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Weighted norm inequalities for maximally modulated singular integral operators

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Abstract. We present a framework that yields a variety of weighted and vector-valued estimates for maximally modulated Calderón-Zygmund singular (and maximal singular) integrals from a single *a priori* weak type unweighted estimate for the maximal modulations of such operators. We discuss two approaches, one based on the good-λ method of Coifman and Fefferman [CF] and an alternative method employing the sharp maximal operator. As an application we obtain new weighted and vector-valued inequalities for the Carleson operator proving that it is controlled by a natural maximal function associated with the Orlicz space $L(\log L)(\log\log\log L)$. This control is in the sense of a good-λ inequality and yields strong and weak type estimates as well as vector-valued and weighted estimates for the operator in question.

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1. Introduction and Main results

In this article we will be concerned with estimates for maximally modulated Calderón-Zygmund singular integrals on $\mathbb{R}^n$. A Calderón-Zygmund operator is a linear operator T which is bounded from $L^2(\mathbb{R}^n)$ into itself such that for $f \in L^\infty_c(\mathbb{R}^n)$ (essentially bounded functions with compact support), we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy, \qquad \text{a.e. } x \in \mathbb{R}^n \setminus \operatorname{supp} f\,.$$

The kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \longrightarrow \mathbb{C}$ is assumed to satisfy the following standard conditions

$$|K(x,y)| \le \frac{c_0}{|x-y|^n}, \qquad x \neq y,$$

and, if $|x-y| > 2\,|y-y'|$,

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \le c_0\, \frac{|y-y'|^\tau}{|x-y|^{n+\tau}},$$

for some c_0, $\tau > 0$. Associated with T there is a truncated operator T_ε and a maximal singular operator $T_\star$ defined as follows:

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y)\, f(y)\, dy, \qquad T_\star f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

Suppose that we are given a family $\Phi = \{\phi_\alpha\}_{\alpha\in\mathcal{A}}$ of measurable real-valued functions indexed by an arbitrary set $\mathcal{A}$. Then we can define maximally modulated versions of T and $T_\star$ associated with Φ. First we define the modulation operator

$$\mathcal{M}^{\phi_\alpha} f(x) = e^{2\,\pi\, i\, \phi_\alpha(x)}\, f(x)$$

and the "Carleson"-type maximal modulated singular integral of T with respect to Φ:

$$T^\Phi f(x) = \sup_{\alpha\in\mathcal{A}} \big| T(\mathcal{M}^{\phi_\alpha} f)(x) \big|.$$

This definition is motivated by the Carleson operator in which T is the Hilbert transform and the family Φ is given by the linear functions $\phi_\alpha(y) = \alpha\, y$ with $\alpha \in \mathbb{R}$. We also define the (maximally) modulated maximal singular integral associated with T and Φ via

$$\begin{aligned} T^\Phi_\star f(x) &= \sup_{\varepsilon>0} \sup_{\alpha\in\mathcal{A}} \big| T_\varepsilon(\mathcal{M}^{\phi_\alpha} f)(x) \big| \\ &= \sup_{\varepsilon>0} \sup_{\alpha\in\mathcal{A}} \Big| \int_{|x-y|>\varepsilon} K(x,y)\, e^{2\,\pi\, i\, \phi_\alpha(y)}\, f(y)\, dy \Big|\,. \end{aligned}$$

The purpose of this article is to present a framework that yields weighted and vector-valued estimates for T^{Φ} and $T^{\Phi}_{\star}$ from a single *a priori* weak type estimate for T^{Φ}. Our main approach is based on the good-λ method of Coifman and Fefferman [CF] although we discuss an alternative approach using the sharp maximal operator. We note that in the special case where T is the Hilbert transform and T^{Φ} is the Carleson operator, boundedness on $L^p(w)$ for w in A_p was obtained by Hunt and Young [HY]. Below we sharpen and extend such weighted estimates to more general maximally modulated operators.

We denote by M the Hardy-Littlewood maximal operator and by $M_r f = M\big(|f|^r\big)^{\frac{1}{r}}$ where $0 < r < \infty$. A non-negative locally integrable function w is said to be in A_p, $1 < p < \infty$, if there exists some constant C such that for every cube Q (with sides parallel to the coordinate axes) we have

$$\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\Big)^{p-1} \le C.$$

Letting $p \to 1$ we analogously define the A_1 class

$$\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\,\|w^{-1}\|_{L^\infty(Q)} \le C.$$

The smallest constant C for which the condition A_p, $1 \le p < \infty$, holds is called the A_p characteristic constant of w. We also recall that $A_\infty = \bigcup_{p>1} A_p$. These classes were introduced by Muckenhoupt in [M] to characterize the boundedness of the Hardy-Littlewood maximal functions on weighted Lebesgue spaces $L^p(w) = L^p(w\,dx)$. The reader is referred to [GR] for a comprehensive account of these topics.

We have the following theorem.

Theorem 1.1. *Let T be a Calderón-Zygmund operator and let Φ a family of measurable real-valued functions. Assume that T^{Φ} maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ for some $1 < r < \infty$ with norm $\|T^{\Phi}\|_{L^r \to L^{r,\infty}}$. Then, for any $w \in A_\infty$ there exist positive constants C_0, ε_0, that depend only on w, and γ_0 that depends on τ, c_0, r, and $\|T^{\Phi}\|_{L^r \to L^{r,\infty}}$, such that for all f in $\bigcup_{1 \le p < \infty} L^p(\mathbb{R}^n)$, for all $0 < \gamma < \gamma_0$, and for all $\lambda > 0$ we have*

$$w\big\{T^{\Phi}_{\star} f > 3\,\lambda,\, M_r f \le \gamma\,\lambda\big\} \le C_0\,\gamma^{r\cdot\varepsilon_0}\,w\big\{T^{\Phi}_{\star} f > \lambda\big\}. \qquad (1)$$

Using (1) and standard techniques we deduce the following weighted estimates.

Corollary 1.2. *Let T and Φ be as before. Assume that T^Φ maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then for every $w \in A_\infty$ and $0 < p < \infty$ there is a constant C that depends on p, w, n, c_0, τ, and $\|T^\Phi\|_{L^r \to L^{r,\infty}}$ such that the estimates below hold*

$$\|T^\Phi f\|_{L^p(w)} \le C\,\|M_r f\|_{L^p(w)}, \tag{2}$$

$$\|T^\Phi f\|_{L^{p,\infty}(w)} \le C\,\|M_r f\|_{L^{p,\infty}(w)}, \tag{3}$$

$$\|T_\star^\Phi f\|_{L^p(w)} \le C\,\|M_r f\|_{L^p(w)}, \tag{4}$$

$$\|T_\star^\Phi f\|_{L^{p,\infty}(w)} \le C\,\|M_r f\|_{L^{p,\infty}(w)}, \tag{5}$$

with the understanding that these estimates hold for all functions f for which the left hand sides of the displayed inequalities are finite. Consequently, it follows that T^Φ and $T_\star^\Phi$ map $L^p(w)$ into $L^p(w)$ for all $p > r$ whenever $w \in A_{p/r}$. Moreover, T^Φ and $T_\star^\Phi$ map $L^r(v)$ into $L^{r,\infty}(v)$ for all $v \in A_1$.

There is a way to obtain Corollary 1.2 bypassing the good-λ inequality of Theorem 1.1. Namely, using the sharp maximal function $M^\#$, one can show that

$$M^\#(T^\Phi f)(x) \le C\,M_r f(x) \tag{6}$$

which implies all the previous estimates of Corollary 1.2. For the sake of completeness, we will discuss this alternative approach as well. The latter idea has been utilized by [RRT] in the study of Carleson-Sjölin operators; the terminology refers to maximally modulated operators in which the family Φ consists of the functions $\phi_a(y) = a \cdot y$, where $a \in \mathbb{R}^n$.

We would like to point out that Corollary 1.2 is weaker than the good-λ inequality contained in Theorem 1.1. Nevertheless, some recent results obtained in [CMP], [CGMP] show that from the single estimate

$$\|T^\Phi f\|_{L^p(w)} \le C_p(w)\,\|M_r f\|_{L^p(w)}, \qquad \text{for all } 0 < p < \infty,\ w \in A_\infty, \tag{7}$$

one can extrapolate and obtain all the conclusions of Corollary 1.2 in the scale of Lorentz, Orlicz spaces, and other rearrangement invariant function spaces. However, here we prefer to deduce these estimates as a corollary of the powerful good-λ inequality of Theorem 1.1. This inequality provides a precise pointwise estimate for the level sets of a maximally modulated singular integrals and it therefore subsumes all possible norm estimates; more importantly, it is of intrinsic interest and yields structural information about such operators.

One advantage of the extrapolation results in [CMP] is that ℓ^q-valued estimates follow from (7) without use of Banach-space theory for Calderón-Zygmund operators, as in [RRT]. Thus, from (2), (4), and [CMP] we obtain the following:

Corollary 1.3. *Let T and Φ be as before. Assume that T^{Φ} maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then for every $w \in A_\infty$ and $0 < p, q < \infty$ there is a constant C such that*

$$\Big\| \Big(\sum_j |T^{\Phi} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C \Big\| \Big(\sum_j (M_r f_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}$$

and

$$\Big\| \Big(\sum_j |T^{\Phi} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^{p,\infty}(w)} \le C \Big\| \Big(\sum_j (M_r f_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^{p,\infty}(w)},$$

for all sequences of functions f_j for which the left hand sides are finite. Consequently, for every $q > r$,

$$\Big\| \Big(\sum_j |T^{\Phi} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C \Big\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}, \quad \text{for } p > r,\ w \in A_{p/r};$$

and

$$\Big\| \Big(\sum_j |T^{\Phi} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^{r,\infty}(w)} \le C \Big\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^r(w)}, \quad \text{for all } w \in A_1$$

for all sequences of functions f_j in $L^p(w)$ (or $L^r(w)$ if $w \in A_1$). Moreover, all the above estimates also hold for $T^{\Phi}_\star$ in place of T^{Φ}.

We note that the two last estimates in Corollary 1.3 could also be obtained as a consequence of the Banach space approach developed in [RRT] suitably adapted to our framework.

Our next goal in this article is to improve the previously known results when estimates near L^1 are known. Let us explain the motivation for this problem. We have seen in Theorem 1.1 and Corollaries 1.2, 1.3 that a maximally modulated singular integral operator T^{Φ} mapping $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ is controlled by the maximal operator M_r. From the proof of the good-λ inequality, or from the approach based on the sharp maximal function, we see that M_r was chosen because T^{Φ} satisfies a weak type estimate in L^r. In general, one would like to replace M_r by a better maximal operator as close as possible to the Hardy-Littlewood maximal operator (which does not control maximally modulated singular integrals.) This would require to study the boundedness of T^{Φ} near L^1.

Let us consider the Carleson operator, that is the operator

$$\mathcal{C}f(x) = \sup_{a \in \mathbb{R}} \Big| \text{p.v.} \int_{\mathbb{R}} \frac{e^{2\pi i\, a\, y}}{x - y}\, f(y)\, dy \Big|$$

acting on functions on the real line. Using the notation previously introduced, $\mathcal{C} = H^{\Psi_1(1)}$ where H is the Hilbert transform and $\Psi_1(1)$ is the family of one-variable real polynomials of degree at most 1. It is well known that $\mathcal{C}$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. Then, we know that $\mathcal{C}$ can be controlled by M_p where p can be taken arbitrarily close to 1. But p cannot be taken equal to 1 as $\mathcal{C}$ is known not to be of weak type $(1,1)$. But there is big gap between M and M_p, $p > 1$; all maximal operators associated with Orlicz spaces between L^1 and L^p, such as $L(\log L)$ or $L(\log L)(\log\log\dots\log L)$, could serve the purpose of controlling $\mathcal{C}$ and other maximally modulated singular integrals in the good-λ sense previously described.

For $\mathcal{C}$ some estimates near L^1, better than L^p, are known. Let us write $S^\star$ for the discrete analog of $\mathcal{C}$ on the torus, that is, for the supremum of the partial sums of a Fourier series in the torus. In [Sj] it was proven that $S^\star$ maps $L(\log L)(\log\log L)$ into $L^{1,\infty}$. There is a general extrapolation result (in the spirit of Yano) which says that a sublinear operator that satisfies a restricted weak L^p estimate with constant $(p-1)^{-m}$ as $p \to 1^+$ is indeed bounded from $L(\log L)^m(\log\log L)$ to $L^{1,\infty}$ (see [So1], [So2]). Lately Antonov [An] sharpened the best known result known for $S^\star$ showing that $\mathcal{C}$ maps the Orlicz space $L(\log L)(\log\log\log L)$ to $L^{1,\infty}$ (see also [Ar]). Recently [SS] have provided a general extrapolation principle that works for several maximal operators which, in particular, gives another proof of the aforementioned result of Antonov concerning $S^\star$ and also yields some positive results for the Walsh-Fourier series and for the *halo conjecture*.

In this work we exploit these kind of ideas to obtain a better maximal operator controlling $\mathcal{C}$. We are going to get a general result for T^Φ only assuming an appropriate growth in the constant of the restricted weak L^p estimate of such an operator. We will see in particular, that the operator $M_{L(\log L)(\log\log\log L)}$ controls $\mathcal{C}$ and also a similar maximally modulated singular integral with quadratic phase (see Section 2 below). Observe that this operator is pointwise smaller than all the M_p, $p > 1$, hence our estimates are better than those previously known.

