

## GREEDY BASES IN VARIABLE LEBESGUE SPACES

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**ABSTRACT.** We compute the right and left democracy functions of admissible wavelet bases in variable Lebesgue spaces defined on  $\mathbb{R}^n$ . As an application we give Lebesgue type inequalities for these wavelet bases. We also show that our techniques can be easily modified to prove analogous results for weighted variable Lebesgue spaces and variable exponent Triebel-Lizorkin spaces.

### 1. INTRODUCTION

Let  $X$  be an infinite dimensional Banach space with norm  $\|\cdot\|_X$  and let  $B = \{b_j\}_{j=1}^\infty$  be a Schauder basis for  $X$ : that is, if  $x \in X$ , then there exists a unique sequence  $\{\lambda_j\}$  such that

$$(1.1) \quad x = \sum_{j=1}^{\infty} \lambda_j b_j.$$

For each  $N = 1, 2, 3, \dots$  we define the best  $N$ -term approximation of  $x \in X$  in terms of  $B$  as follows:

$$\sigma_N(x) = \sigma_N(x; B) := \inf_{y \in \Sigma_N} \|x - y\|_X$$

where  $\Sigma_N$  the set of all  $y \in X$  with at most  $N$  non-zero coefficients in their basis representation.

An important question in approximation theory is the construction of efficient algorithms for  $N$ -term approximation. One algorithm that has been extensively studied in recent years is the so called greedy algorithm. Given  $x \in X$  and the coefficients (1.1), reorder the basis elements so that

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$$\|\lambda_{j_1} b_{j_1}\|_X \geq \|\lambda_{j_2} b_{j_2}\|_X \geq \|\lambda_{j_3} b_{j_3}\|_X \cdots$$

(handling ties arbitrarily); we then define an  $N$ -term approximation by the (non-linear) operator  $G_N : X \rightarrow \Sigma_N$ ,

$$G_N(x) = \sum_{k=1}^N \lambda_{j_k} b_{j_k}.$$

Clearly,  $\sigma_N(x) \leq \|x - G_N(x)\|_X$ . We say that a basis is greedy if the opposite inequality holds up to a constant: there exists  $C > 1$  such that for all  $x \in X$  and  $N > 0$ ,

$$\|x - G_N(x)\|_X \leq C \sigma_N(x).$$

Konyagin and Temlyakov [27] characterized greedy bases as those which are unconditional and democratic. A basis is democratic if given any two index sets  $\Gamma, \Gamma'$ ,  $\text{card}(\Gamma) = \text{card}(\Gamma')$ ,

$$\left\| \sum_{j \in \Gamma} \frac{b_j}{\|b_j\|_X} \right\|_X \approx \left\| \sum_{j \in \Gamma'} \frac{b_j}{\|b_j\|_X} \right\|_X.$$

Wavelet systems form greedy bases in many function and distribution spaces. For example, Temlyakov [33] proved that any wavelet basis  $p$ -equivalent to the Haar basis is a greedy basis in  $L^p(\mathbb{R}^n)$ . However, wavelet bases are not greedy in other function spaces. For example, it was shown in [18] that if  $\Phi$  is a Young function such that the Orlicz space  $L^\Phi$  is not  $L^p$ ,  $1 < p < \infty$ , then wavelet bases are not greedy because they are not democratic. (Earlier, Soardi [32] proved that wavelet bases are unconditional in  $L^\Phi$ .)

In this paper our goal is to extend this result to the variable Lebesgue spaces and other related function spaces. Intuitively, given an exponent function  $p(\cdot)$ , the variable Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^n)$  consists of all functions  $f$  such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

(See Section 2 below for a precise definition.) These spaces are a generalization of the classical  $L^p$  spaces, and have applications in the study of PDEs and variational integrals with non-standard growth conditions. For the history and properties of these spaces we refer to [3, 9].

Many wavelet bases form unconditional bases on the variable Lebesgue spaces: see Theorem 2.1 below. However, unless  $p(\cdot) = p$  is constant, they are never democratic in  $L^{p(\cdot)}(\mathbb{R}^n)$ . When  $n = 1$  this was proved by Kopalani [28] and for general  $n$  it follows from Theorem 1.1 below. In this paper we quantify the failure of democracy by computing precisely the right and left democracy functions of admissible wavelet

bases in  $L^{p(\cdot)}(\mathbb{R}^n)$ . For a basis  $B = \{b_j\}_{j=1}^\infty$  in a Banach space  $X$ , we define the right and left democracy functions of  $X$  (see also [11, 23]) as:

$$h_r(N) = \sup_{\text{card}(\Gamma)=N} \left\| \sum_{j \in \Gamma} \frac{b_j}{\|b_j\|_X} \right\|_X, \quad h_l(N) = \inf_{\text{card}(\Gamma)=N} \left\| \sum_{j \in \Gamma} \frac{b_j}{\|b_j\|_X} \right\|_X$$

To state our main result, given a variable exponent  $p(\cdot)$  define  $p_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x)$  and  $p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x)$ .

**Theorem 1.1.** *Given an exponent function  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . Let  $\Psi$  be an admissible orthonormal wavelet family (see below for the precise definition). The right and left democracy functions of  $\Psi$  in  $L^{p(\cdot)}(\mathbb{R}^n)$  satisfy*

$$h_r(N) \approx N^{1/p_-}, \quad h_l(N) \approx N^{1/p_+}, \quad N = 1, 2, 3, \dots$$

As an immediate corollary to Theorem 1.1 we get a Lebesgue-type estimate for wavelet bases on variable Lebesgue spaces. The proof follows from Wojtaszczyk [34, Theorem 4].

**Corollary 1.2.** *With the hypotheses of Theorem 1.1, we have that for all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$ ,*

$$\|f - G_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C N^{\frac{1}{p_-} - \frac{1}{p_+}} \sigma_N(f, \Psi),$$

*and this estimate is the best possible.*

Our arguments readily adapt to prove analogs of Theorem 1.1 (and so also of Corollary 1.2) for other variable exponent spaces, in particular for the weighted variable Lebesgue spaces [4] and for variable exponent Triebel-Lizorkin spaces [10, 25, 26, 36]. The statement of these results require a number of preliminary definitions, and so we defer these until after the proof of Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2 we give a number of preliminary results regarding the variable Lebesgue spaces that are needed in our main proof. In Section 3 we prove Theorem 1.1. In Sections 4 and 5 we state and prove the corresponding result for weighted variable Lebesgue spaces and variable exponent Triebel-Lizorkin spaces. Throughout this paper, our notation will be standard or defined as needed. If we write  $A \lesssim B$ , we mean that  $A \leq CB$ , where the constant  $C$  depends only on the dimension  $n$  and the underlying exponent function  $p(\cdot)$ . If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$ . The letter  $C$  will denote a constant that may change at each appearance.

## 2. VARIABLE LEBESGUE SPACES

**Basic properties.** We begin with some basic definitions and results about variable Lebesgue spaces. For proofs and further information, see [3, 9, 13, 29].

Let  $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$  be the collection of exponent functions: that is, all measurable functions  $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ . We define the variable Lebesgue space  $L^{p(\cdot)} = L^{p(\cdot)}(\mathbb{R}^n)$  to be the family of all measurable functions  $f$  such that for some  $\lambda > 0$ ,

$$\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This becomes a Banach function space with respect to the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

When  $p(\cdot) = p$  is constant, then  $L^{p(\cdot)} = L^p$  with equality of norms.

To measure the oscillation of  $p(\cdot)$ , given any set  $E \subset \mathbb{R}^n$ , we define

$$p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x), \quad p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x).$$

For brevity we write  $p_+ = p_+(\mathbb{R}^n)$ ,  $p_- = p_-(\mathbb{R}^n)$ .

