THE REGULARITY PROBLEM FOR UNIFORMLY ELLIPTIC OPERATORS IN WEIGHTED SPACES

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ABSTRACT. This paper studies the regularity problem for block uniformly elliptic operators in divergence form with complex bounded measurable coefficients. We consider the case where the boundary data belongs to Lebesgue spaces with weights in the Muckenhoupt classes. Our results generalize those of S. Mayboroda (and those of P. Auscher and S. Stahlhut employing the first order method) who considered the unweighted case. To obtain our main results we use the weighted Hardy space theory associated with elliptic operators recently developed by the last two named authors. One of the novel contributions of this paper is the use of an "inhomogeneous" vertical square function which is shown to be controlled by the gradient of the function to which is applied in weighted Lebesgue spaces.

CONTENTS

1. Introduction

The study of elliptic boundary value problems on upper half spaces and on Lipschitz domains has a long history (see [21] for an introduction of major results for elliptic equations in divergence form with real symmetric coefficients). Recent breakthroughs in the field include [17, 18], where the Dirichlet and Regularity problems with real non-symmetric coefficients were considered. The study of elliptic problems with rough complex coefficients has been arousing great interest, particularly after the solution of Kato's conjecture in [4]. In this direction, the connections between the Dirichlet, regularity and Neumann problems were studied in [24] for the block case. Some of the tools used in the latter reference include a Calderón-Zygmund theory adapted to singular "non-integral" operators (see [1], and also [5, 6, 7]), as well as the a Hardy space theory adapted to elliptic operators (see [19, 20]). The reader is also referred to the work [8] for the robust first order method to deal with more general cases.

The main purpose of the present article is to continue with this line of research and study the regularity problem for elliptic operators with block structure, and with data in weighted Lebesgue spaces, for Muckenhoupt weights. This is a natural problem to consider, in view of a well-established weighted Calderón-Zygmund theory in the series of papers [5, 6, 7] and the weighted Hardy space theory adapted to elliptic operators recently developed in [22, 23, 25]. To state our main results, we need to introduce some background. Some notation is taken from [1, 22, 24]. Let $A = (a_{jk})_{j,k=1}^n$ be an $n \times n$ matrix whose entries are L^∞-valued complex coefficients defined on $\mathbb{R}^n$. We assume that A satisfies the following uniform ellipticity condition: there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \, |\xi|^2 \leq \operatorname{Re} A \, \xi \cdot \bar{\xi} \qquad \text{and} \qquad |A \, \xi \cdot \bar{\zeta}| \leq \Lambda \, |\xi| \, |\zeta|,$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi \cdot \zeta = \xi_1 \zeta_1 + \cdots + \xi_n \zeta_n$ and therefore $\xi \cdot \bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$, hence $A(x) \, \xi \cdot \bar{\zeta} = \sum_{j,k} a_{j,k}(x) \, \xi_k \, \bar{\zeta}_j$. Associated with this matrix, consider the second order divergence form uniformly elliptic operator $L := -\operatorname{div}(A\nabla)$.

Given A and L as above we construct the block matrix

$$\mathbb{A} := \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\mathbb{A}$ is uniformly elliptic in $\mathbb{R}^{n+1}$ and, thus, $\mathbb{A}$ gives rise to the block divergence form uniformly elliptic operator

$$\mathbb{L} := -\operatorname{div}_{x,t}(\mathbb{A}\nabla_{x,t}) = -\operatorname{div}_x(A\nabla_x) - \partial_t^2 = L_x - \partial_t^2.$$

Here and elsewhere the points in $\mathbb{R}^{n+1}$ are written as $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ so that $\nabla_{x,t}$ and $\operatorname{div}_{x,t}$ denote respectively the full gradient and divergence, while ∇_x and div_x stand respectively for the gradient and divergence in the first n variables, and $L_x = L$.

With the previous definition in hand, it is easy to see that whenever $f \in \mathcal{S}$, one has that $u(x,t) := e^{-t\sqrt{L}} f(x)$, where $(x,t) \in \mathbb{R}_+^{n+1}$, is a weak solution of $\mathbb{L}u = 0$ in $\mathbb{R}_+^{n+1}$. By this we mean that $u \in W^{1,2}_{\text{loc}}(\mathbb{R}_+^{n+1})$ satisfies

$$\iint_{\mathbb{R}_+^{n+1}} \mathbb{A}(x)\nabla_{x,t}u(x,t) \cdot \nabla_{x,t}\psi(x,t) \, dx \, dt = 0, \qquad \forall \, \psi \in C_0^\infty(\mathbb{R}_+^{n+1}).$$

Also, $u(\cdot, t) \to f$ in $L^2(\mathbb{R}^n)$ as $t \to 0^+$. We would like to observe that the latter convergence combined with [3, Lemma 8.3] gives non-tangential convergence almost everywhere of the averages of u over Whitney boxes to f (see [3, Remark 2.8] for more details).

Given $u \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ consider the L^2-non-tangential maximal function $\mathcal{N}$ defined as

$$(1.1) \qquad \mathcal{N}u(x) := \sup_{(y,t)\in\Gamma^\kappa(x)} \left(\frac{1}{|D((y,t),\kappa t)|} \iint_{D((y,t),\kappa t)} |u(z,s)|^2 dz ds \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,$$

where $0 < \kappa < 1$ is a fixed small constant, $\Gamma^\kappa(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < \kappa t\}$, and $D((y,t),\kappa t)$ denotes the $\mathbb{R}^{n+1}$-ball centered at (y,t) with radius κt, which is by construction contained in $\mathbb{R}^{n+1}_+$.

Using the previous definitions, the regularity boundary value problem (R_p), $1 < p < \infty$, is said to be solvable for the operator $\mathbb{L}$ in $\mathbb{R}^{n+1}_+$ if, for every $f \in \mathcal{S}$, the weak solution to the equation $\mathbb{L}u = 0$ (with boundary data f) given by the Poisson semigroup $u(x,t) = e^{-t\sqrt{L}}f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, satisfies the non-tangential maximal function estimate

$$(1.2) \qquad \|\mathcal{N}(\nabla_{x,t}u)\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}$$

and $u(\cdot, t) \to f$ in $L^2(\mathbb{R}^n)$ as $t \to 0^+$. As observed above, and since f is a nice function, we already know that the latter convergence holds and hence one also has non-tangential convergence almost everywhere (see [3]). This means that in the solvability of the regularity problem what matters is the estimate (1.2), and for the convergence to boundary data there are several alternative conditions that one can take as part of the definition.

The solvability of (R_p), $1 < p < \infty$, was proved by S. Mayboroda in [24]. In particular, the result is the following:

Theorem 1.3 ([24, Thorem 4.1]). *Let $\mathbb{L}$ be a block elliptic operator in $\mathbb{R}^{n+1}_+$ as above. Then for any p so that $\max\left\{1, \frac{nq_-(L)}{n+q_-(L)}\right\} < p < q_+(L)$, the regularity problem (R_p) is solvable.*

The interval $(q_-(L), q_+(L))$ represents the maximal open interval on which the gradient of the heat semigroup $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ is uniformly bounded on $L^p(\mathbb{R}^n)$. More specifically:

$$(1.4) \qquad q_-(L) := \inf\left\{p \in (1,\infty) : \sup_{t>0}\|\sqrt{t}\nabla e^{-tL}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\},$$

$$(1.5) \qquad q_+(L) := \sup\left\{p \in (1,\infty) : \sup_{t>0}\|\sqrt{t}\nabla e^{-tL}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty\right\}.$$

Similarly, $(p_-(L), p_+(L))$ denotes the maximal open interval of p such that the heat semigroup $\{e^{-tL}\}_{t>0}$ is uniformly bounded on $L^p(\mathbb{R}^n)$. As we shall explain in more detail in Section 2, these two intervals are somehow related, and for instance $p_-(L) = q_-(L)$, see [1, 7] where these intervals were deeply studied.

The goal of this paper is to extend Theorem 1.3 to the context of weighted Lebesgue spaces $L^p(w)$ with w being a Muckenhoupt weight. To set the stage we first give

some definitions and basic properties of Muckenhoupt weights. For further details, see [14, 15, 16]. Given a weight w, that is, a measurable function such that $0 < w < \infty$ a.e. and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we say that $w \in A_p$, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B \left(\fint_B w(x)\,dx \right) \left(\fint_B w(x)^{1-p'}\,dx \right)^{p-1} < \infty,$$

and, when $p = 1$, we say that $w \in A_1$ if

$$[w]_{A_1} := \sup_B \left(\fint_B w(x)\,dx \right) \left(\operatorname*{ess\,sup}_{x \in B} w(x)^{-1} \right) < \infty.$$

Here and below the suprema are taken over all balls $B \subset \mathbb{R}^n$.

We next introduce the reverse Hölder classes. We say that $w \in RH_s$, $1 < s < \infty$, if

$$[w]_{RH_s} := \sup_B \left(\fint_B w(x)\,dx \right)^{-1} \left(\fint_B w(x)^s\,dx \right)^{1/s} < \infty,$$

and, that $w \in RH_\infty$, if

$$[w]_{RH_\infty} := \sup_B \left(\fint_B w(x)\,dx \right)^{-1} \left(\operatorname*{ess\,sup}_{x \in B} w(x) \right) < \infty.$$

Finally, we set

$$A_\infty := \bigcup_{1 \le p < \infty} A_p = \bigcup_{1 < s \le \infty} RH_s.$$

We recall Muckenhoupt's theorem which states that $w \in A_p$, $1 < p < \infty$, if and only if the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) := \sup_{B \ni x} \fint_B |f(y)|\,dy,$$

is bounded on $L^p(w) = L^p(w\,dx)$. In the case that $p = 1$, the boundedness of $\mathcal{M}$ from $L^1(w)$ to $L^{1,\infty}(w)$ is equivalent to the fact that $w \in A_1$.

It is well-known that $w \in A_\infty$ implies that w is a doubling measure. Indeed, for any $w \in A_r$, $1 \le r < \infty$, we have

$$(1.6) \qquad\qquad w(\lambda B) \le [w]_{A_r} \lambda^{n\,r} w(B), \qquad \forall B, \ \forall \lambda > 1.$$

Consequently, $(\mathbb{R}^n, dw, |\cdot|)$ is a space of homogeneous type.

Another important feature of Muckenhoupt classes is their openness. More precisely, the A_p and RH_s classes have the following self-improving property: if $w \in A_p$, there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$, and similarly if $w \in RH_s$, then $w \in RH_{s+\delta}$ for some $\delta > 0$. These facts motivate the following definitions:

$$(1.7) \qquad\qquad r_w := \inf \big\{ p : \ w \in A_p \big\}, \qquad s_w := \inf \big\{ q : \ w \in RH_{q'} \big\}.$$

Note that, according to our definition, s_w is the conjugated exponent of the one defined in [6, Lemma 4.1]. Given $0 \le p_0 < q_0 \le \infty$ and $w \in A_\infty$, [6, Lemma 4.1] implies that

$$(1.8) \qquad \mathcal{W}_w(p_0, q_0) := \left\{ p \in (p_0, q_0) : \ w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p} \right)'} \right\} = \left(p_0 r_w, \frac{q_0}{s_w} \right).$$

If $p_0 = 0$ and $q_0 < \infty$ it is understood that the only condition that stays is $w \in RH_{\left(\frac{q_0}{p}\right)'}$. Analogously, if $0 < p_0$ and $q_0 = \infty$ the only assumption is $w \in A_{\frac{p}{p_0}}$. Finally $\mathcal{W}_w(0, \infty) = (0, \infty)$.

After these observations and definitions, we can introduce the notion of solvability in the weighted context. More precisely, given a weight $w \in A_\infty$ we say that the weighted regularity boundary value problem (R_p^w) is solvable for the operator $\mathbb{L}$ in $\mathbb{R}_+^{n+1}$, if, for every $f \in \mathcal{S}$, the weak solution to the equation $\mathbb{L}u = 0$ with the boundary data f given by the Poisson semigroup $u(x, t) = e^{-t\sqrt{L}}f(x)$, $(x, t) \in \mathbb{R}_+^{n+1}$, satisfies the non-tangential maximal function estimate

$$(1.9) \qquad \|\mathcal{N}(\nabla_{x,t} u)\|_{L^p(w)} \leq C\|\nabla f\|_{L^p(w)}$$

and $u(\cdot, t) \to f$ in $L^2(\mathbb{R}^n)$ as $t \to 0^+$.

Our main result establishes the solvability of (R_p^w):

Theorem 1.10. *Let $\mathbb{L}$ be a block elliptic operator in $\mathbb{R}_+^{n+1}$ as above and let $w \in A_\infty$ be such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \varnothing$. For every p such that*

$$(1.11) \qquad \max\left\{r_w, \frac{nr_w\widehat{q}_-(L)}{nr_w + \widehat{q}_-(L)}\right\} < p < \frac{q_+(L)}{s_w},$$

and for every $f \in \mathcal{S}$, if one sets $u(x, t) = e^{-t\sqrt{L}}f(x)$, $(x, t) \in \mathbb{R}_+^{n+1}$, then

$$(1.12) \qquad \|\mathcal{N}(\nabla_{x,t} u)\|_{L^p(w)} \leq C\|\nabla f\|_{L^p(w)}$$

and (R_p^w) is solvable.

In the previous result $(\widehat{q}_-(L), \widehat{q}_+(L))$ denotes the maximal open interval where $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ is uniformly bounded on $L^p(w)$, or equivalently, where $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ satisfies $L^p(w)$-$L^q(w)$ off-diagonal estimates on balls.

We note that Theorem 1.10 is the natural extension of [24, Thorem 4.1] (see Theorem 1.3 above). Indeed, if $w \equiv 1$ we have $r_w = 1$ and $s_w = 1$. Then, by definition, (see [1] and Section 2 below)

$$(q_-(L), q_+(L)) = (\widehat{q}_-(L), \widehat{q}_+(L)) = \mathcal{W}_w(q_-(L), q_+(L)).$$

Consequently, $\mathcal{W}_w(q_-(L), q_+(L)) \neq \varnothing$, since $q_-(L) < 2 < q_+(L)$. Besides,

$$\max\left\{r_w, \frac{nr_w\widehat{q}_-(L)}{nr_w + \widehat{q}_-(L)}\right\} = \max\left\{1, \frac{nq_-(L)}{n + q_-(L)}\right\}, \qquad \frac{q_+(L)}{s_w} = q_+(L),$$

and thus our range in Theorem 1.10 agrees with that of [24, Thorem 4.1] (see Theorem 1.3 above).

Before explaining our approach to proving Theorem 1.10, we make an observation to clarify the hypothesis requiring that the interval $\mathcal{W}_w(q_-(L), q_+(L))$ is not empty.

