Math. Z. 260 (2008), no. 3, 527--539

WEIGHTED NORM INEQUALITIES, OFF-DIAGONAL ESTIMATES AND ELLIPTIC OPERATORS.

PART IV: RIESZ TRANSFORMS ON MANIFOLDS AND WEIGHTS

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Abstract. This is the fourth article of our series. Here, we study weighted norm inequalities for the Riesz transform of the Laplace-Beltrami operator on Riemannian manifolds and of subelliptic sum of squares on Lie groups, under the doubling volume property and Gaussian upper bounds.

1. Introduction and main results

On $\mathbb{R}^n$, it is well-known that the classical Riesz transforms R_j, $1 \le j \le n$, are bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$ and are of weak-type (1,1) with respect to dx. As a consequence of the weighted theory for classical Calderón-Zygmund operators, the Riesz transforms are also bounded on $L^p(\mathbb{R}^n, w(x)dx)$ for all $w \in A_p(dx)$, $1 < p < \infty$, and are of weak-type (1,1) with respect to $w(x)dx$ for $w \in A_1(dx)$. Furthermore, it can be shown that the A_p condition on the weight is necessary for the weighted L^p boundedness of the Riesz transforms (see, for example, [Gra]).

On a manifold, there has been a number of works discussing the validity of the unweighted L^p theory depending on the geometry of the manifold. Although some progress has been done in this direction, the general picture is far from clear. A difficulty is that one has to leave the class of Calderón-Zygmund operators. In particular, the Riesz transforms on the manifold may not have Calderón-Zygmund kernels, either because one does not have regularity estimates, or worse because one does not even have size estimates. It turns out also that the range of p for which one obtains L^p boundedness may not be $(1, \infty)$. See [ACDH] for a detailed account on all this and Section 2 below.

Here, we wish to develop a weighted theory: we want to obtain weighted L^p estimates for a range of p and for Muckenhoupt weights with respect to the volume form. Of course, they must encompass the unweighted estimates so we shall restrict ourselves to situations where the unweighted theory has been developed. Nothing new will be done on the unweighted case (except the commutator result in Section 4).

Date: December 29, 2006. *Revised*: November 8, 2007.

2000 *Mathematics Subject Classification.* 58J35, 35B65, 35K05, 42B20.

Key words and phrases. Riemannian manifolds, Riesz transforms, Muckenhoupt weights, doubling property, Gaussian upper bounds.

This work was partially supported by the European Union (IHP Network “Harmonic Analysis and Related Problems” 2002-2006, Contract HPRN-CT-2001-00273-HARP). The second author was also supported by MEC “Programa Ramón y Cajal, 2005” and by MEC Grant MTM2007-60952.

We warmly thank T. Coulhon for interest and help in finding appropriate references. We also want to express our deep gratitude to F. Bernicot and J. Zhao for letting us use their unpublished work. This has led us to remove all use of Poincaré inequalities in the revised version. We thank the referee for suggestions to enhance the presentation of this article.

We assume that the volume form is doubling. In that case, we are able to apply a machinery developed in the first article of our series [AM]. In a sense, the results we obtain could have been included in the latter as an application of the theory there, but we preferred a separate article to focus here on the geometric aspect of manifolds and because it also needed technical estimates or ideas from [ACDH]. As a matter of facts, recent developments in [BZ] allowed us to improve and simplify our result. The reader should have both [ACDH] and [AM] handy from now on.

Let M be a complete non-compact Riemannian manifold with d its geodesic distance. Let Δ be the positive Laplace-Beltrami operator on M given by

$$\langle \Delta f, g\rangle = \int_M \nabla f \cdot \nabla g \, d\mu$$

where ∇ is the Riemannian gradient on M and $\cdot$ is an inner product on TM. The Riesz transform is the tangent space valued operator $\nabla\Delta^{-1/2}$ and it is bounded from $L^2(M,\mu)$ into $L^2(M;TM,\mu)$ by construction.

The manifold M verifies the doubling volume property if the volume form is doubling:

$$\mu(B(x,2\,r)) \le C\,\mu(B(x,r)) < \infty,$$

for all $x \in M$ and $r > 0$ where $B(x,r) = \{y \in M : d(x,y) < r\}$. A Riemannian manifold M equipped with the geodesic distance and a doubling volume form is a space of homogeneous type. Non-compactness of M implies infinite diameter, which together with the doubling volume property yields $\mu(M) = \infty$ (see for instance [Mar]).

One says that the heat kernel $p_t(x,y)$ of the semigroup $e^{-t\Delta}$ has Gaussian upper bounds if for some constants $c, C > 0$ and all $t > 0, x, y \in M$,

$$p_t(x,y) \le \frac{C}{\mu(B(x,\sqrt{t}))}\, e^{-c\frac{d^2(x,y)}{t}}.$$

It is known that under doubling it is a consequence of the same inequality only at $y = x$ [Gri, Theorem 1.1]. We recall the theorem proved in [CD].

Theorem 1.1. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and the Gaussian upper bounds. Then*

$$\big\|\,|\nabla\Delta^{-1/2} f|\,\big\|_p \le C_p \|f\|_p \tag{R_p}$$

holds for $1 < p < 2$ and all f bounded with compact support.

Here, $|\cdot|$ is the norm on TM associated with the inner product.

We shall set

$$q_+ = \sup\big\{p \in (1,\infty) \, : \, (R_p) \text{ holds}\big\}.$$

which satisfies $q_+ \ge 2$ under the assumptions of Theorem 1.1. It can be equal to 2 ([CD]). It is bigger than 2 assuming further the stronger L^2-Poincaré inequalities ([AC]). It can be equal to $+\infty$ (see below).

