SHARP WEIGHTED ESTIMATES FOR CLASSICAL OPERATORS

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1

ABSTRACT. We give a general method based on dyadic Calderón-Zygmund theory to prove sharp one and two-weight norm inequalities for some of the classical operators of harmonic analysis: the Hilbert and Riesz transforms, the Beurling-Ahlfors operator, the maximal singular integrals associated to these operators, the dyadic square function and the vector-valued maximal operator.

In the one-weight case we prove the sharp dependence on the A_p constant by finding the best value for the exponent $\alpha(p)$ such that
$$\|Tf\|_{L^p(w)} \leq C_{n,T}\, [w]_{A_p}^{\alpha(p)} \|f\|_{L^p(w)}.$$
For the Hilbert transform, the Riesz transforms and the Beurling-Ahlfors operator the sharp value of $\alpha(p)$ was found by Petermichl and Volberg [47, 48, 49]; their proofs used approximations by the dyadic Haar shift operators, Bellman function techniques, and two-weight norm inequalities. Our proofs again depend on dyadic approximation, but avoid Bellman functions and two-weight norm inequalities. We instead use a recent result due to A. Lerner [34] to estimate the oscillation of dyadic operators. By applying this we get a straightforward proof of the sharp dependence on the A_p constant for any operator that can be approximated by Haar shift operators. In particular, we provide a unified approach for the Hilbert and Riesz transforms, the Beurling-Ahlfors operator (and their corresponding maximal singular integrals), dyadic paraproducts and Haar multipliers. Furthermore, we completely solve the open problem of sharp dependence for the dyadic square functions and vector-valued Hardy-Littlewood maximal function.

In the two-weight case we use the very same techniques to prove sharp results in the scale of A_p bump conditions. For the singular integrals considered above, we show they map $L^p(v)$ into $L^p(u)$, $1 < p < \infty$, if the pair (u,v) satisfies
$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{B,Q} < \infty,$$
where $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$ are Orlicz functions. This condition is sharp. Furthermore, this condition characterizes (in the scale of these A_p bump conditions) the corresponding two-weight norm inequality for the Hardy-Littlewood maximal operator M and its dual: i.e., $M : L^p(v) \longrightarrow L^p(u)$ and $M : L^{p'}(u^{1-p'}) \longrightarrow L^p(v^{1-p'})$. Muckenhoupt and Wheeden conjectured that these two inequalities for M are sufficient for the Hilbert transform to be bounded from $L^p(v)$ into $L^p(u)$. Thus, in the scale of A_p bump conditions, we prove their conjecture. We prove similar, sharp two-weight results for the dyadic square function and the vector-valued maximal operator.

1. Introduction

The problem of proving one and two-weight norm inequalities for the classical operators of harmonic analysis—singular integrals, square functions, maximal operators—has a long and complex history. In the one weight case, the (nearly) universal sufficient and (often) necessary condition for an operator to be bounded on $L^p(w)$ is the A_p condition: given $1 < p < \infty$, a weight w (i.e., a non-negative, locally integrable function) is in A_p if

$$[w]_{A_p} = \sup_Q \left(\fint_Q w(x)\,dx \right) \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $\fint_Q w(x)\,dx = |Q|^{-1} \int_Q w(x)\,dx$. For more on one-weight inequalities we refer the reader to [13, 18, 21].

An important question is to determine the best constant in terms of the A_p constant $[w]_{A_p}$. More precisely, given an operator T, find the smallest power $\alpha(p)$ such that

$$\|Tf\|_{L^p(w)} \leq C_{n,T}[w]_{A_p}^{\alpha(p)}\|f\|_{L^p(w)}.$$

This problem was first investigated by Buckley [3]. More recently, it has attracted renewed attention because of the work of Astala, Iwaniec and Saksman [1]. They proved that sharp regularity results for solutions to the Beltrami equation hold provided that the Beurling-Ahlfors operator satisfies $\alpha(p) = 1$ for $p > 2$.

The problem of characterizing the weights that govern the two-weight norm inequalities for classical operators is still open and there are several approaches to finding sufficient conditions on weights for an operator to be bounded from $L^p(v)$ to $L^p(u)$. One approach is to replace the two-weight A_p condition with the A_p "bump" condition:

$$\sup_Q \|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{B,Q} < \infty,$$

where A and B are Young functions and the norms are localized Orlicz norms slightly larger than the L^p and $L^{p'}$ norms. (Precise definitions will be given below.) Sufficient growth conditions on A and B are known for many operators and this has led to a number of conjectures on sharp sufficient conditions. For the history of this approach we refer the reader to [5, 7, 8, 10, 11].

In this paper we develop a unified approach to both of these problems and the results we get are sharp. We consider one and two-weight norm inequalities for singular integrals, maximal singular integrals, the

dyadic paraproduct, the dyadic square function and the vector-valued maximal operator. The results in the one-weight case for singular integrals are not new, but we believe that our proofs are simpler than existing proofs. The remaining theorems, however, are all new.

We believe that our approach shows that there is a deep connection between sharp results in the one and two-weight case. Further, key to our approach is that the operators are either dyadic or can be approximated by dyadic operators (e.g., by the Haar shift operators defined below). Thus our results will extend to any operator that can be approximated in this way.

Singular integrals. It is conjectured that if T is any Calderón-Zygmund singular integral operator, then for any p, $1 < p < \infty$, and for any $w \in A_p$,

$$(1.1) \qquad \|Tf\|_{L^p(w)} \leq C_{T,n,p} [w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}.$$

This inequality is true if T is the Hilbert transform, a Riesz transform or the Beurling-Ahlfors operator.

Theorem 1.1. *Given p, $1 < p < \infty$, if T is the Hilbert transform, a Riesz transform or the Beurling-Ahlfors operator, then for all $w \in A_p$ inequality (1.1) holds.*

This result was first proved by Petermichl [47, 48] and Petermichl and Volberg [49]. For each operator the proof requires several steps. First, it is enough to prove the case $p = 2$; the other values of p follow from a version of the Rubio de Francia extrapolation theorem with sharp constants due to Dragičević *et al.* [12] (Theorem 2.2 below). Second, for each of the above operators the problem is reduced to proving the weighted L^2 inequality for a corresponding dyadic operator by proving that the given operator can be approximated by integral averages of the dyadic operators (and their analogs defined on translations and dilations of the standard dyadic grid). Finally, the desired inequality was proved for each of these dyadic operators using Bellman function techniques and two-weight norm inequalities.

Recently, Lacey, Petermichl and Reguera-Rodriguez [28] gave a proof of the sharp A_2 constant for a large family of Haar shift operators that includes all of the dyadic operators needed for the above results. Their proof avoids the use of Bellman functions, and instead uses a deep, two-weight "Tb theorem" for Haar shift operators due to Nazarov, Treil and Volberg [40].

We give a different and simpler proof that uses approximation by dyadic Haar shifts but avoids both Bellman functions and two-weight

norm inequalities such as the Tb theorem. Instead, we use a very interesting decomposition argument based on local mean oscillation recently developed by Lerner [30] to prove the corresponding result for dyadic Haar shifts. Intuitively, this decomposition may be thought of as a version of the Calderón-Zygmund decomposition of a function, replacing the mean by the median. (We will make this more precise below.) Theorem 1.1 was announced in [6].

Remark 1.2. After this paper was completed we learned of several other related results. First, Vagharsyakhan [52] has shown that in one dimension, all convolution-type Calderón-Zygmund singular integral operators with sufficiently smooth kernel can be approximated by Haar shifts. Second, Lacey *et al.* [25] used a deep characterization of the one-weight problem in [45] to prove Theorem 1.1 for all singular integrals with sufficiently smooth kernels. Third, Lerner [35] proved Theorem 1.1 for any convolution-type Calderón-Zygmund singular integrals provided $p \geq 3$ or $1 < p \leq 3/2$. Finally, Hytönen [24] proved Theorem 1.1 for all singular integrals and all $p > 1$, thus solving the so-called A_2 conjecture. His proof is extremely technical: it is based on the approach in [45] and a refinement of the arguments in [28]. A simpler proof of the A_2 conjecture based upon the previous three papers appears in [26].

An important advantage of our approach is that it also yields sharp two-weight norm inequalities. To state our result we need a few definitions. A Young function is a function $A : [0, \infty) \to [0, \infty)$ that is continuous, convex and strictly increasing, $A(0) = 0$ and $A(t)/t \to \infty$ as $t \to \infty$. Given a cube Q we define the localized Luxemburg norm by

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \fint_Q A\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

When $A(t) = t^p$, $1 < p < \infty$, we write

$$\|f\|_{p,Q} = \left(\fint_Q |f(x)|^p dx\right)^{1/p}.$$

The associate function of A is the Young function

$$\bar{A}(t) = \sup_{s>0}\{st - A(s)\}.$$

A Young function A satisfies the B_p condition if for some $c > 0$,

$$\int_c^\infty \frac{A(t)}{t^p}\frac{dt}{t} < \infty.$$

Important examples of such functions are of the form $A(t) = t^p \log(e + t)^{-1-\epsilon}$, $\epsilon > 0$, which have associate functions $\bar{A}(t) \approx t^{p'} \log(e+t)^{p'-1+\delta}$, $\delta > 0$.

Theorem 1.3. *Given p, $1 < p < \infty$, let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. Then for any pair of weights (u, v) such that*

$$(1.2) \qquad \sup_Q \|u^{1/p}\|_{A,Q}\|v^{-1/p}\|_{B,Q} < \infty,$$

we have that

$$(1.3) \qquad \|Tf\|_{L^p(u)} \leq C\|f\|_{L^p(v)},$$

where T is the Hilbert transform, a Riesz transform, or the Beurling-Ahlfors operator.

Condition (1.2) is referred to as an A_p bump condition: when $A(t) = t^p$ and $B(t) = t^{p'}$, we get the two-weight A_p condition. Theorem 1.3 was proved in [5] for the Hilbert transform in the special case that $A(t) = t^p \log(e + t)^{p-1+\delta}$, $\delta > 0$ (here $\bar{A} \in B_{p'}$), and for Riesz transforms (indeed, for any Calderón-Zygmund singular integral) given the additional hypothesis that $p > n$. Examples (see [7, 8]) show that in this particular case these results are sharp, since they are false in general if we take $\delta = 0$ (when $\bar{A} \notin B_{p'}$). Theorem 1.3 was proved for the Hilbert transform and general singular integrals when $p > n$ by Lerner [34] by combining his decomposition argument with the arguments in [5].

Two-weight inequalities were first considered by Muckenhoupt [37], who noted that the same proof as in the one-weight case immediately shows that for all p, $1 \leq p < \infty$, $(u, v) \in A_p$ if and only if the maximal operator satisfies the weak (p, p) inequality. However, Muckenhoupt and Wheeden [38] soon showed that while the two-weight A_p condition is necessary for the strong (p, p) inequality for the maximal operator and the strong and weak type inequalities for the Hilbert transform, it is not sufficient. This led Muckenhoupt and Wheeden to focus not on the structural or geometric properties of A_p weights but on their relationship to the maximal operator, in particular, the fact that $w \in A_p$ was necessary and sufficient for the maximal operator to be bounded on $L^p(w)$ and $L^{p'}(w^{1-p'})$. They made the following conjecture that is still open: a sufficient condition for the Hilbert transform to satisfy the strong (p, p) inequality $H : L^p(v) \to L^p(u)$, $1 < p < \infty$, is that the maximal operator satisfies the pair of inequalities

$$(1.4) \qquad M : L^p(v) \to L^p(u), \qquad M : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}).$$

Bump A_p conditions were first considered by Neugebauer [41] who showed the following striking result: a pair of weights (u, v) satisfies (1.2) with power bumps $A(t) = t^{r\,p}$, $A(t) = t^{r\,p'}$ for some $r > 1$ if and only if there exist $w \in A_p$ and positive constants c_1, c_2 such that $c_1 u(x) \leq w(x) \leq c_2 v(x)$. From this condition we immediately get a large number of two-weight norm inequalities as corollaries to the analogous one-weight results. In particular, we get the two inequalities (1.4). An immediate question was whether this condition could be weakened and still get that the maximal operator satisfies $M : L^p(v) \to L^p(u)$. This was answered in [43], where it was shown that a sufficient condition for (1.4) was that the pair of weights satisfies (1.2) with $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. The centrality of these B_p conditions is shown by the fact that they are sharp within the scale of Orlicz bumps as shown in [43]. This led naturally to the following version of the conjecture of Muckenhoupt and Wheeden: a sufficient condition on the pair of weights (u, v) for any singular integral to satisfy $T : L^p(v) \to L^p(u)$ is that (1.2) holds. Progress on this conjecture was made in [11, 5, 34]. Theorem 1.3 completely solves it for the Hilbert and Riesz transforms and the Beurling-Ahlfors operator, and as we noted above it is the best possible result in the scale of B_p bumps. See [7] for further details and references on this topic.