When  $p_+ < \infty$ , we have the following useful integral estimate:  $\|f\|_{p(\cdot)}$  is the unique value such that

$$(2.1) \quad \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\|f\|_{p(\cdot)}} \right)^{p(x)} dx = 1.$$

Given an exponent  $p(\cdot)$ ,  $1 < p_- \leq p_+ < \infty$  we define the conjugate exponent  $p'(\cdot)$  pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then functions  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  and  $g \in L^{p'(\cdot)}(\mathbb{R}^n)$  satisfy Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)};$$

moreover,  $L^{p'(\cdot)}(\mathbb{R}^n)$  is the dual space of  $L^{p(\cdot)}(\mathbb{R}^n)$  and

$$\|f\|_{p(\cdot)} \approx \sup_{\|g\|_{p'(\cdot)}=1} \left| \int_{\mathbb{R}^n} f(x)g(x) dx \right|.$$

**The Hardy-Littlewood maximal operator.** To do harmonic analysis on variable Lebesgue spaces, it is necessary to assume some regularity on the exponent  $p(\cdot)$ . One approach (taken from [3]) is to express this regularity in terms of the boundedness of the Hardy-Littlewood maximal operator. Given a locally integrable function  $f$ , define  $Mf$  by

$$Mf(x) = \sup_{Q \ni x} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q$  with sides parallel to the coordinate axes. If the maximal operator is bounded on  $L^{p(\cdot)}$  we will write  $p(\cdot) \in M\mathcal{P}$ .

The following are basic properties of the maximal operator on variable Lebesgue spaces. For complete information, see [3, 9]. By Chebyshev's inequality, if  $M$  is bounded, then it also satisfies the weak-type inequality

$$(2.2) \quad \|t\chi_{\{x: Mf(x) > t\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \quad t > 0.$$

A necessary condition for  $p(\cdot) \in M\mathcal{P}$  is that  $p_- > 1$ . An important sufficient condition is that  $p(\cdot)$  is log-Hölder continuous locally: there exists  $C_0 > 0$  such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad |x - y| < 1/2;$$

and log-Hölder continuous at infinity: there exists  $p_\infty$  and  $C_\infty > 0$  such that

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

These conditions are not necessary for the maximal operator to be bounded on  $L^{p(\cdot)}$ , but they are sharp in the sense that they are best possible pointwise continuity conditions guaranteeing that  $M$  is bounded on  $L^{p(\cdot)}$ .

**Weighted norm inequalities.** There is a close connection between the variable Lebesgue spaces and the theory of weighted norm inequalities. Here we give some basic information on weights; for more information, see [3, 12, 16].

By a weight we mean a non-negative, locally integrable function. For  $1 < p < \infty$ , we say that a weight  $w$  is in the Muckenhoupt class  $A_p$  if

$$[w]_{A_p} = \sup_Q \left( \int_Q w(x) dx \right) \left( \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where  $\int_Q g(x) dx = |Q|^{-1} \int_Q g(x) dx$ . When  $p = 1$ , we say that  $w \in A_1$  if

$$[w]_{A_1} = \sup_Q \left( \int_Q w(y) dy \right) \operatorname{ess\,sup}_{x \in Q} w(x)^{-1} < \infty.$$

Equivalently,  $w \in A_1$  if  $Mw(x) \leq [w]_{A_1} w(x)$  almost everywhere, where  $M$  is the Hardy-Littlewood maximal operator. Define  $A_\infty = \bigcup_{p \geq 1} A_p$ . If  $w \in A_\infty$ , then there exist constants  $C, \delta > 0$  such that for every cube  $Q$  and  $E \subset Q$ ,

$$\frac{w(E)}{w(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\delta,$$

where  $w(E) = \int_E w(x) dx$ .

**Wavelets.** To state our results precisely we need a few definitions on wavelets; for complete information we refer the reader to [20]. Given the collection of dyadic cubes

$$\mathcal{D} = \{Q_{j,k} = 2^{-j}([0, 1]^n + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

the functions  $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$  form an orthonormal wavelet family if

$$\{\psi_Q^l\} = \{\psi_{Q_{j,k}}^l(x) = 2^{jn/2} \psi^l(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, 1 \leq l \leq L\}$$

is an orthonormal basis of  $L^2(\mathbb{R}^n)$ .

Define the square function

$$\mathcal{W}_\Psi f = \left( \sum_{l=1}^L \sum_{Q \in \mathcal{D}} |\langle f, \psi_Q^l \rangle|^2 |Q|^{-1} \chi_Q \right)^{1/2}.$$

We will say that a wavelet family  $\Psi$  is admissible if for  $1 < p < \infty$  and every  $w \in A_p$ ,

$$\|\mathcal{W}_\Psi f\|_{L^p(w)} \approx \|f\|_{L^p(w)}.$$

Admissible wavelets on the real line include the Haar system [24], spline wavelets [14], the compactly supported wavelets of Daubechies [8], Lemarié-Meyer wavelets [30, 35], and smooth wavelets in the class  $\mathcal{R}^1$  [15, 20].

An important consequence of the boundedness of the maximal operator on  $L^{p(\cdot)}$  is that in this case wavelets form an unconditional basis.

**Theorem 2.1.** *Given  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and  $p(\cdot) \in \mathcal{MP}$ . If  $\Psi$  is an admissible orthonormal wavelet family, then it is an unconditional basis for  $L^{p(\cdot)}(\mathbb{R}^n)$  and*

$$\|\mathcal{W}_\Psi f\|_{p(\cdot)} \approx \|f\|_{p(\cdot)}.$$

Theorem 2.1 was proved in [5, Theorem 4.27] using the theory of Rubio de Francia extrapolation. The result is stated with the stronger hypothesis that  $p(\cdot)$  is log-Hölder continuous, but the extrapolation argument given there works with the weaker assumptions used here (see [3, Corollary 5.32]). This result was also proved by Izuki [22] and by Kopalani [28] on the real line.

## 3. PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1. In order to avoid repeating details in the subsequent sections, we have written the proof in terms of a series of lemmas and propositions; this will allow us to prove our other results by indicating where this proof must be modified.

**Lemma 3.1.** *Given an exponent function  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  such that  $1 < p_- \leq p_+ < \infty$ , then for every cube  $Q$ ,*

$$|Q|^{\frac{1}{p_Q}} \leq 2 \|\chi_Q\|_{p(\cdot)},$$

where

$$\frac{1}{p_Q} = \int_Q \frac{1}{p(x)} dx.$$

When  $p(\cdot) \in M\mathcal{P}$ , this inequality is actually an equivalence: see [9]. For our purposes we only need this weaker result and so we include the short proof.

