WEIGHTED NORM INEQUALITIES, OFF-DIAGONAL ESTIMATES AND ELLIPTIC OPERATORS

PART III: HARMONIC ANALYSIS OF ELLIPTIC OPERATORS

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ABSTRACT. This is the third part of a series of four articles on weighted norm inequalities, off-diagonal estimates and elliptic operators. For L in some class of elliptic operators, we study weighted norm L^p inequalities for singular "non-integral" operators arising from L; those are the operators $\varphi(L)$ for bounded holomorphic functions φ, the Riesz transforms $\nabla L^{-1/2}$ (or $(-\Delta)^{1/2} L^{-1/2}$) and its inverse $L^{1/2}(-\Delta)^{-1/2}$, some quadratic functionals g_L and G_L of Littlewood-Paley-Stein type and also some vector-valued inequalities such as the ones involved for maximal L^p-regularity. For each, we obtain sharp or nearly sharp ranges of p using the general theory for boundedness of Part I and the off-diagonal estimates of Part II. We also obtain commutator results with BMO functions.

CONTENTS

1. INTRODUCTION

In this part, we consider divergence form uniformly elliptic complex operators $L = -\operatorname{div}(A\,\nabla)$ in $\mathbb{R}^n$ and we are interested in weighted L^p estimates for:

(a) $\varphi(L)$ with bounded holomorphic functions φ on sectors (Section 4).

(b) The square root $L^{1/2}$ compared to the ones for ∇ and, in particular, the Riesz transforms $\nabla L^{-1/2}$ (Sections 5, 6).

(*c*) Typical square functions "à la" Littlewood-Paley-Stein: one, g_L, using only functions of L, and the other, G_L, combining functions of L and the gradient operator (Section 7).

(*d*) Vector-valued inequalities for the operators above and the so-called R-boundedness of the analytic semigroup $\{e^{-z\,L}\}$ which is linked to maximal regularity (Section 8).

Let us stress that those operators may not be representable with "usable" kernels: they are "non-integral". But they still are singular in the sense that they are of order 0. Hence, usual methods for singular integrals have to be strengthened. The unweighted L^p estimates are described in [Aus] for the operators in $(a) - (c)$, with emphasis on the sharpness of the ranges of p. The instrumental tools are two criteria for L^p boundedness, valid in spaces of homogeneous type: one was a sharper and simpler version of a theorem by Blunck and Kunstmann [BK1] in the spirit of Hörmander's criterion via the Calderón-Zygmund decomposition, and the other one a criterion of the first author, Coulhon, Duong and Hofmann [ACDH] in the spirit of Fefferman and Stein's sharp maximal function via a good-λ inequality. The main interest of those results were that they yield L^p boundedness of (sub)linear operators on spaces of homogeneous type for p in an arbitrary interval. Such theorems are extended in Part I of our series [AM1] to obtain weighted L^p bounds for the operator itself, its commutators with a BMO function and also vector-valued expressions.

In Part II [AM2], we studied one-parameter families of operators satisfying local $L^p - L^q$ estimates called off-diagonal estimates on balls (the setting is that of a space of homogeneous type). Among other things, such estimates imply uniform L^p-boundedness and are stable under composition. In case of one-parameter semigroups, we showed that as soon as there exists *one* pair (p,q) of indices with $p < q$ for which these local $L^p - L^q$ estimates hold, then they hold for *all* pairs of indices taken in the interior of the range of L^p boundedness. This fact is of utmost importance for applications as we often need to play with exponents. We showed that such estimates pass from the unweighted case to the weighted case. Eventually, we made a thorough study of weighted off-diagonal estimates on balls for the semigroup arising from the operator L above.

Our strategy here has the same two steps in each of the four situations described above. The first step consists in obtaining a first range of exponents p (depending on the weight) by applying the abstract machinery from Part I. This range turns out to be the best possible for both classes of operators and weights.

However, given one operator and one weight, the range of p obtained above may not be sharp, and this leads us to the second step. The sharp range is in fact related to the one for weighted off-diagonal estimates established in Part II. At this point, we use the main results of Part I in the Euclidean space but now equipped with the doubling measure $w(x)\,dx$.

We wish to point out that some of our results can be obtained by different methods (essentially from geometric theory of Banach spaces) once the bounded holomorphic functional calculus is established in (*a*). We give the references in the text.

We wish to say that our proofs are technically simpler than the ones in [Aus] even for the unweighted case, because the notion of off-diagonal estimates used here is more appropriate.

Finally, thanks to the general results in Part I, the same technology allows us to prove in passing weighted L^p estimates for commutators of the operators in $(a) - (c)$ with BMO functions in the same ranges of exponents (see Section 9).

2. General criteria for boundedness and the set $\mathcal{W}_w(p_0, q_0)$

The underlying space is the Euclidean setting $\mathbb{R}^n$ equipped with Lebesgue measure or a doubling measure obtained from an A_∞ weight. We state two results used in this work, referring to [AM1] for statements in stronger form and for references to earlier works.

Given a ball B, we write

$$\fint_B h \, dx = \frac{1}{|B|} \int_B h(x) \, dx.$$

Let us introduce some classical classes of weights. Let w be a weight (that is a non negative locally integrable function) on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\fint_B w \, dx \right) \left(\fint_B w^{1-p'} \, dx \right)^{p-1} \leq C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\fint_B w \, dx \leq C \, w(y), \qquad \text{for a.e. } y \in B.$$

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for any ball B,

$$\left(\fint_B w^q \, dx \right)^{\frac{1}{q}} \leq C \fint_B w \, dx.$$

The endpoint $q = \infty$ is given by the condition $w \in RH_\infty$ whenever there is a constant C such that for any ball B,

$$w(y) \leq C \fint_B w \, dx, \qquad \text{for a.e. } y \in B.$$

The following facts are well-known (see for instance [GR, Gra]).