Let us turn to weighted estimates. Properties of Muckenhoupt weights A_p and reverse Hölder classes RH_s are reviewed in [AM, Section 2]. If $w \in A_\infty(\mu)$, one can define $r_w = \inf\{p > 1 : w \in A_p(\mu)\} \in [1,\infty)$ and $s_w = \sup\{s > 1 : w \in RH_s(\mu)\} \in (1,\infty]$. Given $1 \le p_0 < q_0 \le \infty$, we introduce the (possibly empty) set

$$\mathcal{W}_w(p_0,q_0) = \Big(p_0\, r_w, \frac{q_0}{(s_w)'}\Big) = \big\{p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}}(\mu) \cap RH_{(\frac{q_0}{p})'}(\mu)\big\}.$$

Here, $q' = \frac{q}{q-1}$ is the conjugate exponent to q. And note that RH_1 means no condition on the weight (besides A_∞).

Theorem 1.2. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and Gaussian upper bounds. Let $w \in A_\infty(\mu)$.*

(i) For $p \in \mathcal{W}_w(1, q_+)$, the Riesz transform is of strong-type (p,p) with respect to $w\,d\mu$, that is,

$$\big\| \, |\nabla \Delta^{-1/2} f| \, \big\|_{L^p(M,w)} \le C_{p,w} \, \|f\|_{L^p(M,w)} \tag{1.1}$$

for all f bounded with compact support.

(ii) If $w \in A_1(\mu) \cap RH_{(q_+)'}(\mu)$, then the Riesz transform is of weak-type $(1,1)$ with respect to $w\,d\mu$, that is,

$$\big\| \, |\nabla \Delta^{-1/2} f| \, \big\|_{L^{1,\infty}(M,w)} \le C_{1,w} \, \|f\|_{L^1(M,w)} \tag{1.2}$$

for all f bounded with compact support.

If $q_+ = \infty$ then the Riesz transform is bounded on $L^p(M,w)$ for $r_w < p < \infty$, that is, for $w \in A_p(\mu)$, and we obtain the same weighted theory as for the Riesz transform on $\mathbb{R}^n$:

Corollary 1.3. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and Gaussian upper bounds. Assume that the Riesz transform has strong type (p,p) with respect to $d\mu$ for all $1 < p < \infty$. Then the Riesz transform has strong type (p,p) with respect to $w\,d\mu$ for all $w \in A_p(\mu)$ and $1 < p < \infty$ and it is of weak-type $(1,1)$ with respect to $w\,d\mu$ for all $w \in A_1(\mu)$.*

In [AM, Lemma 4.6], examples of weights in $A_p(\mu) \cap RH_q(\mu)$ are given. The computations are done in the Euclidean setting, but most of them can be carried out in spaces of homogeneous type. In particular, given $f, g \in L^1(M,\mu)$ (or Dirac masses) $1 \le r < \infty$ and $1 < s \le \infty$, we have that $w(x) = M_\mu f(x)^{-(r-1)} + M_\mu g(x)^{1/s} \in A_p(\mu) \cap RH_q(\mu)$ (M_μ is the Hardy-Littlewood maximal function) for all $p > r$ and $q < s$ (and $p = r$ if $r = 1$ and $q = s$ if $s = \infty$). Thus, $r_w \le r$ and $s_w \ge s$.

We next provide some applications, then proof of our main result and eventually we add a short discussion on how to obtain (new) estimates for commutators with bounded mean oscillation functions.

2. Applications

Unweighted L^p bounds for Riesz transforms in different specific situations were reobtained in a unified manner in [ACDH] and the methods used there are precisely those which allowed us to start the weighted theory. Therefore, it is natural to apply this theory in return to those situations. Let us concentrate on five situations (more is done in [ACDH]). Recall the notion of Poincaré inequalities: Let $1 \le p < \infty$. One says that M satisfies the L^p-Poincaré property, we write M satisfies (P_p), if there exists $C > 0$ such that, for every ball B and every f with $f, \nabla f \in L^p_{\text{loc}}(\mu)$,

$$\int_B |f - f_B|^p \, d\mu \le C r(B)^p \int_B |\nabla f|^p \, d\mu. \tag{P_p}$$

Here, $r(B)$ is the radius of B, f_B is the mean value of f over B. It has been proved in [SC1] that the doubling volume property and (P_2) is equivalent to lower and upper

Gaussian estimates on the heat kernel. Recall that (P_p) implies (P_q) when $q > p$ (see for instance [HK]).

2.1. **Example without (P_2).** Consider two copies of $\mathbb{R}^n$ minus the unit ball glued smoothly along their unit circles with $n \geq 2$. It is shown in [CD] that this manifold has doubling volume form and Gaussian upper bounds. (P_2) does not hold: in fact, it satisfies (P_p) if and only if $p > n$ (see [HK] in the case of a double-sided cone in $\mathbb{R}^n$, which is the same). If $n = 2$, (R_p) holds if and only if $p \leq 2$ ([CD]). If $n > 2$, (R_p) holds if and only if $p < n$ ([CCH]). In any case, we have $q_+ = n$. Hence, $\mathcal{W}_w(1, q_+) = (r_w, n/(s_w)')$ is (contained in) the range of L^p boundedness for a given weight provided this is not empty. In other words, if $1 < p < n$ and $w \in A_p(\mu) \cap RH_{(n/p)'}(\mu)$ then one has strong type (p, p) with respect to $w\,d\mu$. For $p = 1$, one has weak-type $(1, 1)$ with respect to $w\,d\mu$ if $w \in A_1(\mu) \cap RH_{n'}(\mu)$.