Next we introduce some notation about Orlicz spaces. For a complete development of this topic the reader is referred to [RR], [BS]. Let $\Upsilon : [0,\infty) \longrightarrow [0,\infty)$ be a Young function, that is, a continuous, convex, increasing function with $\Upsilon(0) = 0$ and such that $\Upsilon(t) \longrightarrow \infty$ as $t \to \infty$. By definition, the Orlicz space L_Υ consists of all measurable functions f such that

$$\int_{\mathbb{R}^n} \Upsilon\left(\frac{|f(x)|}{\lambda}\right) dx < \infty, \qquad \text{for some } \lambda > 0.$$

The space L_Υ is a Banach function space if it is endowed with the Luxemburg norm

$$\|f\|_\Upsilon = \|f\|_{L_\Upsilon} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Upsilon\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\Big\}.$$

For example, if $\Upsilon(t) = t^p$ for $1 < p < \infty$, then $L_\Upsilon = L^p$. Another classical example is given by $\Upsilon(t) = t\,(1 + \log^+ t)$, properly speaking, $\Upsilon(t)$ is the convex majorant of $t\,(1 + \log^+ t)$. In this case L_Υ is the Zygmund space $L\log L$. Let us define the following localized version of the Orlicz norm: for every Q,

$$\|f\|_{\Upsilon,Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Upsilon\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\Big\}.$$

Note that $\|f\|_{\Upsilon,Q} = \|f\|_{L_\Upsilon(Q,\frac{dx}{|Q|})}$. We also define the maximal operator associated to this space as:

$$M_\Upsilon f(x) = \sup_{Q\ni x} \|f\|_{\Upsilon,Q}.$$

For example, if $\Upsilon(t) = t^p$ we have $M_\Upsilon f(x) = M_p f(x)$ since for every cube $\|f\|_{\Upsilon,Q}$ is the L^p-average of f over Q.

We need to introduce a little bit more of notation: for any cube $Q \subset \mathbb{R}^n$ we consider the probability measure $d\mu_Q(x) = \frac{\chi_Q(x)}{|Q|}\,dx$ and we define the localized operator

$$T^\Phi_Q f(x) = T^\Phi(f\ \chi_Q)(x)\ \chi_Q(x).$$

Let us also set $\varphi_m(t) = t\left(1 + \log^+ \frac{1}{t}\right)^m$.

Now, we state the main result of this article. We only work with convolution type operators but analogous results could be obtained for general nonconvolution linear operators.

Theorem 1.4. *Let T be a convolution Calderón-Zygmund operator and $\Phi = \{\phi_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of twice differentiable real-valued functions such that for each $\alpha \in \mathcal{A}$ and for each cube Q we have*

$$\|\phi_\alpha\|_{L^\infty(Q)} + \|\nabla\phi_\alpha\|_{L^\infty(Q)} + \|D^2\phi_\alpha\|_{L^\infty(Q)} \leq C(Q,\alpha) < \infty.$$

Assume that either $\mathcal{A}$ is countable or that there exists a countable subset $\mathcal{A}_0 \subset \mathcal{A}$ such that for almost all $x \in \mathbb{R}^n$ we have

$$T^\Phi f(x) = \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha} f)(x)\big| = \sup_{\alpha\in\mathcal{A}_0} \big|T(\mathcal{M}^{\phi_\alpha} f)(x)\big|.$$

Suppose that for some $C > 0$, $m \geq 0$ and for all $1 < p \leq 2$ and measurable sets A of finite measure, T^{Φ} satisfies the following restricted weak type estimate

$$\big|\{x : T^{\Phi}(\chi_A)(x) > \lambda\}\big|^{\frac{1}{p}} \leq \Big(\frac{C}{p-1}\Big)^m \frac{|A|^{\frac{1}{p}}}{\lambda}. \tag{8}$$

Let $\varUpsilon_m(t) = t\,(1+\log^+ t)^m\,(1+\log^+\log^+\log^+ t)$. Then for all $0 < p < \infty$, $w \in A_\infty$, and all functions f for which the left hand side below is finite we have the estimate

$$\|T^{\Phi} f\|_{L^p(w)} \leq C\,\|M_{\varUpsilon_m} f\|_{L^p(w)}. \tag{9}$$

Moreover, all the estimates (1), (3), (4), (5) *and the vector-valued inequalities contained in Corollary 1.3 hold for $M_{\varUpsilon_m}$ in place of M_r.*

Furthermore, (8) *can be replaced by the weaker condition*

$$\begin{aligned}\mu_Q\big\{x : T^{\Phi}_Q(\chi_A)(x) > \lambda\big\} &= \mu_Q\big\{x \in Q : T^{\Phi}(\chi_A)(x) > \lambda\big\} \\ &\leq \frac{C_0}{\lambda}\,\varphi_m\big(\mu_Q(A)\big).\end{aligned} \tag{10}$$

for any cube $Q \subset \mathbb{R}^n$ and for all measurable sets $A \subset Q$, where C_0 is independent of Q.

The proof of this result is based on some sort of Yano's extrapolation procedure inspired by [SS]; see Theorem 5.3 in Section 5 (Subsection 5.2) for more details. This result will provide the following estimate

$$\|T^{\Phi}_Q f\|_{L^{1,\infty}(Q,\mu_Q)} \leq C\,\|f\|_{\varUpsilon_m,Q},$$

which will be used to yield the corresponding good-λ inequality. For the approach based on the sharp maximal function, the latter estimate will yield a substitute for (6):

$$M^{\#}_{\delta}(T^{\Phi} f)(x) \leq C_\delta\, M_{\varUpsilon_m} f(x), \qquad \text{whenever} \quad 0 < \delta < 1,$$

where $M^{\#}_{\delta} g(x) = M^{\#}\big(|g|^{\delta}\big)(x)^{1/\delta}$.

Remark 1.5. Note that for $t \geq 1$ we have that $\varUpsilon_m(t) \leq t^r$ for $1 < r < \infty$ and therefore $\|f\|_{\varUpsilon_m,Q} \leq 2\,\|f\|_{t^r,Q}$ which gives $M_{\varUpsilon_m} f(x) \leq M_r f(x)$. Thus, Theorem 1.4 is an improvement of previous theorems in which only a single L^r-estimate was assumed.

Remark 1.6. We show that (10) is weaker than (8) and thus it suffices to work with estimate (10) in the proof of Theorem 1.4. To see this we first notice that (8) implies

$$\big|\big\{x : T_Q^{\Phi}(\chi_A)(x) > \lambda\big\}\big|^{\frac{1}{p}} = \big|\big\{x \in Q : T^{\Phi}(\chi_A)(x) > \lambda\big\}\big|^{\frac{1}{p}} \le \left(\frac{C_0}{p-1}\right)^m \frac{|A|^{\frac{1}{p}}}{\lambda}$$

which in terms of the probability measure μ_Q can be written as

$$\Big(\mu_Q\big\{x : T_Q^{\Phi}(\chi_A)(x) > \lambda\big\}\Big)^{\frac{1}{p}} \le \left(\frac{C_0}{p-1}\right)^m \frac{\mu_Q(A)^{\frac{1}{p}}}{\lambda}.$$

Taking in particular $p = 1 + \big(1 - \log \mu_Q(A)\big)^{-1}$ we obtain

$$\begin{aligned}\mu_Q\big\{x : T_Q^{\Phi}(\chi_A)(x) > \lambda\big\} &\le \mu_Q\big\{x : T_Q^{\Phi}(\chi_A)(x) > \lambda\big\}^{\frac{1}{p}} \\ &\le \frac{C_0^m}{\lambda}(p-1)^{-m}\,\mu_Q(A)^{\frac{1}{p}} = \frac{C_0^m}{\lambda}\,\varphi_m\big(\mu_Q(A)\big)\,\mu_Q(A)^{\frac{1}{p}-1} \\ &\le \frac{C_0^m\,e}{\lambda}\,\varphi_m\big(\mu_Q(A)\big)\end{aligned}$$

since $\mu_Q(A)^{\frac{1}{p}-1} \le e$. Therefore we have shown that (8) implies (10).

We organize the paper as follows. In Section 2 we present some applications of our main results. In Section 3 we give a proof of the good-λ estimate in Theorem 1.1. Section 4 contains the alternative approach based on the sharp maximal function. Section 5 is devoted to show Theorem 1.4 and the general extrapolation procedure that leads to it, whose proof is given in Section 6.

2. Applications

Before discussing the proofs of our results, we turn to some applications. Let us denote by $\Psi_1(k)$ the family of all one-variable real polynomials of degree at most k defined on $\mathbb{R}$. Using the notation introduced earlier, the Carleson operator $\mathcal{C}$ is $H^{\Psi_1(1)}$ where H is the Hilbert transform. It is known that $\mathcal{C}$ is bounded on $L^r(\mathbb{R})$ for all $1 < r < \infty$. Then we have

$$\|\mathcal{C}f\|_{L^p(w)} \le C\,\|M_r f\|_{L^p(w)}$$

for all $1 < r < \infty$, all $0 < p < \infty$ and $w \in A_\infty$. We also have the corresponding good-λ estimate in Theorem 1.1, all the estimates in Corollary

1.2 and the vector-valued inequalities in Corollary 1.3. The same estimates are valid for $\mathcal{C}_\star$ in place of $\mathcal{C}$. As a consequence, for $1 < p, q < \infty$ and for $w \in A_p$, by taking $1 < r < \min\{p, q\}$ sufficiently close to 1 so that $w \in A_{p/r}$, we easily obtain

$$\|\mathcal{C}f\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}, \quad \Big\|\Big(\sum_j |\mathcal{C}f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \le C\,\Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)}.$$

The first of these estimates first appeared in [HY] and the second in [RRT].

It is a well known fact [H] that $\mathcal{C}$ satisfies the following restricted weak type (p, p) result

$$\big|\{x : \mathcal{C}(\chi_A)(x) > \lambda\}\big|^{\frac{1}{p}} \le C\,\frac{p^2}{p-1}\,\frac{|A|^{\frac{1}{p}}}{\lambda} \tag{11}$$

for $\lambda > 0$, $1 < p \le 2$. This means that we can apply Theorem 1.4 with $m = 1$. We then obtain the following theorem that improves the results of [HY] and [RRT].

Theorem 2.1. *Let $\mathcal{C}$ be the Carleson operator and consider the Orlicz function $\Upsilon(t) = t\,(1 + \log^+ t)\,(1 + \log^+\log^+\log^+ t)$. Then, for all $0 < p < \infty$ and $w \in A_\infty$ we have*

$$\|\mathcal{C}f\|_{L^p(w)} \le C\,\|M_\Upsilon f\|_{L^p(w)}, \qquad \|\mathcal{C}f\|_{L^{p,\infty}(w)} \le C\,\|M_\Upsilon f\|_{L^{p,\infty}(w)},$$

and

$$\Big\|\Big(\sum_j |\mathcal{C}f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \le C\,\Big\|\Big(\sum_j |M_\Upsilon f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)},$$

$$\Big\|\Big(\sum_j |\mathcal{C}f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p,\infty}(w)} \le C\,\Big\|\Big(\sum_j |M_\Upsilon f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p,\infty}(w)}$$

for all functions f or sequences of functions f_j for which the left hand sides are finite. As a consequence, writing $M^3 = M \circ M \circ M$ we get

$$\|\mathcal{C}f\|_{L^p(w)} \le C\,\|M^3 f\|_{L^p(w)}, \qquad \|\mathcal{C}f\|_{L^{p,\infty}(w)} \le C\,\|M^3 f\|_{L^{p,\infty}(w)},$$

and the associated vector-valued inequalities in $L^p(w)$ and in $L^{p,\infty}(w)$. Furthermore, all these estimates hold with $\mathcal{C}_\star$ in place of $\mathcal{C}$ and the corresponding good-λ inequality is valid.

For the inequalities with M^3 one only needs to observe that

$$M_\Upsilon f(x) \le M_{L(\log L)^2} f(x) \approx M^3 f(x),$$

since $\Upsilon(t) \le t\,(1 + \log^+ t)^2$.

Remark 2.2. Note that in terms of the iterations of the Hardy-Littlewood maximal function, M^3 is, so far, the best known iteration that can be written on the right hand side. As mentioned before, with M such result is not true. Getting M^2 would be equivalent, somehow, to the fact that Fourier series of functions in $L(\log L)$ converge a.e., since $M^2 \approx M_{L(\log L)}$. This remains an open question at the moment.

Remark 2.3. We can obtain a formulation of Theorem 1.4 in terms of iterations of the Hardy-Littlewood maximal function. Note that we have $\Upsilon(t) \le t\,(1+\log^+ t)^{m+1}$ and, as before,

$$M_{\Upsilon_m} f(x) \le M_{L(\log L)^{m+1}} f(x) \approx M^{m+2} f(x),$$

where M^{m+2} is the operator M iterated $m+2$-times. Hence, as a consequence of Theorem 1.4, we also obtain the estimate

$$\|T^{\Phi} f\|_{L^p(w)} \le C\,\|M^{m+2} f\|_{L^p(w)},$$

and all the associated good-λ and vector-valued inequalities.

Assuming the result in [L], which states that the maximally modulated singular integral

$$H^{\Psi_1(2)} f(x) = \sup_{a,b\in\mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} f(y)\, e^{2\,\pi\, i\,(a\,y+b\,y^2)}\, \frac{dy}{x-y} \right|$$

is bounded on $L^p(\mathbb{R})$ for all $1<p<\infty$ with the corresponding restricted weak type inequality (11), one obtains the following result concerning the operator $H^{\Psi_1(2)}$:

Theorem 2.4. *Let* $\Upsilon(t) = t\,(1+\log^+ t)\,(1+\log^+\log^+\log^+ t)$*. Then, for all* $0<p,q<\infty$ *and all* $w\in A_\infty$ *we have*

$$\|H^{\Psi_1(2)} f\|_{L^p(w)} \le C\,\|M_{\Upsilon} f\|_{L^p(w)},$$

and

$$\Big\| \Big(\sum_j |H^{\Psi_1(2)} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \le C\, \Big\| \Big(\sum_j |M_{\Upsilon} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}$$

for all functions or sequences of functions for which the left hand sides are finite. All these inequalities hold for the maximal operator $H_\star^{\Psi_1(2)}$*. Also these estimates are valid with* $L^{p,\infty}(w)$ *in place of* $L^p(w)$ *and the corresponding good-*λ *estimate* (1) *holds. (As before, we can replace* M_Υ *by* M^3*.)*

As a consequence we obtain that for every $1<p,q<\infty$ *and every weight* $w\in A_p$ *we have*

$$\|H^{\Psi_1(2)} f\|_{L^p(w)} \le C\,\|f\|_{L^p(w)}$$

for all f in $L^p(w)$ and also

$$\Big\|\Big(\sum_j |H^{\Psi_1(2)} f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \le C\,\Big\|\Big(\sum_j |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)}$$

for all ℓ^q-valued sequences $\{f_j\}_j$ in $L^p(w)$. The same estimates are valid with $H_\star^{\Psi_1(2)}$.

