Indiana Univ. Math. J. 57 (2008), no. 4, 1845--1870

WEIGHTED NORM INEQUALITIES FOR FRACTIONAL OPERATORS

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Abstract. We prove weighted norm inequalities for fractional powers of elliptic operators together with their commutators with BMO functions, encompassing what is known for the classical Riesz potentials and elliptic operators with Gaussian domination by the classical heat operator. The method relies upon a good-λ method that does not use any size or smoothness estimates for the kernels.

1. Introduction

In [MW] Muckenhoupt and Wheeden resolve the one-weight problem for the classical fractional integrals $I_\alpha = (-\Delta)^{-\alpha/2}$ and fractional maximal operators M_α in $\mathbb{R}^n$ defined by

$$M_\alpha f(x) = \sup r(B)^\alpha \fint_B |f(y)|\, dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ that contain x.

Theorem 1.1 ([MW])**.** *Let w be a weight. Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ and $q = \frac{n\,p}{n-\alpha\,p}$, that is, $1/p - 1/q = \alpha/n$. If $p > 1$, M_α is bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if $w \in A_{p,q}$. If $p = 1$, M_α is bounded from $L^1(w)$ to $L^{q,\infty}(w^q)$ if and only if $w \in A_{1,q}$. Furthermore, the same estimates for the Riesz potential I_α are characterized by the classes $A_{p,q}$.*

The class $A_{p,q}$, whose definition is recalled below, can be equivalently written as $A_{1+1/p'} \cap RH_q$ where A_p and RH_q are the standard Muckenhoupt and reverse Hölder classes. These two operators have intimate relations and the estimates for I_α follow from the ones for M_α. First there is a pointwise domination $M_\alpha f \lesssim I_\alpha(|f|)$ and second, although the pointwise converse does not hold, by means of a good-λ inequality, one has for all $0 < p < \infty$ and $w \in A_\infty$:

$$\int_{\mathbb{R}^n} |I_\alpha f|^p\, w\, dx \lesssim \int_{\mathbb{R}^n} (M_\alpha f)^p\, w\, dx \tag{1.1}$$

Date: March 19, 2007. *Revised*: February 4, 2008.

2000 *Mathematics Subject Classification.* 42B25, 35J15.

Key words and phrases. Muckenhoupt weights, elliptic operators in divergence form, fractional operators, commutators with bounded mean oscillation functions, good-λ inequalities.

This work was partially supported by the European Union (IHP Network "Harmonic Analysis and Related Problems" 2002-2006, Contract HPRN-CT-2001-00273-HARP). Part of this work was carried out while the first author was visiting the Universidad Autónoma de Madrid as a participant of the Centre de Recerca Matemàtica research thematic term "Fourier analysis, geometric measure theory and applications". The second author was also supported by MEC "Programa Ramón y Cajal, 2005", by MEC Grant MTM2007-60952, and by UAM-CM Grant CCG07-UAM/ESP-1664. We warmly thank the anonymous referee for the suggestions that enhanced the presentation of this article.

and also its corresponding $L^{1,\infty} - L^{1,\infty}$ version.

Different authors have studied the commutators of the fractional integrals with BMO functions. Unweighted estimates were considered in [Cha] and the weighted estimates were established in [ST] by means of extrapolation. Another proof based on the sharp maximal function was given in [CF].

Here, we consider operators with the same scaling properties as fractional integrals but which may not be representable by kernels with good estimates and that we call fractional operators. We wish to generalize the part of the theorem concerning I_α but a direct comparison to M_α will not work because we will have a limited range of α. Hence, we are looking for some other technique which could also provide another proof of the sufficiency part of Muckenhoupt-Wheeden theorem for I_α.

Our main example is the fractional power of an elliptic operator L on $\mathbb{R}^n$, given formally by

$$L^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2}\, e^{-t\,L}\, \frac{dt}{t},$$

with $\alpha > 0$ and $Lf = -\operatorname{div}(A\,\nabla f)$, where A is an elliptic $n\times n$ matrix of complex and L^∞-valued coefficients (see Section 3.1 for the precise definition). The operator $-L$ generates a C^0-semigroup $\{e^{-t\,L}\}_{t>0}$ of contractions on $L^2(dx) = L^2(\mathbb{R}^n, dx)$. There exist $p_- = p_-(L)$ and $p_+ = p_+(L)$, $1 \le p_- < 2 < p_+ \le \infty$ such that the semigroup $\{e^{-t\,L}\}_{t>0}$ is uniformly bounded on $L^p(dx)$ for every $p_- < p < p_+$ (see Proposition 3.1 below). The unweighted estimate states as follows.

Theorem 1.2 ([Aus]). *Let $p_- < p < q < p_+$ and $\alpha/n = 1/p - 1/q$. Then $L^{-\alpha/2}$ is bounded from $L^p(dx)$ to $L^q(dx)$.*

Let us observe that the range of α's in Theorem 1.2 is $0 < \alpha < n/p_- - n/p_+$. By (e) in Proposition 3.1 below, for $n = 1$ or $n = 2$ or when L has real coefficients, we have pointwise Gaussian domination of the semigroup, hence $0 < \alpha < n$. In general, by (f) in Proposition 3.1 the range of α's always contains the interval $(0, 2]$.

Our first main result in this paper gives sufficient conditions for the weighted norm inequalities of $L^{-\alpha/2}$.

Theorem 1.3. *Let $p_- < p < q < p_+$ and $\alpha/n = 1/p - 1/q$. Then $L^{-\alpha/2}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $w \in A_{1+\frac{1}{p_-}-\frac{1}{p}} \cap RH_{q\,(\frac{p_+}{q})'}$.*

Notice that if $p_- = 1$ and $p_+ = \infty$ (for instance, when $L = -\Delta$ or under Gaussian domination), then the condition on w becomes $w \in A_{1+1/p'} \cap RH_q$, that is, $w \in A_{p,q}$ (see Proposition 2.1), and our result agrees with that by Muckenhoupt and Wheeden (see Theorem 1.1).

We also obtain estimates for commutators with bounded mean oscillation functions: Let $b \in \mathrm{BMO}$, that is, $\|b\|_{\mathrm{BMO}} = \sup_B \fint_B |b(x) - b_B|\,dx < \infty$, where the supremum is taken over all balls and b_B stands for the average of b on B. Given $f \in L^\infty_c(dx)$, set $(L^{-\alpha/2})^0_b f = L^{-\alpha/2} f$, and for $k \ge 1$, the k-th order commutator

$$(L^{-\alpha/2})^k_b f(x) = L^{-\alpha/2}\big((b(x) - b)^k\, f\big)(x).$$

These commutators can be also defined by recurrence: $(L^{-\alpha/2})^k_b = [b, (L^{-\alpha/2})^{k-1}_b]$ where $[b, T]f(x) = b(x)\,Tf(x) - T(b\,f)(x)$.

We obtain the following weighted estimates:

Theorem 1.4. *Let $p_- < p < q < p_+$ and $\alpha/n = 1/p - 1/q$. Given $k \in \mathbb{N}$, $b \in \mathrm{BMO}$ and $w \in A_{1+\frac{1}{p_-}-\frac{1}{p}} \cap RH_{q\,(\frac{p_+}{q})'}$ we have*

$$\|(L^{-\alpha/2})^k_b f\|_{L^q(w^q)} \le C\,\|b\|^k_{\mathrm{BMO}}\,\|f\|_{L^p(w^p)}.$$

In the particular case $k = 1$ and under Gaussian kernel bounds (as in (e) of Proposition 3.1 below) the unweighted estimates were studied in [DY] using the sharp maximal function introduced in [Mar]. A simpler proof, that also yields the weighted estimates, was obtained in [CMP]: a discretization method inspired by [Pe2] allows the authors to extend (1.1) to $(L^{-\alpha/2})^k_b f$ which is controlled by $M_{L\log L,\alpha} f$ (see the definition below) and then use the weighted estimates for the latter which are studied in [CF].

Theorems 1.3 and 1.4 will be proved in Section 3. They depend on a general statement (Theorem 2.2), interesting on its own, based itself upon a good-λ method in [AM1] developed for operators with the same scaling properties as singular integral operators. This was used in [AM3] for the same class of elliptic operators and also for the Riesz transforms on Riemannian manifolds in [AM4], and we shall see that the very same tools apply as well for fractional operators.

In Section 4 we present a variant of Theorem 2.2 extending earlier results from [AM1] and [She] to the context of fractional operators.

While the good-λ method in [AM1] is valid in all spaces of homogeneous type, the application to fractional operators can be adapted only to those spaces with polynomial growth from below. We comment on this in Section 5.

2. Weighted estimates for general operators

We introduce some notation and recall known facts on weights. We work in $\mathbb{R}^n$. Given a ball B, we write

$$\fint_B h\,dx = \frac{1}{|B|}\int_B h(x)\,dx.$$

2.1. **Muckenhoupt Weights.** Let w be a weight (that is, a non negative locally integrable function) on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\Big(\fint_B w\,dx\Big)\Big(\fint_B w^{1-p'}\,dx\Big)^{p-1} \le C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\fint_B w\,dx \le C\,w(y), \qquad \text{for a.e. } y \in B.$$

Finally, $A_\infty = \cup_{p\ge1} A_p$.

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for any ball B,

$$\Big(\fint_B w^q\,dx\Big)^{\frac{1}{q}} \le C\fint_B w\,dx.$$

The endpoint $q = \infty$ is given by the condition $w \in RH_\infty$ whenever there is a constant C such that for any ball B,

$$w(y) \le C\fint_B w\,dx, \qquad \text{for a.e. } y \in B.$$

We introduce the classes $A_{p,q}$ that characterize the weighted estimates for the fractional operators (see Theorem 1.1). Given $1 \le p \le q < \infty$ we say that $w \in A_{p,q}$ if there exists a constant C such that every ball $B \subset \mathbb{R}^n$,

$$\Big(\fint_B w^q\,dx\Big)^{\frac{1}{q}} \Big(\fint_B w^{-p'}\,dx\Big)^{\frac{1}{p'}} \le C,$$

when $1 < p < \infty$, and

$$\Big(\fint_B w^q\,dx\Big)^{\frac{1}{q}} \le C\,w(x), \qquad \text{for a.e. } x \in B,$$

when $p = 1$.