2.2. **Manifolds with non-negative Ricci curvature.** In this case, the Riesz transform is bounded on (unweighted) L^p for $1 < p < \infty$ ([Ba1], [Ba2]). Thus $q_+ = \infty$. Such manifolds are known to have doubling volume form (see [Cha, Theorem 3.10]), (P_2) and even (P_1) [Bus] (see, for instance, [HK] or [SC2] for other references). By Corollary 1.3, we obtain strong-type (p, p) for $1 < p < \infty$ and $A_p(\mu)$ weights and weak-type (1,1) for $A_1(\mu)$ weights.

2.3. **Co-compact covering manifolds with polynomial growth deck transformation group.** In this case, one has the doubling volume property and (P_2) (see [SC2])*. That the Riesz transform is of unweighted strong type (p, p) for $1 < p \leq 2$ is due to [CD]. For $2 < p < \infty$ this is first done in [Dun] and hence $q_+ = \infty$. By Corollary 1.3, we obtain strong-type (p, p) for $1 < p < \infty$ and $A_p(\mu)$ weights and weak-type (1,1) for $A_1(\mu)$ weights.

2.4. **Conical manifolds with compact basis without boundary.** As mentioned in [ACDH], this is not strictly speaking a smooth manifold but it is stochastically complete and this is what is needed to develop the unweighted theory for the Riesz transform: it is shown in [Li] that q_+ is a finite value related to the bottom of the spectrum on the Laplace operator on the compact basis. Also, one has doubling and (P_2) by [Li] and [CL] (and even (P_1) by using the methods in [GS]). Hence, $\mathcal{W}_w(1, q_+) = (r_w, q_+/(s_w)')$ is (contained in) the range of L^p boundedness for a given weight provided this is not empty. In other words, if $1 < p < q_+$ and $w \in A_p(\mu) \cap RH_{(q_+/p)'}(\mu)$ then one has strong type (p, p) with respect to $w\,d\mu$. For $p = 1$, one has weak-type $(1, 1)$ with respect to $w\,d\mu$ if $w \in A_1(\mu) \cap RH_{(q_+)'}(\mu)$.

2.5. **Lie groups with polynomial volume growth endowed with a sublaplacian.** One starts with left-invariant vector fields X_j satisfying the Hörmander condition and μ is the left (and right) invariant Haar measure. The sublaplacian is $\Delta = -\sum_{j=1}^n X_j^2$. One has the doubling volume property and (P_2) (and even (P_1)) (see [Var] or [HK, p. 70] for a statement and references). The statement of Theorem 1.2 applies with no change to the Riesz transforms $X_j \Delta^{-1/2}$. In this case, $q_+ = \infty$ from [Ale]. By Corollary 1.3, the weighted theory for these Riesz transforms is the

*(P_1) also holds by a discretization method [CS, Théorème 7.2] and Poincaré inequalities for discrete groups (see [HK, p.76]).

same as the ones in $\mathbb{R}^n$ for $1 \le p < \infty$: strong type (p,p) with respect to $w\,d\mu$ holds for $w \in A_p(\mu)$ and $1 < p < \infty$, and weak-type $(1,1)$ with respect to $w\,d\mu$ for $w \in A_1(\mu)$.

3. Proof of the main result

We assume that M satisfies the doubling volume property and Gaussian upper bounds.

We first introduce some notation. Given a ball B we set $C_j(B) = 4B$ for $j = 1$ and $C_j(B) = 2^{j+1}\,B \setminus 2^j\,B$ for $j \ge 2$, where $\lambda\,B$ is the ball co-centered with B and radius $\lambda r(B)$. We use the notation

$$\fint_B h d\mu = \frac{1}{\mu(B)} \int_B h\,d\mu, \qquad \fint_{C_j(B)} h\,d\mu = \frac{1}{\mu(2^{j+1}\,B)} \int_{C_j(B)} h\,d\mu.$$

We write D_μ for the doubling order of μ: $\mu(\lambda B) \le C\,\lambda^{D_\mu}\,\mu(B)$ for every $\lambda > 1$. In the sequel M_μ is the Hardy-Littlewood maximal function with respect to the measure μ on M.

We state a particular case of [AM, Theorem 3.1][†] that is used in the proof of Theorem 1.2, (i).

Theorem 3.1. *Fix $1 < q < \infty$, $a \ge 1$ and $v \in RH_{s'}(\mu)$, $1 < s < q$. Then, there exist C and $K_0 \ge 1$ with the following property: Assume that F, G and H_1 are non-negative measurable functions on M such that for any ball B there exist non-negative functions G_B and H_B with $F(x) \le G_B(x) + H_B(x)$ for a.e. $x \in B$ and, for all $x, \bar{x} \in B$,*

$$\Big(\fint_B H_B^q\,d\mu\Big)^{\frac{1}{q}} \le a\,M_\mu F(x) + H_1(\bar{x}), \qquad \fint_B G_B\,d\mu \le G(x). \tag{3.1}$$

If $1 < r \le q/s$ and $F \in L^1(M,\mu)$ (this assumption being only qualitative) we have

$$\|M_\mu F\|_{L^r(M,v)} \le C\,\|G\|_{L^r(M,v)} + C\,\|H_1\|_{L^r(M,v)}. \tag{3.2}$$