*Proof.* Fix a cube  $Q$ . If we define

$$\frac{1}{p'_Q} = \int_Q \frac{1}{p'(x)} dx,$$

then  $1/p_Q + 1/p'_Q = 1$ . By Jensen's inequality,

$$\left(\frac{1}{|Q|}\right)^{\frac{1}{p'_Q}} = \exp\left(\int_Q \log\left[\left(\frac{1}{|Q|}\right)^{\frac{1}{p'(\cdot)}}\right] dx\right) \leq \int_Q \left(\frac{1}{|Q|}\right)^{\frac{1}{p'(\cdot)}} dx.$$

But then by Hölder's inequality in the scale of variable Lebesgue spaces,

$$|Q|^{\frac{1}{p_Q}} = |Q| \left(\frac{1}{|Q|}\right)^{\frac{1}{p'_Q}} \leq |Q| \int_Q \left(\frac{1}{|Q|}\right)^{\frac{1}{p'(\cdot)}} dx \leq 2 \|\chi_Q\|_{p(\cdot)} \| |Q|^{-1/p'(\cdot)} \chi_Q \|_{p'(\cdot)}.$$

To complete the proof, note that

$$\int_Q (|Q|^{-1/p'(x)})^{p'(x)} dx = 1,$$

and so by (2.1),  $\| |Q|^{-1/p'(\cdot)} \chi_Q \|_{p'(\cdot)} = 1$ . □

**Lemma 3.2.** *Given  $p(\cdot)$ , suppose  $p(\cdot) \in M\mathcal{P}$ . Then for any cube  $Q$  and any set  $E \subset Q$ ,*

$$\frac{|E|}{|Q|} \leq M_0 \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}},$$

where  $M_0$  is the norm of the Hardy-Littlewood operator  $M$  on  $L^{p(\cdot)}(\mathbb{R}^n)$ .

*Proof.* Fix  $Q$  and  $E \subset Q$ . Then for every  $x \in Q$ ,

$$M(\chi_E)(x) \geq \frac{|E|}{|Q|}.$$

Since  $M$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ , by the weak-type inequality with  $t = \frac{|E|}{|Q|}$  (notice that the constant  $C$  in the right hand side of (2.2) can be taken to be  $M_0$ ),

$$\frac{|E|}{|Q|} \|\chi_Q\|_{p(\cdot)} \leq M_0 \|\chi_E\|_{p(\cdot)}.$$

□

**Lemma 3.3.** *Given  $p(\cdot)$ , suppose  $p(\cdot) \in M\mathcal{P}$  and  $1 < p_- \leq p_+ < \infty$ . Then there exist constants  $C, \delta > 0$  such that given any cube  $Q$  and any set  $E \subset Q$ ,*

$$\frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \leq C \left( \frac{|E|}{|Q|} \right)^\delta.$$

*Proof.* Since  $p(\cdot) \in M\mathcal{P}$  and  $1 < p_- \leq p_+ < \infty$ , we have that  $p'(\cdot) \in M\mathcal{P}$  [3, Corollary 4.64]. Therefore, we can define a Rubio de Francia iteration algorithm [5, Section 2.1]:

$$Rg(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{2^k \|M\|_{p'(\cdot)}^k},$$

where  $\|M\|_{p'(\cdot)}$  is the operator norm of the maximal operator on  $L^{p'(\cdot)}$  and  $M^0 g = |g|$ . Then  $g$  and  $Rg$  are comparable in size:  $|g(x)| \leq Rg(x)$  and  $\|Rg\|_{p'(\cdot)} \leq 2\|g\|_{p'(\cdot)}$ . Moreover,  $Rg \in A_1$  and  $[Rg]_{A_1} \leq 2\|M\|_{p'(\cdot)}$ . Therefore, there exist  $C, \delta > 0$  such that given any cube  $Q$  and  $E \subset Q$ ,

$$\frac{Rg(E)}{Rg(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\delta.$$

Now by duality and Hölder's inequality, there exists  $g \in L^{p'(\cdot)}$ ,  $\|g\|_{p'(\cdot)} = 1$ , such that

$$\begin{aligned} \|\chi_E\|_{p(\cdot)} &\leq C \int_{\mathbb{R}^n} \chi_E(x) g(x) dx \leq C Rg(E) \leq C \left( \frac{|E|}{|Q|} \right)^\delta Rg(Q) \\ &\leq C \left( \frac{|E|}{|Q|} \right)^\delta \|\chi_Q\|_{p(\cdot)} \|Rg\|_{p'(\cdot)} \leq 2C \left( \frac{|E|}{|Q|} \right)^\delta \|\chi_Q\|_{p(\cdot)}. \end{aligned}$$

□



We can now prove Theorem 1.1. We first make some reductions, and then divide the proof into three propositions. First, we will do the proof for a single admissible wavelet  $\psi$ , since considering a family of  $L$  wavelets will only introduce an additional finite sum and make the constants depend on  $L$ .

Second, to prove Theorem 1.1 we need to estimate expressions of the form

$$\left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)}.$$

for any finite set  $\Gamma$  of dyadic cubes. By Theorem 2.1 we have

$$\|\psi_Q\|_{p(\cdot)} \approx \| |Q|^{-1/2} \chi_Q \|_{p(\cdot)} = |Q|^{-1/2} \|\chi_Q\|_{p(\cdot)}.$$

Thus, again by Theorem 2.1,

$$(3.1) \quad \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \approx \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{|Q|^{-1/2} \|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \approx \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}^2} \right)^{1/2} \right\|_{p(\cdot)}.$$

Therefore, it will be enough to show that the righthand expression satisfies the desired inequalities. It is illuminating at this point to consider the special case where the cubes in  $\Gamma$  are pairwise disjoint. With this as a model we will then obtain the desired estimate in the general case.

**Proposition 3.4.** *Given an exponent function  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . Then there exist constants such that given any collection  $\Gamma$  of pairwise disjoint dyadic cubes,  $\text{card}(\Gamma) = N$ ,*

$$N^{1/p_+} \lesssim \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}^2} \right)^{1/2} \right\|_{p(\cdot)} \lesssim N^{1/p_-}.$$

*Proof.* We will prove the first inequality; the second is proved in essentially the same way, replacing  $p_+$  by  $p_-$  and reversing the inequalities. Fix a collection  $\Gamma$  with  $\text{Card}(\Gamma) = N$ . Since the cubes in  $\Gamma$  are disjoint, we have that

$$\left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}^2} \right)^{1/2} \right\|_{p(\cdot)} = \left\| \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)}.$$

We now estimate as follows:

$$\int_{\mathbb{R}^n} \left( N^{-1/p_+} \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx$$

$$= \sum_{Q \in \Gamma} \int_Q N^{-p(x)/p_+} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx \geq N^{-1} \sum_{Q \in \Gamma} \int_Q \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx = 1;$$

the last inequality follows from (2.1). Therefore, by the definition of the  $L^{p(\cdot)}$  norm,

$$N^{1/p_+} \leq \left\| \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)}.$$

□

In general, the cubes in the collection  $\Gamma$  will not be disjoint. To overcome this, we will show that we can linearize the square function

$$S_\Gamma(x) := \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}^2} \right)^{1/2}.$$

Such linearization arguments were previously considered in [2, 17, 21]. Here, we will use the technique of “lighted” and “shaded” cubes introduced in [18].