Proposition 2.1.

 (i) $A_1 \subset A_p \subset A_q$ for $1 \leq p \leq q < \infty$.

 (ii) $RH_\infty \subset RH_q \subset RH_p$ for $1 < p \leq q \leq \infty$.

 (iii) If $w \in A_p$, $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in A_q$.

 (iv) If $w \in RH_q$, $1 < q < \infty$, then there exists $q < p < \infty$ such that $w \in RH_p$.

 (v) $A_\infty = \bigcup_{1 \leq p < \infty} A_p = \bigcup_{1 < q \leq \infty} RH_q$

 (vi) If $1 < p < \infty$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.

(vii) If $w \in A_\infty$, then the measure $dw = w\,dx$ is a Borel doubling measure.

If the Lebesgue measure is replaced by a Borel doubling measure μ, then all the above properties remain valid with the notation change [ST].

Given $1 \leq p_0 < q_0 \leq \infty$ and $w \in A_\infty$ (with respect to a Borel doubling measure μ) we define the set

$$\mathcal{W}_w(p_0, q_0) = \left\{ p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'} \right\}.$$

If $w = 1$, then $\mathcal{W}_1(p_0, q_0) = (p_0, q_0)$. As it is shown in [AM1], if not empty, we have

$$\mathcal{W}_w(p_0, q_0) = \left(p_0\, r_w, \frac{q_0}{(s_w)'} \right)$$

where

$$r_w = \inf\{r \geq 1 \,:\, w \in A_r\}, \qquad s_w = \sup\{s > 1 \,:\, w \in RH_s\}.$$

We use the following notation: if B is a ball with radius $r(B)$ and $\lambda > 0$, λB denotes the concentric ball with radius $r(\lambda B) = \lambda\, r(B)$, $C_j(B) = 2^{j+1} B \setminus 2^j B$ when $j \geq 2$, $C_1(B) = 4B$, and

$$\fint_{C_j(B)} h\,d\mu = \frac{1}{\mu(2^{j+1}B)} \int_{C_j(B)} h\,d\mu.$$

Theorem 2.2. *Let μ be a doubling Borel measure on $\mathbb{R}^n$ and $1 \leq p_0 < q_0 \leq \infty$. Let T be a sublinear operator acting on $L^{p_0}(\mu)$, $\{\mathcal{A}_r\}_{r>0}$ a family of operators acting from a subspace $\mathcal{D}$ of $L^{p_0}(\mu)$ into $L^{p_0}(\mu)$ and S an operator from $\mathcal{D}$ into the space of measurable functions on $\mathbb{R}^n$. Assume that*

$$\left(\fint_B |T(I - \mathcal{A}_{r(B)})f|^{p_0}\,d\mu \right)^{\frac{1}{p_0}} \leq \sum_{j \geq 1} g(j) \left(\fint_{2^{j+1}B} |Sf|^{p_0}\,d\mu \right)^{\frac{1}{p_0}}, \qquad (2.1)$$

and

$$\left(\fint_B |T\mathcal{A}_{r(B)}f|^{q_0}\,d\mu \right)^{\frac{1}{q_0}} \leq \sum_{j \geq 1} g(j) \left(\fint_{2^{j+1}B} |Tf|^{p_0}\,d\mu \right)^{\frac{1}{p_0}}, \qquad (2.2)$$

for all $f \in \mathcal{D}$, all ball B where $r(B)$ denotes its radius for some $g(j)$ with $\sum g(j) < \infty$ (with usual changes if $q_0 = \infty$). Let $p \in \mathcal{W}_w(p_0, q_0)$, that is, $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}$. There is a constant C such that for all $f \in \mathcal{D}$

$$\|Tf\|_{L^p(w)} \leq C\,\|Sf\|_{L^p(w)}. \qquad (2.3)$$

An operator acting from A to B is just a map from A to B. Sublinearity means $|T(f+g)| \leq |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda||T(f)|$ for all f, g and $\lambda \in \mathbb{R}$ or $\mathbb{C}$ (although the second property is not needed in this section). Next, $L^p(w)$ is the space of complex valued functions in $L^p(dw)$ with $dw = w\,d\mu$. However, all this extends to functions valued in a Banach space.

Remark 2.3. In the applications below, we have, either $Sf = f$ with $f \in L_c^\infty$ the space of compactly supported bounded functions on $\mathbb{R}^n$, or $Sf = \nabla f$ with $f \in \mathcal{S}$ the Schwartz class on $\mathbb{R}^n$ (see Section 6).

Let us recall that the doubling order D of a doubling measure μ is the smallest number $\kappa \geq 0$ such that there exists $C \geq 0$ for which $\mu(\lambda B) \leq C_\mu \lambda^\kappa \mu(B)$ for every ball B and for any $\lambda > 1$.

The other criterion we are going to use is the following.

Theorem 2.4. *Let μ be a doubling Borel measure on $\mathbb{R}^n$, D its doubling order and $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a sublinear operator bounded on $L^{q_0}(\mu)$ and that $\{\mathcal{A}_r\}_{r>0}$ is family of linear operators acting from L_c^∞ into $L^{q_0}(\mu)$. Assume that for $j \geq 2$,*

$$\left(\fint_{C_j(B)} |T(I - \mathcal{A}_{r(B)})f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \leq g(j) \left(\fint_B |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \tag{2.4}$$

and for $j \geq 1$,

$$\left(\fint_{C_j(B)} |\mathcal{A}_{r(B)}f|^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \leq g(j) \left(\fint_B |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \tag{2.5}$$

for all ball B with $r(B)$ its radius and for all $f \in L_c^\infty$ supported in B. If $\sum_j g(j) \, 2^{Dj} < \infty$ then T is of weak type (p_0, p_0) and hence T is of strong type (p, p) for all $p_0 < p < q_0$. More precisely, there exists a constant C such that for all $f \in L_c^\infty$

$$\|Tf\|_{L^p(\mu)} \leq C \, \|f\|_{L^p(\mu)}.$$

Again, the statement has a vector-valued extension for linear operators acting on and into L^p functions valued in a Banach space.

