ON THE KATO PROBLEM AND EXTENSIONS FOR DEGENERATE ELLIPTIC OPERATORS

DAVID CRUZ-URIBE, OFS, JOSÉ MARÍA MARTELL, AND CRISTIAN RIOS

ABSTRACT. We study the Kato problem for divergence form operators whose ellipticity may be degenerate. The study of the Kato conjecture for degenerate elliptic equations was begun in [16, 17, 19]. In these papers the authors proved that given an operator $L_w = -w^{-1} \operatorname{div}(A\nabla)$, where w is in the Muckenhoupt class A_2 and A is a w-degenerate elliptic measure (that is, A = w B with B(x) an $n \times n$ bounded, complex-valued, uniformly elliptic matrix), then L_w satisfies the weighted estimate $\|\sqrt{L_w}f\|_{L^2(w)} \approx \|\nabla f\|_{L^2(w)}$. In the present paper we solve the L^2 -Kato problem for a family of degenerate elliptic operators. We prove that under some additional conditions on the weight w, the following unweighted L^2 -Kato estimates hold:

$$\|L_w^{1/2}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

This extends the celebrated solution to the Kato conjecture by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian, allowing the differential operator to have some degree of degeneracy in its ellipticity. For example, we consider the family of operators $L_{\gamma} = -|x|^{\gamma} \operatorname{div}(|x|^{-\gamma}B(x)\nabla)$, where B is any bounded, complex-valued, uniformly elliptic matrix. We prove that there exists $\epsilon > 0$, depending only on dimension and the ellipticity constants, such that

$$\|L_{\gamma}^{1/2}f\|_{L^{2}(\mathbb{R}^{n})} \approx \|\nabla f\|_{L^{2}(\mathbb{R}^{n})}, \quad -\epsilon < \gamma < \frac{2n}{n+2}.$$

The case $\gamma = 0$ corresponds to the case of uniformly elliptic matrices. Hence, our result gives a range of γ 's for which the classical Kato square root proved in [4] is an interior point.

Our main results are obtained as a consequence of a rich Calderón-Zygmund theory developed for certain operators naturally associated with L_w . These results, which are of independent interest, establish estimates on $L^p(w)$, and also on $L^p(v dw)$ with $v \in A_{\infty}(w)$, for the associated semigroup, its gradient, the functional calculus, the Riesz transform, and vertical square functions. As an application, we solve some unweighted L^2 -Dirichlet, Regularity and Neumann boundary value problems for degenerate elliptic operators.

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1. INTRODUCTION

We study the degenerate elliptic operators $L_w = -w^{-1} \operatorname{div} A \nabla$, where w is in the Muckenhoupt class A_2 and A(x) is an $n \times n$ complex-valued matrix that satisfies the degenerate ellipticity condition

$$|\lambda w(x)|\xi|^2 \le \operatorname{Re} \langle A(x)\xi,\xi\rangle, \qquad |\langle A(x)\xi,\eta\rangle| \le \Lambda w(x)|\xi||\eta|, \quad \xi,\,\eta\in\mathbb{C}^n, \text{ a.e. } x\in\mathbb{R}^n.$$

Equivalently, A(x) = w(x)B(x), where B is an $n \times n$ complex-valued matrix that satisfies the uniform ellipticity conditions

$$|\lambda|\xi|^2 \le \operatorname{Re} \langle B(x)\xi,\xi\rangle, \qquad |\langle B(x)\xi,\eta\rangle| \le \Lambda|\xi||\eta|, \quad \xi,\,\eta \in \mathbb{C}^n, \text{ a.e. } x \in \mathbb{R}^n.$$

Such operators were first studied (with A a real symmetric matrix) by Fabes, Kenig and Serapioni [21]. When A is complex-valued and uniformly elliptic (i.e. $w \equiv 1$), a landmark result was the proof of the Kato conjecture by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [4]: that for all $f \in H^1$,

$$\|L^{1/2}f\|_2 \approx \|\nabla f\|_2$$

The proof of this long-standing conjecture led naturally to the study of the operators associated with L: the semigroup e^{-tL} , its gradient $\sqrt{t}\nabla e^{-tL}$, the Riesz transform $\nabla L^{-1/2}$, the H^{∞} functional calculus and square functions: for details and complete references, see Auscher [2]. These estimates are interesting in themselves; moreover, it is well known that L^p estimates

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for these operators yield regularity results for boundary value problems for L: for details, see the introduction to [10].

In [19] (see also [16, 17, 9]) the first and third authors solved the Kato problem for degenerate elliptic operators: they showed that if $w \in A_2$ and A satisfies the degenerate ellipticity conditions, then for all $f \in H^1(w)$,

$$\|L_w^{1/2}f\|_{L^2(w)} \approx \|\nabla f\|_{L^2(w)}.$$
(1.1)

In this paper we consider the problem of determining those A_2 weights such that the classical Kato problem can be solved for L_w : that is, finding weights such that L_w satisfies the unweighted estimate

$$\|L_w^{1/2}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)},$$

for f in a class of nice functions (*a posteriori*, by standard density arguments, the estimate can be extended to all $f \in H^1(\mathbb{R}^n)$). We solve this problem in two steps. The first is to prove weighted L^p estimates for some operators associated with L_w (the semigroup, its gradient, the Riesz transform, the functional calculus, and square functions.) These results, which are of interest in their own right, are analogous to those obtained in the uniformly elliptic case. However, a significant technical obstruction is that given a weight $w \in A_2$, while it is the case that there exists $\epsilon > 0$ such that $w \in A_{2-\epsilon}$, it is easy to construct examples to show that ϵ may be arbitrarily small. Therefore, our bounds in the range 1 need to takethis into account.

The second step is to find conditions on the weight w so that these operators satisfy unweighted L^2 estimates. Both steps are carried out simultaneously, and the proofs are intertwined. Our approach is to apply the theory of off-diagonal estimates on balls developed by Auscher and the second author [5, 6, 7, 8]. We will in fact prove weighted estimates on $L^p(v dw)$, where v satisfies Muckenhoupt and reverse Hölder conditions with respect to the measure dw = w dx: $L^p(w)$ estimates are then obtained by taking v = 1, and unweighted estimates by taking $v = w^{-1}$.

The unweighted L^2 estimates are delicate, since they require a careful estimate of the constants that appear. Nevertheless, we are able to give useful sufficient conditions: e.g., $w \in A_1 \cap RH_{\frac{n}{2}+1}$. (For definitions of these classes, see Section 2 below.) For example, we have the following result that is a special case of one of our main results (cf. Theorem 11.11).

Theorem 1.2. Let $L_w = -w^{-1} \operatorname{div} A \nabla$ be a degenerate elliptic operator as above. If $w \in A_1 \cap RH_{\frac{n}{2}+1}$, then the Kato problem can be solved for L_w : for every $f \in H^1(\mathbb{R}^n)$,

$$||L_w^{1/2}f||_{L^2(\mathbb{R}^n)} \approx ||\nabla f||_{L^2(\mathbb{R}^n)}.$$

The implicit constants depend only on the dimension, the ellipticity constants, and the A_1 and $RH_{\frac{n}{2}+1}$ constants of w.

Furthermore, if we define $L_{\gamma} = -|x|^{\gamma} \operatorname{div}(|x|^{-\gamma}B(x)\nabla)$, where B is an $n \times n$ complexvalued matrix that satisfies the uniform ellipticity condition, then there exists $0 < \epsilon < \frac{1}{2}$ small enough (depending only on the dimension and the ratio Λ/λ) such that

$$\|L_{\gamma}^{1/2}f\|_{L^{2}(\mathbb{R}^{n})} \approx \|\nabla f\|_{L^{2}(\mathbb{R}^{n})}, \qquad -\epsilon < \gamma < \frac{2n}{n+2}$$

Remark 1.3. In Theorem 1.2 the operator $L_w^{1/2}$ is a priori only defined on $H^1(w)$; however this means that it is defined on $C_0^{\infty}(\mathbb{R}^n)$ and so by a standard density argument we can extend our results to all $f \in H^1(\mathbb{R}^n)$. Hereafter we will make this extension without further comment.

We emphasize that in Theorem 1.2, when $\gamma = 0$ we are back at the uniformly elliptic case, which is the celebrated solution to the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in [4]. Here we are able to find a range of γ 's for which the same estimates hold and the classical Kato square root problem (i.e., $\gamma = 0$) is an interior point in that range.

These unweighted L^2 estimates have important applications to boundary value problems for degenerate elliptic operators. Consider, for example, the following Dirichlet problem on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0, \infty)$:

$$\begin{cases} \partial_t^2 u - L_w u = 0, & \text{on } \mathbb{R}^{n+1}_+ \\ u = f & \text{on } \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

If $f \in L^2(\mathbb{R}^n)$, then $u(x,t) = e^{-tL_w^{1/2}} f(x)$ is a solution, and if L_w has a bounded H^∞ functional calculus on L^2 , then $\sup_{t>0} ||u(\cdot,t)||_2 \leq ||f||_2$. Similar results hold for the corresponding Neumann and Regularity problems.

Our proofs are unavoidably technical, and the results for each operator considered build upon what was proved previously for other operators. We have organized the material as follows. In Section 2 we gather some essential definitions and results about weights, degenerate elliptic operators, and off-diagonal estimates. Central to all of our subsequent work are Theorems 2.35 and 2.39 (which were proved in [5]).

In Sections 3, 4, and 5 we prove estimates for the semigroup e^{-tL_w} , t > 0, the H^{∞} functional calculus (i.e., operators $\varphi(L_w)$ where $\varphi \in \mathcal{H}^{\infty}$), the vertical square function associated to the semigroup,

$$g_{L_w}f(x) = \left(\int_0^\infty \left| (tL_w)^{1/2} e^{-tL_w} f(x) \right|^2 \frac{dt}{t} \right)^{1/2},$$

and its discrete analog. Here and in subsequent sections we prove both $L^{p}(w)$ estimates and weighted $L^{p}(v dw)$ estimates. In many cases these results are proved simultaneously, with the unweighted results (i.e., in $L^{p}(w)$) following from the weighted results (i.e., in $L^{p}(v dw)$) by taking v = 1.

In Section 6 we prove the so-called reverse inequality, $\|L_w^{1/2}\|_{L^p(w)} \leq \|\nabla f\|_{L^p(w)}$, that generalizes the $L^2(w)$ estimate in (1.1). We note that while the equivalence in (1.1) follows at once from the reverse inequality for p = 2 by duality, the two inequalities behave differently when $p \neq 2$.

In Sections 7 and 8 we prove estimates for the gradient of the semigroup, $\sqrt{t}\nabla e^{-tL_w}$. The proof that there exists $q_+ > 2$ such that this operator satisfies $L^p(w)$ estimates for $2 is quite involved as it requires preliminary estimates for the Riesz transform and the Hodge projection. We note that, as opposed to the non-degenerate case, here we cannot use "global" embeddings, nor can we rescale. Also we cannot expect to obtain that the gradient of the semigroup maps globally <math>L^2(w)$ into $L^p(w)$ for $p \neq 2$. All these difficulties arise naturally from the lack of isotropy of the natural underlying measure w(x) dx and make the typical arguments used in the uniformly elliptic case (cf. [2, Chapter 4]) unusable. We also note that in some sense our result is the best possible: even in the non-degenerate

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case it is known [2] that given any p > 2 there exists a matrix A and operator L such that gradient of the semigroup is not bounded on L^p .

In Section 9 we prove $L^p(w)$ estimates for the Riesz transform $\nabla L^{-1/2}$, and in Section 10 we prove $L^p(w)$ estimates for the square function associated to the gradient of the semigroup,

$$G_{L_w}f(x) = \left(\int_0^\infty |t^{1/2} \nabla e^{-tL_w} f(x)|^2 \frac{dt}{t}\right)^{1/2}$$

In Section 11 we prove unweighted L^2 inequalities for the operators we have considered in previous sections. These are a consequence of the weighted estimates and are obtained by taking $v = w^{-1}$. The main problem is determining conditions on w for these to hold. We essentially have two different kinds of estimates: one for operators that do not involve the gradient, and one for those that do. The latter are more delicate as they involve careful bounds for the parameter q_+ from Section 8 in terms of the weight w. We also show that we get unweighted L^p estimates for p very close to 2.

Finally, in Section 12 we describe in more detail the application of our results to L^2 boundary value problems for degenerate elliptic operators. The results in this section our the culmination of our work as they depend on all the estimates derived in previous sections.

As we were completing this project, we learned that related results had been obtained independently by other authors. In [26] Le studied (among other things) the $L^p(w)$ theory for some of the operators considered here and proved estimates for values of p in the range $(2 - \epsilon, 2 + \epsilon)$. His proofs differ from ours in a number of details. In [24] Hofmann, Le and Morris established some Carleson measure estimates and considered the Dirichlet problem for degenerate elliptic operators. Also, very recently we learned that Yang and Zhang [30] proved Kato type estimates in $L^p(w)$ for p in the range $(p_0, 2]$. Finally, we note that the paper [13] complements our work here as it considers the conical square functions associated to the operator L_w .

2. Preliminaries

Throughout n will denote the dimension of the underlying space \mathbb{R}^n and we will always assume $n \geq 2$. If we write $A \leq B$ we mean that there exists a constant C such that $A \leq CB$. We write $A \approx B$ if $A \leq B$ and $B \leq A$. The constant C in these estimates may depend on the dimension n and other (fixed) parameters that should be clear from the context. All constants, explicit or implicit, may change at each appearance.

Given a ball B, let r(B) denote the radius of B. Let λB denote the concentric ball with radius $r(\lambda B) = \lambda r(B)$.

Weights. By a weight w we mean a non-negative, locally integrable function. For brevity, we will often write dw for w dx. We will use the following notation for averages: given a set E such that $0 < w(E) < \infty$,

$$\int_E f \, dw = \frac{1}{w(E)} \int_E f \, dw,$$

or, if $0 < |E| < \infty$,

$$\int_E f \, dx = \frac{1}{|E|} \int_E f \, dx.$$

We state some definitions and basic properties of Muckenhoupt weights. For further details, see [20, 22]. We say that $w \in A_p$, 1 , if

$$[w]_{A_p} = \sup_Q \oint_Q w(x) \, dx \left(\oint_Q w(x)^{1-p'} \, dx \right)^{p-1} < \infty.$$

When p = 1, we say that $w \in A_1$ if

$$[w]_{A_1} = \sup_{Q} \oint_{Q} w(x) \, dx \, \mathrm{ess} \sup_{x \in Q} w(x)^{-1} < \infty.$$

We say that $w \in RH_s$, $1 < s < \infty$, if

$$[w]_{RH_s} = \sup_{Q} \left(\oint_{Q} w(x) \, dx \right)^{-1} \left(\oint_{Q} w(x)^s \, dx \right)^{1/s} < \infty,$$

and we say that $w \in RH_{\infty}$ if

$$[w]_{RH_{\infty}} = \sup_{Q} \left(\oint_{Q} w(x) \, dx \right)^{-1} \operatorname{ess\,sup}_{x \in Q} w(x) < \infty.$$

Let

$$A_{\infty} = \bigcup_{1 \le p < \infty} A_p = \bigcup_{1 < s \le \infty} RH_s$$

Weights in the A_p and RH_s classes have a self-improving property: if $w \in A_p$, there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$, and similarly if $w \in RH_s$, then $w \in RH_{s+\delta}$ for some $\delta > 0$. Hereafter, given $w \in A_p$, let

$$r_w = \inf\{p : w \in A_p\}, \qquad s_w = \sup\{q : w \in RH_q\}.$$

An important property of A_p weights is that they are doubling: given $w \in A_p$, for all $\tau \geq 1$ and any ball B,

$$w(\tau B) \le [w]_{A_p} \tau^{pn} w(B).$$

In particular, hereafter let $D \leq pn$ be the doubling order of w: that is the smallest exponent such that this inequality holds.

As a consequence of this doubling property, we have that with the ordinary Euclidean distance $|\cdot|$, $(\mathbb{R}^n, dw, |\cdot|)$ is a space of homogeneous type. In this setting we can define the new weight classes $A_p(w)$ and $RH_s(w)$ by replacing Lebesgue measure in the definitions above with dw: e.g., $v \in A_p(w)$ if

$$[v]_{A_p(w)} = \sup_{Q} \oint_{Q} v(x) \, dw \left(\oint_{Q} v(x)^{1-p'} \, dw \right)^{p-1} < \infty$$

It follows at once from these definitions that there is a "duality" relationship between the weighted and unweighted A_p and RH_s conditions: $v = w^{-1} \in A_p(w)$ if and only if $w \in RH_{p'}$ and $v = w^{-1} \in RH_s(w)$ if and only if $w \in A_{s'}$.

Weighted Poincaré-Sobolev inequalities were proved in [21].

Theorem 2.1. Given $w \in A_p$, $p \ge 1$, let $p_w^* = \frac{p n r_w}{n r_w - p}$ if $p < n r_w$ and $p_w^* = \infty$ otherwise. Then for every $p \le q < p_w^*$, ball B and $f \in C_0^{\infty}(B)$,

$$\left(\oint_{B} |f(x)|^{q} dw(x)\right)^{1/q} \leq Cr(B) \left(\oint_{B} |\nabla f(x)|^{p} dw\right)^{1/p}.$$
(2.2)

Moreover, if $f \in C^{\infty}(B)$, then

$$\left(\oint_{B} |f(x) - f_{B,w}|^q \, dw(x)\right)^{1/q} \le C \, r(B) \left(\oint_{B} |\nabla f(x)|^p \, dw\right)^{1/p},\tag{2.3}$$

where $f_{B,w} = \int_B f \, dw$.

Remark 2.4. In the special case when $w \in A_1$ and $1 we can also take <math>q = p_w^* = p^*$, i.e., the regular Sobolev exponent. See Pérez [29, Theorem 2.5.2].

Remark 2.5. If we let $q = \frac{np}{n-1} < p_w^*$, then we can get a sharp estimate for the constant C in (2.2) and (2.3): it is of the form $C(p,n)[w]_{A_p}^{\kappa}$ where $\kappa = \frac{np-1}{np(p-1)}$. This follows from the sharp weighted estimates for the fractional integral operator due to Alberico, Cianchi and Sbordone [1] and the standard pointwise estimates used to prove Poincaré-Sobolev inequalities; see [21] for details.

Remark 2.6. By a standard density argument, once we know that (2.3) holds for smooth functions in B we can easily extend that estimate to any function $f \in L^q(w)$ with $\nabla f \in L^p(w)$. Details are left to the reader.

Degenerate elliptic operators. Given $w \in A_2$ and constants $0 < \lambda \leq \Lambda < \infty$, let $\mathcal{E}_n(w,\lambda,\Lambda)$ denote the class of $n \times n$ matrices $A = (A_{ij}(x))_{i,j=1}^n$ of complex-valued, measurable functions satisfying the degenerate ellipticity condition

$$\lambda w(x)|\xi|^2 \le \operatorname{Re} \langle A\xi, \xi \rangle, \quad |\langle \mathcal{A}\xi, \eta \rangle| \le \Lambda w(x)|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n.$$
(2.7)

Given $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, we define the degenerate elliptic operator in divergence form

$$L_w = -w^{-1} \mathrm{div} A \nabla.$$

These operators were developed in [16] and we refer the reader there for complete details. Here we sketch the key ideas.

Given a weight $w \in A_2$, the space $H^1(w)$ is the weighted Sobolev space that is the completion of C_c^{∞} with respect to the norm

$$||f||_{H^{1}(w)} = \left(\int_{\mathbb{R}^{n}} \left(|f(x)|^{2} + |\nabla f(x)|^{2}\right) dw\right)^{1/2}$$

Note that the space defined above would usually be denoted by $H_0^1(w)$. The space $H^1(w)$ is defined as the set of distributions for which both f and $|\nabla f|$ belong to $L^2(w)$. However, since the underlying domain is \mathbb{R}^n this definition implies that the "boundary" values vanish in the $L^2(w)$ -sense, and both definitions agree [28].

Given a matrix $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, define $\mathfrak{a}(f, g)$ to be the sesquilinear form

$$\mathfrak{a}(f,g) = \int_{\mathbb{R}^n} A(x) \nabla f(x) \cdot \overline{\nabla g(x)} \, dx.$$
(2.8)

Since $w \in A_2$ and A satisfies (2.7), \mathfrak{a} is a closed, maximally accretive, continuous sesquilinear form. Therefore, there exists an operator L_w whose domain $\mathcal{D}(L_w) \subset H^1(w)$ is dense in $L^2(w)$ and such that for every $f \in \mathcal{D}(L_w)$ and every $g \in H^1(w)$,

$$\mathfrak{a}(f,g) = \langle L_w f, g \rangle_w = \int_{\mathbb{R}^n} L_w f(x) \overline{g(x)} \, dw.$$
(2.9)

We note that the operator L_w is one to one. Indeed, if $u, v \in \mathcal{D}(L_w)$ are such that $L_w u = L_w v$, then for all $g \in H^1(w)$

$$0 = \int_{\mathbb{R}^n} A(x)\nabla(u(x) - v(x)) \cdot \overline{\nabla g(x)} \, dx$$

Taking g = u - v implies $\nabla u(x) = \nabla v(x)$ and so u = v.

The properties of the sesquilinear form guarantee that on $L^2(w)$ there exists a bounded, strongly continuous semigroup e^{-tL_w} . Further, it has a holomorphic extension. Let

$$\Sigma_{\omega} = \{ z \in \mathbb{C} : z \neq 0, |\arg(z)| < \omega \}$$

and define $\vartheta, \vartheta^* \in [0, \pi/2)$ by

$$\vartheta = \sup\{|\arg\langle Lf, f\rangle_w| : f \in \mathcal{D}(L_w)\}, \qquad \vartheta^* = \arctan\sqrt{\frac{\Lambda^2}{\lambda^2}} - 1.$$

Then there exists a complex semigroup e^{-zL_w} on $\Sigma_{\pi/2-\vartheta}$ of bounded operators on $L^2(w)$. By the weighted ellipticity condition (2.7), we have that $0 \leq \vartheta \leq \vartheta^* < \pi/2$.

Holomorphic functional calculus. Our operator L_w is "an operator of type ω " with $\omega = \vartheta$, as defined in [27]. Indeed, the ellipticity conditions imply that L_w is closed and densely defined, its spectrum is contained in Σ_{ϑ} , and its resolvent satisfies standard decay estimates [16]. Therefore, we can define an $L^2(w)$ functional calculus as in [27].