We summarize some properties about weights (see [GR], [Gra] and [JN]).

Proposition 2.1.

(i) $A_1 \subset A_p \subset A_q$ *for* $1 \le p \le q < \infty$.

(ii) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \le q \le \infty$.

(iii) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.

(iv) *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.

(v) $A_\infty = \bigcup_{1 \le p < \infty} A_p = \bigcup_{1 < q \le \infty} RH_q$

(vi) *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.

(vii) *If* $1 \le p \le \infty$ *and* $1 < q < \infty$, *then* $w \in A_p \cap RH_q$ *if and only if* $w^q \in A_{q\,(p-1)+1}$.

(viii) *If* $1 \le p \le q < \infty$, *then* $w \in A_{p,q}$ *if and only if* $w^q \in A_{1+q/p'}$ *if and only if* $w \in A_{1+1/p'} \cap RH_q$.

(ix) *If* $1 \le p < q < \infty$ *and* $\alpha/n = 1/p - 1/q$ *then* $w \in A_{p,q}$ *if and only if* $w^q \in A_{q/1^*_\alpha}$ *where* $1^*_\alpha = n/(n-\alpha)$.

2.2. **The general statement.** Our main statement is based on unweighted estimates relating the fractional operators and their commutators with the corresponding fractional maximal functions.

We introduce some notation in order to state our general result in a way that is valid also for sublinear operators. Given a sublinear operator T and $b \in \mathrm{BMO}$, for any $k \in \mathbb{N}$ we define the k-th order commutator as

$$T_b^k f(x) = T\big((b(x) - b)^k f\big)(x), \qquad f \in L^\infty_c(dx), \qquad x \in \mathbb{R}^n.$$

Note that $T_b^0 = T$. We claim that if T is bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$ for some $1 \le p_0 \le s_0 \le \infty$ then $T_b^k f$ is well defined in L^q_{loc} for any $0 < q < s_0$ and for any $f \in L^\infty_c(dx)$: take a cube Q containing the support of f and observe that by sublinearity, for a.e. $x \in \mathbb{R}^n$,

$$|T_b^k f(x)| \le \sum_{m=0}^{k} C_{m,k}\,|b(x) - b_Q|^{k-m}\,\big|T\big((b - b_Q)^m f\big)(x)\big|.$$

John-Nirenberg's inequality implies

$$\int_Q |b(y) - b_Q|^{m\,p_0}\,|f(y)|^{p_0}\,dy \le C\|f\|_{L^\infty}\,\|b\|_{\mathrm{BMO}}^{m\,p_0}\,|Q| < +\infty.$$

Hence, $T\big((b-b_Q)^m f\big) \in L^{s_0}(dx)$ and the claim follows by using again John-Nirenberg's inequality.

Theorem 2.2. *Let $0<\alpha<n$, $1\le p_0<s_0<q_0\le\infty$ such that $1/p_0-1/s_0=\alpha/n$. Suppose that T is a sublinear operator bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$ and that $\{\mathcal{A}_r\}_{r>0}$ is a family of operators acting from $L^\infty_c(dx)$ into $L^{p_0}(dx)$. Assume that*

$$\Big(\fint_B |T(I-\mathcal{A}_{r(B)})f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \le \sum_{j=1}^\infty \alpha_j\, r(2^{j+1}B)^\alpha \Big(\fint_{2^{j+1}B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}, \tag{2.1}$$

and

$$\Big(\fint_B |T\mathcal{A}_{r(B)}f|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \le \sum_{j=1}^\infty \alpha_j \Big(\fint_{2^{j+1}B}|Tf|^{s_0}\,dx\Big)^{\frac{1}{s_0}}, \tag{2.2}$$

for all $f\in L^\infty_c$ and all balls B, where $r(B)$ denotes the radius of B. Let $p_0<p<q<q_0$ be such that $1/p-1/q=\alpha/n$ and $w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_{q\,(\frac{q_0}{q})'}$.

(*a*) *If $\sum_{j\ge1}\alpha_j<\infty$ then T is bounded from $L^p(w^p)$ to $L^q(w^q)$.*

(*b*) *Given $k\in\mathbb{N}$ and $b\in \mathrm{BMO}$, if $\sum_{j\ge1} j^k\,\alpha_j<\infty$ then for every $f\in L^\infty_c(dx)$ we have*

$$\|T_b^k f\|_{L^q(w^q)} \le C\,\|b\|^k_{\mathrm{BMO}}\,\|f\|_{L^p(w^p)}. \tag{2.3}$$

The case $q_0=\infty$ is understood in the sense that the L^{q_0}-average in (2.2) is indeed an essential supremum. Thus, the condition for w turns out to be $w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_q$ for $p>p_0$. Similarly, if (2.2) is satisfied for all $q_0<\infty$ then the conclusions hold for all $p_0<p<\infty$ and $w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_q$.

Remark 2.3. In case (a) the hypotheses can be slightly relaxed. Namely, instead of (2.1) and (2.2), it suffices that

$$\Big(\fint_B |T(I-\mathcal{A}_{r(B)})f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \le C\,M_{\alpha\,p_0}\big(|f|^{p_0}\,dx\big)(x)^{\frac{1}{p_0}}, \qquad \forall\, x\in B, \tag{2.4}$$

$$\Big(\fint_B |T\mathcal{A}_{r(B)}f|^{q_0}\Big)^{\frac{1}{q_0}} \le C\,M\big(|Tf|^{s_0}\big)(x)^{\frac{1}{s_0}}, \qquad \forall\, x\in B. \tag{2.5}$$

It is clear that these estimates follow from (2.1) and (2.2) provided $\sum_j \alpha_j<\infty$. We prove (a) below (which corresponds to (b) with $k=0$) by using these weaker conditions. The proof also shows that the right hand side of (2.5) can be weakened to $C\,M\big(|Tf|^{s_0}\big)(x)^{\frac{1}{s_0}}+C\,M_{\alpha\,p_0}\big(|f|^{p_0}\big)(\bar{x})^{\frac{1}{p_0}}$ where $\bar{x}\in B$ is also arbitrary.

Remark 2.4. Equivalent ways to write the condition $w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_{q\,(\frac{q_0}{q})'}$ are $w^q\in A_{1+\frac{q/p_0}{(p/p_0)'}}\cap RH_{(q_0/q)'}$ or $w^q\in A_{q/(p_0)^*_\alpha}\cap RH_{(q_0/q)'}$ where $(p_0)^*_\alpha=n\,p_0/(n-\alpha\,p_0)$, or $w^{p_0}\in A_{p/p_0,q/p_0}$ and $w^q\in RH_{(q_0/q)'}$ (see ($viii$) and (ix) in Proposition 2.1). Note that when $p_0=1$ and $q_0=\infty$, then this reduces to $w\in A_{1+1/p'}\cap RH_q$ which is equivalent to $w\in A_{p,q}$ by ($viii$) in Proposition 2.1.

Remark 2.5. In the limiting case $\alpha=0$, this result corresponds to a special case of [AM1, Theorems 3.7 and 3.16]. In such a case, we have $p=q$ and the weight w^q turns out to be in $A_{p/p_0}\cap RH_{(q_0/p)'}$. This condition arises naturally when proving weighted norm inequalities for operators with the same scaling properties as singular integral operators —such as those appearing in the functional calculus associated with L, see [AM3]— whose range of unweighted L^p boundedness is (p_0, q_0). Also, these classes of weights admit a variant of the Rubio de Francia extrapolation theorem that is valid for the limited range of exponents (p_0, q_0), see [AM1] .

2.3. **The technical result.** The proof of Theorem 2.2 is a consequence of the following result which appears in a more general form in [AM1] and is based on a two-parameter good-λ inequality.

Theorem 2.6 ([AM1]). *Fix $1 < r \le \infty$, $a \ge 1$ and $w \in RH_{s'}$, $1 \le s < \infty$. Let $1 < p < \frac{r}{s}$. Assume that $F \in L^1(dx)$, G, H_1 and H_2 are non-negative measurable functions on $\mathbb{R}^n$ such that for any cube Q there exist non-negative functions G_Q and H_Q with $F(x) \le G_Q(x) + H_Q(x)$ for a.e. $x \in Q$ and*

$$\Big(\fint_Q H_Q^r\,dx\Big)^{\frac{1}{r}} \le a\,\big(MF(x) + MH_1(x) + H_2(\bar{x})\big), \qquad \forall\, x, \bar{x} \in Q; \tag{2.6}$$

and

$$\fint_Q G_Q\,dx \le G(x), \qquad \forall\, x \in Q. \tag{2.7}$$

Then, there exists a constant $C = C(p, r, n, a, w, s)$ such that

$$\|MF\|_{L^p(w)} \le C\,\big(\|G\|_{L^p(w)} + \|MH_1\|_{L^p(w)} + \|H_2\|_{L^p(w)}\big). \tag{2.8}$$

Note that the assumption $F \in L^1(dx)$ is not used quantitatively. The case $r = \infty$ is the standard one: the L^r-average appearing in the hypothesis is understood as an essential supremum and the $L^p(w)$ estimate holds for any $1 < p < \infty$, no matter the value of s, that is, for any $w \in A_\infty$.