Remark 2.5. Notice the symmetry between (2.1) and (2.4).

3. Off-diagonal estimates

We first introduce the class of elliptic operators considered in this work. Let A be an $n \times n$ matrix of complex and L^∞-valued coefficients defined on $\mathbb{R}^n$. We assume that this matrix satisfies the following ellipticity (or "accretivity") condition: there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \, |\xi|^2 \leq \operatorname{Re} A(x)\,\xi \cdot \bar{\xi} \qquad \text{and} \qquad |A(x)\,\xi \cdot \bar{\zeta}| \leq \Lambda \, |\xi| \, |\zeta|,$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi \cdot \zeta = \xi_1 \zeta_1 + \cdots + \xi_n \zeta_n$ and therefore $\xi \cdot \bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$. Note that then $A(x)\,\xi \cdot \bar{\zeta} = \sum_{j,k} a_{j,k}(x)\,\xi_k \, \bar{\zeta}_j$. Associated with this matrix we define the second order divergence form operator

$$Lf = -\operatorname{div}(A \, \nabla f),$$

which is understood in the standard weak sense as a maximal-accretive operator on $L^2(\mathbb{R}^n, dx)$ with domain $\mathcal{D}(L)$ by means of a sesquilinear form.

The operator $-L$ generates a C^0-semigroup $\{e^{-tL}\}_{t>0}$ of contractions on $L^2(\mathbb{R}^n, dx)$. Define $\vartheta \in [0, \pi/2)$ by,

$$\vartheta = \sup\{|\arg\langle Lf, f\rangle| \, : \, f \in \mathcal{D}(L)\}.$$

Then the semigroup has an analytic extension to a complex semigroup $\{e^{-zL}\}_{z \in \Sigma_{\pi/2 - \vartheta}}$ of contractions on $L^2(\mathbb{R}^n, dx)$. Here we have written for $0 < \theta < \pi$,

$$\Sigma_\theta = \{z \in \mathbb{C}^* : |\arg z| < \theta\}.$$

Let $w \in A_\infty$. Here and thereafter, we write $L^p(w)$ for $L^p(\mathbb{R}^n, w dx)$ and if $w = 1$, we drop w in the notation. We define $\widetilde{\mathcal{J}}_w(L)$ and $\widetilde{\mathcal{K}}_w(L)$ as the (possibly empty) intervals of those exponents $p \in [1, \infty]$ such that $\{e^{-tL}\}_{t>0}$ is a bounded set in $\mathcal{L}(L^p(w))$ and $\{\sqrt{t}\,\nabla e^{-tL}\}_{t>0}$ is a bounded set in $\mathcal{L}(L^p(w))$ respectively (where $\mathcal{L}(X)$ is the space of linear continuous maps on a Banach space X).

We extract from [Aus, AM2] some definitions and results (sometimes in weaker form) on unweighted and weighted off-diagonal estimates. See there for details and more precise statements. Set $d(E, F) = \inf\{|x - y| : x \in E, y \in F\}$ where E, F are subsets of $\mathbb{R}^n$.

Definition 3.1. *Let $1 \leq p \leq q \leq \infty$. We say that a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p - L^q$ full off-diagonal estimates, in short $T_t \in \mathcal{F}(L^p - L^q)$, if for some $c > 0$, for all closed sets E and F, all f and all $t > 0$ we have*[†]

$$\left(\int_F |T_t(\chi_E f)|^q \, dx \right)^{\frac{1}{q}} \lesssim t^{-\frac{1}{2}(\frac{n}{p} - \frac{n}{q})} e^{-\frac{c\, d^2(E,F)}{t}} \left(\int_E |f|^p \, dx \right)^{\frac{1}{p}}. \qquad (3.1)$$

We set $\Upsilon(s) = \max\{s, s^{-1}\}$ for $s > 0$. Given a ball B, recall that $C_j(B) = 2^{j+1} B \setminus 2^j B$ for $j \geq 2$ and if $w \in A_\infty$ we use the notation

$$\fint_B h \, dw = \frac{1}{w(B)} \int_B h \, dw, \fint_{C_j(B)} h \, dw = \frac{1}{w(2^{j+1}B)} \int_{C_j(B)} h \, dw.$$

Definition 3.2. *Given $1 \leq p \leq q \leq \infty$ and any weight $w \in A_\infty$, we say that a family of sublinear operators $\{T_t\}_{t>0}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls, in short $T_t \in \mathcal{O}(L^p(w) - L^q(w))$, if there exist $\theta_1, \theta_2 > 0$ and $c > 0$ such that for every $t > 0$ and for any ball B with radius r and all f,*

$$\left(\fint_B |T_t(\chi_B f)|^q \, dw \right)^{\frac{1}{q}} \lesssim \Upsilon\left(\frac{r}{\sqrt{t}} \right)^{\theta_2} \left(\fint_B |f|^p \, dw \right)^{\frac{1}{p}}; \qquad (3.2)$$

and, for all $j \geq 2$,

$$\left(\fint_B |T_t(\chi_{C_j(B)} f)|^q \, dw \right)^{\frac{1}{q}} \lesssim 2^{j\,\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{c\, 4^j\, r^2}{t}} \left(\fint_{C_j(B)} |f|^p \, dw \right)^{\frac{1}{p}} \qquad (3.3)$$

and

$$\left(\fint_{C_j(B)} |T_t(\chi_B f)|^q \, dw \right)^{\frac{1}{q}} \lesssim 2^{j\,\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{c\, 4^j\, r^2}{t}} \left(\fint_B |f|^p \, dw \right)^{\frac{1}{p}}. \qquad (3.4)$$

Let us make some relevant comments for this work (see [AM2] for further details).