Given $\mu \in (\vartheta, \pi)$, let $\mathcal{H}^{\infty}(\Sigma_{\mu})$ be the collection of bounded holomorphic functions on Σ_{μ} . To define $\varphi(L_w)$ for $\varphi \in \mathcal{H}^{\infty}(\Sigma_{\mu})$ we first consider a smaller class: we say that $\varphi \in \mathcal{H}^{\infty}_0(\Sigma_{\mu})$ if for some c, s > 0 it satisfies

$$|\varphi(z)| \le c|z|^s (1+|z|)^{-2s}, \quad z \in \Sigma_{\mu}.$$

We then have an integral representation of $\varphi(L_w)$. Let Γ_{θ} be the boundary of Σ_{θ} with positive orientation, and let $\vartheta < \theta < \nu < \min(\mu, \pi/2)$; then

$$\varphi(L_w) = \int_{\Gamma_{\pi/2-\theta}} e^{-zL_w} \eta(z) \, dz, \qquad (2.10)$$

where

$$\eta(z) = \frac{1}{2\pi i} \int_{\gamma_{\nu}(z)} e^{\zeta z} \varphi(\zeta) \, d\zeta \tag{2.11}$$

and $\gamma_{\nu}(z) = \mathbb{R}^+ e^{i \operatorname{sign}(\operatorname{Im}(z))\nu}$. Note that

$$|\eta(z)| \lesssim \min(1, |z|^{-s-1}), \quad z \in \Gamma_{\pi/2-\theta},$$

so the representation (2.10) converges in $L^2(w)$, and we have the bound

$$\|\varphi(L_w)f\|_{L^2(w)} \le C \|\varphi\|_{\infty} \|f\|_{L^2(w)}, \qquad f \in \mathcal{H}^{\infty}_0(\Sigma_{\mu}).$$
 (2.12)

Now, since L_w is a one-to-one operator of type ω , it has dense range [14, Theorem 2.3], and so the results in [27] (see also [14, Corollary 2.2]) imply that L_w has an H^{∞} functional calculus and (2.12) extends to all of $\mathcal{H}^{\infty}(\Sigma_{\mu})$. Moreover, in [27, Section 8] the equivalence between the existence of this H^{∞} functional calculus and square function estimates for L_w and L_w^* is established:

$$\left\{\int_{0}^{\infty} \|\varphi(tL_{w})\|_{L^{2}(w)}^{2} \frac{dt}{t}\right\}^{\frac{1}{2}} \leq C \|\varphi\|_{\infty} \|f\|_{L^{2}(w)}, \quad \varphi \in \mathcal{H}_{0}^{\infty}(\Sigma_{\mu}),$$
(2.13)

with similar estimates for L_w^* .

The operators $\varphi(L_w)$ also have the following properties:

- If φ and ψ are bounded holomorphic functions, then we have the operator identity $\varphi(L)\psi(L) = (\varphi\psi)(L)$.
- Given any sequence $\{\varphi_k\}$ of bounded holomorphic functions converging uniformly on compact subsets of Σ_{μ} to φ , we have that $\varphi_k(L_w)$ converges to $\varphi(L_w)$ in the strong operator topology (of operators on $L^2(w)$).

Remark 2.14. The H^{∞} functional calculus can be extended to more general holomorphic functions, such as powers, for which the operators $\varphi(L_w)$ can be defined as unbounded operators: see [23, 27].

Gaffney-type estimates. The semigroup and its gradient satisfy Gaffney-type estimates on $L^2(w)$. Below, we will see that these are a particular case of what we will call full off-diagonal estimates: see Definition 2.33.

Theorem 2.15. Given $w \in A_2$ and $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, for any closed sets E and F, for $f \in L^2(w)$ and for all $z \in \Sigma_{\nu}$, where $0 < \nu < \frac{\pi}{2} - \vartheta$,

(1)
$$\|e^{-zL_w}(f\chi_E)\chi_F\|_{L^2(w)} \le C e^{-\frac{cd(E,F)^2}{|z|}} \|f\chi_E\|_{L^2(w)},$$

(2)
$$\|\sqrt{z}\nabla e^{-zL_w}(f|\chi_E)\chi_F\|_{L^2(w)} \le C e^{-\frac{cd(E,F)^2}{|z|}} \|f\chi_E\|_{L^2(w)},$$

(3)
$$||z L_w e^{-z L_w}(f \chi_E) \chi_F ||_{L^2(w)} \le C e^{-\frac{c d(E,F)^2}{|z|}} ||f \chi_E ||_{L^2(w)}$$

Proof. The semigroup estimate (1) was proved in [16, Theorem 1.6] for real z, but the same proof can be readily modified to prove the analytic version. Alternatively, estimates (1) and (2) follow from the resolvent bounds

$$\|(1+z^2L_w)^{-1}(f\chi_E)\chi_F\|_{L^2(w)} \le Ce^{-\frac{cd(E,F)}{|z|}} \|f\chi_E\|_{L^2(w)},$$
(2.16)

and

$$\|z\nabla(1+z^{2}L_{w})^{-1}(f\chi_{E})\chi_{F}\|_{L^{2}(w)} \leq Ce^{-\frac{cd(E,F)}{|z|}}\|f\chi_{E}\|_{L^{2}(w)},$$
(2.17)

obtained in [19, Lemma 2.10] for $z \in \sum_{\frac{\pi}{2}+\nu}$, together with the integral representation of the semigroup

$$e^{-zL_w}f = \frac{1}{2\pi} \int_{\Gamma} e^{z\zeta} (\zeta + L_w)^{-1} f \, d\zeta,$$

where Γ is the boundary of Σ_{θ} with positive orientation and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \nu - \arg(z)$.

Finally, from (2.16) and (2.17) we obtain the estimate

$$\|z^{2}L_{w}(1+z^{2}L_{w})^{-1}(f\chi_{E})\chi_{F}\|_{L^{2}(w)} \leq Ce^{-\frac{cd(E,F)}{|z|}}\|f\chi_{E}\|_{L^{2}(w)},$$

and then by the same kind of argument we get (3).

The Kato estimate. The starting point for all of our estimates is the $L^2(w)$ Kato estimates for the square root operator $L_w^{1/2}$ proved in [19] (see also [9] for a different proof). This operator is the unique, maximal accretive operator such that $L_w^{1/2}L_w^{1/2} = L_w$. It has the integral representation

$$L_w^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} L_w e^{-tL_w} \frac{dt}{t}$$

(For further details, see [10, 27].)

Theorem 2.18. [19, Theorem 1.1] Given $w \in A_2$ and $A \in \mathcal{E}_n(w, \lambda, \Lambda)$, the domain of L_w is $H^1(w)$ and there exist constants c and C, depending on n, Λ/λ and $[w]_{A_2}$, such that for all $f \in H^1(w)$,

$$c \|\nabla f\|_{L^{2}(w)} \leq \|L_{w}^{1/2} f\|_{L^{2}(w)} \leq C \|\nabla f\|_{L^{2}(w)}.$$
(2.19)

The Riesz transform associated to L_w is the operator $\nabla L_w^{-1/2}$. Formally, by (2.19) we have that the Riesz transform is a bounded operator on $L^2(w, \mathbb{C}^n)$. To legitimize this, we define

$$\nabla L_w^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \nabla e^{-tL_w} \frac{dt}{t}.$$
(2.20)

However, it is not immediate that this integral converges at 0 or ∞ . To rectify this, for $\epsilon > 0$ define

$$S_{\epsilon} = S_{\epsilon}(L_w) = \frac{1}{\sqrt{\pi}} \int_{\epsilon}^{1/\epsilon} \sqrt{t} e^{-tL_w} \frac{dt}{t}.$$
 (2.21)

Since $S_{\epsilon}(z)$ is a uniformly bounded holomorphic function on the right half plane for all $0 < \epsilon < 1$, by the $L^2(w)$ functional calculus described above, $S_{\epsilon}(L_w)$ is uniformly bounded on $L^2(w)$ for that range of ϵ . Further, for $f \in L^{\infty}_c$, $S_{\epsilon}f \in \mathcal{D}(L_w) \subset \mathcal{D}(L^{1/2}_w)$, and so by inequality (2.19) and the functional calculus,

$$\|\nabla S_{\epsilon}f\|_{L^{2}(w)} \lesssim \|L^{1/2}S_{\epsilon}f\|_{L^{2}(w)} = \|\varphi_{\epsilon}(L_{w})f\|_{L^{2}(w)},$$
(2.22)

where

$$\varphi_{\epsilon}(z) = \frac{1}{\sqrt{\pi}} \int_{\epsilon}^{1/\epsilon} \sqrt{t} \sqrt{z} e^{-tz} \frac{dt}{t}$$

The sequence $\{\varphi_{\epsilon}\}$ is uniformly bounded and converges uniformly to 1 on compact subsets of the sector Σ_{μ} , $0 < \mu < \pi/2$. Therefore, $L^{1/2}S_{\epsilon}f \rightarrow f$ strongly in $L^2(w)$. If we combine this fact with (2.22) we see that $\{\nabla S_{\epsilon}f\}$ is Cauchy and so it converges in $L^2(w)$. We therefore define

$$\nabla L^{-1/2} f = \lim_{\epsilon \to 0} \nabla S_{\epsilon} f,$$

where the limit is in $L^2(w)$.

Given this definition, hereafter, when we are proving $L^2(w)$ estimates for the Riesz transform, we should actually prove estimates for ∇S_{ϵ} that are independent of ϵ . These arguments will remain implicit unless there are details we need to emphasize.

Off-diagonal estimates. Off-diagonal estimates as we define them were introduced in [7] and we will refer repeatedly to this paper for further information and results. Throughout this section we will assume that given a weight $w, w \in A_2$.

Given a ball B, for $j \ge 2$ we define the annuli $C_j(B) = 2^{j+1} B \setminus 2^j B$. We let $C_1(B) = 4B$. By a slight abuse of notation, we will define

$$\oint_{C_j(B)} h \, dw = \frac{1}{w(2^{j+1}B)} \int_{C_j(B)} h \, dw.$$

If $w \in A_2$ (as it will be hereafter), then $w(2^{j+1}B) \approx w(C_j(B))$, so this definition is equivalent to the one given above up to a constant. Finally, for s > 0 we set $\Upsilon(s) = \max\{s, s^{-1}\}$.

Definition 2.23. Given $1 \le p \le q \le \infty$, a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls, denoted by

$$T_t \in \mathcal{O}(L^p(w) \to L^q(w))$$

if there exist constants $\theta_1, \theta_2 > 0$ and c > 0 such that for every t > 0 and for any ball B, setting r = r(B),

$$\left(\int_{B} |T_t(\chi_B f)|^q \, dw\right)^{\frac{1}{q}} \lesssim \Upsilon\left(\frac{r}{\sqrt{t}}\right)^{\theta_2} \left(\int_{B} |f|^p \, dw\right)^{\frac{1}{p}}; \tag{2.24}$$

and for all $j \geq 2$,

$$\left(\oint_{B} |T_t(\chi_{C_j(B)} f)|^q dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2} e^{-\frac{c4^j r^2}{t}} \left(\oint_{C_j(B)} |f|^p dw\right)^{\frac{1}{p}}$$
(2.25)

and

$$\left(\int_{C_{j}(B)} |T_{t}(\chi_{B} f)|^{q} dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_{1}} \Upsilon\left(\frac{2^{j} r}{\sqrt{t}}\right)^{\theta_{2}} e^{-\frac{c 4^{j} r^{2}}{t}} \left(\int_{B} |f|^{p} dw\right)^{\frac{1}{p}}.$$
(2.26)

If the family of sublinear operators $\{T_z\}_{z\in\Sigma_{\mu}}$ is defined on a complex sector Σ_{μ} , we say that it satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls in Σ_{μ} if (2.24), (2.25) and (2.26) hold for $z \in \Sigma_{\mu}$ with t replaced by |z| in the righthand terms. We denote this by $T_z \in \mathcal{O}(L^p(w) \to L^q(w), \Sigma_{\mu}).$

We give some basic properties of off-diagonal estimates on balls as a series of lemmas taken from [7, Section 2.2]. The first follows immediately by real interpolation, the second by Hölder's inequality, and the third by duality.

Lemma 2.27. Given $1 \leq p_i \leq q_i \leq \infty$, i = 1, 2, if $T_t \in \mathcal{O}(L^{p_1}(w) \to L^{q_1}(w))$, and $T_t : L^{p_2}(w) \to L^{q_2}(w)$ is uniformly bounded, then $T_t \in \mathcal{O}(L^{p_{\theta}}(w) \to L^{q_{\theta}}(w))$, $0 < \theta < 1$, where

$$\frac{1}{p_{\theta}} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_{\theta}} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

Lemma 2.28. If $1 \le p \le p_1 \le q_1 \le q \le \infty$, then $\mathcal{O}(L^p(w) \to L^q(w)) \subset \mathcal{O}(L^{p_1}(w) \to L^{q_1}(w)).$

Lemma 2.29. If for some $1 \leq p \leq q \leq \infty$, $T_t \in \mathcal{O}(L^p(w) \to L^q(w))$, and the operators T_t are linear, then $T_t^* \in \mathcal{O}(L^{q'}(w) \to L^{p'}(w))$. (Here T_t^* is the dual operator for the inner product $\int_{\mathbb{R}^n} f g \, dw$.)

Lemma 2.30 ([7, Theorem 2.3]).

- (1) If $T_t \in \mathcal{O}(L^p(w) \to L^p(w))$, $1 \le p \le \infty$, then $T_t : L^p(w) \to L^p(w)$ is uniformly bounded.
- (2) If $1 \le p \le q \le r \le \infty$, $T_t \in \mathcal{O}(L^q(w) \to L^r(w))$, and $S_t \in \mathcal{O}(L^p(w) \to L^q(w))$, then $T_t \circ S_t \in \mathcal{O}(L^p(w) \to L^r(w))$.

Remark 2.31. If p < q, then $T_t \in \mathcal{O}(L^p(w) \to L^q(w))$ does not guarantee that T_t is bounded from $L^p(w)$ to $L^q(w)$.

Remark 2.32. Since complex sectors Σ_{μ} , $0 \leq \mu < \pi$, are closed under addition, the proof of Lemma 2.30 extends to give off-diagonal estimates on complex sectors $\mathcal{O}(L^p(w) \rightarrow L^q(w), \Sigma_{\mu})$.

Definition 2.33. Given $1 \le p \le q \le \infty$, a family of operators $\{T_t\}$ satisfies full off-diagonal estimates from $L^p(w)$ to $L^q(w)$, denoted by

$$T_t \in \mathcal{F}(L^p(w) \to L^q(w))$$

if there exist constants $C, c, \theta > 0$ such that given any closed sets E, F,

$$||T_t(f\chi_E)\chi_F||_{L^q(w)} \le Ct^{-\theta}e^{-\frac{cd^2(E,F)}{t}}||f\chi_E||_{L^p(w)}.$$

The connection between full off-diagonal estimates and off-diagonal estimates on balls is given in the following lemma from [7, Section 3.1].

Lemma 2.34. Given $1 \le p \le q \le \infty$:

- (1) if $T_t \in \mathcal{F}(L^p(w) \to L^q(w))$, then $T_t : L^p(w) \to L^q(w)$ is uniformly bounded;
- (2) $T_t \in \mathcal{F}(L^p(w) \to L^p(w))$ if and only if $T_t \in \mathcal{O}(L^p(w) \to L^p(w))$.

The importance of off-diagonal estimates is that they will let us prove weighted norm inequalities for the operators we are interested in. To do so we will make repeated use of two results first proved in [6]; however, we will use special cases of these results as given in [5, Theorems 2.2 and 2.4].

Theorem 2.35. Given $w \in A_2$ and $1 \leq p_0 < q_0 \leq \infty$, let T be a sublinear operator acting on $L^{p_0}(w)$, $\{\mathcal{A}_r\}_{r>0}$ a family of operators acting from a subspace \mathcal{D} of $L^{p_0}(w)$ into $L^{p_0}(w)$, and S an operator from \mathcal{D} into the space of measurable functions on \mathbb{R}^n . Suppose that for every $f \in \mathcal{D}$ and ball B with radius r,

$$\left(\oint_{B} |T(I - \mathcal{A}_{r})f|^{p_{0}} dw\right)^{1/p_{0}} \leq \sum_{j \geq 1} g(j) \left(\oint_{2^{j+1}B} |Sf|^{p_{0}} dw\right)^{1/p_{0}}$$
(2.36)

and

$$\left(\oint_{B} |T\mathcal{A}_{r}f|^{q_{0}} \, dw \right)^{1/q_{0}} \leq \sum_{j \geq 1} g(j) \left(\oint_{2^{j+1}B} |Tf|^{p_{0}} \, dw \right)^{1/p_{0}},\tag{2.37}$$

where $\sum g(j) < \infty$. Then for every $p, p_0 , and weights$

$$v \in A_{p/p_0}(w) \cap RH_{(q_0/p)'}(w),$$

there is a constant C such that for all $f \in \mathcal{D}$,

$$||Tf||_{L^p(v\,dw)} \le C ||Sf||_{L^p(v\,dw)}$$

Remark 2.38. In Theorem 2.35 and Theorem 2.39 below, the case $q_0 = \infty$ is understood in the sense that the $L^{q_0}(w)$ -average is replaced by the essential supremum. Also in Theorem 2.35, if $q_0 = \infty$, then the condition on v becomes $v \in A_{p/p_0}$.

Theorem 2.39. Given $w \in A_2$ with doubling order D, and $1 \leq p_0 < q_0 \leq \infty$, let $T : L^{q_0}(w) \to L^{q_0}(w)$ be a sublinear operator, and $\{\mathcal{A}_r\}_{r>0}$ a family of linear operators acting from L_c^{∞} into $L^{q_0}(w)$. Suppose that for every ball B with radius $r, f \in L_c^{\infty}$ with $\operatorname{supp}(f) \subset B$ and $j \geq 2$,

$$\left(\oint_{C_j(B)} |T(I - \mathcal{A}_r)f|^{p_0} \, dw \right)^{1/p_0} \le g(j) \left(\oint_B |f|^{p_0} \, dw \right)^{1/p_0}.$$
(2.40)

Suppose further that for every $j \ge 1$,

$$\left(\oint_{C_j(B)} |\mathcal{A}_r f|^{q_0} \, dw \right)^{1/q_0} \le g(j) \left(\oint_B |f|^{p_0} \, dw \right)^{1/p_0},\tag{2.41}$$

where $\sum g(j)2^{Dj} < \infty$. Then for all $p, p_0 , there exists a constant <math>C$ such that for all $f \in L_c^{\infty}$,

$$||Tf||_{L^p(w)} \le C ||f||_{L^p(w)}.$$

3. Off-diagonal estimates for the semigroup e^{-tL_w}

In this section we consider off-diagonal estimates for the semigroup associated to L_w . Throughout this and subsequent sections, let $w \in A_2$ and $A \in \mathcal{E}_n(w, \Lambda, \lambda)$ be fixed. Our goal is to characterize the set of pairs $(p, q), p \leq q$ such that these operators are in $\mathcal{O}(L^p(w) \rightarrow L^q(w))$. By Theorem 2.15 we have that

$$e^{-tL_w} \in \mathcal{F}(L^2(w) \to L^2(w)) \subset \mathcal{O}(L^2(w) \to L^2(w)).$$

We will show that in the (p, q)-plane this set contains a right triangle: see Figure 1.

FIGURE 1. (p,q) such that $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$



Let $\widetilde{\mathcal{J}}(L_w) \subset [1,\infty]$ be the set of all exponents p such that $e^{-tL_w} : L^p(w) \to L^p(w)$ is uniformly bounded for all t > 0. By Theorem 2.15 and Lemma 2.34, $2 \in \widetilde{\mathcal{J}}(L_w)$, and if it

contains more than one point, then by interpolation $\widetilde{\mathcal{J}}(L_w)$ is an interval. The set of pairs (p,q) such that $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^p(w))$ is completely characterized by the next result.

Proposition 3.1. There exists an interval $\mathcal{J}(L_w) \subset [1, \infty]$ such that $p, q \in \mathcal{J}(L_w)$ if and only if $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$. Furthermore, $\mathcal{J}(L_w)$ has the following properties:

- (1) $\mathcal{J}(L_w) \subset \widetilde{\mathcal{J}}(L_w);$
- (2) Int $\mathcal{J}(L_w) = \operatorname{Int} \widetilde{\mathcal{J}}(L_w);$
- (3) if $p_{-}(L_w)$ and $p_{+}(L_w)$ are respectively the left and right endpoints of $\mathcal{J}(L_w)$, then $p_{-}(L_w) \leq (2^*_w)'$ and $p_{+}(L_w) \geq 2^*_w$, where 2^*_w is as in Theorem 2.1. In particular, $2 \in \operatorname{Int}(\mathcal{J}(L_w))$.

Remark 3.2. The smaller the value of r_w , the better our bounds on the size of the set $\mathcal{J}(L_w)$. In the limiting case when $w \in A_1$, we have that $p_-(L_w) \leq \frac{2n}{n+2}$ and $p_+(L_w) \geq \frac{2n}{n-2}$. These values should be compared to the estimates in [2, Corollary 4.6] for the non-degenerate case that corresponds to the case w = 1.

We get two corollaries to Proposition 3.1. The first gives us weighted off-diagonal estimates.

Corollary 3.3. Let $p_{-}(L_w) . If <math>v \in A_{p/p_{-}(L_w)}(w) \cap RH_{(p_{+}(L_w)/q)'}(w)$, then $e^{-tL_w} \in \mathcal{O}(L^p(v\,dw) \to L^q(v\,dw))$.

Proof. By Proposition 3.1, if $p_{-}(L_w) , then <math>e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$. Therefore, by [7, Proposition 2.6], if $v \in A_{p/p_{-}(L_w)}(w) \cap RH_{(p_{+}(L_w)/q)'}(w)$, then we have that $e^{-tL_w} \in \mathcal{O}(L^p(v \, dw) \to L^q(v \, dw))$.

As our second corollary we get off-diagonal estimates for the holomorphic extension of the semigroup.

Corollary 3.4. For any ν , $0 < \nu < \frac{\pi}{2} - \vartheta$, and for any $p \leq q$ such that $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$, then for all $m \in \mathbb{N} \cup \{0\}, (zL_w)^m e^{-zL_w} \in \mathcal{O}(L^p(w) \to L^q(w), \Sigma_\nu)$.

Proof. This follows from [7, Theorem 4.3] and the fact that, by Theorem 2.15, for these values of $z, e^{-zL_w} \in \mathcal{F}(L^2(w) \to L^2(w))$.

Proof of Proposition 3.1. Fix $2 < q < 2_w^*$ (If $w \in A_1$ we let $q = 2_w^* = 2^*$.) We will show that $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$. Given this, then we also have that $e^{-tL_w} \in \mathcal{O}(L^{q'}(w) \to L^2(w))$. For if L_w^* is the adjoint of L_w (with respect to $L^2(w)$), then $L_w^* = -w^{-1} \operatorname{div}(A^* \nabla f)$ and the same estimates hold for L_w^* . Hence, $e^{-tL_w^*} \in \mathcal{O}(L^2(w) \to L^q(w))$, and so by Lemma 2.29, $e^{-tL_w} \in \mathcal{O}(L^{q'}(w) \to L^2(w))$. Since e^{-tL_w} is a semigroup, by Lemma 2.30 we have that $e^{-tL_w} \in \mathcal{O}(L^{q'}(w) \to L^q(w))$. Therefore, by [7, Proposition 4.1], we have that there exists an interval $\mathcal{J}(L_w)$ and Properties (1) and (2) hold. Moreover, we have that $[q',q] \subset \mathcal{J}(L_w)$, so if we let $q \to 2_w^*$, then we immediately get Property (3).

