^r\, d\mu(y)\right)^{\frac{q}{r\,p}+\frac{q}{r\,p'}}$$

Raising (7) to the q-th power, for every cube Q,

$$\ell(Q)^{q\,\alpha-\frac{q\,n}{p}}\, u(Q) \left(\frac{1}{\ell(Q)^n} \int_Q \sigma(x)^r\, d\mu(x)\right)^{\frac{q}{r\,p'}} \le C^q,$$

which together with the previous estimate gives

$$\begin{aligned}
\nu(F_\lambda) &\le C \sum_{(j,k)\in F_\lambda} \ell(3\,P_j^k)^{\frac{q\,n}{p}} \left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma^r\, d\mu\right)^{\frac{q}{r\,p}} \\
&\le C \sum_{(j,k)\in F_\lambda} \ell(3\,P_j^k)^{\frac{q\,n}{p}} \left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma^r\, d\mu\right)^{\frac{1}{r}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda\,\ell(P_j^k)^n} \int_{P_j^k} h\,\sigma\, d\mu\right).
\end{aligned}$$

In the last inequality we have used that

$$\left(\frac{1}{\ell(3\,P_j^k)^n} \int_{3\,P_j^k} \sigma(y)^r\, d\mu(y)\right)^{\frac{1}{r}} < \frac{1}{\lambda} \frac{1}{\ell(P_j^k)^n} \int_{P_j^k} h(y)\,\sigma(y)\, d\mu(y),$$

for $(j,k) \in F_\lambda$. We saw in the proof of Theorem 3.1 that the dyadic cubes $\{P_j^k : (j,k) \in F_\lambda\}$ have bounded size and consequently, we can extract a maximal subcollection $\{P_i\}_i$. Thus,

$$\begin{aligned}
\nu(F_\lambda) &\le C \sum_i \sum_{P_j^k \subset P_i} \ell(3\,P_j^k)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \left(\int_{3\,P_j^k} \sigma^r\, d\mu\right)^{\frac{1}{r}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda} \int_{P_j^k} h\,\sigma\, d\mu\right) \\
&\le C \sum_i \left(\int_{3\,P_i} \sigma^r\, d\mu\right)^{\frac{1}{r}\,(\frac{q}{p}-1)} \sum_{m=0}^\infty \sum_{\substack{P_j^k \subset P_i \\ \ell(P_j^k) = 2^{-m}\,\ell(P_i)}} \ell(3\,P_j^k)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda} \int_{P_j^k} h\,\sigma\, d\mu\right).
\end{aligned}$$

Taking into account that there are no repeated cubes in the collection $\{P_j^k : j \in \mathbb{N}, k \in \mathbb{Z}\}$, we obtain

$$\begin{aligned}
&\sum_{m=0}^\infty \sum_{\substack{P_j^k \subset P_i \\ \ell(P_j^k) = 2^{-m}\,\ell(P_i)}} \ell(3\,P_j^k)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda} \int_{P_j^k} h\,\sigma\, d\mu\right) \\
&\quad= C\,\ell(P_i)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \sum_{m=0}^\infty 2^{-m\,\frac{n}{r'}\,(\frac{q}{p}-1)} \sum_{\substack{P_j^k \subset P_i \\ \ell(P_j^k) = 2^{-m}\,\ell(P_i)}} \left(\frac{1}{\lambda} \int_{P_j^k} h\,\sigma\, d\mu\right) \\
&\quad\le C\,\ell(P_i)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda} \int_{P_i} h\,\sigma\, d\mu\right) \sum_{m=0}^\infty 2^{-m\,\frac{n}{r'}\,(\frac{q}{p}-1)} \\
&\quad\le C\,\ell(P_i)^{\frac{n}{r'}\,(\frac{q}{p}-1)} \left(\frac{1}{\lambda} \int_{P_i} h\,\sigma\, d\mu\right),
\end{aligned}$$

since $1<p<q<\infty$. Collecting all these estimates,

$$\begin{aligned}\nu(F_\lambda) &\leq C\sum_i \Big(\int_{3P_i}\sigma^r\,d\mu\Big)^{\frac{1}{r}(\frac{q}{p}-1)}\ell(P_i)^{\frac{n}{r'}(\frac{q}{p}-1)}\Big(\frac{1}{\lambda}\int_{P_i}h\,\sigma\,d\mu\Big)\\ &= C\sum_i \ell(P_i)^{n(\frac{q}{p}-1)}\Big(\frac{1}{\ell(3P_i)^n}\int_{3P_i}\sigma^r\,d\mu\Big)^{\frac{1}{r}(\frac{q}{p}-1)}\Big(\frac{1}{\lambda}\int_{P_i}h\,\sigma\,d\mu\Big).\end{aligned}$$

The cubes $\{P_i\}_i$ are a subcollection of $\{P_j^k:(j,k)\in F_\lambda\}$, so for every i, there exists some $(j,k)\in F_\lambda$ such that $P_i=P_j^k$ and thus,

$$\begin{aligned}\Big(\frac{1}{\ell(3P_i)^n}\int_{3P_i}\sigma^r\,d\mu\Big)^{\frac{1}{r}} &= \Big(\frac{1}{\ell(3P_j^k)^n}\int_{3P_j^k}\sigma^r\,d\mu\Big)^{\frac{1}{r}}<\frac{1}{\lambda}\frac{1}{\ell(P_j^k)^n}\int_{P_j^k}h\,\sigma\,d\mu\\ &= \frac{1}{\lambda}\frac{1}{\ell(P_i)^n}\int_{P_i}h\,\sigma\,d\mu.\end{aligned}$$

Since $1<p<q<\infty$ and the cubes $\{P_i\}_i$ are pairwise disjoint because of their maximality, we conclude that

$$\begin{aligned}\nu(F_\lambda) &\leq C\sum_i \ell(P_i)^{n(\frac{q}{p}-1)}\Big(\frac{1}{\lambda}\frac{1}{\ell(P_i)^n}\int_{P_i}h\,\sigma\,d\mu\Big)^{\frac{q}{p}-1}\Big(\frac{1}{\lambda}\int_{P_i}h\,\sigma\,d\mu\Big)\\ &= C\sum_i\Big(\frac{1}{\lambda}\int_{P_i}h\,\sigma\,d\mu\Big)^{\frac{q}{p}}\leq C\Big(\sum_i\frac{1}{\lambda}\int_{P_i}h\,\sigma\,d\mu\Big)^{\frac{q}{p}}\leq C\Big(\frac{1}{\lambda}\int_{\mathbb{R}^d}h\,\sigma\,d\mu\Big)^{\frac{q}{p}},\end{aligned}$$

where the constant C does not depend on K. □

5.2 Orlicz spaces.

As we have shown in Theorem 4.3, the necessary and sufficient condition for the maximal operator $\mathcal{M}_\alpha$ to be bounded between $L^p(v)$ and $L^{q,\infty}(u)$, with $1\leq p\leq q<\infty$ and $0\leq\alpha<n$, is that the pair of weights (u,v) belongs to the class $\mathcal{A}^\alpha_{p,q}$. By adding a "power-bump" to v, in Theorem 5.1 we have proved that (7) is sufficient for the corresponding strong inequality. By writing this last condition in the following way

$$\ell(Q)^{n(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\Big(\frac{1}{\ell(Q)^n}\int_Q\big(v(x)^{-\frac{1}{p}}\big)^{rp'}\,d\mu(x)\Big)^{\frac{1}{rp'}}\leq C,$$

we are going to see later that the term in v can be read as $\big\|v^{-\frac{1}{p}}\big\|_{\Phi,Q}$, where $\|\cdot\|_{\Phi,Q}$ stands for certain norm localized in Q in the Orlicz space given by the Young function $\Phi(t)=t^{rp'}$. Our aim is to obtain a similar result for more general Orlicz spaces.