- In the Gaussian factors the value of c is irrelevant as long as it remains non negative. We will freely use the same letter from line to line even if its value changes.

- These definitions extend to complex families $\{T_z\}_{z \in \Sigma_\theta}$ with t replaced by $|z|$ in the estimates.

- In both definitions, T_t may only be defined on a dense subspace $\mathcal{D}$ of L^p or $L^p(w)$ $(1 \leq p < \infty)$ that is stable by truncation by indicator functions of measurable sets (for example, $L^p \cap L^2$, $L^p(w) \cap L^2$ or L^∞_c).

[†]Here and thereafter, for two positive quantities A, B, by $A \lesssim B$ we mean that there exists a constant $C > 0$ (independent of the various parameters) such that $A \leq CB$.

- If $q = \infty$, one should adapt the definitions in the usual straightforward way.

- $L^1(w) - L^\infty(w)$ off-diagonal estimates on balls are equivalent to pointwise Gaussian upper bounds for the kernels of T_t.

- Both notions are stable by composition: $T_t \in \mathcal{O}\big(L^q(w) - L^r(w)\big)$ and $S_t \in \mathcal{O}\big(L^p(w) - L^q(w)\big)$ then $T_t \circ S_t \in \mathcal{O}\big(L^p(w) - L^r(w)\big)$ when $1 \le p \le q \le r \le \infty$ and similarly for full off-diagonal estimates.

- When $w = 1$, $L^p - L^q$ off-diagonal estimates on balls are equivalent to $L^p - L^q$ full off-diagonal estimates.

- Notice the symmetry between (3.3) and (3.4).

If I is a subinterval of $[1, \infty]$, Int I denotes the interior in $\mathbb{R}$ of $I \cap \mathbb{R}$.

Proposition 3.3. *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$.*

(a) *There exists a non empty maximal interval in $[1, \infty]$, denoted by $\mathcal{J}(L)$, such that if $p, q \in \mathcal{J}(L)$ with $p \le q$, then $\{(zL)^m e^{-zL}\}_{z \in \Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$. Furthermore, $\mathcal{J}(L) \subset \widetilde{\mathcal{J}}(L)$ and $\text{Int } \mathcal{J}(L) = \text{Int } \widetilde{\mathcal{J}}(L)$.*

(b) *There exists a non empty maximal interval of $[1, \infty]$, denoted by $\mathcal{K}(L)$, such that if $p, q \in \mathcal{K}(L)$ with $p \le q$, then $\{\sqrt{z} \, \nabla (zL)^m e^{-zL}\}_{z \in \Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$. Furthermore, $\mathcal{K}(L) \subset \widetilde{\mathcal{K}}(L)$ and $\text{Int } \mathcal{K}(L) = \text{Int } \widetilde{\mathcal{K}}(L)$.*

(c) *$\mathcal{K}(L) \subset \mathcal{J}(L)$ and, for $p < 2$, we have $p \in \mathcal{K}(L)$ if and only if $p \in \mathcal{J}(L)$.*

(d) *Denote by $p_-(L), p_+(L)$ the lower and upper bounds of $\mathcal{J}(L)$ (hence, of $\text{Int } \widetilde{\mathcal{J}}(L)$ also) and by $q_-(L), q_+(L)$ those of $\mathcal{K}(L)$ (hence, of $\text{Int } \widetilde{\mathcal{K}}(L)$ also). We have $p_-(L) = q_-(L)$ and $(q_-(L))^* \le p_+(L)$.*

(e) *If $n = 1$, $\mathcal{J}(L) = \mathcal{K}(L) = [1, \infty]$.*

(f) *If $n = 2$, $\mathcal{J}(L) = [1, \infty]$ and $\mathcal{K}(L) \supset [1, q_+(L))$ with $q_+(L) > 2$.*

(g) *If $n \ge 3$, $p_-(L) < \frac{2n}{n+2}$, $p_+(L) > \frac{2n}{n-2}$ and $q_+(L) > 2$.*

We have set $q^* = \frac{qn}{n-q}$, the Sobolev exponent of q when $q < n$ and $q^* = \infty$ otherwise.

Proposition 3.4. *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$. Let $w \in A_\infty$.*

(a) *Assume $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$. There is a maximal interval of $[1, \infty]$, denoted by $\mathcal{J}_w(L)$, containing $\mathcal{W}_w\big(p_-(L), p_+(L)\big)$, such that if $p, q \in \mathcal{J}_w(L)$ with $p \le q$, then $\{(zL)^m e^{-zL}\}_{z \in \Sigma_\mu}$ satisfies $L^p(w) - L^q(w)$ off-diagonal estimates on balls and is a bounded set in $\mathcal{L}(L^p(w))$. Furthermore, $\mathcal{J}_w(L) \subset \widetilde{\mathcal{J}}_w(L)$ and $\text{Int } \mathcal{J}_w(L) = \text{Int } \widetilde{\mathcal{J}}_w(L)$.*

(b) *Assume $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$. There exists a maximal interval of $[1, \infty]$, denoted by $\mathcal{K}_w(L)$, containing $\mathcal{W}_w\big(q_-(L), q_+(L)\big)$ such that if $p, q \in \mathcal{K}_w(L)$ with*

$p \leq q$, then $\{\sqrt{z}\, \nabla (zL)^m e^{-z\, L}\}_{z \in \Sigma_\mu}$ *satisfies* $L^p(w) - L^q(w)$ *off-diagonal estimates on balls and is a bounded set in* $\mathcal{L}(L^p(w))$. *Furthermore,* $\mathcal{K}_w(L) \subset \widetilde{\mathcal{K}}_w(L)$ *and* $\mathrm{Int}\, \mathcal{K}_w(L) = \mathrm{Int}\, \widetilde{\mathcal{K}}_w(L)$.