To prove Theorem 2.4 we just need to apply Theorem 1.4 with $m = 1$. The claimed estimates can be easily obtained using the formulation in terms of M^3, noting that M^3 is bounded on $L^p(w)$ for $w \in A_p$ and also satisfies the corresponding weighted vector-inequalities (applying the known estimates for M three times).

Next we turn to higher dimensional analogues of Theorem 2.4. We suppose that Ω is an odd integrable function on S^{n-1} and we introduce a singular integral operator T_Ω by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy$$

for f sufficiently smooth. We denote the family of real polynomials of n variables and degree at most k by

$$\Psi_n(k) = \Big\{P(y) :\ P(y) = \sum_{|\gamma|\le k} c_\gamma y^\gamma\Big\}$$

where c_γ are real coefficients indexed by multi-indices $\gamma = (\gamma_1, \dots, \gamma_n)$ in $\mathbb{R}^n$. We consider the maximally modulated operator $T_\Omega^{\Psi_n(k)}$ and we seek bounds for it. To study this operator we introduce the directional maximally modulated singular integral operator associated with $\Psi_n(k)$ along the direction of a unit vector θ as follows:

$$H_\theta^{\Psi_n(k)} f(x) = \sup_{\psi\in\Psi_n(k)} \left|\text{p.v.} \int_{\mathbb{R}} f(x-r\theta) e^{2\pi i \psi(x-r\theta)} \frac{dr}{r}\right|$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. A simple argument using a suitable orthogonal transformation reduces the $L^p(\mathbb{R}^n)$ boundedness of $H_\theta^{\Psi_n(k)}$ to that of $H_{e_1}^{\Psi_n(k)}$, where $e_1 = (1, 0, \dots, 0)$. For instance in the case $k = 2$, to obtain the boundedness of $H_{e_1}^{\Psi_n(2)}$ we write $\psi(x - re_1) = \phi_{x_2,\dots,x_n}(x_1 - r)$ as a one-variable polynomial of degree at most 2 with coefficients depending on $x_2, \dots, x_n$. Then we have

$$\big\|H_{e_1}^{\Psi_n(2)} f\big\|_{L^p}^p \le \int\!\!\cdots\!\!\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \big(H^{\Psi_1(2)} f(\,\cdot\,, x_2, \dots, x_n)\big)(x_1)^p dx_1 dx_2 \dots dx_n.$$

Assuming the result in [L], and applying it in the first variable, the latter is controlled by a constant multiple of $\|f\|_{L^p}^p$.

We now employ the method of rotations to write

$$T_{\Omega}^{\Psi_n(2)} f(x) \leq \frac{1}{2} \int_{S^{n-1}} |\Omega(\theta)| \, H_{\theta}^{\Psi_n(2)} f(x) \, d\theta \, .$$

We can therefore obtain the boundedness of $T_{\Omega}^{\Psi_n(2)}$ as a consequence of that for $H^{\Psi_1(2)}$. We conclude the following result:

Theorem 2.5. *Let Ω be an odd and integrable function on S^{n-1}. Then for every $1 < p, q < \infty$ and every weight $w \in A_p$ there is a constant $C = C(p,q,w)$ such that*

$$\|T_{\Omega}^{\Psi_n(2)} f\|_{L^p(w)} \leq C \, \|f\|_{L^p(w)}$$

holds for all f in $L^p(w)$ and also

$$\Big\| \Big(\sum_j |T_{\Omega}^{\Psi_n(2)} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \leq C \, \Big\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}$$

holds for all sequences $\{f_j\}_j$ in $L^p_{\ell^q}(w)$. Moreover, these estimates are also satisfied by $(T_{\Omega}^{\Psi_n(2)})_\star$

Finally we consider the operator defined by

$$\widetilde{T}_{\Omega}^{\Psi_n(k)} f(x) = \sup_{\psi \in \Psi_n(k)} \left| \text{p.v.} \int_{\mathbb{R}^n} f(y) e^{2\pi i \psi(x-y)} K(x-y) \, dy \right|$$

where $\frac{\Omega(y/|y|)}{|y|^n} = K(y)$. In view of the trivial estimate

$$\widetilde{T}_{\Omega}^{\Psi_n(k)} f(x) \leq T_{\Omega}^{\Psi_n(k)} f(x)$$

we conclude that all weighted and vector-valued estimates that hold for $T_{\Omega}^{\Psi_n(2)}$ are also valid for $\widetilde{T}_{\Omega}^{\Psi_n(2)}$. At this point it is unclear to us whether boundedness for $\widetilde{T}_{\Omega}^{\Psi_n(k)}$ holds when $k \geq 3$ even in the unweighted case. However, it is worth mentioning a recent theorem of Stein and Wainger [SWa] stating that if the family $\Psi_n(k)$ is replaced by the subfamily $\Psi_n(k)'$ consisting of all polynomials in $\Psi_n(k)$ with no linear term, then the corresponding operator $\widetilde{T}_{\Omega}^{\Psi_n(k)'}$ is $L^p(\mathbb{R}^n)$ bounded for all $1 < p < \infty$. This result should be compared with our observation that the operators $\widetilde{T}_{\Omega}^{\Psi_n(2)}$ are L^p bounded (even on weighted spaces) with no restriction on the linear terms. It is unclear at this point how to combine these two results to obtain the boundedness of $\widetilde{T}_{\Omega}^{\Psi_n(k)}$ for all $k \geq 1$.

Finally we note that Theorem 2.5 can be also extended to the case where Ω is even and of class $L\log L(S^{n-1})$. To achieve this we need to know that for such Ω, $T_\Omega^{\Psi_n(2)}$ is $L^p(\mathbb{R}^n)$ bounded for all $1<p<\infty$. This result requires explicit estimates from the proof of Carleson's theorem [C] and can be obtained by a modification of the proof given for linear phases in Sjölin [Sj].

Naturally Theorems 2.4 and 2.5 can be extended to polynomials of degree k provided an a priori L^p estimate is known to hold for $H^{\Psi_1(k)}$. This would allow one to replace $\Psi_n(k)'$ by $\Psi_n(k)$ in the theorem of Stein and Wainger [SWa]. But this seems to be a difficult task and is rather elusive at present.

3. Proof of the good-λ inequality (1)

We fix f in $\bigcup_{1\le p<\infty} L^p(\mathbb{R}^n)$ and consider the open set

$$\Omega=\{T_\star^\Phi f(x)>\lambda\}=\bigcup_j Q_j\,,$$

where Q_j are the Whitney cubes. We define $Q_j^*=10\sqrt{n}\,Q_j$ and $Q_j^{**}=100\sqrt{n}\,Q_j^*$, where $a\,Q$ denotes the cube with the same center as Q whose sidelength is $a\,l(Q_j)$; here $l(Q_j)$ is the sidelength of Q_j. These Whitney cubes satisfy that the distance from Q_j to Ω^c is at least $2\sqrt{n}\,l(Q_j)$ and at most $4\sqrt{n}\,l(Q_j)$ and therefore Q_j^* must meet Ω^c. We so fix a point $y_j\in\Omega^c\cap Q_j^*$. For each j we write

$$f=f_0^j+f_\infty^j=f\,\chi_{Q_j^{**}}+f\,\chi_{(Q_j^{**})^c}\,.$$

We now claim that it suffices to show that :

$$\big|\big\{x\in Q_j: T_\star^\Phi f(x)>3\lambda,\ M_r f(x)\le\gamma\,\lambda\big\}\big|\le C_n\,\gamma^r\,\big|Q_j\big|. \tag{12}$$

Once the validity of (12) is established, we use standard properties of A_∞ weights and there exist $\varepsilon_0, C_2>0$ (that depend on $[w]_{A_\infty}$ and the dimension n) such that

$$w\big\{x\in Q_j: T_\star^\Phi f(x)>3\lambda,\ M_r f(x)\le\gamma\,\lambda\big\}\le C_2\,C_n^{\varepsilon_0}\,\gamma^{r\cdot\varepsilon_0}\,w(Q_j).$$

Then a simple summation on j gives the desired estimate.

Then we prove (12). We may assume that for each cube Q_j there exists a $z_j\in Q_j$ such that $M_r f(z_j)\le\gamma\,\lambda$, otherwise there is nothing to prove. Then,

$$\big|\big\{x\in Q_j: T_\star^\Phi f(x)>3\lambda,\ M_r f(x)\le\gamma\,\lambda\big\}\big|\le I_0^\lambda+I_\infty^\lambda$$

where

$$I_0^\lambda = \big|\big\{x \in Q_j : T_\star^\Phi f_0^j(x) > \lambda,\, M_r f(x) \le \gamma\,\lambda\big\}\big|,$$
$$I_\infty^\lambda = \big|\big\{x \in Q_j : T_\star^\Phi f_\infty^j(x) > 2\lambda,\, M_r f(x) \le \gamma\,\lambda\big\}\big|.$$

To control the first term we need to observe that $T_\star^\Phi$ also maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. To see that we use that T is a Calderón-Zygmund operator and thus it satisfies the Cotlar estimate $T_\star g(x) \le C\,Mg(x) + C\,M(Tg)(x)$ and then

$$\begin{aligned} T_\star(\mathcal{M}^{\phi_\alpha} f)(x) &\le C\,M(\mathcal{M}^{\phi_\alpha} f)(x) + C\,M(T(\mathcal{M}^{\phi_\alpha} f))(x) \\ &\le C\,Mf(x) + C\,M(T^\Phi f)(x) \end{aligned}$$

which yields $T_\star^\Phi f(x) \le C\,Mf(x) + C\,M(T^\Phi f)(x)$. Using the fact that M maps $L^{r,\infty}(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ and that T^Φ is of weak type (r,r) it follows that $T_\star^\Phi$ is bounded from $L^r(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$.

Now we estimate I_0^λ as follows:

$$\begin{aligned} I_0^\lambda &\le \big|\big\{x \in \mathbb{R}^n : T_\star^\Phi f_0^j(x) > \lambda\big\}\big| \le \frac{C}{\lambda^r} \int_{\mathbb{R}^n} |f_0^j(x)|^r\,dx \\ &\le C\,\frac{|Q_j^{**}|}{\lambda^r}\,\frac{1}{|Q_j^{**}|} \int_{Q_j^{**}} |f(x)|^r\,dx \le C\,\frac{M_r f(z_j)^r}{\lambda^r}\,|Q_j| \le C\,\gamma^r\,|Q_j|. \end{aligned}$$

Next we are going to show that $I_\infty^\lambda = 0$ if we take γ sufficiently small and consequently

$$\big|\big\{x \in Q_j : T_\star^\Phi f(x) > 3\lambda,\, M_r f(x) \le \gamma\,\lambda\big\}\big| \le I_0^\lambda + I_\infty^\lambda \le C\,\gamma^r\,|Q_j|,$$

which yields (12).

Take $\varepsilon > 0$ and $\phi_\alpha \in \Phi$. Then, for every $x \in Q_j$,

$$\big|T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(x) - T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)\big| \le L_1 + L_2 + L_3,$$

where

$$L_1 = \int_{|y_j - y| > \varepsilon} |K(x,y) - K(y_j,y)|\,|f_\infty^j(y)|\,dy,$$
$$L_2 = \int_{|y_j - y| \le \varepsilon < |x-y|} |K(x,y)|\,|f_\infty^j(y)|\,dy,$$
$$L_3 = \int_{|x - y| \le \varepsilon < |y_j-y|} |K(x,y)|\,|f_\infty^j(y)|\,dy.$$

Since $y \notin Q_j^{**}$, $x, z_j \in Q_j$ and $y_j \in Q_j^*$ we have

$$\frac{3}{4} \le \frac{|y-x|}{|y-y_j|} \le \frac{5}{4}, \qquad\qquad \frac{40}{41} \le \frac{|y-z_j|}{|y-x|} \le \frac{40}{39}.$$

We estimate L_1. Note that $|y-z_j| > 49\,n\,\ell(Q_j)$. Besides, the smoothness of K leads to

$$\begin{aligned} L_1 &\le \int_{|y_j-y|>\varepsilon} c_0\,\frac{|x-y_j|^\tau}{|x-y|^{n+\tau}}\,|f_\infty^j(y)|\,dy \\ &\le C\,\ell(Q_j)^\tau \int_{|y-z_j|>49\,n\,\ell(Q_j)} \frac{1}{|y-z_j|^{n+\tau}}\,|f(y)|\,dy \le C\,Mf(z_j), \end{aligned}$$

where for the last inequality one has to break up the integral into dyadic annuli and sum up the geometric series. For L_2, note that $|y-z_j| < 2\,\varepsilon$ and then

$$\begin{aligned} L_2 &\le \int_{|y_j-y|\le\varepsilon<|x-y|} \frac{c_0}{|x-y|^n}\,|f_\infty^j(y)|\,dy \le \frac{C}{\varepsilon^n}\int_{|y-z_j|\le 2\,\varepsilon} |f(y)|\,dy \\ &\le C\,Mf(z_j). \end{aligned}$$