Proof of Theorem 1.2, (i). The argument borrows some ideas from [BZ] which are adapted to the present situation. Fix $w \in A_\infty(\mu)$ and $p \in \mathcal{W}_w(1, q_+)$. Then (see, for instance, [AM, Proposition 2.1]) there exist p_0, q_0 such that

$$1 < p_0 < p < q_0 < q_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}}(\mu) \cap RH_{\left(\frac{q_0}{p}\right)'}(\mu).$$

By [AM, Lemma 4.4] we have that $v = w^{1-p'} \in A_{p'/q_0'}(\mu) \cap RH_{(p_0'/p')'}(\mu)$. We write $\mathcal{T} = \nabla\Delta^{-1/2}$ and observe that the boundedness of $\mathcal{T}$ from $L^p(M,w)$ to $L^p(M;TM,w)$ is equivalent to that of $\mathcal{T}^*$ from $L^{p'}(M;TM,v)$ to $L^{p'}(M,v)$ —we notice that $\mathcal{T}$ takes scalar valued functions on M to functions valued in the tangent space (sections) and $\mathcal{T}^*$ the opposite, TM being equipped with the inner product arising in the definition of the Laplace-Beltrami operator—.

Fix $f \in L^\infty_c(M;TM,\mu)$[‡], and write $h = \mathcal{T}^* f$ and $F = |h|^{q_0'}$. Notice that $F \in L^1(M,\mu)$ since $\mathcal{T}^*$ is bounded from $L^{q_0'}(M;TM,\mu)$ to $L^{q_0'}(M,\mu)$ as $1 < q_0 < q_+$ and $\mathcal{T}$ is bounded from $L^{q_0}(M,\mu)$ to $L^{q_0}(M;TM,\mu)$. We pick $\mathcal{A}_r = I - (I - e^{-r^2\Delta})^m$ with m large enough. Given a ball B we write r_B for its radius. Then,

$$F \le G_B + H_B \equiv 2^{q_0'-1}\,|(I - \mathcal{A}_{r_B})^* h|^{q_0'} + 2^{q_0'-1}\,|\mathcal{A}_{r_B}^* h|^{q_0'}.$$

[†] It is stated in the Euclidean setting, see [AM, Section 5] for the extension to spaces of homogeneous type

[‡] Here and subsequently, the subscript c means with compact support.

We first estimate H_B. Set $q = p_0'/q_0'$ and observe that by duality there exists $g \in L^{p_0}(B, d\mu/\mu(B))$ with norm 1 such that for all $x \in B$

$$\Big(\fint_B H_B^q \, d\mu\Big)^{\frac{1}{q\,q_0'}} \lesssim \mu(B)^{-1} \int_M |h|\, |\mathcal{A}_{r_B} g| \, d\mu$$
$$\lesssim \sum_{j=1}^{\infty} 2^{j\,D_\mu} \Big(\fint_{C_j(B)} |h|^{q_0'} \, d\mu\Big)^{\frac{1}{q_0'}} \Big(\fint_{C_j(B)} |\mathcal{A}_{r_B} g|^{q_0} \, d\mu\Big)^{\frac{1}{q_0}}$$
$$\le M_\mu F(x)^{\frac{1}{q_0'}} \sum_{j=1}^{\infty} 2^{j\,D_\mu} \Big(\fint_{C_j(B)} |\mathcal{A}_{r_B} g|^{q_0} \, d\mu\Big)^{\frac{1}{q_0}}.$$

To estimate the summands we use the Gaussian upper bound on $p_t(x,y)$, so that for any fixed integer m there exist $c, C > 0$ such that for all $j \ge 1$, all ball B, all $g \in L^1(M, \mu)$ supported in B and all $1 \le k \le m$,

$$\sup_{C_j(B)} |e^{-k\,r_B^2\,\Delta} g| \le C\, e^{-c\,4^j} \fint_B |g| \, d\mu. \tag{3.3}$$

Then expanding $\mathcal{A}_r = I - (I - e^{-r^2\,\Delta})^m$ we conclude that

$$\Big(\fint_B H_B^q \, d\mu\Big)^{\frac{1}{q\,q_0'}} \lesssim M_\mu F(x)^{\frac{1}{q_0'}} \sum_{j=1}^{\infty} 2^{j\,D_\mu}\, e^{-c\,4^j} \Big(\fint_B |g|^{p_0} \, d\mu\Big)^{\frac{1}{p_0}} \lesssim M_\mu F(x)^{\frac{1}{q_0'}}. \tag{3.4}$$

We next estimate G_B. Using duality there exists $g \in L^{q_0}(B, d\mu/\mu(B))$ with norm 1 such that for all $x \in B$

$$\Big(\fint_B G_B \, d\mu\Big)^{\frac{1}{q_0'}} \lesssim \mu(B)^{-1} \int_M |f|\, |\mathcal{T}(I - \mathcal{A}_{r_B}) g| \, d\mu$$
$$\lesssim \sum_{j=1}^{\infty} 2^{j\,D_\mu} \Big(\fint_{C_j(B)} |f|^{q_0'} \, d\mu\Big)^{\frac{1}{q_0'}} \Big(\fint_{C_j(B)} |\mathcal{T}(I - \mathcal{A}_{r_B}) g|^{q_0} \, d\mu\Big)^{\frac{1}{q_0}}$$
$$\le M_\mu(|f|^{q_0'})(x)^{\frac{1}{q_0'}} \sum_{j=1}^{\infty} 2^{j\,D_\mu} \Big(\fint_{C_j(B)} |\mathcal{T}(I - \mathcal{A}_{r_B}) g|^{q_0} \, d\mu\Big)^{\frac{1}{q_0}}.$$

To estimate the terms in the sum we use the following auxiliary result whose proof is given below.