It therefore remains to prove that $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$. We first show (2.24). Fix *B* and for brevity write r = r(B) and $C_j = C_j(B)$. By our choice of *q* the Poincaré inequality (2.3) holds. Moreover, as we noted above e^{-tL_w} , $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$; we may assume that the same exponents θ_1 , θ_2 hold for both operators. We thus get that

$$\begin{split} \left(\oint_{B} |e^{-tL_{w}}(\chi_{B} f)|^{q} dw \right)^{\frac{1}{q}} \\ &\leq \left| \left(e^{-tL_{w}}(\chi_{B} f) \right)_{B,w} \right| + \left(\oint_{B} |e^{-tL_{w}}(\chi_{B} f)(x) - \left(e^{-tL_{w}}(\chi_{B} f) \right)_{B,w} |^{q} dw(x) \right)^{\frac{1}{q}} \\ &\lesssim \left(\oint_{B} |e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} + r \left(\oint_{B} |\nabla e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim \left(1 + \frac{r}{\sqrt{t}} \right) \Upsilon \left(\frac{r}{\sqrt{t}} \right)^{\theta_{2}} \left(\oint_{B} |f|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim \Upsilon \left(\frac{r}{\sqrt{t}} \right)^{1+\theta_{2}} \left(\oint_{B} |f|^{2} dw \right)^{\frac{1}{2}}. \end{split}$$

The proof that (2.25) holds is gotten by nearly the same argument:

$$\begin{split} \left(\oint_{B} |e^{-tL_{w}}(\chi_{C_{j}} f)|^{q} dw \right)^{\frac{1}{q}} \\ &\leq |\left(e^{-tL_{w}}(\chi_{C_{j}} f)\right)_{B,w}| + \left(\oint_{B} |e^{-tL_{w}}(\chi_{C_{j}} f)(x) - \left(e^{-tL_{w}}(\chi_{C_{j}} f)\right)_{B,w}|^{q} dw(x) \right)^{\frac{1}{q}} \\ &\lesssim \left(\oint_{B} |e^{-tL_{w}}(\chi_{C_{j}} f)|^{2} dw \right)^{\frac{1}{2}} + r \left(\oint_{B} |\nabla e^{-tL_{w}}(\chi_{C_{j}} f)|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim 2^{j\theta_{1}} \left(1 + \frac{r}{\sqrt{t}} \right) \Upsilon \left(\frac{2^{j}r}{\sqrt{t}} \right)^{\theta_{2}} e^{-\frac{c4^{j}r^{2}}{t}} \left(\oint_{C_{j}} |f|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim 2^{j\theta_{1}} \Upsilon \left(\frac{2^{j}r}{\sqrt{t}} \right)^{1+\theta_{2}} e^{-\frac{c4^{j}r^{2}}{t}} \left(\oint_{C_{j}} |f|^{2} dw \right)^{\frac{1}{2}}. \end{split}$$

Finally, to prove that (2.26) holds we use a covering argument. Fix $j \ge 2$; then we can cover the annulus C_j by a collection of balls $\{B_k\}_{k=1}^N$, $r(B_k) = 2^{j-2}r$, with centers $x_{B_k} \in C_j$. The number of balls required, N, depends only on the dimension. For any such ball, since dw is a doubling measure we have that

$$\begin{split} \left(\oint_{B_{k}} |e^{-tL_{w}}(\chi_{B} f)|^{q} dw \right)^{\frac{1}{q}} \\ &\leq |\left(e^{-tL_{w}}(\chi_{B} f)\right)_{B_{k},w}| + \left(\oint_{B_{k}} |e^{-tL_{w}}(\chi_{B} f)(x) - \left(e^{-tL_{w}}(\chi_{B} f)\right)_{B_{k},w}|^{q} dw(x) \right)^{\frac{1}{q}} \\ &\lesssim \left(\oint_{B_{k}} |e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} + r(B_{k}) \left(\oint_{B_{k}} |\nabla e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim \left(\oint_{2^{j+2}B\setminus 2^{j-1}B} |e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} \end{split}$$

$$+ 2^{j} r \left(\oint_{2^{j+2} B \setminus 2^{j-1} B} |\nabla e^{-t L_{w}} (\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}}.$$

If $j \geq 3$, then $2^{j+2} B \setminus 2^{j-1} B = C_{j+1} \cup C_j \cup C_{j-1}$; then to estimate the last two terms we use the fact that e^{-tL_w} , $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$ and apply (2.26) with p = q = 2in each annulus C_i , $j-1 \le i \le j+1$. (These annuli have comparable measure since dw is a doubling measure so we can divide the average up into three averages). If j = 2, then $2^4 B \setminus 2B = C_3 \cup C_2 \cup (4B \setminus 2B)$. On C_3 and C_2 we argue as before using (2.26). On $4B \setminus B$ we apply [7, Lemma 6.1]. (We note that in the notation there $\widehat{C}_1(B) = 4 B \setminus 2 B$.)

If we combine all of these estimates, we get that for every $j \ge 2$,

$$\left(\oint_{B_k} |e^{-tL_w}(\chi_B f)|^q \, dw \right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \left(1 + \frac{2^j r}{\sqrt{t}} \right) \Upsilon \left(\frac{2^j r}{\sqrt{t}} \right)^{\theta_2} \left(\oint_B |f|^2 \, dw \right)^{\frac{1}{2}}$$
$$\lesssim 2^{j\theta_1} \Upsilon \left(\frac{2^j r}{\sqrt{t}} \right)^{1+\theta_2} e^{-\frac{c4^j r^2}{t}} \left(\oint_B |f|^2 \, dw \right)^{\frac{1}{2}}$$

Since $C_j \subset \bigcup_k B_k$, we can sum in k to get

$$\left(\oint_{C_j(B)} |e^{-tL_w}(\chi_B f)|^q \, dw \right)^{\frac{1}{q}} \lesssim \sum_{k=1}^N \left(\oint_{B_k} |e^{-tL_w}(\chi_B f)|^q \, dw \right)^{\frac{1}{q}}$$

$$\lesssim 2^{j\theta_1} \Upsilon \left(\frac{2^j r}{\sqrt{t}} \right)^{1+\theta_2} e^{-\frac{c4^j r^2}{t}} \left(\oint_B |f|^2 \, dw \right)^{\frac{1}{2}}.$$

$$\text{ and is completes the proof that } e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w)).$$

This completes the proof that $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w)).$

4. The functional calculus

In this section we show that the operator L_w has an $L^p(w)$ holomorphic functional calculus. As we discussed in Section 2 above, we know already that if φ is a bounded holomorphic function on $\Sigma_{\mu}, \mu \in (\vartheta, \pi)$, then $\varphi(L_w)$ is a bounded operator on $L^2(w)$. Recall that for any $\mu \in (\vartheta, \pi)$, we say that $\varphi \in \mathcal{H}_0^\infty(\Sigma_\mu)$ if for some c, s > 0

$$|\varphi(z)| \le c|z|^s (1+|z|)^{-2s}, \qquad z \in \Sigma_{\mu}.$$
 (4.1)

We say that L_w has a bounded holomorphic functional calculus on $L^p(w)$ if for any such φ ,

$$\|\varphi(L_w)f\|_{L^p(w)} \le C \, \|\varphi\|_{\infty} \, \|f\|_{L^p(w)}, \qquad f \in L^p(w) \cap L^2(w), \tag{4.2}$$

where C depends only on p, w, ϑ and μ (but not on the decay of φ). By a standard density argument, (4.2) implies that $\varphi(L_w)$ extends to a bounded operator on all of $L^p(w)$. Furthermore, we then have that this inequality holds if φ is any bounded holomorphic function. For the details of this extension, see [23, 27].

Proposition 4.3. Let
$$p_{-}(L_w) and $\mu \in (\vartheta, \pi)$. Then for any $\varphi \in \mathcal{H}_0^{\infty}(\Sigma_{\mu})$,
 $\|\varphi(L_w)f\|_{L^p(w)} \leq C \, \|\varphi\|_{\infty} \, \|f\|_{L^p(w)}$, (4.4)$$

with C independent of φ and f. Hence, L_w has a bounded holomorphic functional calculus on $L^p(w)$. Moreover, if $v \in A_{p/p_{-}(L_w)}(w) \cap RH_{(p_{+}(L_w)/p)'}(w)$ then L_w also has a bounded holomorphic functional calculus on $L^p(v dw)$:

$$\|\varphi(L_w)f\|_{L^p(v\,dw)} \le C \,\|\varphi\|_{\infty} \,\|f\|_{L^p(v\,dw)},\tag{4.5}$$

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with C independent of φ and f.

Proof. For brevity, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. By density it will suffice to assume that $f \in L_c^{\infty}$. Fix $\varphi \in \mathcal{H}_0^{\infty}(\Sigma_{\mu})$; by linearity we may assume that $\|\varphi\|_{\infty} = 1$.

We divide the proof into two steps. We first obtain (4.4) for $p_- by applying$ $Theorem 2.39 and following the ideas in [2]. To do so, we will pick <math>q_0 = 2$ and $p_0 > p_$ arbitrarily close to p_- . In the second step, using some ideas from [5], we will use Theorem 2.35 to get (4.5); in particular this yields (4.4) for every $2 by taking <math>v \equiv 1$. To apply Theorem 2.35 we will choose $p_0 > p_-$ arbitrarily close to p_- and $q_0 < p_+$ arbitrarily close to p_+ . We will also use the fact that $\varphi(L_w)$ is bounded on $L^{p_0}(w)$; this follows from the first step choosing $p_- < p_0 < 2$.

To apply Theorem 2.39, fix $p_{-} < p_{0} < p < 2$ and let $q_{0} = 2$, $T = \varphi(L_{w})$, and

$$\mathcal{A}_{r}f(x) = \left(I - (I - e^{-r^{2}L_{w}})^{m}\right)f(x), \tag{4.6}$$

where m is a positive integer that will be chosen below. We first show that inequality (2.41) holds. By Proposition 3.1 we have that $e^{-tL_w} \in \mathcal{O}(L^{p_0}(w) \to L^2(w))$. Using

$$\mathcal{A}_{r} = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} e^{-kr^{2}L_{w}}, \qquad (4.7)$$

and that for each fixed m and $1 \le k \le m$

$$\Upsilon\left(\frac{r}{\sqrt{kt}}\right) \le \sqrt{m}\Upsilon\left(\frac{r}{t}\right) \quad \text{and} \quad \exp\left(-\frac{c}{k}\frac{4^{j}r^{2}}{t^{2}}\right) \le \exp\left(-\frac{c}{m}\frac{4^{j}r^{2}}{t^{2}}\right),$$

Proposition 3.1 implies

$$\mathcal{A}_r \in \mathcal{O}(L^p(w) \to L^q(w)), \quad \text{for all } p_-(L_w)
(4.8)$$

In particular, we have that $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^2(w))$. Thus, given any ball B with radius r, if $\operatorname{supp}(f) \subset B$, then for all $j \ge 1$,

$$\left(\int_{C_j(B)} \left|\mathcal{A}_r f\right|^2 dw\right)^{1/2} \lesssim 2^{j\theta_1} \Upsilon \left(2^j\right)^{\theta_2} e^{-c4^j} \left(\int_B \left|f\right|^{p_0} dw\right)^{1/p_0}.$$
(4.9)

This establishes (2.41) with $g(j) = C 2^{j(\theta_1 + \theta_2)} e^{-c4^j}$, for in this case we have that

$$\sum_{j\geq 1} 2^{j(\theta_1+\theta_2+D)}e^{-c4^j}<\infty,$$

where D is the doubling constant of w.

We next prove that (2.40) holds. Since $\varphi(z)(1 - e^{-r^2 z})^m \in \mathcal{H}_0^{\infty}(\Sigma_{\min\{\mu, \pi/2\}})$, by the functional calculus representation (2.10) we have that

$$\varphi(L_w)(I - \mathcal{A}_r) f = \int_{\Gamma} e^{-z L_w} f \eta(z) \, dz,$$

where $\Gamma = \partial \Sigma_{\frac{\pi}{2}-\theta}$, with $0 < \vartheta < \theta < \nu < \min\{\mu, \pi/2\}$, and we choose θ so that the hypotheses of Corollary 3.4 are satisfied for $z \in \Gamma$. Moreover, we have the estimate

$$\left|\eta\left(z\right)\right| \lesssim \frac{r^{2m}}{\left|z\right|^{m+1}};$$

see [2, Section 5.1] for details.

We can now argue as follows: given a ball B with radius r, for each $j \ge 2$, by Minkowski's inequality and Corollary 3.4 (since $p_0 \in \text{Int } \mathcal{J}(L_w)$),

$$\begin{aligned} \left(\oint_{C_{j}(B)} \left| \varphi \left(L_{w} \right) \left(I - \mathcal{A}_{r(B)} \right) f \right|^{p_{0}} dw \right)^{1/p_{0}} \\ &= \left(\oint_{C_{j}(B)} \left| \int_{\Gamma} e^{-z \, L_{w}} f \, \eta(z) \, dz \right|^{p_{0}} dw \right)^{1/p_{0}} \\ &\lesssim \int_{\Gamma} \left(\oint_{C_{j}(B)} \left| e^{-z \, L_{w}} f \right|^{p_{0}} dw \right)^{1/p_{0}} \frac{r^{2m}}{|z|^{m+1}} \left| dz \right| \\ &\lesssim \left(\oint_{B} \left| f \right|^{p_{0}} dw \right)^{1/p_{0}} \int_{\Gamma} \frac{r^{2m}}{|z|^{m+1}} 2^{j\theta_{1}} \Upsilon \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{\theta_{2}} e^{-c\frac{r^{2}}{|z|}4^{j}} \left| dz \right| \\ &= \left(\oint_{B} \left| f \right|^{p_{0}} dw \right)^{1/p_{0}} 2^{j(\theta_{1}-2m)} \int_{0}^{\infty} \sigma^{2m} \Upsilon \left(\sigma \right)^{\theta_{2}} e^{-c\sigma^{2}} \frac{d\sigma}{\sigma} \\ &\lesssim 2^{j(\theta_{1}-2m)} \left(\oint_{B} \left| f \right|^{p_{0}} dw \right)^{1/p_{0}}; \end{aligned}$$

$$(4.10)$$

the final inequality holds (i.e., the integral in σ converges) provided $2m > \theta_2$. Moreover, if we choose $2m > \theta_1 + D$, we have that (2.40) holds with $g(j) = C 2^{(j-1)(\theta_1 - 2m)}$ and

$$\sum_{j\geq 2} g\left(j\right) 2^{jD} \lesssim \sum_{j\geq 2} 2^{j\left(\theta_1 + D - 2m\right)} < \infty$$

We have shown that inequalities (2.40) and (2.41) hold, and so by Theorem 2.39 inequality (4.4) holds for all p such that $p_{-} .$

We will now apply Theorem 2.35 to show that (4.5) holds for $p_- . (In$ $equality (4.4) then follows for <math>2 if we take <math>v \equiv 1$.) Fix $p, p_- and <math>v \in A_{p/p_-}(w) \cap RH_{(p_+/p)'}(w)$. By the openness properties of the A_q and RH_s classes there exist p_0, q_0 such that

$$p_{-} < p_0 < \min\{p, 2\} \le p < q_0 < p_+, \qquad v \in A_{p/p_0}(w) \cap RH_{(q_0/p)'}(w).$$

Let $T = \varphi(L_w)$, $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, S = I, and fix the above values of p_0 and q_0 . By the previous argument we have that $\varphi(L_w)$ is bounded on $L^{p_0}(w)$.

We first show that (2.36) holds. Fix a ball B and decompose f as

$$f = \sum_{j \ge 1} f \chi_{C_j(B)} := \sum_{j \ge 1} f_j.$$
(4.11)

Then, by the same functional calculus argument as given above, we have that for each j,

$$\begin{split} \left(\oint_{B} |\varphi(L_{w})(I - \mathcal{A}_{r})f_{j}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \\ &= \left(\oint_{B} \left| \int_{\Gamma} e^{-z L_{w}} f_{j} \eta(z) dz \right|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \\ &\lesssim \int_{\Gamma} \left(\oint_{B} \left| e^{-z L_{w}} f_{j} \right|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \frac{r^{2m}}{|z|^{m+1}} |dz| \end{split}$$

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$$\lesssim \left(\oint_{C_{j}(B)} |f|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} 2^{j(\theta_{1}-2m)} \int_{\Gamma} \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{2m} \Upsilon \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{\theta_{2}} e^{-\frac{c4^{j}r^{2}}{|z|}} \frac{|dz|}{|z|} \\ \lesssim 2^{j(\theta_{1}-2m)} \left(\oint_{C_{j}(B)} |f|^{p_{0}} dw \right)^{\frac{1}{p_{0}}};$$

the last inequality holds provided $2m > \theta_2$. Hence, since $2^{j+1}B \supset C_j$, by Minkowski's inequality we have (since the sum $\sum f_j$ is finite for $f \in L_c^{\infty}$)

$$\left(\oint_{B} |\varphi(L_{w})(I - \mathcal{A}_{r})f|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \leq \sum_{j \geq 1} \left(\oint_{B} |\varphi(L_{w})(I - \mathcal{A}_{r})f_{j}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}}$$
$$\lesssim \sum_{j \geq 1} 2^{j(\theta_{1} - 2m)} \left(\oint_{2^{j+1}B} |f|^{p_{0}} dw \right)^{\frac{1}{p_{0}}}.$$

This establishes (2.36) with $g(j) = C 2^{j(\theta_1 - 2m)}$. If we take $2m > \max{\{\theta_1, \theta_2\}}$, then $\sum g(j) < \infty$.

We now show that (2.37) holds. Fix a ball B and $j \ge 1$. Since $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$ (see (4.8)),

$$\left(\int_{B} \left|\mathcal{A}_{r}\left(\chi_{C_{j}(B)}\varphi(L_{w})f\right)\right|^{q_{0}}dw\right)^{\frac{1}{q_{0}}} \lesssim 2^{j\theta_{1}}\Upsilon\left(2^{j}\right)^{\theta_{2}}e^{-c4^{j}}\left(\int_{C_{j}(B)} |\varphi\left(L_{w}\right)f|^{p_{0}}d\mu\right)^{\frac{1}{p_{0}}}.$$

Therefore, since $\varphi(L_w)$ and \mathcal{A}_r commute, by Minkowski's inequality we obtain

$$\left(\oint_{B} |\varphi\left(L_{w}\right)\mathcal{A}_{r}f|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \lesssim \sum_{j\geq 1} 2^{j(\theta_{1}+\theta_{2})} e^{-c4^{j}} \left(\oint_{C_{j}(B)} |\varphi\left(L_{w}\right)f|^{p_{0}} d\mu \right)^{\frac{1}{p_{0}}}.$$

This establishes (2.37) with $g(j) = C 2^{j(\theta_1 + \theta_2)} e^{-c4^j}$; again, $\sum g(j) < \infty$. Therefore, our proof is complete.

5. Square function estimates for the semigroup

In this section we prove $L^{p}(w)$ norm inequalities for the vertical square function associated to the semigroup $e^{-tL_{w}}$:

$$g_{L_w}f(x) = \left(\int_0^\infty \left| (tL_w)^{1/2} e^{-tL_w} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}$$

Proposition 5.1. Let $p_{-}(L_w) . Then$

$$\|g_{L_w}f\|_{L^p(w)} \approx \|f\|_{L^p(w)}.$$
(5.2)

Conversely if for some p the equivalence (5.2) holds, then $p \in \tilde{\mathcal{J}}(L_w)$ —i.e., the interior of the interval on which (5.2) holds is $(p_-(L_w), p_+(L_w))$.

Moreover, if $v \in A_{p/p_{-}(L_w)}(w) \cap RH_{(p_{+}(L_w)/p)'}(w)$, then

$$\|g_{L_w}f\|_{L^p(v\,dw)} \approx \|f\|_{L^p(v\,dw)}.$$
(5.3)

We note that the upper bounds in the previous result could be obtained by combining Proposition 4.3 with the operator theory methods developed in [14]. To reach a wider audience we present a self-contained harmonic analysis proof. We will use an auxiliary Hilbert space related to square functions, following the approach in [5]. Let \mathbb{H} denote the Hilbert space $L^2((0,\infty), \frac{dt}{t})$ with norm

$$|\!|\!|h|\!|\!| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

In particular, we have that

$$g_{L_w}f(x) = \|\varphi(L,\cdot)f(x)\|$$

where $\varphi(z,t) = (tz)^{1/2} e^{-tz}$. Furthermore, we define $L^p_{\mathbb{H}}(w)$ to be the space of \mathbb{H} -valued functions with the norm

$$\|h\|_{L^p_{\mathbb{H}}(w)} = \left(\int_{\mathbb{R}^n} \|h(x,\cdot)\|^p \, dw(x)\right)^{\frac{1}{p}}.$$

The following lemma lets us extend scalar valued inequalities to \mathbb{H} -valued inequalities. For a proof, see [5, Lemma 7.4] and the references given there.

Lemma 5.4. Given a Borel measure μ on \mathbb{R}^n , let \mathcal{D} be a subspace of \mathcal{M} , the space of measurable functions in \mathbb{R}^n , and let S, T be linear operators from \mathcal{D} into \mathcal{M} . Fix $1 \leq p \leq q < \infty$ and suppose there exists $C_0 > 0$ such that for all $f \in \mathcal{D}$,

$$||Tf||_{L^{q}(\mu)} \leq C_{0} \sum_{j \geq 1} \alpha_{j} ||Sf||_{L^{p}(F_{j},\mu)},$$

where the F_j are measurable subsets of \mathbb{R}^n and $\alpha_j \geq 0$. Then there is a \mathbb{H} -valued inequality with the same constant: for all $f : \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{C}$ such that for almost all t > 0, $f(\cdot, t) \in \mathcal{D}$,

$$\|Tf\|_{L^q_{\mathbb{H}}(\mu)} \le C_0 \sum_{j\ge 1} \alpha_j \|Sf\|_{L^p_{\mathbb{H}}(F_j,\mu)}.$$

The extension of a linear operator T on \mathbb{C} -valued functions to \mathbb{H} -valued functions is defined for $x \in \mathbb{R}^n$ and t > 0 by $(Th)(x,t) = T(h(\cdot,t))(x)$, that is, t can be considered as a parameter and T acts only on the variable in \mathbb{R}^n .

Proof of Proposition 5.1. We shall first prove the upper bound inequalities. We first claim that the upper bound inequality in (5.2) holds for p = 2. Indeed, since $\varphi(z) = z^{1/2}e^{-z} \in \mathcal{H}_0^{\infty}(\Sigma_{\mu})$, it follows from (2.13) that we have the bound

$$||g_{L_w}f||_{L^2(w)} \lesssim ||f||_{L^2(w)}.$$

For brevity, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. As in previous proofs, we divide our proof into two steps. We will first prove the upper bound in (5.2) for $p_- by applying$ $Theorem 2.39. Fix <math>p_- , and let <math>\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where *m* will be chosen below. Notice that, by (4.8), \mathcal{A}_r is bounded on $L^{q_0}(w)$ for each *m*. Fix $f \in L_c^\infty$; the result for general $f \in L^p(w)$ then follows by a density argument.

