To appear in Int. Math. Res. Not.

WEIGHTED WEAK-TYPE INEQUALITIES AND A CONJECTURE OF SAWYER

D. CRUZ-URIBE, SFO, J. M. MARTELL, AND C. PÉREZ

Abstract. In 1985 Sawyer [15] proved the following weighted weak-type inequality for the Hardy-Littlewood maximal function on $\mathbb{R}$: for all $u, v \in A_1$,

$$uv\{x \in \mathbb{R} : M(fv)(x)v(x)^{-1} > t\} \leq \frac{C}{t}\int_{\mathbb{R}} |f(x)|u(x)v(x)\,dx.$$

He conjectured that the same inequality held for the Hilbert transform.

We give a positive answer to this conjecture. We do so by showing that these inequalities extend to the Hardy-Littlewood maximal function and Calderón-Zygmund operators in $\mathbb{R}^n$, and hold for a larger class of weights. We also extend to higher dimensions the related results of Muckenhoupt and Wheeden in [14]. Our proof uses extrapolation arguments based on techniques developed in [5]; as a consequence we get vector-valued extensions of our results with no extra work.

1. Introduction

In [14], Muckenhoupt and Wheeden proved a pair of weighted norm inequalities for the Hardy-Littlewood maximal operator and the Hilbert transform on the real line.

Theorem 1.1. *If $w \in A_1$, then there is a constant C such that for all $t > 0$,*

$$|\{x \in \mathbb{R} : M(fw^{-1})(x)w(x) > t\}| \leq \frac{C}{t}\int_{\mathbb{R}} |f(x)|\,dx, \tag{1.1}$$

$$|\{x \in \mathbb{R} : |H(fw^{-1})(x)|w(x) > t\}| \leq \frac{C}{t}\int_{\mathbb{R}} |f(x)|\,dx. \tag{1.2}$$

Later, Sawyer [15] developed their techniques and proved results of the same sort for the maximal operator, again on the real line.

Date: May 04, 2005.

1991 *Mathematics Subject Classification.* 42B20,42B25.

Key words and phrases. Weights, maximal functions, singular integral operators, extrapolation, weak-type inequalities.

The first author is partially supported by the Stewart faculty development fund of Trinity College; the second author is partially supported by MEC Grant MTM2004-0678; the third author is partially supported by DGICYT Grant PB980106.

Theorem 1.2. *If $u, v \in A_1$, then there is a constant C such that for all $t > 0$,*

$$uv\left(\left\{x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|u(x)v(x)\,dx. \tag{1.3}$$

In both papers, the motivation for these results was their applicability to proving weighted norm inequalities using the theory of interpolation with change of measure due to Stein and Weiss [16]. In particular, Sawyer showed that Theorem 1.2, combined with the Jones' factorization theorem for A_p weights, yielded a new proof of the boundedness of the maximal operator on $L^p(w)$, $w \in A_p$. In the hope of extending this approach to other operators, Sawyer also conjectured that (1.3) remained true if the maximal operator was replaced by the Hilbert transform.

In this paper we generalize Theorems 1.1 and 1.2 and extend them to higher dimensions. As an immediate corollary we prove Sawyer's conjecture for the Hilbert transform and extend it to other singular integral operators. Further, our method is general enough that it yields similar inequalities for a number of operators and also yields the analogous vector-valued inequalities.

To state our results, recall that a Calderón-Zygmund singular integral is a singular convolution operator whose kernel K is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$, has zero average on the unit sphere, and for all $x \neq 0$,

$$|K(x)| \leq \frac{C}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}.$$

(More generally, we may assume K is a Calderón-Zygmund operator in the sense of Coifman and Meyer; see [9] for a precise definition.)

Given weights u and v, by $v \in A_\infty(u)$ we mean that v satisfies the A_∞ condition defined with respect to the measure $u\,dx$ (as opposed to Lebesgue measure). A more precise definition is given in Section 2 below.

Theorem 1.3. *If $u \in A_1$, and $v \in A_1$ or $v \in A_\infty(u)$, then there is a constant C such that for all $t > 0$,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{M(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x)\,dx, \tag{1.4}$$

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x)\,dx, \tag{1.5}$$

where M is the Hardy-Littlewood maximal operator and T is any Calderón-Zygmund singular integral.

Clearly, (1.4) extends (1.3) to higher dimensions. By applying (1.5) to the Hilbert transform in $\mathbb{R}$ we solve the conjecture made by Sawyer in [15]. To see that Theorem 1.3 also extends Theorem 1.1, fix $w \in A_1$, and let $u = w$ and $v = w^{-1}$. Then

$uv = 1 \in A_\infty$, and this is equivalent to $v \in A_\infty(u)$. (See Lemma 2.1 and Remark 2.2 below.)

The proof of Theorem 1.3 has two parts. The first step is to prove (1.4) with the maximal operator replaced by the dyadic maximal operator.

Theorem 1.4. *If $u \in A_1$, and $v \in A_1$ or $v \in A_\infty(u)$, then there is a constant C such that for all $t > 0$,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{M_d(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|u(x)v(x)\,dx. \tag{1.6}$$

Remark 1.5. We will actually prove Theorem 1.4 assuming that u and v satisfy the corresponding weight conditions on dyadic cubes.

For each condition on v in Theorem 1.4 we give a very different proof. When $v \in A_1$, the proof is adapted from the original proof of Theorem 1.2 in [15], which depends on a very delicate decomposition argument. When $v \in A_\infty(u)$, we use a simpler Calderón-Zygmund decomposition argument partially motivated by an alternative proof of Theorem 1.1 sketched in [14]. However, we believe that there ought to be a proof that works for both cases.

Remark 1.6. We conjecture that Theorem 1.4 remains true with the weaker hypothesis that $u \in A_1$ and $v \in A_\infty$. Note that if $v \in A_\infty(u)$, then $v \in A_\infty$: see Lemmas 2.1 and 2.5. As evidence for the conjecture, consider the following: If $v \in A_\infty$, we can factor it as $v = v_1 v_2$, where $v_1 \in A_1$, and $v_2 \in RH_\infty \subset A_\infty(u)$. (See Section 2 below for definitions of these weight classes.) Further, there exists $s > 1$ such that $v_1^s \in A_1$ and $v_2^{s'} \in RH_\infty$. Therefore, Theorem 1.3 is true for the pairs (u, v_1^s) and $(u, v_2^{s'})$, and the weighted space $L^1(uv)$ is an interpolation space between $L^1(uv_1^s)$ and $L^1(uv_2^{s'})$. (See [3, 16].) Unfortunately, the analogous interpolation result is false in the scale of Lorentz spaces; see Ferreyra [10] and Asekritova, *et al.* [2].

The proof of Theorem 1.4 can be modified to yield a somewhat more general result: Let u be any weight and let $v \in A_\infty(u)$ be such that uv is dyadic doubling. Then there is a constant C such that for all $t > 0$,

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{M_d(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|Mu(x)v(x)\,dx.$$

(Details are left to the reader.) However, this does not shed much light on our conjecture.

The second step in the proof of Theorem 1.3 is to prove an extrapolation theorem using techniques from [5]. Hereafter, $\mathcal{F}$ will denote a family of ordered pairs of non-negative, measurable functions (f, g).