We next recall some definitions and basic facts related with Orlicz spaces. For a complete development of this topic the reader is referred to [RR], [BS]. Let $\Phi:[0,\infty)\longrightarrow[0,\infty)$ be a Young function, that is a continuous, convex, increasing function with $\Phi(0)=0$ and such that $\Phi(t)\longrightarrow\infty$ as $t\to\infty$. By definition, the Orlicz space $L_\Phi=L_\Phi(\mathbb{R}^d,\mu)$ consists of all measurable functions f such that

$$\int_{\mathbb{R}^d}\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)<\infty,\quad\text{for some }\lambda>0.$$

The space L_Φ is a Banach function space if it is endowed with the Luxemburg norm

$$\|f\|_\Phi=\|f\|_{L_\Phi}=\inf\Big\{\lambda>0:\int_{\mathbb{R}^d}\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)\le 1\Big\}.$$

Each Young function Φ has associated to it a complementary Young function $\bar{\Phi}$ which satisfies

$$t\le\Phi^{-1}(t)\,\bar{\Phi}^{-1}(t)\le 2\,t,\qquad\text{for all }t>0.$$

For example, if $\Phi(t)=t^p$ for $1<p<\infty$, then $L_\Phi=L^p(\mu)$ and $\bar{\Phi}(t)=t^{p'}$. Another classical example is given by $\Phi(t)=t\,\log^+t$. In this case L_Φ is the Zygmund space $L\log L$. The complementary function $\bar{\Phi}(t)=t$ for $0\le t\le 1$ and $\bar{\Phi}(t)=\exp(t-1)$ otherwise gives the Zygmund space $L_{\exp}$.

Let us define the following localized version of the Orlicz norm: for every Q,

$$\|f\|_{\Phi,Q}=\inf\Big\{\lambda>0:\frac{1}{\ell(Q)^n}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)\le 1\Big\}.$$

It is an easy exercise, which follows the ideas of [RR, Th. 3, pp. 54–55], to check that $\|\cdot\|_{\Phi,Q}$ provides a norm over $L_\Phi(Q)$: the space of all measurable functions on Q such that there exists $\lambda>0$ for which

$$\frac{1}{\ell(Q)^n}\int_Q\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)<\infty.$$

Furthermore, just as in [RR, Prop. 1, p. 58], the following generalized Hölder's inequality associated with these norms can be proved:

$$\frac{1}{\ell(Q)^n}\int_Q|f(x)\,g(x)|\,d\mu(x)\le 2\,\|f\|_{\Phi,Q}\,\|g\|_{\bar{\Phi},Q}.\tag{11}$$

Next we define a class of Young functions which will be used in the main theorem of this section.

Definition 5.2 *For $1<p<\infty$, a Young function $\Phi\in B_p$, if*

$$\int_c^\infty\frac{\Phi(t)}{t^p}\frac{dt}{t}<\infty,\quad\textit{for some }c>0.$$

It is easy to see that this condition can be also expressed in terms of $\bar{\Phi}$. Namely, if Φ satisfies the doubling property $\Phi(2\,t)\le C\,\Phi(t)$ —this condition can be also found in the literature as the Δ_2 condition, see [BS], [RR]—, then

$$\int_c^\infty\frac{\Phi(t)}{t^p}\frac{dt}{t}\approx\int_c^\infty\left(\frac{t^{p'}}{\bar{\Phi}(t)}\right)^{p-1}\frac{dt}{t},$$

for $c>0$. Once we have introduced some properties of Orlicz spaces we can prove strong inequalities for $\mathcal{M}_\alpha$. For the classical setting the reader is referred to [Pe2].

Theorem 5.3 *Let p, q with $1 < p < q < \infty$ and α with $0 \leq \alpha < n$. Let (u, v) be a pair of weights such that for every cube Q*

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u(x)\,d\mu(x)\Big)^{\frac{1}{q}}\,\big\|v^{-\frac{1}{p}}\big\|_{\Phi,Q} \leq C, \tag{12}$$

where Φ is a Young function whose complementary function $\bar{\Phi} \in B_p$. Then, if $f \in L^p(v)$

$$\Big(\int_{\mathbb{R}^d}(\mathcal{M}_\alpha f(x))^q\,u(x)\,d\mu(x)\Big)^{\frac{1}{q}} \leq C\,\Big(\int_{\mathbb{R}^d}|f(x)|^p\,v(x)\,d\mu(x)\Big)^{\frac{1}{p}}.$$

Remark 5.4 *Theorem* 5.1 *is a consequence of this last result. Indeed, if we take $\Phi(t) = t^{r\,p'}$ with $r > 1$, then*

$$\big\|v^{-\frac{1}{p}}\big\|_{\Phi,Q} \;=\; \inf\Big\{\lambda > 0 : \frac{1}{\ell(Q)^n}\int_Q\left(\frac{v^{-\frac{1}{p}}}{\lambda}\right)^{r\,p'} d\mu \leq 1\Big\} = \Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}},$$

and (12) *becomes* (7)*. Furthermore, since $r > 1$, and $\bar{\Phi} = t^{(r\,p')'}$ satisfies the doubling property, we obtain that $\bar{\Phi} \in B_p$:*

$$\int_1^\infty\left(\frac{t^{p'}}{\Phi(t)}\right)^{p-1}\frac{dt}{t} = \int_1^\infty\left(\frac{t^{p'}}{t^{r\,p'}}\right)^{p-1}\frac{dt}{t} = \int_1^\infty\frac{1}{t^{(r-1)\,p}}\,\frac{dt}{t} < \infty.$$

We have given the proof of the "power-bump" Theorem first because it makes clear how to proceed in the case of Orlicz spaces and it is easier than the one for Orlicz spaces which we are going to give below.

Some other examples which actually relax condition (7) are: $\Phi(t) = t^{p'}\,(\log(1+t))^{p'-1+\beta}$ or $\Phi(t) = t^{p'}\,(\log(1+t))^{p'-1}\,(\log\log(e+t))^{p'-1+\beta}$, for $1 < p < \infty$, $\beta > 0$. It can be proved that their complementary functions belong to B_p and so we can apply the previous Theorem. Note that condition (12) with any of these functions is strictly weaker than (7).

Proof of Theorem 5.3. We proceed like in the proof of Theorem 5.1 just up to (10) and after this point we continue as follows:

$$\begin{aligned}
\mathcal{I}_K \;&\leq\; C\sum_{(j,k)\in\Lambda_K} u(E_j^k)\left(\frac{1}{\ell(P_j^k)^{n-\alpha}}\int_{P_j^k} f(y)\,d\mu(y)\right)^q \\
&\leq\; C\sum_{(j,k)\in\Lambda_K}\ell(P_j^k)^{q\,\alpha}\,u(3\,P_j^k)\left(\frac{1}{\ell(P_j^k)^n}\int_{P_j^k} f(y)\,v(y)^{\frac{1}{p}}\,v(y)^{-\frac{1}{p}}\,d\mu(y)\right)^q \\
&\leq\; C\sum_{(j,k)\in\Lambda_K}\ell(P_j^k)^{q\,\alpha}\,u(3\,P_j^k)\,\big\|v^{-\frac{1}{p}}\big\|_{\Phi,P_j^k}^q\,\big\|f\,v^{\frac{1}{p}}\big\|_{\bar{\Phi},P_j^k}^q,
\end{aligned}$$

where we have used (11) and the fact that $E_j^k \subset 3\,P_j^k$. Let us see that for any function $h \in L_\Phi(3\,Q)$, $\|h\|_{\Phi,Q} \leq 3^n\,\|h\|_{\Phi,3\,Q}$. For this, take $\lambda > \|h\|_{\Phi,3\,Q}$. By the properties of Φ,

$$\frac{1}{\ell(Q)^n}\int_Q\Phi\left(\frac{|h|}{3^n\,\lambda}\right)d\mu \leq \frac{1}{\ell(3\,Q)^n}\int_{3\,Q}\Phi\left(\frac{|h|}{\lambda}\right)d\mu \leq 1$$

and therefore, $\|h\|_{\Phi,Q} \le 3^n \lambda$. Since this happens for every $\lambda > \|h\|_{\Phi,3Q}$, then $\|h\|_{\Phi,Q} \le 3^n \|h\|_{\Phi,3Q}$. By using this inequality, we obtain

$$\mathcal{I}_K \le C \sum_{(j,k)\in\Lambda_K} \ell(3\,P_j^k)^{q\,\alpha}\, u(3\,P_j^k)\, \big\|v^{-\frac{1}{p}}\big\|^q_{\Phi,3\,P_j^k}\, \big\|f\,v^{\frac{1}{p}}\big\|^q_{\bar{\Phi},P_j^k}.$$