In the past decade, a great deal of attention has been focused on proving that "testing conditions" are necessary and sufficient for two-weight norm inequalities for singular integrals. (See Nazarov, Treil and Volberg [39, 53, 40] and the recent preprints by Lacey, Sawyer and Uriarte-Tuero [29, 30].) More precisely, given a singular integral T, it is conjectured that $T : L^p(v) \to L^p(u)$ if and only if for every cube Q,

$$\int_Q |T(v^{1-p'}\chi_Q)(x)|^p u(x)\, dx \leq C \int_Q v(x)^{1-p'}\, dx$$
$$\int_Q |T(u\chi_Q)(x)|^{p'} v(x)^{1-p'}\, dx \leq C \int_Q u(x)\, dx.$$

The necessity of these conditions is immediate. The best known results are for $p = 2$; partial results (with additional hypotheses) are known for other values of p. These results are of great interest not only because of the elegance of this conjecture but also because of their connection with Tb-theorems on non-homogeneous spaces (see [53] and the references it contains).

Testing conditions and A_p bump conditions are not readily comparable: they represent two fundamentally different approaches to the two-weight problem. While both approaches are important, we believe

that bump conditions have several advantages over testing conditions. First, they are universal, geometric conditions: they are independent of the operators and any pair yields norm inequalities for a range of operators. Second, they are much easier to check than the testing conditions, and it is very easy to construct examples of weights that do and do not satisfy a given bump condition. (For many examples and a general technique for constructing them, see [7].) Third, they are not tied to L^2, unlike testing conditions where the transition from $p = 2$ to all p has proved to be very difficult. (In this regard, we note that in [39] it was claimed—without proof—that in the specific case they were considering, testing conditions were not sufficient.)

Maximal singular integrals. Given a singular integral T with convolution kernel K, recall that the associated maximal singular integral is defined by

$$T_* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)| = \sup_{\epsilon > 0} \left| \int_{|y| > \epsilon} K(y) f(x - y)\, dy \right|.$$

Somewhat surprisingly, both Theorem 1.1 and Theorem 1.3 remain true if the singular integral is replaced by the associated maximal singular integral.

Theorem 1.4. *Given p, $1 < p < \infty$, and $w \in A_p$, then inequality (1.1) holds if T is replaced by T_*, where T is the Hilbert transform, a Riesz transform or the Beurling-Ahlfors operator. Similarly, if the pair (u, v) satisfies (1.2), then inequality (1.3) holds if T is replaced by T_*.*

In the one-weight case, Theorem 1.4 was proved very recently by Hytönen *et al.* [23]. Their proof used a very general family of "maximal" dyadic shift operators and a characterization of the two-weight norm inequalities for maximal singular integrals due to Lacey, Sawyer and Uriarte-Tuero [29]. In the two-weight case this result is new. In both the one and two-weight case our approach is to prove the corresponding result for the associated maximal dyadic shift operators.

Remark 1.5. Very recently, Lerner [35] has proved Theorem 1.4 in the one-weight case for general Calderón-Zygmund maximal singular integral operators when $p > 3$.

Dyadic paraproducts and constant Haar multipliers. Let Δ denote the collection of dyadic cubes in $\mathbb{R}$. We consider two operators defined on the real line. A function b is in dyadic BMO, we write

$b \in BMO^d$, if

$$\|b\|_{*,d} = \sup_{I \in \Delta} \left(\fint_I |b(x) - b_I|^2 \, dx \right)^{1/2} < \infty,$$

where $b_I = \fint_I b(x) \, dx$. Given a dyadic interval I, I_+ and I_- are its right and left halves, and the Haar function h_I is defined by

$$h_I(x) = |I|^{-1/2}\big(\chi_{I_-}(x) - \chi_{I_+}(x)\big).$$

Define the dyadic paraproduct π_b by

$$\pi_b f(x) = \sum_{I \in \Delta} f_I \langle b, h_I \rangle h_I(x).$$

For an overview of the history and properties of the dyadic paraproduct, we refer the reader to Pereyra [42].

Theorem 1.6. *Given a function $b \in BMO^d$, and p, $1 < p < \infty$, then for all $w \in A_p$,*

$$\|\pi_b f\|_{L^p(w)} \le C_p \|b\|_{*,d} \, [w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}.$$

Furthermore, given a pair (u, v) that satisfies (1.2), then

$$\|\pi_b f\|_{L^p(u)} \le C \|b\|_{*,d} \|f\|_{L^p(v)}.$$

In the one-weight case, Theorem 1.6 was first proved by Beznosova [2] using Bellman function techniques. A different proof that avoided Bellman functions but used two-weight inequalities was given in [23].

Given a sequence $\alpha = \{\alpha_I\}_{I \in \Delta} \in \ell^\infty$, define the constant Haar multiplier T_α by

$$T_\alpha f(x) = \sum_{I \in \Delta} \alpha_I \langle f, h_I \rangle h_I(x).$$

If $\alpha_I = 1$, then T_α is the identity operator. For more on the properties of these operators, see Pereyra [42]. The analog of Theorem 1.6 is true for constant Haar multipliers.

Theorem 1.7. *Given a sequence $\alpha = \{\alpha_I\}_{I \in \Delta} \in \ell^\infty$, and p, $1 < p < \infty$, then for all $w \in A_p$,*

$$\|T_\alpha f\|_{L^p(w)} \le C_p \|\alpha\|_{\ell^\infty} \, [w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}.$$

Furthermore, given a pair (u, v) that satisfies (1.2), then

$$\|T_\alpha f\|_{L^p(u)} \le C \|\alpha\|_{\ell^\infty} \|f\|_{L^p(v)}.$$

In the special case when $\alpha_I = \pm 1$, Theorem 1.7 was proved by Wittwer [56].

Dyadic square functions. Let Δ denote the collection of dyadic cubes in $\mathbb{R}^n$. Given $Q \in \Delta$, let $\widehat{Q}$ be its dyadic parent: the unique dyadic cube containing Q whose side-length is twice that of Q. The dyadic square function is the operator

$$S_d f(x) = \left(\sum_{Q \in \Delta} (f_Q - f_{\widehat{Q}})^2 \chi_Q(x) \right)^{1/2},$$

where $f_Q = \fint_Q f(x)\,dx$. For the properties of the dyadic square function we refer the reader to Wilson [55].

Theorem 1.8. *Given p, $1 < p < \infty$, then for any $w \in A_p$,*

$$\|S_d f\|_{L^p(w)} \leq C_{n,p}[w]_{A_p}^{\max\left(\frac{1}{2}, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}.$$

Further, the exponent $\max\left(\frac{1}{2}, \frac{1}{p-1}\right)$ *is the best possible.*

The exponent in Theorem 1.8 was first conjectured by Lerner [31] for the continuous square function; he also showed it was the best possible. In [33] he proved that for $p > 2$ the sharp exponent is at most $p'/2 > \max(\frac{1}{2}, \frac{1}{p-1})$. When $p = 2$, Theorem 1.8 was proved by Wittwer [57] and by Hukovic, Treil and Volberg [22]; this was extended to $p < 2$ by extrapolation in [12] and examples were given to show that in this range the exponent is the best possible.

Remark 1.9. Very recently, Lerner [35] proved the analog of Theorem 1.8 for continuous square functions. His proof uses the intrinsic square function introduced by Wilson [55].

Theorem 1.10. *Fix p, $1 < p < \infty$. Suppose $1 < p \leq 2$, and B is a Young function such that $\bar{B} \in B_p$, If the pair (u, v) satisfies*

$$(1.5) \qquad \sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{B,Q} < \infty,$$

then

$$(1.6) \qquad \|S_d f\|_{L^p(u)} \leq C \|f\|_{L^p(v)}.$$

Suppose $2 < p < \infty$, and A and B are Young functions such that $\bar{A} \in B_{(p/2)'}$ and $\bar{B} \in B_p$. If the pair (u, v) satisfies

$$(1.7) \qquad \sup_Q \|u^{2/p}\|_{A,Q}^{1/2} \|v^{-1/p}\|_{B,Q} < \infty,$$

then (1.6) *holds.*

Condition (1.5) is the same condition for the maximal operator to map $L^p(v)$ to $L^p(u)$, whereas condition (1.7) is more similar to the conditions needed for singular integrals, but with a smaller "bump" on the left. This is easier to see in the scale of "log bumps." As we noted above, for a singular integral we need to take $A(t) = t^p \log(e + t)^{p-1+\delta}$, $\delta > 0$, but for the dyadic square function it suffices to take $A(t) = t^{p/2} \log(e + t)^{p/2-1+\delta}$, which after rescaling leads to $t^p \log(e + t)^{p/2-1+\delta}$. This difference in the behavior of the dyadic square function depending on whether $p \leq 2$ or $p > 2$ was first noted in [7]. There we conjectured Theorem 1.10 was true and proved it in some special cases. Furthermore, we proved in [7] that this result is sharp in the scale of log bumps: if we take $p > 2$ and let $A(t) = t^{p/2} \log(e + t)^{p/2-1+\delta}$, then the theorem holds for $\delta > 0$, when $A \in B_{(p/2)'}$, but not for $\delta = 0$, when $A \notin B_{(p/2)'}$.

Remark 1.11. Theorem 1.8 remains true if we replace the A_p condition by the dyadic A_p condition (i.e., defined only with respect to dyadic cubes). Similarly, Theorem 1.10 remains true if the weight conditions are restricted to dyadic cubes. For both theorems this follows by examining the proofs and details are left to the interested reader.

The vector-valued maximal operator. Let M be the Hardy-Littlewood maximal operator. Given a vector-valued function $f = \{f_i\}$, and q, $1 < q < \infty$, define the vector-valued maximal operator $\overline{M}_q$ by

$$\overline{M}_q f(x) = \left(\sum_{i=1}^{\infty} M f_i(x)^q \right)^{1/q}.$$

The vector-valued maximal operator was introduced by C. Fefferman and Stein [14]; for more information see [18].

Similar to the dyadic square function, the behavior of the vector-valued maximal operator depends on the relative sizes of p and q.

Theorem 1.12. *Fix q, $1 < q < \infty$. Given p, $1 < p < \infty$, if $w \in A_p$, then*

$$\|\overline{M}_q f\|_{L^p(w)} \leq C_{p,q,n}[w]_{A_p}^{\max(\frac{1}{q}, \frac{1}{p-1})} \left(\int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p w(x) \, dx \right)^{1/p}.$$

Further, the exponent $\max\left(\frac{1}{q}, \frac{1}{p-1}\right)$ *is the best possible.*

Theorem 1.12 is new. A slightly worse bound than Theorem 1.12 was implicit in the literature, but does not appear to have been stated explicitly. To get this weaker estimate, note that when $q = p$, by the

sharp result for the Hardy-Littlewood maximal operator we have that

$$\|\overline{M}_q f\|_{L^q(w)} \leq C_{q,n}[w]_{A_q}^{\frac{1}{q-1}} \left(\int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^q w(x)\, dx \right)^{1/q}.$$

Then by adapting to this context the sharp version of the Rubio de Francia extrapolation theorem, we get the exponent $\frac{1}{q-1}$ if $p > q$ and $\frac{1}{p-1}$ if $p < q$. Theorem 1.12 improves this bound for p large.