Theorem 1.7. *Given a family $\mathcal{F}$, suppose that for some p, $0 < p < \infty$, and every $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} f(x)^p w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x)\, dx, \tag{1.7}$$

for all $(f, g) \in \mathcal{F}$ such that the lefthand side is finite, and where C depends only on the A_∞ constant of w. Then for all weights $u \in A_1$ and $v \in A_\infty$,

$$\|f v^{-1}\|_{L^{1,\infty}(uv)} \leq C\, \|g v^{-1}\|_{L^{1,\infty}(uv)}, \qquad (f, g) \in \mathcal{F}. \tag{1.8}$$

Remark 1.8. By the A_∞ extrapolation theorem proved in [5], if (1.7) holds for any fixed value of p, then it holds for all values of p, $0 < p < \infty$. In the proof of Theorem 1.7 we will need (1.7) for a particular value of p determined by the weights.

The proof of Theorem 1.3 is now immediate. Consider the pairs $\big(M(fv), M_d(fv)\big)$. Then for $w \in A_\infty$ and all $p > 0$,

$$\int_{\mathbb{R}^n} M(fv)(x)^p w(x)\, dx \leq C \int_{\mathbb{R}^n} M_d(fv)(x)^p w(x)\, dx. \tag{1.9}$$

(This follows by a standard argument from the distribution inequality relating the maximal operator and dyadic maximal operator. See, for example, Duoandikoetxea [9, Lemma 2.12].) We apply Theorem 1.7 to (1.9) and so for all $u \in A_1$ and $v \in A_\infty$ it follows that

$$\|M(f\, v)\, v^{-1}\|_{L^{1,\infty}(uv)} \leq C\, \|M^d(f\, v)\, v^{-1}\|_{L^{1,\infty}(uv)}. \tag{1.10}$$

Theorem 1.3 now follows from Theorem 1.4. If $v \in A_1$, then $v \in A_\infty$, so (1.10) holds and Theorem 1.4 yields (1.4). Similarly, if $v \in A_\infty(u)$, then by Lemmas 2.1 and 2.5 below, $v \in A_\infty$, and again (1.10) holds.

The proof of (1.5) is similar. Let the family $\mathcal{F}$ consist of the pairs $\big(|T(fv)|, M(fv)\big)$. Then by the theorem of Coifman and Fefferman [4], for all $w \in A_\infty$ and all $p > 0$,

$$\int_{\mathbb{R}^n} |T(fv)(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} M(fv)(x)^p w(x)\, dx; \tag{1.11}$$

arguing as before, by Theorem 1.7 and (1.4), we get (1.5).

Remark 1.9. In [5] (using a result from [13]) we give a different proof of (1.11), one which does not use a good-λ inequality.

Remark 1.10. We can extend Theorem 1.3 to any other operator T for which (1.11) holds. For examples of such operators, we refer the interested reader to [1]. Another example is the square function g^*_λ which satisfies (1.11) when $\lambda > 2$; see [7] for details.

By adapting an argument due to Muckenhoupt and Wheeden [14, Theorem 5] we get a necessary condition for inequality (1.4) to hold: if for given weights u and v

the inequality (1.4) holds, then there exists a constant C such that for almost every $y \in \mathbb{R}^n$,

$$\left\| \frac{v(\cdot)^{-1}}{|\cdot - y|^n} \right\|_{L^{1,\infty}(uv)} \leq Cu(y). \tag{1.12}$$

As a first step to proving the conjecture made in Remark 1.6, it would be of interest to show that if $u \in A_1$, $v \in A_\infty$, then (1.12) holds.

Finally, we give some vector-valued extensions of Theorem 1.3.

Corollary 1.11. *If $u \in A_1$, and $v \in A_1$ or $v \in A_\infty(u)$, then for all $1 < q < \infty$ and $t > 0$,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{\left(\sum_j M(f_j\, v)(x)^q\right)^{\frac{1}{q}}}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} \left(\sum_j |f(x)|^q\right)^{\frac{1}{q}} u(x)v(x)\, dx,$$

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{\left(\sum_j |T(f_j\, v)(x)|^q\right)^{\frac{1}{q}}}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} \left(\sum_j |f(x)|^q\right)^{\frac{1}{q}} u(x)v(x)\, dx,$$

where M is the Hardy-Littlewood maximal function and T is a Calderón-Zygmund operator.

Corollary 1.11 follows almost automatically by applying the ideas in [5]. The second estimate follows from the first and from Theorem 1.7. As we showed in [5], by extrapolation (1.11) has a vector-valued extension: for all $0 < p, q < \infty$ and all $w \in A_\infty$,

$$\left\| \left(\sum_j |T(f_j\, v)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j (M(f_j\, v))^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Now define the family of pairs $\big(\|\{T(f_j\, v)\}_j\|_{\ell^q}, \|\{M(f_j\, v)\}_j\|_{\ell^q}\big)$; we can then apply Theorem 1.7 to get

$$\left\| \|\{T(f_j\, v)\}_j\|_{\ell^q} v^{-1} \right\|_{L^{1,\infty}(uv)} \leq \left\| \|\{M(f_j\, v)\}_j\|_{\ell^q} v^{-1} \right\|_{L^{1,\infty}(uv)}.$$

To prove the first inequality in Corollary 1.11, we first note that in [8] it was shown that given $1 < q < \infty$, for all $0 < p < \infty$ and $w \in A_\infty$,

$$\big\| \|\{Mf_j\}_j\|_{\ell^q} \big\|_{L^p(w)} \leq C \left\| M\big(\|\{f_j\}_j\|_{\ell^q}\big) \right\|_{L^p(w)}.$$

Using this result, we apply Theorem 1.7 with the pairs $\big(\|\{Mf_j\}_j\|_{\ell^q}, M\big(\|\{f_j\}_j\|_{\ell^q}\big)\big)$ and (1.4) to get

$$\big\| \|\{M(f_j\, v)\}_j\|_{\ell^q} v^{-1} \big\|_{L^{1,\infty}(uv)} \leq C \left\| M\big(\|\{f_j\}_j\|_{\ell^q} v^{-1}\big) \right\|_{L^{1,\infty}(uv)}$$

$$\leq C \left\| \|\{f_j\}_j\|_{\ell^q} \right\|_{L^1(uv)} = C \int_{\mathbb{R}^n} \Big(\sum_j |f(x)|^q \Big)^{\frac{1}{q}} u(x)v(x)\,dx.$$

The remainder of this paper is organized as follows: In Section 2 we give some preliminary definitions and lemmas about weights which will be used in later sections. In Section 3 we prove Theorem 1.7. In Section 4 we prove Theorem 1.4 in the case $v \in A_\infty(u)$. The proof of the case $v \in A_1$ is quite technical and lengthy. Our argument is an adaptation of Sawyer's original proof on $\mathbb{R}$, and requires many changes that are hidden within the details. So, for the sake of completeness, we have included the whole argument in Appendix B. In Appendix A we sketch the proof of a classical interpolation result in the scale of Lorentz spaces. This result is standard, but we have not been able to find in the literature a proof with the sharp constants needed in our proof.

2. Preliminary Results

In this section we state some basic definitions about weights and prove several lemmas about their structure that we will need in subsequent sections. For complete information, we refer the reader to [9, 11, 12].

Given a doubling measure μ we define the maximal operator M_μ by

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)|\,d\mu(y).$$

For $1 < p < \infty$, given a weight w we say that $w \in A_p(\mu)$ if for all cubes Q,

$$\left(\frac{1}{\mu(Q)} \int_Q w(x)\,d\mu(x) \right) \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'}\,d\mu(x) \right)^{p-1} \leq C.$$

We say that $w \in A_1(\mu)$ if $M_\mu w(x) \leq C w(x)$. We denote the best constant in these inequalities by $[w]_{A_p(\mu)}$, and we denote the union of all the $A_p(\mu)$ classes by $A_\infty(\mu)$.