We have that $(tL_w)^{1/2}e^{-tL_w}(I-\mathcal{A}_r)f = \varphi(L_w,t)f$, where

$$\varphi(z,t) = (tz)^{1/2} e^{-tz} (1 - e^{-r^2 z})^m$$

Moreover, since $\varphi(\cdot, t) \in \mathcal{H}_0^{\infty}(\Sigma_{\min\{\mu, \pi/2\}})$, by the functional calculus representation (2.10) we have that

$$(tL_w)^{1/2} e^{-tL_w} \left(I - \mathcal{A}_r\right) f = \int_{\Gamma} \eta\left(z, t\right) e^{-zL_w} f \, dz,$$

where $\Gamma = \partial \Sigma_{\frac{\pi}{2}-\theta}$, with $0 < \vartheta < \theta < \nu < \min\{\mu, \pi/2\}$, and we choose θ so that the hypotheses of Corollary 3.4 are satisfied for $z \in \Gamma$. Moreover, we have the estimate (see [2, 5])

$$|\eta(z,t)| \lesssim \frac{t^{\frac{1}{2}}r^{2m}}{(|z|+t)^{m+\frac{3}{2}}}, \qquad z \in \Gamma;$$

therefore,

$$\| \eta(z, \cdot) \| = \left(\int_0^\infty |\eta(z, t)|^2 \, \frac{dt}{t} \right)^{1/2} \lesssim \frac{r^{2m}}{|z|^{m+1}}.$$
(5.5)

Now let $f \in L_c^{\infty}$ with supp $(f) \subset B$. For $j \ge 2$, we have

$$\begin{aligned} \left(\oint_{C_{j}(B)} |g_{L_{w}} \left(I - \mathcal{A}_{r} \right) f|^{p} dw \right)^{1/p} \\ &= \left(\oint_{C_{j}(B)} \left| \left(\int_{0}^{\infty} \left| \int_{\Gamma_{\frac{\pi}{2} - \theta}} \eta(z, t) e^{-zL_{w}} f dz \right|^{2} \frac{dt}{t} \right)^{1/2} \right|^{p} dw \right)^{1/p} \\ &\leq \left(\oint_{C_{j}(B)} \left| \int_{\Gamma_{\frac{\pi}{2} - \theta}} |e^{-zL_{w}} f| || \eta(z, \cdot) || |d| z| \right|^{p} dw \right)^{1/p} \\ &\lesssim \int_{\Gamma_{\frac{\pi}{2} - \theta}} \left(\oint_{C_{j}(B)} |e^{-zL_{w}} f|^{p} dw \right)^{1/p} \frac{r^{2m}}{|z|^{m+1}} d|z| \\ &\lesssim 2^{j\theta_{1}} \left(\oint_{B} |f|^{p} dw \right)^{1/p} \int_{\Gamma_{\frac{\pi}{2} - \theta}} \Upsilon \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{\theta_{2}} e^{-\frac{c4^{j}r^{2}}{|z|}} \frac{r^{2m}}{|z|^{m}} \frac{d|z|}{|z|} \\ &\lesssim 2^{j\theta_{1}} 4^{-mj} \left(\oint_{B} |f|^{p} dw \right)^{1/p}; \end{aligned}$$
(5.6)

in the second inequality we applied (5.5) and the off-diagonal estimates for e^{-zL_w} from Corollary 3.4, and the last inequality holds provided $2m > \theta_2$. Thus, if we take $m > \theta_1 + D$, where D is the doubling order of w, the operator g_{L_w} satisfies (2.40) in Theorem 2.39 with $g(j) = C 2^{j(\theta_1 - 2m)}$. Since we already established (2.41) in (4.9) with $g(j) = C 2^{j(\theta_1 + \theta_2)} 4^{-mj}$, the hypotheses of Theorem 2.39 are satisfied if $m > \theta_1 + \theta_2 + D$. Therefore, for each $p_- there exists a constant <math>C$ such that

$$\|g_{L_w}f\|_{L^p(w)} \le C \|f\|_{L^p(w)}.$$
(5.7)

In the second part of the proof we will show that if $p_- and <math>v \in A_{p/p_-}(w) \cap RH_{(p_+/p)'}(w)$, then the upper bound inequality in (5.3) holds. If we take $v \equiv 1$, then we immediately get (5.2). To do so, first note that if we fix p and v, then by the openness properties of weights there exist p_0 , q_0 such that

$$p_- < p_0 < \min\{p, 2\} \le \max\{p, 2\} < q_0 < p_+$$

and $v \in A_{p_0/p_-}(w) \cap RH_{(q_0/p)'}(w).$

We will apply Theorem 2.35 with $T = g_{L_w}$, S = I and $\mathcal{D} = L^{p_0}(w)$ (again, note that by (4.8), \mathcal{A}_r is bounded on $L^{p_0}(w)$). We first prove that inequality (2.36) holds. For each $j \geq 1$, let $f_j = f\chi_{C_j(B)}$; then we can argue exactly as we did in the proof of (5.6), exchanging the roles of B and $C_i(B)$, to get

$$\left(\oint_{B} |g_{L_w}(I-\mathcal{A}_r)f_j|^p \, dw\right)^{\frac{1}{p}} \lesssim 2^{j\theta_1} 4^{-mj} \left(\oint_{2^{j+1}B} |f|^p \, dw\right)^{\frac{1}{p}}.$$

Inequality (2.36) follows if we sum over all j and take $g(j) = 2^{j\theta_1} 4^{-mj}$.

We will now show that inequality (2.37) holds. To do so, we need to prove a vector-valued version of a key inequality. By Proposition 3.1, given a ball B with radius r, then for all $j \ge 1, g$ with $\operatorname{supp}(g) \subset C_j(B)$, and $1 \le k \le m$,

$$\left(\oint_{B} |e^{-kr^{2}L_{w}}g|^{q_{0}} dw\right)^{\frac{1}{q_{0}}} \leq C_{0}2^{j(\theta_{1}+\theta_{2})}e^{-\alpha 4^{j}} \left(\oint_{C_{j}(B)} |g|^{p_{0}} dw\right)^{\frac{1}{p_{0}}}.$$
(5.8)

We now apply Lemma 5.4 with S = I and $T : L^{p_0}(w) \to L^{q_0}(w)$ given by

$$Tg = (C_0 2^{j(\theta_1 + \theta_2)} e^{-\alpha 4^j})^{-1} \frac{w(2^{j+1}B)^{\frac{1}{p_0}}}{w(B)^{\frac{1}{q_0}}} \chi_B e^{-kr^2 L_w} (g\chi_{C_j(B)})$$

This yields the \mathbb{H} -valued extension of (5.8): for all $g \in L^{p_0}_{\mathbb{H}}(w)$ with $\operatorname{supp}(g(\cdot, t)) \subset C_j(B)$, t > 0, we have that

$$\left(\oint_{B} \left\| e^{-kr^{2}L_{w}}g(x,\cdot) \right\|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \leq C_{0}2^{j(\theta_{1}+\theta_{2})}e^{-\alpha 4^{j}} \left(\oint_{C_{j}(B)} \left\| g(x,\cdot) \right\|^{p_{0}} dw \right)^{\frac{1}{p_{0}}}.$$
(5.9)

Given an arbitrary $g \in L^{p_0}_{\mathbb{H}}(w)$, decompose it as

$$g(x,t) = \sum_{j \ge 1} g(x,t) \chi_{C_j(B)}(x) = \sum_{j \ge 1} g_j(x,t).$$

Then inequality (5.9) yields

$$\left(\oint_{B} \| e^{-kr^{2}L_{w}}g(x,\cdot) \| ^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \leq \sum_{j\geq 1} \left(\oint_{B} \| e^{-kr^{2}L_{w}}g_{j}(x,\cdot) \| ^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \\ \lesssim \sum_{j\geq 1} 2^{j(\theta_{1}+\theta_{2})}e^{-\alpha 4^{j}} \left(\oint_{2^{j+1}B} \| g(x,\cdot) \| ^{p_{0}} dw \right)^{\frac{1}{p_{0}}}.$$
(5.10)

Define $g(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$. Then $g_{L_w} f(x) = |||g(x,\cdot)|||$; by our choice of p_0 and the first step of the proof we have that $g \in L^{p_0}_{\mathbb{H}}(w)$. Moreover, since for each t > 0, $(tL_w)^{1/2} e^{-tL_w}$ and $e^{-kr^2L_w}$ commute,

$$g_{L_w}(e^{-kr^2L_w}f)(x) = ||e^{-kr^2L_w}g(x,\cdot)||$$

We can now use (4.7) and (5.10) to get that

$$\left(\oint_{B} |g_{L_{w}} \mathcal{A}_{r} f|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \lesssim \sum_{k=1}^{m} \left(\oint_{B} |\!|\!| e^{-kr^{2}L_{w}} g(x, \cdot) |\!|\!|^{q_{0}} dw \right)^{\frac{1}{q_{0}}}$$

$$\lesssim \sum_{j\geq 1} 2^{j(\theta_1+\theta_2)} e^{-\alpha 4^j} \left(\oint_{2^{j+1}B} |g_{L_w}f|^p \, dw \right)^{1/p_0}$$

This proves (2.37) with $g(j) = C 2^{j(\theta_1 + \theta_2)} e^{-c4^j}$. Therefore, by Theorem 2.35 we get that $\|g_{L_w} f\|_{L^p(v \, dw)} \lesssim \|f\|_{L^p(v \, dw)}$.

It remains to show the reverse inequalities. We will prove the lower bound in (5.3); then the lower bound in (5.2) holds if we take $v \equiv 1$. Fix $p_- and <math>v \in A_{p/p_-(L_w)}(w) \cap RH_{(p_+(L_w)/p)'}(w)$. By the duality properties of weights [6, Lemma 4.4] and since $p_{\pm}(L_w)' = p_{\pm}(L_w^*)$, where L_w^* is the adjoint (on $L^2(w)$) of L_w ,

$$v^{1-p'} \in A_{p'/p_{-}(L^*)}(w) \cap RH_{(p_{+}(L^*)/p')'}(w).$$
 (5.11)

We now proceed as in the proof of [5, Theorem 7.3]. Given $F \in L^p_{\mathbb{H}}(v \, dw) \cap L^2_{\mathbb{H}}(w)$, and $x \in \mathbb{R}^n$ we set

$$T_{L_w}F(x) = \int_0^\infty (t L_w)^{1/2} e^{-t L_w} F(x,t) \frac{dt}{t}.$$
 (5.12)

Recall that $(t L_w)^{1/2} e^{-t L_w} F(x,t) = (t L_w)^{1/2} e^{-t L_w} (F(\cdot,t))(x)$. Hence, T_{L_w} maps \mathbb{H} -valued functions to \mathbb{C} -valued functions. For $h \in L^{p'}(v^{1-p'} dw) \cap L^2(w)$ with $\|h\|_{L^{p'}(v^{1-p'} dw)} = 1$, we have that

$$\begin{split} \left| \int_{\mathbb{R}^n} T_{L_w} F \,\overline{h} \, dw \right| &= \left| \int_{\mathbb{R}^n} \int_0^\infty F(x,t) \,\overline{(t \, L_w^*)^{1/2} \, e^{-t \, L_w^*} h(x)} \, \frac{dt}{t} dw(x) \right| \\ &\leq \int_{\mathbb{R}^n} \left\| F(x,\cdot) \right\| \, g_{L_w^*} h(x) \, dw(x) \\ &\lesssim \|F\|_{L_{\mathbb{H}}^p(v \, dw)} \|g_{L_w^*} h\|_{L^{p'}(v^{1-p'} \, dw)} \\ &\lesssim \|F\|_{L_{\mathbb{H}}^p(v \, dw)}, \end{split}$$

where the last estimate uses the fact that $g_{L_w^*}$ is bounded on $L^{p'}(v^{1-p'} dw)$. This follows from the upper bound in (5.3) (with L_w^* in place of L_w), which we proved above, and (5.11). Taking the supremum over all such functions h and using an standard density argument we have obtained that T_{L_w} is bounded from $L^p_{\mathbb{H}}(v dw)$ to $L^p(v dw)$.

Next, given $f \in L^p(v \, dw) \cap L^2(dw)$, if we define $F(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$, then $F \in L^p_{\mathbb{H}}(v \, dw) \cap L^2_{\mathbb{H}}(w)$ since $\|F\|_{L^p_{\mathbb{H}}(v \, dw)} = \|g_{L_w}f\|_{L^p(v \, dw)}$ and analogously for $L^2(w)$. Also, by the $L^2(w)$ functional calculus we have that

$$f(x) = 2 \int_0^\infty (tL_w)^{1/2} e^{-tL_w} F(x,t) \frac{dt}{t} = 2T_{L_w} F(x).$$
(5.13)

Therefore,

$$\|f\|_{L^{p}(v\,dw)} = 2\|T_{L_{w}}F\|_{L^{p}(v\,dw)} \lesssim \|F\|_{L^{p}_{\mathbb{H}}(v\,dw)} = \|g_{L_{w}}f\|_{L^{p}(v\,dw)},$$

and this completes the proof of (5.3).

To finish the proof of Proposition 5.1 we need to show that the equivalence of norms in (5.2) implies that the semigroup is uniformly bounded. However, this follows immediately from the definition of g_{L_w} and the semigroup property: for any s > 0,

$$g_{L_w}(e^{-sL_w}f)(x) = \left(\int_0^\infty |L_w^{1/2}e^{-(s+t)L_w}f(x)|^2 dt\right)^{1/2} \le g_{L_w}f(x).$$

This completes the proof.

We conclude this section by proving a version of Proposition 5.1 for the "adjoint" of a discrete square function. We will need this estimate in the proof of Proposition 6.1 below.

Proposition 5.14. Define the holomorphic function ψ on the sector $\Sigma_{\pi/2}$ by

$$\psi(z) = \frac{1}{\sqrt{\pi}} \int_{1}^{\infty} z \, e^{-tz} \, \frac{dt}{\sqrt{t}}.$$
(5.15)

If $p_{-}(L_w) , then for any sequence of functions <math>\{\beta_k\}_{k \in \mathbb{Z}}$,

$$\left\|\sum_{k\in\mathbb{Z}}\psi(4^{k}L_{w})\ \beta_{k}\right\|_{L^{p}(w)} \lesssim \left\|\left(\sum_{k\in\mathbb{Z}}|\beta_{k}|^{2}\right)^{\frac{1}{2}}\right\|_{L^{p}(w)}.$$
(5.16)

Proof. By duality and since $p_{\pm}(L_w)' = p_{\mp}(L_w^*)$, it will suffice to show that for every $p_-(L_w^*) ,$

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\overline{\psi}(4^k L_w^*) h|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|h\|_{L^p(w)}, \tag{5.17}$$

The function ψ satisfies $|\psi(z)| \leq C|z|^{1/2}e^{-c|z|}$ uniformly on subsectors Σ_{μ} , $0 \leq \mu < \frac{\pi}{2}$. Thus the operator on the lefthand side of (5.17) is a discrete analog of the square function $g_{L^*_w}$ changing continuous times t to discrete times 4^k and $z^{1/2}e^{-z}$ to $\overline{\psi}(z)$. Since $\overline{\psi}(z)$ has the same quantitative properties as $z^{1/2}e^{-z}$ (decay at 0 and at infinity), we can repeat the previous argument and obtain the desired estimates as in the proof of Proposition 5.1.

Remark 5.18. In Proposition 5.14 we can also get $L^p(v \, dw)$ estimates, but in the proof of Proposition 6.1 below we will only need the unweighted estimates. Further details and the precise statements are left to the interested reader.

6. Reverse inequalities

In this section we will prove $L^p(w)$ estimates of the form $\|L_w^{1/2}f\|_{L^p(w)} \leq C \|\nabla f\|_{L^p(w)}$, which generalize the $L^2(w)$ Kato estimates in Theorem 2.18. These are referred to as reverse inequalities since if we replace f by $L_w^{-1/2}f$, then formally we get a reverse-type inequality for the Riesz transform: $\|f\|_{L^p(w)} \leq C \|\nabla L_w^{-1/2}f\|_{L^p(w)}$.

Since these estimates involve the gradient, in proving them we will rely (implicitly and explicitly) on the weighted Poincaré inequality (2.3). This will require an additional assumption on p when p < 2. To state it simply, define

$$(p_{-}(L_w))_{w,*} = \frac{n r_w p_{-}(L_w)}{n r_w + p_{-}(L_w)} < p_{-}(L_w).$$

Proposition 6.1. Let $\max\{r_w, (p_-(L_w))_{w,*}\} . Then for all <math>f \in S$,

$$\|L_w^{1/2} f\|_{L^p(w)} \le C \, \|\nabla f\|_{L^p(w)}. \tag{6.2}$$

with C independent of f. Furthermore, if $\max\{r_w, p_-(L_w)\} and <math>v \in A_{p/\max\{r_w, p_-(L_w)\}}(w) \cap RH_{(p_+(L_w)/p)'}(w)$, then for all $f \in S$,

$$\|L_w^{1/2} f\|_{L^p(v\,dw)} \le C \,\|\nabla f\|_{L^p(v\,dw)}.$$
(6.3)

Remark 6.4. The quantity $\max\{r_w, (p_-(L_w))_{w,*}\}$ can be equal to either term. For instance, it equals r_w if $p_-(L_w) \le n'r_w$. From Proposition 3.1 we know that $p_-(L_w) < (2_w^*)' = \frac{2nr_w}{nr_w+2}$, but this only implies the previous inequality for some values of n and r_w .

Proof. As before, let $p_- = p_-(L_w)$ and $p_+ = p_+(L_w)$. Fix p, max $\{r_w, (p_-)_{w,*}\} , and <math>f \in \mathcal{S}$. We will first show that

$$\|L_w^{1/2} f\|_{L^{p,\infty}(w)} \lesssim \|\nabla f\|_{L^p(w)}.$$
(6.5)

First note that since $p > r_w$, $w \in A_p$. Therefore, given $\alpha > 0$ we can form the Calderón-Zygmund decomposition given in [5, Lemma 6.6]: there exist a collection of balls $\{B_i\}_i$, smooth functions $\{b_i\}_i$ and a function $g \in L^1_{loc}(w)$ such that

$$f = g + \sum_{i} b_i \tag{6.6}$$

and the following properties hold:

$$|\nabla g(x)| \le C\alpha$$
 for w-a.e. x , (6.7)

$$\operatorname{supp}(b_i) \subset B_i \quad \text{and} \quad \int_{B_i} |\nabla b_i|^p \, dw \le C \alpha^p w(B_i),$$
(6.8)

$$\sum_{i} w(B_i) \le \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw, \tag{6.9}$$

$$\sum_{i} \chi_{B_i} \le N,\tag{6.10}$$

$$\left(\int_{B_i} |b_i|^q \, dw\right)^{\frac{1}{q}} \lesssim C \, \alpha \, r(B_i) \quad \text{for } 1 \le q \le p_w^*, \tag{6.11}$$

where C and N depend only on n, p, q and the doubling constant of w.

To prove (6.5) we will prove the corresponding weak-type estimates with f replaced by g and b_i . For g, we use the $L^2(w)$ Kato estimate (2.19), (6.7), and the fact that p < 2 to get

$$w(\{|L_w^{1/2}g| > \alpha/3\}) \lesssim \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |L_w^{1/2}g|^2 \, dw \lesssim \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |\nabla g|^2 \, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla g|^p \, dw$$
$$\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + \frac{1}{\alpha^p} \int_{\mathbb{R}^n} \left|\sum_i \nabla b_i\right|^p \, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw,$$

where the last estimate follows from (6.10), (6.8), and (6.9).

To prove a weak-type estimate for $L_w^{1/2}(\sum_i b_i)$, let $r_i = 2^k$ if $2^k \leq r(B_i) < 2^{k+1}$. Then for all $i, r_i \sim r(B_i)$. Write

$$L_w^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^{r_i^2} L_w e^{-t L_w} \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_{r_i^2}^{\infty} L_w e^{-t L_w} \frac{dt}{\sqrt{t}} = T_i + U_i;$$

then we have that

$$w\Big(\Big\{\Big|\sum_{i} L_{w}^{1/2}b_{i}\Big| > \frac{2\alpha}{3}\Big\}\Big) \le w\Big(\bigcup_{i} 4B_{i}\Big) + w\Big(\Big\{\Big|\sum_{i} U_{i}b_{i}\Big| > \frac{\alpha}{3}\Big\}\Big) + w\Big(\Big(\mathbb{R}^{n} \setminus \bigcup_{i} 4B_{i}\Big) \bigcap\Big\{\Big|\sum_{i} T_{i}b_{i}\Big| > \frac{\alpha}{3}\Big\}\Big)$$

$$\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + I_1 + I_2,$$

where the last inequality follows from (6.9).

We first estimate I_2 . Since $p > (p_-)_{w,*}$ then $p_w^* > ((p_-)_{w,*})_w^* = p_-$, and we can choose $q \in \mathcal{J}(L_w)$ such that (6.11) is satisfied. By Corollary 3.4, $t L_w e^{-t L_w} \in \mathcal{O}(L^q(w) \to L^q(w))$, and so

$$\begin{split} I_2 &\lesssim \frac{1}{\alpha} \sum_i \sum_{j \ge 2} \int_{C_j(B_i)} |T_i b_i| \, dw \\ &\lesssim \frac{1}{\alpha} \sum_i \sum_{j \ge 2} w(2^j B_i) \int_0^{r_i^2} \int_{C_j(B_i)} |t \, L_w \, e^{-t \, L_w} b_i| \, dw \, \frac{dt}{t^{3/2}} \\ &\lesssim \frac{1}{\alpha} \sum_i \sum_{j \ge 2} 2^{j \, D} \, w(B_i) \int_0^{r_i^2} 2^{j \, \theta_1} \, \Upsilon \left(\frac{2^j \, r_i}{\sqrt{t}}\right)^{\theta_2} \, e^{-\frac{c \, 4^j \, r_i^2}{t}} \frac{dt}{t^{3/2}} \left(\int_{B_i} |b_i|^q \, dw \right)^{\frac{1}{q}} \\ &\lesssim \sum_i \sum_{j \ge 2} 2^{j \, D} \, e^{-c \, 4^j} \, w(B_i) \\ &\lesssim \sum_i w(B_i) \\ &\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw, \end{split}$$

where we have used (6.11) and (6.9), and D is the doubling order of dw.

We will now estimate I_1 . For q as above, by Proposition 4.3 we have an $L^q(w)$ functional calculus for L_w . Therefore, we can write U_i as $r_i^{-1}\psi(r_i^2L_w)$ with ψ defined by (5.15). Let $\beta_k = \sum_{i:r_i=2^k} \frac{b_i}{r_i}$; then,

$$\sum_{i} U_i b_i = \sum_{k \in \mathbb{Z}} \psi(4^k L_w) \left(\sum_{i: r_i = 2^k} \frac{b_i}{r_i}\right) = \sum_{k \in \mathbb{Z}} \psi(4^k L_w) \beta_k.$$

Therefore, by Proposition 5.14, (6.10), (6.11), the fact that $r_i \sim r(B_i)$ and (6.9), we have that

$$\begin{split} I_1 \lesssim \frac{1}{\alpha^q} \left\| \sum_i U_i b_i \right\|_{L^q(w)}^q \lesssim \frac{1}{\alpha^q} \left\| \left(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)}^q \\ \lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n} \sum_i \frac{|b_i|^q}{r_i^q} \, dw \lesssim \sum_i w(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw. \end{split}$$

If we combine all of the estimates we have obtained, we get (6.5) as desired.

To prove (6.2) from the weak-type estimate (6.5) we will use an interpolation argument from [5]. Fix p and r such that max $\{r_w, (p_-)_{w,*}\} < r < p < 2$. Then by (6.5) and (2.19) we have that for every $f \in S$,

$$\|L_w^{1/2}f\|_{L^{r,\infty}(w)} \lesssim \|\nabla f\|_{L^r(w)}, \qquad \|L_w^{1/2}f\|_{L^2(w)} \lesssim \|\nabla f\|_{L^2(w)}.$$
(6.12)

Formally, to apply Marcinkiewicz interpolation, we let $g = \nabla f$ to get a weak (r, r) and strong (2, 2) inequality; this would immediately yield a strong (p, p) inequality. To formalize this we must justify this substitution.