(c) *Let* $n \geq 2$. *Assume* $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$. *Then* $\mathcal{K}_w(L) \subset \mathcal{J}_w(L)$. *Moreover,* $\inf \mathcal{J}_w(L) = \inf \mathcal{K}_w(L)$ *and* $(\sup \mathcal{K}_w(L))_w^* \leq \sup \mathcal{J}_w(L)$.

(d) *If* $n = 1$, *the intervals* $\mathcal{J}_w(L)$ *and* $\mathcal{K}_w(L)$ *are the same and contain* $(r_w, \infty]$ *if* $w \notin A_1$ *and are equal to* $[1, \infty]$ *if* $w \in A_1$.

We have set $q_w^* = \frac{q\, n\, r_w}{n\, r_w - q}$ when $q < n\, r_w$ and $q_w^* = \infty$ otherwise. Recall that $r_w = \inf\{r \geq 1 : w \in A_r\}$ and also that $s_w = \sup\{s > 1 : w \in RH_s\}$.

Note that $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$ means $\frac{p_+(L)}{p_-(L)} > r_w(s_w)'$. This is a compatibility condition between L and w. Similarly, $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$ means $\frac{q_+(L)}{q_-(L)} > r_w(s_w)'$, which is a more restrictive condition on w since $q_-(L) = p_-(L)$ and $q_+(L) \leq p_+(L)$.

In the case of real operators, $\mathcal{J}(L) = [1, \infty]$ in all dimensions because the kernel e^{-tL} satisfies a pointwise Gaussian upper bound. Hence $\mathcal{W}_w\big(p_-(L), p_+(L)\big) = (r_w, \infty)$. If $w \in A_1$, then one has that $\mathcal{J}_w(L) = [1, \infty]$. If $w \notin A_1$, since the kernel is also positive and satisfies a similar pointwise lower bound, one has $\mathcal{J}_w(L) \subset (r_w, \infty]$. Hence, $\mathrm{Int}\, \mathcal{J}_w(L) = \mathcal{W}_w\big(p_-(L), p_+(L)\big)$.

The situation may change for complex operators. But we lack of examples to say whether or not $\mathcal{J}_w(L)$ and $\mathcal{W}_w\big(p_-(L), p_+(L)\big)$ have different endpoints.

Remark 3.5. Note that by density of L_c^∞ in the spaces $L^p(w)$ for $1 \leq p < \infty$, the various extensions of e^{-zL} and ∇e^{-zL} are all consistent. We keep the above notation to denote any such extension. Also, we showed in [AM2] that as long as $p \in \mathcal{J}_w(L)$ with $p \neq \infty$, $\{e^{-tL}\}_{t>0}$ is strongly continuous on $L^p(w)$, hence it has an infinitesimal generator in $L^p(w)$, which is of type ϑ.

From now on, L denotes an operator as defined in this section with the four numbers $p_-(L) = q_-(L)$ *and* $p_+(L)$, $q_+(L)$. *We often drop L in the notation:* $p_- = p_-(L)$, *.... For a given weight $w \in A_\infty$, we set* $\mathcal{W}_w\big(p_-, p_+\big) = (\widetilde{p}_-, \widetilde{p}_+)$ *(when it is not empty) and* $\mathrm{Int}\, \mathcal{J}_w(L) = (\widehat{p}_-, \widehat{p}_+)$. *We have* $\widehat{p}_- \leq \widetilde{p}_- < \widetilde{p}_+ \leq \widehat{p}_+$. *Similarly, we set* $\mathcal{W}_w\big(q_-, q_+\big) = (\widetilde{q}_-, \widetilde{q}_+)$ *(when it is not empty) and* $\mathrm{Int}\, \mathcal{K}_w(L) = (\widehat{q}_-, \widehat{q}_+)$. *We have* $\widehat{q}_- \leq \widetilde{q}_- < \widetilde{q}_+ \leq \widehat{q}_+$.

4. FUNCTIONAL CALCULI

Let $\mu \in (\vartheta, \pi)$ (do not confuse with the measure μ used in Section 2) and φ be a holomorphic function in Σ_μ with the following decay

$$|\varphi(z)| \leq c\, |z|^s\, (1 + |z|)^{-2\, s}, \qquad z \in \Sigma_\mu, \tag{4.1}$$

for some $c,\, s > 0$. Assume that $\vartheta < \theta < \nu < \mu < \pi/2$. Then we have

$$\varphi(L) = \int_{\Gamma_+} e^{-z\, L}\, \eta_+(z)\, dz + \int_{\Gamma_-} e^{-z\, L}\, \eta_-(z)\, dz, \tag{4.2}$$

where $\Gamma_\pm$ is the half ray $\mathbb{R}^+ e^{\pm i\,(\pi/2 - \theta)}$,

$$\eta_\pm(z) = \frac{1}{2\, \pi\, i} \int_{\gamma_\pm} e^{\zeta\, z}\, \varphi(\zeta)\, d\zeta, \qquad z \in \Gamma_\pm, \tag{4.3}$$

with $\gamma_\pm$ being the half-ray $\mathbb{R}^+ e^{\pm i\nu}$ (the orientation of the paths is not needed in what follows so we do not pay attention to it). Note that

$$|\eta_\pm(z)| \lesssim \min(1, |z|^{-s-1}), \qquad z \in \Gamma_\pm, \tag{4.4}$$

hence the representation (4.2) converges in norm in $\mathcal{L}(L^2)$. Usual arguments show the functional property $\varphi(L)\,\psi(L) = (\varphi\,\psi)(L)$ for two such functions φ, ψ.