When μ is doubling, then M_μ is bounded on $L^p(w\,du)$, $1 < p < \infty$, if and only if $w \in A_p(\mu)$. (This is well-known when μ is Lebesgue measure; however the same proof works in the more general setting. See especially [12] for details.)

When μ is the Lebesgue measure we omit the subscript μ and write, simply M, A_p, etc. If $u \in A_p$, $1 \leq p < \infty$, then u is a doubling weight (i.e., $d\mu = u\,dx$ is a doubling measure) and we define the classes $A_p(u) = A_p(u\,dx)$.

A weight $w \in A_p$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|Q|} \int_Q w(x)^s\,dx \right)^{1/s} \leq \frac{C}{|Q|} \int_Q w(x)\,dx,$$

for some $s > 1$ which depends only on $[w]_{A_p}$. We denote this by writing $w \in RH_s$ and the best constant by $[w]_{RH_s}$. We denote by RH_∞ the class of weights such that

$$\frac{C}{|Q|}\int_Q w(x)\,dx \geq \operatorname*{ess\,sup}_Q w(x).$$

For every $s > 1$, $RH_\infty \subset RH_s$. For more information on these classes, see [6].

The next four lemmas give some specific properties of A_p weights.

Lemma 2.1. *If $u \in A_1$ and $v \in A_\infty(u)$, then $uv \in A_\infty$. In particular, if $v \in A_p(u)$, $1 \leq p < \infty$, then $uv \in A_p$.*

Proof. Fix a cube Q. Then, since $u \in A_1$, for almost every $x \in Q$,

$$u_Q = \frac{1}{|Q|}\int_Q u(y)\,dy \leq [u]_{A_1} u(x).$$

Therefore, if $p > 1$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q u(x)v(x)\,dx\left(\frac{1}{|Q|}\int_Q \big(u(x)v(x)\big)^{1-p'}\,dx\right)^{p-1}\\
&\quad\leq \frac{[u]_{A_1}^p}{|Q|}\int_Q u(x)v(x)\,dx\left(\frac{1}{|Q|}\int_Q v(x)^{1-p'}u(x)\,dx\cdot u_Q^{-p'}\right)^{p-1}\\
&\quad= \frac{[u]_{A_1}^p}{u(Q)}\int_Q v(x)u(x)\,dx\left(\frac{1}{u(Q)}\int_Q v(x)^{1-p'}u(x)\,dx\right)^{p-1}\\
&\quad\leq [u]_{A_1}^p [v]_{A_p(u)}.
\end{aligned}$$

Hence, $uv \in A_p$. When $p = 1$ the proof is straightforward. □

Remark 2.2. As we noted above, we have the following equivalence: if $u \in A_1$, then $v \in A_\infty(u)$ if and only if $uv \in A_\infty$. (See [12, Exercise 9.3.6].)

Lemma 2.3. *If $u \in A_1$, $v \in A_p$, $1 \leq p < \infty$, then there exists $0 < \epsilon_0 < 1$ depending only on $[u]_{A_1}$ such that $uv^\epsilon \in A_p$ for all $0 < \epsilon < \epsilon_0$.*

Proof. Since $u \in A_1$, $u \in RH_{s_0}$ for some $s_0 > 1$ depending on $[u]_{A_1}$. Let $\epsilon_0 = 1/s_0'$ and $0 < \epsilon < \epsilon_0$. This implies that $u \in RH_s$ with $s = (1/\epsilon)'$.

We consider first the case when $v \in A_1$. Then since u, $v \in A_1$, for any cube Q and almost every $x \in Q$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q u(y)v(y)^\epsilon\,dy \leq \left(\frac{1}{|Q|}\int_Q u(y)^s\,dy\right)^{1/s}\left(\frac{1}{|Q|}\int_Q v(y)\,dy\right)^{1/s'}\\
&\quad\leq\ [u]_{RH_s}\frac{1}{|Q|}\int_Q u(y)\,dy\left(\frac{1}{|Q|}\int_Q v(y)\,dy\right)^{1/s'} \leq [u]_{RH_s}[u]_{A_1}[v]_{A_1}^\epsilon u(x)v(x)^\epsilon.
\end{aligned}$$

Hence $uv^\epsilon \in A_1$ with $[uv^\epsilon]_{A_1} \le [u]_{RH_s}[u]_{A_1}[v]_{A_1}^\epsilon$.

If $v \in A_p$, $1 < p < \infty$, then for any cube Q,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q u(x)v(x)^\epsilon\,dx\right)\left(\frac{1}{|Q|}\int_Q \big(u(x)v(x)^\epsilon\big)^{1-p'}\,dx\right)^{p-1}\\
&\qquad\le \left(\frac{1}{|Q|}\int_Q u(x)^s\,dx\right)^{1/s}\left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)^{1/s'}\\
&\qquad\qquad\times\left(\frac{1}{|Q|}\int_Q u(x)^{s(1-p')}\,dx\right)^{\frac{p-1}{s}}\left(\frac{1}{|Q|}\int_Q v(x)^{1-p'}\,dx\right)^{\frac{p-1}{s'}}\\
&\qquad\le [u]_{RH_s}\,[u]_{A_1}\,[v]_{A_p}^\epsilon.
\end{aligned}$$

Hence, $uv^\epsilon \in A_p$ with $[uv^\epsilon]_{A_p} \le [u]_{RH_s}[u]_{A_1}[v]_{A_p}^\epsilon$.

□

Lemma 2.4.

- *$w \in A_\infty$ if and only if $w = w_1w_2$, where $w_1 \in A_1$ and $w_2 \in RH_\infty$.*
- *If $w \in A_1$, then $w^{-1} \in RH_\infty$.*
- *If u, $v \in RH_\infty$, then $uv \in RH_\infty$.*

For a proof of this lemma, see [6, Theorems 4.4, 4.8, 5.1].

Lemma 2.5. *If $u \in A_1$ and $uv \in A_\infty$, then $v \in A_\infty$.*

Proof. This result follows by repeatedly applying Lemma 2.4. Since $uv \in A_\infty$, $uv = w_1w_2$, where $w_1 \in A_1$ and $w_2 \in RH_\infty$. As $u \in A_1$, it follows that $u^{-1} \in RH_\infty$ and then $w_2u^{-1} \in RH_\infty$. This and the fact that $w_1 \in A_1$ provides $v = w_1(w_2u^{-1}) \in A_\infty$. □

3. Proof of Theorem 1.7

Fix $u \in A_1$ and $v \in A_\infty$. Define the operator S by

$$Sf(x) = \frac{M(fu)(x)}{u(x)}$$

when $u(x) \neq 0$ and $Sf(x) = 0$ when $u(x) = 0$. (Note that since $u \in A_1$, $u > 0$ a.e.)