Finally, for L_3 we use that $|x-y| > 3\,\varepsilon/4$ and $|y-z_j| \le 40\,\varepsilon/39$. Thus,

$$\begin{aligned} L_3 &\le \int_{|x-y|\le\varepsilon<|y_j-y|} \frac{c_0}{|x-y|^n}\,|f_\infty^j(y)|\,dy \le \frac{C}{\varepsilon^n}\int_{|y-z_j|\le \frac{40}{39}\,\varepsilon} |f(y)|\,dy \\ &\le C\,Mf(z_j). \end{aligned}$$

Collecting the three estimates we therefore obtain

$$\big|T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(x) - T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)\big| \le C\,Mf(z_j)$$

and consequently

$$T_\star^\Phi f_\infty^j(x) \le T_\star^\Phi f_\infty^j(y_j) + C\,Mf(z_j). \tag{13}$$

Now we want to replace the first term in the right hand side by $T_\star^\Phi f(y_j)$. We can do that but we get some extra terms $Mf(z_j)$. We claim that

$$|T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)| \le T_\star^\Phi f(y_j) + C\,Mf(z_j). \tag{14}$$

To show this estimate we first let R_2^j be the smallest number and R_1^j be the largest number so that $B(y_j,R_1^j) \subset Q_j^{**} \subset B(y_j,R_2^j)$. If $\varepsilon \ge R_2^j$ then $Q_j^{**} \subset B(y_j,\varepsilon)$ and hence $T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j) = T_\varepsilon(\mathcal{M}^{\phi_\alpha} f)(y_j)$. In this case (14) is trivial. On the other hand, if $\varepsilon \le R_1^j$, then $T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j) = T_{R_1^j}(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)$. So, it remains to consider the case $R_1^j \le \varepsilon < R_2^j$. We use that $Q_j^{**} \subset B(y_j, R_2^j)$ and hence $T_{R_2^j}(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j) = T_{R_2^j}(\mathcal{M}^{\phi_\alpha} f)(y_j)$. One can easily see that $|y - y_j| \geq \frac{2}{5\sqrt{n}} R_2^j$ and thus

$$\begin{aligned}
|T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)| &= |T_\varepsilon(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j) \pm T_{R_2^j}(\mathcal{M}^{\phi_\alpha} f_\infty^j)(y_j)| \\
&\leq \int_{\varepsilon \leq |y_j - y| \leq R_2^j} \frac{c_0}{|x-y|^n} |f_\infty^j(y)| \, dy + T_\star^\Phi f(y_j) \\
&\leq \int_{\frac{2}{5\sqrt{n}} R_2^j \leq |y_j - y| \leq R_2^j} \frac{c_0}{|x-y|^n} |f_\infty^j(y)| \, dy + T_\star^\Phi f(y_j) \\
&\leq \frac{C}{(R_2^j)^n} \int_{|y - z_j| \leq 2 R_2^j} |f(y)| \, dy + T_\star^\Phi f(y_j) \\
&\leq C \, Mf(z_j) + T_\star^\Phi f(y_j),
\end{aligned}$$

where in the penultimate estimate above we have taken into account that

$$|y - z_j| \leq \frac{40}{39} \frac{5}{4} |y - y_j| \leq 2 R_2^j, \qquad |x - y| \geq \frac{3}{4} |y - y_j| \geq \frac{3}{10\sqrt{n}} R_2^j.$$

Then we have proved (14) which together with (13) yield $T_\star^\Phi f_\infty^j(x) \leq T_\star^\Phi f(y_j) + C_0 \, Mf(z_j)$. Recalling that $y_j \notin \Omega$ and that $M_r f(z_j) \leq \lambda \, \gamma$ we observe that

$$T_\star^\Phi f_\infty^j(x) \leq \lambda + C_0 \, M_r f(z_j) \leq \lambda + C_0 \, \gamma \, \lambda,$$

for every $x \in Q_j$. If $0 < \gamma < \gamma_0 = C_0^{-1}$, then $T_\star^\Phi f_\infty^j(x) < 2\,\lambda$ and so

$$I_\infty^\lambda = \big|\big\{x \in Q_j : T_\star^\Phi f_\infty^j(x) > 2\lambda, \, M_r f(x) \leq \gamma\,\lambda\big\}\big| = |\emptyset| = 0,$$

as desired.

4. An alternative proof of Corollary 1.2.

As we mentioned earlier, Corollary 1.3 can be obtained as a consequence of the good-λ inequality (1) using standard techniques. There is however an alternative approach based on the sharp maximal function that leads to similar estimates. We discuss here this approach.

Proposition 4.1. *If T^Φ maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$, then for every $f \in L_c^\infty(\mathbb{R}^n)$, we have*

$$M^\#(T^\Phi f)(x) \leq C \, M_r f(x).$$

Proof. We fix x and some cube $Q \ni x$. We write x_Q for the center of Q. We split f as follows

$$f(x) = f_1(x) + f_2(x) = f(x)\ \chi_{Q^*}(x) + f(x)\ \chi_{(Q^*)^c}(x).$$

where $Q^* = 2\sqrt{n}\,Q$. Set $a_Q = T^{\Phi} f_2(x_Q)$. Then,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \big|T^{\Phi} f(y) - a_Q\big|\,dy \\
&\quad= \frac{1}{|Q|}\int_Q \big|\sup_{\alpha\in\mathcal{A}} |T(\mathcal{M}^{\phi_\alpha} f)(y)| - \sup_{\alpha\in\mathcal{A}} |T(\mathcal{M}^{\phi_\alpha} f_2)(x_Q)|\big|\,dy \\
&\quad\le \frac{1}{|Q|}\int_Q \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha} f)(y) - T(\mathcal{M}^{\phi_\alpha} f_2)(x_Q)\big|\,dy \\
&\quad\le \frac{1}{|Q|}\int_Q \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha} f_1)(y)\big|\,dy + \\
&\qquad\qquad + \frac{1}{|Q|}\int_Q \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha} f_2)(y) - T(\mathcal{M}^{\phi_\alpha} f_2)(x_Q)\big|\,dy \\
&\quad= I + II.
\end{aligned}$$

For the first term we use the hypothesis on T^{Φ}:

$$\begin{aligned}
I &\le \frac{1}{|Q|}\,\|T^{\Phi} f_1\|_{L^{r,\infty}(\mathbb{R}^n)}\,\|\,\chi_Q\,\|_{L^{r',1}(\mathbb{R}^n)} \le \frac{C}{|Q|}\,\|f_1\|_{L^r(\mathbb{R}^n)}\,|Q|^{\frac{1}{r'}} \\
&= C\Big(\frac{1}{|Q^*|}\int_{Q^*} |f(y)|^r\,dy\Big)^{\frac{1}{r}} \le C\,M_r f(x).
\end{aligned}$$

The estimate for II follows from the smoothness condition assumed on the kernel of T. Let $y \in Q$ and $\alpha \in \mathcal{A}$. Then,

$$\begin{aligned}
&\big|T(\mathcal{M}^{\phi_\alpha} f_2)(y) - T(\mathcal{M}^{\phi_\alpha} f_2)(x_Q)\big| \le \int_{\mathbb{R}^n} |K(y,z) - K(x_Q,z)|\,|f_2(z)|\,dz \\
&\le \int_{(Q^*)^c} c_0\,\frac{|y-x_Q|^{\tau}}{|z-x_Q|^{n+\tau}}\,|f(z)|dz \\
&\le C\,\ell(Q)\int_{|z-x_Q|>\sqrt{n}\,\ell(Q)} \frac{1}{|z-x_Q|^{n+\tau}}\,|f(z)|\,dz \le C\,Mf(x) \le C\,M_r f(x),
\end{aligned}$$

where the penultimate estimate follows by breaking up the integral in dyadic annuli. Therefore, $II \le C\,M_r f(x)$ and

$$\frac{1}{|Q|}\int_Q \big|T^{\Phi} f(y) - a_Q\big|\,dy \le C\,M_r f(x),$$

which leads to the desired estimate. □

Remark 4.2. We point out that the weak type (r, r) assumption on T^Φ has been only used in term I to control $T^\Phi f_1 = T^\Phi(f\,\chi_{Q^*})$ in the cube Q. Indeed, the same estimate remains true if the assumption that T^Φ maps $L^r(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ is replaced by a corresponding local version

$$\|T_Q^\Phi f\|_{L^{r,\infty}(Q)} \le C_0 \|f\|_{L^r(Q)}$$

for all cubes $Q \subset \mathbb{R}^n$ with C_0 independent of Q, where T_Q^Φ denotes the localized operator given by $T_Q^\Phi f(x) = T^\Phi(f\,\chi_Q)(x)\,\chi_Q(x)$. In this case we have that

$$\begin{aligned} I &= \frac{1}{|Q|}\int_Q T^\Phi f_1(y) \le \frac{1}{|Q|}\,\|T_{Q^*}^\Phi f\|_{L^{r,\infty}(Q^*)}\,\|\,\chi_Q\,\|_{L^{r',1}(\mathbb{R}^n)} \\ &\le \frac{C_0}{|Q|}\,\|f\|_{L^r(Q^*)}\,|Q|^{\frac{1}{r'}} \le C\,M_r f(x). \end{aligned}$$

We now deduce Corollary 1.2 from Proposition 4.1. We show that for all $0 < p < \infty$ and $w \in A_\infty$ estimates (2) and (4) hold while we leave the proofs of (3) and (5) to the reader. For these estimates to make sense we should assume that the left-hand sides are finite. As we observed in the proof of the good-λ inequality we have

$$T_\star^\Phi f(x) \le C\,Mf(x) + C\,M(T^\Phi f)(x).$$

Also, $T^\Phi f(x) \le M(T^\Phi f)(x)$ for a.e x, by the Lebesgue differentiation theorem. Therefore, it suffices to show that

$$\int_{\mathbb{R}^n} M(T^\Phi f)(x)^p\,w(x)\,dx \le C\int_{\mathbb{R}^n} M_r f(x)^p\,w(x)\,dx, \tag{15}$$

for all $0 < p < \infty$, $w \in A_\infty$ and $f \in L_c^\infty(\mathbb{R}^n)$. Let us fix $f \in L_c^\infty$ with $\operatorname{supp} f \subset B$ for some ball B, $0 < p < \infty$ and $w \in A_\infty$. Without loss of generality we may assume that w is bounded since otherwise we may replace it by $w_k = \min\{w, k\} \in A_\infty$ whose characteristic constant is uniformly bounded by the A_∞ characteristic constant of w. In this way, w_k is bounded and if we get the desired estimate for w_k with no dependence on k we can let $k \to \infty$ to obtain (15). We will see that this assumption is only used to assure that a certain quantity is finite. We can also suppose that the right hand side of (15) is finite because otherwise there is nothing to prove. Then, by Proposition 4.1 we have

$$\begin{aligned} \int_{\mathbb{R}^n} M(T^\Phi f)(x)^p\,w(x)\,dx &\le C\int_{\mathbb{R}^n} M^\#(T^\Phi f)(x)^p\,w(x)\,dx \\ &\le C\int_{\mathbb{R}^n} M_r f(x)^p\,w(x)\,dx, \end{aligned}$$

whenever the left hand side is finite. Note that the first estimate arises from the well-known Fefferman-Stein's inequality

$$\int_{\mathbb{R}^n} Mg(x)^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M^{\#}g(x)^p\, w(x)\, dx$$

for all $0 < p < \infty$, $w \in A_\infty$, and whenever the left hand side is finite (see [FS]). Therefore, we only have to show that the left hand side of (15) is finite. Recall that we have also assumed that the right hand side is finite which implies

$$\infty > \int_{\mathbb{R}^n} M_r f(x)^p\, w(x)\, dx \ge C(f) \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{\frac{n\,p}{r}}}\, dx. \tag{16}$$

Note that if $y \in (2\,B)^c$ then $T^{\Phi} f(y) \le C(f)/(1+|y|)^n = C(f)\, h(y)$ and then

$$\begin{aligned} I &= \int_{\mathbb{R}^n} M(T^{\Phi} f\, \chi_{(2\,B)^c})(x)^p\, w(x)\, dx \le C(f) \int_{\mathbb{R}^n} Mh(x)^p\, w(x)\, dx \\ &\le C(f) \int_{\mathbb{R}^n} \Big(\frac{1+\log^+|x|}{(1+|x|)^n}\Big)^p w(x)\, dx \le C(f) \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{\frac{n\,p}{r}}}\, dx < \infty, \end{aligned}$$

by (16). On the other hand,

$$\begin{aligned} II &= \int_{(4\,B)^c} M(T^{\Phi} f\, \chi_{2\,B})(x)^p\, w(x)\, dx \le C \int_{(4\,B)^c} \frac{\|T^{\Phi} f\|^p_{L^1(2\,B)}}{(1+|x|)^{n\,p}} w(x)\, dx \\ &\le C\, \|\, \chi_{2\,B}\,\|^p_{L^{r',1}(\mathbb{R}^n)}\, \|T^{\Phi} f\|^p_{L^{r,\infty}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{\frac{n\,p}{r}}}\, dx \\ &\le C\, |B|^{\frac{p}{r'}}\, \|f\|^p_{L^r(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{\frac{n\,p}{r}}}\, dx < \infty. \end{aligned}$$

Finally, since w is bounded, for $q > \max\{r/p, 1\}$,

$$\begin{aligned} III &= \int_{4\,B} M(T^{\Phi} f\, \chi_{2\,B})(x)^p\, w(x)\, dx \\ &\le \|w\|_{L^\infty(\mathbb{R}^n)}\, |4\,B|^{\frac{1}{q'}}\, \|M(T^{\Phi} f\, \chi_{2\,B})\|^p_{L^{p\cdot q}(\mathbb{R}^n)}. \end{aligned}$$

Since $p \cdot q > r > 1$, then M is bounded on $L^{p\cdot q}(\mathbb{R}^n)$. Next, Proposition 4.1 implies that

$$\|T^{\Phi} g\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\, \|g\|_{L^\infty(\mathbb{R}^n)}, \qquad g \in L^\infty_c(\mathbb{R}^n),$$

which by interpolation with $T^\Phi : L^r(\mathbb{R}^n) \longrightarrow L^{r,\infty}(\mathbb{R}^n)$ gives that T^Φ is bounded on $L^s(\mathbb{R}^n)$ for all $r < s < \infty$. In particular, since $p \cdot q > r$, T^Φ is bounded on $L^{p\cdot q}(\mathbb{R}^n)$, which eventually yields

$$III \leq C\, \|w\|_{L^\infty(\mathbb{R}^n)}\, |B|^{\frac{1}{q'}}\, \|f\|^p_{L^{p\cdot q}(\mathbb{R}^n)} < \infty.$$

Collecting these three estimates we conclude as desired that

$$\int_{\mathbb{R}^n} M(T^\Phi f)(x)^p\, w(x)\, dx \leq I + II + III < \infty.$$

5. Proof of Theorem 1.4

As mentioned in the introduction, Theorem 1.4 will be proved by using some Yano's extrapolation type result inspired by [SS]. Unfortunately the results in [SS] cannot be applied to T^Φ or its localized versions. We need to suitably adjust the ideas from the articles [SS], [So1] and [So2] to derive the desired inequality. The following theorem is quite crucial in our work and its proof is postponed until later in this section (Subsection 5.2).