Raising (12) to the q-th power, we get

$$\ell(Q)^{\alpha\,q-\frac{n\,q}{p}}\, u(Q)\, \big\|v^{-\frac{1}{p}}\big\|^q_{\Phi,Q} \le C^q, \quad \text{for every cube } Q.$$

Thus, we can write

$$\mathcal{I}_K \le C \sum_{(j,k)\in\Lambda_K} \ell(3\,P_j^k)^{\frac{n\,q}{p}}\, \big\|f\,v^{\frac{1}{p}}\big\|^q_{\bar{\Phi},P_j^k} = C \int_{\mathcal{Y}} T_K\big(f\,v^{\frac{1}{p}}\big)^q\, d\nu,$$

where we have put $\mathcal{Y} = \mathbb{N}\times\mathbb{Z}$, ν is a measure in $\mathcal{Y}$ defined by $\nu(j,k) = \ell(3\,P_j^k)^{\frac{n\,q}{p}}$ and T_K is an operator given by the expression

$$T_K h(j,k) = \|h\|_{\bar{\Phi},P_j^k}\, \chi_{\Lambda_K}(j,k).$$

If we prove that T_K is uniformly bounded from $L^p(\mathbb{R}^d,\mu)$ to $L^q(\mathcal{Y},\nu)$, we get the desired inequality by doing $K\to\infty$. Let us see how to prove that T_K is a bounded operator. Fix h a bounded function with compact support and define

$$F_\lambda = \{(j,k)\in\mathcal{Y} : T_K h(j,k) > \lambda\} = \{(j,k)\in\Lambda_K : \|h\|_{\bar{\Phi},P_j^k} > \lambda\}.$$

Without loss of generality we can suppose that $\bar{\Phi}$ is normalized so that $\bar{\Phi}(1) = 1$. We split h as follows

$$h(x) = h(x)\,\chi_{\{x:|h(x)|>\frac{\lambda}{2}\}}(x) + h(x)\,\chi_{\{x:|h(x)|\le\frac{\lambda}{2}\}}(x) = h_1(x) + h_2(x).$$

Then,

$$\frac{1}{\ell(Q)^n}\int_Q \bar{\Phi}\left(\frac{|h_2(x)|}{\frac{\lambda}{2}}\right) d\mu(x) \le \frac{1}{\ell(Q)^n}\int_Q \bar{\Phi}(1)\, d\mu(x) = \frac{\mu(Q)}{\ell(Q)^n} \le 1$$

and for every Q, $\|h_2\|_{\bar{\Phi},Q} \le \frac{\lambda}{2}$. If Q is such that $\|h\|_{\bar{\Phi},Q} > \lambda$, by using that $\|\cdot\|_{\bar{\Phi},Q}$ defines a norm, we observe

$$\lambda < \|h\|_{\bar{\Phi},Q} = \|h_1+h_2\|_{\bar{\Phi},Q} \le \|h_1\|_{\bar{\Phi},Q} + \|h_2\|_{\bar{\Phi},Q} \le \|h_1\|_{\bar{\Phi},Q} + \frac{\lambda}{2}$$

and $\|h_1\|_{\bar{\Phi},Q} > \frac{\lambda}{2}$. Thus,

$$F_\lambda = \{(j,k)\in\Lambda_K : \|h\|_{\bar{\Phi},P_j^k} > \lambda\} \subset \left\{(j,k)\in\Lambda_K : \|h_1\|_{\bar{\Phi},P_j^k} > \frac{\lambda}{2}\right\} = \widetilde{F}_\lambda.$$

If $(j,k) \in \widetilde{F}_\lambda$, then

$$\frac{1}{\ell(P_j^k)^n} \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu > 1, \qquad \text{and so} \qquad \ell(P_j^k)^n < \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu.$$

In this way,

$$\nu(F_\lambda) \le \nu(\widetilde{F}_\lambda) = \sum_{(j,k)\in\widetilde{F}_\lambda} \ell(3\,P_j^k)^{\frac{n\,q}{p}} \le 3^{\frac{n\,q}{p}} \sum_{(j,k)\in\widetilde{F}_\lambda} \ell(P_j^k)^{n\,(\frac{q}{p}-1)} \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu.$$

The dyadic cubes $\{P_j^k : (j,k) \in \widetilde{F}_\lambda\}$ have bounded size (because $|k| \le K$) and we can extract a subcollection $\{P_i\}$ which is maximal with respect to the inclusion. Then,

$$\begin{aligned}
\nu(F_\lambda) &\le C \sum_i \sum_{P_j^k \subset P_i} \ell(P_j^k)^{n\,(\frac{q}{p}-1)} \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \\
&= C \sum_i \sum_{m=0}^{\infty} \sum_{\substack{P_j^k \subset P_i \\ \ell(P_j^k) = 2^{-m}\,\ell(P_i)}} \ell(P_j^k)^{n\,(\frac{q}{p}-1)} \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \\
&= C \sum_i \ell(P_i)^{n\,(\frac{q}{p}-1)} \sum_{m=0}^{\infty} 2^{-m\,n\,(\frac{q}{p}-1)} \sum_{\substack{P_j^k \subset P_i \\ \ell(P_j^k) = 2^{-m}\,\ell(P_i)}} \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \\
&\le C \sum_i \ell(P_i)^{n\,(\frac{q}{p}-1)} \Big(\int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \Big) \sum_{m=0}^{\infty} 2^{-m\,n\,(\frac{q}{p}-1)} \\
&\le C \sum_i \ell(P_i)^{n\,(\frac{q}{p}-1)} \int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu,
\end{aligned}$$

where in the next to last inequality we have taken into account that there are no repeated cubes in $\{P_j^k : (j,k) \in \mathbb{N}\times\mathbb{Z}\}$ (see the proof of Theorem 5.1), whereas the last one holds because $1 < p < q < \infty$. Besides, for every i, there exists $(j,k) \in \widetilde{F}_\lambda$ such that $P_i = P_j^k$ and consequently

$$\ell(P_i)^n = \ell(P_j^k)^n < \int_{P_j^k} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu = \int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu.$$

By this, it follows that

$$\begin{aligned}
\nu(F_\lambda) &\le C \sum_i \Big(\int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \Big)^{\frac{q}{p}-1} \int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \\
&= C \sum_i \Big(\int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \Big)^{\frac{q}{p}} \le C \Big(\sum_i \int_{P_i} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \Big)^{\frac{q}{p}} \\
&\le C \Big(\int_{\mathbb{R}^d} \bar{\Phi}\left(\frac{2\,|h_1|}{\lambda}\right) d\mu \Big)^{\frac{q}{p}} = C \Big(\int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{\lambda}{2}\}} \bar{\Phi}\left(\frac{2\,|h|}{\lambda}\right) d\mu \Big)^{\frac{q}{p}}.
\end{aligned}$$