Since $u \in A_1$, S is bounded on $L^\infty(uv)$ with constant $C_1 = [u]_{A_1}$. We will now show that S is bounded on $L^{p_0}(uv)$ for some $1 < p_0 < \infty$. Observe that

$$\int_{\mathbb{R}^n} Sf(x)^{p_0}\,u(x)\,v(x)\,dx = \int_{\mathbb{R}^n} M(fu)(x)^{p_0}\,u(x)^{1-p_0}\,v(x)\,dx.$$

Since $v \in A_\infty$, $v \in A_t$ for some $t > 1$ large. Then by the A_p factorization theorem, there exist $v_1, v_2 \in A_1$ such that $v = v_1\,v_2^{1-t}$; hence,

$$u^{1-p_0}\,v = v_1\left(u\,v_2^{\frac{t-1}{p_0-1}}\right)^{1-p_0}.$$

By Lemma 2.3 there exists $0 < \epsilon_0 < 1$, depending only on $[u]_{A_1}$, such that $u\, v_2^{\epsilon} \in A_1$ for all $v_2 \in A_1$ and $0 < \epsilon < \epsilon_0$. Thus, if we let

$$p_0 = \frac{2(t-1)}{\epsilon_0} + 1,$$

then $u^{1-p_0}\, v \in A_{p_0}$. By Muckenhoupt's theorem, M is bounded on $L^{p_0}(u^{1-p_0} v)$ and therefore S is bounded on $L^{p_0}(uv)$ with some constant C_0. Observe that p_0 depends upon the A_1 constant of u and the A_t constant of v.

Thus by Marcinkiewicz interpolation in the scale of Lorentz spaces, S is bounded on $L^{q,1}(uv)$ for all $p_0 < q < \infty$. In particular, by Proposition A.1 in Appendix A,

$$\|Sf\|_{L^{q,1}(uv)} \leq 2^{1/q} \left(C_0\,(1/p_0 - 1/q)^{-1} + C_1\right) \|f\|_{L^{q,1}(uv)}.$$

Thus, for all $q \geq 2p_0$ we have that $\|Sf\|_{L^{q,1}(uv)} \leq K_0\, \|f\|_{L^{q,1}(uv)}$ with $K_0 = 4p_0\,(C_0 + C_1)$. We emphasize that the constant K_0 is valid for every $q \geq 2p_0$—this will be crucial in the remainder of the proof.

Again by Lemma 2.3, for every weight $W_1 \in A_1$ with $[W_1]_{A_1} \leq 2\,K_0$ there exists $0 < \tilde\epsilon_0 < 1$ (that depends only on K_0) such that $W_1\, W_2^{\epsilon} \in A_1$ for all $W_2 \in A_1$ and $0 < \epsilon < \tilde\epsilon_0$.

Fix $0 < \epsilon < \min\{\tilde\epsilon_0, \frac{1}{2p_0}\}$ and let $r = (1/\epsilon)'$. Then $r' > 2p_0$ and so S is bounded on $L^{r',1}(uv)$ with constant bounded by K_0. Now apply the Rubio de Francia algorithm (see [11, 5]) to define the operator $\mathcal{R}$ on $h \in L^{r',1}(uv)$, $h \geq 0$, by

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{S^k h(x)}{2^k\, K_0^k}.$$

It follows immediately from this definition that:

(*a*) $h(x) \leq \mathcal{R}h(x)$;

(*b*) $\|\mathcal{R}h\|_{L^{r',1}(uv)} \leq 2\,\|h\|_{L^{r',1}(uv)}$;

(*c*) $S(\mathcal{R}h)(x) \leq 2\,K_0\,\mathcal{R}h(x)$.

In particular, it follows from (*c*) and the definition of S that $\mathcal{R}h\, u \in A_1$ with $[\mathcal{R}h\, u]_{A_1} \leq 2\,K_0$. Let $W_1 = \mathcal{R}h\, u$ and $W_2 = v_1 \in A_1$ (recall that $v = v_1\, v_2^{1-t}$); then $W_1\, W_2^{\epsilon} \in A_1$. Hence, $\mathcal{R}h\, uv^{1/r'} = W_1\, W_2^{\epsilon}\, v_2^{\frac{1-t}{r'}} \in A_\infty$.

Fix $(f, g) \in \mathcal{F}$ such that the lefthand side of (1.8) is finite. Then by the duality of $L^{r,\infty}$ and $L^{r',1}$,

$$\left\|f\, v^{-1}\right\|_{L^{1,\infty}(uv)}^{\frac{1}{r}} = \left\|(f\, v^{-1})^{\frac{1}{r}}\right\|_{L^{r,\infty}(uv)} = \sup \int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\, h(x)\, u(x)\, v(x)^{\frac{1}{r'}}\, dx,$$

where the supremum is taken over all non-negative $h \in L^{r',1}(uv)$ with $\|h\|_{L^{r',1}(uv)} = 1$. Fix such a function h. We will now use (1.7) with $p = 1/r$ (see Remark 1.8) and with the weight $\mathcal{R}h\, uv^{1/r'} \in A_\infty$. In order to do so, we must have that the lefthand side is finite, but this follows at once from Hölder's inequality and (b):

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\, \mathcal{R}h(x)\, u(x)\, v(x)^{\frac{1}{r'}}\, dx &\leq \big\|(f\, v^{-1})^{\frac{1}{r}}\big\|_{L^{r,\infty}(uv)}\, \|\mathcal{R}h\|_{L^{r',1}(uv)} \\ &\leq 2\, \big\|f\, v^{-1}\big\|_{L^{1,\infty}(uv)}^{\frac{1}{r}} \|h\|_{L^{r',1}(uv)} < \infty.\end{aligned}$$

Thus by (1.7), (a) and (b),

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\, h(x)\, u(x)\, v(x)^{\frac{1}{r'}}\, dx &\leq \int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\, \mathcal{R}h(x)\, u(x)\, v(x)^{\frac{1}{r'}}\, dx \\ &\leq C \int_{\mathbb{R}^n} g(x)^{\frac{1}{r}}\, \mathcal{R}h(x)\, u(x)\, v(x)^{\frac{1}{r'}}\, dx \\ &\leq C\, \big\|(g\, v^{-1})^{\frac{1}{r}}\big\|_{L^{r,\infty}(uv)}\, \|\mathcal{R}h\|_{L^{r',1}(uv)} \\ &\leq 2\, C\, \big\|g\, v^{-1}\big\|_{L^{1,\infty}(uv)}^{\frac{1}{r}}.\end{aligned}$$

Since C is independent of h, inequality (1.8) follows.

4. Proof of Theorem 1.4 when $v \in A_\infty(u)$

In this section we prove Theorem 1.4 assuming that $v \in A_\infty(u)$.

Fix f; without loss of generality we may assume that f is a bounded, non-negative function with compact support. Since $u \in A_1$, for any dyadic cube Q and $x \in Q$,

$$\frac{1}{|Q|}\int_Q f(y)\, dy = \frac{u(Q)}{|Q|}\frac{1}{u(Q)}\int_Q f(y)\, dy \leq \frac{[u]_{A_1}}{u(Q)}\int_Q f(y)u(y)\, dx \leq [u]_{A_1} M_u^d f(x).$$

Therefore, to prove (1.4) it will suffice to prove

$$uv(\{x \in \mathbb{R}^n : M_u^d(fv)(x)v(x)^{-1} > t\}) \leq \frac{C}{t}\int_{\mathbb{R}^n} f(x)\, u(x)v(x)\, dx. \tag{4.1}$$

Fix $t > 0$. By Lemma 2.1, $uv \in A_\infty$; in particular, uv is a doubling weight. Therefore, we can form the Calderón-Zygmund decomposition of f at height t with respect to the measure $uv\, dx$. This yields a collection of disjoint dyadic cubes $\{Q_j\}$ such that for some $\gamma > 1$,

$$t < \frac{1}{uv(Q_j)}\int_{Q_j} f(x)u(x)v(x)\, dx \leq \gamma t.$$

Further, if we let $\Omega = \bigcup_j Q_j$, then $f(x) \leq t$ for almost every $x \in \mathbb{R}^n \setminus \Omega$.