**Proposition 3.5.** *Given an exponent function  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . Let  $\Gamma$  be any finite collection of dyadic cubes. Then there exists a sub-collection  $\Gamma_{\min} \subset \Gamma$  and a collection of pairwise disjoint sets  $\{\text{Light}(Q)\}_{Q \in \Gamma_{\min}}$ , such that  $\text{Light}(Q) \subset Q$  and*

$$S_\Gamma(x) \approx \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\|\chi_Q\|_{p(\cdot)}}.$$

*In these inequalities the constants are independent of the set  $\Gamma$ .*

*Proof.* Fix a finite collection  $\Gamma$  and let  $\Omega_\Gamma = \bigcup_{Q \in \Gamma} Q$ . For each  $x \in \Omega_\Gamma$ , let  $Q_x \in \Gamma$  be the unique smallest cube that contains  $x$ . We immediately have that for every  $x \in \Omega_\Gamma$ ,

$$S_\Gamma(x)^2 \geq \frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}^2}.$$

We claim that the reverse inequality holds up to a constant. Indeed, let

$$Q_x = Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots$$

be the sequence of all dyadic cubes that contain  $Q_x$ . Then  $|Q_j| = 2^{jn}|Q_0|$ , and by Lemma 3.3,

$$\frac{\|\chi_{Q_0}\|_{p(\cdot)}}{\|\chi_{Q_j}\|_{p(\cdot)}} \leq C \left( \frac{|Q_0|}{|Q_j|} \right)^\delta \leq C 2^{-jn\delta}.$$

Hence,

$$S_\Gamma(x)^2 \leq \sum_{j=0}^{\infty} \frac{1}{\|\chi_{Q_j}\|_{p(\cdot)}^2} \leq \frac{C}{\|\chi_{Q_x}\|_{p(\cdot)}^2} \sum_{j=0}^{\infty} 2^{-2jn\delta} = C \frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}^2}.$$

This gives us the pointwise equivalence

$$(3.2) \quad S_\Gamma(x) \approx \frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}}.$$

Let  $\Gamma_{\min} = \{Q_x : x \in \Omega_\Gamma\}$ ; note that the cubes in  $\Gamma_{\min}$  may still not be pairwise disjoint. To obtain a disjoint family we argue as in [18]. Given  $Q \in \Gamma$ , let  $\text{Shade}(Q) = \bigcup \{R : R \in \Gamma, R \subsetneq Q\}$  and  $\text{Light}(Q) = Q \setminus \text{Shade}(Q)$ . Then (see [18, Section 4.2.2]) we have that  $Q \in \Gamma_{\min}$  if and only if  $\text{Light}(Q) \neq \emptyset$ ,  $x \in \text{Light}(Q_x)$ , the sets  $\text{Light}(Q)$  are pairwise disjoint, and

$$\bigcup_{Q \in \Gamma} Q = \bigcup_{Q \in \Gamma_{\min}} \text{Light}(Q).$$

If we combine this analysis with (3.2) we get

$$(3.3) \quad S_\Gamma(x) \approx \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\|\chi_Q\|_{p(\cdot)}},$$

where in the righthand sum there is at most one non-zero term for any  $x \in \Omega_\Gamma$ .  $\square$

We can now estimate the square function  $S_\Gamma$  for an arbitrary finite set of dyadic cubes  $\Gamma$ .

**Proposition 3.6.** *Given an exponent function  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . If  $\Gamma$  is a finite set of dyadic cubes,  $\text{card}(\Gamma) = N$ , then*

$$N^{1/p_+} \lesssim \|S_\Gamma\|_{p(\cdot)} \lesssim N^{1/p_-}.$$

*Proof.* By Proposition 3.5, to prove the righthand inequality it suffices to show that

$$\left\| \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq N^{1/p_-}.$$

By the definition of the  $L^{p(\cdot)}$  norm, this follows from the fact that

$$\begin{aligned} \int_{\mathbb{R}^n} \left( N^{-1/p_-} \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx &= \sum_{Q \in \Gamma_{\min}} \int_{\text{Light}(Q)} N^{-p(x)/p_-} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx \\ &\leq \frac{1}{N} \sum_{Q \in \Gamma_{\min}} \int_Q \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx = \frac{1}{N} \text{card}(\Gamma_{\min}) \leq 1, \end{aligned}$$

where we have used that the sets  $\text{Light}(Q)$  are disjoint,  $p(x) \geq p_-$  and (2.1).

We now prove the lefthand inequality; again by Proposition 3.5 it suffices to show that

$$\left\| \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \geq CN^{1/p^+}.$$

where  $C = M_0^{-1}2^{-n} \left(\frac{2^n-1}{2^n}\right)^{1/p_-}$  and  $M_0$  is the norm of the maximal operator on  $L^{p(\cdot)}$ . In fact, we will prove this inequality with  $\Gamma_{\min}$  replaced by a sub-collection  $\Gamma_L$ .

Given a cube  $Q \in \Gamma$ , we say  $Q$  is *lighted* if  $|\text{Light}(Q)| \geq |Q|/2^n$ . Let  $\Gamma_L$  be the collection of lighted cubes. Observe that  $\Gamma_L \subset \Gamma_{\min}$ . As was proved in [18, Lemma 4.3], for every finite set  $\Gamma$  of dyadic cubes,

$$\frac{2^n - 1}{2^n} \text{card}(\Gamma) \leq \text{card}(\Gamma_L) \leq \text{card}(\Gamma_{\min}) \leq \text{card}(\Gamma).$$

Hence, by Lemma 3.2, if  $Q \in \Gamma_L$ , then

$$\frac{\|\chi_{\text{Light}(Q)}\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \geq \frac{1}{M_0} \frac{|\text{Light}(Q)|}{|Q|} \geq \frac{1}{2^n M_0}.$$

We can now estimate as follows: since  $p_- \leq p(x) \leq p_+$ , the sets  $\text{Light}(Q)$ ,  $Q \in \Gamma_L$ , are disjoint and (2.1),

$$\begin{aligned} & \int_{\mathbb{R}^n} \left( C^{-1} N^{-1/p_+} \sum_{Q \in \Gamma_L} \frac{\chi_{\text{Light}(Q)}(x)}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx \\ &= \sum_{Q \in \Gamma_L} \int_{\text{Light}(Q)} C^{-p(x)} N^{-p(x)/p_+} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx \\ &\geq \frac{1}{N} \sum_{Q \in \Gamma_L} \int_{\text{Light}(Q)} C^{-p(x)} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx \\ &\geq \frac{1}{N} \sum_{Q \in \Gamma_L} \int_{\text{Light}(Q)} C^{-p(x)} \frac{1}{(2^n M_0)^{p(x)}} \|\chi_{\text{Light}(Q)}\|_{p(\cdot)}^{-p(x)} dx \\ &= \frac{1}{N} \sum_{Q \in \Gamma_L} \int_{\text{Light}(Q)} \left( \frac{2^n - 1}{2^n} \right)^{-p(x)/p_-} \|\chi_{\text{Light}(Q)}\|_{p(\cdot)}^{-p(x)} dx \\ &\geq \frac{2^n}{2^n - 1} \frac{1}{N} \sum_{Q \in \Gamma_L} \int_{\text{Light}(Q)} \|\chi_{\text{Light}(Q)}\|_{p(\cdot)}^{-p(x)} dx \\ &= \frac{2^n}{2^n - 1} \frac{1}{N} \text{card}(\Gamma_L) \\ &\geq \frac{1}{N} \text{card}(\Gamma) = 1. \end{aligned}$$

The desired inequality now follows by the definition of the  $L^{p(\cdot)}$  norm.  $\square$

To finish the proof of Theorem 1.1 we need to show that the bounds given in Proposition 3.6 are sharp. This is an immediate consequence of the following result: since the constants in it are independent of  $\epsilon$ , we can let  $\epsilon \rightarrow 0$ .

**Proposition 3.7.** *Given an exponent function  $p(\cdot)$ , suppose  $1 < p_- \leq p_+ < \infty$  and the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ . Fix  $\epsilon > 0$  and  $N \in \mathbb{N}$ ; then there exists families  $\Gamma_1, \Gamma_2$ , of pairwise disjoint dyadic cubes such that*

$$\left\| \sum_{Q \in \Gamma_1} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \geq C_1 N^{\frac{1}{p_- + \epsilon}}$$

and

$$\left\| \sum_{Q \in \Gamma_2} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq C_2 N^{\frac{1}{p_+ - \epsilon}}.$$

Moreover, the constants  $C_1$  and  $C_2$  are independent of  $\epsilon$  and  $N$ .