Remark 1.13. The fact that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \varnothing$ means that $\frac{q_+(L)}{q_-(L)} > r_w s_w$. This is a compatibility condition between L and w. It appears naturally in the theory developed in [5, 6, 7] and forces the weight to be sufficiently good, depending on how close $q_-(L)$ and $q_+(L)$ are to each other. As it was shown in [1], $q_-(L) = p_-(L) < \frac{2n}{n+2}$ while, in general, we only know that $q_+(L) > 2$, actually, there are examples showing that $q_+(L)$ can be found arbitrarily close to two. Thus, $\frac{q_+(L)}{q_-(L)} > 1 + \frac{2}{n}$,

and if $r_w s_w < 1 + \frac{2}{n}$, we always have that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$, for every L as above. Of course, this is a very restrictive condition since we would expect to have wider classes of weights as the operator L gets "nicer". A very illustrative example is that of power weights of the form $\omega_\alpha(x) := |x|^\alpha$. The weight ω_α belongs to A_∞, if and only if, $\alpha > -n$. It is not hard to see that $r_w = \max\{1, 1 + \alpha/n\}$ and $s_w = \max\{1, (1 + \alpha/n)^{-1}\}$ (see the definitions of r_w and s_w in (1.7)) and hence $\frac{q_+(L)}{q_-(L)} > r_w s_w$ implies that $-n\left(1 - \frac{q_-(L)}{q_+(L)}\right) < \alpha < n\left(\frac{q_+(L)}{q_-(L)} - 1\right)$. Note that this range of α always contains the interval $[-\frac{2n}{n+2}, 2]$ and gets bigger as $q_-(L)$ decreases to one, and/or $q_+(L)$ increases to infinity.

To proof Theorem 1.10 we independently consider the estimate of the weighted norm of the non-tangential maximal function for the time derivative $\partial_t e^{-t\sqrt{L}} f$, and the one for the spatial derivatives $\nabla e^{-t\sqrt{L}} f$. For the former we use ideas from [24, Theorem 4.1] combined with the weighted Hardy space theory associated with elliptic operators developed in [22, 23, 25].[1]

Regarding, the spatial derivatives we follow a different path, for which we need to introduce the following "inhomogeneous" vertical square function

$$(1.14) \qquad \widetilde{\mathrm{G}}_{\mathrm{H}} f(x) := \left(\int_0^\infty \left| t^2 \nabla L e^{-t^2 L} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Note that compared with the usual vertical square functions (or Littlewood-Paley-Stein functionals), the power of t does not seem to be consistent with the order of the operator in the integral. Indeed, the power of t corresponding to the usual square function would be three instead of two. That is why we use the terminology "inhomogeneous". This has the effect of getting an estimate for $\widetilde{\mathrm{G}}_{\mathrm{H}} f$ in terms of ∇f in $L^q(w)$, rather than obtaining an estimate from $L^q(w)$ to $L^q(w)$ (which is the natural one for the usual square functions). We would like to mention that in the case that L is the Laplace operator, this inhomogeneous square function (as well as its relation to reverse Riesz transform estimates), was studied in [12] in the setting of Riemannian manifolds.

As we shall see below, the estimate regarding the non-tangential maximal function for the spatial derivatives can be inferred from the following estimate of the inhomogeneous vertical square function introduced in (1.14). This approach is one of the novel contributions of this paper even in the unweighted case.

Theorem 1.15. *Let $w \in A_\infty$ and assume that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. Given q so that* $\max\left\{ r_w, \frac{n r_w \widehat{q}_-(L)}{n r_w + \widehat{q}_-(L)} \right\} < q < \widehat{q}_+(L)$ *there holds*

$$(1.16) \qquad \qquad \|\widetilde{\mathrm{G}}_{\mathrm{H}} f\|_{L^q(w)} \lesssim \|\nabla f\|_{L^q(w)},$$

for every $f \in \mathcal{S}$.

The plan of the paper is as follows. In Section 2 we present some preliminaries and some auxiliary results needed in Sections 3 and 4. As for these two last sections, the

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[1]We note that there is an alternative and independent method developed in the forthcoming paper [2] to deal with the regularity problem for degenerate elliptic operators which is based on some adapted Calderón-Zygmund theory, and does not require to deal with the highly technicality of Hardy spaces. The corresponding result in [2] gives however a more restricted range.

former is devoted to the proof of Theorem 1.15, and the later to proving Theorem 1.10.

2. Preliminaries and Auxiliary results

Throughout the paper the letters c, C, θ or $\widetilde{\theta}$ will represent any harmless constant that will not depend on any relevant parameter of the corresponding computation, and that will may be different in both sides of an inequality. Moreover, we shall write $\nabla = (\partial_{x_1}, \ldots \partial_{x_n})$ to denote the gradient in $\mathbb{R}^n$. We will typically write the elements in $\mathbb{R}^{n+1}_+$ as (x, t) with $x \in \mathbb{R}^n$ and $t > 0$ and hence $\nabla_{x,t} = (\nabla, \partial t)$ stands for the gradient in $\mathbb{R}^{n+1}$. Given a ball B, we use the notation $C_1(B) = 4B$ and $C_j(B) = 2^{j+1}B \backslash 2^j B$ for $j \geq 2$. If $w \in A_\infty$, we write

$$\fint_B h \, dw = \frac{1}{w(B)} \int_B h \, dw, \qquad \fint_{C_j(B)} h \, dw = \frac{1}{w(2^{j+1}(B))} \int_{C_j(B)} h \, dw.$$

We recall the notion of full off-diagonal estimates:

Definition 2.1. Let $\{T_t\}_{t>0}$ be a family of sublinear operators and let $1 \leq p \leq q \leq \infty$. We say that $\{T_t\}_{t>0}$ satisfies L^p-L^q full off-diagonal estimates, denoted by $T_t \in \mathcal{F}(L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n))$, if there exists a constant $c > 0$ such that for all closed sets E and F, all f and all $t > 0$ we have

$$(2.2) \qquad \|T_t(\mathbf{1}_E f)\mathbf{1}_F\|_{L^q(\mathbb{R}^n)} \lesssim t^{-\frac{1}{2}(\frac{n}{p} - \frac{n}{q})} e^{-\frac{cd(E,F)^2}{t}} \|f\mathbf{1}_E\|_{L^p(\mathbb{R}^n)},$$

where $d(E, F) = \inf\{|x - y| : x \in E, y \in F\}$.

Definition 2.3. Given $1 \leq p \leq q \leq \infty$ and any weight $w \in A_\infty$. We say that a family of sublinear operators $\{T_t\}_{t>0}$ satisfies L^p-L^q off-diagonal estimates on balls, denoted by $T_t \in \mathcal{O}(L^p(w) - L^q(w))$, if there exist $\theta_1, \theta_2 > 0$ and $c > 0$ such that for all $t > 0$ and for any ball B with radius r_B,

$$(2.4) \qquad \left(\fint_B |T_t(f\mathbf{1}_B)|^q \, dw \right)^{\frac{1}{q}} \lesssim \Upsilon\left(\frac{r_B}{\sqrt{t}} \right)^{\theta_2} \left(\fint_B |f|^p dw \right)^{\frac{1}{p}},$$

and, for all $j \geq 2$,

$$(2.5) \qquad \left(\fint_B |T_t(f\mathbf{1}_{C_j(B)})|^q \, dw \right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r_B}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{c4^j r_B^2}{t}} \left(\fint_{C_j(B)} |f|^p dw \right)^{\frac{1}{p}},$$

and

$$(2.6) \qquad \left(\fint_{C_j(B)} |T_t(f\mathbf{1}_B)|^q \, dw \right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r_B}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{c4^j r_B^2}{t}} \left(\fint_B |f|^p dw \right)^{\frac{1}{p}}.$$

In the previous definition $\Upsilon(s)$ is defined as $\Upsilon(s) := \max\{s, s^{-1}\}$ for $s > 0$. Besides note that if $q = \infty$ we would consider the corresponding $L^\infty(\mathbb{R}^n) = L^\infty(w)$ norms, in place of the $L^q(w)$-norms.

For the following results we recall that $(p_-(L), p_+(L))$ (resp. $(q_-(L), q_+(L))$) denotes the maximal open interval on which $\{e^{-tL}\}_{t>0}$ (resp. $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$) is uniformly bounded on $L^p(\mathbb{R}^n)$.

Proposition 2.7 ([1, Chapter 3])**.**

(i) If $p_-(L) < p \leq q < p_+(L)$, then $\{(tL)^m e^{-tL}\}_{t>0} \in \mathcal{F}(L^p - L^q)$ for every $m \in \mathbb{N}$.

(ii) If $q_-(L) < p \leq q < q_+(L)$, then $\{\sqrt{t}\nabla(tL)^m e^{-tL}\}_{t>0} \in \mathcal{F}(L^p - L^q)$ for every $m \in \mathbb{N}$. In addition, we have $p_-(L) = q_-(L)$ and $(q_-(L))^* \leq p_+(L)$.

(iii) $p_-(L), q_+(L) \leq (q_+(L))^* \leq p_+(L)$, and $q_+(L) > 2$.

(iv) $p_-(L) = 1$ and $p_+(L) = \infty$ if $n = 1, 2$; $q_+(L) = \infty$ if $n = 1$; and $p_-(L) < \frac{2n}{n+2}$, $p_+(L) > \frac{2n}{n-2}$ if $n \geq 3$.

Proposition 2.8 ([7, Proposition 5.9]). *Let $w \in A_\infty$.*

(i) *Assume $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$. There exists a maximal open interval denoted by $(\widehat{p}_-(L), \widehat{p}_+(L))$ containing $\mathcal{W}_w(p_-(L), p_+(L))$ such that if $\widehat{p}_-(L) < p \leq q < \widehat{p}_+(L)$, then $\{(tL)^m e^{-tL}\}_{t>0} \in \mathcal{O}(L^p(w) - L^q(w))$ and is a bounded set in $\mathcal{L}(L^p(w))$.*

(ii) *Assume $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. There exists a maximal open interval denoted by $(\widehat{q}_-(L), \widehat{q}_+(L))$ containing $\mathcal{W}_w(q_-(L), q_+(L))$ such that if $\widehat{q}_-(L) < p \leq q < \widehat{q}_+(L)$, then $\{\sqrt{t}\nabla(tL)^m e^{-tL}\}_{t>0} \in \mathcal{O}(L^p(w) - L^q(w))$ and is a bounded set in $\mathcal{L}(L^p(w))$.*

(iii) $\widehat{p}_-(L) = \widehat{q}_-(L)$ and $(\widehat{q}_+(L))_w^* \leq \widehat{p}_+(L)$.

In the above proposition $(q_+(L))^*$ and $(\widehat{q}_+(L))_w^*$ are defined as follows: for all $0 < q < \infty$,

$$(2.9) \qquad (q)_w^* := \begin{cases} \frac{qnr_w}{nr_w - q}, & nr_w > q, \\ \infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad (q)^* := \begin{cases} \frac{qn}{n-q}, & n > q, \\ \infty, & \text{otherwise.} \end{cases}$$

The following result, which appears in a more general way in [25, Proposition 4.1], and it is a weighted version of [23, (5.12)] (see also [19]), contains some off-diagonal estimates for the family $\{\mathcal{T}_{t,s}\}_{s,t>0} := \{(e^{-t^2 L} - e^{-(t^2+s^2)L})^M\}_{s,t>0}$, where $M \in \mathbb{N}$.

Proposition 2.10. *Let $w \in A_\infty$, $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, $p \in (\widehat{p}_-(L), \widehat{p}_+(L))$, and let $0 < t, s < \infty$. Given $M \in \mathbb{N}$, for all sets $E_1, E_2 \subset \mathbb{R}^n$ and $f \in L^p(w)$ such that $\mathrm{supp}(f) \subset E_1$, we have that $\{\mathcal{T}_{t,s}\}_{s,t>0} := \{(e^{-t^2 L_w} - e^{-(t^2+s^2)L_w})^M\}_{s,t>0}$ satisfies the following $L^p(w)$-$L^p(w)$ off-diagonal estimates:*

$$(2.11) \qquad \|\mathbf{1}_{E_2} \mathcal{T}_{t,s} f\|_{L^p(w)} \lesssim \left(\frac{s^2}{t^2}\right)^M e^{-c\frac{d(E_1, E_2)^2}{t^2+s^2}} \|f\mathbf{1}_{E_1}\|_{L^p(w)}.$$

In particular,

$$(2.12) \qquad \|\mathcal{T}_{t,s} f\|_{L^p(w)} \lesssim \left(\frac{s^2}{t^2}\right)^M \|f\|_{L^p(w)}.$$

We introduced before the Hardy-Littlewood maximal function and here we present some weighted maximal version which will be used throughout the paper. Given $w \in A_\infty$ and $0 < q < \infty$ we set

$$(2.13) \qquad \mathcal{M}_q^w f(x) := \sup_{B \ni x} \left(\fint_B |f(y)|^q \, dw\right)^{\frac{1}{q}}, \quad 1 \leq q < \infty.$$

Since $w \in A_\infty$ implies that w is a doubling measure (see (1.6)) then $\mathcal{M}_q^w$ is bounded on $L^p(w)$ for every $q < p \leq \infty$ and is bounded from $L^q(w)$ to $L^{q,\infty}(w)$.

The following result contains a Calderón-Zygmund decomposition where the function is split according to the level sets of its gradient and to its norm in weighted Lebesgue spaces.

Lemma 2.14 ([5, Lemma 6.6], [6, Proposition 9.1]). *Let $n \geq 1$, $w \in A_\infty$, $\mu := wdx$, and $r_w < p_0 < \infty$ (with the possibility of taking $p_0 = 1$ if $r_w = 1$). Assume that $h \in \mathcal{S}$, and let $\alpha > 0$. Then, one can find a collection of balls $\{B_i\}_{i \in \mathbb{N}}$ (with radii r_{B_i}), smooth functions b_i, and a function $g \in L^1_{loc}(w)$ such that*

$$h = g + \sum_{i \in \mathbb{N}} b_i, \qquad in \ L^{p_0}(w),$$

and the following properties hold

$$(2.15) \qquad\qquad |\nabla g(x)| \leq C\alpha, \ for \ \mu\text{-}a.e. \ x,$$

$$(2.16) \qquad \operatorname{supp} b_i \subset B_i \quad and \quad \int_{B_i} |\nabla b_i(x)|^{p_0} w(x) dx \leq C\alpha^{p_0} w(B_i),$$

$$(2.17) \qquad\qquad \sum_{i \in \mathbb{N}} w(B_i) \leq \frac{C}{\alpha^{p_0}} \int_{\mathbb{R}^n} |\nabla h(x)|^{p_0} w(x) dx,$$

$$(2.18) \qquad\qquad \sum_{i \in \mathbb{N}} \mathbf{1}_{B_i} \leq N,$$

where C and N depend only on the dimension, the doubling constant of μ, and p_0. In addition, for $1 \leq q < (p_0)^*_w$, where $(p_0)^*_w$ is defined in (2.9), we have

$$(2.19) \qquad\qquad \left(\fint_{B_i} |b_i(x)|^q dw \right)^{\frac{1}{q}} \lesssim \alpha r_{B_i}.$$

The estimates contained in the following auxiliary result follows easily from Hölder's inequality along with the definitions of the A_p and RH_s classes:

Lemma 2.20. *For every $0 < p \leq q < \infty$, every ball $B \subset \mathbb{R}^n$ and every $j \geq 1$, there hold*

$$\left(\fint_{C_j(B)} |f(x)|^p dx \right)^{\frac{1}{p}} \lesssim \left(\fint_{C_j(B)} |f(x)|^q dw \right)^{\frac{1}{q}}, \qquad \forall \, w \in A_{\frac{q}{p}},$$

and

$$\left(\fint_{C_j(B)} |f(x)|^p dw \right)^{\frac{1}{p}} \lesssim \left(\fint_{C_j(B)} |f(x)|^q dx \right)^{\frac{1}{q}}, \qquad \forall \, w \in RH_{\left(\frac{q}{p}\right)'},$$

where the implicit constants are independent of j.