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For every $q > r_w$ by [5, Lemma 6.7] we have that

$$\mathcal{E} = \left\{ (-\Delta)^{1/2} f : f \in \mathcal{S}, \operatorname{supp} \widehat{f} \subset \mathbb{R}^n \setminus \{0\} \right\}$$

is dense in $L^q(w)$, where \widehat{f} denotes the Fourier transform of f. Moreover, since $r > r_w$, $w \in A_r$ and the Riesz transforms, $R_j = \partial_j (-\Delta)^{-1/2}$, are bounded on $L^r(w)$ [22]. It follows from this and the identity $-I = R_1^2 + \cdots + R_n^2$ that for $g \in L^r(w)$,

$$\|g\|_{L^r(w)} \sim \|\nabla(-\Delta)^{-1/2}g\|_{L^r(w)}$$

Thus, for $g \in \mathcal{E}$, $L_w^{1/2}(-\Delta)^{-1/2}g = L_w^{1/2}f$ if $f = (-\Delta)^{-1/2}g$ and $\|\nabla f\|_{L^r(w)} \sim \|g\|_{L^r(w)}$ for $r > r_w$. Thus (6.12) becomes weighted weak (r, r) and strong (2, 2) inequalities for $T = L_w^{1/2}(-\Delta)^{-1/2}$, and this operator is defined a priori on \mathcal{E} . Since \mathcal{E} is dense in each $L^q(w)$, we can extend T by density in both cases and their restrictions to the space of simple functions agree. Hence, we can apply Marcinkiewicz interpolation and conclude, again by density, that (6.2) holds for all p with r . Since <math>r is arbitrary, we get (6.2) in the range max $\{r_w, (p_-)_{w,*}\} .$

For the second step of the proof we will prove (6.3) using Theorem 2.35. Inequality (6.2) for its full range of exponents then follows by letting v = 1. Define $\tilde{p}_{-} = \max\{r_w, p_{-}\} < 2$, and fix $\tilde{p}_{-} and <math>v \in A_{p/\tilde{p}_{-}}(w) \cap RH_{(p_{+}/p)'}(w)$. By the openness properties of A_q and RH_s weights, there exist p_0, q_0 such that

$$\tilde{p}_{-} < p_0 < \min\{p, 2\} \le p < q_0 < p_+, \qquad v \in A_{p/p_0}(w) \cap RH_{(q_0/p)'}(w).$$

To apply Theorem 2.35, let $T = L_w^{1/2}$, $S = \nabla$, and $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$ where the value of m will be fixed below. We will first show that (2.37) holds. By (4.8) we have that $\mathcal{A}_r \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$ since $p_0, q_0 \in \mathcal{J}(L_w)$. Let $h = L_w^{1/2} f$ and decompose h as we decomposed f in (4.11). Then, since $L_w^{1/2}$ and \mathcal{A}_r commute, it follows that

$$\left(\oint_{B} \left| L_{w}^{1/2} \mathcal{A}_{r} f \right|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \lesssim \sum_{j \ge 1} \left(\oint_{B} \left| \mathcal{A}_{r} h_{j} \right|^{q_{0}} dw \right)^{\frac{1}{q_{0}}}$$
$$\lesssim \sum_{j \ge 1} 2^{j\theta_{1}} \Upsilon \left(2^{j} \right)^{\theta_{2}} e^{-c4^{j}} \left(\oint_{C_{j}} \left| h \right|^{p_{0}} dw \right)^{\frac{1}{p_{0}}}$$
$$\leq \sum_{j \ge 1} 2^{j(\theta_{1} + \theta_{2})} e^{-c4^{j}} \left(\oint_{2^{j+1} B} \left| L_{w}^{1/2} f \right|^{p_{0}} dw \right)^{\frac{1}{p_{0}}}.$$

This gives us (2.37) with $g(j) = C 2^{j(\theta_1 + \theta_2)} e^{-c4^j}$; clearly, $\sum g(j) < \infty$.

We now prove that (2.36) holds. Fix $f \in \mathcal{S}$ and let $\varphi(z) = z^{1/2}(1 - e^{-r^2 z})^m$ so that $\varphi(L_w)f = L_w^{1/2}(I - e^{-r^2 L_w})^m f$. By the conservation property (see [19] or [2, Section 2.5]),

$$\varphi(L_w) f = \varphi(L_w) \left(f - f_{4B,w} \right) = \sum_{j \ge 1} \varphi(L_w) h_j, \tag{6.13}$$

where $h_j = (f - f_{4B,w}) \phi_j$, $\phi_j = \chi_{C_j(B)}$ for $j \ge 3$, ϕ_1 is a smooth function with support in $4B, 0 \le \phi_1 \le 1, \phi_1 = 1$ in 2B and $\|\nabla \phi_1\|_{\infty} \le C/r$, and ϕ_2 is chosen so that $\sum_{j\ge 1} \phi_j = 1$.

We estimate each term in the righthand side of (6.13) separately. When j = 1, since $p_{-} < p_{0} < p_{+}$, by the bounded holomorphic functional calculus on $L^{p_{0}}(w)$ (Proposition 4.3) and the fact that $\varphi(L_{w}) h_{1} = (I - e^{-r^{2} L_{w}})^{m} L_{w}^{1/2} h_{1}$, we have that

$$\|\varphi(L_w) h_1\|_{L^{p_0}(w)} \lesssim \|L_w^{1/2} h_1\|_{L^{p_0}(w)}$$

uniformly in r. By the above argument we have that (6.2) holds for $p = p_0$ since $\tilde{p}_- < p_0 < 2$. Further, since $f \in S$, $h_1 \in S$ by our choice of ϕ_1 . This, together with the $L^{p_0}(w)$ -Poincaré inequality (2.3) (since $p_0 > r_w$, $w \in A_{p_0}$) and the definition of h_1 yield

$$\begin{aligned} \|L_w^{1/2}h_1\|_{L^{p_0}(w)} &\lesssim \|\nabla h_1\|_{L^{p_0}(w)} \\ &\lesssim \|(\nabla f)\,\chi_{4B}\,\|_{L^{p_0}(w)} + r^{-1}\,\|(f - f_{4B,w})\,\chi_{4B}\,\|_{L^{p_0}(w)} \lesssim \|(\nabla f)\,\chi_{4B}\,\|_{L^{p_0}(w)}. \end{aligned}$$

Therefore,

$$\left(\oint_{B} |\varphi(L_w) h_1|^{p_0} \, dw \right)^{\frac{1}{p_0}} \lesssim \left(\oint_{4B} |\nabla f|^{p_0} \, dw \right)^{\frac{1}{p_0}}$$

When $j \ge 3$, the functions η associated with φ by (2.11) satisfy

$$|\eta(z)| \lesssim \frac{r^{2m}}{|z|^{m+3/2}}, \qquad z \in \Gamma_{\pi/2-\theta}.$$

Since $p_0 \in \mathcal{J}(L_w)$, by Corollary 3.4, $e^{-z L_w} \in \mathcal{O}(L^{p_0}(w) \to L^{p_0}(w), \Sigma_{\mu})$. This, together with the representation (2.10), gives us that

$$\begin{split} \left(\oint_{B} |\varphi(L_{w})h_{j}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \\ & \leq \int_{\Gamma_{\pi/2-\theta}} \left(\int_{B} |e^{-zL}h_{j}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} |\eta(z)| |dz| \\ & \lesssim 2^{j\theta_{1}} \int_{\Gamma_{\pi/2-\theta}} \Upsilon \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{\theta_{2}} e^{-\frac{\alpha 4^{j}r^{2}}{|z|}} \frac{r^{2m}}{|z|^{m+3/2}} |dz| \left(\int_{C_{j}(B)} |h_{j}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \\ & \lesssim 2^{j(\theta_{1}-2m-1)} r^{-1} \left(\int_{2^{j+1}B} |f - f_{4B,w}|^{p_{0}} dw \right)^{\frac{1}{p_{0}}} \\ & \lesssim 2^{j(\theta_{1}-2m-1)} \sum_{l=1}^{j} 2^{l} \left(\int_{2^{l+1}B} |\nabla f|^{p_{0}} dx \right)^{\frac{1}{p_{0}}}, \end{split}$$

provided $2m+1 > \theta_2$. The last estimate follows from $L^{p_0}(w)$ -Poincaré inequality (2.3) (here we again use that $p_0 > r_w$ and so $w \in A_{p_0}$):

$$\left(\oint_{2^{j+1}B} |f - f_{4B,w}|^{p_0} dw \right)^{\frac{1}{p_0}}$$

$$\leq \left(\oint_{2^{j+1}B} |f - f_{2^{j+1}B,w}|^{p_0} dw \right)^{\frac{1}{p_0}} + \sum_{l=2}^{j} |f_{2^{l}B,w} - f_{2^{l+1}B,w}|^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\lesssim \sum_{l=1}^{j} \left(\oint_{2^{l+1}B} |f - f_{2^{l+1}B}|^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\lesssim r \sum_{l=1}^{j} 2^{l} \left(\oint_{2^{l+1}B} |\nabla f|^{p_{0}} dx \right)^{\frac{1}{p_{0}}}.$$
(6.14)

When j = 2 we can argue similarly, using the fact that

$$|h_2| \le |f - f_{4B,w}| \ \chi_{8B\setminus 2B} \le |f - f_{2B,w}| \ \chi_{8B\setminus 2B} + |f_{4B,w} - f_{2B,w}| \ \chi_{8B\setminus 2B} + |f_{4B,w} - f_{2B,w}| \ \chi_{8B\setminus 2B} + |f_{8B,w}| \ \chi_{8B\setminus 2B} + |f_{8B,w}| \ \chi_{8B\setminus 2B} + |f_{8B,w}| \ \chi_{8B,w}| \ \chi$$

If we combine these estimates, then by (6.13) and Minkowski's inequality we get

$$\left(\oint_{B} |\varphi(L_w)h|^{p_0} dw\right)^{\frac{1}{p_0}} \lesssim \sum_{j\geq 1} \left(\oint_{B} |\varphi(L_w)h_j|^{p_0} dw\right)^{\frac{1}{p_0}} \leq \sum_{j\geq 1} g(j) \left(\oint_{B} |\nabla f|^{p_0} dw\right)^{\frac{1}{p_0}}$$

with $g(j) = C_m 2^{j(\theta_1 - 2m)}$ provided $2m + 1 > \theta_2$. If we further assume that $2m > \theta_1$, then $\sum_j g(j) < \infty$. This proves that (2.36) holds. Therefore, by Theorem 2.35 we get (6.3) as desired.

7. The gradient of the semigroup $\sqrt{t}\nabla e^{-tL_w}$

Let $\widetilde{\mathcal{K}}(L_w) \subset [1,\infty]$ be the set of all exponents p such that $\sqrt{t}\nabla e^{-tL_w} : L^p(w) \to L^p(w)$ is uniformly bounded for all t > 0. By Theorem 2.15 and Lemma 2.34, $2 \in \widetilde{\mathcal{K}}(L_w)$ and if it contains more than one point, then by interpolation $\widetilde{\mathcal{K}}(L_w)$ is an interval. In this section we give a partial description of the set of (p,q) such that $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$.

Proposition 7.1. There exists an interval $\mathcal{K}(L_w)$ such that if $p, q \in \mathcal{K}(L_w)$, $p \leq q$, then $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$. Moreover, $\mathcal{K}(L_w)$ has the following properties:

- (1) $\mathcal{K}(L_w) \subset \widetilde{\mathcal{K}}(L_w);$
- (2) if $q_{-}(L_w)$ and $q_{+}(L_w)$ are the left and right endpoints of $\mathcal{K}(L_w)$, then $q_{-}(L_w) = p_{-}(L_w), 2 \leq q_{+}(L_w) \leq (q_{+}(L_w))_w^* \leq p_{+}(L_w)$. In particular, $2 \in \mathcal{K}(L_w)$ and $\mathcal{K}(L_w) \subset \mathcal{J}(L_w)$;
- (3) If $q \geq 2$ and p < q, and if $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$, then $p, q \in \mathcal{K}(L_w)$;
- (4) $\sup \mathcal{K}(L_w) = q_+(L_w).$

Remark 7.2. Unlike in the unweighted case (see [7]) we are unable to give a complete characterization of $\mathcal{K}(L_w)$. More precisely, if we have an off-diagonal estimate and p < q < 2, then we cannot prove that $p, q \in \mathcal{K}(L_w)$.

Remark 7.3. In Section 8 below we will show that $q_+(L_w) > 2$; in particular, this gives that $2 \in \operatorname{Int} \mathcal{K}(L_w)$.

As an immediate consequence of Proposition 7.1 we get weighted inequalities for the gradient of the semigroup. The proof is identical to the proof of Corollaries 3.3 and 3.4.

Corollary 7.4. Let $q_{-}(L_w) . If <math>v \in A_{p/q_{-}(L_w)}(w) \cap RH_{(q_{+}(L_w)/q)'}(w)$, then $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(v \, dw) \to L^q(v \, dw))$ and $\sqrt{z} \nabla e^{-zL_w} \in \mathcal{O}(L^p(v \, dw) \to L^q(v \, dw), \Sigma_{\nu})$ for all $\nu, 0 < \nu < \frac{\pi}{2} - \vartheta$.

The proof of Proposition 7.1 requires two lemmas.

Lemma 7.5. Given $w \in A_{\infty}$ and a family of sublinear operators $\{T_t\}_{t>0}$ such that $T_t \in$ $\mathcal{O}(L^p(w) \to L^q(w))$, with $1 \leq p < q \leq \infty$, there exist $\alpha, \beta > 0$ such that for any ball B with radius r and for any t > 0,

$$\left(\oint_{B} |T_{t}(\chi_{B} f)|^{q} dw\right)^{\frac{1}{q}} \lesssim \max\left\{\left(\frac{r}{\sqrt{t}}\right)^{\alpha}, \left(\frac{r}{\sqrt{t}}\right)^{\beta}\right\} \left(\oint_{B} |f|^{p} dw\right)^{\frac{1}{p}}.$$
(7.6)

Proof. This result is implicit in [7, Proof of Proposition 2.4, p. 306]; here we reprove it with a small improvement in the constant. There it was shown that in Definition 2.23 it is sufficient to consider the case where $r \approx \sqrt{t}$. But in this case we get that $\Upsilon(r/\sqrt{t}) \approx 1$ and for all j > 2, $\Upsilon(2^j r/\sqrt{t}) \approx 2^j$. The argument in [7, p. 306] shows that if we assume that (2.24), (2.25), (2.26) hold when $r \approx \sqrt{t}$, then (2.24) holds in general with constant $\max\{1, (r/\sqrt{t})^{\alpha}\}\$ for some $\alpha > 0$ depending on p, q and w. In this maximum the 1 occurs when $r \leq \sqrt{t}$; therefore, to prove (7.6) we need to show that if $r \leq \sqrt{t}$, then we can replace 1 by the better constant $(r/\sqrt{t})^{\beta}$ for some $\beta > 0$.

Fix $r \leq \sqrt{t}$. If B = B(x, r), then $B \subset B_t = B(x, \sqrt{t})$. As in [7, p. 306] we apply (2.24) to T_t and B_t ; this yields

$$\left(\oint_{B} |T_{t}(\chi_{B} \ f)|^{q} \ dw \right)^{\frac{1}{q}} \leq \left(\frac{w(B_{t})}{w(B)} \right)^{\frac{1}{q}} \left(\oint_{B_{t}} |T_{t}(\chi_{B} \ f)|^{q} \ dw \right)^{\frac{1}{q}}$$

$$\lesssim \left(\frac{w(B_{t})}{w(B)} \right)^{\frac{1}{q}} \left(\oint_{B_{t}} |\chi_{B} \ f|^{p} \ dw \right)^{\frac{1}{p}} \leq \left(\frac{w(B)}{w(B_{t})} \right)^{\frac{1}{p} - \frac{1}{q}} \left(\oint_{B} |f|^{p} \ dw \right)^{\frac{1}{p}}.$$

Since $w \in A_{\infty}$, we have that for some $\theta > 0$,

$$\frac{w(B)}{w(B_t)} \lesssim \left(\frac{|B|}{|B_t|}\right)^{\theta} = \left(\frac{r}{\sqrt{t}}\right)^{\theta n}$$

Since p < q we have that

$$\left(\oint_{B} |T_t(\chi_B \ f)|^q \ dw\right)^{\frac{1}{q}} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)\theta n} \left(\oint_{B} |f|^p \ dw\right)^{\frac{1}{p}}.$$

Therefore, if we combine this with the argument from [7, p. 306] described above, we get that (7.6) holds with $\beta = (1/p - 1/q) \theta n$. \square

The second lemma gives the close connection between off-diagonal estimates for e^{-tL_w} and $\sqrt{t}\nabla e^{-tL_w}$ for p < 2.

Lemma 7.7. Given $1 \le p < 2$ the following are equivalent:

- (1) $e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w)).$
- (2) $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w)).$ (3) $tL_w e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w)).$

Proof. We follow the proof of [7, Lemma 5.3]. To prove that (1) implies (2), note that by Theorem 2.15, $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$. If we compose this with (1), by Lemma 2.30 and the semigroup property, we get (2).

To prove that (2) implies (3), define $S_t \vec{f} = \sqrt{t} e^{-t L_w} (w^{-1} \operatorname{div}(A\vec{f}))$. By duality, we have that

$$\begin{split} \langle S_t \vec{f}, g \rangle_{L^2(w)} &= \langle w^{-1} \operatorname{div}(A\vec{f})), \sqrt{t} e^{-t L_w^*} g \rangle_{L^2(w)} = \langle \operatorname{div}(A\vec{f})), \sqrt{t} e^{-t L_w^*} g \rangle_{L^2} \\ &= -\langle \vec{f}, A^* \sqrt{t} \nabla e^{-t L_w^*} g \rangle_{L^2} = \langle \vec{f}, w^{-1} A^* \sqrt{t} \nabla e^{-t L_w^*} g \rangle_{L^2(w)}. \end{split}$$

The matrix $w^{-1}A^*$ is uniformly elliptic, and so multiplication by it is bounded on $L^2(w)$. Furthermore, $\sqrt{t} \nabla e^{-t L_w^*} \in \mathcal{O}(L^2(w) \to L^2(w))$. Therefore, it follows that $S_t \in \mathcal{O}(L^2(w) \to L^2(w))$. If we combine this with (2), we get that $-t L_w e^{-2t L_w} = S_t \circ \sqrt{t} \nabla e^{-t L_w} \in \mathcal{O}(L^p(w) \to L^2(w))$. This proves (3).

Finally we show that (3) implies (1). We first prove (2.24). Fix B and f, g such that $(f_B | f|^p dw)^{\frac{1}{p}} = (f_B | g|^2 dw)^{\frac{1}{2}} = 1$, and assume also that $f \in L^2(B, dw)$. Define

$$h(t) = \int_{B} e^{-t L_{w}} (\chi_{B} f)(x) g(x) dw(x).$$

By duality it will suffice to show that $|h(t)| \leq \Upsilon(r/\sqrt{t})^{\theta}$. (Note that our assumption implies that t h'(t) satisfies such a bound.) First, we claim that

$$\lim_{t \to \infty} h(t) = 0$$

To see this we use the fact (discussed in Section 2) that L_w has a bounded holomorphic functional calculus on $L^2(w)$. Given this, since $z \mapsto e^{-tz}$ converges to 0 uniformly on compact subsets of $\operatorname{Re} z > 0$, we get the desired limit.

Hence, we can write $h(t) = -\int_t^{\infty} h'(s) ds$. Notice that $|t h'(t)| \leq \Upsilon(r/\sqrt{t})^{\theta_2}$ but this does not give a convergent integral. However, if we apply Lemma 7.5 to $t L_w e^{-t L_w} \in \mathcal{O}(L^p(w) \to L^2(w))$, we get that $|t h'(t)| \leq \widetilde{\Upsilon}(r/\sqrt{t})$ with $\widetilde{\Upsilon}(s) = \max\{s^{\alpha}, s^{\beta}\}$. It follows from this estimate that

$$|h(t)| \le \int_t^\infty |h'(s)| \, ds \lesssim \int_t^\infty \widetilde{\Upsilon}\left(\frac{r}{\sqrt{s}}\right) \frac{ds}{s} \approx \int_0^{\frac{r}{\sqrt{t}}} \widetilde{\Upsilon}(s) \frac{ds}{s} \lesssim \widetilde{\Upsilon}\left(\frac{r}{\sqrt{t}}\right) \lesssim \Upsilon\left(\frac{r}{\sqrt{t}}\right)^{\alpha+\beta}$$

To prove (2.25) we argue as before, but with $(\int_{C_j(B)} |f|^p dw)^{\frac{1}{p}} = (\int_B |g|^2 dw)^{\frac{1}{2}} = 1$ and

$$h(t) = \int_{B} e^{-t L_{w}} (\chi_{C_{j}(B)} f)(x) g(x) dw(x).$$

Since $d(B, C_j(B)) > 0$, by Theorem 2.15 and Hölder's inequality, $h(t) \to 0$ as $t \to 0$. Therefore, $h(t) = \int_0^t h'(s) \, ds$. Since $t \, L_w \, e^{-t \, L_w} \in \mathcal{O}(L^p(w) \to L^2(w))$, we have that

$$\begin{split} h(t) &\leq \int_0^t |h'(s)| \, ds \lesssim 2^{j\,\theta_1} \int_0^t \Upsilon \left(\frac{2^j \, r}{\sqrt{s}}\right)^{\theta_2} \, e^{-\frac{c\,4^j \, r^2}{s}} \frac{ds}{s} \\ &\approx 2^{j\,\theta_1} \int_{\frac{2^j \, r}{\sqrt{t}}}^\infty \Upsilon(s)^{\theta_2} \, e^{-c\,s^2} \frac{ds}{s} \lesssim 2^{j\,\theta_1} \Upsilon \left(\frac{2^j \, r}{\sqrt{t}}\right)^{\theta_2} \, e^{-\frac{c\,4^j \, r^2}{t}}. \end{split}$$

This is (2.25).

Finally, the proof of (2.26) is essentially the same and we omit the details. This completes the proof that (3) implies (1).

Proof of Proposition 7.1. Define the sets $\mathcal{K}_{-}(L_w)$ and $\mathcal{K}_{+}(L_w)$ to be

$$\mathcal{K}_{-}(L_w) = \{ p \in [1,2] : \sqrt{t} \, \nabla e^{-t \, L_w} \in \mathcal{O}\big(L^p(w) \to L^2(w)\big) \}$$

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 $\mathcal{K}_+(L_w) = \{ p \in [2,\infty] : \sqrt{t} \, \nabla e^{-t \, L_w} \in \mathcal{O}\big(L^2(w) \to L^p(w)\big) \},\$

and let $\mathcal{K}(L_w) = \mathcal{K}_-(L_w) \cup \mathcal{K}_+(L_w)$. The set is non-empty, since $2 \in \mathcal{K}(L_w)$. By Lemma 2.28 it is an interval. Now fix $p, q \in \mathcal{K}(L_w)$ with p < q. If $p < q \leq 2$ or $2 \leq p < q$, then by Lemma 2.28, $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$ since $p, q \in \mathcal{K}_-(L_w)$ or $p, q \in \mathcal{K}_+(L_w)$. If $p \leq 2 < q$, then $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^2(w) \to L^q(w))$ and by Lemma 7.7, $e^{-tL} \in \mathcal{O}(L^p(w) \to L^2(w))$. Hence, by Lemma 2.30 and the semigroup property, $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$. Thus, in every case we get the desired off-diagonal estimate.

We now prove (1)-(4). By Lemma 2.30, off-diagonal estimates on balls imply uniform boundedness, and so $\mathcal{K}(L_w) \subset \widetilde{\mathcal{K}}(L_w)$. This proves (1).