We decompose f as $g+b$, where

$$g(x) = \begin{cases} \dfrac{1}{uv(Q_j)} \displaystyle\int_{Q_j} f(x)u(x)v(x)\,dx & x \in Q_j \\ f(x) & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

and $b = \sum_j b_j$, with

$$b_j(x) = \left(f(x) - \frac{1}{uv(Q_j)} \int_{Q_j} f(x)u(x)v(x)\,dx \right) \chi_{Q_j}(x).$$

It follows from these definitions that $g(x) \le \gamma t$ a.e., and $\int_{Q_j} b_j(x)u(x)v(x)\,dx = 0$.

If Q is a dyadic cube, then for all $x \in Q$,

$$\begin{aligned} \frac{1}{u(Q)} \int_Q f(x)\,u(x)\,v(x)\,dx &= \frac{1}{u(Q)} \int_Q g(x)\,u(x)\,v(x)\,dx + \frac{1}{u(Q)} \int_Q b(x)\,u(x)\,v(x)\,dx \\ &\le M_u(gv)(x) + \widetilde{M}_u(bv)(x), \end{aligned}$$

where

$$\widetilde{M}_u(h)(x) = \sup_{x \in Q} \left| \frac{1}{u(Q)} \int_Q h(y)\,u(y)\,dy \right|,$$

and the supremum is taken over all dyadic cubes containing x. Hence,

$$M_u(fv)(x) \le M_u(gv)(x) + \widetilde{M}_u(bv)(x).$$

Using this inequality, we argue as follows:

$$\begin{aligned} &uv(\{x \in \mathbb{R}^n : M_u^d(fv)(x)v(x)^{-1} > t\}) \\ &\quad \le uv(\{x \in \mathbb{R}^n : M_u^d(gv)(x)v(x)^{-1} > t/2\}) \\ &\qquad + uv(\{x \in \Omega : \widetilde{M}_u(bv)(x)v(x)^{-1} > t/2\}) \\ &\qquad + uv(\{x \in \mathbb{R}^n \setminus \Omega : \widetilde{M}_u(bv)(x)v(x)^{-1} > t/2\}) \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

We estimate the measure of each set in turn. Since $v \in A_\infty(u)$, there exists $p > 1$ (close to 1) such that $v \in A_{p'}(u)$, or equivalently, $v^{1-p} \in A_p(u)$. Hence, M_u^d is bounded on $L^p(uv^{1-p})$. Therefore,

$$\begin{aligned} I_1 &\le \frac{2^p}{t^p} \int_{\mathbb{R}^n} M_u(gv)(x)^p u(x)v(x)^{1-p}\,dx \\ &\le \frac{C}{t^p} \int_{\mathbb{R}^n} g(x)^p u(x)v(x)\,dx \\ &\le \frac{C\gamma^{p-1}}{t} \int_{\mathbb{R}^n} g(x)u(x)v(x)\,dx \end{aligned}$$

$$= \frac{C}{t}\int_{\mathbb{R}^n\setminus\Omega} f(x)u(x)v(x)\,dx + \frac{C}{t}\sum_j \frac{1}{uv(Q_j)}\int_{Q_j} f(x)u(x)v(x)\,dx\cdot uv(Q_j)$$
$$= \frac{C}{t}\int_{\mathbb{R}^n} f(x)u(x)v(x)\,dx.$$

The estimate for I_2 follows immediately from the properties of the cubes Q_j:

$$I_2 \le uv(\Omega) = \sum_j uv(Q_j) \le \frac{1}{t}\sum_j \int_{Q_j} f(x)u(x)v(x)\,dx \le \frac{C}{t}\int_{\mathbb{R}^n} f(x)u(x)v(x)\,dx.$$

Finally, we will show that $I_3 = 0$. To see this, fix $x \in \mathbb{R}^n \setminus \Omega$. Since $\operatorname{supp}(b) \subset \Omega$, to compute $\widetilde{M}_u(bv)(x)$ we only need to consider dyadic cubes Q which intersect Ω. Fix such a cube Q; then for each j either $Q_j \subset Q$ or $Q_j \cap Q = \emptyset$. Therefore, since $\int_{Q_j} b_j(x)u(x)v(x)\,dx = 0$,

$$\frac{1}{u(Q)}\int_Q b(x)u(x)v(x)\,dx = \frac{1}{u(Q)}\sum_j \int_{Q\cap Q_j} b_j(x)u(x)v(x)\,dx$$
$$= \frac{1}{u(Q)}\sum_{Q_j\subset Q}\int_{Q_j} b_j(x)u(x)v(x)\,dx = 0.$$

This completes the proof.

Appendix A. An interpolation result

The Marcinkiewicz interpolation theorem is a well-known result for proving that an operator is bounded on Lorentz spaces. In Section 3 we used a version of it to prove that a particular operator is bounded on $L^{p,1}$. However, for our purposes we need better control on the constant than is available in published versions of this theorem (cf. [12, 17]). Therefore, here we give a precise statement and sketch a proof.

We begin with some notation: given a measurable function f, $\mu_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$ and the decreasing rearrangement of f is the function $f^*(t) = \inf\{s \ge 0 : \mu_f(s) \le t\}$.

Proposition A.1. *Given p_0, $1 < p_0 < \infty$, let T be a sublinear operator such that*

$$\|Tf\|_{L^{p_0,\infty}} \le C_0\,\|f\|_{L^{p_0,1}} \qquad \text{and} \qquad \|Tf\|_{L^\infty} \le C_1\,\|f\|_{L^\infty}.$$

Then for all $p_0 < p < \infty$,

$$\|Tf\|_{L^{p,1}} \le 2^{1/p}\left(C_0\,(1/p_0 - 1/p)^{-1} + C_1\right)\|f\|_{L^{p,1}}.$$

Remark A.2. If T is bounded on L^{p_0} then it maps $L^{p_0,1}$ into $L^{p_0,\infty}$ (since $L^{p_0,1} \hookrightarrow L^{p_0} \hookrightarrow L^{p_0,\infty}$) with the same constant. This is what we used above in Section 3.

Proof. We adopt the notation in Grafakos [12, p. 62]. Fix $t > 0$ and write
$$f = f_t + f^t = f\ \chi_{\{x:|f(x)|\le f^*(t)\}} + f\ \chi_{\{x:|f(x)|>f^*(t)\}}.$$
Then
$$\begin{aligned}\|Tf\|_{L^{p,1}} &= \int_0^\infty t^{\frac{1}{p}}\,(Tf)^*(t)\,\frac{dt}{t}\\ &\le 2^{\frac{1}{p}}\Big(\int_0^\infty t^{\frac{1}{p}}\,(Tf_t)^*(t)\,\frac{dt}{t} + \int_0^\infty t^{\frac{1}{p}}\,(Tf^t)^*(t)\,\frac{dt}{t}\Big) = 2^{\frac{1}{p}}\,(I + II).\end{aligned}$$
Since
$$(Tf_t)^*(t) \le (Tf_t)^*(0) = \|Tf_t\|_{L^\infty} \le C_1\,\|f_t\|_{L^\infty} \le C_1\,f^*(t),$$
we get that
$$I \le \int_0^\infty t^{\frac{1}{p}}\,f^*(t)\,\frac{dt}{t} = C_1\,\|f\|_{L^{p,1}}.$$
To bound II, observe that $|f^t(x)| \le |f(x)|$; hence, $(f^t)^*(s) \le f^*(s)$ for all $0 < s < \infty$. Furthermore, $\mu_{f^t}(0) = \mu_f(f^*(t)) \le t$, which implies $(f^t)^*(t) = 0$; since f^* is decreasing we thus have that $f^*(s) = 0$ for $s \ge t$. Hence, since $p_0 < p$,
$$\begin{aligned}II &\le \int_0^\infty t^{\frac{1}{p}-\frac{1}{p_0}}\,\|Tf^t\|_{L^{p_0,\infty}}\,\frac{dt}{t}\\ &\le C_0\int_0^\infty t^{\frac{1}{p}-\frac{1}{p_0}}\,\|f^t\|_{L^{p_0,1}}\,\frac{dt}{t}\\ &\le C_0\int_0^\infty t^{\frac{1}{p}-\frac{1}{p_0}}\int_0^t s^{\frac{1}{p_0}}\,f^*(s)\,\frac{ds}{s}\,\frac{dt}{t}\\ &= C_0\,(1/p_0 - 1/p)^{-1}\int_0^\infty s^{\frac{1}{p}}\,f^*(s)\,\frac{ds}{s}\\ &= C_0\,(1/p_0 - 1/p)^{-1}\,\|f\|_{L^{p,1}}.\end{aligned}$$
□