2.4. **Proof of Theorem 2.2, Part** (a)**.** As mentioned in Remark 2.3, we can relax the hypotheses by assuming (2.4) and (2.5), which we do. We consider the case $q_0 < \infty$, the other one is left to the reader. Let $f \in L^\infty_c(dx)$, so $F = |Tf|^{s_0} \in L^1(dx)$. We fix a cube Q (we switch to cubes for the proof). As T is sublinear, we have

$$F \le G_Q + H_Q \equiv 2^{s_0-1}\,|T(I - \mathcal{A}_{r(Q)})f|^{s_0} + 2^{s_0-1}\,|T\mathcal{A}_{r(Q)}f|^{s_0}.$$

Then (2.4) and (2.5) yield respectively (2.7) and (2.6) with $r = q_0/s_0$, $H_1 = H_2 \equiv 0$, $a = 2^{s_0-1}\,C^{s_0}$ and $G = 2^{s_0-1}\,C^{s_0}\,M_{\alpha\,p_0}\big(|f|^{p_0}\big)^{s_0/p_0}$. By Remark 2.4, $w^q \in RH_{(q_0/q)'}$ and one can pick $1 < s < q_0/q$ so that $w^q \in RH_{s'}$. Thus, Theorem 2.6 with q/s_0 in place of p (notice that $1 < q/s_0 < r/s$) yields

$$\begin{aligned}\|Tf\|_{L^q(w^q)}^{s_0} &\le \|MF\|_{L^{\frac{q}{s_0}}(w^q)} \le C\,\|G\|_{L^{\frac{q}{s_0}}(w^q)} = C\,\big\|M_{\alpha\,p_0}\big(\,|f|^{p_0}\big)\big\|_{L^{\frac{q}{p_0}}((w^{p_0})^{q/p_0})}^{\frac{s_0}{p_0}} \\ &\le C\,\big\||f|^{p_0}\big\|_{L^{\frac{p}{p_0}}((w^{p_0})^{p/p_0})}^{\frac{s_0}{p_0}} = C\,\|f\|_{L^p(w^p)}^{s_0}.\end{aligned}$$

In the last estimate we have used that $M_{\alpha\,p_0}$ maps $L^{\frac{p}{p_0}}((w^{p_0})^{p/p_0}))$ into $L^{\frac{q}{p_0}}((w^{p_0})^{q/p_0}))$ by Theorem 1.1 from $w^{p_0} \in A_{p/p_0, q/p_0}$ (see Remark 2.4) and the easily checked conditions $0 < \alpha\,p_0 < n$, $1 < p/p_0 < n/(\alpha\,p_0)$ and $1/(p/p_0) - 1/(q/p_0) = \alpha\,p_0/n$.

2.5. **Proof of Theorem 2.2, Part** (b)**.** Before starting the proof, let us introduce some notation (see [BS] for more details). Let ϕ be a Young function: $\phi : [0,\infty) \longrightarrow [0,\infty)$ is continuous, convex, increasing and satisfies $\phi(0+) = 0$, $\phi(\infty) = \infty$. Given a cube Q we define the localized Luxemburg norm

$$\|f\|_{\phi,Q} = \inf\left\{\lambda > 0 : \fint_Q \phi\left(\frac{|f|}{\lambda}\right) \le 1\right\},$$

and then the maximal operator

$$M_\phi f(x) = \sup_{Q\ni x} \|f\|_{\phi,Q}.$$

In the definition of $\|\cdot\|_{\phi,Q}$, if the probability measure $dx/|Q|$ is replaced by dx and Q by $\mathbb{R}^n$, then one has the Luxemburg norm $\|\cdot\|_\phi$ which allows one to define the Orlicz space L^ϕ.

Some specific examples needed here are $\phi(t) \approx e^{t^r}$ for $t \ge 1$ which gives the classical space $\exp L^r$ and $\phi(t) = t\,(1+\log^+ t)^\alpha$ with $\alpha > 0$ that gives the space $L\,(\log L)^\alpha$. In the particular case $\alpha = k-1$ with $k \ge 1$, it is well known that $M_{L(\log L)^{k-1}} f \approx M^k f$ where M^k is the k-iteration of M.

We also need fractional maximal operators associated with an Orlicz space: given $0 < \alpha < n$ we define

$$M_{\phi,\alpha} f(x) = \sup_{Q\ni x} \ell(Q)^\alpha\, \|f\|_{\phi,Q}.$$

John-Nirenberg's inequality implies that for any function $b \in \mathrm{BMO}$ and any cube Q we have $\|b - b_Q\|_{\exp L,Q} \lesssim \|b\|_{\mathrm{BMO}}$. This yields the following estimates: First, for each cube Q and $x \in Q$

$$\begin{aligned} \fint_Q |b-b_Q|^{k\,s_0}\,|f|^{s_0} &\le \|b-b_Q\|_{\exp L,Q}^{k\,s_0}\, \big\||f|^{s_0}\big\|_{L\,(\log L)^{k\,s_0},Q} \\ &\lesssim \|b\|_{\mathrm{BMO}}^{k\,s_0}\, M_{L\,(\log L)^{k\,s_0}}\big(|f|^{s_0}\big)(x) \lesssim \|b\|_{\mathrm{BMO}}^{k\,s_0}\, M^{[k\,s_0]+2}\big(|f|^{s_0}\big)(x), \end{aligned} \tag{2.9}$$

where $[s]$ is the integer part of s (if $k\,s_0 \in \mathbb{N}$, one can take $M^{[k\,s_0]+1}$). Second, for each $j \ge 1$ and each Q,

$$\begin{aligned} \|b - b_{2\,Q}\|_{\exp L,2^j\,Q} &\le \|b - b_{2^j\,Q}\|_{\exp L,2^j\,Q} + |b_{2^j\,Q} - b_{2\,Q}| \lesssim \|b\|_{\mathrm{BMO}} + \sum_{l=1}^{j-1} |b_{2^{l+1}\,Q} - b_{2^l\,Q}| \\ &\lesssim \|b\|_{\mathrm{BMO}} + \sum_{l=1}^{j-1} \fint_{2^{l+1}\,Q} |b - b_{2^{l+1}\,Q}| \lesssim j\,\|b\|_{\mathrm{BMO}}. \end{aligned} \tag{2.10}$$

The following auxiliary result allows us to assume further that $b \in L^\infty(dx)$. The proof is postponed until the end of this section.

Lemma 2.7. *Let $1 \le p_0 < s_0 < \infty$, $p_0 < p < q < \infty$, $k \in \mathbb{N}$ and $w^q \in A_\infty$. Let T be a sublinear operator bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$.*

(i) *If $b \in \mathrm{BMO} \cap L^\infty(dx)$ and $f \in L^\infty_c(dx)$, then $T^k_b f \in L^{s_0}(dx)$.*

(ii) *Assume that for any $b \in \mathrm{BMO} \cap L^\infty(dx)$ and for any $f \in L^\infty_c(dx)$ we have that*

$$\|T^k_b f\|_{L^q(w^q)} \le C_0\, \|b\|_{\mathrm{BMO}}^k\, \|f\|_{L^p(w^p)}, \tag{2.11}$$

where C_0 does not depend on b and f. Then for all $b \in \mathrm{BMO}$, (2.11) holds with constant $2^k\, C_0$ instead of C_0.

Part (ii) in this result ensures that it suffices to consider the case $b \in L^\infty(dx)$ (provided the constants obtained do not depend on b). So from now on we assume that $b \in L^\infty(dx)$ and obtain (2.11) with C_0 independent of b and f. Note that by homogeneity we can also assume that $\|b\|_{\mathrm{BMO}} = 1$.

We proceed by induction. Note that the case $k = 0$ corresponds to (a). We write the case $k = 1$ in full detail and indicate how to pass from $k-1$ to k as the argument is essentially the same. Let us fix $p_0 < p < q_0$ and $w^q \in A_{1+\frac{q/p_0}{(p/p_0)'}} \cap RH_{(q_0/q)'}$ (see Remark 2.4). We assume that $q_0 < \infty$, for $q_0 = \infty$ the main ideas are the same and details are left to the interested reader.

Case $k=1$: We use the ideas in [AM1] (see also [Pe1]). Let $f \in L^\infty_c(dx)$ and set $F = |T_b^1 f|^{s_0}$. Note that $F \in L^1(dx)$ by (i) in Lemma 2.7 (this is the only place in this step where we use that $b \in L^\infty(dx)$). Given a cube Q, we set $f_{Q,b} = (b_{4\,Q} - b)\, f$ and decompose T_b^1 as follows:

$$
\begin{aligned}
|T_b^1 f(x)| &= \big|T\big((b(x)-b)\,f\big)(x)\big| \le |b(x)-b_{4\,Q}|\,|Tf(x)| + \big|T\big((b_{4\,Q}-b)\,f\big)(x)\big| \\
&\le |b(x)-b_{4\,Q}|\,|Tf(x)| + |T(I-\mathcal{A}_{r(Q)})f_{Q,b}(x)| + |T\mathcal{A}_{r(Q)} f_{Q,b}(x)|.
\end{aligned}
$$

With the notation of Theorem 2.6, we observe that $F \le G_Q + H_Q$ where

$$
G_Q = 4^{s_0-1}\big(G_{Q,1}+G_{Q,2}\big) = 4^{s_0-1}\left(|b-b_{4\,Q}|^{s_0}\,|Tf|^{s_0} + |T(I-\mathcal{A}_{r(Q)})f_{Q,b}|^{s_0}\right)
$$

and $H_Q = 2^{s_0-1}\,|T\mathcal{A}_{r(Q)} f_{Q,b}|^{s_0}$.