Theorem 1.13. *Fix q, $1 < q < \infty$. Suppose $1 < p \leq q$, and B is a Young function such that $\bar{B} \in B_p$. If the pair (u, v) satisfies*

$$(1.8) \qquad \sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{B,Q} < \infty,$$

then

$$(1.9) \qquad \|\overline{M}_q f\|_{L^p(u)} \leq C \left(\int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p v(x)\, dx \right)^{1/p}.$$

Suppose $q < p < \infty$, and A and B are Young functions such that $\bar{A} \in B_{(p/q)'}$ and $\bar{B} \in B_p$. If the pair (u, v) satisfies

$$(1.10) \qquad \sup_Q \|u^{q/p}\|_{A,Q}^{1/q} \|v^{-1/p}\|_{B,Q} < \infty,$$

then (1.9) *holds.*

When $p > q$, Theorem 1.13 is sharp in the scale of log bumps. The example in [9] shows that if $A(t) = t^{p/q} \log(e+t)^{p/q-1+\delta}$, then the result fails if $\delta = 0$, when $A \notin B_{(p/q)'}$. (If $\delta > 0$, $A \in B_{(p/q)'}$). In the case $p \leq q$ Theorem 1.13 is not new: it was first proved in [44]; for a different proof see [7]. We include it here for completeness and to highlight the similarity to the dyadic square function. The case $p > q$ is new; it was first conjectured in [7] where a few special cases were proved.

Remark 1.14. To obtain Theorems 1.12 and 1.13 we first consider the corresponding dyadic vector-valued maximal operator and establish both results for it. In such a case, as observed before for the dyadic square function, we ca replace the A_p condition by the dyadic A_p condition, and in (1.8), (1.10) the sup can be taken over all dyadic cubes. Further details are left to the interested reader.

Organization. The remainder of this paper is organized as follows. In Section 2 we gather some basic results, primarily about weighted norm inequalities, that are needed in subsequent sections. In Section 3 we give some preliminary material about the local mean oscillation of a function and state the decomposition theorem of Lerner. In Section 4

we define the Haar shift operators and prove the key estimate we need to apply Lerner's results. In Sections 5–8 we prove our main results. In Section 5 we prove our results for singular integrals by proving the corresponding results for Haar shift operators. As a corollary to these theorems we get our results for dyadic paraproducts and constant Haar multipliers. After the proof of Theorems 1.1 and 1.3 we will briefly discuss the technical obstructions which prevent us from applying our approach directly to a Calderón-Zygmund singular integral. The proof for maximal singular integrals is very similar to the proof for singular integrals, but since we introduce a new family of dyadic operators we give these results in Section 6. The proofs for square functions and vector-valued maximal operators are also very similar to those for singular integrals, so we will only sketch the proofs of these results in Sections 7 and 8, highlighting the key changes.

We would like to thank Andrei Lerner and Michael Wilson for a clarifying discussion about the results in Section 3.

2. Preliminary results

In this section we state some basic results that we will need in our proofs. The first is the sharp one-weight bound for the Hardy-Littlewood maximal operator. This result is due to Buckley [3]; for an elementary proof, see Lerner [32].

Theorem 2.1. *Given* p, $1 < p < \infty$, *and any* $w \in A_p$,

$$\|Mf\|_{L^p(w)} \le C_n \, (p')^{1/p}(p)^{1/p'} \, [w]_{A_p}^{\frac{1}{p-1}} \, \|f\|_{L^p(w)}.$$

The next result is the sharp version of the Rubio de Francia extrapolation theorem due to Dragičević *et al.* [12].

Theorem 2.2. *Suppose that for some* p_0, $1 < p_0 < \infty$, *there exists* $\alpha(p_0) > 0$ *such that for every* $w \in A_{p_0}$, *a sublinear operator* T *satisfies*

$$\|Tf\|_{L^{p_0}(w)} \le C_{n,T,p_0}[w]_{A_{p_0}}^{\alpha(p_0)}\|f\|_{L^{p_0}(w)}.$$

Then for every p, $1 < p < \infty$,

$$\|Tf\|_{L^p(w)} \le C_{n,T,p_0,p}[w]_{A_p}^{\alpha(p_0)\,\max\left(1,\frac{p_0-1}{p-1}\right)}\|f\|_{L^p(w)}.$$

Third, we need a norm inequality for a weighted dyadic maximal operator.

Lemma 2.3. *Let σ be a locally integrable function such that $\sigma > 0$ a.e., and define the weighted dyadic maximal operator*

$$M_\sigma^d f(x) = \sup_{\substack{Q \in \Delta \\ x \in Q}} \frac{1}{\sigma(Q)} \int_Q |f(y)| \sigma(y) \, dy.$$

Then for all p, $1 < p < \infty$,

$$\|M_\sigma^d f\|_{L^p(\sigma)} \leq p' \|f\|_{L^p(\sigma)}.$$

In particular, the constant is independent of σ.

This result is well known and follows by standard arguments. Clearly, M_σ^d is bounded on $L^\infty(\sigma)$ with constant 1. It is also of weak-type $(1,1)$ (with respect to σ) with constant 1; this follows from the dyadic structure. Then by Marcinkiewicz interpolation we get the desired estimate (see [20, Chapter 1, Exercise 1.3.3]).

In the two-weight case we will need a norm inequality for Orlicz maximal operators. This result was proved in [43].

Theorem 2.4. *Given p, $1 < p < \infty$, suppose that A is a Young function such that $A \in B_p$. Then the Orlicz maximal operator*

$$M_A f(x) = \sup_{x \ni Q} \|f\|_{A,Q}$$

is bounded on $L^p(\mathbb{R}^n)$.

We will also need a two-weight norm inequality for the maximal operator, also from [43].

Theorem 2.5. *Given p, $1 < p < \infty$, suppose that B is a Young function such that $\bar{B} \in B_p$. Then for any pair (u,v) that satisfies*

$$\sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{B,Q} < \infty,$$

we have that

$$\|Mf\|_{L^p(u)} \leq C \|f\|_{L^p(v)}.$$

To apply Theorem 2.5 we will need to use two facts about Young functions and Orlicz norms. First, if A is a Young function such that $\bar{A} \in B_{p'}$, then $\bar{A}(t) \leq C \, t^{p'}$ for $t \geq 1$, so that $t^p \leq A(C\,t)$ for $t \geq 1$, and therefore, $\|u^{1/p}\|_{p,Q} \leq C \|u^{1/p}\|_{A,Q}$. Second, given a Young function A we have the generalized Hölder's inequality,

$$\fint_Q |f(x)g(x)| \, dx \leq 2\|f\|_{A,Q} \|g\|_{\bar{A},Q}.$$

See [7] for more details.

3. Local mean oscillation

To state Lerner's decomposition argument we must first make some definitions and give a few basic results. We follow the terminology and notation in [34], which in turn is based on Fujii [16, 17] and Jawerth and Torchinsky [27]. We note in passing that many of the underlying ideas originated in the work of Carleson [4] and Garnett and Jones [19].

Hereafter we assume that all functions f are measurable and finite-valued almost everywhere. Given a cube Q and λ, $0 < \lambda < 1$, define the local mean oscillation of f on Q by

$$\omega_\lambda(f, Q) = \inf_{c \in \mathbb{R}} \left((f - c)\chi_Q \right)^*(\lambda|Q|),$$

where f^* is the (left-continuous) non-increasing rearrangement of f:

$$f^*(t) = \inf \left\{ \alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t \right\}.$$

The local sharp maximal function of f relative to Q is then defined by

$$M_{\lambda,Q}^\# f(x) = \sup_{\substack{Q' \ni x \\ Q' \subset Q}} \omega_\lambda(f, Q).$$

The local sharp maximal function is significantly smaller than the C. Fefferman-Stein sharp maximal function: for all $\lambda > 0$ sufficiently small, $M(M_{\lambda,Q}^\# f)(x) \leq C(n, \lambda) M^\# f(x)$. (See [27].)

A median value of f on Q is a (possibly not unique) number $m_f(Q)$ such that

$$\max \left(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}| \right) \leq \frac{|Q|}{2}.$$

(A different but functionally equivalent definition is given in [17].) The median plays the same role for the local sharp maximal function as the mean does for the C. Fefferman-Stein sharp maximal function. More precisely, for each λ, $0 < \lambda \leq 1/2$,

$$\omega_\lambda(f, Q) \leq \left((f - m_f(Q))\chi_Q \right)^*(\lambda|Q|) \leq 2\omega_\lambda(f, Q).$$

(The first inequality is immediate; the second follows from (3.3) below and the fact that for any constant c, $m_f(Q) - c = m_{f-c}(Q)$; see Lerner [36].)

To estimate the median and the local mean oscillation we need several properties. For the convenience of the reader we gather them as a lemma and sketch their proofs.

Lemma 3.1. *Given a measurable function f and a cube Q, then for all λ, $0 < \lambda < 1$, and p, $0 < p < \infty$,*

(3.1) $$(f\chi_Q)^*(\lambda|Q|) \leq \lambda^{-1/p} \|f\|_{L^{p,\infty}(Q,|Q|^{-1}dx)},$$

(3.2)
$$(f\chi_Q)^*(\lambda|Q|) \leq \left(\frac{1}{\lambda|Q|}\int_Q |f|^p\,dx\right)^{1/p}.$$

Furthermore,

(3.3)
$$|m_f(Q)| \leq (f\chi_Q)^*(|Q|/2);$$

in particular, if $f \in L^p$ for any $p > 0$, then $m_f(Q) \to 0$ as $|Q| \to \infty$.

Proof. Inequality (3.2) follows immediately from (3.1). To prove this inequality, fix $\alpha < (f\chi_Q)^*(\lambda|Q|)$. Then

$$\lambda^{-1/p}\|f\|_{L^{p,\infty}(Q,|Q|^{-1}dx)} \geq \lambda^{-1/p}|Q|^{-1/p}\alpha|\{x\in Q : |f(x)| > \alpha\}|^{1/p} \geq \alpha.$$

Since this is true for all such α, (3.1) follows at once.

To prove (3.3) we consider two cases. Suppose $m_f(Q) \geq 0$. Define

$$m_+ = \sup\{\beta \geq m_f(Q) : |\{x \in Q : f(x) < \beta\}| \leq |Q|/2\};$$

then $0 \leq m_f(Q) \leq m_+$ so it will be enough to prove $m_+ \leq f^*(|Q|/2)$. Take any $\alpha > 0$ such that

$$|\{x \in Q : |f(x)| > \alpha\}| < |Q|/2.$$

Then

$$|\{x \in Q : f(x) \leq \alpha\}| \geq |\{x \in Q : |f(x)| \leq \alpha\}| > |Q|/2.$$

Hence, for any $\beta > \alpha$,

$$|\{x \in Q : f(x) < \beta\}| \geq |\{x \in Q : f(x) \leq \alpha\}| > |Q|/2,$$

and so $\beta \geq m_+$. Since this is true for all $\beta > \alpha$, we have that $\alpha \geq m_+$, and taking the infimum of all such α we get that $(f\chi_Q)^*(|Q|/2) \geq m_+$.

Finally if $m_f(Q) < 0$, define $g(x) = -f(x)$. Then $-m_f(Q)$ is a median of g and the previous case yields $|m_f(Q)| = -m_f(Q) \leq g^*(|Q|/2) = f^*(|Q|/2)$. $\qquad\square$

Remark 3.2. Inequality (3.1) is central to our proofs as it allows us to use weak $(1,1)$ inequalities directly in our estimates. By way of comparison, in [5] a key technical difficulty resulted from having to use Kolmogorov's inequality rather than the weak $(1,1)$ inequality for a singular integral. Overcoming this is the reason the results there were limited to log bumps.

To state Lerner's decomposition theorem, we generalize our notation slightly: given a cube Q_0, let $\Delta(Q_0)$ be the collection of dyadic cubes relative to Q_0. Given $Q \in \Delta(Q_0)$, $Q \neq Q_0$, let $\widehat{Q}$ be its dyadic parent: the unique cube in $\Delta(Q_0)$ containing Q whose side-length is twice that of Q.