We also need the following off-diagonal estimate on Sobolev spaces, which can be proved as [1, (4.6)]. Nevertheless, we include the proof for the sake of completeness.

Proposition 2.21. *Let $w \in A_\infty$. Given p, q with $\max\{r_w, \widehat{q}_-(L)\} < p \le q < \widehat{q}_+(L)$, $x \in \mathbb{R}^n$, $s > 0$, $\lambda \ge 1$, and H defined in $\mathbb{R}_+^{n+1}$ there hold*

$$\left(\fint_{B(x,\lambda s)} |\nabla e^{-s^2 L} H(\cdot, s)|^q dw \right)^{\frac{1}{q}} \lesssim e^{-c\lambda^2} \sum_{j=1}^\infty e^{-c4^j} \left(\fint_{B(x, 2^{j+1}\lambda s)} |\nabla H(\cdot, s)|^p dw \right)^{\frac{1}{p}},$$

whenever $\nabla H(\cdot, s) \in L^p(w)$ and

$$\left(\fint_{B(x,\lambda s)} |\nabla e^{-s^2 L} H(\cdot, s)|^q dw \right)^{\frac{1}{q}} \lesssim e^{-c\lambda^2} \sum_{j=1}^\infty e^{-c4^j} \left(\fint_{B(x, 2^{j+1}\lambda s)} |\nabla e^{-\frac{s^2}{2} L} H(\cdot, s)|^p dw \right)^{\frac{1}{p}},$$

whenever $H(\cdot, s) \in L^p(w)$.

Proof. It is straightforward to see that it suffices to prove the first estimate. Indeed assuming that, we can write $t = s/\sqrt{2}$ and $\widetilde{H}(\cdot, t) = e^{-t^2 L} H(\cdot, \sqrt{2}t)$ to easily get

$$\left(\fint_{B(x,\lambda s)} |\nabla e^{-s^2 L} H(\cdot, s)|^q dw \right)^{\frac{1}{q}} = \left(\fint_{B(x,\lambda\sqrt{2}t)} |\nabla e^{-t^2 L} \widetilde{H}(\cdot, t)|^q dw \right)^{\frac{1}{q}}$$

$$\lesssim e^{-2c\lambda^2} \sum_{j=1}^\infty e^{-c4^j} \left(\fint_{B(x, 2^{j+1}\lambda\sqrt{2}t)} |\nabla \widetilde{H}(\cdot, t)|^p dw \right)^{\frac{1}{p}}$$

$$= e^{-2c\lambda^2} \sum_{j=1}^\infty e^{-c4^j} \left(\fint_{B(x, 2^{j+1}\lambda s)} |\nabla e^{-\frac{s^2}{2} L} H(\cdot, s)|^p dw \right)^{\frac{1}{p}}.$$

In order to prove the first estimate, fix $(x, s) \in \mathbb{R}_+^{n+1}$ and define $B := B(x, \lambda s)$. Writing $H_s(\cdot) := H(\cdot, s)$, by the conservation property ($e^{-sL} 1 = 1$, $s > 0$) we have that

$$\nabla e^{-s^2 L} H(\cdot, s) = \nabla e^{-s^2 L}(H_s - (H_s)_{4B}),$$

where $(H_s)_{4B} := \fint_{4B} H(\cdot, s) dw$. Consequently, applying the $L^p(w)$-$L^q(w)$ off-diagonal estimates on balls satisfied by $\sqrt{\tau} \nabla e^{-\tau L}$,

$$\left(\fint_{B(x,\lambda s)} |\nabla e^{-s^2 L} H(\cdot, s)|^q dw \right)^{\frac{1}{q}}$$

$$\lesssim \sum_{j\ge1} 2^{j(\theta_2 + \theta_1)} \lambda^{\theta_2} \frac{e^{-c4^j \lambda^2}}{s} \left(\fint_{B(x, 2^{j+1}\lambda s)} |H_s - (H_s)_{4B}|^p dw \right)^{\frac{1}{p}}$$

$$\lesssim e^{-c\lambda^2} \sum_{j\ge1} \frac{e^{-c4^j}}{s} \left(\left(\fint_{2^{j+1}B} |H_s - (H_s)_{2^{j+1}B}|^p dw \right)^{\frac{1}{p}} + \sum_{l=2}^j |(H_s)_{2^{l+1}B} - (H_s)_{2^l B}| \right)$$

$$\lesssim e^{-c\lambda^2} \sum_{j\ge1} \frac{e^{-c4^j}}{s} \sum_{l=2}^j \left(\fint_{2^{l+1}B} |H_s - (H_s)_{2^{l+1}B}|^p dw \right)^{\frac{1}{p}}.$$

Hence, by Poincaré inequality, which holds since $p > r_w$ and thus $w \in A_p$, we conclude that

$$\left(\fint_{B(x,\lambda s)} |\nabla e^{-s^2 L} H(\cdot, s)|^q dw \right)^{\frac{1}{q}} \lesssim e^{-c\lambda^2} \sum_{j\ge1} \frac{e^{-c4^j}}{s} \sum_{l=2}^j \left(\fint_{2^{l+1}B} |H_s - (H_s)_{2^{l+1}B}|^p dw \right)^{\frac{1}{p}}$$

$$\lesssim e^{-c\lambda^2} \sum_{j\geq 1} e^{-c4^j} \sum_{l=2}^{j} 2^{l+1} \left(\fint_{2^{l+1}B} |\nabla H_s|^p dw \right)^{\frac{1}{p}}$$

$$\lesssim e^{-c\lambda^2} \sum_{l\geq 1} e^{-c4^l} \left(\fint_{2^{l+1}B} |\nabla H_s|^p dw \right)^{\frac{1}{p}}.$$

This finishes the proof. $\qquad\square$

3. Proof of Theorem 1.15

In this section, we shall prove Theorem 1.15, which establishes weighted norm estimates for $\widetilde{G}_H$ (defined in (1.14)). To this end, we introduce two results that will be used in that proof.

Theorem 3.1 ([5, Theorem 6.2]). *Let $w \in A_\infty$ be such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$. Given p, $\max\left\{ r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)} \right\} < p < \widehat{p}_+(L)$, there holds*

$$\|L^{\frac{1}{2}}f\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}, \tag{3.2}$$

for every $f \in \mathcal{S}$.

Note that recalling (1.8), and using Propositions 2.7 and 2.8, it is not difficult to see that the range where (3.2) holds contains $\mathcal{W}_w(p_-(L), p_+(L))$ and hence also $\mathcal{W}_w(q_-(L), q_+(L))$.

The following result deals with some vertical square functions. It can be proved with an argument similar to that of [5, Theorem 7.2], or also combining [13, Proof of Proposition 10.1] and [22, Theorem 1.12]). Further details are left to the interested reader.

Lemma 3.3. *Let $w \in A_\infty$ be such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$. If $\widehat{q}_-(L) < p < \widehat{q}_+(L)$ then for all $f \in L_c^\infty(\mathbb{R}^n)$ we have*

$$\left\| \left(\int_0^\infty |t^2 \nabla \sqrt{L} e^{-t^2 L} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p(w)} + \left\| \left(\int_0^\infty |t^3 \nabla L e^{-t^2 L} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

We next define some auxiliary square functions that will be also useful in the proof of Theorem 1.15. In particular, consider

$$G_H f = \left(\int_0^\infty \fint_{B(x,t)} |t\nabla t\sqrt{L} e^{-t^2 L} f(y)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}}$$

and, for $m = 1, 2$,

$$\mathcal{S}_{m,H} f = \left(\int_0^\infty \fint_{B(x,t)} |(t\sqrt{L})^m e^{-t^2 L} f(y)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}}. \tag{3.4}$$

3.1. **Proof of Theorem 1.15.** First of all fix $f \in \mathcal{S}$ and consider $T_t := \nabla e^{-tL}$ and $F(y,t) := t^2 L f(y)$, for $(y,t) \in \mathbb{R}^{n+1}_+$. Then, fix $(x,t) \in \mathbb{R}^{n+1}_+$, and note that for $B := B(x,t)$ and by the second estimate in Proposition 2.21 with $w = 1$ and $\lambda = 1$, we have that, for all $2 < q_0 < q_+(L)$,

$$\left(\fint_{B(x,t)} |T_{t^2} F(y,t)|^{q_0} dy \right)^{\frac{1}{q_0}} \lesssim \sum_{j \geq 1} e^{-c4^j} \left(\fint_{B(x,2^{j+1}t)} |T_{t^2/2} F(y,t)|^2 dy \right)^{\frac{1}{2}}.$$

Besides note that for any constant $c > 0$, $F(y,ct) = c^2 F(y,t)$. Hence, we can apply [26, Proposition 4.2, (b)] to obtain that, for all $0 < q < \frac{q_+(L)}{s_w}$,

$$\left\| \widetilde{\mathrm{G}}_{\mathrm{H}} f \right\|_{L^q(w)} \lesssim \left\| \mathrm{G}_{\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)}.$$

Moreover, by [26, Proposition 4.16, (a)] and [26, Proposition 4.5, (b)], we have that, for all $0 < q < \frac{(p_+(L))^*}{s_w}$

$$(3.5) \qquad \left\| \mathrm{G}_{\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)} \lesssim \left\| \mathcal{S}_{1,\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)} \lesssim \left\| \mathcal{S}_{2,\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)}.$$

Thus, since $\frac{q_+(L)}{s_w} \leq \frac{(p_+(L))^*}{s_w}$, we obtain that, for all $0 < q < \frac{q_+(L)}{s_w}$,

$$(3.6) \qquad \left\| \widetilde{\mathrm{G}}_{\mathrm{H}} f \right\|_{L^q(w)} \lesssim \left\| \mathcal{S}_{2,\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)}.$$

Next, note that $h := \sqrt{L} f$ is in the Hardy space $\mathbb{H}^q_{\nabla L^{-\frac{1}{2}}, p}(w)$ (see [25]) for any $p \in \mathcal{W}_w(q_-(L), q_+(L))$, since, by Theorem 3.1, $h \in L^p(w)$, and $\|\nabla L^{-\frac{1}{2}} h\|_{L^q(w)} = \|\nabla f\|_{L^q(w)} < \infty$. Therefore, by [25, Proposition 9.1] we obtain that, for all

$$\max \left\{ r_w, \frac{n r_w \widehat{p}_-(L)}{n r_w + \widehat{p}_-(L)} \right\} < q < \frac{p_+(L)}{s_w},$$

it holds that

$$(3.7) \qquad \left\| \mathcal{S}_{2,\mathrm{H}}(\sqrt{L} f) \right\|_{L^q(w)} = \|\mathcal{S}_{2,\mathrm{H}} h\|_{L^q(w)} \lesssim \|\nabla L^{-\frac{1}{2}} h\|_{L^q(w)} = \|\nabla f\|_{L^q(w)}.$$

This, together with (3.6), allows us to conclude, for all $\max \left\{ r_w, \frac{n r_w \widehat{q}_-(L)}{n r_w + \widehat{q}_-(L)} \right\} < q < \frac{q_+(L)}{s_w}$, (recall that $\widehat{q}_-(L) = \widehat{p}_-(L)$ and $q_+(L) \leq p_+(L)$),

$$\|\widetilde{\mathrm{G}}_{\mathrm{H}} f\|_{L^q(w)} \lesssim \|\nabla f\|_{L^q(w)}.$$

To finish the proof, note that in view of Lemma 3.3 and Theorem 3.1, we can improve the upper bound of the interval where the above inequality holds up to $\widehat{q}_+(L)$ (assuming that $q_+(L)/s_w < \widehat{q}_+(L)$, otherwise there is nothing to prove). Indeed, we just need to observe that for q such that $q_+(L)/s_w \leq q < \widehat{q}_+(L)$, we have that q falls within the scope of those results. This follows since we assume that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \emptyset$, and by definition, it holds that $\mathcal{W}_w(q_-(L), q_+(L)) \subset \mathcal{W}_w(p_-(L), p_+(L))$. Then, we have that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$ and

$$\max \left\{ r_w, \frac{n r_w \widehat{p}_-(L)}{n r_w + \widehat{p}_-(L)} \right\} \leq \max \{r_w, \widehat{p}_-(L)\} = \max \{r_w, \widehat{q}_-(L)\}$$

$$\leq \max \{r_w, r_w q_-(L)\} = r_w q_-(L) < \frac{q_+(L)}{s_w} \leq q < \widehat{q}_+(L) \leq \widehat{p}_+(L).$$

This completes the prof. $\qquad \square$

Remark 3.8. The result given by Theorem 1.15 is, as far as we know, new even in the unweighted case, that is, when $w \equiv 1$. In that scenario it says that for every q such that $\max\left\{1, \frac{nq_-(L)}{n+q_-(L)}\right\} < q < q_+(L)$ and $f \in \mathcal{S}$, there holds

$$\|\widetilde{\mathrm{G}}_{\mathrm{H}}f\|_{L^q(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^q(\mathbb{R}^n)}.$$

The condition $\mathcal{W}_w(q_-(L), q_+(L)) \neq \varnothing$ always holds when $w \equiv 1$, since $q_-(L) < 2 < q_+(L)$, and, by definition, $(q_-(L), q_+(L)) = \mathcal{W}_w(q_-(L), q_+(L))$.

4. Proof of Theorem 1.10

In this section we prove Theorem 1.10. Given $f \in \mathcal{S}$ we set $u(x,t) := e^{-t\sqrt{L}}f(x)$, for each $(x,t) \in \mathbb{R}^{n+1}_+$. It is well-known that $u(\cdot, t) \in L^2(\mathbb{R}^n)$ uniformly in $t > 0$ since the Poisson semigroup is uniformly bounded on $L^2(\mathbb{R}^n)$ (this latter fact can be seen directly from the subordination formula (4.24) below along with the uniform L^2-boundedness of the heat semigroup, see Section 2). This and Caccioppoli's inequality readily imply that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and also $\mathbb{L}u = 0$ in the weak sense. Using standard holomorphic functional calculus techniques one can also see that $u(\cdot, t) \to f$ in $L^2(\mathbb{R}^n)$ as $t \to 0^+$. Thus we are left with showing (1.12) and to this end, it suffices to individually bound the operators $\mathcal{N}(\partial_t e^{-t\sqrt{L}}f)$ and $\mathcal{N}(\nabla e^{-t\sqrt{L}}f)$.