Appendix B. Proof of Theorem 1.4 when $v \in A_1$

In this section we prove Theorem 1.4 assuming that the weight $v \in A_1$. (Recall that we also assume that $u \in A_1$.) Our argument is an adaption of Sawyer's original proof on the real line [15]. While in some ways the adaptation is straightforward, the proof itself is quite subtle and the changes required are buried in the details. Therefore, for the convenience of the reader we include the full proof and we adopt the notation of the original proof.

Fix $t > 0$ and let $g = fv/t$; then we need to prove that
$$uv(\{x \in \mathbb{R}^n : M_d g(x) > v(x)\}) \le C\int_{\mathbb{R}^n} |g(x)|u(x)\,dx. \tag{B.1}$$

Without loss of generality we may assume that g is bounded and has compact support.

Fix $a > 2^n$. For each $k \in \mathbb{Z}$, let $\{I_j^k\}$ be the collection of maximal, disjoint dyadic cubes whose union is the set

$$\Omega_k = \{x \in \mathbb{R}^n : M_d v(x) > a^k\} \cap \{x \in \mathbb{R}^n : M_d g(x) > a^k\}.$$

This decomposition exists since g is bounded and has compact support, so the second set is contained in the union of maximal dyadic cubes. Define

$$\Gamma = \{(k,j) : |I_j^k \cap \{x : v(x) \le a^{k+1}\}| > 0\}.$$

Since $v \in A_1$, $Mv(x) \le [v]_{A_1} v(x)$ almost everywhere. Hence, for $(k,j) \in \Gamma$,

$$\begin{aligned} \text{(B.2)} \quad \frac{a^k}{[v]_{A_1}} &\le [v]_{A_1}^{-1} \operatorname*{ess\,inf}_{x \in I_j^k} M_d v(x) \le \operatorname*{ess\,inf}_{x \in I_j^k} v(x) \\ &\le \frac{1}{|I_j^k|} \int_{I_j^k} v(x)\,dx \le [v]_{A_1} \operatorname*{ess\,inf}_{x \in I_j^k} v(x) \le [v]_{A_1} a^{k+1}. \end{aligned}$$

(Intuitively, if $(k,j) \in \Gamma$, then I_j^k behaves like a cube from the Calderón-Zygmund decomposition of v at height a^k.) Then up to a set of measure zero we have the following inclusions: for each k,

$$\begin{aligned} &\{x \in \mathbb{R}^n : a^k < v(x) \le a^{k+1}\} \cap \{x \in \mathbb{R}^n : M_d g(x) > v(x)\} \\ &\qquad \subset \{x : M_d v(x) > a^k\} \cap \{x : v(x) \le a^{k+1}\} \cap \{x : M_d g(x) > a^k\} \subset \bigcup_{j:(k,j)\in\Gamma} I_j^k. \end{aligned}$$

Combining this with (B.2) we get that

$$\begin{aligned} uv(\{x \in \mathbb{R}^n : M_d g(x) > v(x)\}) &= \sum_k uv(\{M_d g(x) > v(x)\} \cap \{a^k < v(x) \le a^{k+1}\}) \\ &\le a \sum_k a^k u(\{a^k < v(x) \le a^{k+1}\} \cap \{M_d g(x) > v(x)\}) \\ &\le a \sum_k a^k \sum_{j:(k,j)\in\Gamma} u(I_j^k) \\ &\le a[v]_{A_1} \sum_{(k,j)\in\Gamma} |I_j^k|^{-1} v(I_j^k) u(I_j^k). \end{aligned}$$

Fix $N < 0$ and define $\Gamma_N = \{(k,j) \in \Gamma : k \ge N\}$. We will show that

$$\sum_{(k,j)\in\Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \le C \int_{\mathbb{R}^n} |g(x)| u(x)\,dx$$

with a constant independent of N; (B.1) then follows if we take the limit as $N \to -\infty$.

To prove this, we are going to replace the set of cubes $\{I_j^k\}$ by a subset with better properties. First, since $v \in A_1 \subset A_\infty$, there exist $C, \epsilon > 0$ such that given any cube I and $E \subset I$,

$$\frac{v(E)}{v(I)} \leq C \left(\frac{|E|}{|I|}\right)^\epsilon. \tag{B.3}$$

Fix δ such that $0 < \delta < \epsilon$.

Define $\Delta_N = \{I_j^k : (k,j) \in \Gamma_N\}$. The cubes in Δ_N are all dyadic, so they are either pairwise disjoint or one is contained in the other. For $k > t$, since $\Omega_k \subset \Omega_t$, and since the cubes I_j^k are maximal in Ω_k, if $I_s^t \cap I_j^k \neq \emptyset$, then $I_j^k \subset I_s^t$. In particular, each cube $I_j^k \in \Delta_N$ is contained in $\bigcup_j I_j^N \subset \{x : M_d g(x) > a^N\}$; as we noted above, the last set is bounded, so Δ_N contains a maximal disjoint subcollection of cubes.

We form a sequence of sets $\{G_n\}$ by induction. Let G_0 be the set of all pairs $(k,j) \in \Gamma_N$ such that I_j^k is maximal in Δ_N. For $n \geq 0$, given the set G_n, define the set G_{n+1} to be the set of pairs $(k,j) \in \Gamma_N$ such that there exists $(t,s) \in G_n$ with $I_j^k \subsetneq I_s^t$ (by the maximality of I_j^k, $k > t$), and

$$\frac{1}{|I_j^k|} \int_{I_j^k} u(x)\,dx > a^{(k-t)\delta} \frac{1}{|I_s^t|} \int_{I_s^t} u(x)\,dx, \tag{B.4}$$

$$\frac{1}{|I_i^l|} \int_{I_i^l} u(x)\,dx \leq a^{(l-t)\delta} \frac{1}{|I_s^t|} \int_{I_s^t} u(x)\,dx \tag{B.5}$$

whenever $(l,i) \in \Gamma_N$ and $I_j^k \subsetneq I_i^l \subset I_s^t$. (In other words, the cubes I_j^k are maximal among all cubes contained in I_s^t for which (B.4) holds.)

Let $P = \bigcup_{n\geq 0} G_n$. Given $(s,t) \in P$, we refer to the cube I_s^t as a principal cube. Since every cube in Δ_N is contained in a maximal cube, every cube in Δ_N is contained in one or more principal cubes.

To continue, we divide the proof into several steps.