To prove (2), we first note that if p < 2, then by Lemma 7.7, $p \in \mathcal{J}(L_w)$ if and only if $p \in \mathcal{K}_-(L_w)$. Thus $\mathcal{J}(L_w) \cap [1, 2] = \mathcal{K}_-(L_w)$ and so $q_-(L_w) = p_-(L_w)$. To show that $(q_+(L_w))_w^* \leq p_+(L_w)$, first note that if $q_+(L_w) = 2$, then by Proposition 3.1 we have that $(q_+(L_w))_w^* = 2_w^* \leq p_+(L_w)$. If $q_+(L_w) > 2$, then we proceed as in the proof of this proposition. Let $2 and <math>p < q < p_w^*$. Then by (2.3), $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^2(w))$, and $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^p(w))$, we get that

$$\begin{split} \left(\oint_{B} |e^{-tL_{w}}(\chi_{B} f)|^{q} dw \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_{B} |e^{-tL_{w}}(\chi_{B} f)|^{2} dw \right)^{\frac{1}{2}} + r \left(\int_{B} |\nabla e^{-tL_{w}}(\chi_{B} f)|^{p} dw \right)^{\frac{1}{p}} \\ &\lesssim \Upsilon \left(\frac{r}{\sqrt{t}} \right)^{1+\theta_{2}} \left(\int_{B} |f|^{2} dw \right)^{\frac{1}{2}}. \end{split}$$

This gives us inequality (2.24). The other two inequalities in Definition 2.23 can be proved in exactly the same way. Thus $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$ which implies $q \leq p_+(L_w)$. Letting $p \nearrow q_+(L_w)$ and $q \nearrow p_w^*$ we conclude that $(q_+(L_w))_w^* \leq p_+(L_w)$.

The last estimate implies in particular that $q_+(L_w) \leq p_+(L_w)$. If $q_+(L_w) < \infty$ we clearly have that $q_+(L_w) < p_+(L_w)$ and so $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$. Otherwise, $p_+(L) = \infty$ and again we have that $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$. This completes the proof of (2).

To prove (3), suppose first that $2 \leq p < q$ and $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^p(w) \to L^q(w))$. We will show that $p, q \in \mathcal{K}(L_w)$. Since we also have that $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^2(w) \to L^2(w))$, by interpolation (Lemma 2.27), $\sqrt{t} \nabla e^{-tL} \in \mathcal{O}(L^{p_{\theta}}(w) \to L^{q_{\theta}}(w))$ where $1/p_{\theta} = (1-\theta)/p + \theta/2$, $1/q_{\theta} = (1-\theta)/q + \theta/2$ and $\theta \in (0,1)$. If $p \notin \mathcal{K}_+(L_w)$, then $q > \sup \mathcal{K}_+(L_w)$. We can choose θ such that $p_{\theta} < \sup \mathcal{K}_+(L_w) < q_{\theta}$. Since $\mathcal{K}_+(L_w) \subset \mathcal{J}(L_w)$, $p_{\theta} \in \mathcal{J}(L_w)$: i.e., $e^{-tL} \in \mathcal{O}(L^2(w) \to L^{p_{\theta}}(w))$. By composition and the semigroup property, $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^{q_{\theta}}(w))$; hence, $q_{\theta} \in \mathcal{K}_+(L_w)$, a contradiction. Therefore, $p \in \mathcal{K}_+(L_w)$. As we have $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$ by assumption and $e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^p(w))$ since $p \in \mathcal{J}(L_w)$, by composition and the semigroup property, $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$.

The case $p < 2 \leq q$ is straightforward. Since $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^q(w))$, by Lemma 2.28 we have that $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^q(w))$ and $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^p(w) \to L^2(w))$. Hence, $p \in \mathcal{K}_-(L_w)$ and $q \in \mathcal{K}_+(L_w)$. Finally, we prove (4). Suppose to the contrary that $\sup \widetilde{\mathcal{K}}(L_w) > q_+(L_w)$. Then there exist p, q such that $q_+(L_w) . Fix <math>r$ such that $p_-(L_w) = q_-(L_w) < r < 2$. Then we have that $\sqrt{t}\nabla e^{-tL_w}$ is uniformly bounded on $L^q(w)$ and in $\mathcal{O}(L^r(w) \to L^2(w))$. By Lemma 2.27 we can interpolate between these to get that $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}(L^s(w) \to L^p(w))$ for some s < p. But then by the above converse, we have that $p \in \mathcal{K}(L_w)$ which is a contradiction.

8. An upper bound for $\mathcal{K}(L_w)$

In this section we will prove that $q_+(L_w) > 2$: that is, the set $\mathcal{K}(L_w)$ contains 2 in its interior. In general, all we can say is that $q_+(L_w) > 2$: as noted in [2, Section 4.5], even in the unweighted case this is the best possible bound, since given any $\epsilon > 0$ it is possible to find an operator L such that $q_+(L) < 2 + \epsilon$. In Section 11 below we will give some estimates for $q_+(L_w)$ in terms of $[w]_{A_2}$.

We have broken the proof that $q_+(L_w) > 2$ into a series of discrete steps where we borrow some ideas from [3]. We first prove a reverse Hölder inequality and use Gehring's inequality to get a higher integrability estimate. We then prove that the Hodge projection is bounded on $L^q(w)$ for a range of q > 2 and use this to prove the Riesz transform is also bounded for exponents greater than 2. (In Section 9 we give a more complete discussion of the Riesz transform.) From this we deduce that $q_+(L_w) > 2$.

A reverse Hölder inequality. Fix a ball B_0 and let $u \in H^1(w)$ be any weak solution of $L_w u = 0$ in $4B_0$. Then for any ball B such that $3B \subset 4B_0$, we can again prove via a standard argument a Caccioppoli inequality:

$$\left(\int_{B} |\nabla u|^2 dw\right)^{1/2} \le \frac{C_1}{r} \left(\int_{2B} |u - u_{2B,w}|^2 dw\right)^{1/2},$$

where $C_1 = C(n, \Lambda/\lambda)[w]_{A_2}^{1/2} \ge 1$. Fix q such that

$$\max\left\{\frac{2(n-1)}{n}, r_w, \frac{2nr_w}{2+nr_w}\right\} < q < 2;$$
(8.1)

such a q exists since $r_w < 2$. Our choice of q guarantees that $2 < q_w^*$ and also that 2 < n q/(q-1). Then, by the weighted Poincaré inequality, Theorem 2.1,

$$\frac{1}{r} \left(\int_{2B} |u - u_{2B,w}|^2 \, dw \right)^{1/2} \le C_2 \left(\int_{2B} |\nabla u|^q \, dw \right)^{1/q},\tag{8.2}$$

where $C_2 = C(n)[w]_{A_2}^{\kappa} \ge 1$ and $\kappa = \frac{nq-1}{nq(q-1)}$. (By our choice of q we can get this sharp estimate: see Remark 2.5. Since q < 2 we could write $[w]_{A_q}$, but we use that $[w]_{A_q} \le [w]_{A_2}$.) If we combine these inequalities, we get a reverse Hölder inequality:

$$\left(\oint_{B} |\nabla u|^2 \, dw\right)^{1/2} \le C_1 C_2 \left(\oint_{2B} |\nabla u|^q \, dw\right)^{1/q}$$

We now apply Gehring's lemma in the setting of spaces of homogeneous type (see Björn and Björn [12, Theorem 3.22]) to get that there exists $p_0 > 2$ such that for every such B,

$$\left(\oint_{B} |\nabla u|^{p_{0}} dw\right)^{1/p_{0}} \leq C_{0} \left(\oint_{2B} |\nabla u|^{2} dw\right)^{1/2}.$$
(8.3)

Moreover, we can take the following values: $C_0 = 8C_1^2C_2^2[w]_{A_2}^{31}$ and

$$p_0 = 2 + \frac{2 - q}{2^{4/q+1} C_1^2 C_2^2 [w]_{A_2}^{6/q+17}}.$$
(8.4)

In Section 11 below we will need these precise values. Here, it suffices to note that in inequality (8.3) we have $p_0 > 2$.

The Hodge projection. Define the Hodge projection operator by

$$T = \nabla L_w^{-1/2} (\nabla (L_w^*)^{-1/2})^*$$

where the adjoint operators are defined with respect to the inner product in $L^2(w)$. As we noted in Section 2, the Riesz transform is bounded on $L^2(w)$; hence, the Hodge projection is also bounded. By duality, $(\nabla(L_w^*)^{-1/2})^*\vec{f} = -L_w^{-1/2}(w^{-1}\operatorname{div}(w\vec{f}))$, and so

$$T\vec{f} = -\nabla L_w^{-1/2} L_w^{-1/2} (w^{-1} \operatorname{div}(w\vec{f})) = -\nabla L_w^{-1} (w^{-1} \operatorname{div}(w\vec{f})).$$

Now fix $\vec{f} \in L^2(w, \mathbb{C}^n) \cap L^{p_0}(w, \mathbb{C}^n)$ such that $\operatorname{supp}(\vec{f}) \subset \mathbb{R}^n \setminus 4B_0$. Let $u \in H^1(w)$ be a solution to the equation

$$L_w u = w^{-1} \operatorname{div}(w\vec{f});$$

by a standard Lax-Milgram argument because A satisfies (2.7) (cf. [21, Theorem 2.2]), we know u exists. Then

$$T\vec{f} = -\nabla L_w^{-1} L_w u = -\nabla u,$$

where equality is in the sense of distributions. In particular, since f = 0 on $4B_0$, $L_w u = 0$ on $4B_0$. Therefore, we can apply (8.3) to u: on any ball B such that $3B \subset 4B_0$,

$$\left(f_{B}|T\vec{f}|^{p_{0}}\,dw\right)^{1/p_{0}} = \left(f_{B}|\nabla u|^{p_{0}}\,dw\right)^{1/p_{0}} \le C_{0}\left(f_{2B}|\nabla u|^{2}\,dw\right)^{1/2} = \left(f_{2B}|T\vec{f}|^{2}\,dw\right)^{1/2}.$$

As a consequence of this inequality, we have by [6, Theorem 3.14] (see also Section 5 of the same paper) that for all $q, 2 \leq q < p_0, T : L^q(w, \mathbb{C}^n) \to L^q(w, \mathbb{C}^n)$.

Boundedness of the Riesz transform. To show that the Riesz transform $\nabla L_w^{-1/2}$ is bounded, fix q such that

$$\max\left(p_{-}(L_{w}^{*}), r_{w}, p_{0}'\right) = \max\left(p_{-}(L_{w}^{*}), r_{w}, p_{0}', \frac{nr_{w}p_{-}(L_{w}^{*})}{nr_{w} + p_{-}(L_{w}^{*})}\right) < q' < 2.$$

(The reason for including $p_{-}(L_{w}^{*})$ will be made clear below.) By the above argument we have that T^{*} is bounded on $L^{q'}(w)$, where $T^{*}\vec{f} = -\nabla(L_{w}^{*})^{-1}(w^{-1}\operatorname{div}(w\vec{f}))$. Furthermore, by Proposition 6.1, we have that

$$\|(L_w^*)^{1/2}f\|_{L^{q'}(w)} \le C \|\nabla f\|_{L^{q'}(w)}$$

Therefore,

$$\begin{aligned} \| (\nabla L_w^{-1/2})^* \vec{f} \|_{L^{q'}(w)} &= \| (L_w^*)^{-1/2} (w^{-1} \operatorname{div}(w\vec{f})) \|_{L^{q'}(w)} \\ &= \| (L_w^*)^{1/2} (L_w^*)^{-1} (w^{-1} \operatorname{div}(w\vec{f})) \|_{L^{q'}(w)} \\ &\lesssim \| \nabla (L_w^*)^{-1} (w^{-1} \operatorname{div}(w\vec{f})) \|_{L^{q'}(w)} \\ &= \| T^* \vec{f} \|_{L^{q'}(w)} \end{aligned}$$

$$\lesssim \|\vec{f}\|_{L^{q'}(w)}$$

Hence, by duality we have that $\nabla L_w^{-1/2} : L^q(w) \to L^q(w)$ for all q such that

$$2 < q < \min\left(p_+(L_w), r'_w, p_0\right) = q_w;$$

here we have used the fact that by duality, $p_{-}(L_{w}^{*})' = p_{+}(L_{w})$.

Boundedness of the gradient of the semigroup. Finally, we show that if $2 < q < q_w$, then $\sqrt{t}\nabla e^{-tL_w} : L^q(w) \to L^q(w)$. The desired estimate for $q_+(L_w)$ follows from this: by Proposition 7.1, part (4),

$$q_+(L_w) = \sup \mathcal{K}(L_w) \ge q_w > 2.$$

Fix such a q; then by the above estimate for the Riesz transform,

$$\begin{aligned} \|\sqrt{t}\nabla e^{-tL_w}f\|_{L^q(w)} &= \|\nabla L_w^{-1/2}(tL_w)^{1/2}e^{-tL_w}f\|_{L^q(w)} \\ &\lesssim \|(tL_w)^{1/2}e^{-tL_w}f\|_{L^q(w)} = \|\varphi_t(L_w)f\|_{L^q(w)}, \end{aligned}$$

where $\varphi_t(z) = (tz)^{1/2} e^{-tz}$. For all t > 0 this is a uniformly bounded holomorphic function in the right half plane. Therefore, since $2 < q < p_+(L_w)$, by Proposition 4.3 we have that

$$\|\sqrt{t}\nabla e^{-tL_w}f\|_{L^q(w)} \lesssim \|\varphi_t\|_{\infty} \|f\|_{L^q(w)} \lesssim \|f\|_{L^q(w)}$$

and the bound is independent of t. This completes the proof that $q_+(L_w) > 2$.

9. Riesz transform estimates

In this section we prove $L^p(w)$ norm inequalities for the Riesz transform $\nabla L_w^{-1/2}$. We have already proved such inequalities for a small range of values q > 2 in Section 8. Here we prove the following result.

Proposition 9.1. Let $q_{-}(L_w) . Then there exists a constant C such that$

$$\|\nabla L_w^{-1/2} f\|_{L^p(w)} \le C \|f\|_{L^p(w)}.$$
(9.2)

Furthermore, if $v \in A_{p/q_{-}(L_w)}(w) \cap RH_{(q_{+}(L_w)/p)'}(w)$, then

$$\|\nabla L_w^{-1/2} f\|_{L^p(v\,dw)} \le C \|f\|_{L^p(v\,dw)}.$$
(9.3)

To prove Proposition 9.1 we would like to follow the same outline as the proof of Proposition 4.3. The first step—i.e., proving (9.2) holds when $q_-(L_w) — does work with$ the appropriate changes. However, the second step (i.e., the proof that (9.3) holds) runs $into difficulties since <math>\nabla L_w^{-1/2}$ and the auxiliary operators \mathcal{A}_r do not commute. One approach to overcoming this obstacle would be to adapt the proof in [5] (see also [2]). In this case we would need to use an $L^{p_0}(w)$ -Poincaré inequality which may not hold unless we assume $w \in A_{p_0}$. This would yield estimates in the range $\max\{r_w, q_-(L_w)\} , analogous$ to those in Proposition 6.1.

There is, however, an alternative approach. In [8] the authors considered Riesz transforms associated with the Laplace-Beltrami operator of a complete, non-compact Riemannian manifold. Their proof avoids Poincaré inequalities for p close to 1 as these may not hold. Instead, they use a duality argument based on ideas in [11]; this requires that they first prove that the Riesz transform is bounded for p > 2 in the appropriate range of values. This reverses the order used in the proof of Proposition 4.3.

Proof of Proposition 9.1. For brevity, let $q_- = q_-(L_w)$ and $q_+ = q_+(L_w)$. To implement the approach sketched above, we divide the proof in two steps. First we will prove that (9.2) holds when $2 . We do so using Theorem 2.35 and some ideas from [2, 5]. We note that since the Riesz transform and <math>\mathcal{A}_r$ do not commute, we will use an $L^2(w)$ -Poincaré inequality. This holds since $w \in A_2$: the problem with using the Poincaré inequality only occurs with exponents less than 2. The second step is to prove that (9.3) holds by adapting the proof in [8]. Here we will use duality and a result from [6] that is based on good- λ inequalities. Inequality (9.2) then holds when $q_- by taking <math>v \equiv 1$.

To apply Theorem 2.35, fix $2 and let <math>T = \nabla L_w^{-1/2}$, S = I and $\mathcal{D} = L_c^{\infty}$. Let $p_0 = 2$ and fix q_0 such that $2 . As before we take <math>\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where *m* will be chosen below. We first show that (2.36) holds. Let $f \in L_c^{\infty}$ and decompose it as in (4.11); then we have

$$\left(\oint_{B} |\nabla L_{w}^{-1/2} (I - e^{-r^{2} L_{w}})^{m} f|^{2} dw \right)^{\frac{1}{2}} \leq \sum_{j \geq 1} \left(\oint_{B} |\nabla L_{w}^{-1/2} (I - e^{-r^{2} L_{w}})^{m} f_{j}|^{2} dw \right)^{\frac{1}{2}}$$

To estimate the first term, note that $\nabla L_w^{-1/2}$ and $e^{-r^2 L_w}$ are bounded on $L^2(w)$ by Theorems 2.15 and 2.18. Hence,

$$\left(\int_{B} |\nabla L_{w}^{-1/2} (I - e^{-r^{2} L_{w}})^{m} f_{1}|^{2} dw\right)^{\frac{1}{2}} \lesssim \left(\int_{4B} |f|^{2} dw\right)^{\frac{1}{2}}.$$
(9.4)

Fix $j \ge 2$; to get the desired L^2 estimates we will use the L^2 bounds for the gradient of the square function. If $h \in L^2(w)$, by (2.20)

$$\nabla L_w^{-1/2} (I - e^{-r^2 L_w})^m h = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \, \nabla \varphi(L_w, t) h \, \frac{dt}{t}, \tag{9.5}$$

where $\varphi(z,t) = e^{-tz} (1 - e^{-r^2 z})^m \in \mathcal{H}_0^{\infty}(\Sigma_{\mu})$. We can therefore use the integral representation (2.10) for $\varphi(\cdot, t)$. The function $\eta(\cdot, t)$ in this representation satisfies

$$|\eta(z,t)| \lesssim \frac{r^{2m}}{(|z|+t)^{m+1}}, \qquad z \in \Gamma, \ t > 0.$$

By Theorem 2.15, $\sqrt{z} \nabla e^{-z L_w} \in \mathcal{O}(L^2(w) \to L^2(w))$; hence,

$$\left(\oint_{B} \left| \int_{\Gamma} \eta(z) \sqrt{t} \nabla e^{-zL_{w}} f_{j} dz \right|^{2} dw \right)^{\frac{1}{2}} \leq \int_{\Gamma} \left(\int_{B} |\sqrt{z} \nabla e^{-zL_{w}} f_{j}|^{2} dw \right)^{\frac{1}{2}} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)| |dz| \\ \lesssim 2^{j\theta_{1}} \int_{\Gamma} \Upsilon \left(\frac{2^{j}r}{\sqrt{|z|}} \right)^{\theta_{2}} e^{-\frac{\alpha 4^{j}r^{2}}{|z|}} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)| |dz| \left(\int_{C_{j}(B)} |f|^{2} dw \right)^{\frac{1}{2}} \\ \lesssim 2^{j\theta_{1}} \int_{0}^{\infty} \Upsilon \left(\frac{2^{j}r}{\sqrt{s}} \right)^{\theta_{2}} e^{-\frac{\alpha 4^{j}r^{2}}{s}} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2m}}{(s+t)^{m+1}} ds \left(\int_{C_{j}(B)} |f|^{2} dw \right)^{\frac{1}{2}}. \quad (9.6)$$

When $2m > \theta_2$,

$$\int_{0}^{\infty} \int_{0}^{\infty} \Upsilon \left(\frac{2^{j} r}{\sqrt{s}}\right)^{\theta_{2}} e^{-\frac{\alpha 4^{j} r^{2}}{s}} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2m}}{(s+t)^{m+1}} ds \frac{dt}{t} = C 4^{-jm}.$$
(9.7)

If we insert this into the representation (2.10) we get

$$\left(\oint_{B} |\nabla e^{-tL_{w}} (I - e^{-r^{2}L_{w}})^{m} f_{j}|^{2} dw \right)^{\frac{1}{2}} \lesssim \int_{0}^{\infty} \left(\oint_{B} |\sqrt{t} \nabla \varphi(L_{w}, t) f_{j}|^{2} dw \right)^{\frac{1}{2}} \frac{dt}{t}$$
$$\lesssim 2^{j(\theta_{1} - 2m)} \left(\oint_{C_{j}(B)} |f|^{2} dw \right)^{\frac{1}{2}}.$$
(9.8)

If we now combine (9.4) and (9.8) we get (2.36) with $g(j) = C_m 2^{j(\theta_1 - 2m)}$; if we also fix $2m > \theta_1$, we get that $\sum g(j) < \infty$.

We now show that (2.37) holds. As we remarked above, the Riesz transform does not commute with \mathcal{A}_r . To overcome this obstacle, we will prove an off-diagonal estimate for the gradient of the semigroup (using the $L^2(w)$ -Poincaré inequality), and then use an approximation argument to get the desired estimate for the Riesz transform.

More precisely, we claim that for every $f \in H^1(w)$ and $1 \le k \le m$,

$$\left(\oint_{B} |\nabla e^{-k r^{2} L_{w}} f|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \leq \sum_{j \geq 1} g(j) \left(\oint_{2^{j+1} B} |\nabla f|^{2} dw \right)^{\frac{1}{2}}, \tag{9.9}$$

where $g(j) = C_m 2^j \sum_{l \ge j} 2^{l\theta} e^{-\alpha 4^l}$. Assume for the moment that (9.9) holds. Then for every $\epsilon > 0$ we can apply this estimate to $S_{\varepsilon}f$ (defined by (2.21)) since $S_{\varepsilon}f \in H^1(w)$. Moreover, we have that \mathcal{A}_r and S_{ε} commute, and so if we expand $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ and apply (9.9), we get

$$\left(\oint_{B} |\nabla S_{\varepsilon} \mathcal{A}_{r} f|^{q_{0}} dw\right)^{\frac{1}{q_{0}}} \leq C_{m} \sum_{j \geq 1} g(j) \left(\oint_{2^{j+1} B} |\nabla S_{\varepsilon} f|^{2} dw\right)^{\frac{1}{2}}.$$

If we let ε go to 0, we obtain (2.37). (The justification of this uses the observations made in Section 2 after (2.21) and is left to the reader.) Moreover, we have that $\sum_{j\geq 1} g(j) < \infty$, and so by Theorem 2.35 with $v \equiv 1$ (which trivially satisfies $v \in A_{p/2}(w) \cap RH_{(q_0/p)'}(w)$) we have that (9.2) holds for $f \in L_c^{\infty}$ and for every 2 .