Theorem 3.3. ([34]) *Given a measurable function f and a cube Q_0, for each $k \geq 1$ there exists a (possibly empty) collection of pairwise disjoint cubes $\{Q_j^k\} \subset \Delta(Q_0)$ such that if $\Omega_k = \bigcup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$ and $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$. Furthermore, for almost every $x \in Q_0$,*

$$|f(x) - m_f(Q_0)| \leq 4M_{\frac{1}{4},Q_0}^{\#}f(x) + 4\sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f, \widehat{Q}_j^k)\chi_{Q_j^k}(x).$$

Remark 3.4. If for all j and k we define $E_j^k = Q_j^k \setminus \Omega_{k+1}$, then the sets E_j^k are pairwise disjoint and $|E_j^k| \geq \frac{1}{2}|Q_j^k|$.

Remark 3.5. Though it is not explicit in [34], it follows at once from the proof that we can replace $M_{\frac{1}{4},Q_0}^{\#}$ by the corresponding dyadic operator $M_{\frac{1}{4},Q_0}^{\#,d}$, where

$$M_{\lambda,Q}^{\#,d}f(x) = \sup_{x\in Q'\in\Delta(Q)} \omega_\lambda(f, Q').$$

Intuitively, one may think of the cubes $\{Q_j^k\}$ as being the analog of the Calderón-Zygmund cubes for the function $f - m_f(Q_0)$ but defined with respect to the median instead of the mean. The cubes Q_j^k are maximal dyadic cubes with respect to a dyadic local sharp maximal operator. The terms on the right-hand side of the above inequality then play a role like that of the good and bad parts of the Calderón-Zygmund decomposition. A key difference, of course, is that while the Calderón-Zygmund decomposition is done at one "scale," the above theorem requires that we estimate the local mean oscillation of f at all scales.

4. The Haar shift operators

To prove Theorems 1.1 and 1.3 we need to prove the corresponding inequalities for certain dyadic operators that can be used to approximate the Hilbert transform, the Riesz transforms and the Beurling-Ahlfors operator. We follow the approach used in [28] and consider simultaneously a family of dyadic operators—the Haar shift operators—that contains all the operators we are interested in.

Let Δ be the set of dyadic cubes in $\mathbb{R}^n$. For our arguments we properly need to consider the sets $\Delta_{s,t}$, $s \in \mathbb{R}^n$, $t > 0$, of translations and dilations of dyadic cubes. However, it will be immediate that all of our arguments for dyadic cubes extend to these more general families, so without loss of generality we will restrict ourselves to dyadic cubes.

We define a Haar function on a cube $Q \in \Delta$ to be a function h_Q such that

(a) $\mathrm{supp}(h_Q) \subset Q$;
(b) if $Q' \in \Delta$ and $Q' \subsetneq Q$, then h_Q is constant on Q';
(c) $\|h_Q\|_\infty \le |Q|^{-1/2}$;
(d) $\int_Q h_Q(x)\,dx = 0$.

Given an integer $\tau \ge 0$, a Haar shift operator of index τ is an operator of the form

$$H_\tau f(x) = \sum_{Q \in \Delta} \sum_{\substack{Q',Q'' \in \Delta(Q) \\ 2^{-\tau n}|Q| \le |Q'|,|Q''|}} a_{Q',Q''}\langle f, h_{Q'}\rangle h_{Q''}(x),$$

where $a_{Q',Q''}$ is a constant such that

$$|a_{Q',Q''}| \le C\left(\frac{|Q'|}{|Q|}\frac{|Q''|}{|Q|}\right)^{1/2}.$$

We say that H_τ is a CZ Haar shift operator if it is bounded on L^2.

An important example of a Haar shift operator when $n = 1$ is the Haar shift (also known as the dyadic Hilbert transform) H^d, defined by

$$H^d f(x) = \sum_{I \in \Delta}\langle f, h_I\rangle\big(h_{I_-}(x) - h_{I_+}(x)\big),$$

where, as before, given a dyadic interval I, I_+ and I_- are its right and left halves, and

$$h_I(x) = |I|^{-1/2}\big(\chi_{I_-}(x) - \chi_{I_+}(x)\big).$$

Clearly h_I is a Haar function on I and one can write H^d as a Haar shift operator of index $\tau = 1$ with $a_{I',I''} = \pm 1$ for $I' = I$, $I'' = I_\pm$ and $a_{I',I''} = 0$ otherwise. These are the operators used by Petermichl [46, 47] to approximate the Hilbert transform. More precisely, she used the family of operators $H^d_{s,t}$, $s \in \mathbb{R}$, $t > 0$, which are defined as above but with the dyadic grid replaced by its translation by s and dilation by t. The Hilbert transform is then the limit of integral averages of these operators, so norm inequalities for H follow from norm inequalities for $H^d_{s,t}$ by Fatou's lemma and Minkowski's inequality. Similar approximations hold for the Riesz transforms and Beurling-Ahlfors operator, and we refer the reader to [48, 49] for more details.

To apply Theorem 3.3 to the Haar shift operators we need two lemmas. The first is simply that CZ Haar shift operators satisfy a weak $(1,1)$ inequality. The proof of this is known but we could not find it in the literature, except when $\tau = 0$—in this case H_τ is a constant Haar multiplier and the proof is given in [42]. Therefore, we provide a brief

sketch of the details. Here and below we will use the following nota-
tion: given an integer $\tau \geq 0$ and a dyadic cube Q, let Q^τ denote its τ-th
generation "ancestor": that is, the unique dyadic cube Q^τ containing
Q such that $|Q^\tau| = 2^{\tau n}|Q|$.

Lemma 4.1. *Given an integer $\tau \geq 0$, there exists a constant $C_{\tau,n}$ such
that for every $t > 0$,*

$$|\{x \in \mathbb{R}^n : |H_\tau f(x)| > t\}| \leq \frac{C_{\tau,n}}{t} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Proof. Fix $t > 0$ and form the Calderón-Zygmund decomposition of
f at height t. Decompose f as the sum of the good and bad parts,
$g + b$. The estimate for g is standard. For b, since Lebesgue measure
is doubling, it suffices to show that the set

$$|\{x \in \mathbb{R}^n \setminus (\cup_j Q_j^\tau) : |H_\tau b(x)| > t/2\}|$$

has measure 0. Fix j and $x \in \mathbb{R}^n \setminus Q_j^\tau$; then we would be done if
we could show that $H_\tau b_j(x) = 0$. Fix a term $a_{Q',Q''}\langle b_j, h_{Q'}\rangle h_{Q''}(x)$ in
the sum defining $H_\tau b_j(x)$. If this is non-zero, then $h_{Q''}(x) \neq 0$, so
$Q'' \cap \mathbb{R}^n \setminus Q_j^\tau \neq \emptyset$. Since $Q'' \subset Q$, $Q \cap \mathbb{R}^n \setminus Q_j^\tau \neq \emptyset$. On the other
hand, since $\int_{Q_j} b_j(x)\, dx = 0$, $\langle b_j, h_{Q'}\rangle \neq 0$ only if $Q' \subset Q_j$, which in
turn implies that $Q \subset Q_j^\tau$, a contradiction. $\qquad\square$

Our second lemma is a key estimate that is a sharper variant of a
result known for Calderón-Zygmund singular integrals (see [27]) and
whose proof is similar. For completeness we include the details.

Lemma 4.2. *Given $\tau \geq 0$, let H_τ be a CZ Haar shift operator. Fix λ,
$0 < \lambda < 1$. Then for any function f, every dyadic cube Q_0, and every
$x \in Q_0$,*

$$\omega_\lambda(H_\tau f, Q_0) \leq \frac{C_{\tau,n}}{\lambda} \fint_{Q_0^\tau} |f(x)|\, dx,$$

$$M_{\lambda,Q_0}^{\#,d}(H_\tau f)(x) \leq \frac{C_{\tau,n}}{\lambda} M^d f(x).$$

Proof. It suffices to prove the first inequality; the second follows imme-
diately from the definition of $M_{\lambda,Q_0}^{\#,d}$. Fix Q_0 and write H_τ as the sum
of two operators:

$$H_\tau f(x) = H_\tau(f\chi_{Q_0^\tau})(x) + H_\tau(f\chi_{\mathbb{R}^n \setminus Q_0^\tau})(x).$$

We claim the second term is constant for all $x \in Q_0$. Let Q be
any dyadic cube. Then the corresponding term in the sum defining

$H_\tau(f\chi_{\mathbb{R}^n\setminus Q_0^\tau})(x)$ is

$$(4.1) \qquad \sum_{\substack{Q',Q''\in\Delta(Q) \\ 2^{-\tau n}|Q|\le|Q'|,|Q''|}} a_{Q',Q''}\langle f\chi_{\mathbb{R}^n\setminus Q_0^\tau}, h_{Q'}\rangle h_{Q''}(x).$$

We may assume that $Q'' \cap Q_0 \neq \emptyset$ (otherwise we get a zero term); since $Q'' \subset Q$, this implies that $Q \cap Q_0^\tau \neq \emptyset$. Similarly, we have $Q \cap (\mathbb{R}^n \setminus Q_0^\tau) \neq \emptyset$. Therefore, $Q_0^\tau \subsetneq Q$, so $|Q_0| < 2^{-\tau n}|Q| \le |Q''|$. Hence, $Q_0 \subsetneq Q''$ and $h_{Q''}$ is constant on Q_0. Thus, (4.1) does not depend on x and so is constant on Q_0.

Denote this constant by $H_\tau f(Q_0)$; then

$$|\{x \in Q_0 : |H_\tau f(x) - H_\tau f(Q_0)| > t\}| = |\{x \in Q_0 : |H_\tau(f\chi_{Q_0^\tau})(x)| > t\}|.$$

Since H_τ is a CZ Haar shift operator it is weak $(1,1)$. Therefore, by inequality (3.1),

$$\omega_\lambda(H_\tau f, Q_0) \le \big((H_\tau f - H_\tau f(Q_0))\chi_{Q_0}\big)^*(\lambda|Q_0|)$$

$$\le \lambda^{-1}\|H_\tau(f\chi_{Q_0^\tau})\|_{L^{1,\infty}(Q_0,|Q_0|^{-1}dx)} \le \frac{C_{\tau,n}}{\lambda}\fint_{Q_0^\tau} |f(x)|\,dx.$$

$\square$

5. Singular integrals, paraproducts and Haar multipliers

In this section we prove Theorems 1.1, 1.3, 1.6 and 1.7. The principal results are the first two for singular integrals; the results for paraproducts and constant Haar multipliers are variations of these and we will only sketch the changes. We will also indicate the technical obstacles in attempting to apply our results to general Calderón-Zygmund singular integrals.

One weight inequalities: proof of Theorem 1.1. As we discussed in the previous section, to prove Theorem 1.1 it will suffice to establish the analogous result for Haar shift operators.

Theorem 5.1. *Given an integer $\tau \ge 0$ and a CZ Haar shift operator H_τ, and given p, $1 < p < \infty$, then for any $w \in A_p$,*

$$\|H_\tau f\|_{L^p(w)} \le C_{\tau,n,p}[w]_{A_p}^{\max\left(1,\frac{1}{p-1}\right)}\|f\|_{L^p(w)}.$$

Proof of Theorem 5.1. By Theorem 2.2 it will suffice to prove that

$$\|H_\tau f\|_{L^2(w)} \le C_{\tau,n}[w]_{A_2}\|f\|_{L^2(w)}.$$

Fix $w \in A_2$ and fix f. By a standard approximation argument we may assume without loss of generality that f is bounded and has compact

support. Let $\mathbb{R}^n_j$, $1 \leq j \leq 2^n$, denote the n-dimensional quadrants in $\mathbb{R}^n$: that is, the sets $I^{\pm} \times I^{\pm} \times \cdots \times I^{\pm}$ where $I^+ = [0, \infty)$ and $I^- = (-\infty, 0)$.