Lemma 3.2. *For all $\beta \in [1, \tilde{q}_+) \cup [1, 2]$, one has the following estimate: for all $m \ge 1$, there exists $C > 0$ such that for all $j \ge 2$, all ball B, all $g \in L^1(M, \mu)$ with support in B,*

$$\Big(\fint_{C_j(B)} |\nabla \Delta^{-1/2} (I - e^{-r(B)^2\,\Delta})^m g|^\beta \, d\mu\Big)^{\frac{1}{\beta}} \le C 4^{-jm} \fint_B |g| \, d\mu. \tag{3.5}$$

Here, $\tilde{q}_+$ is defined as the supremum of those $p \in (1, \infty)$ such that for all $t > 0$,

$$\big\| \, |\nabla e^{-t\,\Delta} f| \, \big\|_p \le C\, t^{-1/2} \|f\|_p. \tag{3.6}$$

By analyticity of the heat semigroup, one always have $\tilde{q}_+ \ge q_+$. Under the doubling volume property and (P_2), it is shown in [ACDH, Theorem 1.3] that $q_+ = \tilde{q}_+$. We do not know if the equality holds or not under doubling and Gaussian upper bounds.

This lemma allows us to conclude right away with the terms where $j \geq 2$. When $j = 1$, we use that $\mathcal{T}$ is bounded from $L^{q_0}(M,\mu)$ to $L^{q_0}(M;TM,\mu)$ as $1 < q_0 < q_+$. Also, applying (3.3) it follows easily that

$$\begin{aligned}\fint_{4B} |\mathcal{T}(I-\mathcal{A}_{r_B})g|^{q_0}\,d\mu &\lesssim \frac{1}{\mu(4B)}\Big(\int_B |g|^{q_0}\,d\mu + \sum_{j=1}^{\infty}\int_{C_j(B)} |\mathcal{A}_{r_B}g|^{q_0}\,d\mu\Big)\\ &\lesssim \fint_B |g|^{q_0}\,d\mu \sum_{j=1}^{\infty} 2^{j\,D_\mu}\,e^{-c\,4^j} \lesssim \fint_B |g|^{q_0}\,d\mu. \end{aligned}\tag{3.7}$$

Using this and (3.5) (with $\beta = q_0 < q_+ \leq \tilde{q}_+$) we conclude the estimate for G_B:

$$\begin{aligned}\Big(\fint_B G_B\,d\mu\Big)^{\frac{1}{q_0'}} &\lesssim M_\mu(|f|^{q_0'})(x)^{\frac{1}{q_0'}} \sum_{j=1}^{\infty} 2^{j\,D_\mu} 4^{-j\,m}\Big(\fint_B |g|^{q_0}\,d\mu\Big)^{\frac{1}{q_0}}\\ &\leq C\,M_\mu(|f|^{q_0'})(x)^{\frac{1}{q_0'}} = G(x)^{\frac{1}{q_0'}},\end{aligned}\tag{3.8}$$

provided $m > D_\mu/2$. With these estimates in hand we can use Theorem 3.1 with $r = p'/q_0'$, $q = p_0'/q_0'$ and $H_1 \equiv 0$. Notice that $v \in RH_{s'}(\mu)$ with $s = p_0'/p'$, $1 < s < q < \infty$ and $r = q/s$. Hence, using $v \in A_r(\mu)$ we obtain the desired estimate

$$\|\mathcal{T}^* f\|_{L^{p'}(M,v)}^{q_0'} \leq \|M_\mu F\|_{L^r(M,v)} \lesssim \|M_\mu(|f|^{q_0'})\|_{L^r(M,v)} \lesssim \||f|\|_{L^{p'}(M,v)}^{q_0'}. \tag{3.9}$$

□

Proof of Lemma 3.2. First, this estimate is known for $\beta = 2$ (see [ACDH]). Also, the inequality for a fixed β_0 implies the same one for all β with $1 \leq \beta \leq \beta_0$. It suffices to treat the case $\beta > 2$, which happens only if $\tilde{q}_+ > 2$.

We use a trick from [ACDH, Proof of Lemma 3.1]. Fix a ball B, with radius r, and $f \in L^\infty(M,\mu)$ supported in B. We have

$$\nabla\Delta^{-1/2}(I - e^{-r^2\Delta})^m f = \int_0^\infty g_r(t)\,\nabla e^{-t\Delta} f\,dt$$

where $g_r\colon \mathbb{R}^+ \to \mathbb{R}$ is a function such that

$$\int_0^\infty |g_r(t)|\,e^{-\frac{c\,4^j\,r^2}{t}}\,\frac{dt}{\sqrt{t}} \leq C_m\,4^{-j\,m}. \tag{3.10}$$

By definition of $\tilde{q}_+$ and the argument of [ACDH, p. 944] we have

$$\Big(\int_M |\nabla_x p_t(x,y)|^\beta\,e^{\gamma\frac{d^2(x,y)}{t}}\,d\mu(x)\Big)^{1/\beta} \leq \frac{C}{\sqrt{t}\,\big[\mu(B(y,\sqrt{t}))\big]^{1-1/\beta}},$$

for all $t > 0$ and $y \in M$, with $\gamma > 0$ depending on β. This implies that for all $j \geq 2$, $y \in B$ and all $t > 0$,

$$\Big(\fint_{C_j(B)} |\nabla_x p_t(x,y)|^\beta\,d\mu(x)\Big)^{1/\beta} \lesssim \frac{1}{\sqrt{t}}\,e^{-\frac{c\,4^j\,r^2}{t}}\,\frac{1}{\mu(B(y,\sqrt{t}))^{1-1/\beta}\,\mu(2^{j+1}B)^{1/\beta}}.$$