Theorem 5.1. *Under the hypotheses of Theorem 1.4, for every cube Q and every function in $L_\Upsilon(Q)$ we have that*

$$\|T^\Phi_Q f\|_{L^{1,\infty}(Q,\mu_Q)} \leq C\, \|f\|_{\Upsilon_m,Q},$$

where C does not depend on Q.

To prove Theorem 1.4, we will first focus on the approach with the sharp maximal function. Afterwards, we will sketch the proof of the good-λ approach.

We are now going to obtain the pointwise estimate for $M^\#_\delta(T^\Phi f)$ when $0 < \delta < 1$, where $M^\#_\delta g(x) = M^\#(|g|^\delta)(x)^{1/\delta}$.

Proposition 5.2. *Let $f \in L^\infty_c(\mathbb{R}^n)$. Then*

$$M^\#_\delta(T^\Phi f)(x) \leq C_\delta\, M_{\Upsilon_m} f(x), \qquad 0 < \delta < 1.$$

The proof of this result is given below in Subsection 5.1.

We now deduce Theorem 1.4. We want to obtain (9) and the analog for $T^\Phi_\star$. Since T is a Calderón-Zygmund operator we have the following Cotlar estimate

$$T_\star g(x) \leq C(\delta)\, \big(Mg(x) + M_\delta(Tg)(x)\big), \qquad 0 < \delta \leq 1.$$

We apply this estimate to $g = \mathcal{M}^{\phi_\alpha} f$ and take the supremum over $\alpha \in \mathcal{A}$. This gives

$$T_\star^\Phi f(x) \le C(\delta)\big(M_{\Upsilon_m} f(x) + M_\delta(T^\Phi f)(x)\big) \tag{17}$$

since $t \le \Upsilon(t)$ for all $t \ge 0$ and so $Mf(x) \le M_{\Upsilon_m} f(x)$. In view of the Lebesgue differentiation theorem we have $T^\Phi f(x) \le M_\delta(T^\Phi f)(x)$ a.e.. It therefore suffices to show that $M_\delta(T^\Phi f)$ is controlled by $M_{\Upsilon_m} f$ (in norm). Then, if $0 < \delta < 1$, $0 < p < \infty$ and $w \in A_\infty$, we use Proposition 5.2 to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} M_\delta(T^\Phi f)(x)^p\, w(x)\, dx &= \int_{\mathbb{R}^n} M\big(|T^\Phi f|^\delta\big)(x)^{\frac{p}{\delta}}\, w(x)\, dx \\ &\le C \int_{\mathbb{R}^n} M^\#\big(|T^\Phi f|^\delta\big)(x)^{\frac{p}{\delta}}\, w(x)\, dx \\ &= C \int_{\mathbb{R}^n} M_\delta^\#(T^\Phi f)(x)^p\, w(x)\, dx \\ &\le C \int_{\mathbb{R}^n} M_{\Upsilon_m} f(x)^p\, w(x)\, dx.\end{aligned}$$

From this we eventually deduce the required estimates

$$\|T^\Phi f\|_{L^p(w)} \le C\, \|M_{\Upsilon_m} f\|_{L^p(w)}, \qquad \|T_\star^\Phi f\|_{L^p(w)} \le C\, \|M_{\Upsilon_m} f\|_{L^p(w)}.$$

Then by the extrapolation results in [CMP], [CGMP] we get the analogs in $L^{p,\infty}(w)$ as well as the vector-valued estimates. These extrapolation results also yield estimates in Lorentz spaces, Orlicz spaces, and in other rearrangement invariant function spaces. This completes the proof with the approach based on the sharp maximal function.

The good-λ approach relies on a slight modification of the argument given in Section 3. We follow the same steps changing $M_r f$ by $M_{\Upsilon_m} f$ at any place it occurs adopting the same notation. Note that at the very end of the estimate of I_∞^λ we used before that $Mf(x) \le M_r f(x)$. In this case $Mf(x) \le M_{\Upsilon_m} f(x)$ since $t \le \Upsilon(t)$, and so $I_\infty^\lambda = 0$. Thus, we only need to change the estimate of I_0^λ since this is the only place where the boundedness of the operator T^Φ is used.

We fix $0 < \delta < 1$ and use (17) to observe

$$\begin{aligned} I_0^\lambda &\le \big|\{x \in \mathbb{R}^n : T_\star^\Phi f_0^j(x) > \lambda\}\big| \\ &\le \big|\{x \in \mathbb{R}^n : C(\delta)\, \big(Mf_0^j(x) + M_\delta(T^\Phi f_0^j)(x)\big) > \lambda\}\big| \\ &\le \big|\{x : C(\delta)\, Mf_0^j(x) > \lambda/2\}\big| + \big|\{x : C(\delta) M_\delta(T^\Phi f_0^j)(x) > \lambda/2\}\big| \\ &= |E_\lambda| + |F_\lambda|.\end{aligned}$$

We estimate the first term:

$$|E_\lambda| \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f_0^j(x)|\, dx \le \frac{C}{\lambda} |Q_j|\, Mf(z_j) \le \frac{C}{\lambda} |Q_j|\, M_{\Upsilon_m} f(z_j) \le C\, \gamma\, |Q_j|,$$

where we have used that $z_j \in Q_j$ with $M_{\Upsilon_m} f(z_j) \le \gamma\,\lambda$. For $|F_\lambda|$, a little bit more of work is required

$$\begin{aligned} |F_\lambda| &= \big|\big\{x\in\mathbb{R}^n : M\big(|T^{\Phi} f_0^j|^\delta\big)(x) > C\,\lambda^\delta\big\}\big| \le \frac{C}{\lambda^\delta}\int_{F_\lambda} |T^{\Phi} f_0^j(y)|^\delta\,dy \\ &= \frac{C}{\lambda^\delta}\Big(\int_{F_\lambda\cap 2\,Q_j^{**}} |T^{\Phi} f_0^j(y)|^\delta\,dy + \int_{F_\lambda\setminus 2\,Q_j^{**}} |T^{\Phi} f_0^j(y)|^\delta\,dy\Big) \\ &= \frac{C}{\lambda^\delta}\,(B_1+B_2). \end{aligned}$$

To treat term B_1 we pass to the local operator $T^{\Phi}_{2\,Q_j^{**}}$:

$$\begin{aligned} B_1 &= C\int_{F_\lambda} |T^{\Phi}_{2\,Q_j^{**}} f_0^j(y)|^\delta\,dy \le C_\delta\,|F_\lambda|^{1-\delta}\,\big\|T^{\Phi}_{2\,Q_j^{**}} f_0^j\big\|^\delta_{L^{1,\infty}(\mathbb{R}^n)} \\ &= C\,|F_\lambda|^{1-\delta}\,|2\,Q_j^{**}|^\delta\,\big\|T^{\Phi}_{2\,Q_j^{**}} f_0^j\big\|^\delta_{L^{1,\infty}(2\,Q_j^{**},\mu_{2\,Q_j^{**}})} \\ &\le C\,|Q_j|^\delta\,|F_\lambda|^{1-\delta}\,\|f\|^\delta_{\Upsilon_m,2\,Q_j^{**}} \\ &\le C\,|Q_j|^\delta\,|F_\lambda|^{1-\delta}\,M_{\Upsilon_m} f(z_j)^\delta \\ &\le C\,|Q_j|^\delta\,|F_\lambda|^{1-\delta}\,(\gamma\,\lambda)^\delta, \end{aligned}$$

where we have used Kolmogorov's inequality and Theorem 5.1. As before, it is crucial to note that the previous constant C does not depend on the cubes. We deal with term B_2 as follows: for $y \in F_\lambda \setminus 2\,Q_j^{**}$ we have

$$\begin{aligned} T^{\Phi} f_0^j(y) &\le \int_{Q_j^{**}} |K_j(y-z)|\,|f(z)|\,dz \le \int_{Q_j^{**}} \frac{c_0}{|y-z|^n}\,|f(z)|\,dz \\ &\le \frac{2^n\,c_0}{\ell(Q_j^{**})^n}\int_{Q_j^{**}} |f(z)|\,dz \le C\,Mf(z_j) \le C\,M_{\Upsilon_m} f(z_j) \le C\,\gamma\,\lambda, \end{aligned}$$

since $z_j \in Q_j \subset Q_j^{**}$. Thus $B_2 \le C\,|F_\lambda|\,(\gamma\,\lambda)^\delta$. Collecting the estimates for B_1 and B_2 we eventually obtain

$$|F_\lambda| \le \frac{C}{\lambda^\delta}\,(B_1+B_2) \le C_1\,|Q_j|^\delta\,|F_\lambda|^{1-\delta}\,\gamma^\delta + C_2\,|F_\lambda|\,\gamma^\delta.$$

Take γ such that $C_2\,\gamma^\delta \le 1/2$. Since $|F_\lambda| < \infty$ (this follows by using that M and T^{Φ} are bounded in $L^p(\mathbb{R}^n)$, here we may have to consider functions that are in some $L^p(\mathbb{R}^n)$ for $p > 1$), we get

$$|F_\lambda| \le 2\,C_1\,|Q_j|^\delta\,|F_\lambda|^{1-\delta}\,\gamma^\delta$$

which yields $|F_\lambda| \le C\,\gamma\,|Q_j|$ and therefore $I_0^\lambda \le C\,\gamma\,|Q_j|$. This completes the proof of Theorem 1.4 using the good-λ approach.

5.1. Proof of Proposition 5.2

We proceed as in the proof of Proposition 4.1. We fix x and a cube $Q \ni x$. We split $f = f_1 + f_2$ as in that proof and define a_Q in the same way. Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \big||T^{\Phi}f(y)|^{\delta} - |a_Q|^{\delta}\big|\,dy \le \frac{1}{|Q|}\int_Q \big|T^{\Phi}f(y) - a_Q\big|^{\delta}\,dy \\
&\le \frac{1}{|Q|}\int_Q T^{\Phi}f_1(y)^{\delta}dy + \frac{1}{|Q|}\int_Q \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha}f_2)(y) - T(\mathcal{M}^{\phi_\alpha}f_2)(x_Q)\big|^{\delta}dy \\
&\le \frac{1}{|Q|}\int_Q T^{\Phi}f_1(y)^{\delta}dy + \\
&\qquad\qquad + \Big(\frac{1}{|Q|}\int_Q \sup_{\alpha\in\mathcal{A}} \big|T(\mathcal{M}^{\phi_\alpha}f_2)(y) - T(\mathcal{M}^{\phi_\alpha}f_2)(x_Q)\big|dy\Big)^{\delta} \\
&= I_\delta + (II)^{\delta},
\end{aligned}$$

For II, as in Proposition 4.1, we observe that $II \le C\,Mf(x)$. Notice that $Mf(x) \le M_{\Upsilon_m}f(x)$ and then we get the desired estimate. Now we have to analyze I_δ and this is the only part where the boundedness of the operator is used. Note that $f_1 = f\,\chi_{Q^*}$ and therefore

$$\begin{aligned}
I_\delta &\le (2\sqrt{n}\,)^n \int_{Q^*} T^{\Phi}_{Q^*}f(y)^{\delta}\,d\mu_{Q^*}(y) \le \frac{C}{1-\delta}\,\|T^{\Phi}_{Q^*}f\|^{\delta}_{L^{1,\infty}(Q^*,\mu_{Q^*})} \\
&\le C_\delta\,\|f\|^{\delta}_{\Upsilon_m,Q^*} \le C_\delta\,M_{\Upsilon_m}f(x)^{\delta},
\end{aligned}$$

where we have used that $0 < \delta < 1$, Kolmogorov's inequality, Theorem 5.1, and also that $x \in Q \subset Q^*$. (Note that the constant is independent of the cube Q.) Thus we have seen that

$$\frac{1}{|Q|}\int_Q \big|T^{\Phi}f(y)^{\delta} - |a_Q|^{\delta}\big|\,dy \le I_\delta + (II)^{\delta} \le C\,M_{\Upsilon_m}f(x)^{\delta},$$

which yields the desired estimate. □

5.2. Proof of Theorem 5.1

Behind the main estimate claimed in Theorem 5.1 there is a general result in the spirit of [SS]. We state it precisely:

Theorem 5.3. *Let $S^\star f(x) = \sup_j |S_j f(x)|$ be a maximal operator such that each S_j is a singular integral operator given by a kernel $s_j(x,y)$ which is defined away from the diagonal $x = y$. For any cube Q, we set $S^\star_Q f = S^\star(f\,\chi_Q)\,\chi_Q$ and in an analogous way we define $S_{j,Q}$ whose kernel is $s^Q_j(x,y) = s_j(x,y)\,\chi_{Q\times Q}(x,y)$. For any cube Q, we assume* *(a)* $s_j^Q(x,\cdot) \in L^1(Q,\mu_Q)$ *for a.e.* $x \in Q$.