For each j, $1 \leq j \leq 2^n$, and for each $N > 0$ let $Q_{N,j}$ be the dyadic cube adjacent to the origin of side length 2^N that is contained in $\mathbb{R}^n_j$. Since $Q_{N,j} \in \Delta$, $\Delta(Q_{N,j}) \subset \Delta$. Because H^{τ} is a CZ shift operator, it is bounded on L^2. Thus, since $f \in L^2$, by (3.3) and (3.2), $m_{H_{\tau}f}(Q_{N,j}) \to 0$ as $N \to \infty$. Therefore, by Fatou's lemma and Minkowski's inequality,

$$\|H_{\tau}f\|_{L^2(w)} \leq \liminf_{N \to \infty} \sum_{j=1}^{2^n} \left(\int_{Q_{N,j}} |H_{\tau}f(x) - m_{H_{\tau}f}(Q_{N,j})|^2 w(x)\, dx \right)^{1/2}.$$

Hence, it will suffice to prove that each term in the sum on the right is bounded by $C_{\tau,n}[w]_{A_2}\|f\|_{L^2(w)}$.

Fix j and let $Q_N = Q_{N,j}$. By Theorem 3.3 and Lemma 4.2, for every $x \in Q_N$ we have that

(5.1) $|H_{\tau}f(x) - m_{H_{\tau}f}(Q_N)|$

$$\leq 4\, M^{\#,d}_{\frac{1}{4},Q_N}(H_{\tau}f)(x) + 4 \sum_{j,k} \omega_{\frac{1}{2^{n+2}}}(H_{\tau}f, \widehat{Q}^k_j) \chi_{Q^k_j}(x)$$

$$\leq C_{\tau,n}\, Mf(x) + C_{\tau,n} \sum_{j,k} \left(\fint_{P^k_j} |f(x)|\, dx \right) \chi_{Q^k_j}(x)$$

$$= C_{\tau,n}\, Mf(x) + C_{\tau,n}\, F(x),$$

where $P^k_j = (\widehat{Q}^k_j)^{\tau}$. We get the desired estimate for the first term from Theorem 2.1 with $p = 2$:

$$\|Mf\|_{L^2(Q_N,w)} \leq \|Mf\|_{L^2(w)} \leq C_n\,[w]_{A_2}\|f\|_{L^2(w)}.$$

To estimate F we use duality. Fix a non-negative function $h \in L^2(w)$ with $\|h\|_{L^2(w)} = 1$; then by Remark 3.4 and Lemma 2.3 we have that

$$\int_{Q_N} F(x)\, h(x)\, w(x)\, dx = C_{\tau,n} \sum_{j,k} \fint_{P^k_j} |f(x)|\, dx \int_{Q^k_j} w(x)h(x)\, dx$$

$$\leq 2 \cdot 2^{(\tau+1)n} \sum_{j,k} \frac{w(P^k_j)}{|P^k_j|} \frac{w^{-1}(P^k_j)}{|P^k_j|} |E^k_j|$$

$$\times \frac{1}{w^{-1}(P^k_j)} \int_{P^k_j} |f(x)|w(x)w(x)^{-1}\, dx$$

$$\times \frac{1}{w(Q^k_j)} \int_{Q^k_j} h(x)w(x)\, dx$$

$$\leq C_{\tau,n}\,[w]_{A_2} \sum_{j,k} \int_{E_j^k} M_{w^{-1}}^d(fw)(x) M_w^d h(x)\,dx$$

$$\leq C_{\tau,n}\,[w]_{A_2} \int_{\mathbb{R}^n} M_{w^{-1}}^d(fw)(x) M_w^d h(x)\,dx$$

$$\leq C_{\tau,n}\,[w]_{A_2} \left(\int_{\mathbb{R}^n} M_{w^{-1}}^d(fw)(x)^2 w(x)^{-1}\,dx \right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^n} M_w^d h(x)^2 w(x)\,dx \right)^{1/2}$$

$$\leq C_{\tau,n}\,[w]_{A_2} \left(\int_{\mathbb{R}^n} |f(x)w(x)|^2 w(x)^{-1}\,dx \right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^n} h(x)^2 w(x)\,dx \right)^{1/2}$$

$$= C_{\tau,n}\,[w]_{A_2} \left(\int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx \right)^{1/2}.$$

If we take the supremum over all such functions h, we conclude that

$$\|F\|_{L^2(Q_N,w)} \leq C_{\tau,n}\,[w]_{A_2} \|f\|_{L^2(w)}.$$

Combining our estimates we have that

$$\left(\int_{Q_N} |H_\tau f(x) - m_{H_\tau f}(Q_N)|^2 w(x)\,dx \right)^{1/2} \leq C_{\tau,n}\,[w]_{A_2} \|f\|_{L^2(w)},$$

and this completes the proof. $\qquad\square$

Two weight inequalities: Proof of Theorem 1.3. To prove Theorem 1.3 it will suffice to establish the corresponding result for the Haar shift operators. We record this as a separate result.

Theorem 5.2. *Given an integer $\tau \geq 0$, let H_τ be a CZ Haar shift operator. Given p, $1 < p < \infty$, and let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. Then for any pair (u,v) such that (1.2) holds we have that*

$$\|H_\tau f\|_{L^p(u)} \leq C\|f\|_{L^p(v)}.$$

Proof of Theorem 5.2. The proof is very similar to the proof of Theorem 5.1 replacing the A_2 estimates with an argument from the second half of the proof of the main theorem in [5]; therefore, we omit many of the details.

We argue as in the one-weight case and with the same notation; it will suffice to prove

$$\left(\int_{Q_N} |H_\tau f(x) - m_{H_\tau f}(Q_N)|^p u(x)\, dx\right)^{1/p} \leq C\|f\|_{L^p(v)}$$

and we use (5.1). The estimate of the term containing the maximal operator is straightforward: by Theorem 2.5 we have $M : L^p(v) \to L^p(u)$ since the pair (u,v) satisfies (1.2). Therefore, by duality (this time with respect to Lebesgue measure) it is enough to show that for every non-negative $h \in L^{p'}(\mathbb{R}^n)$ with $\|h\|_{L^{p'}} = 1$,

$$I = \int_{Q_N} F(x)\, u(x)^{1/p}\, h(x)\, dx \leq C\|f\|_{L^p(v)}.$$

We apply Remark 3.4 and the generalized Hölder's inequality to get

$$I \leq C \sum_{j,k} \fint_{P_j^k} |f(x)|\, dx \fint_{Q_j^k} u(x)^{1/p} h(x)\, dx\, |E_j^k|$$

$$\leq C \sum_{j,k} \|fv^{1/p}\|_{\bar{B},P_j^k} \|v^{-1/p}\|_{B,P_j^k} \|u^{1/p}\|_{A,Q_j^k} \|h\|_{\bar{A},Q_j^k}\, |E_j^k|.$$

By convexity, $\|u^{1/p}\|_{A,Q_j^k} \leq 2^{n(\tau+1)} \|u^{1/p}\|_{A,P_j^k}$, so since the pair (u,v) satisfies (1.2),

$$I \leq C \sum_{j,k} \int_{E_j^k} M_{\bar{B}}(fv^{1/p})(x) M_{\bar{A}}h(x)\, dx$$

$$\leq C \int_{\mathbb{R}^n} M_{\bar{B}}(fv^{1/p})(x) M_{\bar{A}}h(x)\, dx.$$

Since $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$, by Theorem 2.4, $M_{\bar{B}}$ is bounded on L^p and $M_{\bar{A}}$ is bounded in $L^{p'}$. The desired estimate now follows by Hölder's inequality. $\qquad\square$

General Calderón-Zygmund singular integrals. Key to the proofs of Theorems 5.1 and 5.2 are the sharp estimates for the local mean oscillation in Lemma 4.2. If we were to try to extend these proofs to an arbitrary Calderón-Zygmund singular integral T, then we would have to estimate the local mean oscillation by a sum (see [27]):

$$\omega_\lambda(Tf, Q) \leq C \sum_{i=0}^\infty 2^{-i} \fint_{2^i Q} |f(x)|\, dx.$$

If we use this estimate in the proof of Theorem 5.1, then we still get that T is bounded (since the sum is bounded by $2\inf_{x \in Q} Mf(x)$), but we get an additional factor of $[w]_{A_2}$. The proof of Theorem 5.2

can be modified to handle this sum, but to get convergence you need the additional assumption that $p > n$. This is the approach used by Lerner [34]. This (seemingly artificial) restriction $p > n$ also appears in [5]. It would be very interesting to find a refinement of Theorem 3.3 that would let us remove this restriction. Alternatively, it is tempting to conjecture that the estimate above could be improved by replacing 2^{-i} by $2^{-(n+\epsilon)i}$, which would be sufficient to adapt the proofs in both the one and two-weight case. However, it is not clear that such an inequality is true, even for singular integrals with smooth kernels.

Dyadic paraproducts and Haar multipliers: Proof of Theorems 1.6 and 1.7. The proof of Theorem 1.6 is essentially identical to the proof of the corresponding results for singular integrals once we prove the analog of Lemma 4.2:

$$\omega_\lambda(\pi_b f, Q) \leq \frac{C\|b\|_{*,d}}{\lambda} \fint_Q |f(x)| \, dx.$$

The proof follows as before. The dyadic paraproduct is a local operator, since for any $I \in \Delta$, h_I is constant on proper dyadic sub-intervals of I, and so, given a fixed dyadic interval I_0, $\pi_b(f\chi_{\mathbb{R}\setminus I_0})$ is constant on I_0. Furthermore, π_b is bounded on L^p and satisfies a weak $(1,1)$ inequality: for every $t > 0$,

$$|\{x \in \mathbb{R} : |\pi_b f(x)| > t\}| \leq \frac{C\|b\|_{*,d}}{t} \int_{\mathbb{R}} |f(x)| \, dx.$$

For a proof, see Pereyra [42].

Theorem 1.7 is actually a special case of Theorems 5.1 and 5.2, since the constant Haar multipliers are clearly Haar shift operators of index $\tau = 0$. The dependence on $\|\alpha\|_{\ell^\infty}$ follows at once by linearity.

6. Maximal singular integrals

In this section we prove Theorem 1.4. To do so, we will follow the approach used by Hytönen *et al.* [23] and actually prove the corresponding result for a family of "maximal" dyadic shift operators. The underlying dyadic operators are a generalization of the Haar shift operators defined in Section 4. As noted in [23], the results for the maximal singular integrals associated to the Hilbert transform, the Riesz transforms and the Beurling-Ahlfors operator are gotten by the same approximation arguments as we discussed above.

We begin by defining the appropriate shift operators. To distinguish them from the operators defined above, we will refer to them as generalized Haar shift operators. (In [23] they are simply referred to as

Haar shift operators, but our change in terminology should not cause any confusion.) We say that an operator T is a generalized Haar shift operator of index $\tau \geq 0$ if

$$Tf = \sum_{Q \in \Delta} \langle f, g_Q \rangle \, \gamma_Q,$$

where the functions γ_Q are such that:

(a) $\operatorname{supp}(\gamma_Q) \subset Q$;
(b) if $Q' \in \Delta$ and $Q' \subset Q$ with $|Q'| \leq 2^{-\tau n} |Q|$, then γ_Q is constant on Q';
(c) $\|\gamma_Q\|_\infty \leq |Q|^{-1/2}$.

The functions g_Q also have these properties. Finally we assume that the functions γ_Q, g_Q are such that T extends to a bounded operator on L^2. Together, these hypotheses imply that T is of weak-type $(1,1)$ (see [23]). Examples of generalized Haar shift operators include the dyadic paraproducts and their adjoints.

Associated with a generalized Haar shift operator T is the maximal Haar shift operator

$$T_* f = \sup_{\epsilon > 0} |T_\epsilon f| = \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \Delta \\ |Q| \geq \epsilon^n}} \langle f, g_Q \rangle \, \gamma_Q \right|,$$

We again have that T_* is bounded on L^2 and is of weak-type $(1,1)$ (see [23]).

Our main result for maximal Haar shift operators is the following.