Use again that $\frac{q}{p} > 1$ to obtain

$$\begin{aligned}
\int_{\mathcal{Y}} T_K h^q \, d\nu &= q \int_0^\infty \lambda^q \, \nu\{(j,k) \in \mathcal{Y} : T_K h(j,k) > \lambda\} \frac{d\lambda}{\lambda} = q \int_0^\infty \lambda^q \, \nu(F_\lambda) \frac{d\lambda}{\lambda} \\
&\leq C \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\lambda^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{\lambda}{2}\}} \bar{\Phi}\left(\frac{2\,|h|}{\lambda}\right) d\mu\right)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \\
&\leq C \sum_{k\in\mathbb{Z}} \left((2^{k+1})^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{2^k}{2}\}} \bar{\Phi}\left(\frac{2\,|h|}{2^k}\right) d\mu\right)^{\frac{q}{p}} \\
&\leq C \left(\sum_{k\in\mathbb{Z}} (2^{k+1})^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{2^k}{2}\}} \bar{\Phi}\left(\frac{2\,|h|}{2^k}\right) d\mu\right)^{\frac{q}{p}}.
\end{aligned}$$

Inside the parenthesis we can recover the integral in the following way

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} (2^{k+1})^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{2^k}{2}\}} \bar{\Phi}\left(\frac{2\,|h(x)|}{2^k}\right) d\mu(x) \\
&\quad = \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (2^{k+1})^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{2^k}{2}\}} \bar{\Phi}\left(\frac{2\,|h(x)|}{2^k}\right) d\mu(x) \frac{d\lambda}{2^k} \\
&\quad \leq \int_0^\infty (2\,\lambda)^p \int_{\{x\in\mathbb{R}^d : |h(x)| > \frac{\lambda}{4}\}} \bar{\Phi}\left(\frac{4\,|h(x)|}{\lambda}\right) d\mu(x) \frac{2\,d\lambda}{\lambda} \\
&\quad = 2^{p+1} \int_{\mathbb{R}^d} \int_0^{4\,|h(x)|} \lambda^p \, \bar{\Phi}\left(\frac{4\,|h(x)|}{\lambda}\right) \frac{d\lambda}{\lambda} \, d\mu(x) \\
&\quad = 2^{p+1} \int_{\mathbb{R}^d} \int_1^\infty \left(\frac{4\,|h(x)|}{s}\right)^p \bar{\Phi}(s) \frac{ds}{s} \, d\mu(x) \\
&\quad = 2^{3\,p+1} \left(\int_{\mathbb{R}^d} |h(x)|^p \, d\mu(x)\right) \left(\int_1^\infty \frac{\bar{\Phi}(s)}{s^p} \frac{ds}{s}\right) \\
&\quad \leq C \int_{\mathbb{R}^d} |h(x)|^p \, d\mu(x).
\end{aligned}$$

Note that in the last inequality we have used that $\bar{\Phi} \in B_p$. In short we have just proved that

$$\int_{\mathcal{Y}} T_K h^q \, d\nu \leq \left(\int_{\mathbb{R}^d} |h(x)|^p \, d\mu(x)\right)^{\frac{q}{p}}.$$

□

6 Fractional integrals.

Let $0 < \alpha < n$. For $f \in L^\infty(\mu)$ a boundedly supported function we define the fractional integral of order α as

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y).$$

To see that this operator is well defined, we need to study the convergence of the integral. Since the support of f is bounded, there is no problem of integrability at infinity. Although the kernel of the operator is singular at the diagonal $x = y$, we have the following

$$\begin{aligned}
\int_{|x-y|\le 1} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) &\le \|f\|_{L^\infty(\mu)} \sum_{k=0}^{\infty} \int_{2^{-k-1}\le|x-y|<2^{-k}} \frac{1}{|x-y|^{n-\alpha}}\, d\mu(y)\\
&\le \|f\|_{L^\infty(\mu)} \sum_{k=0}^{\infty} \frac{\mu(Q(x,2^{-k+1}))}{2^{(-k-1)\,(n-\alpha)}}\\
&\le 2^{2\,n-\alpha}\, \|f\|_{L^\infty(\mu)} \sum_{k=0}^{\infty} 2^{-k\,\alpha} < \infty,
\end{aligned}$$

and hence the integral which defines I_α is absolutely convergent.

Theorem 6.1 (Hedberg's inequality) *Let $0 < \alpha < n$ and f be a bounded function with compact support. Then, for $1 \le p < \frac{n}{\alpha}$, the following inequality holds*

$$|I_\alpha f(x)| \le C\, \|f\|_{L^p(\mu)}^{\frac{p\,\alpha}{n}}\, \mathcal{M}f(x)^{1-\frac{p\,\alpha}{n}}.$$

This result was proved by Hedberg in the classical setting (see [Hed]).

Proof. Let $s > 0$. Then,

$$|I_\alpha f(x)| \le \int_{|x-y|<s} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) + \int_{|x-y|\ge s} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) = I + II.$$

For the first term we have

$$\begin{aligned}
I &= \sum_{k=0}^{\infty} \int_{2^{-k-1}\,s\le|x-y|<2^{-k}\,s} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y)\\
&\le 2^{2\,n-\alpha}\, s^\alpha \sum_{k=0}^{\infty} 2^{-k\,\alpha} \frac{1}{(2^{-k+1}\,s)^n} \int_{Q(x,2^{-k+1}\,s)} |f(y)|\, d\mu(y)\\
&= C\, s^\alpha\, \mathcal{M}f(x).
\end{aligned}$$

On the other hand, if $p = 1$, then

$$II = \int_{|x-y|\ge s} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) \le \frac{1}{s^{n-\alpha}} \int_{|x-y|\ge s} |f(y)|\, d\mu(y) \le s^{-(n-\alpha)}\, \|f\|_{L^1(\mu)}.$$

For $1 < p < \frac{n}{\alpha}$, put $\beta = p'\,(n-\alpha) - n$. Then, $\beta > 0$ and we can conclude

$$II = \int_{|x-y|\ge s} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) \le \|f\|_{L^p(\mu)} \Big(\int_{|x-y|\ge s} \frac{d\mu(y)}{|x-y|^{p'\,(n-\alpha)}} \Big)^{\frac{1}{p'}}.$$

Besides,

$$\begin{aligned}\int_{|x-y|\geq s} \frac{d\mu(y)}{|x-y|^{p'(n-\alpha)}} &= \sum_{k=0}^{\infty} \int_{2^k s\leq|x-y|<2^{k+1} s} \frac{d\mu(y)}{|x-y|^{n+\beta}} \leq \sum_{k=0}^{\infty} \frac{\mu(Q(x,2^{k+2}s))}{(2^k s)^{n+\beta}} \\ &\leq C\, s^{-\beta},\end{aligned}$$

and therefore

$$II \leq C\, s^{-\frac{\beta}{p'}}\, \|f\|_{L^p(\mu)} = C\, s^{-(\frac{n}{p}-\alpha)}\, \|f\|_{L^p(\mu)}.$$

Observe that by putting $p=1$ and $C=1$, this inequality becomes the one obtained for the case $p=1$. Thus, for $1 \leq p < \frac{n}{\alpha}$, all these estimates provide

$$|I_\alpha f(x)| \leq I + II \leq C\left(s^\alpha\, \mathcal{M}f(x) + s^{-(\frac{n}{p}-\alpha)}\, \|f\|_{L^p(\mu)}\right), \quad \text{for any } s>0.$$

Minimizing this expression in the variable s we get the desired inequality. □

With the previous estimate we can obtain some inequalities for I_α.