(b) $s_j^Q(\cdot,y) \in L^1(Q,\mu_Q)$ *uniformly in* $y \in Q$.

(c) Given $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, j, Q)$ *such that*

$$\int_Q |s_j^Q(x,y_1) - s_j^Q(x,y_2)|\, d\mu_Q(x) \le \varepsilon, \qquad |y_1 - y_2| < \delta,\ y_1, y_2 \in Q.$$

(d) For all $A \subset Q$ *and* $\lambda > 0$, *and for some* $m \ge 1$,

$$\mu_Q\{S_Q^\star(\chi_A)(x) > \lambda\} \le \frac{C_0}{\lambda}\, \varphi_m\big(\mu_Q(A)\big) = \frac{C_0}{\lambda}\, \mu_Q(A)\bigg(1 + \log^+ \frac{1}{\mu_Q(A)}\bigg)^m$$

where the constant C_0 *does not depend on the cube* Q.

Then, there exists a constant C, *independent of* Q, *such that for all functions* f *in* $L_{\Upsilon_m}(Q)$ *we have*

$$\|S_Q^\star f\|_{L^{1,\infty}(Q,\mu_Q)} \le C\, \|f\|_{\Upsilon_m,Q} = C\, \|f\|_{L(\log L)^m \log\log\log L, Q}.$$

We will give a proof of this result in the next section. In the rest of this section, we apply Theorem 5.3 to derive the desired estimate in Theorem 5.1.

We would like to use this extrapolation result for the maximal operator $T_Q^\Phi f(x)$. Notice that the kernels are $e^{2\pi i \phi_\alpha(y)}\, K(x-y)\, \chi_{Q\times Q}(x,y)$ which, because of their singularity, may not belong to $L^1(Q)$. We avoid this difficulty by defining

$$\widetilde{T}^\Phi f(x) = \sup_{\alpha\in\mathcal{A}} \big|T\big((e^{i\phi_\alpha(\cdot)} - e^{i\phi_\alpha(x)})\, f(\cdot)\big)(x)\big|,$$

and analogously its localized version $\widetilde{T}_Q^\Phi$. Note that we have incorporated the factor 2π into the function ϕ_α to make the computations cleaner. Using the notation of Theorem 5.3, $S^\star = \widetilde{T}^\Phi$ and, by hypothesis, we can assume that $\mathcal{A}$ is countable. Then, we have

$$|T^\Phi f(x) - \widetilde{T}^\Phi f(x)| \le \sup_{\alpha\in\mathcal{A}} \big|T\big(e^{i\phi_\alpha(x)}\, f(\cdot)\big)(x)\big| = |Tf(x)|, \tag{18}$$

and the same holds for the localized operators. Since T is a Calderón-Zygmund operator, it is of weak type $(1,1)$. Then, for any cube Q and for every $A \subset Q$ we have

$$\begin{aligned}\mu_Q\{x : |T_Q(\chi_A)(x)| > \lambda\} &\le \frac{1}{|Q|}\, \big|\{x \in \mathbb{R}^n : |T(\chi_A)(x)| > \lambda\}\big| \\ &\le \frac{C}{\lambda}\, \frac{|A|}{|Q|} \le \frac{C}{\lambda}\, \varphi_m\big(\mu_Q(A)\big),\end{aligned}$$

where C does not depend on Q. This estimate, (10) and (18) yield

$$\mu_Q\big\{x : \widetilde{T}_Q^{\Phi}(\chi_A)(x) > \lambda\big\} \le \frac{C}{\lambda}\,\varphi_m\big(\mu_Q(A)\big),$$

where C is independent of the cube Q. This means that property (d) in Theorem 5.3 holds.

Let us fix some arbitrary cube Q_0 and $\alpha \in \mathcal{A}$. The kernel of $\widetilde{T}_{Q_0}^{\Phi}$ is

$$K_\alpha(x,y) = K_\alpha^{Q_0}(x,y) = (e^{i\,\phi_\alpha(y)} - e^{i\,\phi_\alpha(x)})\,K(x-y)\ \chi_{Q_0\times Q_0}(x,y).$$

Next, we have to check that $K_\alpha(x,y)$ satisfies (a), (b), (c). The first two go as follows: if $x \in Q_0$, then

$$\begin{aligned}\int_{Q_0} |K_\alpha(x,y)|\,d\mu_{Q_0}(y) &\le c_0\,\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,|Q_0|^{-1}\int_{Q_0}|x-y|^{1-n}\,dy\\ &\le c_0\,\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,c_n\,|Q_0|^{-1+\frac{1}{n}} < \infty,\end{aligned}$$

which gives (a). In the same way, we get

$$\sup_{y\in Q_0}\int_{Q_0}|K_\alpha(x,y)|\,d\mu_{Q_0}(x) \le c_0\,\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,c_n\,|Q_0|^{-1+\frac{1}{n}} < \infty,$$

and (b) holds. The verification of (c) requires more work. We fix ε and we have to find $\delta > 0$ such that

$$\int_{Q_0}|K_\alpha(x,y_1) - K_\alpha(x,y_2)|\,d\mu_{Q_0}(x) \le \varepsilon, \qquad |y_1-y_2| < \delta,\ y_1,y_2\in Q_0.$$

Note that it is sufficient to prove the same estimate with dx in place of μ_{Q_0}, and that δ might depend on the cube Q_0. We fix $\varepsilon > 0$, set $B_0 = B(0,\sqrt{n}\,\ell(Q_0))$ and define the function $H(x) = |x|\,K(x)\ \chi_{B_0}(x) \in L^1(B_0)$ because $|K(x)| \le c_0\,|x|^{-n}$. Then, there exists $h \in C_0^\infty(B_0)$ such that

$$\|H-h\|_{L^1(B_0)} < \frac{\varepsilon}{8\,\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}}.$$

Setting $C_{\phi_\alpha} = \|\nabla\phi_\alpha\|^2_{L^\infty(Q_0)} + \|D^2\phi_\alpha\|_{L^\infty(Q_0)}$, we take

$$0 < \varepsilon_1 < \frac{\varepsilon}{16}\,\frac{1}{\|h\|_{L^\infty(B_0)}\,C_{\phi_\alpha}\,|Q_0|}.$$

Then, for y_1, $y_2 \in Q_0$,

$$\begin{aligned}&\int_{Q_0}|K_\alpha(x,y_1)-K_\alpha(x,y_2)|\,dx\\ &\quad= \int_{Q_A}|K_\alpha(x,y_1)-K_\alpha(x,y_2)|\,dx + \int_{Q_B}|K_\alpha(x,y_1)-K_\alpha(x,y_2)|\,dx\\ &\quad= A+B,\end{aligned}$$

where

$$Q_A = Q_0 \cap B(y_1, \varepsilon_1) \cap B(y_2, \varepsilon_1), \qquad Q_B = Q_0 \setminus (B(y_1, \varepsilon_1) \cap B(y_2, \varepsilon_1)).$$

We start with A and define

$$O(x,y) = \frac{e^{i\,\phi_\alpha(y)} - e^{i\,\phi_\alpha(x)}}{|x-y|}\ \chi_{Q_0 \times Q_0}(x,y).$$

Hence,

$$\begin{aligned} K_\alpha(x,y) &= O(x,y)\,H(x-y) \\ &= O(x,y)\,\big(H(x-y) - h(x-y)\big) + O(x,y)\,h(x-y) \\ &= I(x,y) + II(x,y). \end{aligned}$$

Note that for any $y \in Q_0$, we have

$$\begin{aligned} \int_{Q_0} |I(x,y)|\,dx &\le \|\nabla \phi_\alpha\|_{L^\infty(Q_0)} \int_{Q_0} |H(x-y) - h(x-y)|\,dx \\ &\le \|\nabla \phi_\alpha\|_{L^\infty(Q_0)} \int_{B_0} |H(z) - h(z)|\,dz < \frac{\varepsilon}{8}, \end{aligned}$$

and therefore, for $y_1, y_2 \in Q_0$,

$$\int_{Q_A} |I(x,y_1) - I(x,y_2)|\,dx < \frac{\varepsilon}{4}. \tag{19}$$

On the other hand, for $y_1, y_2 \in Q_0$

$$\begin{aligned} &\int_{Q_A} |II(x,y_1) - II(x,y_2)|\,dx \\ &\quad \le \int_{Q_0} |O(x,y_1)|\,|h(x-y_1) - h(x-y_2)|\,dx + \\ &\qquad\qquad + \int_{Q_A} |h(x-y_2)|\,|O(x,y_1) - O(x,y_2)|\,dx \\ &\quad \le \|\nabla \phi_\alpha\|_{L^\infty(Q_0)}\,\|\nabla h\|_{L^\infty(B_0)}\,|Q_0|\,|y_1 - y_2| + \\ &\qquad\qquad + \|h\|_{L^\infty(B_0)} \int_{Q_A} |O(x,y_1) - O(x,y_2)|\,dx. \end{aligned} \tag{20}$$

Then we have to estimate the last displayed integral. Let $x \notin \{y_1, y_2\}$ such that $x \in Q_A = Q_0 \cap B(y_1, \varepsilon_1) \cap B(y_2, \varepsilon_1)$. Using the order 1 Taylor expansion for the function $g(y) = e^{i\,\phi_\alpha(y)}$ centered at x we obtain

$$\begin{aligned} &|O(x,y_1) - O(x,y_2)| \\ &\le |\nabla g_\alpha(x)| \left| \frac{y_1 - x}{|y_1 - x|} - \frac{y_2 - x}{|y_2 - x|} \right| + \frac{1}{2}\,\|D^2 g_\alpha\|_{L^\infty(Q_0)}\,\big(|y_1 - x| + |y_2 - x|\big) \\ &\le \|\nabla g_\alpha\|_{L^\infty(Q_0)}\,|m(y_1 - x) - m(y_2 - x)| + \|D^2 g_\alpha\|_{L^\infty(Q_0)}\,\varepsilon_1 \\ &\le \|\nabla \phi_\alpha\|_{L^\infty(Q_0)}\,|m(y_1 - x) - m(y_2 - x)| + C_{\phi_\alpha}\,\varepsilon_1, \end{aligned}$$

where $m(x) = \chi_{B_0}(x) \cdot x/|x| \in L^1(\mathbb{R}^n)$. Then,

$$\begin{aligned}
&\int_{Q_A} |O(x,y_1) - O(x,y_2)|\,dx \\
&\qquad \le \|\nabla\phi_\alpha\|_{L^\infty(Q_0)} \int_{Q_0} |m(y_1 - x) - m(y_2 - x)|\,dx + C_{\phi_\alpha}\,\varepsilon_1\,|Q_0| \\
&\qquad \le \|\nabla\phi_\alpha\|_{L^\infty(Q_0)} \int_{\mathbb{R}^n} |m(z + y_2 - y_1) - m(z)|\,dz + C_{\phi_\alpha}\,\varepsilon_1\,|Q_0|.
\end{aligned}$$

We fix

$$\varepsilon_2 = \frac{\varepsilon}{16}\,\frac{1}{\|h\|_{L^\infty(B_0)}\,\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}}.$$

Since $m \in L^1(\mathbb{R}^n)$ we can use the properties of the translation operator in this space and there exists $\delta_1 > 0$ such that

$$\int_{\mathbb{R}^n} |m(z + \Delta z) - m(z)|\,dz < \varepsilon_2, \qquad \text{whenever} \quad |\Delta z| < \delta_1.$$

Thus, if $|y_1 - y_2| < \delta_1$, we have

$$\int_{Q_A} |O(x,y_1) - O(x,y_2)|\,dx \le \|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,\varepsilon_2 + C_{\phi_\alpha}\varepsilon_1\,|Q_0| < \frac{\varepsilon}{8\|h\|_{L^\infty(B_0)}}.$$

We set

$$\delta_A = \min\left\{\delta_1,\ \frac{\varepsilon}{8}\,\frac{1}{\|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,\|\nabla h\|_{L^\infty(B_0)}\,|Q_0|}\right\}$$

and, for $|y_1 - y_2| < \delta_A$, the latter estimate plugged into (20) yields

$$\begin{aligned}
&\int_{Q_A} |II(x,y_1) - II(x,y_2)|\,dx \\
&\quad \le \|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,\|\nabla h\|_{L^\infty(B_0)}\,|Q_0|\,|y_1 - y_2| + \|h\|_{L^\infty(Q_0)}\,\frac{\varepsilon}{8\,\|h\|_{L^\infty(B_0)}} \\
&\quad < \frac{\varepsilon}{4}.
\end{aligned}$$

This and (19) provide

$$A \le \int_{Q_A} |I(x,y_1) - I(x,y_2)|\,dx + \int_{Q_A} |II(x,y_1) - II(x,y_2)|\,dx < \frac{\varepsilon}{2},$$

whenever $|y_1 - y_2| < \delta_A$.