Using the doubling property, $\mu(2^{j+1}B) \sim \mu(B(y,2^{j+1}r))$ uniformly in $y \in B$ and

$$\frac{\mu(B(y,2^{j+1}r))}{\mu(B(y,\sqrt{t}))} \lesssim \max\Big\{1, \frac{2^j\,r}{\sqrt{t}}\Big\}^{D_\mu}.$$

Hence, with another $c>0$,

$$\Big(\fint_{C_j(B)}|\nabla_x\, p_t(x,y)|^\beta\,d\mu(x)\Big)^{1/\beta}\lesssim\frac{1}{\sqrt t}\,e^{-\frac{c\,4^j\,r^2}{t}}\,\frac{1}{\mu(2^{j+1}\,B)}\le\frac{1}{\sqrt t}\,e^{-\frac{c\,4^j\,r^2}{t}}\,\frac{1}{\mu(B)}.$$

We conclude using Minkowski's integral inequality and (3.10) that the left hand side of (3.5) is bounded by

$$\int_0^\infty|g_r(t)|\int_B|f(y)|\,\Big(\fint_{C_j(B)}|\nabla_x\,p_t(x,y)|^\beta\,d\mu(x)\Big)^{1/\beta}d\mu(y)\,dt$$
$$\lesssim\int_0^\infty|g_r(t)|\frac{1}{\sqrt t}\,e^{-\frac{c\,4^j\,r^2}{t}}\,dt\,\fint_B|f|\,d\mu\lesssim 4^{-j\,m}\fint_B|f|\,d\mu.$$

□

The following result, used to prove Theorem 1.2, (ii), is taken from [AM, Theorem 8.8 & Remark 8.10] (see also [AM, Section 8.4] for the extension to spaces of homogeneous type)

Theorem 3.3. *Let $1\le p_0<q_0\le\infty$ and $w\in A_\infty(\mu)$. Let T be a sublinear operator defined on $L^2(M,\mu)$ and $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $L^\infty_c(M,\mu)$ into $L^2(M,\mu)$. Assume the following conditions:*

- (a) *There exists $q\in\mathcal{W}_w(p_0,q_0)$ such that T is bounded from $L^q(M,w)$ to $L^{q,\infty}(M,w)$.*
- (b) *For all $j\ge1$, there exist a constant α_j such that for any ball B with $r(B)$ its radius and for any $f\in L^\infty_c(M,\mu)$ supported in B,*
$$\Big(\fint_{C_j(B)}|\mathcal{A}_{r(B)}f|^{q_0}\,d\mu\Big)^{\frac{1}{q_0}}\le\alpha_j\,\Big(\fint_B|f|^{p_0}\,d\mu\Big)^{\frac{1}{p_0}}.\tag{3.11}$$
- (c) *There exists $\beta>(s_w)'$, i.e. $w\in RH_{\beta'}(\mu)$, with the following property: for all $j\ge2$, there exist a constant α_j such that for any ball B with $r(B)$ its radius and for any $f\in L^\infty_c(M,\mu)$ supported in B and for $j\ge2$,*
$$\Big(\fint_{C_j(B)}|T(I-\mathcal{A}_{r(B)})f|^\beta\,d\mu\Big)^{\frac{1}{\beta}}\le\alpha_j\,\Big(\fint_B|f|^{p_0}\,d\mu\Big)^{\frac{1}{p_0}}.\tag{3.12}$$
- (d) *$\sum_j\alpha_j\,2^{D_w\,j}<\infty$ for α_j in (b) and (c), where D_w is the doubling constant of $w\,d\mu$.*

If $w\in A_1(\mu)\cap RH_{(q_0/p_0)'}(\mu)$ then T is of weak-type (p_0,p_0) with respect to $w\,d\mu$, that is, for all $f\in L^\infty_c(M,\mu)$,

$$\|Tf\|_{L^{p_0,\infty}(M,w)}\le C\,\|f\|_{L^{p_0}(M,w)}.$$

Proof of Theorem 1.2, (ii). We are going to apply Theorem 3.3 with $p_0=1$ and $q_0=q_+$. Thus we need to check that the four items hold. Fix $w\in A_1(\mu)\cap RH_{(q_+)'}(\mu)$. By [AM, Proposition 2.1], there exists $1<q<q_+$ such that $w\in A_q(\mu)\cap RH_{(q_+/q)'}(\mu)$. This means that $q\in\mathcal{W}_w(1,q_+)$, therefore by (i), $T=|\nabla\Delta^{-1/2}|$ is bounded on $L^q(M,w)$ and so (a) holds.