Proposition 6.2 *Let* $0<\alpha<n$*.*

(i) If $1<p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$*, then* $I_\alpha : L^p(\mu) \longrightarrow L^q(\mu)$*.*

(ii) If $\frac{1}{q}=1-\frac{\alpha}{n}$*, then* $I_\alpha : L^1(\mu) \longrightarrow L^{q,\infty}(\mu)$*.*

Remark 6.3 *Let* $0<\alpha<n$*,* $x\in\mathbb{R}^d$ *and* $Q \ni x$*. Then, for every* $y\in Q$ *we have* $|x-y|\leq\sqrt{d}\,\ell(Q)$ *and so*

$$\frac{1}{\ell(Q)^{n-\alpha}} \int_Q |f(y)|\, d\mu(y) \leq d^{\frac{n-\alpha}{2}} \int_Q \frac{1}{|x-y|^{n-\alpha}}\, |f(y)|\, d\mu(y) \leq d^{\frac{n-\alpha}{2}}\, I_\alpha(|f|)(x).$$

By taking the supremum over all cubes Q *containing* x*, we obtain that*

$$\mathcal{M}_\alpha f(x) \leq d^{\frac{n-\alpha}{2}}\, I_\alpha(|f|)(x).$$

From this inequality and the previous result, some of the estimates in Proposition 2.1 *can be obtained in a different way.*

In [Mat] there is another estimate involving I_α and $\mathcal{M}_\alpha$. Namely, the author obtains a good-λ inequality in terms of Hausdorff measure, which states that I_α can be controlled by M_α. Estimates between Lipschitz and BMO spaces and further results for I_α can also be found in [GG].

Proof of Proposition 6.2. For (i), observe that Hedberg's inequality implies that

$$\begin{aligned}\Big(\int_{\mathbb{R}^d} |I_\alpha f(x)|^q\, d\mu(x)\Big)^{\frac{1}{q}} &\leq C\, \|f\|_{L^p(\mu)}^{\frac{p\,\alpha}{n}} \Big(\int_{\mathbb{R}^d} \mathcal{M}f(x)^{q\,(1-\frac{p\,\alpha}{n})}\, d\mu(x)\Big)^{\frac{1}{q}} \\ &= C\, \|f\|_{L^p(\mu)}^{\frac{p\,\alpha}{n}} \Big(\int_{\mathbb{R}^d} \mathcal{M}f(x)^p\, d\mu(x)\Big)^{\frac{1}{q}} \\ &\leq C\, \|f\|_{L^p(\mu)}.\end{aligned}$$

Let us see what happens in the case $p=1$. In this case, Hedberg's inequality becomes

$$|I_\alpha f(x)| \le C\,\|f\|_{L^1(\mu)}^{\frac{\alpha}{n}}\,\mathcal{M}f(x)^{1-\frac{\alpha}{n}} = C\,\|f\|_{L^1(\mu)}^{\frac{\alpha}{n}}\,\mathcal{M}f(x)^{\frac{1}{q}}.$$

Use that $\mathcal{M}$ is of weak type $(1,1)$, to conclude

$$\begin{aligned}
\mu\{x\in\mathbb{R}^d : |I_\alpha f(x)|>\lambda\} &\le \mu\left\{x\in\mathbb{R}^d : \mathcal{M}f(x) > \left(\frac{\lambda}{C\,\|f\|_{L^1(\mu)}^{\frac{\alpha}{n}}}\right)^q\right\} \\
&\le \left(\frac{C\,\|f\|_{L^1(\mu)}^{\frac{\alpha}{n}}}{\lambda}\right)^q \|f\|_{L^1(\mu)} = \left(\frac{C\,\|f\|_{L^1(\mu)}}{\lambda}\right)^q
\end{aligned}$$

□

The following result extends an inequality, proved in the classical setting in [Wel], to our case. This inequality will be the key to obtain two-weight norm inequalities for fractional integrals.

Theorem 6.4 (Welland's inequality) *Let $0<\alpha<n$ and $0<\varepsilon<\min\{\alpha, n-\alpha\}$. Then for any bounded function with bounded support f we have*

$$|I_\alpha f(x)| \le C\left(\mathcal{M}_{\alpha+\varepsilon}f(x)\,\mathcal{M}_{\alpha-\varepsilon}f(x)\right)^{\frac{1}{2}},$$

where C only depends on n, α and ε.

Proof. We take $s>0$ and split I_α like in the proof Theorem 6.1. For I we have

$$\begin{aligned}
I &= \sum_{k=0}^{\infty}\int_{2^{-k-1}\,s\le|x-y|<2^{-k}\,s}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y) \\
&\le 2^{2\,(n-\alpha)}\sum_{k=0}^{\infty}\frac{1}{(2^{-k+1}\,s)^{-\varepsilon}}\,\frac{1}{(2^{-k+1}\,s)^{n-(\alpha-\varepsilon)}}\int_{Q(x,2^{-k+1}\,s)}|f(y)|\,d\mu(y) \\
&\le C\,s^{\varepsilon}\,\mathcal{M}_{\alpha-\varepsilon}f(x)\sum_{k=0}^{\infty}2^{-k\,\varepsilon} = C\,s^{\varepsilon}\,\mathcal{M}_{\alpha-\varepsilon}f(x).
\end{aligned}$$

For II we obtain

$$\begin{aligned}
II &= \sum_{k=0}^{\infty}\int_{2^{k}\,s\le|x-y|<2^{k+1}\,s}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y) \\
&\le \sum_{k=0}^{\infty}\frac{1}{(2^{k}\,s)^{n-\alpha}}\int_{|x-y|<2^{k+1}\,s}|f(y)|\,d\mu(y) \\
&\le C\,s^{-\varepsilon}\,\mathcal{M}_{\alpha+\varepsilon}f(x)\sum_{k=0}^{\infty}2^{-k\,\varepsilon} = C\,s^{-\varepsilon}\,\mathcal{M}_{\alpha+\varepsilon}f(x).
\end{aligned}$$

Note that this choice of ε assures that $0 < \alpha - \varepsilon < \alpha < \alpha + \varepsilon < n$. Collecting both estimates,

$$|I_\alpha f(x)| \le C\left(s^\varepsilon\, \mathcal{M}_{\alpha-\varepsilon} f(x) + s^{-\varepsilon}\, \mathcal{M}_{\alpha+\varepsilon} f(x)\right), \quad \text{for any } s > 0.$$

To complete the proof we just have to minimize this expression in the variable s. $\square$

This inequality combined with the results we have already obtained for $\mathcal{M}_\alpha$ will allow us to get two-weight inequalities for fractional integrals. In the homogeneous case, similar results were obtained by different methods in [SW] and also, with Orlicz norms, in [Pe2].

Theorem 6.5 *Let p, q with $1 < p < q < \infty$ and α with $0 < \alpha < n$. Let (u, v) be a pair of weights for which there exists $r > 1$ such that for every cube Q,*

$$\ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\left(\frac{1}{\ell(Q)^n}\int_Q u(x)^r\, d\mu(x)\right)^{\frac{1}{r\,q}}\left(\frac{1}{\ell(Q)^n}\int_Q v(x)^{r\,(1-p')}\, d\mu(x)\right)^{\frac{1}{r\,p'}} \le C. \tag{13}$$

Then for every $f \in L^p(v)$,

$$\left(\int_{\mathbb{R}^d} |I_\alpha f(x)|^q\, u(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^d} |f(x)|^p\, v(x)\, d\mu(x)\right)^{\frac{1}{p}}.$$

Proof. Let $0 \le f \in L^\infty(\mu)$ a boundedly supported function. Choose ε such that

$$0 < \varepsilon < \min\left\{\alpha, n-\alpha, \frac{n}{q}, n\left(\frac{1}{p}-\frac{1}{q}\right), \frac{n}{q\, r'}\right\}.$$

By Theorem 6.4, we observe that

$$\left(\int_{\mathbb{R}^d} |I_\alpha f|^q\, u\, d\mu\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^d} \left(\mathcal{M}_{\alpha+\varepsilon} f\cdot \mathcal{M}_{\alpha-\varepsilon} f\right)^{\frac{q}{2}}\, u\, d\mu\right)^{\frac{1}{q}} = C\left(\int_{\mathbb{R}^d} F\cdot G\, d\mu\right)^{\frac{1}{q}},$$

where

$$F(x) = \left(\mathcal{M}_{\alpha+\varepsilon} f(x)\, u(x)^{\frac{1}{q}}\right)^{\frac{q}{2}} \quad \text{and} \quad G(x) = \left(\mathcal{M}_{\alpha-\varepsilon} f(x)\, u(x)^{\frac{1}{q}}\right)^{\frac{q}{2}}.$$