*Proof.* We first construct  $\Gamma_1$ . Let  $G_\epsilon = \{x : p(x) \leq p_- + \epsilon\}$ . By the definition of  $p_-$ ,  $|G_\epsilon| > 0$ . Let  $x \in G_\epsilon$  be a Lebesgue point of the function  $\chi_{G_\epsilon}$ ; by the Lebesgue differentiation theorem, if  $\{Q_k\}$  is a sequence of dyadic cubes of decreasing side-length such that  $\bigcap_k Q_k = \{x\}$ , then

$$\lim_{k \rightarrow \infty} \int_{Q_k} \chi_{G_\epsilon}(x) dx = 1.$$

Therefore, we can find a dyadic cube  $Q_x$  containing  $x$  such that

$$(3.4) \quad \frac{|G_\epsilon \cap Q_x|}{|Q_x|} \geq \frac{1}{2}.$$

(Here, the choice of  $1/2$  is arbitrary: any constant  $0 < c < 1$  would suffice.) Moreover, we can choose the side-length of  $Q_x$  to be arbitrarily small.

By fixing  $N$  such Lebesgue points, we can form a family  $\Gamma_1$  of disjoint cubes  $Q$  such that  $|Q \cap G_\epsilon| > \frac{1}{2}|Q|$ . By Lemma 3.2,

$$(3.5) \quad \frac{\|\chi_{G_\epsilon \cap Q}\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \geq \frac{1}{2M_0}.$$

(Again,  $M_0$  is the bound of the maximal operator on  $L^{p(\cdot)}$ .)

We can now estimate as follows: since the cubes in  $\Gamma_1$  are disjoint and using (3.5),

$$\int_{\mathbb{R}^n} \left( 2M_0 N^{\frac{-1}{p_- + \epsilon}} \sum_{Q \in \Gamma_1} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx = \sum_{Q \in \Gamma_1} \int_Q (2M_0)^{p(x)} N^{\frac{-p(x)}{p_- + \epsilon}} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx$$

$$\begin{aligned}
&\geq \sum_{Q \in \Gamma_1} \int_{G_\epsilon \cap Q} N^{\frac{-p(x)}{p_- + \epsilon}} \|\chi_{G_\epsilon \cap Q}\|_{p(\cdot)}^{-p(x)} dx \\
&\geq \sum_{Q \in \Gamma_1} N^{-1} \int_{G_\epsilon \cap Q} \|\chi_{G_\epsilon \cap Q}\|_{p(\cdot)}^{-p(x)} dx = 1.
\end{aligned}$$

In the second inequality we use the fact that  $p(x) \leq p_- + \epsilon$  a.e. in  $G_\epsilon$  and the last inequality follows from (2.1). By the definition of the norm, this gives us the first inequality with  $C_1 = 2M_0$ .

The construction of  $\Gamma_2$  is similar but requires a more careful selection of Lebesgue points. Let  $H_\epsilon = \{x : p(x) \geq p_+ - \epsilon\}$ ; again we have that  $|H_\epsilon| > 0$ . Let  $x$  be a Lebesgue point of the function  $\chi_{H_\epsilon}$  contained in the set  $H_{\epsilon/2}$  and also such that  $x$  is a Lebesgue point of the locally integrable function  $p(\cdot)^{-1}$ . Then by the Lebesgue differentiation theorem we can find an arbitrarily small dyadic cube  $Q$  containing  $x$  such that

$$(3.6) \quad \frac{|H_\epsilon \cap Q|}{|Q|} > 1 - \frac{1}{2N}.$$

Moreover, since (again by the Lebesgue differentiation theorem)

$$\oint_Q \frac{1}{p(y)} dy \rightarrow \frac{1}{p(x)} \leq \frac{1}{p_+ - \epsilon/2},$$

we may also choose  $Q$  so small that

$$(3.7) \quad \frac{1}{p_Q} = \oint_Q \frac{1}{p(y)} dy < \frac{1}{p_+ - \epsilon}.$$

Finally, choose  $N$  such Lebesgue points and take the cubes  $Q$  small enough that they are pairwise disjoint and so that  $|Q| \leq 1$ . This gives us our family  $\Gamma_2$ .

Fix a constant  $C_0 > 1$ ; the exact value will be determined below. We can now estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} \left( (2C_0N)^{\frac{-1}{p_+ - \epsilon}} \sum_{Q \in \Gamma_2} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx &= \sum_{Q \in \Gamma_2} \int_Q (2C_0N)^{\frac{-p(x)}{p_+ - \epsilon}} \|\chi_Q\|_{p(\cdot)}^{-p(x)} dx \\
&= \sum_{Q \in \Gamma_2} \int_{H_\epsilon \cap Q} + \int_{Q \setminus H_\epsilon} \\
&= I_1 + I_2.
\end{aligned}$$

The estimate for  $I_1$  is immediate: since  $p(x) \geq p_+ - \epsilon$  in  $H_\epsilon$  we have that

$$I_1 \leq \frac{1}{2C_0N} \sum_{Q \in \Gamma_2} \int_{H_\epsilon \cap Q} \|\chi_{H_\epsilon \cap Q}\|_{p(\cdot)}^{-p(x)} dx = \frac{1}{2C_0} < \frac{1}{2}.$$

To estimate  $I_2$ , note that by Lemma 3.1,  $\|\chi_Q\|_{p(\cdot)}^{-1} \leq 2|Q|^{-\frac{1}{p_Q}}$ . Then, since  $2N \geq 1$  and by the definitions of  $p_-$  and  $p_+$  we have that

$$I_2 \leq \sum_{Q \in \Gamma_2} \int_{Q \setminus H_\epsilon} (2N)^{\frac{-p(x)}{p_+ - \epsilon}} C_0^{\frac{-p(x)}{p_+ - \epsilon}} 2^{p(x)} |Q|^{\frac{-p(x)}{p_Q}} dx \leq 2^{p_+} C_0^{\frac{-p_-}{p_+ - \epsilon}} \sum_{Q \in \Gamma_2} \int_{Q \setminus H_\epsilon} |Q|^{\frac{-p(x)}{p_Q}} dx.$$

In  $Q \setminus H_\epsilon$ ,  $p(x) < p_+ - \epsilon < p_Q$  by (3.7). Thus,  $|Q|^{\frac{-p(x)}{p_Q}} < |Q|^{-1}$  since  $|Q| < 1$ . Furthermore, by (3.6) we have that  $\frac{|Q \setminus H_\epsilon|}{|Q|} < \frac{1}{2N}$ . Hence,

$$I_2 \leq 2^{p_+} C_0^{\frac{-p_-}{p_+ - \epsilon}} \sum_{Q \in \Gamma_2} \frac{|Q \setminus H_\epsilon|}{|Q|} \leq 2^{p_+} C_0^{\frac{-p_-}{p_+ - \epsilon}} \frac{1}{2N} N = \frac{1}{2},$$

where the last equality holds if we choose  $C_0$  such that

$$2^{p_+} = C_0^{\frac{p_-}{p_+ - \epsilon}}.$$

Since  $I_1 + I_2 \leq 1$ , again by the definition of the  $L^{p(\cdot)}$  norm we have that

$$\left\| \sum_{Q \in \Gamma_2} \frac{\chi_Q}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq (2C_0 N)^{\frac{1}{p_+ - \epsilon}} = 2^{\frac{1}{p_+ - \epsilon}} 2^{\frac{p_+}{p_-}} N^{\frac{1}{p_+ - \epsilon}}.$$

This completes the proof of Proposition 3.7 with  $C_2 = 2^{\frac{p_+}{p_-} + 1}$ .  $\square$

#### 4. WEIGHTED VARIABLE LEBESGUE SPACES

We begin with some preliminary definitions and results on weighted variable Lebesgue spaces. For proofs and further information, see [4]. Given an exponent  $p(\cdot)$  we say that a weight  $w \in A_{p(\cdot)}$  if

$$[w]_{A_{p(\cdot)}} = \sup_Q |Q|^{-1} \|w \chi_Q\|_{p(\cdot)} \|w^{-1} \chi_Q\|_{p'(\cdot)} < \infty.$$

This definition generalizes the Muckenhoupt  $A_p$  classes to the variable setting. We define the weighted variable Lebesgue space  $L^{p(\cdot)}(w)$  to the set of all measurable functions  $f$  such that  $\|fw\|_{p(\cdot)} < \infty$ .