We pick $\mathcal{A}_r=I-(I-e^{-r^2\,\Delta})^m$ with m large enough to be chosen. Notice that expanding $\mathcal{A}_r$, (3.3) yields (b) with $\alpha_j=C\,e^{-c\,4^j}$. To see (c) we apply Lemma 3.2 with $(s_w)'<\beta$ —notice that such β exist: we have $q_+\le\tilde q_+$ and $w\in RH_{(q_+)'}(\mu)$ implies $(s_w)'<q_+$—. Then, we obtain (c) with $\alpha_j=C\,4^{-j\,m}$. Finally, we pick $m>D_w/2$ so that (d) holds and therefore Theorem 3.3 gives the weak-type $(1,1)$ with respect to $w\,d\mu$. □

4. Commutators

Let us write again $\mathcal{T}=\nabla\Delta^{-1/2}$ and take $b\in \mathrm{BMO}(M,\mu)$ (the space of bounded mean oscillation functions on M). We define the first order commutator $\mathcal{T}_b^1 g = [b,\mathcal{T}]g = b\,\mathcal{T} g - \mathcal{T}(b\,g)$, and for $k\geq 2$ the k-th order commutator is $\mathcal{T}_b^k=[b,\mathcal{T}_b^{k-1}]$. Here g,b are scalar valued and $\mathcal{T}_b^k g$ is valued in the tangent space.

Theorem 4.1. *Under the assumptions of Theorem 1.2, $\mathcal{T}_b^k$ satisfies (1.1) for each $k\geq 1$, that is, it is bounded from $L^p(M,w)$ into $L^p(M;TM,w)$ under the same conditions on w,p.*

This theorem applies in particular to the five situations described in Section 2. Note that even the unweighted L^p estimates for the commutators are new.

Proof. The proof is similar to that of Theorem 1.2 using again the ideas in [BZ] and we point out the main changes. We only consider the case $k=1$: the general case follows by induction and the details are left to the reader (see [AM, Section 6.2] for similar arguments). As in [AM, Lemma 6.1] it suffices to assume qualitatively $b\in L^\infty(M,\mu)$ and quantitatively $\|b\|_{\mathrm{BMO}(M,\mu)}=1$ and get uniform bounds.

We proceed as before working with $\mathcal{T}_b^1$ in place of $\mathcal{T}$. Write $F=|(\mathcal{T}_b^1)^*f|^{q_0'}$ with $f\in L_c^\infty(M;TM,\mu)$ and observe that $F\in L^1(M,\mu)$ as $b\in L^\infty(M,\mu)$ and $\mathcal{T}^*$ is bounded from $L^{q_0'}(M;TM,\mu)$ into $L^{q_0'}(M,\mu)$ (as $1<q_0<q_+$) —we observe that this is the only place where we use that $b\in L^\infty(M,\mu)$—. Fixing B we write $\hat{b}=b-b_B$ and decompose $\mathcal{T}_b^1$ as $\mathcal{T}_b^1 g=-\mathcal{T}(\hat{b}g)+\hat{b}\,\mathcal{T} g$. Using this equality one sees $(\mathcal{T}_b^1)^*=-(\mathcal{T}^*)_b^1$. Then we have

$$
\begin{aligned}
F&=|(\mathcal{T}_b^1)^*f|^{q_0'}=|(\mathcal{T}^*)_b^1 f|^{q_0'}\leq 2^{q_0'-1}\,|\hat{b}\,\mathcal{T}^*f|^{q_0'}+2^{q_0'-1}\,|\mathcal{T}^*(\hat{b}\,f)|^{q_0'}\\
&\leq \big(2^{q_0'-1}\,|\hat{b}\,\mathcal{T}^*f|^{q_0'}+4^{q_0'-1}\,|(I-\mathcal{A}_{r_B})^*\mathcal{T}^*(\hat{b}\,f)|^{q_0'}\big)+4^{q_0'-1}\,|\mathcal{A}_{r_B}^*\mathcal{T}^*(\hat{b}\,f)|^{q_0'}\\
&=G_B+H_B.
\end{aligned}
$$

We estimate H_B. By duality we pick g as before and obtain

$$
\begin{aligned}
\Big(\fint_B H_B^q\,d\mu\Big)^{\frac{1}{q\,q_0'}}&=C\,\mu(B)^{-1}\int_M \mathcal{T}^*(\hat{b}\,f)\,\mathcal{A}_{r_B}g\,d\mu\\
&=C\,\mu(B)^{-1}\int_M\big(-(\mathcal{T}^*)_b^1 f+\hat{b}\,\mathcal{T}^*f\big)\,\mathcal{A}_{r_B}g\,d\mu\\
&\lesssim \mu(B)^{-1}\int_M|(\mathcal{T}^*)_b^1 f|\,|\mathcal{A}_{r_B}g|\,d\mu+\mu(B)^{-1}\int_M|\hat{b}|\,|\mathcal{T}^*f|\,|\mathcal{A}_{r_B}g|\,d\mu=I+II.
\end{aligned}
$$

The estimate for I follows as in (3.4) by using (3.3): for all $x\in B$,

$$
I\lesssim\sum_{j=1}^\infty 2^{j\,D_\mu}\Big(\fint_{C_j(B)}F\,d\mu\Big)^{\frac{1}{q_0'}}\Big(\fint_{C_j(B)}|\mathcal{A}_{r_B}g|^{q_0}\,d\mu\Big)^{\frac{1}{q_0}}\lesssim M_\mu F(x)^{\frac{1}{q_0'}}.
$$