Let

$$\frac{1}{q_\varepsilon^+} = \frac{1}{q} - \frac{\varepsilon}{n}, \quad \frac{1}{q_\varepsilon^-} = \frac{1}{q} + \frac{\varepsilon}{n}, \quad q^+ = 2\,\frac{q_\varepsilon^+}{q} \quad \text{and} \quad q^- = 2\,\frac{q_\varepsilon^-}{q}.$$

Due to the way we choose ε, we have

$$1 < p < q_\varepsilon^- < q < q_\varepsilon^+ < \infty, \quad 1 < q^- < q^+ < \infty \quad \text{and} \quad \frac{1}{q^+} + \frac{1}{q^-} = 1.$$

We use Hölder's inequality to get,

$$\left(\int_{\mathbb{R}^d} F\cdot G\, d\mu\right)^{\frac{1}{q}} \le \|F\|_{L^{q^+}(\mu)}^{\frac{1}{q}}\, \|G\|_{L^{q^-}(\mu)}^{\frac{1}{q}} = \left\|\mathcal{M}_{\alpha+\varepsilon} f\right\|_{L^{q_\varepsilon^+}(u^+)}^{\frac{1}{2}}\, \left\|\mathcal{M}_{\alpha-\varepsilon} f\right\|_{L^{q_\varepsilon^-}(u^-)}^{\frac{1}{2}},$$

where $u^+ = u^{\frac{q_\varepsilon^+}{q}}$ and $u^- = u^{\frac{q_\varepsilon^-}{q}}$. For the first term, since $1 < \frac{q_\varepsilon^+}{q} < r$,

$$\begin{aligned}
&\ell(Q)^{n\,(\frac{1}{q_\varepsilon^+}+\frac{\alpha+\varepsilon}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^+\,d\mu\Big)^{\frac{1}{q_\varepsilon^+}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}}\\
&\quad= \ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^{\frac{q_\varepsilon^+}{q}}\,d\mu\Big)^{\frac{1}{q_\varepsilon^+}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}}\\
&\quad\le \ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^{r}\,d\mu\Big)^{\frac{1}{r\,q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}} \le C,
\end{aligned}$$

and so, the pair of weights (u^+, v) satisfies (7) with $1 < p < q_\varepsilon^+ < \infty$ and $\alpha + \varepsilon$. Then, Theorem 5.1 says that $\mathcal{M}_{\alpha+\varepsilon}$ is bounded from $L^p(v)$ to $L^{q_\varepsilon^+}(u^+)$. Now, for the second term it is easier to prove the estimate for the weights, since $\frac{q_\varepsilon^-}{q} < 1 < r$, and

$$\begin{aligned}
&\ell(Q)^{n\,(\frac{1}{q_\varepsilon^-}+\frac{\alpha-\varepsilon}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^-\,d\mu\Big)^{\frac{1}{q_\varepsilon^-}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}}\\
&\quad= \ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^{\frac{q_\varepsilon^-}{q}}\,d\mu\Big)^{\frac{1}{q_\varepsilon^-}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}}\\
&\quad\le \ell(Q)^{n\,(\frac{1}{q}+\frac{\alpha}{n}-\frac{1}{p})}\Big(\frac{1}{\ell(Q)^n}\int_Q u^{r}\,d\mu\Big)^{\frac{1}{r\,q}}\Big(\frac{1}{\ell(Q)^n}\int_Q v^{r\,(1-p')}\,d\mu\Big)^{\frac{1}{r\,p'}} \le C.
\end{aligned}$$

In this way, the pair of weights (u^-, v) verifies (7) with $1 < p < q_\varepsilon^- < \infty$ and $\alpha - \varepsilon$. By Theorem 5.1, $\mathcal{M}_{\alpha-\varepsilon}$ is a bounded operator from $L^p(v)$ to $L^{q_\varepsilon^-}(u^-)$. Pasting these estimates together, we get

$$\|I_\alpha f\|_{L^q(u)} \le C\,\big\|\mathcal{M}_{\alpha+\varepsilon} f\big\|^{\frac{1}{2}}_{L^{q_\varepsilon^+}(u^+)}\,\big\|\mathcal{M}_{\alpha-\varepsilon} f\big\|^{\frac{1}{2}}_{L^{q_\varepsilon^-}(u^-)} \le C\,\|f\|^{\frac{1}{2}}_{L^p(v)}\,\|f\|^{\frac{1}{2}}_{L^p(v)} = C\,\|f\|_{L^p(v)}.$$

□

7 The case $\alpha = 0$. The Hardy-Littlewood radial maximal function.

In this section, we shall pay special attention to the Hardy-Littlewood radial maximal function, which corresponds to the case $\alpha = 0$. For simplicity, we shall write

$$\mathcal{M} f(x) = \sup_{Q \ni x} \frac{1}{\ell(Q)^n} \int_Q |f(y)|\,d\mu(y).$$

Proposition 2.1 states that this operator is of weak type $(1,1)$ and of strong type (p,p), $1 < p \le \infty$. Actually, these inequalities can be also obtained by using that $\mathcal{M}$ is controlled by the centered Hardy-Littlewood maximal function. Since the underlying space is $\mathbb{R}^d$, the Besicovitch covering lemma provides the weak type $(1,1)$, and hence the strong type (p,p) for $1 < p \le \infty$.

For $1 < p < \infty$, we shall say that the pair of weights $(u, v) \in \mathcal{S}_p$, if the following Sawyer type condition holds:

$$(\mathcal{S}_p) \qquad \int_Q (\mathcal{M}(v^{1-p'} \chi_Q)(x))^p \, u(x) \, d\mu(x) \leq C \int_Q v(x)^{1-p'} \, d\mu(x)$$

for every cube $Q \subset \mathbb{R}^d$. Theorem 3.1 assures that this condition is equivalent to the boundedness of $\mathcal{M}$ between $L^p(v)$ and $L^p(u)$. When $u = v = w$, we shall simply write $w \in \mathcal{S}_p$. On the other hand, if $1 \leq p < \infty$, the radial Muckenhoupt classes are given by Definition 4.1. In particular, the pair of weights $(u, v) \in \mathcal{A}_p$ if for every cube Q,

$$(\mathcal{A}_p) \qquad \Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{1-p'} \, d\mu(x)\Big)^{p-1} \leq C,$$

when $1 < p < \infty$. For $p = 1$ we shall write

$$(\mathcal{A}_1) \qquad \Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \leq C \, v(x), \qquad \text{for } \mu\text{-almost every } x \in Q,$$

or equivalently, $\mathcal{M}u(x) \leq C \, v(x)$ for μ-almost every $x \in \mathbb{R}^d$ (see Lemma 4.2). Observe that these classes of weights correspond to those defined previously in the following way: $\mathcal{A}_p = \mathcal{A}^0_{p,p}$ for $1 \leq p < \infty$. Thus, Theorem 4.3 says in particular that for $1 \leq p < \infty$, $\mathcal{M}$ is bounded from $L^p(v)$ to $L^{p,\infty}(u)$, if and only if, $(u, v) \in \mathcal{A}_p$. Again, when we deal with the same problem but only with one weigh, that is, $u = v = w$, we shall put $w \in \mathcal{A}_p$.

It is clear that these classes of weights verify that for $1 < p < \infty$, $\mathcal{S}_p \subset \mathcal{A}_p$. The next result contains another properties about them.