We first estimate the average of G_Q on Q. Fix any $x \in Q$. By (2.9) with $k=1$,

$$
\fint_Q G_{Q,1} = \fint_Q |b-b_{4\,Q}|^{s_0}\,|Tf|^{s_0} \lesssim \|b\|_{\mathrm{BMO}}^{s_0}\, M^{[s_0]+2}\big(|Tf|^{s_0}\big)(x).
$$

Using (2.1), (2.9) and (2.10),

$$
\begin{aligned}
\Big(\fint_Q G_{Q,2}\Big)^{\frac{1}{s_0}} &= \Big(\fint_Q |T(I-\mathcal{A}_{r(Q)})f_{Q,b}|^{s_0}\Big)^{\frac{1}{s_0}} \lesssim \sum_{j=1}^\infty \alpha_j\, \ell(2^{j+1}\,Q)^{\alpha}\,\Big(\fint_{2^{j+1}\,Q} |f_{Q,b}|^{p_0}\Big)^{\frac{1}{p_0}} \\
&\le \sum_{j=1}^\infty \alpha_j\, \|b-b_{4\,Q}\|_{\exp L, 2^{j+1}\,Q}\, M_{L\,(\log L)^{p_0},\alpha\,p_0}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x) \\
&\lesssim \|b\|_{\mathrm{BMO}}\, M_{L\,(\log L)^{p_0},\alpha\,p_0}\big(|f|^{p_0}\big)(x)^{\frac{1}{p_0}} \sum_{j=1}^\infty \alpha_j\, j \\
&\lesssim M_{L\,(\log L)^{p_0},\alpha\,p_0}\big(|f|^{p_0}\big)^{\frac{1}{p_0}}(x),
\end{aligned}
$$

since $\sum_j \alpha_j\, j < \infty$. Hence, for any $x \in Q$

$$
\fint_Q G_Q \le C\left(M^{[s_0]+2}\big(|Tf|^{s_0}\big)(x) + M_{L\,(\log L)^{p_0},\alpha\,p_0}\big(|f|^{p_0}\big)(x)^{\frac{s_0}{p_0}}\right) \equiv G(x).
$$

We next estimate the average of H_Q^r on Q with $r = q_0/s_0$. Using (2.2) and proceeding as before

$$\begin{aligned}\Big(\fint_Q H_Q^r\Big)^{\frac{1}{q_0}} &= 2^{(s_0-1)/s_0}\Big(\fint_Q |T\mathcal{A}_{r(Q)} f_{Q,b}|^{q_0}\Big)^{\frac{1}{q_0}} \lesssim \sum_{j=1}^{\infty} \alpha_j\Big(\fint_{2^{j+1}Q} |Tf_{Q,b}|^{s_0}\Big)^{\frac{1}{s_0}} \\ &\le \sum_{j=1}^{\infty} \alpha_j\Big(\fint_{2^{j+1}Q} |T_b^1 f|^{s_0}\Big)^{\frac{1}{s_0}} + \sum_{j\ge1} \alpha_j\Big(\fint_{2^{j+1}Q} |b-b_{4Q}|^{s_0}|Tf|^{s_0}\Big)^{\frac{1}{s_0}} \\ &\lesssim (MF)^{\frac{1}{s_0}}(x) + \sum_{j=1}^{\infty}\alpha_j\,\|b-b_{4Q}\|_{\exp L,2^{j+1}Q}\, M^{[s_0]+2}\big(|Tf|^{s_0}\big)^{\frac{1}{s_0}}(\bar{x}) \\ &\lesssim (MF)^{\frac{1}{s_0}}(x) + M^{[s_0]+2}\big(|Tf|^{s_0}\big)^{\frac{1}{s_0}}(\bar{x})\sum_{j=1}^{\infty}\alpha_j\, j \\ &\lesssim (MF)^{\frac{1}{s_0}}(x) + M^{[s_0]+2}\big(|Tf|^{s_0}\big)^{\frac{1}{s_0}}(\bar{x}),\end{aligned}$$

for any x, $\bar{x} \in Q$, where we have used that $\sum_j \alpha_j\, j < \infty$. Thus we have obtained

$$\Big(\fint_Q H_Q^r\Big)^{\frac{1}{r}} \le C\,\big(MF(x) + M^{[s_0]+2}\big(|Tf|^{s_0}\big)(\bar{x})\big) \equiv C\,\big(MF(x) + H_2(\bar{x})\big).$$

As mentioned before $F \in L^1$. Since $w^q \in RH_{(q_0/q)'}$, we can choose $1 < s < q_0/q$ so that $w^q \in RH_{s'}$. Thus, Theorem 2.6 with q/s_0 in place of p (notice that $1 < q/s_0 < r/s$) yields

$$\begin{aligned}\|T_b^1 f\|_{L^q(w^q)}^{s_0} \le \|MF\|_{L^{\frac{q}{s_0}}(w^q)} &\lesssim \|G\|_{L^{\frac{q}{s_0}}(w^q)} + \|H_2\|_{L^{\frac{q}{s_0}}(w^q)} \\ &\lesssim \Big\|M_{L\,(\log L)^{p_0},\alpha\, p_0}\big(|f|^{p_0}\big)^{\frac{s_0}{p_0}}\Big\|_{L^{\frac{q}{s_0}}(w^q)} + \Big\|M^{[s_0]+2}\big(|Tf|^{s_0}\big)\Big\|_{L^{\frac{q}{s_0}}(w^q)} \\ &\lesssim I + II.\end{aligned}$$

We estimate each term in turn. For I, we claim that $M_{L\,(\log L)^{p_0},\alpha\, p_0}$ maps $L^{\frac{p}{p_0}}(w^p)$ into $L^{\frac{q}{p_0}}(w^q)$. This implies that

$$I = \Big\|M_{L\,(\log L)^{p_0},\alpha\, p_0}\big(|f|^{p_0}\big)\Big\|_{L^{\frac{q}{p_0}}(w^q)}^{\frac{s_0}{p_0}} \lesssim \|f\|_{L^p(w^p)}^{s_0}.$$

Let us show our claim. We observe that $1 + \frac{q/p_0}{(p/p_0)'} = \frac{q}{s_0} = q\,(\frac{1}{p_0} - \frac{\alpha}{n})$. Then, by (iii) in Proposition 2.1, there exists $1 < s < p/p_0$ so that $w^q \in A_{q\,(\frac{1}{s\,p_0}-\frac{\alpha}{n})}$. Let us observe that the choice of s guarantees that $q\,(\frac{1}{s\,p_0} - \frac{\alpha}{n}) > 1$.

We set $\tilde{\alpha} = s\,p_0\,\alpha$, $\tilde{p} = p/(p_0\,s)$ and $\tilde{q} = q/(p_0\,s)$. Let us observe that $0 < \tilde{\alpha} < n$, $1 < \tilde{p} < n/\tilde{\alpha}$ and $1/\tilde{p} - 1/\tilde{q} = \tilde{\alpha}/n$. Besides by (ix) of Proposition 2.1 we have that $\tilde{w} = w^{p_0\,s} \in A_{\tilde{p},\tilde{q}}$. Therefore, by Theorem 1.1 it follows that $M_{\tilde{\alpha}}$ maps $L^{\tilde{p}}(\tilde{w}^{\tilde{p}})$ into $L^{\tilde{q}}(\tilde{w}^{\tilde{q}})$.

Notice that as $s > 1$ we have that $t\,(1 + \log^+ t)^{p_0} \lesssim t^s$ for every $t \ge 1$. Thus,

$$\begin{aligned}M_{L\,(\log L)^{p_0},\alpha\, p_0}\, g(x) &= \sup_{Q\ni x} \ell(Q)^{\alpha\, p_0}\,\|g\|_{L\,(\log L)^{p_0},Q} \lesssim \sup_{Q\ni x} \ell(Q)^{\alpha\, p_0}\,\|g\|_{L^s,Q} \\ &= M_{\alpha\, p_0\, s}\big(|g|^s\big)(x)^{\frac{1}{s}} = M_{\tilde{\alpha}}\big(|g|^s\big)(x)^{\frac{1}{s}},\end{aligned}$$

and therefore we conclude the desired estimate

$$\|M_{L\,(\log L)^{p_0},\alpha\,p_0} g\|_{L^{\frac{q}{p_0}}(w^q)} \lesssim \big\|M_{\tilde\alpha}\big(|g|^s\big)^{\frac1s}\big\|_{L^{\frac{q}{p_0}}(w^q)} = \big\|M_{\tilde\alpha}\big(|g|^s\big)\big\|^{\frac1s}_{L^{\tilde q}(\tilde w^{\tilde q})}$$
$$\lesssim \big\||g|^s\big\|^{\frac1s}_{L^{\tilde p}(\tilde w^{\tilde p})} = \|g\|_{L^{\frac{p}{p_0}}(w^p)}.$$

For II as before we observe that $1+\frac{q/p_0}{(p/p_0)'} = \frac{q}{s_0}$. Besides, $1/p-1/q=\alpha/n=1/p_0-1/s_0$ implies that $1/s_0-1/q=1/p_0-1/p>0$ and therefore $q/s_0>1$. Consequently, M (hence, $M^2, M^3,\dots$) is bounded on $L^{\frac{q}{s_0}}(w^q)$ which gives

$$II = \big\|M^{[s_0]+2}\big(|Tf|^{s_0}\big)\big\|_{L^{\frac{q}{s_0}}(w^q)} \lesssim \|Tf\|^{s_0}_{L^q(w^q)} \lesssim \|f\|^{s_0}_{L^p(w^p)},$$

where in the last inequality we have used (a) (which is the case $k=0$).