Theorem 6.1. *Let T be a generalized Haar shift operator of index $\tau \geq 0$, and let T_* be the corresponding maximal Haar shift operator. Then, for every p, $1 < p < \infty$, and for all $w \in A_p$,*

$$\|T_* f\|_{L^p(w)} \leq C_{\tau,n,p} [w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p(w)}.$$

Furthermore, if the pair of weights (u, v) satisfies (1.2), then

$$\|T_* f\|_{L^p(u)} \leq C \|f\|_{L^p(v)}.$$

The proof of Theorem 6.1 is very much the same as the proofs of Theorem 5.1 and 5.2, so we will only describe the differences between the two arguments. First, if f is bounded and has compact support, $\sup_{\epsilon > 0} |m_{T_\epsilon f}(Q)| \to 0$ as $|Q| \to \infty$. Indeed, by (3.3) and (3.2),

$$\sup_{\epsilon > 0} |m_{T_\epsilon f}(Q)| \leq 2^{1/p} \sup_{\epsilon > 0} \left(\frac{1}{|Q|} \int_Q |T_\epsilon f|^2 \, dx \right)^{1/2} \leq |Q|^{-1/2} \|T_* f\|_{L^2},$$

and the right-hand term tends to 0 as $|Q| \to \infty$ since T_* is bounded on L^2. Now fix j, $1 \leq j \leq n$, and as before let $Q_N = Q_{N,j}$. Then in the one-weight case by Fatou's lemma we have that

$$\int_{\mathbb{R}_j^n} |T_* f(x)|^2 w(x)\, dx$$

$$\leq \liminf_{N \to \infty} \int_{Q_N} \left| \sup_{\epsilon > 0} |T_\epsilon f(x)| - \sup_{\epsilon > 0} |m_{T_\epsilon f}(Q_N)| \right|^2 w(x)\, dx$$

$$\leq \liminf_{N \to \infty} \int_{Q_N} \sup_{\epsilon > 0} |T_\epsilon f(x) - m_{T_\epsilon f}(Q_N)|^2 w(x)\, dx.$$

In the two-weight case we get the same inequality with 2 replaced by p and w with the weight u.

Fix $\epsilon > 0$ and apply Theorem 3.3. To continue the proof we need an analog of Lemma 4.2 that takes into account the supremum. This in turn reduces to showing the following: given λ, $0 < \lambda < 1$, and a dyadic cube Q_0, for every $x \in Q_0$

$$(6.1) \qquad \sup_{\epsilon > 0} \omega_\lambda(T_\epsilon f, Q_0) \leq \frac{C_{\tau,n}}{\lambda} \fint_{Q_0^\tau} |f(x)|\, dx.$$

Given inequality (6.1) the remainder of the proof in both the one and two-weight case proceeds exactly as before.

To prove (6.1) we proceed as in the proof of Lemma 4.2. Fix $\epsilon > 0$; since T_ϵ is linear,

$$T_\epsilon f(x) = T_\epsilon(f \chi_{Q_0^\tau})(x) + T_\epsilon(f \chi_{\mathbb{R}^n \setminus Q_0^\tau})(x)$$

We claim that the second term is constant for all $x \in Q_0$; denote it by $T_\epsilon(Q_0)$. Assuming this for the moment, we have that

$$|\{x \in Q_0 : |T_\epsilon f(x) - T_\epsilon f(Q_0)| > t\}| = |\{x \in Q_0 : |T_\epsilon(f \chi_{Q_0^\tau})(x)| > t\}|$$

$$\leq |\{x \in Q_0 : T_*(f \chi_{Q_0^\tau})(x) > t\}|$$

$$\leq \frac{C}{t} \int_{Q_0^\tau} |f(x)|\, dx,$$

where we have used that T_* is of weak-type $(1,1)$. Inequality (6.1) follows at once from this and (3.1):

$$\sup_{\epsilon > 0} \omega_\lambda(T_\epsilon f, Q_0)$$

$$\leq \sup_{\epsilon > 0} \left((T_\epsilon f - T_\epsilon f(Q_0)) \chi_{Q_0} \right)^* (\lambda |Q_0|) \leq \frac{C_{\tau,n}}{\lambda} \fint_{Q_0^\tau} |f(x)|\, dx.$$

It remains to show that $T_\epsilon f(Q_0)$ is indeed a constant. Fix $x \in Q_0$; then

$$T_\epsilon(f\chi_{\mathbb{R}^n \setminus Q_0^\tau})(x) = \sum_{\substack{Q \in \Delta \\ |Q| \geq \epsilon^n}} \langle f\chi_{\mathbb{R}^n \setminus Q_0^\tau}, g_Q \rangle \gamma_Q(x).$$

We may restrict the sum to those cubes Q satisfying $Q \cap Q_0 \neq \emptyset$ and $Q \cap (\mathbb{R}^n \setminus Q_0^\tau) \neq \emptyset$ since otherwise we get terms equal to 0. In this case, $Q_0^\tau \subsetneq Q$, and consequently $Q_0 \subset Q$ with $|Q_0| < 2^{-\tau n}|Q|$. This implies that γ_Q is constant on Q_0 which proves our claim.

7. The dyadic square function

To prove our results for the dyadic square function we must first give a version of Lemma 4.2. The key change, however, is that we prove it not for $S_d f$ but for $(S_d f)^2$.

Lemma 7.1. *Fix λ, $0 < \lambda < 1$. Then for any function f, every dyadic cube Q_0, and every $x \in Q_0$,*

$$\omega_\lambda((S_d f)^2, Q_0) \leq \frac{C_n}{\lambda^2} \left(\fint_{Q_0} |f(x)|\, dx \right)^2,$$

$$M_{\lambda,Q_0}^{\#,d}((S_d f)^2)(x) \leq \frac{C_n}{\lambda^2} M^d f(x)^2.$$

Proof. It suffices to prove the first inequality; the second follows immediately from the definition of $M_{\lambda,Q_0}^{\#,d}$. Fix Q_0; then for every $x \in Q_0$ we can decompose $S_d f(x)^2$ as

$$S_d f(x)^2 = \sum_{\substack{Q \in \Delta \\ Q \subsetneq Q_0}} |f_Q - f_{\widehat{Q}}|^2 \chi_Q(x) + \sum_{\substack{Q \in \Delta \\ Q \supset Q_0}} |f_Q - f_{\widehat{Q}}|^2.$$

The second term is a constant; denote it by $S_d f(Q_0)^2$. Furthermore, we have that for $x \in Q_0$,

$$0 \leq S_d f(x)^2 - S_d f(Q_0)^2 = \sum_{\substack{Q \in \Delta \\ Q \subsetneq Q_0}} |f_Q - f_{\widehat{Q}}|^2 \chi_Q(x) \leq S_d(f\chi_{Q_0})(x)^2.$$

Hence, since S_d is weak $(1,1)$ (see, for instance, [55]), for every $t > 0$ we have that

$$|\{x \in Q_0 : |S_d f(x)^2 - S_d f(Q_0)^2| > t\}|$$

$$\leq |\{x \in Q_0 : |S_d(f\chi_{Q_0})(x)| > t^{1/2}\}| \leq \frac{C_n}{t^{1/2}} \int_{Q_0} |f(x)|\, dx.$$

Therefore, by (3.1) with $p = 1/2$,

$$\omega_\lambda((S_df)^2, Q_0)$$

$$\leq \left(((S_df)^2 - S_df(Q_0)^2)\chi_{Q_0}\right)^*(\lambda|Q_0|) \leq \frac{C_n}{\lambda^2}\left(\fint_{Q_0} |f(x)|\,dx\right)^2.$$

$\square$

One weight inequalities: Proof of Theorem 1.8. The proof is a variation of the proof of Theorem 5.1 and we describe the main changes. The first is that rather than proving this result for $p = 2$, we choose $p = 3$, so that $1/2 = (p-1)^{-1}$. Then by Theorem 2.2 it will suffice to prove that for any $w \in A_3$,

$$\|S_df\|_{L^3(w)} \leq C_n[w]_{A_3}^{\frac{1}{2}}\|f\|_{L^3(w)}.$$

Fix $w \in A_3$ and j, $1 \leq j \leq 2^n$. As before, let $Q_N = Q_{N,j}$. Then,

$$\left(\int_{\mathbb{R}_j^n} |S_df(x)|^3 w(x)\,dx\right)^{2/3}$$

$$\leq \liminf_{N\to\infty}\left(\int_{Q_N} |S_df(x)^2 - m_{(S_df)^2}(Q_N)|^{3/2}w(x)\,dx\right)^{2/3}.$$

By Theorem 3.3 and Lemma 7.1, for every $x \in Q_N$ we have

$$(7.1) \quad |S_df(x)^2 - m_{(S_df)^2}(Q_N)|$$

$$\leq C_n\,Mf(x)^2 + C_n\sum_{j,k}\left(\fint_{\widehat{Q}_j^k}|f(x)|\,dx\right)^2\chi_{Q_j^k}(x)$$

$$= C_n\,Mf(x)^2 + C_n\,F(x).$$

To estimate the first term we use Theorem 2.1 with $p = 3$:

$$\|(Mf)^2\|_{L^{3/2}(Q_N,w)} \leq \|Mf\|_{L^3(w)}^2 \leq C_n\,[w]_{A_3}\|f\|_{L^3(w)}^2.$$

To estimate F we use duality. Fix a non-negative function $h \in L^3(w)$ with $\|h\|_{L^3(w)} = 1$; then Remark 3.4 and Lemma 2.3 yield

$$\int_{Q_N} F(x)\,h(x)\,dx = \sum_{j,k}\left(\fint_{\widehat{Q}_j^k}|f(x)|\,dx\right)^2\int_{Q_j^k} w(x)h(x)\,dx$$

$$\leq 2^{n+1}\sum_{j,k}\frac{w(\widehat{Q}_j^k)}{|\widehat{Q}_j^k|}\left(\frac{w^{-1/2}(\widehat{Q}_j^k)}{|\widehat{Q}_j^k|}\right)^2|E_j^k|$$

$$\times\left(\frac{1}{w^{-1/2}(\widehat{Q}_j^k)}\int_{\widehat{Q}_j^k}|f(x)|w(x)^{1/2}w(x)^{-1/2}\,dx\right)^2$$

$$\times \frac{1}{w(Q_j^k)} \int_{Q_j^k} h(x)w(x)\,dx$$

$$\leq C_n[w]_{A_3} \sum_{j,k} \int_{E_j^k} M_{w^{-1/2}}^d(fw^{1/2})(x)^2 M_w^d h(x)\,dx$$

$$\leq C_n[w]_{A_3} \int_{\mathbb{R}^n} M_{w^{-1/2}}^d(fw^{1/2})(x)^2 M_w^d h(x)\,dx$$

$$\leq C_n[w]_{A_3} \left(\int_{\mathbb{R}^n} M_{w^{-1/2}}^d(fw^{1/2})(x)^3 w(x)^{-1/2}\,dx \right)^{2/3}$$

$$\times \left(\int_{\mathbb{R}^n} M_w^d h(x)^3 w(x)\,dx \right)^{1/3}$$

$$\leq C_n[w]_{A_3} \left(\int_{\mathbb{R}^n} |f(x)w(x)^{1/2}|^3 w(x)^{-1/2}\,dx \right)^{2/3}$$

$$\times \left(\int_{\mathbb{R}^n} h(x)^3 w(x)\,dx \right)^{1/3}$$

$$= C_n[w]_{A_3} \|f\|_{L^3(w)}^2.$$

Taking the supremum over all such functions h we conclude that

$$\|F\|_{L^{3/2}(Q_N,w)} \leq C_n[w]_{A_3} \|f\|_{L^3(w)}^2.$$

If we combine the two estimates we get

$$\left(\int_{Q_N} |S_d f(x)^2 - m_{(S_d f)^2}(Q_N)|^{3/2} w(x)\,dx \right)^{1/3} \leq C_n\,[w]_{A_3}^{1/2} \|f\|_{L^3(w)},$$

and the desired inequality follows as before.

The exponent $\max\left(\frac{1}{2}, \frac{1}{p-1}\right)$ is the best possible. As we noted above, for $p \leq 2$ specific examples were constructed by Dragičević *et al.* [12]. For $p > 2$, a proof was sketched by Lerner [31], adapting an argument for singular integrals due to R. Fefferman and Pipher [15]. For completeness we give the details.