Proposition 7.1

(i) *If $1 < p < q < \infty$, then $\mathcal{A}_1 \subset \mathcal{S}_p \subset \mathcal{A}_p \subset \mathcal{S}_q \subset \mathcal{A}_q$. Thus, for $(u, v) \in \mathcal{A}_p$, with $1 \leq p < \infty$, and for $p < q < \infty$, we have $\mathcal{M} : L^q(v) \longrightarrow L^q(u)$.*

(ii) *Consider $1 < p < \infty$ and (u, v) a pair of weights for which there exists $r > 1$ such that, for every cube Q,*

$$\Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{r\,(1-p')} \, d\mu(x)\Big)^{\frac{p-1}{r}} \leq C.$$

Then, there exists some q, $1 < q < p$, such that $(u, v) \in \mathcal{A}_q$. Consequently, $(u, v) \in \mathcal{S}_p$, that is, for every $f \in L^p(v)$,

$$\int_{\mathbb{R}^d} (\mathcal{M}f(x))^p \, u(x) \, d\mu(x) \leq C \int_{\mathbb{R}^d} |f(x)|^p \, v(x) \, d\mu(x).$$

(iii) *If $0 < \varepsilon < 1$ and $(u, v) \in \mathcal{A}_p$ with $1 < p < \infty$, then $(u^\varepsilon, v^\varepsilon) \in \mathcal{A}_{\varepsilon p + 1 - \varepsilon}$.*

(iv) *Let $1 < p < \infty$. Then, $(u, v) \in \mathcal{A}_p$, if and only if, $(v^{1-p'}, u^{1-p'}) \in \mathcal{A}_{p'}$.*

(*v*) *Let* $w(x) \geq 0$*,* $w \in L^1_{\mathrm{loc}}(\mu)$*, then for* $1 < p < \infty$*:*

$$\begin{aligned} w\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\} &\leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(x)| \, \mathcal{M}w(x) \, d\mu(x), \\ \int_{\mathbb{R}^d} (\mathcal{M}f(x))^p \, w(x) \, d\mu(x) &\leq C \int_{\mathbb{R}^d} |f(x)|^p \, \mathcal{M}w(x) \, d\mu(x), \end{aligned}$$

or what is the same, the pair of weights $(w, \mathcal{M}w)$ *belongs to* $\mathcal{A}_1$ *and so to* $\mathcal{S}_p$*.*

Proof. The proof uses standard arguments, explained in detail in [GR, Chapter IV]. For the first part, note that if $u(E) > 0$ then $\mu(E) > 0$ and if $\mu(E) > 0$ then $v(E) > 0$ (because $v^{1-p'} \in L^1_{\mathrm{loc}}(\mu)$ and thereby $v > 0$ μ-almost everywhere). Thus,

$$\|\mathcal{M}f\|_{L^\infty(u)} \leq \|\mathcal{M}f\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(v)},$$

that is, $\mathcal{M} : L^\infty(v) \longrightarrow L^\infty(u)$. Marcinkiewicz interpolation theorem and Theorems 3.1, 4.3 bring these inclusions. For (*ii*), by taking $q = 1 + \frac{p-1}{r}$, we have $1 < q < p < \infty$. Furthermore,

$$\begin{aligned} &\Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{1-q'} \, d\mu(x)\Big)^{q-1} \\ &\quad= \Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{r\,(1-p')} \, d\mu(x)\Big)^{\frac{p-1}{r}} \leq C, \end{aligned}$$

and so $(u, v) \in \mathcal{A}_q \subset \mathcal{S}_p$ by (*i*). Let us see what happens with (*iii*). Let $(u, v) \in \mathcal{A}_p$ and $r = \varepsilon\, p + 1 - \varepsilon$. We want to show that $(u^\varepsilon, v^\varepsilon) \in \mathcal{A}_r$. Use Hölder's inequality with $\frac{1}{\varepsilon} > 1$ and the fact that $(u, v) \in \mathcal{A}_p$ to obtain

$$\begin{aligned} &\Big(\frac{1}{\ell(Q)^n} \int_Q u(x)^\varepsilon \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{\varepsilon\,(1-r')} \, d\mu(x)\Big)^{r-1} \\ &\quad\leq \Big\{\Big(\frac{1}{\ell(Q)^n} \int_Q u(x) \, d\mu(x)\Big) \Big(\frac{1}{\ell(Q)^n} \int_Q v(x)^{1-p'} \, d\mu(x)\Big)^{p-1}\Big\}^\varepsilon \leq C, \end{aligned}$$

that is, $(u^\varepsilon, v^\varepsilon) \in \mathcal{A}_r$. Part (*iv*) is trivial. Finally, observe that $(u, v) = (w, \mathcal{M}w)$ is a pair of weights in $\mathcal{A}_1$. This is obtained by means of the second characterization of this class, which trivially holds with constant 1. In this way, the first inequality follows, whereas the second one arises as a consequence of (*i*). □

One can not expect that $\mathcal{A}_p = \mathcal{S}_p$, or in other words, two-weight weak type inequalities do not imply in general strong inequalities. A counterexample comes from the classical setting. Take μ the Lebesgue measure in $\mathbb{R}^d$, which is "d-dimensional". Then $\mathcal{M}$ is the (classical) Hardy-Littlewood maximal function which will be denoted by M. Like we can find in [GR, p. 395], the fact that $(u, v) \in A_p$, $1 < p < \infty$, is not sufficient for M to be bounded between $L^p(v)$ and $L^p(u)$. However, in this classical setting when one is dealing with the same problem but only with one weight —that is, $u = v = w$— the weights have better properties and $A_p = S_p$. We would like to consider the one-weight problem for $\mathcal{M}$; in particular, we shall see that some properties of the classical setting may fail in this new context.

Theorem 7.2 *The following statements are* ***false*** *in general:*

(*a*) *If* $w \in \mathcal{A}_p$, $1 < p < \infty$, *then* w *satisfies a reverse Hölder's inequality* (*RHI*), *that is, there exists* $\varepsilon > 0$ *such that*

$$\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{1+\varepsilon}\,d\mu(x)\Big)^{\frac{1}{1+\varepsilon}} \le \frac{C}{\ell(Q)^n}\int_Q w(x)\,d\mu(x)$$

holds for every cube Q.

(*b*) *If* $w \in \mathcal{A}_p$, $1 < p < \infty$, *then there exits* $\varepsilon > 0$ *such that* $w^{1+\varepsilon} \in \mathcal{A}_p$.

(*c*) *If* $w \in \mathcal{A}_p$, $1 < p < \infty$, *then there exists* $\varepsilon > 0$ *in such a way that* $w \in \mathcal{A}_{p-\varepsilon}$.

(*d*) *If* $w \in \mathcal{A}_p$, $1 < p < \infty$, *then* $\mathcal{M}$ *is bounded on* $L^p(w)$.

Before proving this result, we are going to see the connections of these properties. Actually, we can obtain that

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d).$$

Fix $w \in \mathcal{A}_p$, $1 < p < \infty$. For the first implication, by Proposition 7.1, part (*iv*), $w^{1-p'} \in \mathcal{A}_{p'}$. By applying (*a*) to w and $w^{1-p'}$, we get $\varepsilon_1, \varepsilon_2 > 0$ such that the RHI with ε_1 and ε_2 holds respectively for w and $w^{1-p'}$. Then, if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, by Hölder's inequality and (*a*) we have

$$\begin{aligned}
&\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{1+\varepsilon}\,d\mu(x)\Big)\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{(1+\varepsilon)\,(1-p')}\,d\mu(x)\Big)^{p-1}\\
&\le \Big\{\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{1+\varepsilon_1}\,d\mu(x)\Big)^{\frac{1}{1+\varepsilon_1}}\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{(1+\varepsilon_2)\,(1-p')}\,d\mu(x)\Big)^{\frac{p-1}{1+\varepsilon_2}}\Big\}^{1+\varepsilon}\\
&\le \Big\{\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)\,d\mu(x)\Big)\Big(\frac{1}{\ell(Q)^n}\int_Q w(x)^{1-p'}\,d\mu(x)\Big)^{p-1}\Big\}^{1+\varepsilon} \le C.
\end{aligned}$$

Now, we check that (*c*) follows from (*b*). Note that Hölder's inequality and (*b*) imply that the pair of weights (w, w) satisfy (*ii*) of Proposition 7.1 with $r = 1+\varepsilon$. Thus, $(w, w) \in \mathcal{A}_q$ for some $1 < q < p$. Finally, (*c*) $\Longrightarrow$ (*d*) arises as a consequence of Proposition 7.1, part (*i*).