To complete this step we need to prove (9.9). Fix $1 \leq k \leq m$ and $f \in H^1(w)$. Let $h = f - f_{4B,w}$, where $f_{4B,w} = f_{4B} f dw$. Then by the conservation property (see [19], or the proof in [2, Section 2.5]), $e^{-tL_w} 1 = 1$ for all t > 0, and so

$$\nabla e^{-k r^2 L_w} f = \nabla e^{-k r^2 L_w} (f - f_{4B,w}) = \nabla e^{-k r^2 L_w} h = \sum_{j \ge 1} \nabla e^{-k r^2 L_w} h_j,$$

where $h_j = h \chi_{C_i(B)}$. Hence,

$$\left(\oint_{B} |\nabla e^{-k r^{2} L_{w}} f|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \leq \sum_{j \geq 1} \left(\oint_{B} |\nabla e^{-k r^{2} L_{w}} h_{j}|^{q_{0}} dw \right)^{\frac{1}{q_{0}}}.$$

Since $2 < q_0 < q_+$, by Proposition 7.1, $\sqrt{t} \nabla e^{-tL_w} \in \mathcal{O}(L^2(w) \to L^{q_0}(w))$. If we apply this and the $L^2(w)$ -Poincaré inequality (see Remark 2.6 with p = q = 2), then for each $j \ge 1$ we

get

$$\begin{split} \left(\oint_{B} |\nabla e^{-k r^{2} L_{w}} h_{j}|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \\ &\lesssim \frac{2^{j (\theta_{1} + \theta_{2})} e^{-\alpha 4^{j}}}{r} \left(\int_{C_{j}(B)} |h_{j}|^{2} dw \right)^{\frac{1}{2}} \\ &\leq \frac{2^{j (\theta_{1} + \theta_{2})} e^{-\alpha 4^{j}}}{r} \left(\int_{2^{j+1} B} |f - f_{4B,w}|^{2} dw \right)^{\frac{1}{2}} \\ &\leq \frac{2^{j (\theta_{1} + \theta_{2})} e^{-\alpha 4^{j}}}{r} \left(\left(\int_{2^{j+1} B} |f - f_{2^{j+1} B,w}|^{2} dw \right)^{\frac{1}{2}} + \sum_{l=2}^{j} |f_{2^{l} B,w} - f_{2^{l+1} B,w}| \right) \\ &\lesssim \frac{2^{j (\theta_{1} + \theta_{2})} e^{-\alpha 4^{j}}}{r} \sum_{l=1}^{j} \left(\int_{2^{l+1} B} |f - f_{2^{l+1} B,w}|^{2} dw \right)^{\frac{1}{2}} \\ &\lesssim 2^{j (\theta_{1} + \theta_{2})} e^{-\alpha 4^{j}} \sum_{l=1}^{j} 2^{l} \left(\int_{2^{l+1} B} |\nabla f|^{2} dw \right)^{\frac{1}{2}}. \end{split}$$

If we combine these two estimates and exchange the order of summation we get (9.9) with $\theta = \theta_1 + \theta_2$. This completes the proof that (9.2) holds when 2 .

For the second step of our proof we show that (9.3) holds for all $p, q_- and <math>v \in A_{p/q_-}(w) \cap RH_{(q_+/p)'}(w)$. Fix such a p and v; then by the openness properties of A_q and RH_s weights, there exist p_0, q_0 such that

$$q_{-} < p_0 < \min\{p, 2\} \le \max\{p, 2\} < q_0 < q_+$$
 and $v \in A_{p/p_0}(w) \cap RH_{(q_0/p)'}(w)$

By the duality properties of weights [6, Lemma 4.4],

$$u = v^{1-p'} \in A_{p'/q'_0}(w) \cap RH_{(p'_0/p')'}(w).$$

Let $T = \nabla L_w^{-1/2}$; then T is bounded from $L^p(\mathbb{R}^n, v \, dw)$ to $L^p(\mathbb{R}^n; \mathbb{C}^n, v \, dw)$ if and only if T^* is bounded from $L^{p'}(\mathbb{R}^n; \mathbb{C}^n, u \, dw)$ to $L^{p'}(\mathbb{R}^n; u \, dw)$. (Note that T takes scalar valued functions to vector functions valued and T^* the opposite.)

Therefore, it will suffice to prove the boundedness of T^* . We will do so using a particular case of [6, Theorem 3.1]. This result is stated there in the Euclidean setting but it extends to spaces of homogeneous type. Here we give the weighted version we need: see [6, Section 5].

Theorem 9.10. Fix $1 < q < \infty$, $a \ge 1$ and $u \in RH_{s'}(w)$, $1 < s < \infty$. Then there exists C > 1 with the following property: suppose $F \in L^1(w)$ and G are non-negative measurable functions such that for any ball B there exist non-negative functions G_B and H_B with $F(x) \le G_B(x) + H_B(x)$ for a.e. $x \in B$ and, for all $x \in B$,

$$\left(\oint_{B} H_{B}^{q} dw\right)^{\frac{1}{q}} \le a M_{w} F(x), \qquad \qquad \oint_{B} G_{B} dw \le G(x), \tag{9.11}$$

where M_w is the Hardy-Littlewood maximal function with respect to dw. Then for 1 < t < q/s,

$$\|M_w F\|_{L^t(u\,dw)} \le C \,\|G\|_{L^t(u\,dw)}.\tag{9.12}$$

To apply Theorem 9.10, fix $\vec{f} \in L^{\infty}_{c}(\mathbb{R}^{n};\mathbb{C}^{n})$, and let $h = T^{*}\vec{f}$ and $F = |h|^{q'_{0}}$. Then $F \in L^{1}(w)$: by the argument above, since $2 < q_{0} < q_{+}$, T is bounded from $L^{q_{0}}(\mathbb{R}^{n},w)$ to $L^{q_{0}}(\mathbb{R}^{n};\mathbb{C}^{n},w)$, thus T^{*} is bounded from $L^{q'_{0}}(\mathbb{R}^{n};\mathbb{C}^{n},w)$ to $L^{q'_{0}}(\mathbb{R}^{n},w)$.

Now let $\mathcal{A}_r = I - (I - e^{-r^2 L_w})^m$, where m > 0 will be fixed below. Given a ball B with radius r, we define

$$F \le 2^{q'_0 - 1} |(I - \mathcal{A}_r)^* h|^{q'_0} + 2^{q'_0 - 1} |\mathcal{A}_r^* h|^{q'_0} \equiv G_B + H_B,$$

where, as before, the adjoint is with respect to $L^2(w)$. To complete the proof, suppose for the moment that we could prove (9.11) with $q = p'_0/q'_0$ and $G = M_w(|\vec{f}|^{q'_0})$. Since $u \in RH_{(p'_0/p')'}(w)$, by the openness property of reverse Hölder weights, $u \in RH_{s'}(w)$ for some $s < p'_0/p'$. Then if we let $t = p'/q'_0 = (p'_0/q'_0)/(p'_0/p') < q/s$, we have $u \in A_t(w)$, and so M_w is bounded on $L^t(u \, dw)$. Therefore, by (9.12),

$$\|T^*\vec{f}\|_{L^{p'}(u\,dw)}^{q'_0} \le \|M_wF\|_{L^t(u\,dw)} \le C \,\|G\|_{L^t(u\,dw)} = C \,\|M_w(|\vec{f}|^{q'_0})\|_{L^t(u\,dw)} \lesssim \|\vec{f}\|_{L^{p'}(u\,dw)}^{q'_0}$$

To complete the proof we need to show that (9.11) holds. We first estimate H_B . By duality there exists $g \in L^{p_0}(B, dw/w(B))$ with norm 1 such that for all $x \in B$,

$$\left(\oint_{B} H_{B}^{q} dw \right)^{\frac{1}{qq_{0}}} \lesssim w(B)^{-1} \int_{\mathbb{R}^{n}} |h| |\mathcal{A}_{r}g| dw$$

$$\lesssim \sum_{j=1}^{\infty} 2^{jD} \left(\oint_{C_{j}(B)} |h|^{q_{0}'} dw \right)^{\frac{1}{q_{0}'}} \left(\oint_{C_{j}(B)} |\mathcal{A}_{r}g|^{q_{0}} dw \right)^{\frac{1}{q_{0}'}}$$

$$\lesssim M_{w}F(x)^{\frac{1}{q_{0}'}} \sum_{j=1}^{\infty} 2^{j(D+\theta_{1}+\theta_{2})} e^{-\alpha 4^{j}} \left(\oint_{B} |g|^{p_{0}} dw \right)^{\frac{1}{p_{0}'}}$$

$$\lesssim M_{w}F(x)^{\frac{1}{q_{0}'}},$$

where in the second to last inequality we used the fact that by our choice of $p_0, q_0, e^{-tL_w} \in \mathcal{O}(L^{p_0}(w) \to L^{q_0}(w))$, and so \mathcal{A}_r is as well.

We now estimate G_B . Again by duality there exists $g \in L^{q_0}(B, dw/w(B))$ with norm 1 such that for all $x \in B$,

$$\left(\oint_{B} G_{B} dw \right)^{\frac{1}{q_{0}^{\prime}}} \lesssim w(B)^{-1} \int_{\mathbb{R}^{n}} |\vec{f}| |T(I - \mathcal{A}_{r})g| dw$$

$$\lesssim \sum_{j=1}^{\infty} 2^{jD} \left(\oint_{C_{j}(B)} |\vec{f}|^{q_{0}^{\prime}} dw \right)^{\frac{1}{q_{0}^{\prime}}} \left(\oint_{C_{j}(B)} |T(I - \mathcal{A}_{r})g|^{q_{0}} dw \right)^{\frac{1}{q_{0}^{\prime}}}$$

$$\leq M_{w} (|\vec{f}|^{q_{0}^{\prime}})(x)^{\frac{1}{q_{0}^{\prime}}} \sum_{j=1}^{\infty} 2^{jD} \left(\oint_{C_{j}(B)} |T(I - \mathcal{A}_{r})g|^{q_{0}} d\mu \right)^{\frac{1}{q_{0}^{\prime}}}.$$
(9.13)

To estimate each term in the sum we argue as in the first half of the proof. When j = 1, $\nabla L_w^{-1/2}$ and $e^{-r^2 L_w}$ are bounded on $L^{q_0}(w)$ by the first part of the proof and Theorem 2.15. Hence,

$$\left(\oint_{4B} |\nabla L_w^{-1/2} (I - e^{-r^2 L_w})^m g|^{q_0} \, dw \right)^{\frac{1}{q_0}} \lesssim \left(\oint_B |g|^{q_0} \, dw \right)^{\frac{1}{q_0}} = 1. \tag{9.14}$$

For $j \geq 2$ we use the integral representation (9.5). If we estimate as in (9.6), with the roles of B and $C_j(B)$ switched and using the fact that $\sqrt{z} \nabla e^{-z L_w} \in \mathcal{O}(L^{q_0}(w) \to L^{q_0}(w))$ since $2 < q_0 < q_+$, we see that

$$\begin{split} \left(\oint_{C_j(B)} \left| \int_{\Gamma} \eta(z) \sqrt{t} \, \nabla e^{-z \, L_w} g \, dz \right|^{q_0} dw \right)^{\frac{1}{q_0}} \\ &\leq \int_{\Gamma} \left(\oint_{C_j(B)} |\sqrt{z} \, \nabla e^{-z \, L_w} g|^{q_0} \, dw \right)^{\frac{1}{q_0}} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)| \, |dz| \\ &\lesssim 2^{j \, \theta_1} \int_{\Gamma} \Upsilon \left(\frac{2^j \, r}{\sqrt{|z|}} \right)^{\theta_2} e^{-\frac{\alpha \, 4^j \, r^2}{|z|}} \frac{\sqrt{t}}{\sqrt{|z|}} |\eta(z)| \, |dz| \left(\oint_B |g|^{q_0} \, dw \right)^{\frac{1}{2}} \\ &\lesssim 2^{j \, \theta_1} \int_{0}^{\infty} \Upsilon \left(\frac{2^j \, r}{\sqrt{s}} \right)^{\theta_2} e^{-\frac{\alpha \, 4^j \, r^2}{s}} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2 \, m}}{(s+t)^{m+1}} \, ds. \end{split}$$

If we take $2m > \theta_2$, we can combine this with (9.7). We can then insert this estimate into the representation (2.10) to get that for every $j \ge 2$,

$$\left(\oint_{C_{j}(B)} |\nabla e^{-t L_{w}} (I - e^{-r^{2} L_{w}})^{m} g|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \\ \lesssim \int_{0}^{\infty} \left(\oint_{C_{j}(B)} |\sqrt{t} \nabla \varphi(L_{w}, t) g|^{q_{0}} dw \right)^{\frac{1}{q_{0}}} \frac{dt}{t} \lesssim 2^{j (\theta_{1} - 2m)}.$$
(9.15)

Taken together, (9.13), (9.14) and (9.15) yield

$$\left(\oint_B G_B \, dw \right)^{\frac{1}{q'_0}} \lesssim M_w(|\vec{f}|^{q'_0})(x)^{\frac{1}{q'_0}} \sum_{j=1}^{\infty} 2^{j \, (D+\theta_1-2\,m)} \lesssim M_w(|\vec{f}|^{q'_0})(x)^{\frac{1}{q'_0}} = G(x)^{\frac{1}{q'_0}},$$

provided we take *m* large enough so that $D + \theta_1 - 2m < 0$. This completes the estimate of H_B and G_B and so completes our proof.

10. Square function estimates for the gradient of the semigroup

In this section we prove $L^{p}(w)$ estimates for the vertical square function

$$G_{L_w}f(x) = \left(\int_0^\infty |t^{1/2} \nabla e^{-tL_w} f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Proposition 10.1. *Let* $q_{-}(L_w)$ *. Then*

$$|G_{L_w}f||_{L^p(w)} \lesssim ||f||_{L^p(w)}.$$
(10.2)

 $\|G_{L_w}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$ Furthermore, if $v \in A_{p/q_-(L_w)}(w) \cap RH_{(q_+(L_w)/p)'}(w)$, then

$$\|G_{L_w}f\|_{L^p(v\,dw)} \lesssim \|f\|_{L^p(v\,dw)} \,. \tag{10.3}$$

We can also prove a reverse inequality for G_{L_w} . To do so we need to introduce an auxiliary operator. Define the weighted Laplacian by $\Delta_w = -w^{-1} \operatorname{div} w \nabla$: i.e., Δ_w is the operator L_w if we take the matrix A to be wI, where I is the identity matrix.

Proposition 10.4. Let $q_+(\Delta_w)' . Then$

$$\|f\|_{L^{p}(w)} \lesssim \|G_{L_{w}}f\|_{L^{p}(w)}.$$
(10.5)

Furthermore, if $v \in A_{p/q_+(\Delta_w)'}(w)$, then

$$\|f\|_{L^p(v\,dw)} \lesssim \|G_{L_w}f\|_{L^p(v\,dw)}.\tag{10.6}$$

Proof of Proposition 10.1. The proof could be done in a way similar to those for the square function g_{L_w} in Section 5. However, we will give a shorter proof that uses the Riesz transform estimates from Section 9.

Let $q_{-} = q_{-}(L_w)$ and $q_{+} = q_{+}(L_w)$. Fix p,

$$q_{-} = p_{-}(L_w)$$

and $v \in A_{p/q_-}(w) \cap RH_{(q_+/p)'}(w)$. Then by Proposition 9.1, the Riesz transform is bounded on $L^p(v \, dw)$, and so by Lemma 5.4 it has a bounded extension to $L^p_{\mathbb{H}}(v \, dw)$: i.e., if $g(x,t) \in L^p_{\mathbb{H}}(v \, dw)$, then $\|\nabla L^{-1/2}_w g\|_{L^p_{\mathbb{H}}(v \, dw)} \lesssim \|g\|_{L^p_{\mathbb{H}}(v \, w)}$, where the extension of $\nabla L^{-1/2}_w$ to \mathbb{H} -valued functions is defined for $x \in \mathbb{R}^n$ and t > 0 by $(\nabla L^{-1/2}_w g)(x,t) = \nabla L^{-1/2}_w (g(\cdot,t))(x)$.

Define $g_f(x,t) = (tL_w)^{1/2} e^{-tL_w} f(x)$ and $G_f(x,t) = t^{1/2} \nabla e^{-tL_w} f(x)$; then we clearly have $\|g_{L_w}f\|_{L^p(v\,dw)} = \|g_f\|_{L^p_{\mathbb{H}}(v\,dw)}$ and $\|G_{L_w}f\|_{L^p(v\,dw)} = \|G_f\|_{L^p_{\mathbb{H}}(v\,dw)}$. Furthermore, $G_f(x,t) = \nabla L_w^{-1/2}(g_f(\cdot,t))(x) = (\nabla L_w^{-1/2}g_f)(x,t)$. Hence,

$$\begin{aligned} \|G_{L_w}f\|_{L^p(v\,dw)} &= \|G_f\|_{L^p_{\mathbb{H}}(v\,dw)} = \left\|\nabla L^{-1/2}_w g_f\right\|_{L^p_{\mathbb{H}}(v\,dw)} \\ &\lesssim \|g_f\|_{L^p_{\mathbb{H}}(v\,dw)} = \|g_{L_w}f\|_{L^p(v\,dw)} \lesssim \|f\|_{L^p(v\,dw)}. \end{aligned}$$

To prove the last inequality we used Proposition 5.1; we also used the fact that $q_- = p_-(L_w) and <math>v \in A_{p/q_-}(w) \cap RH_{(q_+/p)'}(w)$, which together imply that $v \in A_{p/p_-(L_w)}(w) \cap RH_{(p_+(L_w)/p)'}(w)$. This proves (10.3). To prove inequality (10.2), we take $v \equiv 1$.

To prove Proposition 10.4 we need the following identity relating G_{L_w} and Δ_w . It is a straightforward extension of a similar unweighted result given in [2, Section 7.1]. For completeness we include the proof.

Lemma 10.7. If $f, g \in L_c^{\infty}(w)$ then

$$\left| \int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dw \right| \le (\Lambda + 1) \int_{\mathbb{R}^n} G_{L_w} f(x) \, \overline{G_{\Delta_w} g(x)} \, dw.$$

Proof. By the definition and properties of the operators L_w and Δ_w we have that

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} \, dw$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} e^{-\varepsilon L_w} f(x)\overline{e^{-\varepsilon \Delta_w}g(x)} \, dw - \lim_{R \uparrow \infty} \int_{\mathbb{R}^n} e^{-RL_w} f(x)\overline{e^{-R\Delta_w}g(x)} \, dw$$

$$= -\int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} e^{-tL_w} f(x)\overline{e^{-t\Delta_w}g(x)} \, dw \, dt$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \left(L_w e^{-tL_w} f(x)\overline{e^{-t\Delta_w}g(x)} + e^{-tL_w} f(x)\overline{\Delta_w e^{-t\Delta_w}g(x)} \right) \, dw \, dt$$

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$$= \int_0^\infty \int_{\mathbb{R}^n} \left(A(x)w(x)^{-1} + I \right) \left(\nabla e^{-tL_w} f(x) \ \overline{\nabla e^{-t\Delta_w} g(x)} \right) dw \, dt$$

Since $||Aw^{-1}||_{\infty} \leq \Lambda$, if we apply Hölder's inequality in the t variable we get the desired result.

Proof of Proposition 10.4. As a consequence of the Gaussian estimate for weighted operators with real symmetric coefficients that were proved in [16, 18], we have that $\Delta_w \in \mathcal{O}(L^1(w) \to L^{\infty}(w))$. In particular, $q_-(\Delta_w) = p_-(L_{\Delta_w}) = 1$. Further, by the results in Section 8 we have that $q_+(\Delta_w) > 2$.

Therefore, by Proposition 10.1, if $1 < p' < q_+(\Delta_w)$, and

$$u \in A_{p'}(w) \cap RH_{(q_+(\Delta_w)/p')'}(w),$$
 (10.8)

then

$$\|G_{\Delta_w}f\|_{L^{p'}(u\,dw)} \lesssim \|f\|_{L^{p'}(u\,dw)}.$$
(10.9)

We want to apply inequality (10.9) with $u = v^{1-p'}$. By [6, Lemma 4.4], the condition (10.8) is equivalent to $v \in A_{p/q_+(w)'}(w)$.

Now fix $f, g \in L_c^{\infty}$, and a weight $v \in A_{p/q_+(w)'}(w)$. Then by Lemma 10.7, for $q_+(\Delta_w)' ,$

$$\left| \int_{\mathbb{R}^{n}} f(x)g(x) \, dw \right| \leq (\Lambda + 1) \int_{\mathbb{R}^{n}} G_{L_{w}}f(x) \, G_{\Delta_{w}}g(x) \, dw$$

= $(\Lambda + 1) \int_{\mathbb{R}^{n}} G_{L_{w}}f(x) \, G_{\Delta_{w}}g(x) \, v^{1/p} \, v^{-1/p} \, dw$
 $\leq (\Lambda + 1) \, \|G_{L_{w}}f\|_{L^{p}(v \, dw)} \, \|G_{\Delta_{w}}g\|_{L^{p'}(v^{1-p'} \, dw)}$
 $\lesssim \|G_{L_{w}}f\|_{L^{p}(v \, dw)} \, \|g\|_{L^{p'}(v^{1-p'} \, dw)};$

the last inequality follows from (10.9). If we take $g = \operatorname{sign}(f) |f|^{p-1} v$, we get

$$\|f\|_{L^{p}(v\,dw)}^{p} \lesssim \|G_{L_{w}}f\|_{L^{p}(v\,w)} \, \||f|^{p-1} \, v\|_{L^{p'}(v^{1-p'}\,dw)} = \|G_{L_{w}}f\|_{L^{p}(v\,dw)} \, \|f\|_{L^{p}(v\,dw)}^{p/p'} \, .$$

This immediately gives us the desired inequality.

11. UNWEIGHTED L^2 KATO ESTIMATES

In this section we prove unweighted L^2 estimates for the operators we have considered in the previous sections. These will all be consequences of the weighted $L^p(v \, dw)$ estimates we have already proved: it will only be necessary to find further conditions on $w \in A_2$ so that the weight $v = w^{-1}$ satisfies the requisite conditions.

We are particularly interested in power weights and we recall some well-known facts about them. Define $w_{\alpha}(x) = |x|^{\alpha}$, $\alpha > -n$; this restriction guarantees that w_{α} is locally integrable. We can exactly determine the Muckenhoupt A_p and reverse Hölder RH_s classes of these weights in terms of α : if $-n < \alpha \leq 0$, then $w \in A_1$; for $1 , <math>w \in A_p$ if $-n < \alpha < n (p-1)$. Furthermore, if $0 \leq \alpha < \infty$, $w \in RH_{\infty}$; for $1 < q < \infty$, $w \in RH_q$. if $-n/q < \alpha < \infty$. Hence, we easily see that

$$r_{w_{\alpha}} = \max\{1, 1 + \alpha/n\}, \qquad s_{w_{\alpha}} = \left(\max\{1, (1 + \alpha/n)^{-1}\}\right)'.$$
 (11.1)

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We first consider the semigroup e^{-tL_w} , the functional calculus, and the square function g_{L_w} , since these estimates will depend on $p_-(L_w)$ and $p_+(L_w)$ and we have good estimates for these quantities.

Theorem 11.2. Given a weight $w \in A_2$, suppose $1 \leq r_w < 1 + \frac{2}{n}$ and $s_w > \frac{n}{2}r_w + 1$. Then $e^{-tL_w} : L^2 \to L^2$ is uniformly bounded for all t > 0. Similarly, $\varphi(L_w) : L^2 \to L^2$, where φ is any bounded holomorphic function on Σ_{μ} , $\mu \in (\vartheta, \pi)$, and $g_{L_w} : L^2 \to L^2$.