Let us get a similar estimate for B. We break up Q_B as follows

$$\begin{aligned}
Q_B &= Q_0 \setminus (B(y_1,\varepsilon_1) \cap B(y_2,\varepsilon_1)) = \big(Q_0 \setminus B(y_1,\varepsilon_1)\big) \cup \big(Q_0 \setminus \cap B(y_2,\varepsilon_1)\big) \\
&= Q_B^1 \cup Q_B^2.
\end{aligned}$$

and then

$$B \le \int_{Q_B^1} |K_\alpha(x,y_1) - K_\alpha(x,y_2)|\,dx + \int_{Q_B^2} |K_\alpha(x,y_1) - K_\alpha(x,y_2)|\,dx$$
$$= B_1 + B_2.$$

We we are going to get an estimate for B_1, for B_2 we only have to switch y_1 and y_2. Let $y_1, y_2 \in Q_0$ be such that $|y_1 - y_2| < \varepsilon_1/2$, $x \in Q_B^1$. Then $|x-y_2| > \varepsilon_1/2$ and so $x \ne y_1, y_2$. We estimate the difference of the kernels:

$$\begin{aligned}
&|K_\alpha(x,y_1) - K_\alpha(x,y_2)| \\
&\le \left|e^{i\,\phi_\alpha(x)} - e^{i\,\phi_\alpha(y_2)}\right| |K(x-y_1) - K(x-y_2)| + \\
&\qquad\qquad + \left|e^{i\,\phi_\alpha(y_2)} - e^{i\,\phi_\alpha(y_1)}\right| |K(x-y_1)| \\
&\le 2\,|K(x-y_1) - K(x-y_2)| + \|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,|y_1-y_2|\,|K(x-y_1)|.
\end{aligned}$$

Note that $|x-y_1| > \varepsilon_1 > 2\,|y_1-y_2|$ and therefore we can use the regularity assumed on the Calderón-Zygmund kernel K to obtain

$$|K(x-y_1) - K(x-y_2)| \le c_0\,\frac{|y_1-y_2|^\tau}{|x-y_1|^{n+\tau}} \le c_0\,\frac{|y_1-y_2|^\tau}{\varepsilon_1^{n+\tau}}.$$

On the other hand, by the size condition of K we have

$$|K(x-y_1)| \le \frac{c_0}{|x-y_1|^n} < \frac{c_0}{\varepsilon_1^n}$$

Hence,

$$\begin{aligned}
B_1 &\le 2\int_{Q_1^B} |K(x-y_1) - K(x-y_2)|\,dx + \\
&\qquad\qquad + \|\nabla\phi_\alpha\|_{L^\infty(Q_0)}\,|y_1-y_2| \int_{Q_1^B} |K(x-y_1)|\,dx \\
&\le \frac{2\,c_0\,|Q_0|}{\varepsilon_1^{n+\tau}}\,|y_1-y_2|^\tau + \frac{\|\nabla\phi_\alpha\|_{L^\infty(Q_0)} c_0\,|Q_0|}{\varepsilon_1^n}\,|y_1-y_2|
\end{aligned}$$

whenever $|y_1 - y_2| < \varepsilon_1/2$. If we set

$$\delta_B = \min\left\{\frac{\varepsilon_1}{2},\ \frac{\varepsilon}{8}\,\frac{\varepsilon_1^n}{\|\nabla\phi_\alpha\|_{L^\infty(Q_0)} c_0\,|Q_0|},\ \left[\frac{\varepsilon}{8}\,\frac{\varepsilon_1^{n+\tau}}{2\,c_0\,|Q_0|}\right]^{1/\tau}\right\}$$

and we assume that $|y_1 - y_2| < \delta_B$, we have proved that $B_1 < \varepsilon/4$. In a similar way we also obtain that $B_2 < \varepsilon/4$ which yields

$$B \le B_1 + B_2 < \frac{\varepsilon}{2}, \qquad \text{whenever} \quad |y_1 - y_2| < \delta_B.$$

Putting all together we have shown that given $\varepsilon > 0$ there exists $\delta = \min\{\delta_A, \delta_B\}$ (that depends on α, Q_0, ε) such that, for y_1, $y_2 \in Q_0$ we have

$$\int_{Q_0} |K_\alpha(x, y_1) - K_\alpha(x, y_2)|\, dx \leq A + B < \varepsilon, \quad \text{whenever} \quad |y_1 - y_2| < \delta.$$

This gives (c) in Theorem 5.3. This concludes the proof that all conditions of Theorem 5.3 apply to T_Q^{Φ}. The conclusion of this theorem therefore yields

$$\|T_Q^{\Phi} f\|_{L^{1,\infty}(Q,\mu_Q)} \leq C\, \|f\|_{\Upsilon_m,Q},$$

where C does not depend on Q. □

6. Proof of Theorem 5.3

The proof of this theorem is inspired by [SS]. The exact formulation of the theorem in [SS] does not exactly apply to our setting and in this section we suitably modify the arguments given in [SS] to obtain the proof of Theorem 5.3. It is crucial for us to obtain localized estimates that do not depend on the cube, otherwise our arguments will not work.

We fix an arbitrary cube Q_0. Given $N \geq 1$ we write $S_N^\star f(x) = \sup_{1\leq j\leq N} |S_j f(x)|$ and we use the notation $S_{N,Q_0}^\star$ for the corresponding localized operators.

The proof splits in the following steps:

Step 1: Given a bounded function $0 \leq f \in L^1(Q_0, \mu_{Q_0})$, the sequence $\{a_k\}_k$ with $a_0 = 0$ and $a_k = 2^{2^k}$ for $k \geq 1$, and $\varepsilon > 0$ there exists $k_0 \geq 1$, so that $f(x) \leq a_{k_0}$, and a simple function h:

$$h = \sum_{k=1}^{k_0} a_k\, \chi_{F_k} = \sum_{k=1}^{\infty} a_k\, \chi_{F_k}$$

such that

(*i*) $F_k \subset G_k = \{x \in Q_0 : \;\; a_{k-1} < f(x) \leq a_k\}$, $k \geq 1$ (note that $F_k = G_k = \emptyset$ for $k > k_0$).

(*ii*) $\displaystyle\int_{G_k} f(x)\, d\mu_{Q_0}(x) = \int_{G_k} h(x)\, d\mu_{Q_0}(x) = a_k\, \mu_{Q_0}(F_k)$.

(*iii*) $\displaystyle\int_{Q_0} S_{N,Q_0}^\star (f - h)(x)\, d\mu_{Q_0}(x) \leq \varepsilon$.

Step 2: For h as above

$$\mu_{Q_0}\{x : S^\star_{Q_0}h(x) > \lambda\} \\ \leq \frac{C}{\lambda}\Big(1 + \int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0}\Big)\Big(1 + \log\Big(1 + \int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0}\Big)\Big),$$

where C does not depend on Q_0 nor on ε.

Step 3: $\|S^\star_{Q_0}f\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \leq C\,\|f\|_{\varUpsilon_m,Q}$, for all $f \in L_{\varUpsilon_m}(Q, \mu_Q)$ and where C does not depend on Q_0.

Proof of Step 3: By the monotone convergence theorem, it is enough to fix $N \geq 1$ and obtain estimates for $S^\star_{N,Q_0}$ with C independent of N. Notice that $\|\cdot\|_{L^{1,\infty}(Q_0,\mu_{Q_0})}$ is a quasi-norm and $\|\cdot\|_{\varUpsilon_m,Q_0}$ is a norm for the Banach space $L_{\varUpsilon_m}(Q_0, \mu_{Q_0})$. On the other hand, $S^\star_{N,Q_0}$ is a sublinear operator, thus it suffices to show

$$\|S^\star_{Q_0}f\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \leq C$$

(with C independent of Q_0) for any function $0 \leq f \in L_{\varUpsilon_m}(Q_0, \mu_{Q_0})$ with $\|f\|_{\varUpsilon_m,Q} < 1$. On the other hand, without lost of generality we can assume that f is also bounded: set $f^M(x) = f(x)$ when $f(x) \geq M$ with M some fixed large number, then by (b),

$$\begin{aligned}\int_{Q_0} S^\star_{N,Q_0}f^M(x)\, d\mu_{Q_0}(x) &\leq \sum_{j=1}^{N}\int_{Q_0} |S_{j,Q}f^M(x)|\, d\mu_{Q_0}(x) \\ &\leq \sum_{j=1}^{N}\sup_{y\in Q_0}\Big(\int_{Q_0} |s_j^{Q_0}(x,y)|\, d\mu_{Q_0}(x)\Big)\int_{Q_0} |f^M(y)|\, d\mu_{Q_0}(y) \\ &\leq C(Q_0, N)\,\|f^M\|_{L^1(Q_0,\mu_{Q_0})} \longrightarrow 0, \qquad \text{as} \quad M \to \infty.\end{aligned}$$

So, we take $0 \leq f \in L^\infty(Q_0)$ such that $\|f\|_{\varUpsilon_m,Q_0} < 1$ which implies

$$\int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0} \leq 1. \tag{21}$$

Let $\varepsilon > 0$ to be chosen. Let $\lambda > 0$ and $0 < \eta < \lambda$. We apply **Step 1** with $\widetilde{\varepsilon} = \eta\,\varepsilon$. Then $S^\star_{N,Q_0}f(x) \leq S^\star_{N,Q_0}(f-h)(x) + S^\star_{N,Q_0}h(x)$ and

$$\begin{aligned}&\mu_{Q_0}\{x : S^\star_{N,Q_0}f(x) > \lambda\} \\ &\quad\leq \mu_{Q_0}\{x : S^\star_{N,Q_0}(f-h)(x) > \eta\} + \mu_{Q_0}\{x : S^\star_{N,Q_0}h(x) > \lambda - \eta\} \\ &\quad\leq \frac{1}{\eta}\,\|S^\star_{N,Q_0}(f-h)\|_{L^1(Q_0,\mu_{Q_0})} + \frac{C}{\lambda-\eta} \\ &\quad\leq \frac{1}{\eta}\,\widetilde{\varepsilon} + \frac{C_0}{\lambda-\eta} = \varepsilon + \frac{C_0}{\lambda-\eta} \longrightarrow \frac{C_0}{\lambda}, \qquad \text{as} \quad \eta, \varepsilon \to 0.\end{aligned}$$

Note that in the second estimate we have used Chebichev's inequality for the first term and **Step 2**, with (21), for the second. The third estimate is (*iii*) in **Step 1**. We would like to point out that C_0 is independent of Q_0. In this way, we have shown that

$$\|S^\star_{Q_0} f\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \le C_0$$

as desired. □

Proof of Step 2: Let us recall that $\varphi_m(t) = t\,\big(1+\log^+\frac{1}{t}\big)^m$. We are going to use the log-convexity of the $L^{1,\infty}$ norm (see [SWe], [K]), namely, if $\{g_j\}$ is a sequence of functions such that $\|g_k\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \le C_0$ and $\{\beta_k\}_k$ is a sequence of non-negative numbers then

$$\Big\|\sum_k \beta_k\, g_k\Big\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \le 6\,C_0\,\mathcal{N}(\{\beta_k\}_k) \tag{22}$$

where

$$\mathcal{N}(\{\beta_k\}_k) = \sum_k \beta_k \left(1+\log\frac{\sum_j \beta_j}{\beta_k}\right).$$

Note that

$$\left\|S^\star_{Q_0}\left(\frac{\chi_{F_k}}{\varphi_m(\mu_{Q_0}(F_k))}\right)\right\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} = \frac{\|S^\star_{Q_0}(\chi_{F_k})\|_{L^{1,\infty}(Q_0,\mu_{Q_0})}}{\varphi_m(\mu_{Q_0}(F_k))} \le C_0$$

where we have used (d) and C_0 does not depend on Q_0. Thus, writing $b_k = \mu_{Q_0}(F_k)$, we can use (22) to get

$$\begin{aligned}\|S^\star_{Q_0} h\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} &\le \left\|\sum_k a_k\,\varphi_m(b_k)\,S^\star_{Q_0}\left(\frac{\chi_{F_k}}{\varphi_m(b_k)}\right)\right\|_{L^{1,\infty}(Q_0,\mu_{Q_0})}\\ &\le 6\,C_0\,\mathcal{N}\big(\big\{a_k\,\varphi_m(b_k)\big\}_k\big).\end{aligned}$$

We are going to modify the functional $\mathcal{N}$ in the following way: let $\{\beta_k\}_k$ be a non-identically-zero sequence of non-negative numbers. Let $\beta = \sum_k \beta_k > 0$. Then using the submultiplicativity of φ_1,

$$\begin{aligned}\mathcal{N}(\{\beta_k\}_k) &= \beta\sum_k \frac{\beta_k}{\beta}\left(1+\log\frac{1}{\beta_k/\beta}\right) = \beta\sum_k \varphi_1\Big(\frac{\beta_k}{\beta}\Big)\\ &\le \beta\,\varphi_1\Big(\frac{1}{\beta}\Big)\sum_k \varphi_1(\beta_k) = \Big(1+\log^+\sum_k \beta_k\Big)\,\widetilde{\mathcal{N}}(\{\beta_k\}_k)\end{aligned}$$

where $\widetilde{\mathcal{N}}(\{\beta_k\}_k) = \sum_k \varphi_1(\beta_k)$. In this way,

$$\|S^\star_{Q_0} h\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} \le 6\,C_0\,\Big(1+\log^+\sum_k a_k\,\varphi_m(b_k)\Big)\,\widetilde{\mathcal{N}}(\{a_k\,\varphi_m(b_k)\}_k).$$

Lemma 6.1. *Let $m \geq 1$, there exist C_1, C_2 —that only depend on m— such that for any sequence $\{\beta_k\}$ with $0 \leq \beta_k \leq 1$ we have*

$$\sum_{k=1}^{\infty} 2^{2^k} \varphi_m(\beta_k) \leq C_1 \Big(1 + \sum_{k=1}^{\infty} 2^{2^k} 2^{k\,m} \beta_k\Big),$$

$$\sum_{k=1}^{\infty} \varphi_1\big(2^{2^k} \varphi_m(\beta_k)\big) \leq C_2 \Big(1 + \sum_{k=1}^{\infty} 2^{2^k} 2^{k\,m}(1 + \log k)\, \beta_k\Big).$$