We first deal with $\mathcal{N}(\partial_t e^{-t\sqrt{L}}f)$. In the unweighted case, in [24], the estimate of this operator relies on the characterization of Hardy spaces associated with L via the non-tangential maximal function associated with the Poisson semigroup and the Riesz transform (characterization established in [19, 20]). Recently, the weighted Hardy spaces have been carefully studied in [10, 11, 23, 25], including various characterization of weighted Hardy spaces via molecules, square functions, non-tangential maximal functions, Riesz transform etc. This weighted Hardy space theory enables us to treat the weighted estimate of $\mathcal{N}(\partial_t e^{-t\sqrt{L}}f)$ by following the path laid down in [24]. More precisely we obtain the following result whose proof is given in Section 4.1:

Proposition 4.1. *Let $w \in A_\infty$ be such that $\mathcal{W}_w(p_-(L), p_+(L)) \neq \varnothing$, and let p be chosen so that $\max\left\{r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)}\right\} < p < \frac{p_+(L)}{s_w}$. Then, for any $f \in \mathcal{S}$,*

$$(4.2) \qquad \|\mathcal{N}(\partial_t e^{-t\sqrt{L}}f)\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}.$$

Regarding the spatial derivatives, the following result establish the desired bound for $\mathcal{N}(\nabla e^{-t\sqrt{L}}f)$:

Proposition 4.3. *Let $w \in A_\infty$ be such that $\mathcal{W}_w(q_-(L), q_+(L)) \neq \varnothing$. Then, for $\max\left\{r_w, \frac{nr_w\widehat{q}_-(L)}{nr_w+\widehat{q}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$ and $f \in \mathcal{S}$, there holds*

$$(4.4) \qquad \|\mathcal{N}(\nabla e^{-t\sqrt{L}}f)\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}.$$

The proof of this result is in Section 4.2. Our method differs from the one in [24] and it is of independent interest. More precisely, when $w \equiv 1$, our proof provides an alternative approach (correcting some flaw) to [24, Proof of Theorem 4.1, Steps II–VIII], in which matters can be essentially reduced to estimate the inhomogeneous vertical square function $\widetilde{\mathrm{G}}_{\mathrm{H}}$ (see Theorem 1.15) along with some similar "homogeneous" conical square function estimates proved in [26] (see (3.5) and (3.7)).

4.1. **Proof of Proposition 4.1.** We first introduce the non-tangential maximal function with respect to the Poisson semigroup defined by

$$(4.5) \qquad \mathcal{N}_{\mathrm{P}}(g)(x) := \sup_{t>0} \left(\fint_{B(x,t)} |e^{-t\sqrt{L}}g(z)|^2 dz \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

The weighted norm inequalities for $\mathcal{N}_{\mathrm{P}}$ can be found in [23, Proposition 7.1 (b)] and [26, Theorem 3.7]:

Lemma 4.6. *Given* $w \in A_\infty$ *such that* $\mathcal{W}_w(p_-(L), p_+(L)) \neq \emptyset$, *then* $\mathcal{N}_{\mathrm{P}}$ *is bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_-(L), (p_+(L))^*)$.

We are now ready to prove Proposition 4.1. First, it was shown in [24, (4.25)] that

$$(4.7) \qquad \mathcal{N}(\partial_t e^{-t\sqrt{L}}f)(x) \lesssim \mathcal{N}_{\mathrm{P}}(\sqrt{L}f)(x), \quad \forall x \in \mathbb{R}^n.$$

Then by Lemma 4.6, we have, for $p \in \mathcal{W}_w(p_-(L), (p_+(L))^*)$,

$$(4.8) \qquad \|\mathcal{N}_{\mathrm{P}}(\sqrt{L}f)\|_{L^p(w)} \lesssim \|\sqrt{L}f\|_{L^p(w)}.$$

Besides, [25, Theorem 1.1] shows that for any $p, q \in \mathcal{W}_w(p_-(L), p_+(L))$, the weighted Lebesgue space $L^p(w)$ and the Hardy space $H^p_{\mathcal{S}_{\mathrm{H}},q}(w)$ (see the definition in [25]) are isomorphic with equivalent norms. This and Lemma 4.6 readily give that for any fixed $q \in \mathcal{W}_w(p_-(L), p_+(L))$,

$$(4.9) \qquad \mathcal{N}_{\mathrm{P}} : H^p_{\mathcal{S}_{\mathrm{H}},q}(w) \to L^p(w), \quad \forall p \in \mathcal{W}_w(p_-(L), p_+(L)).$$

Furthermore, by [23, Theorems 3.9 and 3.11] we also have that

$$(4.10) \qquad \mathcal{N}_{\mathrm{P}} : H^1_{\mathcal{S}_{\mathrm{H}},q}(w) \to L^1(w).$$

Hence, in view of the interpolation result [25, Theorem 5.1], by (4.9) and (4.10), we get

$$\mathcal{N}_{\mathrm{P}} : H^p_{\mathcal{S}_{\mathrm{H}},q}(w) \to L^p(w), \quad 1 \leq p < \frac{p_+(L)}{s_w},$$

and thus

$$(4.11) \qquad \|\mathcal{N}_{\mathrm{P}}(\sqrt{L}f)\|_{L^p(w)} \lesssim \|\sqrt{L}f\|_{H^p_{\mathcal{S}_{\mathrm{H}},q}(w)}, \quad 1 \leq p < \frac{p_+(L)}{s_w}.$$

Next, from the weighted Hardy space Riesz transform characterization (see [25, Propositions 9.1]), it follows that, for all $f \in \mathcal{S}$,

$$(4.12) \qquad \|\sqrt{L}f\|_{H^p_{\mathcal{S}_{\mathrm{H}},q}(w)} \lesssim \|\nabla f\|_{L^p(w)}, \quad \max\left\{ r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w + \widehat{p}_-(L)} \right\} < p < \frac{p_+(L)}{s_w}.$$

Finally, combining (4.7), (4.11), and (4.12), we obtain that, for $\max\left\{ r_w, \frac{nr_w\widehat{p}_-(L)}{nr_w+\widehat{p}_-(L)} \right\} < p < \frac{p_+(L)}{s_w}$,

$$\|\mathcal{N}(\partial_t e^{-t\sqrt{L}}f)\|_{L^p(w)} \lesssim \|\mathcal{N}_{\mathrm{P}}(\sqrt{L}f)\|_{L^p(w)} \lesssim \|\sqrt{L}f\|_{H^p_{\mathcal{S}_{\mathrm{H}},q}(w)} \lesssim \|\nabla f\|_{L^p(w)},$$

for all $f \in \mathcal{S}$. This completes the proof. $\qquad\qquad\square$

4.2. **Proof of Proposition 4.3.** We split the proof into two steps. In **Step 1** we obtain (4.4) for all $\max\{r_w, \widehat{q}_-(L)\} < p < \frac{q_+(L)}{s_w}$; and, in **Step 2** we show that the same estimate holds in a bigger range, namely for all $\max\left\{r_w, \frac{nr_w\widehat{q}_-(L)}{nr_w+\widehat{q}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$.

The following claims are common to both steps, so we start by proving them.

Claim 4.13. *Under the hypothesis of Proposition 4.3, for any* $\max\{r_w, \widehat{q}_-(L)\} < q_0 < q_+(L)/s_w$, *there holds*

$$\left(\fint_{B(x,t)} |\nabla e^{-t^2L}f|^2 dz\right)^{\frac{1}{2}} \leq \sum_{l\geq 1} e^{-c4^l} \left(\fint_{B(x,2^{l+1}t)} |\nabla S_{t/\sqrt{2}}f|^{q_0} dw\right)^{\frac{1}{q_0}},$$

where $S_{t/\sqrt{2}}$ *could be equal to* $e^{-\frac{t^2}{4}L}$ *or the identity.*

Claim 4.14. *Under the hypothesis of Proposition 4.3, for any* $\max\{r_w, \widehat{q}_-(L)\} < q_0 < q_+(L)/s_w$, *there holds*

$$\int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z)|^2 dz\right)^{\frac{1}{2}} \frac{du}{u}$$
$$\lesssim \sum_{l\geq 1} e^{-c2^l} \left(\fint_{B(x,2^{l+1}t)} |\widetilde{\mathbb{G}}_{\mathrm{H}}f|^{q_0} dw\right)^{\frac{1}{q_0}}.$$

Claim 4.15. *Under the hypothesis of Proposition 4.3, there holds*

$$\int_{\frac{1}{4}}^\infty e^{-u}u^{\frac{1}{2}} \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z)|^2 dz\right)^{\frac{1}{2}} \frac{du}{u} \lesssim \int_{\frac{1}{4}}^\infty u\, e^{-u}\mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}f)(x)\frac{du}{u},$$

where

$$\mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}f)(x) := \left(\fint_{B(x,2\sqrt{u}s)} \int_0^\infty |s\nabla s\sqrt{L}e^{-s^2L}\sqrt{L}f(z)|^2 \frac{dz\,ds}{s^{n+1}}\right)^{\frac{1}{2}}.$$

In order to prove these claims, fix p_0 and q such that $q_-(L) < p_0 < 2$, $r_w p_0 < q_+(L)/s_w$, and $\max\{q_0, r_w p_0\} < q < q_+(L)/s_w$.

Proof of Claim 4.13. First, apply the second estimate in Proposition 2.21 with $w \equiv 1$, $\lambda = 1$, $s = t$, $q = 2$, $p = p_0$, and $H(z,t) = f(z)$ for all $(z,t) \in \mathbb{R}_+^{n+1}$. Then

$$(4.16) \quad \left(\fint_{B(x,t)} |\nabla e^{-t^2L}f(z)|^2 dz\right)^{\frac{1}{2}} \lesssim \sum_{j\geq 1} e^{-c4^j} \left(\fint_{B(x,2^{j+1}t)} |\nabla e^{-\frac{t^2}{2}L}f(z)|^{p_0} dz\right)^{\frac{1}{p_0}}.$$

Besides, note that since $w \in A_{\frac{q}{p_0}}$, by Lemma 2.20, we have

$$(4.17) \quad \left(\fint_{B(x,2^{j+1}t)} |\nabla e^{-\frac{t^2}{2}L}f(z)|^{p_0} dz\right)^{\frac{1}{p_0}} \lesssim \left(\fint_{B(x,2^{j+1}t)} |\nabla e^{-\frac{t^2}{2}L}f(z)|^q dw\right)^{\frac{1}{q}}.$$

Also, since $\max\{r_w, \widehat{q}_-(L)\} < q_0 < q < q_+(L)/s_w \leq \widehat{q}_+(L)$, again by Proposition 2.21 (we use the first or the second estimate depending on whether $S_{t/\sqrt{2}}$ is the identity

or $e^{-\frac{t^2}{4}L}$ respectively) with $\lambda = 2^{j+1}\sqrt{2}$ and $s = t/\sqrt{2}$, we obtain

$$\left(\fint_{B(x,2^{j+1}t)} |\nabla e^{-\frac{t^2}{2}L}f(z)|^q dw\right)^{\frac{1}{q}} \lesssim e^{-c4^j}\sum_{i\geq 1} e^{-c4^i}\left(\fint_{B(x,2^{j+i+2}t)}|\nabla S_{t/\sqrt{2}}f(z)|^{q_0}dw\right)^{\frac{1}{q_0}}.$$

This, (4.16) and (4.17) imply

$$\left(\fint_{B(x,t)} |\nabla e^{-t^2L}f(z)|^2 dz\right)^{\frac{1}{2}} \lesssim \sum_{j\geq 1} e^{-c4^j}\sum_{i\geq 1} e^{-c4^i}\left(\fint_{B(x,2^{j+i+2}t)}|\nabla S_{t/\sqrt{2}}f(z)|^{q_0}dw\right)^{\frac{1}{q_0}}$$

$$\lesssim \sum_{l\geq 1} e^{-c2^l}\left(\fint_{B(x,2^{l+1}t)}|\nabla S_{t/\sqrt{2}}f(z)|^{q_0}dw\right)^{\frac{1}{q_0}},$$

which gives the desired estimate. $\qquad\square$

Proof of Claim 4.14. We proceed as in Claim 4.13. First apply the second estimate in Proposition 2.21 with $w \equiv 1$, $\lambda = \sqrt{2}$, $s = t/\sqrt{2}$, $q = 2$, and $p = p_0$. Next, apply Lemma 2.20; and finally, apply the first estimate in Proposition 2.21, with $\lambda = 2^{j+2}$, and $s = t/2$, to obtain for every $0 < u < 1/4$

$$(4.18)\quad \left(\fint_{B(x,t)}|\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z)|^2 dz\right)^{\frac{1}{2}}$$

$$= \left(\fint_{B(x,t)}|\nabla e^{-\frac{t^2}{2}L}(e^{-\left(\frac{t^2}{4u}-\frac{t^2}{2}\right)L} - e^{-\frac{t^2}{2}L})f(z)|^2 dz\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j}\left(\fint_{B(x,2^{j+1}t)}|\nabla e^{-\frac{t^2}{4}L}(e^{-\left(\frac{t^2}{4u}-\frac{t^2}{2}\right)L} - e^{-\frac{t^2}{2}L})f(z)|^{p_0} dz\right)^{\frac{1}{p_0}}$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j}\left(\fint_{B(x,2^{j+1}t)}|\nabla e^{-\frac{t^2}{4}L}(e^{-\left(\frac{t^2}{4u}-\frac{t^2}{2}\right)L} - e^{-\frac{t^2}{2}L})f(z)|^q dw\right)^{\frac{1}{q}}$$

$$\lesssim \sum_{j\geq 1} e^{-c4^j}\sum_{i\geq 1} e^{-c4^i}\left(\fint_{B(x,2^{j+i+2}t)}|\nabla(e^{-\left(\frac{t^2}{4u}-\frac{t^2}{2}\right)L} - e^{-\frac{t^2}{2}L})f(z)|^{q_0} dw\right)^{\frac{1}{q_0}}.$$

Moreover, for $0 < u < 1/4$,

$$(4.19)\quad \left|\nabla\left(e^{-(\frac{1}{4u}-\frac{1}{2})t^2L} - e^{-\frac{t^2}{2}L}\right)f(z)\right| \leq \int_{\frac{t}{\sqrt{2}}}^{t\sqrt{\frac{1}{4u}-\frac{1}{2}}}|\partial_s\nabla e^{-s^2L}f(z)|\,ds$$

$$\lesssim \int_{\frac{t}{\sqrt{2}}}^{t\sqrt{\frac{1}{4u}}}|s^2\nabla Le^{-s^2L}f(z)|\frac{ds}{s} \lesssim \left(\int_{\frac{t}{2}}^{\infty}|s^2\nabla Le^{-s^2L}f(z)|^2\frac{ds}{s}\right)^{\frac{1}{2}}\left(\log u^{-1}\right)^{\frac{1}{2}}$$

$$\lesssim \left(\log u^{-1}\right)^{\frac{1}{2}}\widetilde{G}_{H,t}(f)(z),$$

where $\widetilde{G}_{H,t}$ is defined as

$$(4.20)\quad \widetilde{G}_{H,t}f(x) := \left(\int_{\frac{t}{2}}^{\infty}|s^2\nabla Le^{-s^2L}f(z)|^2\frac{ds}{s}\right)^{\frac{1}{2}}.$$

Therefore, by (4.18) and (4.19),

$$(4.21) \quad \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2 L})f(z)|^2 dz \right)^{\frac{1}{2}}$$

$$\lesssim \sum_{j \geq 1} \sum_{i \geq 1} e^{-c2^{j+i+1}} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\log u^{-1} \right)^{\frac{1}{2}} \frac{du}{u} \left(\fint_{B(x,2^{j+i+2}t)} |\widetilde{G}_{H,t}f(z)|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim \sum_{l \geq 1} e^{-c2^l} \left(\fint_{B(x,2^{l+1}t)} |\widetilde{G}_{H,t}f(z)|^{q_0} dw \right)^{\frac{1}{q_0}}.$$

This completes the proof of the present claim. □

Proof of Claim 4.15. First, note that for $1/4 \leq u < \infty$,

$$\left| \nabla \left(e^{-\frac{t^2}{4u}L} - e^{-t^2 L} \right) f(z) \right| \leq \int_{\frac{t}{2\sqrt{u}}}^{\frac{t}{\sqrt{2}}} \left| \partial_s \nabla e^{-s^2 L} f(z) \right| ds$$

$$\lesssim u^{\frac{1}{2}} \left(\int_{\frac{t}{2\sqrt{u}}}^{\frac{t}{\sqrt{2}}} \left| s \nabla s \sqrt{L} e^{-s^2 L} \sqrt{L} f(z) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}}.$$

Thus,

$$\int_{\frac{1}{4}}^{\infty} e^{-u} u^{\frac{1}{2}} \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2 L})f(z)|^2 dz \right)^{\frac{1}{2}} \frac{du}{u}$$

$$\lesssim \int_{\frac{1}{4}}^{\infty} u \, e^{-u} \left(\fint_{B(x,2\sqrt{u}s)} \int_0^{\infty} \left| s \nabla s \sqrt{L} e^{-s^2 L} \sqrt{L} f(z) \right|^2 \frac{dz \, ds}{s^{n+1}} \right)^{\frac{1}{2}} \frac{du}{u}$$

$$= \int_{\frac{1}{4}}^{\infty} u \, e^{-u} G_H^{2\sqrt{u}} (\sqrt{L}f)(x) \frac{du}{u},$$

which is the desired estimate. □

Once we have proved the claims, we can start discussing the two cases into which we split the proof of Proposition 4.3.