We pick $1<s<\infty$ and use Hölder's inequality to obtain that for all $\bar{x}\in B$

$$
II\lesssim\sum_{j=1}^\infty 2^{j\,D_\mu}\Big(\fint_{C_j(B)}|\mathcal{T}^*f|^{q_0'}\,d\mu\Big)^{\frac{1}{q_0'}}\Big(\fint_{C_j(B)}|\mathcal{A}_{r_B}g|^{q_0\,s}\,d\mu\Big)^{\frac{1}{q_0\,s}}\Big(\fint_{2^{j+1}B}|\hat{b}|^{q_0\,s'}\,d\mu\Big)^{\frac{1}{q_0\,s'}}
$$

$$\lesssim \|b\|_{\mathrm{BMO}(M,\mu)} M_\mu(|\mathcal{T}^* f|^{q_0'})(\bar{x})^{\frac{1}{q_0'}} \sum_{j=1}^\infty 2^{j\,D_\mu} e^{-c\,4^j}\,(1+j) \lesssim M_\mu(|\mathcal{T}^* f|^{q_0'})(\bar{x})^{\frac{1}{q_0'}},$$

where we have used (3.3) and John-Nirenberg's inequality. Collecting I and II we conclude the first estimate in (3.1) with $H_1 = M_\mu(|\mathcal{T}^* f|^{q_0'})$. Let us write $G_{B,1}$ and $G_{B,2}$ for each of the terms that define G_B and we estimate them in turn. Take $\delta > 1$ to be chosen and use John-Nirenberg's inequality: for any $x \in B$ we have

$$\begin{aligned}
\fint_B G_{B,1}\,d\mu = C \fint_B |\hat{b}\,\mathcal{T}^* f|^{q_0'}\,d\mu &\lesssim \Big(\fint_B |\mathcal{T}^* f|^{q_0'\,\delta}\,d\mu\Big)^{\frac{1}{\delta}} \Big(\fint_B |b - b_B|^{q_0'\,\delta'}\,d\mu\Big)^{\frac{1}{\delta'}} \\
&\lesssim \|b\|_{\mathrm{BMO}(M,\mu)}^{q_0'} M_\mu(|\mathcal{T}^* f|^{q_0'\,\delta})(\bar{x})^{\frac{1}{\delta}}.
\end{aligned}$$

To estimate $G_{B,2}$ we proceed as with G_B in the proof of Theorem 1.2. Let g be the corresponding dual function and use again John-Nirenberg's inequality

$$\begin{aligned}
\Big(\fint_B G_{B,2}\,d\mu\Big)^{\frac{1}{q_0'}} &= C\Big(\fint_B |(I - \mathcal{A}_{r_B})^*\,\mathcal{T}^*(\hat{b} f)|^{q_0'}\,d\mu\Big)^{\frac{1}{q_0'}} \\
&\lesssim \mu(B)^{-1} \int_M |\hat{b}|\,|f|\,|\mathcal{T}(I - \mathcal{A}_{r_B})g|\,d\mu \\
&\lesssim \sum_{j=1}^\infty 2^{j\,D_\mu} \Big(\fint_{2^{j+1}B} |b - b_B|^{q_0'\,\delta'}\,d\mu\Big)^{\frac{1}{q_0'\delta'}} \Big(\fint_{C_j(B)} |f|^{q_0'\,\delta}\,d\mu\Big)^{\frac{1}{q_0'\,\delta}} \\
&\qquad\qquad \Big(\fint_{C_j(B)} |\mathcal{T}(I - \mathcal{A}_{r_B})g|^{q_0}\,d\mu\Big)^{\frac{1}{q_0}} \\
&\lesssim M_\mu(|f|^{\delta\,q_0'})(x)^{\frac{1}{\delta\,q_0'}} \sum_{j=1}^\infty 2^{j\,D_\mu}\,(1+j) \Big(\fint_{C_j(B)} |\mathcal{T}(I - \mathcal{A}_{r_B})g|^{q_0}\,d\mu\Big)^{\frac{1}{q_0}} \\
&\lesssim M_\mu(|f|^{\delta\,q_0'})(x)^{\frac{1}{\delta\,q_0'}}.
\end{aligned}$$

where in the last estimate we have proceeded as in (3.8) using (3.5) for $j \geq 2$ and (3.7) for $j = 1$. Gathering what has been obtained for $G_{B,1}$ and $G_{B,2}$ we conclude the second estimate in (3.1) with $G = M_\mu(|\mathcal{T}^* f|^{q_0'\,\delta})^{\frac{1}{\delta}} + M_\mu(|f|^{\delta\,q_0'})^{\frac{1}{\delta}}$.

We apply Theorem 3.1 as in the proof of Theorem 1.2. In this case, we observe that as $v \in A_r(\mu)$, there exists $1 < \delta < r$ so that $v \in A_{r/\delta}(\mu)$. Then, the desired estimate follows

$$\begin{aligned}
\|(\mathcal{T}_b^1)^* f\|_{L^{p'}(M,v)}^{q_0'} \leq \|M_\mu F\|_{L^r(M,v)} &\lesssim \|G\|_{L^r(M,v)} + \|H_1\|_{L^r(M,v)} \\
&\leq \|M_\mu(|f|^{\delta\,q_0'})^{\frac{1}{\delta}}\|_{L^r(M,v)} + \|M_\mu(|\mathcal{T}^* f|^{\delta\,q_0'})^{\frac{1}{\delta}}\|_{L^r(M,v)} \\
&\lesssim \|f\|_{L^{p'}(M;TM,v)}^{q_0'} + \|\mathcal{T}^* f\|_{L^{p'}(M,v)}^{q_0'} \lesssim \|f\|_{L^{p'}(M;TM,v)}^{q_0'}
\end{aligned}$$

where we have used that $\mathcal{T}^*$ is bounded from $L^{p'}(M;TM,v)$ into $L^{p'}(M,v)$ (see (3.9)). □

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