Lemma 6.2. *Let h be the function defined in **Step 1**. Then,*

$$2^{-m} \int_{Q_0} \varUpsilon_m\big(h(x)\big)\, d\mu_{Q_0}(x) \leq \sum_{k=1}^{\infty} \mu_{Q_0}(F_k)\, 2^{2^k} 2^{k\,m} (1 + \log k)$$

$$\leq 4^{m+1} \int_{Q_0} \varUpsilon_m\big(f(x)\big)\, d\mu_{Q_0}(x).$$

We will prove these auxiliary results later. Recall that we took $a_k = 2^{2^k}$ and wrote $b_k = \mu_{Q_0}(F_k)$, which satisfies $0 \leq b_k \leq 1$ since μ_{Q_0} is a probability measure. We use the first estimate in Lemma 6.1, and Lemma 6.2 to get

$$\begin{aligned}\sum_k a_k\, \varphi_m(b_k) &\leq C_1 \Big(1 + \sum_{k=1}^{\infty} 2^{2^k} 2^{k\,m} b_k\Big)\\ &\leq C_1 \Big(1 + \sum_{k=1}^{\infty} 2^{2^k} 2^{k\,m} (1 + \log k)\, b_k\Big)\\ &\leq C_1\, 4^{m+1} \Big(1 + \int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0}\Big).\end{aligned}$$

On the other hand, by the second estimate in Lemma 6.1, and by Lemma 6.2,

$$\begin{aligned}\widetilde{\mathcal{N}}(\{a_k\, \varphi_m(b_k)\}_k) &= \sum_{k=1}^{\infty} \varphi_1\big(2^{2^k} \varphi_m(b_k)\big)\\ &\leq C_2 \Big(1 + \sum_{k=1}^{\infty} 2^{2^k} 2^{k\,m}(1 + \log k)\, b_k\Big)\\ &\leq C_2\, 4^{m+1} \Big(1 + \int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0}\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\|S^\star_{Q_0} h\|_{L^{1,\infty}(Q_0,\mu_{Q_0})} &\le 6\, C_0 \Big(1+\log^+ \sum_k a_k\, \varphi_m(b_k)\Big)\, \widetilde{\mathcal{N}}(\{a_k\,\varphi_m(b_k)\}) \\ &\le C\left(1+\log\Big(1+\int_{Q_0} \varUpsilon_m(f)\, d\mu_{Q_0}\Big)\right)\Big(1+\int_{Q_0}\varUpsilon_m(f)\, d\mu_{Q_0}\Big),\end{aligned}$$

as desired.

Proof (Lemma 6.1). Both estimates are proved in a similar way. Set

$$I=\Big\{k\ge 1: \beta_k \le \frac{1}{2^{2^k}\, 2^{k\, m}\, k^2}\Big\}, \qquad II=\Big\{k\ge 1: \beta_k > \frac{1}{2^{2^k}\, 2^{k\, m}\, k^2}\Big\}.$$

Then, using that φ_1 is an increasing function,

$$\begin{aligned}\varSigma^I &= \sum_{k\in I} 2^{2^k}\, \varphi_m(\beta_k) \le \sum_{k\in I} 2^{2^k}\, \varphi_m\Big(\frac{1}{2^{2^k}\, 2^{k\, m}\, k^2}\Big) \\ &= \sum_{k=1}^\infty 2^{2^k}\, \frac{1}{2^{2^k}\, 2^{k\, m}\, k^2}\, \big(1+\log^+(2^{2^k}\, 2^{k\, m}\, k^2)\big)^m \le C_m \sum_{k=1}^\infty \frac{1}{k^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varSigma^{II} &= \sum_{k\in II} 2^{2^k}\, \varphi_m(\beta_k) = \sum_{k\in II} 2^{2^k}\, \beta_k\, \Big(1+\log^+ \frac{1}{\beta_k}\Big)^m \\ &\le \sum_{k=1}^\infty 2^{2^k}\, \beta_k\, \Big(1+\log^+(2^{2^k}\, 2^{k\, m}\, k^2)\Big)^m \le C_m \sum_{k=1}^\infty 2^{2^k}\, 2^{k\, m}\, \beta_k.\end{aligned}$$

Thus,

$$\sum_{k=1}^\infty 2^{2^k}\, \varphi_m(\beta_k) = \varSigma^I + \varSigma^{II} \le C_1 \Big(1+\sum_{k=1}^\infty 2^{2^k}\, 2^{k\, m}\, \beta_k\Big),$$

where C_1 only depends on m.

We indicate how to obtain the latter estimate. Since both φ_1, φ_m are increasing we have

$$\begin{aligned}\varSigma^I &= \sum_{k\in I} \varphi_1\big(2^{2^k}\, \varphi_m(\beta_k)\big) \le \sum_{k=1}^\infty \varphi_1\left(2^{2^k}\, \varphi_m\Big(\frac{1}{2^{2^k}\, 2^{k\, m}\, k^2}\Big)\right) \\ &\le \sum_{k=1}^\infty \varphi_1\Big(\frac{C_m}{k^2}\Big) \le C_m \sum_{k=1}^\infty \frac{1+\log k}{k^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Sigma^{II} &= \sum_{k\in II} \varphi_1\big(2^{2^k}\,\varphi_m(\beta_k)\big) = \sum_{k\in II} \varphi_1\bigg(2^{2^k}\,\beta_k\,\Big(1+\log^+\frac{1}{\beta_k}\Big)^m\bigg) \\
&\le \sum_{k\in II} \varphi_1\bigg(2^{2^k}\,\beta_k\,\big(1+\log^+(2^{2^k}\,2^{k\,m}\,k^2)\big)^m\bigg) \\
&\le \sum_{k\in II} \varphi_1\big(C_m\,2^{2^k}\,2^{k\,m}\,\beta_k\big) \\
&\le C_m \sum_{k\in II} 2^{2^k}\,2^{k\,m}\,\beta_k\,\Big(1+\log^+\frac{1}{2^{2^k}\,2^{k\,m}\,\beta_k}\Big) \\
&\le C_m \sum_{k=0}^{\infty} 2^{2^k}\,2^{k\,m}\,\beta_k\,(1+\log^+ k).
\end{aligned}$$

Collecting the estimates for Σ^I and Σ^{II} we get the desired inequality. □

Proof (Lemma 6.2). The first inequality is trivial: the sets F_k are pairwise disjoint (since the G_k's are) and therefore

$$\begin{aligned}
2^{-m}&\int_{Q_0} \Upsilon_m(h)\,d\mu_{Q_0} = 2^{-m}\int_{Q_0} \Upsilon_m\Big(\sum_{k=1}^{\infty} a_k\,\chi_{F_k}\Big)\,d\mu_{Q_0} \\
&= 2^{-m}\sum_{k=1}^{\infty} \Upsilon_m(a_k)\,\mu_{Q_0}(F_k) \\
&= 2^{-m}\sum_{k=1}^{\infty} \mu_{Q_0}(F_k)\,2^{2^k}\,\big(1+\log^+ 2^{2^k}\big)^m\,\big(1+\log^+\log^+\log^+ 2^{2^k}\big) \\
&\le \sum_{k=1}^{\infty} \mu_{Q_0}(F_k)\,2^{2^k}\,2^{k\,m}\,(1+\log k).
\end{aligned}$$

For the second estimate we use (ii) in **Step 1**:

$$\sum_{k=1}^{\infty} \mu_{Q_0}(F_k)\,2^{2^k}\,2^{k\,m}\,(1+\log k) = \sum_{k=1}^{\infty}\int_{G_k} f(x)\,d\mu_{Q_0}(x)\,2^{k\,m}\,(1+\log k).$$

To finish we only have to notice that $G_1 = \{x\in Q_0 : 0 < f(x)\le 4\}$ and

$$G_k = \{x\in Q_0 : 2^{2^{k-1}} < f(x) \le 2^{2^k}\},\quad k\ge 2.$$

Thus, for $x\in G_k$, we have

$$2^k \le 4\,\big(1+\log^+ f(x)\big),\qquad 1+\log k \le 4\,\big(1+\log^+\log^+\log^+ f(x)\big).$$

Hence, since the sets G_k are pairwise disjoint

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mu_{Q_0}(F_k)\, 2^{2^k}\, 2^{k\,m}\,(1+\log k) = \sum_{k=1}^{\infty} \int_{G_k} f(x)\, 2^{k\,m}\,(1+\log k)\, d\mu_{Q_0}(x) \\
&\le 4^{m+1} \sum_{k=1}^{\infty} \int_{G_k} f(x)\,\big(1+\log^+ f(x)\big)^m \big(1+\log^+\log^+\log^+ f(x)\big)\, d\mu_{Q_0}(x) \\
&\le 4^{m+1} \int_{Q_0} \Upsilon_m\big(f(x)\big)\, d\mu_{Q_0}(x).
\end{aligned}$$

□

Proof of Step 1: This step is an extension of an approximation lemma in [SS]. We will prove it for completeness. It is clear that since f is bounded, $f(x) \le a_{k_0}$ for all $x \in Q_0$ and for some $k_0 \ge 1$. Fix $\varepsilon > 0$. Let $\{Q_l\}$ be a finite family of dyadic subcubes of Q_0, which all are of the same fixed generation (that is, they all have the same side length). The generation that we are considering is taken in such a way that the length of the diagonal of any of them is smaller than δ, for some $\delta > 0$ to be chosen. Let $G_k = \{x \in Q_0 :\ a_{k-1} < f(x) \le a_k\}$. Note that $G_k = \emptyset$ for $k \ge k_0+1$. Since,

$$\int_{G_k\cap Q_l} f(x)\, d\mu_{Q_0}(x) \le a_k\, \mu_{Q_0}(G_k \cap Q_l),$$

there exists $F_k^l \subset \mathrm{Int}(Q_l) \cap G_k$ such that

$$\int_{G_k\cap Q_l} f(x)\, d\mu_{Q_0}(x) = a_k\, \mu_{Q_0}(F_k^l).$$

We define $F_k = \cup_l F_k^l$ and note that $F_k = \emptyset$ for $k \ge k_0 + 1$. Our simple function is

$$h(x) = \sum_{k=1}^{\infty} a_k\ \chi_{F_k} = \sum_{k=1}^{k_0} a_k\ \chi_{F_k}\,.$$

Conclusion (i) holds by construction. The same occurs for (ii):

$$\begin{aligned}
\int_{G_k} f(x)\ d\mu_{Q_0}(x) &= \sum_l \int_{G_k\cap Q_l} f(x)\, d\mu_{Q_0}(x) = \sum_l a_k\, \mu_{Q_0}(F_k^l) \\
&= a_k\, \mu_{Q_0}(F_k) = \int_{G_k} h(x)\ d\mu_{Q_0}(x),
\end{aligned}$$

by the disjointness of the sets $\{F_k^l\}_l$ and also of $\{G_k\}_k$. So we only have to prove (iii). We proceed as follows

$$\int_{Q_0} S^{\star}_{N,Q_0}(f-h)(x)\,d\mu_{Q_0}(x) \le \sum_{j=1}^{N}\int_{Q_0} |S_{j,Q_0}(f-h)(x)|\,d\mu_{Q_0}(x)$$
$$= \sum_{j=1}^{N}\int_{Q_0}\Big|\int_{Q_0} s_j(x,y)\,\big(f(y)-h(y)\big)\,dy\Big|\,d\mu_{Q_0}(x)$$
$$\le \sum_{j=1}^{N}\sum_{k=1}^{k_0}\sum_{l}\int_{Q_0}\Big|\int_{G_k\cap Q_l} s_j(x,y)\,\big(f(y)-a_k\ \chi_{F_k^l(y)}\big)\,dy\Big|\,d\mu_{Q_0}(x)$$
$$\le \sum_{j=1}^{N}\sum_{k=1}^{k_0}\sum_{l}\int\limits_{G_k\cap Q_l}|f(y)-a_k\ \chi_{F_k^l}(y)|dy\int\limits_{Q_0}\big|s_j(x,y)-s_j(x,y_l)\big|d\mu_{Q_0}(x)$$

where y_l is the center of Q_l and we have used that

$$\int\limits_{G_k\cap Q_l}\big(f(y)-a_k\ \chi_{F_k^l}(y)\big)\,dy = |Q_0|\Big(\int\limits_{G_k\cap Q_l} f(y)\,d\mu_{Q_0}(y)-a_k\mu_{Q_0}(F_k^l)\Big) = 0.$$

Take,

$$\varepsilon_0 = \frac{\varepsilon}{2\ N\,|Q_0|\,\|f\|_{L^1(Q_0,\mu_{Q_0})}}.$$

By (c), for each $j=1,\dots,N$, there exists δ_j such that

$$\int_{Q_0}|s_j(x,y_1)-s_j(x,y_2)|\,d\mu_{Q_0}(x)\le\varepsilon_0,\quad |y_1-y_2|<\delta_j,\ y_1,y_2\in Q_0.$$

We choose $\delta=\min\{\delta_1,\dots,\delta_N\}$. Since y_l is the center of Q_l and $y\in Q_l$ we have that $|y-y_l|<\operatorname{diag}(Q_l)\le\delta$ and so

$$\int_{Q_0}|s_j(x,y)-s_j(x,y_l)|\,d\mu_{Q_0}(x)\le\varepsilon_0,\qquad j=1,\dots,N.$$

Thus,

$$\int_{Q_0} S^{\star}_{N,Q_0}(f-h)(x)\,d\mu_{Q_0}(x)\le\varepsilon_0\,N\sum_{k=1}^{k_0}\sum_{l}\int\limits_{G_k\cap Q_l}\big(f(y)+a_k\ \chi_{F_k^l}(y)\big)\,dy$$
$$=2\,\varepsilon_0\,N\sum_{k=1}^{k_0}\sum_{l}\int_{G_k\cap Q_l} f(y)\,dy\le 2\,\varepsilon_0\,N\,|Q_0|\,\|f\|_{L^1(Q_0,\mu_{Q_0})}=\varepsilon.$$

□

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