Any L as above is maximal-accretive and so it has a bounded holomorphic functional calculus on L^2. Given any angle $\mu \in (\vartheta, \pi)$:

(a) For any function φ, holomorphic and bounded in Σ_μ, the operator $\varphi(L)$ can be defined and is bounded on L^2 with

$$\|\varphi(L)f\|_2 \le C\,\|\varphi\|_\infty\,\|f\|_2$$

where C only depends on ϑ and μ.

(b) For any sequence φ_k of bounded and holomorphic functions on Σ_μ converging uniformly on compact subsets of Σ_μ to φ, we have that $\varphi_k(L)$ converges strongly to $\varphi(L)$ in $\mathcal{L}(L^2)$.

(c) The product rule $\varphi(L)\,\psi(L) = (\varphi\,\psi)(L)$ holds for any two bounded and holomorphic functions φ, ψ in Σ_μ.

Let us point out that for more general holomorphic functions (such as powers), the operators $\varphi(L)$ can be defined as unbounded operators.

Given a functional Banach space X, we say that L has a bounded holomorphic functional calculus on X if for any $\mu \in (\vartheta, \pi)$, for any φ holomorphic and satisfying (4.1) in Σ_μ one has

$$\|\varphi(L)f\|_X \le C\,\|\varphi\|_\infty\,\|f\|_X, \qquad f \in X \cap L^2, \tag{4.5}$$

where C depends only on X, ϑ and μ (but not on the decay of φ).

If $X = L^p(w)$ as below, then (4.5) implies that $\varphi(L)$ extends to a bounded operator on X by density. That (a), (b) and (c) hold with L^2 replaced by X for all bounded holomorphic functions in Σ_μ, follow from the theory in [McI] using the fact that on those X, the semigroup $\{e^{-tL}\}_{t>0}$ has an infinitesimal generator which is of type ϑ (see the last remark of previous section). We skip such classical arguments of functional calculi.

Theorem 4.1. [BK1, Aus] *The interior of the set of exponents $p \in (1, \infty)$ such that L has a bounded holomorphic functional calculus on L^p is equal to $\mathrm{Int}\,\mathcal{J}(L)$ defined in Proposition 3.3.*

Our first result is a weighted version of this theorem. We mention [Mar] where similar weighted estimates are proved under kernel upper bounds assumptions.

Theorem 4.2. *Let $w \in A_\infty$ be such that $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \ne \varnothing$. Let $p \in \mathrm{Int}\,\mathcal{J}_w(L)$ and $\mu \in (\vartheta, \pi)$. For any φ holomorphic on Σ_μ satisfying (4.1), we have*

$$\|\varphi(L)f\|_{L^p(w)} \le C\,\|\varphi\|_\infty\,\|f\|_{L^p(w)}, \qquad f \in L_c^\infty, \tag{4.6}$$

with C independent of φ and f. Hence, L has a bounded holomorphic functional calculus on $L^p(w)$.

Remark 4.3. Fix $w \in A_\infty$ with $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$. If $1 < p < \infty$ and L has a bounded holomorphic functional calculus on $L^p(w)$, then $p \in \widetilde{\mathcal{J}}_w(L)$. Indeed, take $\varphi(z) = e^{-z}$. As $\operatorname{Int} \widetilde{\mathcal{J}}_w(L) = \operatorname{Int} \mathcal{J}_w(L)$ by Proposition 3.3, this shows that the range obtained in the theorem is optimal up to endpoints.

Proof. It is enough to assume $\mu < \pi/2$. Note that the operators e^{-zL} are uniformly bounded on $L^p(w)$ when $z \in \Sigma_\mu$, hence, by (4.4), the representation (4.2) converges in norm in $\mathcal{L}(L^p(w))$. Of course, this simple argument does not yield the right estimate, (4.6), which is our goal. It is no loss of generality to assume that $\|\varphi\|_\infty = 1$. We split the argument into three cases: $p \in (\widetilde{p}_-, \widetilde{p}_+)$, $p \in (\widetilde{p}_-, \widehat{p}_+)$, $p \in (\widehat{p}_-, \widetilde{p}_+)$.

Case $p \in (\widetilde{p}_-, \widetilde{p}_+)$. By (iii) and (iv) in Proposition 2.1, there exist p_0, q_0 such that

$$p_- < p_0 < p < q_0 < p_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

The desired bound (4.6) follows on applying Theorem 2.2 for the underlying measure dx and weight w to $T = \varphi(L)$ with p_0, q_0, $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ where $m \geq 1$ is an integer to be chosen and $S = I$. As $\varphi(L)$ and $(I - e^{-r^2 L})^m$ are bounded on L^{p_0} (uniformly with respect to r for the latter) by Proposition 3.3 and Theorem 4.1, it remains to checking both (2.1) and (2.2) on $\mathcal{D} = L_c^\infty$.