Step 1: $\max\{r_w, \widehat{q}_-(L)\} < p < \frac{q_+(L)}{s_w}$.

First of all, note that proceeding similarly as in [24, (4.25)] one can show that for every fixed $x \in \mathbb{R}^n$

$$(4.22) \quad \mathcal{N}(\nabla e^{-t\sqrt{L}} f)(x) \lesssim \sup_{t>0} \left(\fint_{B(x,t)} |\nabla e^{-t\sqrt{L}} f(z)|^2 dz \right)^{\frac{1}{2}}$$

$$\leq \sup_{t>0} \left(\fint_{B(x,t)} |\nabla e^{-t^2 L} f(z)|^2 dz \right)^{\frac{1}{2}}$$

$$+ \sup_{t>0} \left(\fint_{B(x,t)} |\nabla(e^{-t\sqrt{L}} - e^{-t^2 L})f(z)|^2 dz \right)^{\frac{1}{2}}$$

$$=: I + II.$$

Since $\max\{r_w, \widehat{q}_-(L)\} < p < \frac{q_+(L)}{s_w}$, we can choose q_0 so that $\max\{r_w, \widehat{q}_-(L)\} < q_0 < p < \frac{q_+(L)}{s_w}$. Hence by Claim 4.13 with S_t equal to the identity, we obtain

$$(4.23) \qquad I \lesssim \sup_{t>0} \sum_{l \geq 1} e^{-c2^l} \left(\fint_{B(x,2^{l+1}t)} |\nabla f(z)|^{q_0} dw \right)^{\frac{1}{q_0}} \lesssim \mathcal{M}_{q_0}^w(\nabla f)(x).$$

On the other hand, note that applying the following subordination formula:

$$(4.24) \qquad e^{-t\sqrt{L}}f = C \int_0^\infty u^{\frac{1}{2}} e^{-u} e^{-\frac{t^2}{4u}L} f \frac{du}{u},$$

and Minkowski's integral inequality, we have

$$(4.25) \quad II \lesssim \sup_{t>0} \int_0^{\frac{1}{4}} u^{\frac{1}{2}} \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z)|^2 dz \right)^{\frac{1}{2}} \frac{du}{u}$$

$$+ \sup_{t>0} \int_{\frac{1}{4}}^\infty e^{-u} u^{\frac{1}{2}} \left(\fint_{B(x,t)} |\nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z)|^2 dz \right)^{\frac{1}{2}} \frac{du}{u} =: II_1 + II_2.$$

In order to estimate II_1, we apply Claim 4.14:

$$(4.26) \quad II_1 \lesssim \sup_{t>0} \sum_{l \geq 1} e^{-c2^l} \left(\fint_{B(x,2^{l+1}t)} |\widetilde{G}_{H,t}f(z)|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim \sup_{t>0} \sum_{l \geq 1} e^{-c2^l} \left(\fint_{B(x,2^{l+1}t)} |\widetilde{G}_H f(z)|^{q_0} dw \right)^{\frac{1}{q_0}} \lesssim \mathcal{M}_{q_0}^w(\widetilde{G}_H f)(x),$$

where $\widetilde{G}_H$ is the inhomogeneous vertical square function defined in (1.14). On the other hand, for $1/4 \leq u < \infty$, one can see that

$$\left| \nabla(e^{-\frac{t^2}{4u}L} - e^{-t^2L})f(z) \right| \leq \int_{\frac{t}{2\sqrt{u}}}^{\frac{t}{\sqrt{2}}} \left| \partial_s \nabla e^{-s^2L}f(z) \right| ds$$

$$\lesssim \left(\int_{\frac{t}{2\sqrt{u}}}^{\frac{t}{\sqrt{2}}} \left| s\nabla s\sqrt{L}e^{-s^2L}\sqrt{L}f(z) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \left(\log u^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

Next, applying Claim 4.15 we get

$$II_2 \lesssim \int_{\frac{1}{4}}^\infty u\, e^{-u} G_H^{2\sqrt{u}}(\sqrt{L}f)(x) \frac{du}{u}.$$

This, (4.22), (4.23), (4.25), and (4.26) imply that, for $\max\{r_w, \widehat{q}_-(L)\} < q_0 < p$,
(4.27)

$$\mathcal{N}(\nabla e^{-t\sqrt{L}}f)(x) \leq \mathcal{M}_{q_0}^w(\nabla f)(x) + \mathcal{M}_{q_0}^w(\widetilde{G}_H f)(x) + \int_{\frac{1}{4}}^\infty u\, e^{-u} G_H^{2\sqrt{u}}(\sqrt{L}f)(x) \frac{du}{u}.$$

Consequently, for all $\max\{r_w, \widehat{q}_-(L)\} < p < \frac{q_+}{s_w}$, by the boundedness of $\mathcal{M}_{q_0}^w$ on $L^p(w)$, recall that $p/q_0 > 1$, and change of angles (see [22, Proposition 3.2]), we conclude that

$$\|\mathcal{N}(\nabla e^{-t\sqrt{L}}f)\|_{L^p(w)}$$

$$\lesssim \|\mathcal{M}_{q_0}^w(\nabla f)\|_{L^p(w)} + \|\mathcal{M}_{q_0}^w(\widetilde{\mathrm{G}}_{\mathrm{H}}f)\|_{L^p(w)} + \int_{\frac{1}{4}}^{\infty} u\, e^{-u} \|\mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}f)\|_{L^p(w)}\frac{du}{u}$$

$$\lesssim \|\nabla f\|_{L^p(w)} + \|\widetilde{\mathrm{G}}_{\mathrm{H}}f\|_{L^p(w)} + \|\mathrm{G}_{\mathrm{H}}(\sqrt{L}f)\|_{L^p(w)}$$

$$\lesssim \|\nabla f\|_{L^p(w)},$$

where the last inequality follows from Theorem 1.15, (3.5), and (3.7). This completes the proof of **Step 1**.

Step 2: $\max\left\{r_w, \frac{nr_w\widehat{q}_-(L)}{nr_w+\widehat{q}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$.

We start by observing that **Step 1** leads to

$$(4.28) \qquad \mathcal{N}(\nabla e^{-t\sqrt{L}}) : \dot{W}^{1,q}(w) \to L^q(w), \quad \max\{r_w, \widehat{q}_-(L)\} < q < \frac{q_+(L)}{s_w},$$

where $\dot{W}^{1,q}(w) = \overline{\{f \in \mathcal{S} : \nabla f \in L^q(w)\}}$, with the closure taken with respect to the seminorm $f \to \|\nabla f\|_{L^q(w)}$. With this in hand, by interpolation (see [9]), we can conclude (4.4) for all $\max\left\{r_w, \frac{r_w n\widehat{q}_-(L)}{r_w n+\widehat{q}_-(L)}\right\} < p < \frac{q_+(L)}{s_w}$ provided that for every $\widetilde{p}$ and $r_0 > r_w$ such that $\widehat{q}_-(L) < \widetilde{p} < q_+(L)/s_w$, $r_w q_-(L) < r_0 q_-(L) < q_+(L)/s_w$, and $\widetilde{q} := \max\left\{r_0, \frac{nr_0\widetilde{p}}{nr_0+\widetilde{p}}\right\}$, we show that

$$(4.29) \qquad \mathcal{N}(\nabla e^{-t\sqrt{L}}) : \dot{W}^{1,\widetilde{q}}(w) \to L^{\widetilde{q},\infty}(w).$$

In order to prove (4.29) fix $f \in \mathcal{S}$, and fix $\widetilde{p}$ and r_0 satisfying the above restrictions. Furthermore, fix q_2 and p_1 so that $q_-(L) < q_2 < 2$, $r_0 q_2 < \frac{q_+(L)}{s_w}$, and $\max\{\widetilde{p}, r_0 q_2\} < p_1 < q_+(L)/s_w$. Note that in particular

$$(4.30) \qquad w \in A_{\frac{p_1}{q_2}} \cap RH_{\left(\frac{q_+(L)}{p_1}\right)'}, \quad \text{and} \quad p_1 > \widetilde{q}.$$

Moreover, note by Claim 4.13 with $S_t = e^{-\frac{t^2}{4}L}$, and Claims 4.14 and 4.15, we obtain that, for any q_0 such that $\max\{r_w, \widehat{q}_-(L)\} < q_0 < \max\{r_0, \widetilde{p}\} < q_+(L)/s_w$ and any function $h \in L^2(\mathbb{R}^n)$,

$$(4.31) \quad \mathcal{N}(\nabla e^{-t\sqrt{L}}h)(x) \lesssim \sum_{l\geq 1} e^{-c2^l} \sup_{t>0}\left(\fint_{B(x,2^{l+2}t)}|\nabla e^{-t^2L}h(z)|^{q_0}dw\right)^{\frac{1}{q_0}}$$

$$+ \sum_{l\geq 1} e^{-c2^l} \sup_{t>0}\left(\fint_{B(x,2^{l+2}t)}|\widetilde{\mathrm{G}}_{\mathrm{H},t}h(z)|^{q_0}dw\right)^{\frac{1}{q_0}} + \int_{\frac{1}{4}}^{\infty} u\, e^{-u}\mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}h)(x)\frac{du}{u}$$

$$=: \sum_{l\geq 1} e^{-c2^l}\mathfrak{D}_{1,l}h(x) + \sum_{l\geq 1} e^{-c2^l}\mathfrak{D}_{2,l}h(x) + \mathfrak{D}_3 h(x).$$

Next we observe that to obtain (4.29) it suffices to prove for every $\alpha > 0$ that

$$(4.32) \qquad w\left(\left\{x \in \mathbb{R}^n : \mathcal{N}(\nabla e^{-t\sqrt{L}}f)(x) > \alpha\right\}\right) \lesssim \frac{1}{\alpha^{\widetilde{q}}}\int_{\mathbb{R}^n}|\nabla f(x)|^{\widetilde{q}}dw.$$

In order to obtain this inequality, fix $\alpha > 0$ and consider the Calderón-Zygmund decomposition for the function f at height α given by Lemma 2.14 with $p_0 = \widetilde{q}$. Let

$\{B_i\}_i$ be the corresponding collection of balls, and define, for $M \in \mathbb{N}$ arbitrarily large,

$$\mathcal{B}_{r_{B_i}} := (I - e^{-r_{B_i}^2 L})^M, \qquad \mathcal{A}_{r_{B_i}} := I - \mathcal{B}_{r_{B_i}} = \sum_{k=1}^{M} C_{k,M} e^{-kr_{B_i}^2 L}.$$

Hence,

$$(4.33) \qquad f = g + \sum_i \mathcal{A}_{r_{B_i}} b_i + \sum_i \mathcal{B}_{r_{B_i}} b_i =: g + \widetilde{b} + \sum_i \widehat{b}_i =: g + \widetilde{b} + \widehat{b}.$$

and then
(4.34)

$$
\begin{aligned}
w\big(\{x \in \mathbb{R}^n : \mathcal{N}\big(\nabla e^{-t\sqrt{L}}f\big)(x) > \alpha\}\big) &\leq w\big(\{x \in \mathbb{R}^n : \mathcal{N}\big(\nabla e^{-t\sqrt{L}}g\big)(x) > \alpha/5\}\big) \\
&+ w\big(\{x \in \mathbb{R}^n : \mathcal{N}\big(\nabla e^{-t\sqrt{L}}\widetilde{b}\big)(x)\alpha/5\}\big) \\
&+ \sum_{m=1}^{2} w\big(\{x \in \mathbb{R}^n : C\sum_{l\geq 1} e^{-c2^l}\mathfrak{D}_{m,l}\widehat{b}(x) > \alpha/5\}\big) \\
&+ w\big(\{x \in \mathbb{R}^n : C\mathfrak{D}_3\widehat{b}(x) > \alpha/5\}\big) \\
&=: \mathcal{I} + \mathcal{II} + \sum_{m=1}^{2} \mathcal{III}_m + \mathcal{IV}.
\end{aligned}
$$

Before starting with the estimate of the above terms, we make a couple of observations for later use. First, take $1 < \mathfrak{q} < \infty$ and $h \in L^{\mathfrak{q}'}(w)$ such that $\|h\|_{L^{\mathfrak{q}'}(w)} = 1$ and recall the definition of $\mathcal{M}^w$ in (2.13). Then, using a Kolmogorov type inequality (see [16, Exercise 2.1.5]), the fact that $\mathcal{M}^w$ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ since $w \in A_\infty$ and hence it gives rise to a doubling measure, and (2.17), we have that

$$(4.35) \quad \Big(\sum_{i\in\mathbb{N}} \int_{B_i} \mathcal{M}^w(|h|^{\mathfrak{q}'})(x)^{\frac{1}{\mathfrak{q}'}} w(x)dx\Big)^{\mathfrak{q}} \lesssim \Big(\int_{\cup_{i\in\mathbb{N}}B_i} \mathcal{M}^w(|h|^{\mathfrak{q}'})(x)^{\frac{1}{\mathfrak{q}'}} w(x)dx\Big)^{\mathfrak{q}}$$

$$\lesssim w(\cup_{i\in\mathbb{N}}B_i)\|h\|_{L^{\mathfrak{q}'}(w)}^{\mathfrak{q}} \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f(x)|^{\widetilde{q}} w(x)dx.$$

Second, note that for $p_2 := \max\{r_0, \widetilde{p}\}$, we have that $q_0 < p_2$. Assuming momentarily that $p_2 < (\widetilde{q})_w^*$, by (2.19), we get that

$$(4.36) \qquad \Big(\fint_{B_i} |b_i(x)|^{p_2} dw\Big)^{\frac{1}{p_2}} \lesssim \alpha r_{B_i}.$$