If  $1 < p_- \leq p_+ < \infty$ ,  $p(\cdot)$  is log-Hölder continuous locally and at infinity, and  $w \in A_{p(\cdot)}$ , then the maximal operator is bounded on  $L^{p(\cdot)}(w)$ : there exists a constant  $C$  such that

$$\|(Mf)w\|_{p(\cdot)} \leq C \|fw\|_{p(\cdot)}.$$

Note that with these hypotheses, we have that  $p'(\cdot)$  is also log-Hölder continuous and  $w^{-1} \in A_{p'(\cdot)}$ ; thus, the maximal operator is also bounded on  $L^{p'(\cdot)}(w^{-1})$ . Because of this, we make the following definition: given an exponent  $p(\cdot)$  and a weight  $w \in A_{p(\cdot)}$ , we say that  $(p(\cdot), w)$  is an  $M$ -pair if the maximal operator is bounded on  $L^{p(\cdot)}(w)$  and  $L^{p'(\cdot)}(w^{-1})$ .

We can now state the analog of Theorem 1.1 for weighted variable Lebesgue spaces.

**Theorem 4.1.** *Given an exponent function  $p(\cdot)$ ,  $1 < p_- \leq p_+ < \infty$ , and a weight  $w \in A_{p(\cdot)}$ , suppose  $(p(\cdot), w)$  is an  $M$ -pair. Let  $\Psi$  be an admissible orthonormal wavelet family. The right and left democracy functions of  $\Psi$  in  $L^{p(\cdot)}(w)$  satisfy*

$$h_r(N) \approx N^{1/p_-}, \quad h_l(N) \approx N^{1/p_+}, \quad N = 1, 2, 3, \dots$$

*Proof.* The proof of Theorem 4.1 is nearly identical to the proof of Theorem 1.1: here we describe the changes.

First, we need the analog of Theorem 2.1 for the weighted variable Lebesgue spaces. Theorem 2.1 was proved in [5] using Rubio de Francia extrapolation in the scale of variable Lebesgue spaces. Extrapolation can also be used to prove norm inequalities in the weighted space  $L^{p(\cdot)}(w)$  provided that  $(p(\cdot), w)$  is an  $M$ -pair: this was proved recently in [6]. Therefore, the same proof as in [5] yields

$$(4.1) \quad \|(\mathcal{W}_\Psi f)w\|_{p(\cdot)} \approx \|fw\|_{p(\cdot)}.$$

We replace Lemma 3.1 with its weighted version:

$$(4.2) \quad W(Q)^{\frac{1}{p_{Q,w}}} \leq 2\|\chi_Q w\|_{p(\cdot)},$$

where we set  $W(x) = w(x)^{p(x)}$  and

$$\frac{1}{p_{Q,w}} = \frac{1}{W(Q)} \int_Q \frac{1}{p(x)} W(x) dx = \int_Q \frac{1}{p(x)} dW.$$

The proof follows that of the unweighted version replacing  $dx$  by  $dW$ . Before using Hölder's inequality we divide and multiply by  $w$  and at the last step we use that  $\|W(Q)^{-1/p'(\cdot)} W w^{-1}\|_{p'(\cdot)} = 1$  by (2.1).

The weighted versions of Lemmas 3.2 and 3.3 hold:

$$(4.3) \quad \frac{|E|}{|Q|} \leq M_w \frac{\|\chi_E w\|_{p(\cdot)}}{\|\chi_Q w\|_{p(\cdot)}}$$

$$(4.4) \quad \frac{\|\chi_E w\|_{p(\cdot)}}{\|\chi_Q w\|_{p(\cdot)}} \leq C \left( \frac{|E|}{|Q|} \right)^\delta,$$

where  $M_w$  is the norm of the maximal operator on  $L^{p(\cdot)}(w)$ . The proofs follow the same steps, using the fact that since  $(p(\cdot), w)$  is an  $M$ -pair, the maximal operator is bounded on  $L^{p(\cdot)}(w)$  and  $L^{p'(\cdot)}(w^{-1})$ . In the proof of (4.4) the following changes are required. First, we construct the Rubio de Francia iteration algorithm using the norm of the maximal operator on  $L^{p(\cdot)}(w^{-1})$  so that  $\|(Rg)w^{-1}\|_{p(\cdot)'} \leq 2\|g w^{-1}\|_{p(\cdot)'}$ . Second, we replace  $Rg$  by  $R(gw)$ . Third, before applying Hölder's inequality we multiply and divide by  $w$ .

To modify the proof of Theorem 4.1 proper we use (4.1) to replace (3.1) with



$$\begin{aligned}
(4.5) \quad & \left\| \sum_{Q \in \Gamma} \frac{\psi_Q w}{\|\psi_Q w\|_{p(\cdot)}} \right\|_{p(\cdot)} \\
& \approx \left\| \sum_{Q \in \Gamma} \frac{\psi_Q w}{|Q|^{-1/2} \|\chi_Q w\|_{p(\cdot)}} \right\|_{p(\cdot)} \approx \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q w\|_{p(\cdot)}^2} \right)^{1/2} w \right\|_{p(\cdot)}.
\end{aligned}$$

The proof of the weighted version of Proposition 3.4 is exactly the same, replacing  $dx$  by  $dW$  and using the fact that by (2.1), for any set  $E$ ,

$$(4.6) \quad \int_E \|\chi_E w\|_{p(\cdot)}^{-p(x)} dW = \int_E \left( \frac{w(x)}{\|\chi_E w\|_{p(\cdot)}} \right)^{p(x)} dx = 1.$$

The linearization estimate in Proposition 3.5 is the same, but defining

$$S_\Gamma(x) = \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\|\chi_Q w\|_{p(\cdot)}^2} \right)^{1/2},$$

replacing  $dx$  by  $dW$  and using (4.4) instead of Lemma 3.3. The proof of Proposition 3.6 is the same, replacing  $dx$  by  $dW$  and Lemma 3.2 by (4.3) and using (4.6): the properties of lighted and shaded cubes are geometric and so remain unchanged.

Finally, the proof of Proposition 3.7 requires the following changes. We construct  $\Gamma_1$  much as before (in particular the Lebesgue differentiation theorem is used in exactly the same manner). The proof then proceeds the same way with  $dW$  in place of  $dx$ , with (4.3) replacing Lemma 3.2 and by using at the end (4.6). To construct  $\Gamma_2$  we consider the same set  $H_\epsilon$  but now the Lebesgue differentiation theorem is applied to  $p_{Q,w}$  with respect to the measure  $dW$  and for dyadic cubes. (Recall that the dyadic Hardy-Littlewood maximal function defined with respect to the measure  $dW$  is of weak-type  $(1, 1)$  with respect to  $dW$  since  $0 < W(Q) < \infty$  for every dyadic cube  $Q$ .) In particular we obtain  $W(H_\epsilon \cap Q)/W(Q) > 1 - (2N)^{-1}$  and  $1/p_{Q,w} < (p_+ - \epsilon)^{-1}$  which we use to replace (3.6) and (3.7), respectively. Also, the cubes  $Q$  are taken so small that  $W(Q) \leq 1$ . Given these changes the remainder of the proof is the same *mutatis mutandis*, replacing  $dx$  by  $dW$ ,  $p_Q$  by  $p_{Q,w}$ , Lemma 3.1 by (4.2), and using again (4.6).  $\square$

## 5. VARIABLE EXPONENT TRIEBEL-LIZORKIN SPACES

The theory of (nonhomogeneous) Triebel-Lizorkin spaces with variable exponents has been developed by Diening, *et al.* [10] and Kempka [25, 26]. (Also see Xu [36].) We refer the reader to these papers for complete information. Here, we sketch the essentials.