Proof of Theorem 7.2. In view of the facts above, we only have to find an example for which (*d*) fails. However, we shall give two examples: in the first one, it is easy to check that (*a*), (*b*) and (*c*) do not hold. Whereas the second one, which is less natural, will be used to get that (*d*) fails. In $\mathbb{R}$, consider $d\mu(x) = \gamma(x)\,dx$ with $\gamma(x) = e^{-x^2}$. Since $0 < \gamma(x) \le 1$, this measure is "1-dimensional". The Muckenhoupt condition becomes:

$$(\mathcal{A}_p) \qquad \Big(\frac{1}{|I|}\int_I w(t)\,\gamma(t)\,dt\Big)\Big(\frac{1}{|I|}\int_I w(t)^{1-p'}\,\gamma(t)\,dt\Big)^{p-1} \le C,$$

for every bounded interval I and for $1 < p < \infty$. Take $w(t) = \gamma(t)^{p-1}$. Then,

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_I w(t)\,\gamma(t)\,dt\Big)\Big(\frac{1}{|I|}\int_I w(t)^{1-p'}\,\gamma(t)\,dt\Big)^{p-1}\\
&\qquad = \Big(\frac{1}{|I|}\int_I e^{-p\,t^2}\,dt\Big)\Big(\frac{1}{|I|}\int_I 1\,dt\Big)^{p-1} \le 1,
\end{aligned}$$

and hence $w \in \mathcal{A}_p$. We show that $w \notin \mathcal{A}_q$ for every $q < p$. Take $1 < q < p$ and put $\theta = (q'-1)\,(p-1) > 1$. Then,

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_I w(t)\,\gamma(t)\,dt\Big)\Big(\frac{1}{|I|}\int_I w(t)^{1-q'}\,\gamma(t)\,dt\Big)^{q-1}\\
&\qquad = \Big(\frac{1}{|I|}\int_I e^{-p\,t^2}\,dt\Big)\Big(\frac{1}{|I|}\int_I e^{(\theta-1)\,t^2}\,dt\Big)^{q-1}.
\end{aligned}$$

In particular, for $I = [-r, r]$ with $r > 1$, this amount is bigger than

$$\Big(\frac{1}{2\,r}\int_{-1}^{1} e^{-p\,t^2}\,dt\Big)\Big(\frac{1}{2\,r}\int_{-r}^{r} e^{(\theta-1)\,t^2}\,dt\Big)^{q-1} = C\,\Big(\frac{1}{r^{q'}}\int_0^r e^{(\theta-1)\,t^2}\,dt\Big)^{q-1} \longrightarrow \infty$$

as $r \to \infty$, since $\theta > 1$. Therefore, $w \notin \mathcal{A}_q$. Then, (a) and (b) fail. In fact, one can see that $w^{1-p'} \in \mathcal{A}_{p'}$ does not satisfy (a) and that w neither verifies (b).

For the other example, we shall work again in $\mathbb{R}$. Take g a continuous function which is positive everywhere and integrable with respect to Lebesgue measure. Let us recall that M denotes the (classical) Hardy-Littlewood maximal function in $\mathbb{R}$. Consider

$$\gamma(x) = \left(\frac{g(x)}{Mg(x)}\right)^{\frac{p-1}{p}} > 0.$$

Note that this function is well defined, since Mg never vanishes. Besides, $\gamma(x) \le 1$ almost everywhere (with respect to dx) and so $d\mu(x) = \gamma(x)\,dx$ is an "1-dimensional" measure. Consider the weight

$$w(x) = \Big(g(x)^{\frac{1}{p}}\,Mg(x)^{\frac{1}{p'}}\Big)^{-(p-1)}.$$

First of all, we see that $w \in \mathcal{A}_p$:

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_I w(t)\,\gamma(t)\,dt\Big)\Big(\frac{1}{|I|}\int_I w(t)^{1-p'}\,\gamma(t)\,dt\Big)^{p-1}\\
&\qquad = \Big(\frac{1}{|I|}\int_I Mg(t)^{1-p}\,dt\Big)\Big(\frac{1}{|I|}\int_I g(t)\,dt\Big)^{p-1}\\
&\qquad = \Big\{\Big(\frac{1}{|I|}\int_I g(t)\,dt\Big)\Big(\frac{1}{|I|}\int_I Mg(t)^{1-p}\,dt\Big)^{p'-1}\Big\}^{p-1}.
\end{aligned}$$

This amount is bounded if the pair of weights $(g, Mg) \in A_{p'}$. We know that $(g, Mg) \in A_1 \subset A_{p'}$. Thus, we have obtained that $w \in \mathcal{A}_p$. Assume that $\mathcal{M}$ is bounded on $L^p(w\,d\mu)$. Since $\mathcal{M}f(x) = M(f\,\gamma)(x)$, this inequality can be seen in this fashion

$$\int_{\mathbb{R}} (Mf(x))^p\,w(x)\,\gamma(x)dx \le C\,\int_{\mathbb{R}} |f(x)|^p\,w(x)\,\gamma(x)^{1-p}\,dx,$$

that is,

$$\int_{\mathbb{R}} (Mf(x))^p\,(Mg(x))^{1-p}\,dx \le C \int_{\mathbb{R}} |f(x)|^p\,g(x)^{1-p}\,dx.$$

By setting $f = g$, it follows

$$\int_{\mathbb{R}} (Mg(x))^p\,(Mg(x))^{1-p}\,dx \le C \int_{\mathbb{R}} g(x)^p\,g(x)^{1-p}\,dx = C \int_{\mathbb{R}} g(x)\,d\mu(x) < \infty$$

and consequently, $Mg \in L^1(dx)$, something that only happens if $g \equiv 0$. So, $\mathcal{M}$ is not bounded in $L^p(w\,d\mu)$. $\square$

Note that in the previous examples the underlying measure is non-doubling —as a matter of fact, if it were doubling Muckenhoupt weights would verify a RHI (see [ST])—. Actually, if some weight satisfies a RHI, then the measure must be doubling.

Proposition 7.3 *If there exists $w > 0$ μ-almost everywhere, $w \in L^1_{\mathrm{loc}}(\mu)$, which satisfies a RHI, that is,*

$$\Big(\frac{1}{\ell(Q)^n} \int_Q w(x)^{1+\varepsilon}\,d\mu(x)\Big)^{\frac{1}{1+\varepsilon}} \le \frac{C}{\ell(Q)^n} \int_Q w(x)\,d\mu(x),$$

for every Q and for some $\varepsilon > 0$, then $\mu(Q) \ge C\,\ell(Q)^n$ for every cube with $\mu(Q) > 0$ and thus μ is doubling.

Proof. It is enough to apply Hölder's inequality with exponent $1+\varepsilon$:

$$\begin{aligned} w(Q) &= \int_Q w(x)\,d\mu(x) \le \Big(\int_Q w(x)^{1+\varepsilon}\,d\mu(x)\Big)^{\frac{1}{1+\varepsilon}} \mu(Q)^{\frac{1}{(1+\varepsilon)'}} \\ &\le C\,\ell(Q)^{\frac{n}{1+\varepsilon}} \Big(\frac{1}{\ell(Q)^n} \int_Q w(x)\,d\mu(x)\Big)\,\mu(Q)^{\frac{1}{(1+\varepsilon)'}} \\ &= C\,\left(\frac{\mu(Q)}{\ell(Q)^n}\right)^{\frac{1}{(1+\varepsilon)'}} w(Q). \end{aligned}$$

Then for every Q with $\mu(Q) > 0$, we have $w(Q) > 0$ and thus $\mu(Q) \ge C\,\ell(Q)^n$. $\square$

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