In order to see that $p_2 < (\widetilde{q})_w^*$, we first consider the case $p_2 = r_0$. Then, $\widetilde{q} = \max\left\{r_0, \frac{nr_0\widetilde{p}}{nr_0+\widetilde{p}}\right\} = r_0$, and thus $p_2 = r_0 < (r_0)_w^* = (\widetilde{q})_w^*$. On the other had, if $p_2 = \widetilde{p}$, we may assume that $nr_w > \widetilde{q}$ —otherwise we trivially have $\widetilde{p} = p_2 < (\widetilde{q})_w^* = \infty$. Besides,

$$\frac{1}{(\widetilde{q})_w^*} = \frac{1}{\widetilde{q}} - \frac{1}{nr_w} < \frac{nr_0 + \widetilde{p}}{nr_0\widetilde{p}} - \frac{1}{nr_w} = \frac{1}{\widetilde{p}} + \frac{1}{nr_0} - \frac{1}{nr_w}$$

$$= \frac{1}{\widetilde{p}} + \frac{1}{n}\left(\frac{1}{r_0} - \frac{1}{r_w}\right) \leq \frac{1}{\widetilde{p}} = \frac{1}{p_2}.$$

Now, we are ready to estimate the terms in (4.34). In order to estimate $\mathcal{I}$, first recall that $\widetilde{q} < p_1$ and $\max\{r_w, \widehat{q}_-(L)\} < p_1 < q_+(L)/s_w$ (see (4.30)). Then, by Chebyshev's inequality, (4.28) and Lemma 2.14, we get

$$(4.37) \qquad \mathcal{I} \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} \mathcal{N}\big(\nabla e^{-t\sqrt{L}}g\big)^{p_1} dw \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g|^{p_1} dw \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

Next we estimate $\mathcal{II}$. Consider p_2 defined as in (4.36) and apply Chebyshev's inequality and (4.28) (recall that $\max\{r_w, \widehat{q}_-(L)\} < p_2 < p_1 < q_+(L)/s_w$). Thus

$$(4.38) \qquad \mathcal{II} \lesssim \frac{1}{\alpha^{p_2}} \int_{\mathbb{R}^n} \mathcal{N}\big(\nabla e^{-t\sqrt{L}}\widetilde{b}\big)^{p_2} dw \lesssim \frac{1}{\alpha^{p_2}} \int_{\mathbb{R}^n} |\nabla \widetilde{b}|^{p_2} dw.$$

We claim that

$$(4.39) \qquad \int_{\mathbb{R}^n} |\nabla \widetilde{b}|^{p_2} dw \lesssim \frac{\alpha^{p_2}}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

Indeed, by the definition of $\mathcal{A}_{r_{B_i}}$,

$$\int_{\mathbb{R}^n} |\nabla \widetilde{b}|^{p_2} dw \lesssim \int_{\mathbb{R}^n} \left| \nabla \Big(\sum_{i \in \mathbb{N}} \sum_{k=1}^{M} C_{k,M} e^{-kr_{B_i}^2 L} b_i \Big) \right|^{p_2} dw$$

$$\lesssim \sup_{\|h\|_{L^{p_2'}(w)}=1} \left(\sum_{k=1}^{M} \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} \left| \sqrt{k} r_{B_i} \nabla e^{-kr_{B_i}^2 L} \Big(\frac{b_i}{r_{B_i}} \Big) \right| |h| dw \right)^{p_2}.$$

Besides, using that $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$ satisfies $L^{p_2}(w)$-$L^{p_2}(w)$ off-diagonal estimates on balls; and by (1.6) and (4.36), we have

$$\int_{\mathbb{R}^n} \left| \sqrt{k} r_{B_i} \nabla e^{-kr_{B_i}^2 L} \Big(\frac{b_i}{r_{B_i}} \Big) \right| |h| dw$$

$$\lesssim \sum_{j \geq 1} w(2^{j+1} B_i) \Big(\fint_{C_j(B_i)} \Big| \sqrt{k} r_{B_i} \nabla e^{-kr_{B_i}^2 L} \Big(\frac{b_i}{r_{B_i}} \Big) \Big|^{p_2} dw \Big)^{\frac{1}{p_2}} \Big(\fint_{C_j(B_i)} |h|^{p_2'} dw \Big)^{\frac{1}{p_2'}}$$

$$\lesssim \sum_{j \geq 1} e^{-c 4^j} w(B_i) \Big(\fint_{B_i} \Big| \frac{b_i(x)}{r_{B_i}} \Big|^{p_2} dw \Big)^{\frac{1}{p_2}} \inf_{x \in B_i} \mathcal{M}^w(|h|^{p_2'})(x)^{\frac{1}{p_2'}}$$

$$\lesssim \alpha \int_{B_i} \mathcal{M}^w(|h|^{p_2'})^{\frac{1}{p_2'}} dw.$$

Hence, by (4.35) with $\mathfrak{q} = p_2$, we have

$$\int_{\mathbb{R}^n} |\nabla \widetilde{b}|^{p_2} dw \lesssim \alpha^{p_2} \sup_{\|h\|_{L^{p_2'}(w)}=1} \Big(\int_{B_i} \mathcal{M}^w(|h|^{p_2'})^{\frac{1}{p_2'}} dw \Big)^{p_2} \lesssim \frac{\alpha^{p_2}}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f(x)|^{\widetilde{q}} dw.$$

Consequently, (4.39) holds. In view of (4.38), that implies

$$(4.40) \qquad \mathcal{II} \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f(x)|^{\widetilde{q}} dw.$$

We continue by estimating $\mathcal{IV}$. To this end, write $b = \sum_i b_i$ so that $\widehat{b} = b - \widetilde{b}$, and note that

$$\mathcal{IV} \le w\big(\big\{x \in \mathbb{R}^n : C\mathfrak{O}_3 b(x) > \alpha/10\big\}\big)$$
$$+ w\big(\big\{x \in \mathbb{R}^n : C\mathfrak{O}_3 \widetilde{b}(x) > \alpha/10\big)\big\} =: \mathcal{IV}_1 + \mathcal{IV}_2.$$

In order to estimate $\mathcal{IV}_1$ apply Chebyshev's inequality, Minkowski's integral inequality, change of angles ([22, Proposition 3.2]), (3.5), and (3.7), to get

$$\mathcal{IV}_1 \lesssim \frac{1}{\alpha^{\widetilde{q}}} \bigg(\int_{\frac{1}{4}}^{\infty} e^{-cu} \big\| \mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}\, b) \big\|_{L^{\widetilde{q}}(w)} \frac{du}{u} \bigg)^{\widetilde{q}}$$
$$\lesssim \frac{1}{\alpha^{\widetilde{q}}} \big\| \mathrm{G}_{\mathrm{H}}(\sqrt{L}\, b) \big\|_{L^{\widetilde{q}}(w)}^{\widetilde{q}} \lesssim \frac{1}{\alpha^{\widetilde{q}}} \|\nabla b\|_{L^{\widetilde{q}}(w)}^{\widetilde{q}}.$$

Besides, note that by (2.16) and (2.17)

$$\|\nabla b\|_{L^{\widetilde{q}}(w)}^{\widetilde{q}} \lesssim \sum_i \int_{B_i} |\nabla b_i|^{\widetilde{q}} dw \lesssim \alpha^{\widetilde{q}} \sum_i w(B_i) \lesssim \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

Hence,

$$\mathcal{IV}_1 \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

As for the estimate of $\mathcal{IV}_2$, apply again Chebyshev's inequality, Minkowski's integral inequality, change of angles, (3.5), and (3.7), to get

$$\mathcal{IV}_2 \lesssim \frac{1}{\alpha^{p_2}} \bigg(\int_{\frac{1}{4}}^{\infty} e^{-cu} \big\| \mathrm{G}_{\mathrm{H}}^{2\sqrt{u}}(\sqrt{L}\, \widetilde{b}) \big\|_{L^{p_2}(w)} \frac{du}{u} \bigg)^{p_2}$$
$$\lesssim \frac{1}{\alpha^{p_1}} \big\| \mathrm{G}_{\mathrm{H}}(\sqrt{L}\, \widetilde{b}) \big\|_{L^{p_2}(w)}^{p_2} \lesssim \frac{1}{\alpha^{p_2}} \|\nabla \widetilde{b}\|_{L^{p_2}(w)}^{p_2}$$

Thus, by (4.39),

$$\mathcal{IV}_2 \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

Collecting the estimates for $\mathcal{IV}_1$ and $\mathcal{IV}_2$ we conclude that

$$(4.41) \qquad \mathcal{IV} \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

It remains to estimate $\mathcal{III}_m$, for $m = 1, 2$. Note that by (1.6) and (2.17),

(4.42)

$$\mathcal{III}_m \le w\Big(\bigcup_i 16 B_i \Big) + w\Big(\Big\{ x \in \mathbb{R}^n \setminus \cup_i 16 B_i : C \sum_{l \ge 1} e^{-c2^l} \mathfrak{O}_{m,l} \widehat{b}(x) > \alpha/5 \Big\} \Big)$$

$$\lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw + \sum_{l \ge 1} w\Big(\Big\{ x \in \mathbb{R}^n \setminus \cup_i 16 B_i : \mathfrak{O}_{m,l} \widehat{b}(x) > \frac{e^{c2^l} \alpha}{C2^l} \Big\} \Big)$$

$$=: \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw + \sum_{l \ge 1} \mathcal{III}_{m,l}.$$

Applying Chebyshev's inequality, duality, and Hölder's inequality, it follows that

$$(4.43) \quad \mathcal{III}_{m,l} \lesssim \frac{e^{-c2^l}}{\alpha^{p_2}} \int_{\mathbb{R}^n \setminus \cup_i 16 B_i} |\mathfrak{O}_{m,l} \widehat{b}|^{p_2} dw$$

$$\lesssim \frac{e^{-c2^l}}{\alpha^{p_2}} \left(\sup_{\|h\|_{L^{p_2'}(w)}=1} \sum_i \sum_{j\geq 4} \left(\int_{C_j(B_i)} |\mathfrak{D}_{m,l}\widehat{b}_i|^{p_2} dw \right)^{\frac{1}{p_2}} \|h\mathbf{1}_{C_j(B_i)}\|_{L^{p_2'}(w)} \right)^{p_2}$$

$$=: \frac{e^{-c2^l}}{\alpha^{p_2}} \left(\sup_{\|h\|_{L^{p_2'}(w)}=1} \sum_i \sum_{j\geq 4} I_{m,l}^{ij} \|h\mathbf{1}_{C_j(B_i)}\|_{L^{p_2'}(w)} \right)^{p_2}.$$

Claim 4.44. *There exist* $\Theta, M_0 > 1$ *such that for every* $2M > \max\{\Theta, M_0\}$

$$I_{m,l}^{ij} \leq C 2^{lC_M} \alpha w(2^{j+1}B_i)^{\frac{1}{p_2}} 2^{-j(2M-\Theta)}, \quad m = 1, 2.$$

Assuming this momentarily, in view of (4.43), for $2M > \max\{\Theta + nr_w, M_0\}$, and by (4.35) with $\mathfrak{q} = p_2$ we get, for $m = 1, 2$,

$$\mathcal{III}_{m,l} \lesssim e^{-c2^l} \left(\sum_i \int_{B_i} \mathcal{M}^w(|h|^{p_2'})^{\frac{1}{p_2'}} dw \right)^{p_2} \lesssim e^{-c2^l} \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

Therefore, by (4.42), for $m = 1, 2$,

$$\mathcal{III}_m \lesssim \sum_{l\geq 1} e^{-c2^l} \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw \lesssim \frac{1}{\alpha^{\widetilde{q}}} \int_{\mathbb{R}^n} |\nabla f|^{\widetilde{q}} dw.$$

By the above inequality, (4.34), (4.37),(4.40), and (4.41), we get (4.32). This leads to (4.29) what in turn, as we have already observed, finishes the proof modulo Claim 4.44. $\qquad\square$

Proof of Claim 4.44, $m = 1$. Note that

$$I_{1,l}^{ij} \lesssim \left(\int_{C_j(B_i)} \left(\sup_{0<t<2^{j-l-3}r_{B_i}} \left(\fint_{B(x,2^{l+2}t)} |\nabla e^{-t^2 L}\widehat{b}_i|^{q_0} dw \right)^{\frac{1}{q_0}} \right)^{p_2} dw \right)^{\frac{1}{p_2}}$$

$$+ \left(\int_{C_j(B_i)} \left(\sup_{t\geq 2^{j-l-3}r_{B_i}} \left(\fint_{B(x,2^{l+2}t)} |\nabla e^{-t^2 L}\widehat{b}_i|^{q_0} dw \right)^{\frac{1}{q_0}} \right)^{p_2} dw \right)^{\frac{1}{p_2}}$$

$$=: \mathfrak{C}_1 + \mathfrak{C}_2.$$

In order to estimate $\mathfrak{C}_1$, we use functional calculus. Take $\phi(z,t) := e^{-t^2 z}(1 - e^{-r_{B_i}^2 z})^M$, then $\phi(z,t)$ is holomorphic in the open sector $\Sigma_\mu = \{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \mu\}$ and satisfies $|\phi(z,t)| \lesssim |z|^M (1+|z|)^{-2M}$ (with implicit constant depending on μ, $t > 0$, r_{B_i}, and M) for every $z \in \Sigma_\mu$. We can check that for every $z \in \Gamma = \partial \sum_{\frac{\pi}{2}-\theta}$,

$$|\eta(z,t)| \lesssim \frac{r_{B_i}^{2M}}{(|z| + t^2)^{M+1}}.$$

Now fix $x \in C_j(B_i)$, $j \geq 4$, and $0 < t < 2^{j-l-3}r_{B_i}$, then $B(x, 2^{l+2}t) \subset 2^{j+2}B_i \setminus 2^{j-1}B_i$. This and Minkowski's integral inequality imply

$$\left(\fint_{B(x,2^{l+2}t)} |\nabla e^{-t^2 L}\widehat{b}_i|^{q_0} dw \right)^{\frac{1}{q_0}} = \left(\fint_{B(x,2^{l+2}t)} |\nabla \phi(L,t)b_i|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim \int_\Gamma \left(\fint_{B(x,2^{l+2}t)} |z^{\frac{1}{2}} \nabla e^{-zL}b_i|^{q_0} dw \right)^{\frac{1}{q_0}} \frac{r_{B_i}^{2M}}{|z|^{\frac{1}{2}}(|z| + t^2)^{M+1}} |dz|$$

$$\lesssim \int_{\Gamma} \left(\fint_{B(x,2^{l+2}t)} \left| \mathbf{1}_{2^{j+2}B_i \setminus 2^{j-1}B_i} z^{\frac{1}{2}} \nabla e^{-zL} b_i \right|^{q_0} dw \right)^{\frac{1}{q_0}} \frac{r_{B_i}^{2M}}{|z|^{M+\frac{3}{2}}} |dz|$$

$$\lesssim \int_{\Gamma} \mathcal{M}_{q_0}^w \left(\mathbf{1}_{2^{j+2}B_i \setminus 2^{j-1}B_i} z^{\frac{1}{2}} \nabla e^{-zL} b_i \right)(x) \frac{r_{B_i}^{2M}}{|z|^{M+\frac{3}{2}}} |dz|.$$