Collecting the obtained estimates for I and II we conclude as desired

$$\|T^1_b f\|^{s_0}_{L^q(w^q)} \lesssim \|f\|^{s_0}_{L^p(w^p)}.$$

Case k: We now sketch the induction argument. Assume that we have already proved the cases $m=0,\dots,k-1$. Let $f\in L^\infty_c(dx)$. Given a cube Q, write $f_{Q,b}=(b_{4\,Q}-b)^k f$ and decompose T^k_b as follows:

$$\begin{aligned}|T^k_b f(x)| &= |T\big((b(x)-b)^k f\big)(x)|\\ &\le \sum_{m=0}^{k-1} C_{k,m}|b(x)-b_{4\,Q}|^{k-m}|T^m_b f(x)| + |T\big((b_{4\,Q}-b)^k f\big)(x)|\\ &\lesssim \sum_{m=0}^{k-1} |b(x)-b_{4\,Q}|^{k-m}|T^m_b f(x)| + |T(I-\mathcal{A}_{r(Q)})f_{Q,b}(x)| + |T\mathcal{A}_{r(Q)}f_{Q,b}(x)|.\end{aligned}$$

Following the notation of Theorem 2.6, we set $F=|T^k_b f|^{s_0}\in L^1(dx)$ by (i) in Lemma 2.7. Observe that $F\le G_Q+H_Q$ where

$$G_Q = 4^{s_0-1}\,C\,\bigg(\Big(\sum_{m=0}^{k-1}|b-b_{4\,Q}|^{k-m}|T^m_b f|\Big)^{s_0} + |T(I-\mathcal{A}_{r(Q)})f_{Q,b}|^{s_0}\bigg)$$

and $H_Q=2^{s_0-1}\,|T\mathcal{A}_{r(Q)}f_{Q,b}|^{s_0}$. Proceeding as before we obtain for any $x\in Q$

$$\fint_Q G_Q \le C\,\Big(\sum_{m=0}^{k-1} M^{[(k-m)\,s_0]+2}\big(|T^m_b f|^{s_0}\big)(x) + M_{L\,(\log L)^{k\,p_0},\alpha\,p_0}\big(|f|^{p_0}\big)(x)^{\frac{s_0}{p_0}} \equiv G(x),$$

and for $r=q_0/s_0$

$$\Big(\fint_Q H_Q^r\Big)^{\frac1r} \le C\,\Big(MF(x)+\sum_{m=0}^{k-1}M^{[(k-m)\,s_0]+2}\big(|T^m_b f|^{s_0}\big)(\bar x)\Big) \equiv C\,\big(MF(x)+H_2(\bar x)\big).$$

Therefore, as $F\in L^1$, Theorem 2.6 gives us as before

$$\begin{aligned}\|T^k_b f\|^{s_0}_{L^q(w^q)} &\le \|MF\|_{L^{\frac{q}{s_0}}(w^q)} \lesssim \|G\|_{L^{\frac{q}{s_0}}(w^q)} + \|H_2\|_{L^{\frac{q}{s_0}}(w^q)}\\ &\lesssim \big\|M_{L\,(\log L)^{k\,p_0},\alpha\,p_0}\big(|f|^{p_0}\big)^{\frac{s_0}{p_0}}\big\|_{L^{\frac{q}{s_0}}(w^q)} + \sum_{m=0}^{k-1}\big\|M^{[(k-m)\,s_0]+2}\big(|T^m_b f|^{s_0}\big)\big\|_{L^{\frac{q}{s_0}}(w^q)}\end{aligned}$$

$$\lesssim \|f\|_{L^p(w^p)}^{s_0} + \sum_{m=0}^{k-1} \|T_b^m f\|_{L^q(w^q)}^{s_0} \lesssim \|f\|_{L^p(w^p)}^{s_0},$$

where have proceeded as in the estimates of I and II in the case $k = 1$ and we have used the induction hypothesis on T_b^m, $m = 0, \dots, k-1$. Let us point out again that none of the constants involved in the proof depend on b and f.

Proof of Lemma 2.7. We use an argument similar to that in [AM1] (see also [Pe1]). Fix $f \in L_c^\infty(dx)$. Note that (i) follows easily observing that

$$|T_b^k f(x)| \lesssim \sum_{m=0}^{k} |b(x)|^{m-k} |T(b^m f)(x)| \le \sum_{m=0}^{k} \|b\|_{L^\infty}^{m-k} |T(b^m f)(x)| \in L^{s_0}(dx),$$

since $b \in L^\infty(dx)$, $f \in L_c^\infty(dx)$ imply that $b^m f \in L_c^\infty(dx) \subset L^{p_0}(dx)$ and, by assumption, $T(b^m f) \in L^{s_0}(dx)$.

To obtain (ii), we fix $b \in \mathrm{BMO}$ and $f \in L_c^\infty(dx)$. Let Q_0 be a cube such that $\operatorname{supp} f \subset Q_0$. We may assume that $b_{Q_0} = 0$ since otherwise we can work with $\widetilde{b} = b - b_{Q_0}$ and observe that $T_b^k = T_{\widetilde{b}}^k$ and $\|b\|_{\mathrm{BMO}} = \|\widetilde{b}\|_{\mathrm{BMO}}$. Note that for all $m = 0, \dots, k$, we have that $|b^m f|$ and $\big|T(b^m f)\big|$ are finite almost everywhere since $|b^m f| \in L^{p_0}(dx)$ and $\big|T(b^m f)\big| \in L^{s_0}(dx)$. Let $N > 0$ and define b_N as follows: $b_N(x) = b(x)$ when $-N \le b(x) \le N$, $b_N(x) = N$ when $b(x) > N$ and $b(x) = -N$ when $b(x) < -N$. Then, it is immediate to see that $|b_N(x) - b_N(y)| \le |b(x) - b(y)|$ for all x, y. Thus, $\|b_N\|_{\mathrm{BMO}} \le 2\,\|b\|_{\mathrm{BMO}}$. As $b_N \in L^\infty(dx)$ we can use (2.11) and

$$\|T_{b_N}^k f\|_{L^q(w^q)} \le C_0\,\|b_N\|_{\mathrm{BMO}}^k\,\|f\|_{L^p(w^p)} \le C_0\,2^k\,\|b\|_{\mathrm{BMO}}^k\,\|f\|_{L^p(w^p)} < \infty.$$

To conclude, by Fatou's lemma, it suffices to show that $|T_{b_{N_j}} f(x)| \longrightarrow |T_b^k f(x)|$ for a.e. $x \in \mathbb{R}^n$ and for some subsequence $\{N_j\}_j$ such that $N_j \to \infty$.

As $|b_N| \le |b| \in L^p(Q_0)$ for any $1 \le p < \infty$, the dominated convergence theorem yields that $(b_N)^m f \longrightarrow b^m f$ in $L^{p_0}(dx)$ as $N \to \infty$ for all $m = 0, \dots, k$. Therefore, the fact that T is bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$ yields $T\big((b_N)^m f - b^m f\big) \longrightarrow 0$ in $L^{s_0}(dx)$. Thus, there exists a subsequence $N_j \to \infty$ such that $T\big((b_{N_j})^m f - b^m f\big)(x) \longrightarrow 0$ for a.e. $x \in \mathbb{R}^n$ and for all $m = 1, \dots, k$. In this way we obtain

$$\big||T_{b_{N_j}}^k f(x)| - |T_b^k f(x)|\big| \lesssim \big|T\big(\big[(b_{N_j}(x) - b_{N_j})^k - (b(x) - b)^k\big]\,f\big)(x)\big|$$

$$\lesssim \sum_{m=0}^{k} |b_{N_j}(x)|^{k-m}\,\big|T\big((b_{N_j})^m f - b^m f\big)(x)\big| + \big|b_{N_j}(x)^{k-m} - b(x)^{k-m}\big|\,\big|T(b^m f)(x)\big|,$$

and as desired we get that $|T_{b_{N_j}} f(x)| \longrightarrow |T_b^k f(x)|$ for a.e. $x \in \mathbb{R}^n$. □

3. Proof of Theorems 1.3 and 1.4

We first introduce our class of elliptic operators and state some needed properties. Then we present an auxiliary lemma which leads us to prove the weighted estimates for $L^{-\alpha/2}$ and the corresponding commutators.

3.1. **The Class of Elliptic Operators.** Let A be an $n \times n$ matrix of complex and L^∞-valued coefficients defined on $\mathbb{R}^n$. We assume that this matrix satisfies the following ellipticity (or "accretivity") condition: there exist $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda\,|\xi|^2 \le \operatorname{Re} A(x)\,\xi\cdot\bar{\xi} \qquad \text{and} \qquad |A(x)\,\xi\cdot\bar{\zeta}| \le \Lambda\,|\xi|\,|\zeta|,$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi\cdot\zeta = \xi_1\,\zeta_1 + \cdots + \xi_n\,\zeta_n$ and therefore $\xi\cdot\bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$. Note that then $A(x)\,\xi\cdot\bar{\zeta} = \sum_{j,k} a_{j,k}(x)\,\xi_k\,\bar{\zeta}_j$. Associated with this matrix we define the second order divergence form operator

$$Lf = -\operatorname{div}(A\,\nabla f),$$

which is understood in the standard weak sense as a maximal-accretive operator on $L^2(dx)$ with domain $\mathcal{D}(L)$ by means of a sesquilinear form.

The operator $-L$ generates a C^0-semigroup $\{e^{-t\,L}\}_{t>0}$ of contractions on $L^2(dx)$. Define $\vartheta \in [0, \pi/2)$ by,

$$\vartheta = \sup\big\{\big|\arg\langle Lf, f\rangle\big| \,:\, f \in \mathcal{D}(L)\big\}.$$

Then the semigroup has an analytic extension to a complex semigroup $\{e^{-z\,L}\}_{z\in\Sigma_{\pi/2-\vartheta}}$ of contractions on $L^2(dx)$. Here we have written for $0 < \theta < \pi$,

$$\Sigma_\theta = \{z \in \mathbb{C}^* : |\arg z| < \theta\}.$$

We need to recall some properties of the generated semigroup $\{e^{-t\,L}\}_{t>0}$ (the reader is referred to [Aus] and [AM2] for more details and complete statements). In what follows we set $d(E,F) = \inf\{|x-y| \,:\, x \in E, y \in F\}$ where E, F are subsets of $\mathbb{R}^n$.