Let  $\mathcal{P}_0$  be the set of all measurable exponent functions  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ . Then with the same definitions and notation as used above, we can define the spaces  $L^{p(\cdot)}$ ;

if  $p_- < 1$ , then  $\|\cdot\|_{p(\cdot)}$  is a quasi-norm and  $L^{p(\cdot)}$  is a quasi-Banach space. The maximal operator will no longer be bounded on such spaces; a useful substitute is the assumption that there exists  $p_0$ ,  $0 < p_0 < p_-$ , such that the maximal operator is bounded on  $L^{p(\cdot)/p_0}(\mathbb{R}^n)$ . This is the case if, for instance, if  $0 < p_- \leq p_+ < \infty$  and  $p(\cdot)$  is log-Hölder continuous locally and at infinity. (For further information on these spaces, see [7], where they were used to define variable Hardy spaces.)

To define the variable exponent Triebel-Lizorkin spaces we need three exponent functions,  $p(\cdot)$ ,  $q(\cdot)$ , and  $s(\cdot)$ . We let  $p(\cdot), q(\cdot) \in \mathcal{P}_0$  be such that  $0 < p_- \leq p_+ < \infty$ ,  $0 < q_- \leq q_+ < \infty$ , and  $p(\cdot), q(\cdot)$  are log-Hölder continuous locally and at infinity (see Section 2). We assume that  $s(\cdot)$ , the “smoothness” parameter, is in  $L^\infty$  and is locally log-Hölder continuous. (We note that in [10] it was assumed that  $s_- \geq 0$ , but this hypothesis was removed in [25, 26].) Given these exponents, the nonhomogeneous variable exponent Triebel-Lizorkin space  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  is defined using an approximation of the identity on  $\mathbb{R}^n$ : for a precise definition, see [10, Definition 3.3] or [25, Section 4]. These spaces have many properties similar to those of the usual (constant exponent) Triebel-Lizorkin spaces. In particular, if  $1 < p_- \leq p_+ < \infty$ ,  $F_{p(\cdot), 2}^0(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ . For  $p_- > 0$ ,  $F_{p(\cdot), 2}^0(\mathbb{R}^n) = h^{p(\cdot)}(\mathbb{R}^n)$ , the local Hardy spaces with variable exponent introduced by Nakai and Sawano [31]. When  $s \geq 0$  is constant,  $F_{p(\cdot), 2}^s(\mathbb{R}^n) = \mathfrak{L}^{s, p(\cdot)}(\mathbb{R}^n)$ , the variable exponent Bessel potential spaces introduced in [1, 19]. When  $s \in \mathbb{N}$  these become the variable exponent Sobolev spaces,  $W^{s, p(\cdot)}$  (see [3, Chapter 6]).

A wavelet decomposition of variable exponent Triebel-Lizorkin spaces was proved in [26]. Let  $\mathcal{D}^+$  be the collection of all dyadic cubes  $Q$  such that  $|Q| \leq 1$ . Given an orthonormal wavelet family  $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$  with appropriate smoothness and zero-moment conditions (determined by the exponent functions  $p(\cdot), q(\cdot), s(\cdot)$ ) we have that  $f \in F_{p(\cdot), q(\cdot)}^{s(\cdot)}$  if and only if

$$(5.1) \quad f = \sum_{l=1}^L \sum_{Q \in \mathcal{D}^+} \langle f, \psi_Q^l \rangle \psi_Q^l,$$

and this series converges unconditionally in  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}$ . Moreover, if we define

$$\mathcal{W}_\Psi^{s(\cdot), q(\cdot)} f(x) = \left( \sum_{l=1}^L \sum_{Q \in \mathcal{D}^+} \left( |\langle f, \psi_Q^l \rangle| |Q|^{-\frac{s(x)}{n} - \frac{1}{2}} \chi_Q(x) \right)^{q(x)} \right)^{\frac{1}{q(x)}},$$

then

$$(5.2) \quad \|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} \approx \|\mathcal{W}_\Psi^{s(\cdot), q(\cdot)} f\|_{p(\cdot)}.$$

We want to stress that the above result is only known for the nonhomogeneous, variable exponent Triebel-Lizorkin spaces, and it remains an open problem to define and prove the basic properties of variable exponent Triebel-Lizorkin spaces in the homogeneous case. (See [10, Remark 2.4].) Nevertheless, we can define the space  $\dot{F}_{p(\cdot),q(\cdot)}^{s(\cdot)}$  with norm

$$(5.3) \quad \|f\|_{\dot{F}_{p(\cdot),q(\cdot)}^{s(\cdot)}} = \|\dot{\mathcal{W}}_{\Psi}^{s(\cdot),q(\cdot)} f\|_{p(\cdot)}.$$

where we define  $\dot{\mathcal{W}}_{\Psi}^{s(\cdot),q(\cdot)}$  exactly as in (5.1) except that the sum is taken over all  $Q \in \mathcal{D}$ .

The arguments given in Section 3 let us extend Theorem 1.1 to the variable exponent Triebel-Lizorkin spaces. We first consider the homogeneous case  $\dot{F}_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  with a constant smoothness parameter.

**Theorem 5.1.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0$  be two exponent functions that are log-Hölder continuous locally and at infinity and that satisfy  $0 < p_- \leq p^+ < \infty$ ,  $0 < q_- \leq q^+ < \infty$ . Let  $s \in \mathbb{R}$ . Suppose that  $\Psi$  is an orthonormal wavelet family with sufficient smoothness. Then the right and left democracy functions of  $\Psi$  in  $\dot{F}_{p(\cdot),q(\cdot)}^s(\mathbb{R}^n)$  satisfy*

$$h_r(N) \approx N^{1/p_-}, \quad h_l(N) \approx N^{1/p_+}, \quad N = 1, 2, 3, \dots$$

*Proof.* To modify the proof of Theorem 1.1 we must first give variants of Lemmas 3.1, 3.2, and 3.3. Fix  $p_0$ ,  $0 < p_0 < p_-$ . Then, as we noted above, the maximal operator is bounded on  $L^{p(\cdot)/p_0}$ . Moreover, by a change of variable in the definition of the  $L^{p(\cdot)}$  norm, we have that for any set  $E \subset \mathbb{R}^n$  and  $\tau > 0$ ,

$$\|\chi_E\|_{p(\cdot)} = \|\chi_E^\tau\|_{p(\cdot)} = \|\chi_E\|_{\tau p(\cdot)}^\tau.$$

Therefore, if we apply Lemma 3.1 to the exponent  $p(\cdot)/p_0$ , we get that

$$(5.4) \quad |Q|^{\frac{p_0}{p_Q}} \leq 2\|\chi_Q\|_{p(\cdot)/p_0} = 2\|\chi_Q\|_{p(\cdot)}^{p_0}.$$