We start by showing (2.2). We fix $f \in L_c^\infty$ and a ball B. We will use several times the following decomposition of any given function h:

$$h = \sum_{j \geq 1} h_j, \qquad h_j = h \, \chi_{C_j(B)}. \tag{4.7}$$

Fix $1 \leq k \leq m$. Since $p_0 \leq q_0$ and $p_0, q_0 \in \mathcal{J}(L)$, we have $e^{-tL} \in \mathcal{O}\big(L^{p_0} - L^{q_0}\big)$ (we are using the equivalence between the two notions of off-diagonal estimates for the Lebesgue measure), hence

$$\left(\fint_B |e^{-k r^2 L} h_j|^{q_0} \, dx \right)^{\frac{1}{q_0}} \lesssim 2^{j(\theta_1 + \theta_2)} e^{-\alpha 4^j} \left(\fint_{2^{j+1} B} |h|^{p_0} \, dx \right)^{\frac{1}{p_0}}$$

and by Minkowski's inequality

$$\left(\fint_B |e^{-k r^2 L} h|^{q_0} \, dx \right)^{\frac{1}{q_0}} \lesssim \sum_{j \geq 1} g(j) \left(\fint_{2^{j+1} B} |h|^{p_0} \, dx \right)^{\frac{1}{p_0}} \tag{4.8}$$

with $g(j) = 2^{j(\theta_1 + \theta_2)} e^{-\alpha 4^j}$ for any $h \in L^{p_0}$. This estimate with $h = \varphi(L) f \in L^{p_0}$ yields (2.2) since, by the commutation rule, $\varphi(L) e^{-k r^2 L} f = e^{-k r^2 L} h$.

We next show (2.1). Let $f \in L_c^\infty$ and B be a ball. Write $f = \sum_{j \geq 1} f_j$ as before. For $j = 1$, we use the L^{p_0} boundedness of $\varphi(L)$ and $(I - e^{-r^2 L})^m$, hence

$$\left(\fint_B |\varphi(L)(I - e^{-r^2 L})^m f_1|^{p_0} \, dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4B} |f|^{p_0} \, dx \right)^{\frac{1}{p_0}}. \tag{4.9}$$

For $j \geq 2$, the functions $\eta_\pm$ associated with $\psi(z) = \varphi(z)(1 - e^{-r^2 z})^m$ by (4.3) satisfy

$$|\eta_\pm(z)| \lesssim \frac{r^{2m}}{|z|^{m+1}}, \qquad z \in \Gamma_\pm.$$

Since $p_0 \in \mathcal{J}(L)$, $\{e^{-zL}\}_{z \in \Gamma_\pm} \in \mathcal{O}(L^{p_0} - L^{p_0})$ and so

$$\left(\fint_B \left| \int_{\Gamma_+} \eta_+(z)\, e^{-zL} f_j\, dz \right|^{p_0} dx \right)^{\frac{1}{p_0}} \le \int_{\Gamma_+} \left(\fint_B |e^{-zL} f_j|^{p_0}\, dx \right)^{\frac{1}{p_0}} |\eta_+(z)|\, |dz|$$

$$\lesssim\ 2^{j\,\theta_1} \int_{\Gamma_+} \Upsilon\!\left(\frac{2^j\, r}{\sqrt{|z|}} \right)^{\theta_2} e^{-\frac{\alpha\, 4^j\, r^2}{|z|}} \frac{r^{2m}}{|z|^{m+1}} |dz| \left(\fint_{C_j(B)} |f|^{p_0}\, dx \right)^{\frac{1}{p_0}}$$

$$\lesssim\ 2^{j\,(\theta_1 - 2m)} \left(\fint_{C_j(B)} |f|^{p_0}\, dx \right)^{\frac{1}{p_0}}$$

provided $2m > \theta_2$. We have used, after a change of variable, that

$$\int_0^\infty \Upsilon(s)^{\theta_2}\, e^{-c s^2}\, s^{2m}\, \frac{ds}{s} < \infty.$$

The same is obtained when one deals with the term corresponding to Γ_-. Plugging both estimates into the representation of $\psi(L)$ given by (4.2) one obtains

$$\left(\fint_B |\varphi(L)(I - e^{-r^2 L})^m f_j|^{p_0}\, dx \right)^{\frac{1}{p_0}} \lesssim 2^{j\,(\theta_1 - 2m)} \left(\fint_{C_j(B)} |f|^{p_0}\, dx \right)^{\frac{1}{p_0}}, \qquad (4.10)$$

therefore, (2.1) holds when $2m > \max\{\theta_1, \theta_2\}$ since $C_j(B) \subset 2^{j+1} B$.

Case $p \in (\widetilde{p}_-, \widehat{p}_+)$: Take p_0, q_0 such that $\widetilde{p}_- < p_0 < \widetilde{p}_+$ and $p_0 < p < q_0 < \widehat{p}_+$. Let $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some large enough $m \ge 1$. Remark that by the previous case, $\varphi(L)$ has the right norm in $\mathcal{L}(L^{p_0}(w))$ and so does $\mathcal{A}_r$ by Proposition 3.4. We apply Theorem 2.2 with the Borel doubling measure dw and no weight. Thus, it is enough to see that $\varphi(L)$ satisfies (2.1) and (2.2) for dw on $\mathcal{D} = L_c^\infty \subset L^{p_0}(w)$. But this follows by adapting the preceding argument replacing everywhere dx by dw and observing that $e^{-zL} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$ since $p_0, q_0 \in \mathrm{Int}\,\mathcal{J}_w(L)$ and $p_0 \le q_0$. We skip details.