Step 1: We claim that

$$\sum_{(k,j)\in\Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C \sum_{(k,j)\in P} |I_j^k|^{-1} v(I_j^k) u(I_j^k). \tag{B.6}$$

To prove this, fix $(t,s) \in P$ and let $Q = Q(t,s)$ be the set of indices $(k,j) \in \Gamma_N$ such that $I_j^k \subset I_s^t$ and I_s^t is the smallest principal cube containing I_j^k. In particular, each I_j^k is not a principal cube unless it equals I_s^t. So by (B.5) and since $I_j^k \subset \{x : M_d v(x) > a^k\}$,

$$\sum_{(k,j)\in Q} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C \sum_{(k,j)\in Q} a^{(k-t)\delta} |I_s^t|^{-1} u(I_s^t) v(I_j^k)$$

$$
\begin{aligned}
&= |I_s^t|^{-1} u(I_s^t) \sum_{k\geq t} \sum_{j:(k,j)\in Q} a^{(k-t)\delta} v(I_j^k);\\
&\leq |I_s^t|^{-1} u(I_s^t) \sum_{k\geq t} a^{(k-t)\delta} v(I_s^t \cap \{x : M_d v(x) > a^k\}).
\end{aligned}
$$

By (B.3), (B.2), and since $v \in A_1$,

$$
\begin{aligned}
v(I_s^t \cap \{x : M_d v(x) > a^k\}) &\leq C\, v(I_s^t) \left(\frac{|I_s^t \cap \{x : M_d v(x) > a^k\}|}{|I_s^t|} \right)^{\epsilon}\\
&\leq C\, v(I_s^t) \left(\frac{Ca^{-k}}{|I_s^t|} \int_{I_s^t} M_d v(x)\, dx \right)^{\epsilon}\\
&\leq C\, v(I_s^t) \left(\frac{Ca^{-k}}{|I_s^t|} \int_{I_s^t} v(x)\, dx \right)^{\epsilon}\\
&\leq C\, v(I_s^t) a^{(t-k)\epsilon}.
\end{aligned}
$$

Combining these inequalities, we see that

$$
\sum_{(k,j)\in Q} |I_j^k|^{-1} v(I_j^k) u(I_j^k) \leq C |I_s^t|^{-1} u(I_s^t) v(I_s^t) \sum_{k\geq t} a^{(k-t)\delta} a^{(t-k)\epsilon} \leq C |I_s^t|^{-1} u(I_s^t) v(I_s^t);
$$

the second inequality holds since $\epsilon - \delta > 0$ so the series converges. If we now sum over all $(t,s) \in P$, we get (B.6) since $\bigcup_{(t,s)\in P} Q(t,s) = \Gamma_N$. This completes Step 1.

Step 2: For each k, let $\{J_i^k\}$ be the collection of maximal disjoint dyadic cubes whose union is $\{x : M_d g(x) > a^k\}$. Then

$$
a^k < \frac{1}{|J_i^k|} \int_{J_i^k} g(x)\, dx.
$$

For each j, $I_j^k \subset \{M_d g(x) > a^k\}$, so there exists a unique $i = i(j,k)$ such that $I_j^k \subset J_i^k$. Hereafter, the index i will always be this function of (k,j). Hence, by (B.6) and by (B.2),

$$
\begin{aligned}
\sum_{(k,j)\in\Gamma_N} |I_j^k|^{-1} v(I_j^k) u(I_j^k) &\leq C \sum_{(k,j)\in P} |I_j^k|^{-1} v(I_j^k) u(I_j^k)\\
&\leq C \sum_{(k,j)\in P} a^k u(I_j^k)\\
&\leq C \sum_{(k,j)\in P} |J_i^k|^{-1} g(J_i^k) u(I_j^k)\\
&= C \int_{\mathbb{R}^n} \Bigg[\sum_{(k,j)\in P} |J_i^k|^{-1} u(I_j^k) \chi_{J_i^k}(x) \Bigg] g(x)\, dx
\end{aligned}
$$

$$= C \int_{\mathbb{R}^n} h(x)g(x)\,dx.$$

To complete the proof we will show that for each x, $h(x) \le Cu(x)$. Fix $x \in \mathbb{R}^n$; without loss of generality we may assume that $u(x)$ is finite. For each k there exists at most one cube J_b^k such that $x \in J_b^k$. If it exists, denote this cube by J^k. Define $P_k = \{(k,j) \in P : I_j^k \subset J^k\}$, and $G = \{k : P_k \ne \emptyset\}$. We form a sequence $\{k_n\}$ by induction. If $k \in G$, then $k \ge N$, so let k_0 be the least integer in G. Given k_m, $m \ge 0$, choose $k_{m+1} > k_m$ in G such that

$$\frac{1}{|J^{k_{m+1}}|} \int_{J^{k_{m+1}}} u(y)\,d > \frac{2}{|J^{k_m}|} \int_{J^{k_m}} u(y)\,dy, \tag{B.7}$$

$$\frac{1}{|J^l|} \int_{J^l} u(y)\,dy \le \frac{2}{|J^{k_m}|} \int_{J^{k_m}} u(y)\,dy, \quad k_m \le l < k_{m+1}, \quad l \in G. \tag{B.8}$$

Since $u(x)$ is finite, the sequence $\{k_m\}$ only contains a finite number of terms. Otherwise, for all $m \ge 0$,

$$0 < \frac{2^m}{|J^{k_0}|} \int_{J^{k_0}} u(y)\,dy < \frac{1}{|J^{k_m}|} \int_{J^{k_m}} u(y)\,dy \le Mu(x) \le [u]_{A_1} u(x),$$

and we get a contradiction when $m \to \infty$.

Given the sequence $\{k_m\}$, we have that

$$\begin{aligned} h(x) &= \sum_{(k,j)\in P} |J_i^k|^{-1} u(I_j^k) \chi_{J_i^k}(x) \\ &= \sum_{(k,j)\in P} |J^k|^{-1} u(I_j^k) \\ &= \sum_{(k,j)\in P} \frac{u(I_j^k)}{u(J^k)} \left(\frac{1}{|J^k|} \int_{J^k} u(y)\,dy \right) \\ &= \sum_m \sum_{\substack{l\in G \\ k_m \le l < k_{m+1}}} \left(\frac{1}{|J^l|} \int_{J^l} u(y)\,dy \right) \sum_{(l,j)\in P_l} \frac{u(I_j^l)}{u(J^l)} \\ &\le 2 \sum_n \left(\frac{1}{|J^{k_m}|} \int_{J^{k_m}} u(y)\,dy \right) \sum_{\substack{l\in G \\ k_m \le l < k_{m+1}}} \sum_{(l,j)\in P_l} \frac{u(I_j^l)}{u(J^l)}. \end{aligned}$$

We claim that

$$\sum_{\substack{l\in G \\ k_m \le l < k_{m+1}}} \sum_{(l,j)\in P_l} \frac{u(I_j^l)}{u(J^l)} \le C. \tag{B.9}$$

Given this, we would be done: since the sequence $\{k_m\}$ is finite, let m be the largest index. Then by (B.7) and (B.9),

$$\begin{aligned}h(x) \le C\sum_{n=0}^{m}\frac{1}{|J^{k_m}|}\int_{J^{k_m}}u(y)\,dy \le \frac{C}{|J^{k_m}|}\int_{J^{k_m}}u(y)\,dy\sum_{n=0}^{m}2^{n-m} \\ \le \frac{C}{|J^{k_m}|}\int_{J^{k_m}}u(y)\,dy \le CMu(x) \le Cu(x).\end{aligned}$$

Therefore, to complete the proof we must show (B.9). We do this in two steps.