Similarly, we can conclude from Lemmas 3.2 and 3.3 that if  $E \subset Q$ , then

$$(5.5) \quad \frac{|E|}{|Q|} \leq M_0 \left( \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \right)^{p_0}$$

and

$$(5.6) \quad \left( \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \right)^{p_0} \leq C \left( \frac{|E|}{|Q|} \right)^\delta.$$

Turning to the proof proper, we may first assume, as in the proof of Theorem 1.1, that  $L = 1$ . We then need to prove the lower and upper bounds for

$$\left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{\dot{F}_{p(\cdot), q(\cdot)}^s}} \right\|_{\dot{F}_{p(\cdot), q(\cdot)}^s},$$

where  $\Gamma$  is a finite set of dyadic cubes with  $\text{card}(\Gamma) = N$ . By (5.3),

$$(5.7) \quad \|\psi_Q\|_{\dot{F}_{p(\cdot), q(\cdot)}^s} = |Q|^{-\frac{s}{n} - \frac{1}{2}} \|\chi_Q\|_{p(\cdot)},$$

and so

$$(5.8) \quad \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{\dot{F}_{p(\cdot), q(\cdot)}^s}} \right\|_{\dot{F}_{p(\cdot), q(\cdot)}^s} = \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{|Q|^{-\frac{s}{n} - \frac{1}{2}} \|\chi_Q\|_{p(\cdot)}} \right\|_{\dot{F}_{p(\cdot), q(\cdot)}^s} = \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}^{q(x)}} \right)^{1/q(x)} \right\|_{p(\cdot)}.$$

When the cubes in  $\Gamma$  are pairwise disjoint, the proof of Proposition 3.4 is unchanged. To modify the proof of Proposition 3.5, define

$$S_\Gamma^{p(\cdot), q(\cdot)}(x) = \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}^{q(x)}} \right)^{1/q(x)}.$$

Then

$$(5.9) \quad \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{\dot{F}_{p(\cdot), q(\cdot)}^s}} \right\|_{\dot{F}_{p(\cdot), q(\cdot)}^s} = \|S_\Gamma^{p(\cdot), q(\cdot)}\|_{p(\cdot)}.$$

With the same notation as before, we clearly have that

$$\frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}} \leq S_\Gamma^{p(\cdot), q(\cdot)}(x).$$

We prove the opposite inequality almost as before, using (5.6) instead of Lemma 3.3:

$$\begin{aligned} S_\Gamma^{p(\cdot), q(\cdot)}(x) &\leq \left( \sum_{Q \supset Q_x} \frac{1}{\|\chi_Q\|_{p(\cdot)}^{q(x)}} \right)^{1/q(x)} \\ &\leq \left( \sum_{j=0}^{\infty} \frac{C}{\|\chi_{Q_x}\|_{p(\cdot)}^{q(x)}} 2^{-jn\delta q(x)/p_0} \right)^{1/q(x)} = C \frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}}; \end{aligned}$$

in the last inequality we use the fact that  $q(x) \geq q_- > 0$ . Therefore,

$$S_\Gamma^{p(\cdot), q(\cdot)}(x) \approx \frac{\chi_{Q_x}(x)}{\|\chi_{Q_x}\|_{p(\cdot)}}.$$

From here, the proof of Theorem 5.1 is exactly the same as that of Theorem 1.1: the proofs of Propositions 3.6 and 3.7 are the same since  $S_\Gamma(x) \approx S_\Gamma^{p(\cdot), q(\cdot)}(x)$ . We only note that because we use (5.4) in place of Lemma 3.1 and (5.5) instead of Lemma 3.2, some of the constants which appear must be adjusted to account for the exponent  $p_0$ .  $\square$

In the nonhomogeneous case we may take  $s(\cdot)$  to be variable.

**Theorem 5.2.** *Let  $p(\cdot), q(\cdot) \in \mathcal{P}_0$  be two exponent functions that are log-Hölder continuous locally and at infinity and that satisfy  $0 < p_- \leq p^+ < \infty$ ,  $0 < q_- \leq q^+ < \infty$ . Let  $s(\cdot) \in L^\infty$  be locally log-Hölder continuous. Suppose that  $\Psi$  is an orthonormal wavelet family with sufficient smoothness (i.e., so that (5.1) and (5.2) hold). Then the right and left democracy functions of  $\Psi$  in  $F_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$  satisfy*

$$h_r(N) \approx N^{1/p_-}, \quad h_l(N) \approx N^{1/p_+}, \quad N = 1, 2, 3, \dots$$

*Proof.* The proof is nearly identical to the proof of Theorem 5.1. The key difference is in equalities (5.7) and (5.8). In (5.7) we used the fact that  $s$  was constant in order to pull the term  $|Q|^{-\frac{s}{n}-\frac{1}{2}}$  out of the  $L^{p(\cdot)}$  norm. We can no longer do this if  $s(\cdot)$  is a function.

However, we can use local log-Hölder continuity and the fact that  $|Q| \leq 1$  to prove the analog of (5.8). A very important consequence of the log-Hölder continuity of  $s(\cdot)$  is that there exists  $C > 1$  such that for every cube  $Q$ ,

$$|Q|^{s_-(Q)-s_+(Q)} \leq C.$$

(See [3, Lemma 3.24].) In particular, for any  $x \in Q$  with  $|Q| \leq 1$ ,

$$\begin{aligned} |Q|^{-s(x)} &= |Q|^{-s(x)+s_-(Q)} |Q|^{-s_-(Q)} \leq C |Q|^{-s_-(Q)}, \\ |Q|^{-s(x)} &= |Q|^{-s(x)+s_+(Q)} |Q|^{-s_+(Q)} \geq C^{-1} |Q|^{-s_+(Q)}. \end{aligned}$$

Therefore, by (5.2) we have that

$$\|\psi_Q\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} \approx \| |Q|^{-\frac{s(\cdot)}{n}-\frac{1}{2}} \chi_Q \|_{p(\cdot)} \lesssim |Q|^{-\frac{s_-(Q)}{n}-\frac{1}{2}} \|\chi_Q\|_{p(\cdot)}$$

and

$$\|\psi_Q\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} \gtrsim |Q|^{-\frac{s_+(Q)}{n}-\frac{1}{2}} \|\chi_Q\|_{p(\cdot)}.$$

Moreover, because every  $Q \in \Gamma$  is such that  $|Q| \leq 1$ , we have that

$$\begin{aligned} \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}}} \right\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} &\lesssim \left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{|Q|^{-\frac{s_+(Q)}{n}-\frac{1}{2}} \|\chi_Q\|_{p(\cdot)}} \right\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} \\ &\lesssim \left\| \left( \sum_{Q \in \Gamma} \left[ \frac{|Q|^{-\frac{s(x)}{n}-\frac{1}{2}} \chi_Q(x)}{|Q|^{-\frac{s_+(Q)}{n}-\frac{1}{2}} \|\chi_Q\|_{p(\cdot)}} \right]^{q(x)} \right)^{1/q(x)} \right\|_{p(\cdot)} \lesssim \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}^{q(x)}} \right)^{1/q(x)} \right\|_{p(\cdot)}. \end{aligned}$$

In the same way, and again using strongly that  $|Q| \leq 1$ , we have

$$\left\| \sum_{Q \in \Gamma} \frac{\psi_Q}{\|\psi_Q\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}}} \right\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot)}} \gtrsim \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{\|\chi_Q\|_{p(\cdot)}^{q(x)}} \right)^{1/q(x)} \right\|_{p(\cdot)}.$$

Given this equivalence, the proof now continues exactly as in the proof of Theorem 5.1.  $\square$

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