In particular, these L^2 estimates hold if we assume that $w \in A_1 \cap RH_{1+\frac{n}{2}}$, or more generally if $w \in A_r \cap RH_{\frac{n}{2}r+1}$ for $1 < r \leq 1 + \frac{2}{n}$, or if we take the power weights

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{2n}{n+2} < \alpha < 2.$$

Proof. Let p = q = 2, $p_0 = (2_w^*)'$, $q_0 = 2_w^*$, and let $v = w^{-1}$. Then by Proposition 3.1, Corollary 3.3 and the nesting properties of weights, $e^{-tL_w} \in \mathcal{O}(L^2 \to L^2)$ if $w^{-1} \in A_{2/p_0}(w) \cap RH_{(q_0/2)'}(w)$; in particular, by Lemma 2.30, $e^{-tL_w} : L^2 \to L^2$ is uniformly bounded. However, this weight condition is equivalent to

$$w \in RH_{(2/p_0)'} \cap A_{q_0/2}.$$

A straightforward computation shows that

$$\frac{q_0}{2} = \frac{nr_w}{nr_w - 2}, \qquad \left(\frac{2}{p_0}\right)' = \frac{n}{2}r_w + 1.$$

Since $r_w < 1 + \frac{2}{n}$, we have that $r_w < \frac{nr_w}{nr_w-2}$, so we automatically have that $w \in A_{q_0/2}$. Therefore, the desired bounds hold if we have $s_w > \frac{n}{2}r_w + 1$. If $w \in A_r \cap RH_{\frac{n}{2}r+1}$ with $1 \le r \le 1 + \frac{2}{n}$, then $r_w \le r$ and $s_w > \frac{n}{2}r + 1 \ge \frac{n}{2}r_w + 1$. The desired conclusion for power weights follows at once from (11.1).

The same argument holds for $\varphi(L_w)$ and g_{L_w} , using Proposition 4.3 or Proposition 5.1, respectively.

It is straightforward to construct weights more general than power weights that satisfy the conditions on r_w and s_w in the above theorems. For instance, $w \in A_{1+\frac{2}{n}} \cap RH_{2+\frac{n}{2}}$ (which corresponds to the choice $r = 1 + \frac{2}{n}$) if and only if there exist $u_1, u_2 \in A_1$ such that

$$w = u_1^{\frac{2}{n+4}} u_2^{-\frac{2}{n}}.$$

This follows from the Jones factorization theorem and the properties of A_1 weights: cf. [15].

Remark 11.3. We can modify the proof of Theorem 11.2 to get unweighted L^p estimates for values of p close to 2. We leave the details to the interested reader.

For the reverse inequalities we need to take into account the slightly stronger hypotheses in Proposition 6.1; otherwise, the proof of the following result follows exactly as in the proof of Theorem 11.2.

Theorem 11.4. Given a weight $w \in A_2$, suppose that

$$1 \le r_w < 1 + \frac{2}{n} \quad and \ s_w > \max\left\{ \left(\frac{2}{r_w}\right)', \frac{n}{2}r_w + 1 \right\}.$$
$$\|L_w^{1/2}f\|_{L^2} \le C \|\nabla f\|_{L^2}, \qquad f \in \mathcal{S}.$$
(11.5)

Then

In particular, this is the case if we either assume that $w \in A_1 \cap RH_{1+\frac{n}{2}}$, or more generally that $w \in A_r \cap RH_{\max\{(\frac{2}{r})', \frac{n}{2}r+1\}}$, with $1 < r \leq 1 + \frac{2}{n}$, or for power weights if we take

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{2n}{n+2} = -\min\left\{\frac{n}{2}, \frac{2n}{n+2}\right\} < \alpha < 2.$$

Remark 11.6. Note that $\max\{(\frac{2}{r})', \frac{n}{2}r+1\} = \frac{n}{2}r+1$ provided $r \le 2-\frac{2}{n}$ and this always holds if $n \ge 4$ as $1+\frac{2}{n} \le 2-\frac{2}{n}$. In this case, the conditions in the second part of Theorem 11.4 simplify to the same conditions as in Theorem 11.2.

Remark 11.7. We note that in Theorems 11.2 and 11.4 we can replace $1 \leq r_w < 1 + \frac{2}{n}$ with the possibly weaker condition $1 \leq r_w < \frac{p_+(L_w)}{2}$. The proof only requires us to take $q_0 = p_+(L_w)$.

For the gradient of the semigroup $\sqrt{t}\nabla e^{-tL_w}$, the Riesz transform $\nabla L_w^{-1/2}$, and the square function G_{L_w} , our estimates depend on $q_+(L_w)$.

Theorem 11.8. Given a weight $w \in A_2$, suppose $1 \leq r_w < \frac{q_+(L_w)}{2}$ and $s_w > \frac{n}{2}r_w + 1$. Then $\sqrt{t}\nabla e^{-tL_w} : L^2 \to L^2$ is uniformly bounded for all t > 0. Similarly, we have that $\nabla L_w^{-1/2} : L^2 \to L^2$ and $G_{L_w} : L^2 \to L^2$.

In particular, this is the case if we assume that $w \in A_1 \cap RH_{\frac{n}{2}+1}$. Furthermore, these L^2 estimates hold if the following is true: given $\Theta \geq 1$ there exists $\epsilon_0 = \epsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \epsilon_0 \leq \frac{1}{2n}$, such that $w \in A_{1+\epsilon} \cap RH_{\frac{n}{2}(1+\epsilon)+1}$, $0 \leq \epsilon < \epsilon_0$, and $[w]_{A_2} \leq \Theta$.

For power weights, there exists $\epsilon_1 = \epsilon_1(n, \Lambda/\lambda)$, $0 < \epsilon_1 \leq \frac{1}{2}$, such that these estimate holds for

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{2n}{n+2} < \alpha < \epsilon_1.$$

Proof. We will prove this result for $\sqrt{t}\nabla e^{-tL_w}$ using Proposition 7.1. The proof for $\nabla L_w^{-1/2}$ or G_{L_w} is exactly the same, using Proposition 9.1 or Proposition 10.1.

By Proposition 7.1, $\sqrt{t}\nabla e^{-tL_w}$: $L^2 \to L^2$ if $w^{-1} = v \in A_{2/q_-(L_w)}(w) \cap RH_{q_+(L_w)/2)'}(w)$, which is equivalent to

$$w \in RH_{(2/q_{-}(L_w))'} \cap A_{q_{+}(L_w)/2}$$

Therefore, we need $r_w < q_+(L_w)/2$. Furthermore, since we have that $q_-(L_w) = p_-(L_w) \le (2_w^*)'$, we can take

$$s_w > \left(\frac{2}{(2_w^*)'}\right)' = \frac{n}{2}r_w + 1$$

To get the particular examples stated in the theorem, note first that if we let $r_w = 1$, then it clearly suffices to assume $w \in A_1 \cap RH_{\frac{n}{2}+1}$, since we showed in Section 8 that $q_+(L_w) > 2$ for every $w \in A_2$.

We now prove the condition for weights $w \in A_{1+\epsilon}$. In this case it is more difficult to satisfy the condition $r_w < q_+(L_w)/2$ since the righthand side can be very close to 1 depending on w. Assume then that $w \in A_{1+\epsilon} \cap RH_{\frac{n}{2}(1+\epsilon)+1}$, with $0 \le \epsilon < \epsilon_0 \le \frac{1}{2n}$, $[w]_{A_2} \le \Theta$, and with $\epsilon_0 > 0$ to be fixed below. Then we have that

$$s_w > \frac{n}{2} (1+\epsilon) + 1 \ge \frac{n}{2} r_w + 1.$$

Therefore, in order to apply the first half of the theorem we need to show that we can choose ϵ_0 sufficiently small so that $r_w < q_+(L_w)/2$. To do so we will use the notation and computations from Section 8. There we showed that $q_+(L_w) \ge q_w$, and so it will suffice to show that

$$2r_w < q_w = \min(r'_w, p_+(L_w), p_0).$$
(11.9)

We will compare r_w to each term in the minimum in turn.

The first two terms are straightforward. First, we have that $r_w < 1 + \epsilon < 1 + \frac{1}{2n} < \frac{3}{2}$ and so $2r_w < r'_w$. Second, $r_w < 1 + \frac{1}{2n} < 1 + \frac{2}{n}$, and it follows at once from this that $2r_w < 2^*_w$. By Proposition 3.1, $2^*_w \le p_+(L_w)$ and so $2r_w < p_+(L_w)$.

Finally, we estimate p_0 , the exponent from the higher integrability condition (8.3). We will use the formula (8.4). First, we need to fix the exponent q from the Poincaré inequality (8.2). Let q = 2 - 1/n; this value satisfies (8.1) since $r_w < 1 + \frac{1}{2n} < 1 + \frac{1}{n}$. With this choice of q(that only depends on n), we have that

$$p_0 = 2 + \frac{2 - q}{2^{4/q+1} C_1^2 C_2^2 [w]_{A_2}^{6/q+17}} = 2 + \frac{1}{n C(n, \Lambda/\lambda) [w]_{A_2}^{\theta_n}}$$

where $C(n, \Lambda/\lambda) \geq 1$ depends only on n and the ratio Λ/λ of the ellipticity constants of the matrix A used to define L_w , and where $\theta_n \geq 1$ depends only on n. Then, since we also assumed that $[w]_{A_2} \leq \Theta$, we get that

$$p_0 = 2 + \frac{1}{n C(n, \Lambda/\lambda) [w]_{A_2}^{\theta_n}} \ge 2 + \frac{1}{n C(n, \Lambda/\lambda) \Theta^{\theta_n}} = 2 + 2\epsilon_0,$$

and $\epsilon_0 = (2 n C(n, \Lambda/\lambda) \Theta^{\theta_n})^{-1}$ is such that $0 < \epsilon_0 \leq \frac{1}{2n}$. Thus $2r_w < 2(1+\epsilon) < 2(1+\epsilon_0) \leq p_0$ and so $2r_w < p_0$. This completes the proof that (11.9) is satisfied, and so the L^2 estimates hold for weights that satisfy $w \in A_{1+\epsilon} \cap RH_{\frac{n}{2}(1+\epsilon)+1}$.

Finally, we consider power weights. First, it is easy to see that

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad \qquad \frac{-2n}{n+2} < \alpha \le 0$$

yields the desired estimates, since in this case $r_w = 1$ and $s_w > \frac{n}{2} + 1 = \frac{n}{2}r_w + 1$.

Now consider the case $\alpha > 0$. If we assume that $\alpha < \frac{1}{2}$, then $w \in A_{1+\frac{1}{2n}} \cap RH_{\infty}$. Moreover, it is straightforward to show that for all such α , there exists Θ , depending only on n, such that $[w_{\alpha}]_{A_2} \leq \Theta$. Now apply the above argument to find $\epsilon_0 \in (0, \frac{1}{2n}]$; this value will only depend on n and the ratio Λ/λ . If we let $\epsilon_1 = n \epsilon_0$ and assume that $0 < \alpha < \epsilon_1$, then $\alpha < \frac{1}{2}$ and $w_{\alpha} \in A_{1+\epsilon}$ for some $\epsilon < \epsilon_0$ as desired.

To find examples of weights other than power weights to which Theorem 11.8 apply, we argue as before. If $u_1 \in A_1$, then

$$w = u_1^{\frac{2}{n+2}} \in A_1 \cap RH_{\frac{n}{2}+1}$$

To get weights that are not in A_1 , take $u \in A_2$ and let $w = u^{\theta}$. If θ is sufficiently small (depending on n, the ratio Λ/λ and $[u]_{A_2}$) we can show that w satisfies the final conditions given in Theorem 11.8. Details are left to the interested reader.

Remark 11.10. To get the unweighted lower estimate

$$||f||_{L^2} \le C ||G_{L_w}f||_{L^2},$$

we note that by (10.6) we need $w^{-1} \in A_{2/q_+(\Delta_w)'}(w)$, or equivalently, $w \in RH_{(2/q_+(\Delta_w)')'}$. Hence, it suffices to assume

$$s_w > 1 + \frac{q_+(\Delta_w)}{q_+(\Delta_w) - 2}.$$

Arguing as above we can construct weights that satisfy this condition; details are left to the interested reader.

If we combine Theorems 11.4, 11.8, and Remark 11.7 we solve the Kato square root problem for degenerate elliptic operators.

Theorem 11.11. Let $L_w = -w^{-1} \operatorname{div} A \nabla$ be a degenerate elliptic operator with $w \in A_2$. If

$$1 \le r_w < \frac{q_+(L_w)}{2}$$
 and $s_w > \max\left\{\left(\frac{2}{r_w}\right)', \frac{n}{2}r_w + 1\right\}$

then the Kato problem can be solved for L_w : that is, for every $f \in H^1(\mathbb{R}^n)$.

$$\|L_w^{1/2}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)},\tag{11.12}$$

where the implicit constants depend only on the dimension, the ellipticity constants λ , Λ , and w.

In particular, (11.12) holds if $w \in A_1 \cap RH_{\frac{n}{2}+1}$. Further, (11.12) holds if the following is true: given $\Theta \ge 1$ there exists $\epsilon_0 = \epsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \epsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\epsilon} \cap RH_{\max\{(\frac{2}{1+\epsilon})', \frac{n}{2}(1+\epsilon)+1\}}$, $0 \le \epsilon < \epsilon_0$, and $[w]_{A_2} \le \Theta$.

For power weights, there exists $\epsilon_1 = \epsilon_1(n, \Lambda/\lambda)$, $0 < \epsilon_1 \leq \frac{1}{2}$, such that inequality (11.12) holds (with w_{α} in place of w) if

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad \qquad -\frac{2n}{n+2} < \alpha < \epsilon_1.$$

We can restate the final part of Theorem 11.11 as follows: consider the family of operators $L_{\gamma} = -|x|^{\gamma} \operatorname{div}(|x|^{-\gamma}B(x)\nabla)$, where B is an $n \times n$ complex-valued matrix that satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \le \operatorname{Re} \langle B(x)\xi,\xi\rangle, \qquad |\langle B(x)\xi,\eta\rangle| \le \Lambda |\xi| |\eta|, \quad \xi, \eta \in \mathbb{C}^n, \text{ a.e. } x \in \mathbb{R}^n.$$

Then,

$$\|L_{\gamma}^{1/2}f\|_{L^{2}(\mathbb{R}^{n})} \approx \|\nabla f\|_{L^{2}(\mathbb{R}^{n})}, \qquad -\epsilon_{1} < \gamma < \frac{2n}{n+2}.$$
(11.13)

When $\gamma = 0$ we get the classical Kato square root problem solved by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [4]. Inequality (11.13) shows that we can find an open interval containing 0 such that if γ is in this interval, the same estimate holds.

12. Applications to L^2 boundary value problems

In this section we apply the results from the previous section to some L^2 boundary value problems involving the degenerate elliptic operator L_w . We follow the ideas in [10] and consider semigroup solutions: for the Dirichlet or Regularity problems we let $u(x,t) = e^{-tL_w^{1/2}}f(x)$; for the Neumann problem we let $u(x,t) = -L_w^{-1/2}e^{-tL_w^{1/2}}f(x)$. In each case, for t > 0 fixed $L_w u(\cdot, t)$ makes sense in a weak sense since $u(\cdot, t)$ is in the domain of L_w . Further, derivatives in t are well defined because of the semigroup properties. Finally, note that by the strong continuity of the semigroup and the off-diagonal estimates, in the context of the following results we have that $e^{-tL_w^{1/2}}f \to f$ as $t \to 0^+$ in L^2 ; see [7, Section 4.2]. Further details are left to the interested reader.

We first consider the Dirichlet problem on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0, \infty)$:

$$\begin{cases} \partial_t^2 u - L_w u = 0, & \text{on } \mathbb{R}^n \\ u \big|_{\partial \mathbb{R}^{n+1}_+} = f & \text{on } \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$
(12.1)

Theorem 12.2. Given a weight $w \in A_2$, suppose $1 \le r_w < 1 + \frac{2}{n}$ and $s_w > \frac{n}{2}r_w + 1$. Then for any $f \in L^2(\mathbb{R}^n)$, $u(x,t) = e^{-tL_w^{1/2}}f(x)$ is a solution of (12.1) with convergence to the boundary data as $t \to 0^+$ in the L²-sense. Furthermore, we have that

$$\sup_{t>0} \|u(\cdot,t)\|_{L^2} \le C \|f\|_{L^2}.$$
(12.3)

In particular, this is the case if we assume that $w \in A_1 \cap RH_{1+\frac{n}{2}}$, or $w \in A_r \cap RH_{\frac{n}{2}r+1}$ with $1 < r \leq 1 + \frac{2}{n}$, or if we take the power weights

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{2n}{n+2} < \alpha < 2.$$

Proof. Formally, it is clear that u is a solution to (12.1), and this formalism can be justified by appealing to the theory of maximal accretive operators: see Kato [25]. Alternatively, the weighted estimates for the functional calculus in Proposition 4.3 show that both $\frac{\partial^2}{\partial t^2}u(\cdot,t)$ and $L_wu(\cdot,t)$ belong to L^2 for each t > 0 and that they are equal in the L^2 -sense. To see that inequality (12.3) holds, it suffices to let $\varphi_t(z) = e^{-t\sqrt{z}}$. Then φ_t is a bounded holomorphic function on Σ_{μ} , and so by Theorem 11.2 we get the desired bound.

Remark 12.4. Note that as observed in Remark 11.7, in the previous result we can replace $1 \le r_w < 1 + \frac{2}{n}$ with the possibly weaker condition $1 \le r_w < \frac{p_+(L_w)}{2}$. Also, by Proposition 4.3 we also have that for u as in Theorem 12.2 and all $k \ge 1$

$$\sup_{t>0} \left\| t^k \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_{L^2} = \sup_{t>0} \left\| (t^k L_w^{1/2})^k e^{-tL_w^{1/2}} f(\cdot) \right\|_{L^2} \le C \|f\|_{L^2}.$$
(12.5)

For the regularity problem we have the following.

Theorem 12.6. Given a weight $w \in A_2$, suppose

$$1 \le r_w < \frac{q_+(L_w)}{2}$$
 and $s_w > \max\left\{\left(\frac{2}{r_w}\right)', \frac{n}{2}r_w + 1\right\}$

Then for any $f \in H^1(\mathbb{R}^n)$, $u(x,t) = e^{-tL_w^{1/2}}f(x)$ is a solution of (12.1) with convergence to the boundary data as $t \to 0^+$ in the L²-sense. Furthermore, we have that

$$\sup_{t>0} \|\nabla_{x,t} u(\cdot,t)\|_{L^2} \le C \|\nabla f\|_{L^2}.$$
(12.7)

In particular, (12.7) holds if we assume that $w \in A_1 \cap RH_{1+\frac{n}{2}}$. Furthermore, it holds if the following is true: given $\Theta \ge 1$ there exists $\epsilon_0 = \epsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \epsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\epsilon} \cap RH_{\max\{(\frac{2}{1+\epsilon})', \frac{n}{2}(1+\epsilon)+1\}}$, $0 \le \epsilon < \epsilon_0$, and $[w]_{A_2} \le \Theta$.

For power weights, there exists $\epsilon_1 = \epsilon_1(n, \Lambda/\lambda), \ 0 < \epsilon_1 \leq \frac{1}{2}$, such that (12.7) holds if

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{n}{2} < \alpha < \epsilon_1.$$

Proof. Arguing as before, it suffices to prove that (12.7) holds. For any t > 0 we have, by Theorem 11.11

$$\begin{aligned} \|\nabla_{x,t}u(\cdot,t)\|_{L^{2}} &\leq \|\nabla L_{w}^{-1/2}L_{w}^{1/2}e^{-tL_{w}^{1/2}}f\|_{L^{2}} + \|L_{w}^{1/2}e^{-tL_{w}^{1/2}}f\|_{L^{2}} \\ &\lesssim \|L_{w}^{1/2}e^{-tL_{w}^{1/2}}f\|_{L^{2}} = \|e^{-tL_{w}^{1/2}}L_{w}^{1/2}f\|_{L^{2}} \lesssim \|L_{w}^{1/2}f\|_{L^{2}} \lesssim \|\nabla f\|_{L^{2}}. \end{aligned}$$

Note that under the hypothesis of Theorem 12.6, and as observed in Remark 12.4, we have that $u(\cdot, t) = e^{-tL_w^{1/2}} f$ satisfies (12.3) and (12.5). Additionally, from the functional calculus estimates on L^2 it follows that

$$\sup_{t>0} \|t\nabla_{x,t}u(\cdot,t)\|_{L^2} \lesssim \|tL_w^{1/2}e^{-tL_w^{1/2}}f\|_{L^2} \lesssim \|f\|_{L^2}.$$
(12.8)

Finally, we consider the Neumann problem

$$\begin{cases} \partial_t^2 u - L_w u = 0, & \text{on } \mathbb{R}^n \\ \partial_t u \Big|_{\partial \mathbb{R}^{n+1}_+} = f & \text{on } \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$
(12.9)

Theorem 12.10. Given a weight $w \in A_2$, suppose $1 \le r_w < \frac{q_+(L_w)}{2}$ and $s_w > \frac{n}{2}r_w + 1$. Then for any $f \in L^2(\mathbb{R}^n)$, $u(x,t) = -L_w^{-1/2}e^{-tL_w^{1/2}}f(x)$ is a solution of (12.9) with convergence of $\partial_t u(\cdot,t) \to f$ as $t \to 0^+$ in the L²-sense. Furthermore, we have that

$$\sup_{t>0} \|\nabla_{x,t} u(\cdot,t)\|_{L^2} \le C \|f\|_{L^2}.$$
(12.11)

In particular, (12.11) holds if we assume that $w \in A_1 \cap RH_{1+\frac{n}{2}}$. Furthermore, it holds if the following is true: given $\Theta \ge 1$ there exists $\epsilon_0 = \epsilon_0(\Theta, n, \Lambda/\lambda)$, $0 < \epsilon_0 \le \frac{1}{2n}$, such that $w \in A_{1+\epsilon} \cap RH_{\frac{n}{2}(1+\epsilon)+1}$, $0 \le \epsilon < \epsilon_0$, and $[w]_{A_2} \le \Theta$.

For power weights, there exists $\epsilon_1 = \epsilon_1(n, \Lambda/\lambda), \ 0 < \epsilon_1 \leq \frac{1}{2}$, such that (12.11) holds if

$$w_{\alpha}(x) = |x|^{\alpha}, \qquad -\frac{2n}{n+2} < \alpha < \epsilon_1.$$

Proof. Again, u is clearly a formal solution of (12.9); see [25]. The proof that (12.11) holds is similar to the proof of (12.7):

$$\|\nabla_{x,t}u(\cdot,t)\|_{L^2} \le \|\nabla L_w^{-1/2}e^{-tL_w^{1/2}}f\|_{L^2} + \|e^{-tL_w^{1/2}}f\|_{L^2} \le \|e^{-tL_w^{1/2}}f\|_{L^2} \le \|f\|_{L^2}$$

where we have used Theorem 11.8 (for the Riesz transform) and Theorem 11.2 (for the functional calculus with $\varphi(z) = e^{-t\sqrt{z}}$).

Remark 12.12. As we noted in Remark 11.3, we can also get unweighted L^p bounds for these operators for values of p close to 2. As a consequence we can also get estimates L^p boundary value problems for the same values of p. Details are left to the reader.

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DAVID CRUZ-URIBE, OFS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487-0350, USA

E-mail address: dcruzuribe@ua.edu

JOSÉ MARÍA MARTELL, INSTITUTO DE CIENCIAS MATEMÁTICAS CSIC-UAM-UC3M-UCM, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, C/ NICOLÁS CABRERA, 13-15, E-28049 MADRID, SPAIN *E-mail address*: chema.martell@icmat.es

AND

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA *E-mail address*: martellj@missouri.edu

CRISTIAN RIOS, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, 2500 UNIVERSITY DRIVE NW, CALGARY, AB, T2N-1N4, CANADA

E-mail address: crios@ucalgary.ca