Proposition 3.1. *Given L as above, there exist $p_- = p_-(L)$ and $p_+ = p_+(L)$, $1 \le p_- < 2 < p_+ \le \infty$ such that:*

(a) *The semigroup $\{e^{-t\,L}\}_{t>0}$ is uniformly bounded on $L^p(dx)$ for every $p_- < p < p_+$.*

(b) *The semigroup $\{e^{-t\,L}\}_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates for every $p_- < p \le q < p_+$: For $1 \le p \le q \le \infty$, $L^p - L^q$ off-diagonal estimates mean that for some $c > 0$, for all closed sets E and F, all f and all $t > 0$ we have*

$$\Big(\int_F |e^{-t\,L}(\chi_E\, f)|^q\,dx\Big)^{\frac{1}{q}} \lesssim t^{-\frac{1}{2}(\frac{n}{p}-\frac{n}{q})}\,e^{-\frac{c\,d^2(E,F)}{t}}\Big(\int_E |f|^p\,dx\Big)^{\frac{1}{p}}. \tag{3.1}$$

(c) *For every $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$, the complex family $\{(zL)^m e^{-z\,L}\}_{z\in\Sigma_\mu}$ is uniformly bounded on $L^p(dx)$ for $p_- < p < p_+$ and satisfies $L^p - L^q$ off-diagonal estimates for every $p_- < p \le q < p_+$ (in (3.1) one replaces t by $|z|$).*

(d) *The interval (p_-, p_+) is maximal for any of the properties above up to end-points, that is, none of them can hold outside $[p_-, p_+]$.*

(e) *If $n = 1$ or 2, or L has real coefficients, then $p_- = 1$ and $p_+ = \infty$. In those cases, one has the stronger Gaussian domination $|e^{-tL} f| \le C e^{ct\Delta}|f|$ for all $f \in L^1(dx) \cup L^\infty(dx)$ and $t > 0$ with constants $c, C > 0$. This implies uniform boundedness and off-diagonal estimates in the whole interval $[1, \infty]$. Other instances of a Gaussian domination occur for complex, continuous and periodic coefficients in any dimension, see [ERS].*

(f) *If $n \ge 3$, $p_- < \frac{2n}{n+2}$ and $p_+ > \frac{2n}{n-2}$.*

Let us make some relevant comments. In the Gaussian factors of the off-diagonal estimates the value of c is irrelevant as long as it remains positive. When $q=\infty$ in (3.1), one should adapt the definitions in the usual straightforward way. One can prove that L^1-L^∞ off-diagonal estimates are equivalent to pointwise Gaussian upper bounds for the kernels of the family (see [AM2]). In dimensions $n\ge 3$, it is not clear what happens at the endpoints for either boundedness or off-diagonal estimates.

3.2. **Auxiliary Lemma.** The proofs of Theorems 1.3 and 1.4 will use the following auxiliary lemma.

Lemma 3.2. *Let $p_-<p_0<s_0<q_0<p_+$ so that $1/p_0-1/s_0=\alpha/n$. Fix a ball B with radius r. For $f\in L^\infty_c(dx)$ and m large enough we have*

$$\Big(\fint_B |L^{-\alpha/2}(I-e^{-r^2 L})^m f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \le \sum_{j=1}^\infty g_1(j)\,(2^{j+1}\,r)^\alpha \Big(\fint_{2^{j+1}B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}, \tag{3.2}$$

and for $1\le l\le m$

$$\Big(\fint_B |L^{-\alpha/2}e^{-l\,r^2 L} f|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \le \sum_{j=1}^\infty g_2(j)\Big(\fint_{2^{j+1}B}|L^{-\alpha/2}f|^{s_0}\,dx\Big)^{\frac{1}{s_0}}, \tag{3.3}$$

where $g_1(j)=C\,2^{-j\,(2\,m-n/s_0)}$ and $g_2(j)=C\,e^{-c\,4^j}$.

Proof. We first obtain (3.3). We fix $f\in L^\infty_c(dx)$ and a ball B. We decompose any given function h as

$$h=\sum_{j\ge 1}h_j, \qquad h_j=h\,\chi_{C_j(B)}, \tag{3.4}$$

where $C_j(B)=2^{j+1}\,B\setminus 2^j\,B$ when $j\ge 2$ and $C_1(B)=4B$.

Fix $1\le l\le m$. Since $p_-<s_0<q_0<p_+$ by Proposition 3.1 part (b) we have

$$\Big(\fint_B |e^{-l\,r^2 L}h_j|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \lesssim r^{-\frac{n}{q_0}}\,(l\,r^2)^{-\frac12\,(\frac{n}{s_0}-\frac{n}{q_0})}\,e^{-\frac{c\,d^2(C_j(B),B)}{l\,r^2}}\Big(\int_{C_j(B)}|h|^{s_0}\,dx\Big)^{\frac{1}{s_0}}$$
$$\lesssim 2^{j\,n/s_0}\,e^{-c\,4^j}\Big(\fint_{2^{j+1}B}|h|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \lesssim e^{-c\,4^j}\Big(\fint_{2^{j+1}B}|h|^{s_0}\,dx\Big)^{\frac{1}{s_0}}$$

and by Minkowski's inequality

$$\Big(\fint_B |e^{-k\,r^2 L}h|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \lesssim \sum_{j\ge 1} g(j)\Big(\fint_{2^{j+1}B}|h|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \tag{3.5}$$

with $g(j)=e^{-c\,4^j}$ for any $h\in L^{s_0}(dx)$. This estimate with $h=L^{-\alpha/2}f\in L^{s_0}(dx)$ —here we use that $f\in L^\infty_c(dx)$ and Theorem 1.2— yields (3.3) since, by the commutation rule, $L^{-\alpha/2}e^{-l\,r^2 L}f=e^{-l\,r^2 L}h$.

Next we obtain (3.2). We decompose $f=\sum_{j\ge1}f_j$ as in (3.4). For $j=1$, we use that $L^{-\alpha/2}$ maps $L^{p_0}(dx)$ into $L^{s_0}(dx)$ by Theorem 1.2, and that $(I-e^{-r^2 L})^m$ is bounded on L^{p_0} uniformly on r by (a) in Proposition 3.1 as $p_-<p_0<p_+$. Hence,

$$\Big(\fint_B |L^{-\alpha/2}(I-e^{-r^2 L})^m f_1|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \lesssim |B|^{-1/s_0}\Big(\int_{\mathbb{R}^n}|(I-e^{-r^2 L})^m f_1|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$
$$\lesssim |B|^{-1/s_0}\Big(\int_{4B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}} \lesssim (4\,r)^\alpha\Big(\fint_{4B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}. \tag{3.6}$$

Next we estimate the terms $j \geq 2$. We first write

$$L^{-\alpha/2}\,(1-e^{-r^2\,L})^m f_j = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2}\,e^{-t\,L}(1-e^{-r^2\,L})^m f_j\,\frac{dt}{t}$$
$$= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2}\,\varphi(t,L) f_j\,\frac{dt}{t}, \tag{3.7}$$

where $\varphi(t,z) = e^{-t\,z}(1-e^{-r^2\,z})^m$. The argument will show that the integral in t converges strongly in $L^{s_0}(B)$. Let $\mu \in (\vartheta,\pi)$ and assume that $\vartheta < \theta < \nu < \mu < \pi/2$. Then we have

$$\varphi(t,L) = \int_{\Gamma_+} e^{-z\,L}\,\eta_+(t,z)\,dz + \int_{\Gamma_-} e^{-z\,L}\,\eta_-(t,z)\,dz, \tag{3.8}$$

where $\Gamma_\pm$ is the half ray $\mathbb{R}^+\,e^{\pm i\,(\pi/2-\theta)}$,

$$\eta_\pm(t,z) = \frac{1}{2\,\pi\,i}\int_{\gamma_\pm} e^{\zeta\,z}\,\varphi(t,\zeta)\,d\zeta, \qquad z\in\Gamma_\pm,$$

with $\gamma_\pm$ being the half-ray $\mathbb{R}^+\,e^{\pm i\,\nu}$ (the orientation of the paths is not needed in what follows so we do not pay attention to it). It is easy to see (see for instance [Aus]) that

$$|\eta_\pm(t,z)| \lesssim \frac{r^{2\,m}}{(|z|+t)^{m+1}}, \qquad z\in\Gamma_\pm.$$

Then, since $p_- < p_0 < s_0 < p_+$ by (c) in Proposition 3.1 we have

$$\Big(\fint_B \Big|\int_{\Gamma_+}\eta_+(t,z)\,e^{-z\,L}f_j\,dz\Big|^{s_0}dx\Big)^{\frac{1}{s_0}} \leq \int_{\Gamma_+}\Big(\fint_B |e^{-z\,L}f_j|^{s_0}\,dx\Big)^{\frac{1}{s_0}}|\eta_+(t,z)|\,|dz|$$
$$\lesssim \int_{\Gamma_+} r^{-\frac{n}{s_0}}\,|z|^{-\frac12\,(\frac{n}{p_0}-\frac{n}{s_0})}\,e^{-\frac{c\,4^j\,r^2}{|z|}}\Big(\int_{C_j(B)}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}|\eta_+(t,z)|\,|dz|$$
$$\lesssim 2^{j\,n/s_0}\Big(\fint_{2^{j+1}\,B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}\int_0^\infty\Big(\frac{2^j\,r}{\sqrt{s}}\Big)^\alpha e^{-\frac{c\,4^j\,r^2}{s}}\,\frac{r^{2\,m}}{(s+t)^{m+1}}\,ds.$$

The same is obtained when one deals with the term corresponding to Γ_-. We plug both estimates into the representation of $\varphi(t,L)$ and use Minkowski's inequality for the integral in the t variable in (3.7) to obtain

$$\Big(\fint_B |L^{-\alpha/2}(I-e^{-r^2\,L})^m f_j|^{s_0}\,dx\Big)^{\frac{1}{s_0}}$$
$$\lesssim 2^{j\,n/s_0}\Big(\fint_{2^{j+1}\,B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}\int_0^\infty t^{\alpha/2}\int_0^\infty\Big(\frac{2^j\,r}{\sqrt{s}}\Big)^\alpha e^{-\frac{c\,4^j\,r^2}{s}}\,\frac{r^{2\,m}}{(s+t)^{m+1}}\,ds\,\frac{dt}{t}$$
$$\lesssim 2^{j\,n/s_0}\,4^{-j\,m}\,(2^{j+1}\,r)^\alpha\Big(\fint_{2^{j+1}\,B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}, \tag{3.9}$$

since, after changing variables and taking $m+1>\alpha/2$,

$$\int_0^\infty\!\!\int_0^\infty t^{\alpha/2}\Big(\frac{2^j\,r}{\sqrt{s}}\Big)^\alpha e^{-\frac{c\,4^j\,r^2}{s}}\,\frac{r^{2\,m}}{(s+t)^{m+1}}\,\frac{dt}{t}\,ds$$
$$= 2\cdot 4^{-j\,m}\,(2^j\,r)^\alpha\Big(\int_0^\infty e^{-c\,s^2}\,s^{2\,m}\,\frac{ds}{s}\Big)\Big(\int_0^\infty\frac{t^{\alpha/2}}{(1+t)^{m+1}}\,\frac{dt}{t}\Big) \lesssim 4^{-j\,m}\,(2^{j+1}\,r)^\alpha.$$

Gathering (3.6) and (3.9) it follows that

$$\Big(\fint_B |L^{-\alpha/2}(I-e^{-r^2 L})^m f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \lesssim \sum_{j\ge 1} g(j)\,(2^{j+1}\,r)^\alpha \Big(\fint_{2^{j+1}B} |f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$

with $g(j)=2^{-j(2\,m-n/s_0)}$. □

3.3. **The proofs.** We are going to apply Theorem 2.2 to the linear operator $T=L^{-\alpha/2}$. Part (a) yields Theorem 1.3 and part (b) gives the estimates of the commutators in Theorem 1.4. Thus, it suffices to establish (2.1) and (2.2) with a sequence $\{\alpha_j\}_j$ that decays fast enough.