Applying again Minkowski's integral inequality, and recalling that $\mathcal{M}_{q_0}^w$ is bounded on $L^{p_2}(w)$ since $q_0 < p_2$, we get

$$\mathfrak{C}_1 \lesssim \int_{\Gamma} \left(\int_{C_j(B_i)} \mathcal{M}_{q_0}^w \left(\mathbf{1}_{2^{j+2}B_i \setminus 2^{j-1}B_i} z^{\frac{1}{2}} \nabla e^{-zL} b_i \right)^{p_2} dw \right)^{\frac{1}{p_2}} \frac{r_{B_i}^{2M}}{|z|^{M+\frac{3}{2}}} |dz|$$

$$\lesssim \int_{\Gamma} \left(\int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla e^{-zL} b_i \right|^{p_2} dw \right)^{\frac{1}{p_2}} \frac{r_{B_i}^{2M}}{|z|^{M+\frac{3}{2}}} |dz|.$$

Observe that $2^{j+2}B_i \setminus 2^{j-1}B_i = \cup_{l=1}^{3} C_{l+j-2}(B_i)$. Then by the fact that $z^{\frac{1}{2}} \nabla e^{-zL} \in \mathcal{O}(L^{p_2}(w) - L^{p_2}(w))$, (4.36), and changing the variable s into $\frac{4^j r_{B_i}^2}{s^2}$,

$$\mathfrak{C}_1 \lesssim w(2^{j+1}B_i)^{\frac{1}{p_2}} 2^{j\theta_1} \left(\fint_{B_i} |b_i|^{p_2} dw \right)^{\frac{1}{p_2}} \int_0^\infty \Upsilon\left(\frac{2^j r_{B_i}}{s^{\frac{1}{2}}} \right)^{\theta_2} e^{-c\frac{4^j r_{B_i}^2}{s}} \frac{s r_{B_i}^{2M}}{s^{M+\frac{3}{2}}} \frac{ds}{s}$$

$$\lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_2}} 2^{-j(2M+1-\theta_1)} \int_0^\infty \Upsilon(s)^{\theta_2} e^{-cs^2} s^{2M+1} \frac{ds}{s}$$

$$\lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_2}} 2^{-j(2M+1-\theta_1)},$$

provided $2M + 1 > \theta_2$.

We continue by estimating $\mathfrak{C}_2$. To this end, first change the variable t into $t\sqrt{M+1} =: t\theta_M$. Next, for any $x \in C_j(B_i)$, $j \geq 4$, and $t \geq \frac{2^{j-3} r_{B_i}}{2^l \theta_M}$, note that $B_i \subset B(x_{B_i}, \theta_M 2^l t) =: B_i^l \subset B(x, \theta_M 2^{l+5} t)$ (x_{B_i} denotes the center of B_i). Then,

$$\mathfrak{C}_2 \lesssim \left(\int_{C_j(B_i)} \left(\sup_{t \geq \frac{2^{j-3} r_{B_i}}{2^l \theta_M}} \fint_{B(x,\theta_M 2^{l+2}t)} \left| \nabla e^{-t^2 L} \mathcal{T}_{t,r_{B_i}} (\mathbf{1}_{B_i^l} b_i) \right|^{q_0} dw \right)^{\frac{p_2}{q_0}} dw \right)^{\frac{1}{p_2}}$$

$$\lesssim \left(\int_{C_j(B_i)} \left(\sup_{t \geq \frac{2^{j-3} r_{B_i}}{2^l \theta_M}} w(B(x,\theta_M 2^{l+2}t))^{-1} \int_{\mathbb{R}^n} \left| \nabla e^{-t^2 L} \mathcal{T}_{t,r_{B_i}} (\mathbf{1}_{B_i^l} b_i) \right|^{q_0} dw \right)^{\frac{p_2}{q_0}} dw \right)^{\frac{1}{p_2}},$$

where $\mathcal{T}_{t,r_{B_i}} := \left(e^{-t^2 L} - e^{-(t^2 + r_{B_i}^2)L} \right)^M.$

In the above setting, first recall that $\max\{r_w, \widehat{q}_-(L)\} < q_0 < p_2 < q_+(L)/s_w$, consequently $t\nabla e^{-t^2 L}$ is bounded on $L^{q_0}(w)$. Besides, applying the $L^{q_0}(w)$-$L^{q_0}(w)$ off-diagonal estimates that $\mathcal{T}_{t,r_{B_i}}$ satisfies (see (2.11)), and (1.6) to obtain

$$(4.45) \qquad \left(\int_{\mathbb{R}^n} \left| \nabla e^{-t^2 L} \mathcal{T}_{t,r_{B_i}} (\mathbf{1}_{B_i^l} b_i) \right|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim 2^l r_{B_i}^{-1} \left(\int_{\mathbb{R}^n} \left| \mathcal{T}_{t,r_{B_i}} (\mathbf{1}_{B_i^l} b_i) \right|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim 2^l \sum_{N \geq 1} w(C_N(B_i^l))^{\frac{1}{q_0}} \left(\fint_{C_N(B_i^l)} \left| \mathcal{T}_{t,r_{B_i}} \left(\mathbf{1}_{B_i^l} \frac{b_i}{r_{B_i}} \right) \right|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim 2^{l\theta} w(B_i^l)^{\frac{1}{q_0}} \sum_{N \geq 1} e^{-c4^N} \left(\frac{r_{B_i}^2}{t^2} \right)^M \left(\fint_{B_i^l} \left| \frac{b_i}{r_{B_i}} \right|^{q_0} dw \right)^{\frac{1}{q_0}}$$

$$\lesssim 2^{l\theta} w(B_i)^{\frac{1}{q_0}} \alpha \left(\frac{r_{B_i}^2}{t^2} \right)^M,$$

where in the last inequality we have used (4.36) since $q_0 < p_2$. Consequently,

$$\mathfrak{C}_2 \lesssim 2^{l\theta} \alpha \left(\int_{C_j(B_i)} \left(\sup_{t \geq \frac{2^{j-l-3} r_{B_i}}{\theta_M}} \left(\frac{r_{B_i}^2}{t^2} \right)^M \right)^{p_2} dw \right)^{\frac{1}{p_2}} \lesssim \alpha w(2^{j+1} B_i)^{\frac{1}{p_2}} 2^{-j2M} 2^{l(2M+\theta)},$$

where in the first inequality, we have used that $w(B(x, \theta_M 2^{l+2} t))^{-1} w(B_i) \leq C$, since $B_i \subset B(x, \theta_M 2^{l+5} t)$ and by (1.6).

Collecting the estimates obtained for $\mathfrak{C}_1$ and $\mathfrak{C}_2$, we conclude that, for $M \in \mathbb{N}$ such that $2M + 1 > \theta_2$,

$$(4.46) \qquad I_{1,l}^{ij} \lesssim \alpha \, w(2^{j+1} B_i)^{\frac{1}{p_2}} 2^{-j(2M-\theta_1)} 2^{l(2M+\theta)}.$$

This completes the proof of Claim 4.44 for $m = 1$. $\qquad \square$

Proof of Claim 4.44, $m = 2$. Splitting the supremum in t, we have

$$I_{2,l}^{ij} \lesssim \left(\int_{C_j(B_i)} \sup_{0 < t < 2^{j-l-3} r_{B_i}} \left(\fint_{B(x, 2^{l+2} t)} \widetilde{G}_{\mathrm{H}} (\mathcal{B}_{r_{B_i}} b_i)^{q_0} dw \right)^{\frac{p_2}{q_0}} dw \right)^{\frac{1}{p_2}}$$

$$+ \left(\int_{C_j(B_i)} \sup_{t \geq 2^{j-l-3} r_{B_i}} \left(\fint_{B(x, 2^{l+2} t)} \widetilde{G}_{\mathrm{H},t} (\mathcal{B}_{r_{B_i}} b_i)^{q_0} dw \right)^{\frac{p_2}{q_0}} dw \right)^{\frac{1}{p_2}}$$

$$=: D_1^{ij} + D_2^{ij}.$$

Estimate for D_1^{ij}: We claim that, for some fixed constants $\widetilde{\theta}_1 > 0$, $\widetilde{\theta}_2 > 0$, and for M large enough the following holds

$$(4.47) \qquad D_1^{ij} \lesssim \alpha w(2^{j+1} B_i)^{\frac{1}{p_2}} 2^{-j\left(2M - \widetilde{\theta}_1 - 2\widetilde{\theta}_2 \right)}.$$

To show this, first note that as before, for $0 < t < 2^{j-l-3} r_{B_i}$ and $x \in C_j(B_i)$, we have that $B(x, 2^{l+2} t) \subset 2^{j+2} B_i \setminus 2^{j-1} B_i$. Next recall that $\mathcal{M}_{q_0}^w$ is bounded on $L^{p_2}(w)$ since $q_0 < p_2$. Hence,

$$D_1^{ij} \lesssim \left(\int_{C_j(B_i)} \mathcal{M}_{q_0}^w \left(\mathbf{1}_{2^{j+2} B_i \setminus 2^{j-1} B_i} \widetilde{G}_{\mathrm{H}} \widehat{b}_i \right)^{p_2} dw \right)^{\frac{1}{p_2}}$$

$$\lesssim w(2^{j+1} B_i)^{\frac{1}{p_2}} \left(\fint_{2^{j+2} B_i \setminus 2^{j-1} B_i} \left| \widetilde{G}_{\mathrm{H}} \widehat{b}_i \right|^{p_2} dw \right)^{\frac{1}{p_2}}.$$

Now, note that since $p_2 < q_+(L)/s_w$, we can take $\max\{2, p_2\} < q < q_+(L)$ so that $w \in RH_{\left(\frac{q}{p_2}\right)'}$. Hence by Lemma 2.20 and Minkowski's integral inequality, we have

$$(4.48) \qquad D_1^{ij} \lesssim w(2^{j+1}B_i)^{\frac{1}{p_2}} \left(\int_0^\infty \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| r\nabla r L e^{-r^2 L} \widehat{b_i} \right|^q dx \right)^{\frac{2}{q}} \frac{dr}{r} \right)^{\frac{1}{2}}.$$

In order to estimate the integral in x, we use functional calculus with the same choice of ϕ as in the estimate of $\mathfrak{C}_1$. Then

$$(4.49) \quad \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| r\nabla r L e^{-r^2 L} \widehat{b_i} \right|^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \int_\Gamma \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla z L e^{-zL} b_i \right|^q dx \right)^{\frac{1}{q}} \frac{r^2 r_{B_i}^{2M}}{|z|^{\frac{1}{2}}(|z|+r^2)^{M+1}} \frac{|dz|}{|z|}.$$

Split the integral in x as follows

$$(4.50) \quad \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla z L e^{-zL} b_i \right|^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \sum_{l=1}^{j-3} \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla z L e^{-\frac{z}{2}L}(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i) \right|^q dx \right)^{\frac{1}{q}}$$
$$+ \sum_{l \geq j-2} \left(\fint_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla z L e^{-\frac{z}{2}L}(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i) \right|^q dx \right)^{\frac{1}{q}} =: \mathfrak{A} + \mathfrak{B}.$$

Note now that since $j \geq 4$, for $1 \leq l \leq j-3$ we have that $d(2^{j+2}B_i \setminus 2^{j-1}B_i, C_l(B_i)) \geq 2^{j-2}r_{B_i} \geq 2^{l+1}r_{B_i}$. Then, in that case, first applying the fact that $\sqrt{\tau}\nabla \tau L e^{-\tau L}$ satisfies $L^{q_2}(\mathbb{R}^n)$- $L^q(\mathbb{R}^n)$ off-diagonal estimates and split the exponential term (recall that $l \leq j-3$). Next apply Lemma 2.20 since $w \in A_{\frac{p_1}{q_2}}$ (see (4.30)) to get

$$\mathfrak{A} \lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l=1}^{j-3} \left(\int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left| z^{\frac{1}{2}} \nabla z L e^{-\frac{z}{2}L}(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i) \right|^q dx \right)^{\frac{1}{q}}$$
$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l=1}^{j-3} \left(\int_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i \right|^{q_2} dx \right)^{\frac{1}{q_2}} e^{-c\frac{4^j r_{B_i}^2}{|z|}} |z|^{-\frac{n}{2}(\frac{1}{q_2} - \frac{1}{q})}$$
$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l=1}^{j-3} |2^{l+1}B_i|^{\frac{1}{q_2}} \left(\fint_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i \right|^{p_1} dw \right)^{\frac{1}{p_1}} e^{-c\frac{4^j r_{B_i}^2}{|z|}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} |z|^{-\frac{n}{2}(\frac{1}{q_2} - \frac{1}{q})}.$$

Now, since $e^{-\frac{z}{2}L} \in \mathcal{O}(L^{p_2}(w) - L^{p_1}(w))$ and by (4.36)

$$\left(\fint_{C_l(B_i)} \left| e^{-\frac{z}{2}L} b_i \right|^{p_1} dw \right)^{\frac{1}{p_1}} \lesssim 2^{l\widetilde{\theta_1}} \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}} \right)^{\widetilde{\theta_2}} \left(\fint_{B_i} |b_i|^{p_2} dw \right)^{\frac{1}{p_2}}$$
$$\lesssim \alpha r_{B_i} 2^{l\widetilde{\theta_1}} \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}} \right)^{\widetilde{\theta_2}}.$$

Hence,

$$\mathfrak{A} \lesssim \alpha r_{B_i} |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l=1}^{j-3} 2^{l\widetilde{\theta_1}} \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}}\right)^{\widetilde{\theta_2}} |2^{l+1}B_i|^{\frac{1}{q_2}} e^{-c\frac{4^j r_{B_i}^2}{|z|}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)}.$$

If we now consider $l \geq j - 2$, in this case, we do not have distance between $2^{j+2}B_i \setminus 2^{j-1}B_i$ and $C_l(B_i)$, but we do have it between $C_l(B_i)$ and B_i. Indeed, since $l \geq j - 2 \geq 2$ (recall that $j \geq 4$), we have that $d(C_l(B_i), B_i) > 2^{l-1}r_{B_i} \geq 2^{j-3}r_{B_i}$. Hence, proceeding as in the above computation, we obtain

$$\mathfrak{B} \lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l\geq j-2} \left(\int_{2^{j+2}B_i \setminus 2^{j-1}B_i} \left|z^{\frac{1}{2}}\nabla z L e^{-\frac{z}{2}L}\left(\mathbf{1}_{C_l(B_i)} e^{-\frac{z}{2}L} b_i\right)\right|^q dy\right)^{\frac{1}{q}}$$

$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l\geq j-2} \left(\int_{C_l(B_i)} \left|e^{-\frac{z}{2}L} b_i(y)\right|^{q_2} dy\right)^{\frac{1}{q_2}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)}$$