If $2 < p \leq 3$, then the sharpness of this exponent follows at once by extrapolation. For suppose there existed p_0 in this range such that the best possible exponent satisfied $\alpha(p_0) < \frac{1}{p_0-1}$. Then by Theorem 2.2, we get that the exponent in the weighted L^2 inequality is

$$\alpha(p_0) \max\left(1, \frac{p_0-1}{2-1} \right) < 1,$$

contradicting the fact that the best possible exponent is 1.

We now consider the case $p > 3$. Suppose to the contrary that there exists a non-decreasing function ϕ such that $\phi(t)/t^{1/2} \to 0$ as $t \to \infty$, and suppose that for some $p_0 > 2$,

$$(7.2) \qquad \|S_d f\|_{L^{p_0}(w)} \le C_{n,p_0} \phi([w]_{A_{p_0}}) \|f\|_{L^{p_0}(w)}.$$

We will show that this implies for all $p > p_0$ that

$$(7.3) \qquad \|S_d f\|_{L^p} \le C_1 \phi(C_2 p) \|f\|_{L^p}.$$

Below we will give an example to show that this is a contradiction.

To prove (7.3), fix $p > p_0$ and fix a non-negative function $h \in L^{(p/p_0)'}$, $\|h\|_{L^{(p/p_0)'}(\mathbb{R}^n)} = 1$. Define the Rubio de Francia iteration algorithm (see [7])

$$Rh = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|_{L^{(p/p_0)'}(\mathbb{R}^n)}^k}.$$

Then it follows from this definition that $\|Rh\|_{L^{(p/p_0)'}(\mathbb{R}^n)} \le 2$ and

$$[Rh]_{A_{p_0}} \le [Rh]_{A_1} \le 2 \|M\|_{L^{(p/p_0)'}(\mathbb{R}^n)} \le C_{n,p_0} p.$$

Therefore, by (7.2) and Hölder's inequality,

$$\int_{\mathbb{R}^n} S_d f(x)^{p_0} h(x)\, dx \le \int_{\mathbb{R}^n} S_d f(x)^{p_0} Rh(x)\, dx$$

$$\le C_{n,p_0} \phi([Rh]_{A_{p_0}})^{p_0} \int_{\mathbb{R}^n} f(x)^{p_0} Rh(x)\, dx \le C_{n,p_0} \phi(C_{n,p_0} p)^{p_0} \|f\|_{L^p(\mathbb{R}^n)}^{p_0}.$$

Inequality (7.3) now follows by duality, giving us the desired contradiction.

It remains to show that (7.3) cannot hold. This result is known: see, for instance, Wang [54]. For completeness, here we construct a simple example of a function f on the real line such that $\|S_d f\|_p \ge c p^{1/2} \|f\|_p$. Define the function f on $\mathbb{R}$ by

$$f(x) = \sum_{j=0}^{\infty} \chi_{(2^{-2j-1}, 2^{-2j})}(x).$$

Then

$$\|f\|_p = \left(\sum_{j=0}^{\infty} 2^{-2j} - 2^{-2j-1} \right)^{1/p} = \left(\frac{1}{2} \sum_{j=0}^{\infty} 2^{-2j} \right)^{1/p} = \left(\frac{2}{3} \right)^{1/p} \le 1.$$

To estimate the norm of $S_d f$, let $F_i = f_{Q_i}$, $i \ge 1$, denote the average of f on the interval $Q_i = [0, 2^{-i})$. Then repeating the above calculation

shows that

$$F_{2i} = 2^{2i} \sum_{j=i}^{\infty} 2^{-2j} - 2^{-2j-1} = \frac{2}{3}.$$

Since the integrals of f on Q_{2i} and Q_{2i-1} are the same, $F_{2i-1} = \frac{1}{3}$. Therefore, given $i \geq 2$, if $2^{-2i-1} < x < 2^{-2i}$,

$$S_d f(x)^2 \geq \sum_{1 \leq j \leq i} \left| F_{2j} - F_{2j-1} \right|^2 = \frac{i}{9} \geq c \log(1/x).$$

The same estimate (with a smaller constant c) holds when $2^{-2i} < x < 2^{-2i+1}$. Therefore,

$$\|S_d f\|_p \geq c \left(\int_0^{2^{-3}} \log(1/x)^{p/2} \, dx \right)^{1/p} \geq c \left(\sum_{k=3}^{\infty} k^{p/2} e^{-k} \right)^{1/p} \geq c p^{1/2},$$

where to get the last estimate we drop all the terms in the sum except for $k = [p] + 2$. Combining these two estimates, we see that

$$\|S_d f\|_p \geq c p^{1/2} \|f\|_p,$$

which is what we wanted to prove.

Two weight inequalities: Proof of Theorem 1.10. Fix p, $1 < p < \infty$. Then, arguing as before it suffices to show that

$$\left(\int_{Q_N} |S_d f(x)^2 - m_{(S_d f)^2}(Q_N)|^{p/2} u(x) \, dx \right)^{2/p} \leq C \|f\|_{L^p(v)}^2.$$

We again use (7.1). To estimate the term containing M, note that we have

$$\|(Mf)^2\|_{L^{p/2}(Q_N, u)} \leq \|Mf\|_{L^p(u)}^2 \leq C \|f\|_{L^p(v)}^2,$$

where we have used Theorem 2.5 and the fact that (u, v) satisfies (1.5) when $1 < p \leq 2$ or (1.7) when $p > 2$.

To estimate F we consider two cases. Suppose first that $1 < p \leq 2$. Then we use that $p/2 \leq 1$, inequality (1.5), and Theorem 2.4 and the fact that $\bar{B} \in B_p$ to get

$$\int_{Q_N} F(x)^{p/2} u(x) \, dx \leq \sum_{j,k} \left(\fint_{\widehat{Q}_j^k} |f(x)| \, dx \right)^p u(Q_j^k)$$

$$\leq C \sum_{j,k} \fint_{Q_j^k} u(x) \, dx \, \|v^{-1/p}\|_{B,\widehat{Q}_j^k}^p \, \|f v^{1/p}\|_{\bar{B},\widehat{Q}_j^k}^p |E_j^k|$$

$$\leq C \sum_{j,k} \int_{E_j^k} M_{\bar{B}}(f v^{1/p})(x)^p \, dx$$

$$\leq C \int_{\mathbb{R}^n} M_{\bar{B}}(fv^{1/p})(x)^p \, dx$$
$$\leq C\|f\|_{L^p(v)}^p.$$

Combining these two estimates we get the desired inequality.

Now suppose that $p > 2$. In this case the proof is very similar to the proof of Theorem 5.2 and we highlight the changes. To use duality with respect to Lebesgue measure, fix a non-negative function $h \in L^{(p/2)'}(\mathbb{R}^n)$ with $\|h\|_{L^{(p/2)'}} = 1$. Then (1.7) gives

$$\int_{Q_N} F(x) \, u(x)^{2/p} \, h(x) \, dx$$

$$\leq C \sum_{j,k} \left(\fint_{\widehat{Q}_j^k} |f(x)| \, dx \right)^2 \fint_{Q_j^k} u(x)^{2/p} h(x) \, dx \, |E_j^k|$$

$$\leq C \sum_{j,k} \|fv^{1/p}\|_{\bar{B},\widehat{Q}_j^k}^2 \|v^{-1/p}\|_{B,\widehat{Q}_j^k}^2 \|u^{2/p}\|_{A,Q_j^k} \|h\|_{\bar{A},Q_j^k} \, |E_j^k|$$

$$\leq C \sum_{j,k} \int_{E_j^k} M_{\bar{B}}(fv^{1/p})(x)^2 M_{\bar{A}}h(x) \, dx$$

$$\leq C \int_{\mathbb{R}^n} M_{\bar{B}}(fv^{1/p})(x)^2 M_{\bar{A}}h(x) \, dx$$

$$\leq C \, \|M_{\bar{B}}(fv^{1/p})(x)\|_{L^p}^2 \, \|M_{\bar{A}}h\|_{L^{(p/2)'}}$$

$$\leq C \, \|f\|_{L^p(v)}^2,$$

where we have used Hölder's inequality, Theorem 2.4 and the fact that $\bar{A} \in B_{(p/2)'}$ and $\bar{B} \in B_p$. The desired estimate follows at once if we take the supremum over all such functions h.

8. THE VECTOR-VALUED MAXIMAL OPERATOR

Our two results for the vector-valued maximal operator are exact parallels of our results for the dyadic square function. Formally, the change only requires replacing "2" by "q", $1 < q < \infty$, and in fact, the proofs do adapt readily as we will sketch below.

As with singular integral operators, in order to prove sharp results for vector-valued maximal operator, we need to consider a dyadic operator. Recall that the dyadic maximal operator is defined by

$$M^d f(x) = \sup_{\substack{Q \in \Delta \\ Q \ni x}} \fint_Q |f(y)| \, dy.$$

Given $q > 1$ and $f = \{f_i\}$, define the dyadic vector valued maximal operator by

$$\overline{M}_q^d f(x) = \left(\sum_{i=1}^{\infty} M^d f_i(x)^q \right)^{1/q}.$$

By an argument that goes back to C. Fefferman and Stein [14] (see also [50] and [18]), the maximal operator can be approximated by the dyadic maximal operator and the analogous operator defined on all translates of the dyadic grid. Therefore, by a straightforward argument using Fatou's lemma and Minkowski's inequality, to prove weighted norm inequalities for the vector-valued maximal operator it suffices to prove them for $\overline{M}_q^d$. (For the details of this argument, see [7].)

Again like the dyadic square function, the key estimate for the dyadic vector-valued maximal operator is to control the local mean oscillation of $(\overline{M}_q^d f)^q$.

Lemma 8.1. *Fix λ, $0 < \lambda < 1$, and q, $1 < q < \infty$. Then for any function $f = \{f_i\}$, every dyadic cube Q_0, and every $x \in Q_0$,*

$$\omega_\lambda((\overline{M}_q^d f)^q, Q_0) \le \frac{C_{n,q}}{\lambda^q} \left(\fint_{Q_0} \|f(x)\|_{\ell^q} \, dx \right)^q,$$

$$M_{\lambda,Q_0}^{\#,d}((\overline{M}_q^d f)^q)(x) \le \frac{C_{n,q}}{\lambda^q} M^d(\|f(\cdot)\|_{\ell^q})(x)^q.$$

Proof. The second estimate again follows from the first. To prove the first, fix Q_0. Then for every $x \in Q_0$ and every $i \ge 1$, we observe that

$$M^d f_i(x) = \max \left(M^d(f_i \chi_{Q_0})(x), \sup_{\substack{Q \in \Delta \\ Q_0 \subset Q}} \fint_Q |f_i(y)| \, dy \right).$$

The second term on the right is constant; using this we define

$$K_0 = \left(\sum_{i=1}^{\infty} \left(\sup_{\substack{Q \in \Delta \\ Q_0 \subset Q}} \fint_Q |f_i(y)| \, dy \right)^q \right)^{1/q}.$$

For $x \in Q_0$, $\overline{M}_q^d f(x)^q \ge K_0^q$. We also have the following elementary inequality: for every $a, b \ge 0$, $0 \le \max(a,b) - b \le a$. Combining these facts we get that

$$0 \le \overline{M}_q^d f(x)^q - K_0^q \le \sum_{i=1}^{\infty} M^d(f_i \chi_{Q_0})(x)^q = \overline{M}_q^d(f \chi_{Q_0})(x)^q.$$

Since the vector-valued maximal operator is weak $(1, 1)$ (see [14]), for any $t > 0$,

$$|\{x \in Q_0 : |\overline{M}_q^d f(x)^q - K_0^q| > t\}|$$
$$\leq |\{x \in Q_0 : \overline{M}_q^d(f\chi_{Q_0})(x) > t^{1/q}\}| \leq \frac{C_{n,q}}{t^{1/q}} \int_{Q_0} \|f(x)\|_{\ell^q} \, dx.$$