We fix $p_-<p<q<p_+$, α so that $\alpha/n=1/p-1/q$, and $w\in A_{1+\frac{1}{p_-}-\frac{1}{p}}\cap RH_{q\,(\frac{p_+}{q})'}$. By (iii) and (iv) in Proposition 2.1 there exist p_0, q_0, s_0 such that $1/p_0-1/s_0=\alpha/n$,

$$p_-<p_0<s_0<q_0<p_+, \qquad p_0<p<q<q_0 \qquad \text{and} \qquad w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_{q\,(\frac{q_0}{q})'}.$$

Notice that as $1\le p_-<p_+\le\infty$ we have that $1<p_0<s_0<q_0<\infty$. By Theorem 1.2, $T=L^{-\alpha/2}$ maps $L^{p_0}(dx)$ into $L^{s_0}(dx)$. We take $\mathcal{A}_r=I-(I-e^{-r^2 L})^m$ where $m\ge 1$ is an integer to be chosen. By the property (a) of the semigroup in Proposition 3.1, it follows that the family $\{\mathcal{A}_r\}_{r>0}$ is uniformly bounded on $L^{p_0}(dx)$ (as $p_-<p_0<p_+$) and so acts from $L^\infty_c(dx)$ into $L^{p_0}(dx)$. We apply Lemma 3.2. Note that (3.2) is (2.1). Also, (2.2) follows from (3.3) after expanding $\mathcal{A}_r=I-(I-e^{-r^2 L})^m$. Then, we have that $\sum_{j\ge1} j^k\,g_i(j)<\infty$ for $i=1,2$ by choosing $2\,m>n/s_0$. Consequently applying Theorem 2.2, part (a) if $k=0$ and part (b) otherwise, we conclude that T_b^k maps $L^p(w^p)$ into $L^q(w^q)$ as desired.

4. A variant of Theorem 2.2

The next result is an extension to the context of fractional operators of [AM1, Theorem 3.14], itself inspired greatly by [She, Theorem 3.1].

Theorem 4.1. *Let $0<\alpha<n$, $1\le p_0<s_0<q_0\le\infty$ such that $1/p_0-1/s_0=\alpha/n$. Suppose that T is a sublinear operator bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$. Assume that there exist constants $\alpha_2>\alpha_1>1$, $C>0$ such that*

$$\Big(\fint_B |Tf|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \le C\left\{\Big(\fint_{\alpha_1 B} |Tf|^{s_0}\,dx\Big)^{\frac{1}{s_0}} + M_{\alpha\,p_0}\big(|f|^{p_0}\big)(x)^{\frac{1}{p_0}}\right\}, \tag{4.1}$$

for all balls B, $x\in B$ and all $f\in L^\infty(dx)$ with compact support in $\mathbb{R}^n\setminus\alpha_2\,B$. Let $p_0<p<q<q_0$ with $1/p-1/q=\alpha/n$ and $w\in A_{1+\frac{1}{p_0}-\frac{1}{p}}\cap RH_{q\,(\frac{q_0}{q})'}$. Then, there is a constant C such that

$$\|Tf\|_{L^q(w^q)}\le C\,\|f\|_{L^p(w^p)}$$

for all $f\in L^\infty_c(dx)$.

Proof. A straightforward modification of Theorem 2.2 is to replace the family $\{\mathcal{A}_r\}_{r>0}$ indexed by radii of balls by $\{\mathcal{A}_B\}_B$ indexed by balls. For any ball B, let $\mathcal{A}_B f=(1-\chi_{\alpha_2 B})\,f$. With this choice, we check (2.4) and the weakened version of (2.5). Fix $f\in L^\infty_c(dx)$, a ball B and $x,\bar x\in B$. Using that T is bounded from $L^{p_0}(dx)$ to $L^{s_0}(dx)$ we have

$$\Big(\fint_{\alpha_1 B} |T(I-\mathcal{A}_B)f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} \lesssim r(B)^\alpha \Big(\fint_{\alpha_2 B}|f|^{p_0}\,dx\Big)^{\frac{1}{p_0}} \lesssim M_{\alpha\,p_0}\big(|f|^{p_0}\big)(x)^{\frac{1}{p_0}}. \tag{4.2}$$

In particular (2.4) holds since $\alpha_1 > 1$. Next, by (4.1) and since $|\mathcal{A}_B f| \le |f|$ we have

$$\Big(\fint_B |T\mathcal{A}_B f|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \le C\left\{\Big(\fint_{\alpha_1 B} |T\mathcal{A}_B f|^{s_0}\,dx\Big)^{\frac{1}{s_0}} + M_{\alpha\,p_0}\big(|f|^{p_0}\big)(\bar{x})^{\frac{1}{p_0}}\right\}.$$

Using (4.2) and the sublinearity of T, it follows that

$$\Big(\fint_B |T\mathcal{A}_B f|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \le C\,M\big(|Tf|^{s_0}\big)^{\frac{1}{s_0}}(x) + C\,M_{\alpha\,p_0}\big(|f|^{p_0}\big)(\bar{x})^{\frac{1}{p_0}},$$

which is the weakened version of (2.5). We conclude on applying the above mentioned variant of Theorem 2.2. □

5. Spaces of homogeneous type

As Theorem 2.6 passes entirely to spaces of homogeneous type —a (quasi-)metric space $(\mathcal{X}, d)$ equipped with a Borel doubling measure μ— one may wonder whether Theorems 2.2 and 4.1 can be extended to this setting.

In the Euclidean setting, the classical Riesz potential I_α or the fractional maximal operator M_α are bounded from $L^p(dx)$ to $L^q(dx)$ necessarily when $1/p - 1/q = \alpha/n$. This is caused by the homogeneity of these operators plus the dilation structure of the Lebesgue measure (that is, $|B| = c\,r(B)^n$). Concerning the weighted estimates, the boundedness of M_α and I_α from $L^p(w^p)$ to $L^q(w^q)$ are modeled by suitable modifications of the Muckenhoupt conditions which are vacuous unless $1/p - 1/q = \alpha/n$.

Let $(\mathcal{X}, d, \mu)$ be an space of homogeneous type where it is assumed that d is a distance (see [MS]). We also impose that $\mu(B) \ge c\,r(B)^n$ for some $n > 0$ —with this assumption, the fractional operators defined below are bounded with the same restriction in p and q as above. In this setting one can define the classes A_p, RH_q and $A_{p,q}$ by simply replacing the Lebesgue measure by μ. All the properties in Proposition 2.1 hold (to avoid some technicalities we assume that the weights are doubling). Here and in the sequel we understand that the averages are taken with respect to the measure μ.

We consider the following fractional operators that appear, for instance, in the study of subelliptic equations (see [Nag], [SW], [PW] and the references therein):

$$T_\alpha f(x) = \int_{\mathcal{X}} \frac{d(x,y)^\alpha}{\mu(B(x, d(x,y)))}\, f(y)\,d\mu(y)$$

for $0 < \alpha < n$. The associated maximal operator is

$$M_\alpha f(x) = \sup_{B \ni x} r(B)^\alpha \fint_B |f(y)|\,d\mu(y).$$

As mentioned before there is a version of Theorem 2.6 in spaces of homogeneous type. Thus, in order to extend Theorem 2.2 and, therefore, Theorem 4.1, one only needs to study the boundedness of the fractional maximal operators M_α defined above.

Proposition 5.1. *Let $0 < \alpha < n$, $1 \le p < n/\alpha$ and $1/q = 1/p - \alpha/n$. For every $w \in A_{p,q}$, M_α maps $L^p(w^p)$ into $L^q(w^q)$ if $p > 1$ and $L^1(w)$ into $L^{q,\infty}(w^q)$ if $p = 1$.*

Proof. The proof follows the classical scheme in [MW] and we give just a few details. Given α, p, q and w as above, using Hölder's inequality and that $w \in A_{p,q}$ one can easily obtain that for every $0 \le g \in L^p(w^p)$

$$\Big(r(B)^\alpha \fint_B g\,d\mu\Big)^q w^q(B) \lesssim \frac{r(B)^{\alpha q}}{\mu(B)^{\frac{q}{p}-1}} \Big(\int_B g^p\,w^p\,d\mu\Big)^{\frac{q}{p}} \lesssim \Big(\int_B g^p\,w^p\,d\mu\Big)^{\frac{q}{p}}, \tag{5.1}$$

where in the last estimate we have used that $\mu(B) \gtrsim r(B)^n$.