$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l\geq j-2} (2^l r_{B_i})^{\frac{n}{q_2}} \left(\fint_{C_l(B_i)} \left|e^{-\frac{z}{2}L} b_i(y)\right|^{p_1} dw\right)^{\frac{1}{p_1}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)}$$

$$\lesssim |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l\geq j-2} 2^{l\widetilde{\theta_1}} (2^l r_{B_i})^{\frac{n}{q_2}} \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}}\right)^{\widetilde{\theta_2}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} \left(\fint_{B_i} |b_i|^{p_2} dw\right)^{\frac{1}{p_2}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)}$$

$$\lesssim \alpha r_{B_i} |2^{j+1}B_i|^{-\frac{1}{q}} \sum_{l\geq j-2} 2^{l\widetilde{\theta_1}} (2^l r_{B_i})^{\frac{n}{q_2}} \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}}\right)^{\widetilde{\theta_2}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} e^{-c\frac{4^j r_{B_i}^2}{|z|}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)}.$$

Thus, from the estimates of $\mathfrak{A}$ and $\mathfrak{B}$, in view of (4.48), (4.49), and (4.50), we have

(4.51)

$$D_1^{ij} \lesssim \alpha r_{B_i} |2^{j+1}B_i|^{-\frac{1}{q}} w(2^{j+1}B_i)^{\frac{1}{p_2}} \sum_{l\geq 1} 2^{l\widetilde{\theta_1}} (2^l r_{B_i})^{\frac{n}{q_2}}$$

$$\left(\int_0^\infty \left(\int_\Gamma \Upsilon\left(\frac{2^{l+1}r_{B_i}}{|z|^{\frac{1}{2}}}\right)^{\widetilde{\theta_2}} e^{-c\frac{4^l r_{B_i}^2}{|z|}} e^{-c\frac{4^j r_{B_i}^2}{|z|}} |z|^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)} \frac{|z|^{-\frac{1}{2}} r^2 r_{B_i}^{2M}}{(|z|+r^2)^{M+1}} \frac{|dz|}{|z|}\right)^2 \frac{dr}{r}\right)^{\frac{1}{2}}.$$

Doing the change of variables s into $4^j r_{B_i}^2/s^2$, we obtain

(4.52)

$$\int_0^\infty \Upsilon\left(\frac{2^{l+1}r_{B_i}}{s^{\frac{1}{2}}}\right)^{\widetilde{\theta_2}} e^{-c\frac{4^l r_{B_i}^2}{s}} e^{-c\frac{4^j r_{B_i}^2}{s}} s^{-\frac{n}{2}\left(\frac{1}{q_2}-\frac{1}{q}\right)-\frac{1}{2}} \frac{r^2 r_{B_i}^{2M}}{(s+r^2)^{M+1}} \frac{ds}{s}$$

$$\lesssim (2^j r_{B_i})^{-n\left(\frac{1}{q_2}-\frac{1}{q}\right)-1} \int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta_2}} e^{-c\frac{4^l s^2}{4^j}} e^{-s^2} s^{n\left(\frac{1}{q_2}-\frac{1}{q}\right)+1} \frac{r^2 r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+1}} \frac{ds}{s}.$$

Besides, changing the variable r into $2^j r_{B_i} r$, we have

(4.53)

$$\left(\int_0^\infty \left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta_2}} s^{n\left(\frac{1}{q_2}-\frac{1}{q}\right)+1} e^{-cs^2} e^{-c\frac{4^l s^2}{4^j}} r^2 \frac{r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+1}} \frac{ds}{s}\right)^2 \frac{dr}{r}\right)^{\frac{1}{2}}$$

$$\lesssim 2^{-j2M}\left(\int_0^\infty r^4\left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta}_2}\frac{s^{n(\frac{1}{q_2}-\frac{1}{q})+1}e^{-cs^2}e^{-c\frac{4^l s^2}{4^j}}}{(1/s^2+r^2)^{M+1}}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}.$$

In order to bound the above integral, take $\widetilde{M}=\frac{1}{2}(\widetilde{\theta}_2+\widetilde{\theta}_1+\frac{n}{q_2}+1)$ and $M\geq 2$ so that $2M+n(\frac{1}{q_2}-\frac{1}{q})-2\widetilde{M}-\widetilde{\theta}_2-1>0$. Then,

$$\left(\int_0^1 r^4\left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta}_2}s^{n(\frac{1}{q_2}-\frac{1}{q})+1}e^{-cs^2}e^{-c\frac{4^l s^2}{4^j}}\frac{1}{(1/s^2+r^2)^{M+1}}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}$$

$$\lesssim 2^{-l(2\widetilde{M}-\widetilde{\theta}_2)}2^{j(2\widetilde{M}+\widetilde{\theta}_2)}\left(\left(\int_0^1 r^4\left(\int_0^1 s^{2M+3+n(\frac{1}{q_2}-\frac{1}{q})-2\widetilde{M}-\widetilde{\theta}_2}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_0^1 r^4\left(\int_1^\infty s^{2M+3+n(\frac{1}{q_2}-\frac{1}{q})-2\widetilde{M}+\widetilde{\theta}_2}e^{-cs^2}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}\right)$$

$$\lesssim 2^{-l(2\widetilde{M}-\widetilde{\theta}_2)}2^{j(2\widetilde{M}+\widetilde{\theta}_2)}.$$

And, since $=(s^{-2}+r^2)^{-M-1}=(s^{-2}+r^2)^{-M+1}(s^{-2}+r^2)^{-2}\leq s^{2(M-1)}r^{-4}$,

$$\left(\int_1^\infty r^4\left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta}_2}s^{n(\frac{1}{q_2}-\frac{1}{q})+1}e^{-cs^2}e^{-c\frac{4^l s^2}{4^j}}\frac{1}{(1/s^2+r^2)^{M+1}}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_1^\infty r^{-4}\left(\int_0^\infty \Upsilon\left(\frac{2^l s}{2^j}\right)^{\widetilde{\theta}_2}s^{2M+n(\frac{1}{q_2}-\frac{1}{q})-1}e^{-cs^2}e^{-c\frac{4^l s^2}{4^j}}\frac{ds}{s}\right)^2\frac{dr}{r}\right)^{\frac{1}{2}}$$

$$\lesssim 2^{-l(2\widetilde{M}-\widetilde{\theta}_2)}2^{j(2\widetilde{M}+\widetilde{\theta}_2)}.$$

Hence, by our choice of M and $\widetilde{M}$

$$D_1^{ij}\lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_2}}2^{-j(2M+1+\frac{n}{q_2}-2\widetilde{M}-\widetilde{\theta}_2)}\sum_{l\geq 1}2^{-l(2\widetilde{M}-\widetilde{\theta}_2-\widetilde{\theta}_1-\frac{n}{q_2})}$$

$$\lesssim \alpha w(2^{j+1}B_i)^{\frac{1}{p_2}}2^{-j(2M-\widetilde{\theta}_1-2\widetilde{\theta}_2)}.$$

Estimate for D_2^{ij}: We claim that, for some fixed constant $\widetilde{\theta}>0$ and for M large enough the following holds

$$(4.54)\qquad\qquad D_2^{ij}\lesssim 2^{l\left(2M+\frac{n}{q_2}+\widetilde{\theta}\right)}\alpha w(2^{j+1}B_i)^{\frac{1}{p_2}}2^{-2jM}.$$

For any $t\geq 2^{j-l-3}r_{B_i}$ and $f\in L^2(\mathbb{R}^n)$, we have that

$$\widetilde{G}_{H,t}f(x)=\left(\int_{\frac{t}{2}}^\infty |r\nabla rLe^{-r^2L}f(x)|^2\frac{dr}{r}\right)^{\frac{1}{2}}\leq\left(\int_{2^{j-l-4}r_{B_i}}^\infty |r\nabla rLe^{-r^2L}f(x)|^2\frac{dr}{r}\right)^{\frac{1}{2}}.$$

Take q_1 such that $q<q_1<q_+(L)$, and recall that $q_0<p_2$. Consequently, we get that $w\in RH_{\left(\frac{q}{p_2}\right)'}\subset RH_{\left(\frac{q_1}{p_2}\right)'}$ and $w\in RH_{\left(\frac{q}{p_2}\right)'}\subset RH_{\left(\frac{q}{q_0}\right)'}$. Hence, by Lemma 2.20 we have that $\mathcal{M}_{q_0}^w h\leq\mathcal{M}_q h$ and the third inequality below. Next, we use the boundedness of $\mathcal{M}_q$ on $L^{q_1}(\mathbb{R}^n)$, since $q_1>q$, and Minkowski's integral inequality to

obtain

(4.55)

$$
D_2^{ij} \lesssim \left(\int_{C_j(B_i)} \mathcal{M}_{q_0}^w \big(\widetilde{G}_{H,2^{j-l-3}r_{B_i}}(\widehat{b}_i) \big)^{p_2} dw \right)^{\frac{1}{p_2}}
$$

$$
\lesssim \left(\int_{C_j(B_i)} \mathcal{M}_q \big(\widetilde{G}_{H,2^{j-l-3}r_{B_i}}(\widehat{b}_i) \big)^{p_2} dw \right)^{\frac{1}{p_2}}
$$

$$
\lesssim w(2^{j+1}B_i)^{\frac{1}{p_2}} \left(\fint_{C_j(B_i)} \mathcal{M}_q \big(\widetilde{G}_{H,2^{j-l-3}r_{B_i}}(\widehat{b}_i) \big)^{q_1} dx \right)^{\frac{1}{q_1}}
$$

$$
\lesssim w(2^{j+1}B_i)^{\frac{1}{p_2}} |2^{j+1}B_i|^{-\frac{1}{q_1}} \left(\int_{\mathbb{R}^n} \left(\int_{2^{j-l-4}r_{B_i}}^{\infty} \big| r \nabla r L e^{-r^2 L} \widehat{b}_i \big|^2 \frac{dr}{r} \right)^{\frac{q_1}{2}} dx \right)^{\frac{1}{q_1}}
$$

$$
\lesssim w(2^{j+1}B_i)^{\frac{1}{p_2}} |2^{j+1}B_i|^{-\frac{1}{q_1}} \left(\int_{\frac{2^{j-l-4}r_{B_i}}{\theta_M}}^{\infty} \left(\int_{\mathbb{R}^n} \big| r \nabla r L e^{-r^2 L} \mathcal{T}_{r,r_{B_i}}(\mathbf{1}_{B_i^l} b_i) \big|^{q_1} dx \right)^{\frac{2}{q_1}} \frac{dr}{r} \right)^{\frac{1}{2}},
$$

where in the last inequality we have changed the variable r into $r\theta_M := r\sqrt{M+1}$, used that $B_i \subset B(x_{B_i}, \theta_M 2^l r) =: B_i^l$, for $r > \frac{2^{j-l-4}r_{B_i}}{\theta_M}$ and $j \geq 4$ (x_{B_i} denotes the center of B_i), and we recall that $\mathcal{T}_{r,r_{B_i}} := (e^{-r^2 L} - e^{-(r^2 + r_{B_i}^2)L})^M$.

By the $L^{q_2}(\mathbb{R}^n)$-$L^{q_1}(\mathbb{R}^n)$ off-diagonal estimates of $\tau \nabla \tau^2 L e^{-\tau^2 L}$, and (2.12) (with $w \equiv 1$ and $p = q_2$), (recall the choice of q_2 in (4.30) and that $2 < q_1 < q_+(L)$)

$$
\left(\int_{\mathbb{R}^n} \big| r \nabla r L e^{-r^2 L} \mathcal{T}_{r,r_{B_i}}(\mathbf{1}_{B_i^l} b_i) \big|^{q_1} dx \right)^{\frac{1}{q_1}}
$$

$$
\lesssim 2^l r_{B_i}^{-1} r^{-n(\frac{1}{q_2} - \frac{1}{q_1})} \left(\int_{\mathbb{R}^n} \big| \mathcal{T}_{r,r_{B_i}} e^{-\frac{r^2}{2}L}(\mathbf{1}_{B_i^l} b_i) \big|^{q_2} dx \right)^{\frac{1}{q_2}}
$$

$$
\lesssim 2^l r_{B_i}^{-1} \left(\frac{r_{B_i}^2}{r^2} \right)^M r^{-n(\frac{1}{q_2} - \frac{1}{q_1})} \left(\int_{\mathbb{R}^n} \big| e^{-\frac{r^2}{2}L}(\mathbf{1}_{B_i^l} b_i) \big|^{q_2} dx \right)^{\frac{1}{q_2}}.
$$

Since $w \in A_{\frac{p_1}{q_2}}$ (see (4.30)), by Lemma 2.20, and the $L^{p_2}(w)$-$L^{p_1}(w)$ off-diagonal estimates on balls satisfied by $e^{-\tau L}$, we have that

$$
\left(\int_{\mathbb{R}^n} \big| e^{-\frac{r^2}{2}L}(\mathbf{1}_{B_i^l} b_i) \big|^{q_2} dx \right)^{\frac{1}{q_2}} \lesssim \sum_{N \geq 1} |2^{N+1} B_i^l|^{\frac{1}{q_2}} \left(\fint_{C_N(B_i^l)} \big| e^{-\frac{r^2}{2}L}(\mathbf{1}_{B_i^l} b_i) \big|^{q_2} dx \right)^{\frac{1}{q_2}}
$$

$$
\lesssim \sum_{N \geq 1} |2^{N+1} B_i^l|^{\frac{1}{q_2}} \left(\fint_{C_N(B_i^l)} \big| e^{-\frac{r^2}{2}L}(\mathbf{1}_{B_i^l} b_i) \big|^{p_1} dw \right)^{\frac{1}{p_1}}
$$

$$
\lesssim 2^{l\widetilde{\theta}} |B_i^l|^{\frac{1}{q_2}} \left(\fint_{B_i^l} |b_i|^{p_2} dw \right)^{\frac{1}{p_2}}
$$

$$
\lesssim \alpha r_{B_i} 2^{l\widetilde{\theta}} |B_i^l|^{\frac{1}{q_2}},
$$

where in the last inequality we have used that for $r > \frac{2^{j-l-4} r_{B_i}}{\theta_M}$ and $j \geq 4$, $B_i \subset B_i^l$. Plugging this into (4.55) leads to

$$D_2^{ij} \lesssim 2^{l\left(\widetilde{\theta} + \frac{n}{q_2}\right)} \alpha w(2^{j+1} B_i)^{\frac{1}{p_2}} |2^{j+1} B_i|^{-\frac{1}{q_1}} \left(\int_{\frac{2^{j-l-5} r_{B_i}}{\theta_M}}^{\infty} \left(\frac{r_{B_i}^2}{r^2} \right)^{2M} r^{\frac{2n}{q_1}} \frac{dr}{r} \right)^{\frac{1}{2}}$$

$$\lesssim 2^{l\left(\widetilde{\theta} + \frac{n}{q_2} + M\right)} \alpha w(2^{j+1} B_i)^{\frac{1}{p_2}} 2^{-j2M},$$

provided $2M > \frac{n}{q_1}$.

Gather (4.47) and (4.54), then for $M \in \mathbb{N}$ such that $2M > \max\{2\widetilde{\theta}_2 + \widetilde{\theta}_1 + 2 + n/q, n/q_1\}$,

$$I_{2,l}^{ij} \lesssim 2^{l\widetilde{\theta}} \alpha w(2^{j+1} B_i)^{\frac{1}{p_2}} 2^{-j(2M - 2\widetilde{\theta}_2 - \widetilde{\theta}_1)}.$$

This completes the proof of Claim 4.44 in the case $m = 2$. $\square$

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