Step 3: We first prove that if $(l,j)\in P_l$, $k_m \le l < k_{m+1}$, then

$$\frac{1}{|I^l_j|}\int_{I^l_j}u(y)\,dy > \frac{a^{(l-k_m)\delta}}{2[u]_{A_1}}\frac{1}{|J^l|}\int_{J^l}u(y)\,dy. \tag{B.10}$$

Since $\Omega_l \subset \Omega_{k_m}$, by maximality there exists a unique s such that $I^l_j \subset I^{k_m}_s$. We will first show that $(k_m,s)\in\Gamma_N$. If $(k_m,s)\in P\subset\Gamma_N$, there is nothing to show, so we may assume that $(k_m,s)\notin P$. Since $k_m\in G$, J^{k_m} contains a cube $I^{k_m}_r$ with $(k_m,r)\in P$. Since $r\neq s$, $I^{k_m}_s$ and $I^{k_m}_r$ are disjoint, and in particular we must have that $I^{k_m}_s \subsetneq J^{k_m}$. Since J^{k_m} is a maximal subcube of $\{x : M_d g(x) > a^{k_m}\}$, and since $I^{k_m}_s \subsetneq J^{k_m}$, it is not a maximal subcube for this set. However, it is a maximal subcube of Ω_{k_m}, and so it follows that $I^{k_m}_s$ is a maximal dyadic subcube of $\{x : M_d v(x) > a^{k_m}\}$. Therefore, by the properties of such cubes (and since $a > 2^n$),

$$\frac{1}{|I^{k_m}_s|}\int_{I^{k_m}_s}v(y)\,dy \le 2^n a^{k_m} \le a^{k_m+1}.$$

Hence, $|I^{k_m}_s \cap \{x : v(x) \le a^{k_m+1}\}| > 0$, and so $(k_m,s)\in\Gamma_N$.

Since $(k_m,s)\in\Gamma_N$, $I^{k_m}_s$ is contained in at least one principal cube. Let I^k_σ be the smallest principal cube which contains $I^{k_m}_s$. Since I^l_j is also principal, by (B.4) we have that

$$\frac{1}{|I^l_j|}\int_{I^l_j}u(y)\,dy \ge \frac{a^{(l-k)\delta}}{|I^k_\sigma|}\int_{I^k_\sigma}u(y)\,dy.$$

Similarly, by (B.5) we have that

$$\frac{1}{|I^{k_m}_s|}\int_{I^{k_m}_s}u(y)\,dy \le \frac{a^{(k_m-k)\delta}}{|I^k_\sigma|}\int_{I^k_\sigma}u(y)\,dy.$$

By (B.8) and since $u\in A_1$,

$$\frac{1}{|J^l|}\int_{J^l}u(y)\,dy \le \frac{2}{|J^{k_m}|}\int_{J^{k_m}}u(y)\,dy$$

$$\leq 2\,[u]_{A_1} \operatorname*{ess\,inf}_{y\in J^{k_m}} u(y) \leq 2[u]_{A_1} \operatorname*{ess\,inf}_{y\in I_s^{k_m}} u(y) \leq \frac{2\,[u]_{A_1}}{|I_s^{k_m}|}\int_{I_s^{k_m}} u(y)\,dy.$$

If we combine these three inequalities we get (B.10). This completes Step 3.

Step 4: We will now prove (B.9). By (B.10) and again since $u \in A_1$, if $y \in I_j^l$, then

$$u(y) > \frac{a^{(l-k_m)\delta}}{2[u]_{A_1}^2}\frac{u(J^l)}{|J^l|} = \lambda;$$

hence,

$$\bigcup_{j:(l,j)\in P_l} I_j^l \subset \{x \in J^l : u(x) > \lambda\}.$$

For l fixed the cubes I_j^l are disjoint. Therefore, since $u \in A_1 \subset A_\infty$ there exist $C, \nu > 0$ such that

$$\begin{aligned}
\sum_{j:(l,j)\in P_l} u(I_j^l) &= u\Bigg(\bigcup_{j:(l,j)\in P_l} I_j^l\Bigg)\\
&\leq u(\{x \in J^l : u(x) > \lambda\})\\
&\leq C\,u(J^l)\left(\frac{|\{x \in J^l : u(x) > \lambda\}|}{|J^l|}\right)^\nu\\
&\leq C\,u(J^l)\left(\frac{\lambda^{-1}}{|J^l|}\int_{J^l} u(y)\,dy\right)^\nu\\
&\leq C\,u(J^l)a^{(k_m-l)\delta\nu}.
\end{aligned}$$

Therefore, we have that

$$\sum_{\substack{l\in G\\ k_m\leq l<k_{m+1}}}\ \sum_{(l,j)\in P_l} \frac{u(I_j^l)}{u(J^l)} \leq C \sum_{\substack{l\in G\\ k_m\leq l<k_{m+1}}} a^{(k_m-l)\delta\nu} \leq C\sum_{l\geq k_m} a^{(k_m-l)\delta\nu} \leq C.$$

This shows that (B.9) holds, and the proof is complete.

References

[1] J. Alvarez and C. Pérez, *Estimates with A_∞ weights for various singular integral operators*, Bollettino U.M.I. (7) 8-A (1994), 123-133.

[2] I. Asekritova, N. Krugljak, L. Maligranda, L. Nikolova, L. Persson, *Lions-Peetre reiteration formulas for triples and their applications*, Studia Math. 145 (2001), 219-254.

[3] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Grund. der math. Wissenschaften 223, Springer-Verlag, New York, 1976.

[4] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241-250.

[5] D. Cruz-Uribe, SFO, J. M. Martell, and C. Pérez, *Extrapolation from A_∞ weights with applications*, J. Funct. Anal. 213 (2004), 412-439.
[6] D. Cruz-Uribe, SFO, and C.J. Neugebauer, *The structure of the reverse Hölder classes*, Trans. Amer. Math. Soc. 347 (1995), 2941-2960.
[7] D. Cruz-Uribe, SFO, and C. Pérez, *On the two-weight problem for singular integral operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. I (2002), 821-849.
[8] G. Curbera, J. García-Cuerva, J. M. Martell and C. Pérez, *Extrapolation with weights, rearrangement invariant function spaces, modular inequalities and applications to singular integrals*, Preprint 2004.
[9] J. Duoandikoetxea, *Fourier Analysis*, Grad. Studies Math. 29, Amer. Math. Soc., Providence, 2000.
[10] E. Ferreyra, *On a negative result concerning interpolation with change of measures for Lorentz spaces*, Proc. Amer. Math. Soc. 125 (1997), 1413-1417.
[11] J. García-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North Holland Math. Studies 116, North Holland, Amsterdam, 1985.
[12] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.
[13] A.K. Lerner, *Weighted norm inequalities for the local sharp maximal function*, Journal of Fourier Analyis and Applications, 10 (2004) 465-474.
[14] B. Muckenhoupt and R. Wheeden, *Some weighted weak-type inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Indiana Math. J. 26 (1977), 801-816.
[15] E. Sawyer, *A weighted weak type inequality for the maximal function*, Proc. Amer. Math. Soc. 93 (1985), 610-614.
[16] E.M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), 159-172.
[17] E. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971.

Department of Mathematics, Trinity College, Hartford, CT 06106-3100, USA
E-mail address: david.cruzuribe@trincoll.edu

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail address: chema.martell@uam.es

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Sevilla, 41080 Sevilla, Spain
E-mail address: carlosperez@us.es