Given $f \in L^p(w^p)$ and $\lambda > 0$, Vitali's covering lemma yields

$$E_\lambda = \{x \in \mathcal{X} : M_\alpha f(x) > \lambda\} \subset \bigcup_j 5\,B_j$$

where $\{B_j\}_j$ is a family of pairwise disjoint balls such that $r(B_j)^\alpha \fint_{B_j} |f|\,d\mu > c\,\lambda$. Using (5.1) with $B = 5\,B_j$ and $g = |f|\,\chi_{B_j}$, and that $p < q$ we obtain

$$\begin{aligned} w^q(E_\lambda) &\le \sum_j w^q(5\,B_j) \lesssim \sum_j \Big(\int_{B_j} |f|^p\,w^p\,d\mu\Big)^{\frac{q}{p}} \Big(r(B_j)^\alpha \fint_{B_j} |f|\,d\mu\Big)^{-q} \\ &\lesssim \lambda^{-q} \Big(\sum_j \int_{B_j} |f|^p\,w^p\,d\mu\Big)^{\frac{q}{p}} \le \lambda^{-q} \Big(\int_{\mathcal{X}} |f|^p\,w^p\,d\mu\Big)^{\frac{q}{p}}. \end{aligned}$$

This shows that M_α maps $L^p(w^p)$ into $L^{q,\infty}(w^q)$ for every $w \in A_{p,q}$. When $p = 1$, this is the desired estimate.

To conclude that M_α is of strong type when $p > 1$ we use an interpolation argument in [MW]. Having fixed p, q and $w \in A_{p,q}$, we have $\tilde{w} = w^q \in A_r$ with $r = 1 + q/p'$ (see $(viii)$ in Proposition 2.1). We define a new operator $S_\alpha g = M_\alpha\big(g\,\tilde{w}^{\alpha/n}\big)$. By Proposition 2.1 part (iii), there exists $1 < p_1 < p$ such that $\tilde{w} \in A_{r_1}$ where $r_1 = 1 + q_1/p_1' < r$ and $1/q_1 = 1/p_1 - \alpha/n$. Thus, $\tilde{w}^{1/q_1} \in A_{p_1,q_1}$ and the argument above shows that S_α is bounded from $L^{p_1}(\tilde{w})$ to $L^{q_1,\infty}(\tilde{w})$. On the other hand we can find $p < p_2 < \alpha/n$, then we define $1/q_2 = 1/p_2 - \alpha/n$ and $r_2 = 1 + q_2/p_2' > r$ and so $\tilde{w} \in A_{r_2}$. Thus, $\tilde{w}^{1/q_2} \in A_{p_2,q_2}$ and as before we conclude that S_α maps $L^{p_2}(\tilde{w})$ into $L^{q_2,\infty}(\tilde{w})$. By Marcinkiewicz's interpolation theorem, it follows that S_α is bounded from $L^p(\tilde{w})$ into $L^q(\tilde{w})$ which in turn gives the desired weighted norm inequality M_α. □

Once we have obtained the weighted norm estimates for M_α, the proofs of Theorems 2.2 and 4.1 can be carried out in $\mathcal{X}$. The precise proofs and formulations are left to the interested reader. As a consequence, we show weighted estimates for T_α.

Corollary 5.2. *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then, T_α maps $L^p(w^p)$ into $L^q(w^q)$ for all $w \in A_{p,q}$.*

Proof. We first notice that it suffices work with the sublinear operator $f \longmapsto T_\alpha(|f|)$. Abusing on the notation, we write T_α for this new operator. Note that in that case $T_\alpha f \ge 0$.

We claim that T_α maps $L^p(\mu)$ into $L^q(\mu)$ for every $1 < p < q < \infty$ such that $1/p - 1/q = \alpha/n$, and also that

$$\sup_{x\in B} T_\alpha f(x) \lesssim \fint_B T_\alpha f(x)\,d\mu(x) \tag{5.2}$$

for every ball B, and $f \in L_c^\infty$ with $\operatorname{supp} f \subset \mathcal{X} \setminus 4B$.

Assuming this, we obtain the desired estimate. Fix p, q, and $w \in A_{p,q}$. Note that $w \in A_{1+1/p'} \cap RH_q$ and by (iii) in Proposition 2.1 there exists $1 < p_0 < p$ such that

$A_{1+\frac{1}{p_0}-\frac{1}{p}} \cap RH_q$. We take s_0 so that $1/p_0 - 1/s_0 = \alpha/n$, and $q_0 = \infty$. Then, (5.2) clearly implies (4.1) and thus T_α is bounded from $L^p(w^p)$ to $L^q(w^q)$.

To finish we need to show our claims. First, we obtain the boundedness of T_α. Fixed p, q, let $0 < s < \infty$ to be chosen. Then, as $\alpha > 0$,

$$T_\alpha\big(f\,\chi_{B(x,s)}\big)(x) = \sum_{k=0}^\infty \int_{2^{-k-1}\,s\le d(x,y)<2^{-k}\,s} \frac{(2^{-k}\,s)^\alpha}{\mu(B(x,2^{-k-1}\,s))}\,|f(y)|\,d\mu(y)$$

$$\lesssim s^\alpha \sum_{k=0}^\infty 2^{-\alpha\,k} \fint_{B(x,2^{-k}\,s)} |f|\,d\mu \lesssim s^\alpha\,Mf(x).$$

On the other hand, since $1 < p < n/\alpha$,

$$T_\alpha\big(f\,\chi_{\mathcal{X}\setminus B(x,s)}\big)(x) \le \|f\|_{L^p(\mu)} \left(\sum_{k=0}^\infty \int_{2^k\,s\le d(x,y)<2^{k+1}\,s} \frac{(2^{k+1}\,s)^{\alpha\,p'}}{\mu(B(x,2^k\,s))^{p'}}\,d\mu(y)\right)^{\frac{1}{p'}}$$

$$\lesssim \|f\|_{L^p(\mu)} \left(\sum_{k=0}^\infty \frac{(2^k\,s)^{\alpha\,p'}}{\mu(B(x,2^k\,s))^{p'-1}}\right)^{\frac{1}{p'}}$$

$$\lesssim \|f\|_{L^p(\mu)} \left(\sum_{k=0}^\infty (2^k\,s)^{\alpha\,p'-n\,(p'-1)}\right)^{\frac{1}{p'}} \lesssim \|f\|_{L^p(\mu)}\,s^{\alpha-\frac{n}{p}}.$$

Collecting the obtained estimates and choosing $s = \big(\|f\|_{L^p(\mu)}/Mf(x)\big)^{p/n}$ we conclude

$$T_\alpha f(x) \lesssim \|f\|_{L^p(\mu)}^{\frac{p\,\alpha}{n}}\,Mf(x)^{1-\frac{p\,\alpha}{n}} = \|f\|_{L^p(\mu)}^{\frac{p\,\alpha}{n}}\,Mf(x)^{\frac{p}{q}}.$$

Let us point out that this estimate in the classical setting was shown by Hedberg [Hed]. From here, that T_α maps $L^p(\mu)$ into $L^q(\mu)$ follows from the boundedness of M on $L^p(\mu)$.

Next, we show (5.2). Let B be a ball and f supported on $\mathcal{X}\setminus 4\,B$. For every x, $z\in B$ and $y\notin 4\,B$ we have that $d(x,y)\approx d(z,y)$ and the doubling condition yields $\mu(B(x,d(x,y)))\approx\mu(B(z,d(z,y)))$. Therefore $T_\alpha f(x)\approx T_\alpha f(z)$ for every x, $z\in B$ and this readily leads to (5.2). □

References

[Aus] P. Auscher, *On necessary and sufficient conditions for L^p estimates of Riesz transform associated elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc. **186** (871) (2007).

[AM1] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225–276.

[AM2] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part II: Off-diagonal estimates on spaces of homogeneous type*, J. Evol. Equ. **7** (2007), no. 2, 265–316.

[AM3] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part III: Harmonic analysis of elliptic operators*, J. Funct. Anal. **241** (2006), no. 2, 703–746.

[AM4] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part IV: Riesz transforms on manifolds and weights*, To appear in Math. Z. (2008), doi:10.1007/s00209-007-0286-1. Available at `http://www.uam.es/chema.martell`

[BS] C. Bennett & R.C. Sharpley, *Interpolation of Operators*, Pure and Appl. Math. 129, Academic Press, 1988.

[Cha] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.

[CF] D. Cruz-Uribe & A. Fiorenza, *Endpoint estimates and weighted norm inequalities for commutators of fractional integrals*, Publ. Mat. **47** (2003), 103–131.

[CMP] D. Cruz-Uribe, J.M. Martell & C. Pérez, *Extrapolation results for A_∞ weights and applications*, J. Funct. Anal. **213** (2004), 412–439.

[DY] X.T. Duong & L. Yan, *On commutators of fractional integrals*, Proc. Amer. Math. Soc. **132** (2004), no. 12, 3549–3557.

[ERS] A.F.M. ter Elst, D. Robinson & A. Sikora, *On second-order periodic elliptic operators in divergence form*, Math. Z. **238** (2001), 569–637.

[GR] J. García-Cuerva & J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North Holland Math. Studies 116, North Holland, Amsterdam, 1985.

[Gra] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.

[Hed] L.I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[JN] R. Johnson & C.J. Neugebauer, *Change of variable results for A_p-and reverse Hölder RH_r-classes*, Trans. Amer. Math. Soc. **328** (1991), no. 2, 639–666.

[MS] R.A. Macías & C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), no. 3, 257–270.

[Mar] J.M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. **161** (2004), 113–145.

[MW] B. Muckenhoupt & R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

[Nag] A. Nagel, *Vector fields and nonisotropic metrics*, Beijing lectures in harmonic analysis (Beijing, 1984), 241–306, Ann. of Math. Stud., **112**, Princeton Univ. Press, Princeton, NJ, 1986.

[Pe1] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), 163–185.

[Pe2] C. Pérez, *Sharp L^p-weighted Sobolev inequalities*, Ann. Inst. Fourier (Grenoble) 45 (1995), no. 3, 809–824.

[PW] C. Pérez & R.L. Wheeden, *Uncertainty principle estimates for vector fields*, J. Funct. Anal. **181** (2001), no. 1, 146–188.

[SW] E.T. Sawyer & R.L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

[ST] C. Segovia & J.L. Torrea, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat. **35** (1991), 209–235.

[She] Z. Shen, *Bounds of Riesz transforms on L^p spaces for second order elliptic operators*, Ann. Inst. Fourier **55** (2005), no. 1, 173–197.

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