Therefore, by (3.1) with $p = 1/q$,

$$\omega_\lambda((\overline{M}_q^d f)^q, Q_0)$$
$$\leq \left(((\overline{M}_q^d f)^q - K_0^q)\chi_{Q_0}\right)^*(\lambda|Q_0|) \leq \frac{C_{n,q}}{\lambda^q} \left(\fint_{Q_0} \|f(x)\|_{\ell^q} \, dx\right)^q.$$

$\square$

One weight inequalities: Proof of Theorem 1.12. As we noted above, the proof is very similar to the proof of Theorem 1.8, and so we briefly sketch the key details. By Theorem 2.2 it will suffice to prove it for the special case when $p = q + 1$. For this value of p we have that $(p/q)' = p$ and $1 - p' = -1/q$. As before, fix $w \in A_p$ and Q_N; we will show that

$$\left(\int_{Q_N} |\overline{M}_q^d f(x)^q - m_{(\overline{M}_q^d f)^q}(Q_N)|^{p/q} w(x) \, dx\right)^{q/p}$$
$$\leq C_{n,q}[w]_{A_p} \left(\int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p w(x) \, dx\right)^{q/p}.$$

By Theorem 3.3 and Lemma 8.1, for every $x \in Q_N$,

$$(8.1) \quad |\overline{M}_q^d f(x)^q - m_{(\overline{M}_q^d f)^q}(Q_N)|$$

$$\leq C_{n,q} M(\|f(\cdot)\|_{\ell^q})(x)^q + C_{n,q} \sum_{j,k} \left(\fint_{\widehat{Q}_j^k} \|f(x)\|_{\ell^q} \, dx\right)^q \chi_{Q_j^k}(x)$$
$$= C_{n,q} M(\|f(\cdot)\|_{\ell^q})(x)^q + C_{n,q} F(x).$$

To estimate the first term we use Theorem 2.1. The estimate for F uses duality: fix a non-negative function $h \in L^p(w)$ with $\|h\|_{L^p(w)} = 1$ (recall that $(p/q)' = p$). Then, proceeding as before, we use the definition of $A_p = A_{q+1}$ to show that

$$\int_{Q_N} F(x) h(x) w(x) \, dx$$
$$\leq C_n[w]_{A_p} \int_{\mathbb{R}^n} M_{w^{-1/q}}^d(\|f(\cdot)\|_{\ell^q} w^{1/q})(x)^q w(x)^{-1/p} M_w^d h(x) w(x)^{1/p} dx.$$

Finally, we use Hölder's inequality, Theorem 2.3 and then take the supremum over all such functions h to get the desired estimate.

To prove that the exponent $\max\left(\frac{1}{q}, \frac{1}{p-1}\right)$ is the best possible, we consider two cases. If $p \leq q + 1$, then the exponent is $\frac{1}{p-1}$, which is the same as the sharp exponent for the scalar maximal function. Therefore, the examples given by Buckley [3] immediately adapt to the vector-valued maximal operator.

If $p > q+1$, then we can argue exactly as we did for the dyadic square function, replacing the exponent $1/2$ by $1/q$. Therefore, to show that the exponent $1/q$ is sharp we need to show that there exists a vector-valued function $f = \{f_i\}$ such that $\|\overline{M}_q f\|_p \geq cp^{1/q}\|f\|_p$. But such a function is given by Stein [51, p. 75].

Two weight inequalities: Proof of Theorem 1.13. The proof is again nearly the same as the proof of Theorem 1.10 for the dyadic square function, so we only sketch the highlights. Fix p, $1 < p < \infty$; then it suffices to show that

$$\int_{Q_N} |\overline{M}_q^d f(x)^q - m_{(\overline{M}_q^d f)^q}(Q_n)|^{p/q} u(x)\,dx \leq C \int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p\, v(x)\,dx.$$

We use (8.1). We estimate the term involving M using Theorem 2.5 and the fact that (u, v) satisfies (1.8) when $1 < p \leq q$ or (1.10) when $p > q$.

To estimate F we consider two cases. Suppose first that $1 < p \leq q$, then

$$\int_{Q_N} F(x)^{p/q}\, u(x)\,dx \leq \sum_{j,k} \left(\fint_{\widehat{Q}_j^k} \|f(x)\|_{\ell^q}\,dx\right)^p u(Q_j^k),$$

and this term is estimated exactly as before. Combining these two estimates we get the desired inequality.

When $p > q$, we use duality with respect to Lebesgue measure and consider a non-negative function h such that $\|h\|_{L^{(p/q)'}} = 1$. Then,

$$\int_{Q_N} F(x)\, u(x)^{q/p}\, h(x)\,dx$$
$$\leq C_{n,q} \sum_{j,k} \left(\fint_{\widehat{Q}_j^k} \|f(x)\|_{\ell^q}\,dx\right)^q \fint_{Q_j^k} u(x)^{q/p} h(x)\,dx\, |E_j^k|.$$

From here we follow the argument in the proof of Theorem 1.10, replacing 2 by q.

References

[1] K. Astala, T. Iwaniec, and E. Saksman. Beltrami operators in the plane. *Duke Math. J.*, 107(1):27–56, 2001.

[2] O. V. Beznosova. Linear bound for the dyadic paraproduct on weighted Lebesgue space $L_2(w)$. *J. Funct. Anal.*, 255(4):994–1007, 2008.

[3] S. M. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

[4] L. Carleson. Two remarks on H^1 and BMO. *Advances in Math.*, 22(3):269–277, 1976.

[5] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture. *Adv. Math.*, 216(2):647–676, 2007.

[6] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp weighted estimates for approximating dyadic operators. *Electron. Res. Announc. Math. Sci.*, 17:12–19, 2010.

[7] D. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia.* Birkhäuser, Basel, 2011.

[8] D. Cruz-Uribe and C. Pérez. Sharp two-weight, weak-type norm inequalities for singular integral operators. *Math. Res. Lett.*, 6(3-4):417–427, 1999.

[9] D. Cruz-Uribe and C. Pérez. Two weight extrapolation via the maximal operator. *J. Funct. Anal.*, 174(1):1–17, 2000.

[10] D. Cruz-Uribe and C. Pérez. Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators. *Indiana Univ. Math. J.*, 49(2):697–721, 2000.

[11] D. Cruz-Uribe and C. Pérez. On the two-weight problem for singular integral operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(4):821–849, 2002.

[12] O. Dragičević, L. Grafakos, M. Pereyra, and S. Petermichl. Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[13] J. Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by D. Cruz-Uribe.

[14] C. Fefferman and E. M. Stein. Some maximal inequalities. *Amer. J. Math.*, 93:107–115, 1971.

[15] R. Fefferman and J. Pipher. Multiparameter operators and sharp weighted inequalities. *Amer. J. Math.*, 119(2):337–369, 1997.

[16] N. Fujii. A proof of the Fefferman-Stein-Strömberg inequality for the sharp maximal functions. *Proc. Amer. Math. Soc.*, 106(2):371–377, 1989.

[17] N. Fujii. A condition for a two-weight norm inequality for singular integral operators. *Studia Math.*, 98(3):175–190, 1991.

[18] J. García-Cuerva and J.L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[19] J.B. Garnett and P.W. Jones. BMO from dyadic BMO. *Pacific J. Math.*, 99(2):351–371, 1982.

[20] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[21] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.

[22] S. Hukovic, S. Treil, and A. Volberg. The Bellman functions and sharp weighted inequalities for square functions. In *Complex analysis, operators, and related topics*, volume 113 of *Oper. Theory Adv. Appl.*, pages 97–113. Birkhäuser, Basel, 2000.

[23] P. T. Hytönen, M. Lacey, M. C. Reguera, and A. Vagharshakyan. Weak and strong-type estimates for Haar shift operators: Sharp power on the A_p characteristic. *Preprint*, 2009.

[24] T. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Preprint*, 2010.

[25] T. Hytönen, M. Lacey, C. Reguera, E. Sawyer, I. Uriarte-Tuero, and A. Vagharshakyan. Weak and strong type A_p estimates for Calderón-Zygmund operators. *Preprint*, 2010.

[26] T. Hytönen, C. Pérez, S. Treil, and A. Volberg. Sharp weighted estimates of the dyadic shifts and A_2 conjecture. *Preprint*, 2010.

[27] B. Jawerth and A. Torchinsky. Local sharp maximal functions. *J. Approx. Theory*, 43(3):231–270, 1985.

[28] M. Lacey, S. Petermichl, and M. Reguera. Sharp A_2 inequality for Haar shift operators. *Math. Ann.*, 348(1):127–141, 2010.

[29] M. Lacey, E. Sawyer, and I. Uriarte-Tuero. A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure. *Preprint*, 2010.

[30] M. Lacey, E. Sawyer, and I. Uriarte-Tuero. A two weight inequality for the Hilbert transform assuming an energy hypothesis. *Preprint*, 2010.

[31] A. Lerner. On some sharp weighted norm inequalities. *J. Funct. Anal.*, 232(2):477–494, 2006.

[32] A. Lerner. An elementary approach to several results on the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 136(8):2829–2833, 2008.

[33] A. Lerner. On some weighted norm inequalities for Littlewood-Paley operators. *Illinois J. Math.*, 52(2):653–666, 2009.

[34] A. Lerner. A pointwise estimate for local sharp maximal function with applications to singular integrals. *Bull. London Math. Soc.*, 42:843–856, 2010.

[35] A. Lerner. Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals. *Preprint*, 2010.

[36] A. K. Lerner. Weighted norm inequalities for the local sharp maximal function. *J. Fourier Anal. Appl.*, 10(5):465–474, 2004.

[37] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[38] B. Muckenhoupt and R. L. Wheeden. Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform. *Studia Math.*, 55(3):279–294, 1976.

[39] F. Nazarov, S. Treil, and A. Volberg. The Bellman functions and two-weight inequalities for Haar multipliers. *J. Amer. Math. Soc.*, 12(4):909–928, 1999.

[40] F. Nazarov, S. Treil, and A. Volberg. Two weight inequalities for individual Haar multipliers and other well localized operators. *Math. Res. Lett.*, 15(3):583–597, 2008.

[41] C. J. Neugebauer. Inserting A_p-weights. *Proc. Amer. Math. Soc.*, 87(4):644–648, 1983.

[42] M. Pereyra. Lecture notes on dyadic harmonic analysis. In *Second Summer School in Analysis and Mathematical Physics (Cuernavaca, 2000)*, volume 289 of *Contemp. Math.*, pages 1–60. Amer. Math. Soc., Providence, RI, 2001.

[43] C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.

[44] C. Pérez. Sharp weighted inequalities for the vector-valued maximal function. *Trans. Amer. Math. Soc.*, 352(7):3265–3288, 2000.

[45] C. Pérez, S. Treil, and A. Volberg. On A_2 conjecture and corona decomposition of weights. *Preprint*, 2010.

[46] S. Petermichl. Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(6):455–460, 2000.

[47] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic. *Amer. J. Math.*, 129(5):1355–1375, 2007.

[48] S. Petermichl. The sharp weighted bound for the Riesz transforms. *Proc. Amer. Math. Soc.*, 136(4):1237–1249, 2008.

[49] S. Petermichl and A. Volberg. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.*, 112(2):281–305, 2002.

[50] E.T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.

[51] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[52] A. Vagharsyakhan. Recovering singular integrals from the Haar shifts. *Proc. Amer. Math. Soc.*, to appear.

[53] A. Volberg. *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, volume 100 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2003.

[54] G. Wang. Sharp inequalities for the conditional square function of a martingale. *Ann. Probab.*, 19(4):1679–1688, 1991.

[55] J. M. Wilson. *Weighted Littlewood-Paley theory and exponential-square integrability*, volume 1924 of *Lecture Notes in Math.* Springer, Berlin, 2007.

[56] J. Wittwer. A sharp estimate on the norm of the martingale transform. *Math. Res. Lett.*, 7(1):1–12, 2000.

[57] J. Wittwer. A sharp estimate on the norm of the continuous square function. *Proc. Amer. Math. Soc.*, 130(8):2335–2342 (electronic), 2002.

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