Case $p \in (\widehat{p}_-, \widetilde{p}_+)$: Take p_0, q_0 such that $\widetilde{p}_- < q_0 < \widetilde{p}_+$ and $\widehat{p}_- < p_0 < p < q_0$. Set $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some integer $m \ge 1$ to be chosen later. Since $q_0 \in (\widetilde{p}_-, \widetilde{p}_+)$, by the first case, $\varphi(L)$ has the right norm in $\mathcal{L}(L^{q_0}(w))$ and so does $\mathcal{A}_r$ by Proposition 3.4. We apply Theorem 2.4 with underlying measure dw. It is enough to show (2.4) and (2.5). Fix a ball B and $f \in L_c^\infty$ supported in B.

We begin with (2.5) for $\mathcal{A}_r$. It is enough to show it for $e^{-k r^2 L}$ with $1 \le k \le m$. Since $p_0, q_0 \in \mathcal{J}_w(L)$ and $p_0 \le q_0$ we have $e^{-tL} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$, hence

$$\left(\fint_{C_j(B)} |e^{-k r^2 L} f|^{q_0}\, dw \right)^{\frac{1}{q_0}} \lesssim 2^{j\,(\theta_1 + \theta_2)} e^{-c\, 4^j} \left(\fint_B |f|^{p_0}\, dw \right)^{\frac{1}{p_0}}. \qquad (4.11)$$

This implies (2.5) with $g(j) = C\, 2^{j\,(\theta_1 + \theta_2)} e^{-c\, 4^j}$ and $\sum_{j \ge 1} g(j)\, 2^{D j} < \infty$ holds where D is the doubling order of dw.

We turn to (2.4). Let $j \ge 2$. The argument is the same as the one for (4.10) by reversing the roles of $C_j(B)$ and B, and using dw and $e^{-zL} \in \mathcal{O}(L^{p_0}(w) - L^{p_0}(w))$ (since $p_0 \in \mathcal{J}_w(L)$) instead of dx and $e^{-zL} \in \mathcal{O}(L^{p_0} - L^{p_0})$. We obtain

$$\left(\fint_{C_j(B)} |\varphi(L)(I - e^{-r^2 L})^m f|^{p_0}\, dw \right)^{\frac{1}{p_0}} \lesssim 2^{j\,(\theta_1 - 2m)} \left(\fint_B |f|^{p_0}\, dw \right)^{\frac{1}{p_0}}$$

provided $2\,m > \theta_2$ and it remains to impose further $2\,m > \theta_1 + D$ to conclude. $\qquad \square$

Remark 4.4. If $\mathcal{W}_w(p_-, p_+) \neq \emptyset$, the last part of the proof yields weighted weak-type $(\widehat{p}_-, \widehat{p}_-)$ of $\varphi(L)$ provided $\widehat{p}_- \in \mathcal{J}_w(L)$. To do so, one only has to take $p_0 = \widehat{p}_-$.

5. Riesz transforms

The Riesz transforms associated to L are $\partial_j L^{-1/2}$, $1 \leq j \leq n$. Set $\nabla L^{-1/2} = (\partial_1 L^{-1/2}, \ldots, \partial_n L^{-1/2})$. The solution of the Kato conjecture [AHLMcT] implies that this operator extends boundedly to L^2 (we ignore the $\mathbb{C}^n$-valued aspect of things). This allows the representation

$$\nabla L^{-1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-t\,L} f \, \frac{dt}{\sqrt{t}}, \tag{5.1}$$

in which the integral converges strongly in L^2 both at 0 and ∞ when $f \in L^2$. Note that for an arbitrary $f \in L^2$, $h = L^{-1/2} f$ makes sense in the homogeneous Sobolev space $\dot{H}^1$ which is the completion of $C_0^\infty(\mathbb{R}^n)$ for the semi-norm $\|\nabla h\|_2$ and ∇ becomes the extension of the gradient to that space. This construction can be forgotten if $n \geq 3$ as $\dot{H}^1 \subset L^{2^*}$ but not if $n \leq 2$. To circumvent this technical difficulty, we introduce $S_\varepsilon = \frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} e^{-t\,L} \frac{dt}{\sqrt{t}}$ for $0 < \varepsilon < 1$ and, in fact, ∇S_ε are uniformly bounded on L^2 and converge strongly in L^2. This defines $\nabla L^{-1/2}$.

Theorem 5.1 ([Aus]). *The maximal interval of exponents $p \in (1, \infty)$ for which $\nabla L^{-1/2}$ has a bounded extension to L^p is equal to $\mathrm{Int}\,\mathcal{K}(L)$ defined in Proposition 3.3 and for $p \in \mathrm{Int}\,\mathcal{K}(L)$, $\|\nabla f\|_p \sim \|L^{1/2} f\|_p$ for all $f \in \mathcal{D}(L^{1/2}) = H^1$ (the Sobolev space).*

Again, the operators ∇S_ε are uniformly bounded on L^p and converge strongly in L^p as $\varepsilon \to 0$. Indeed, for $f \in L_c^\infty$, $S_\varepsilon f \in \mathcal{D}(L) \subset \mathcal{D}(L^{1/2})$ and $\|\nabla S_\varepsilon f\|_p \lesssim \|L^{1/2} S_\varepsilon f\|_p$. Observe that $L^{1/2} S_\varepsilon = \varphi_\varepsilon(L)$, where φ_ε is a bounded holomorphic function in Σ_μ for any $0 < \mu < \pi/2$ with $\sup_\varepsilon \|\varphi_\varepsilon\|_\infty < \infty$ and $\{\varphi_\varepsilon\}_\varepsilon$ converges uniformly to 1 on compact subsets of Σ_μ as $\varepsilon \to 0$. The claim follows by Theorem 4.1 and density.

We turn to weighted norm inequalities. Remark that by Proposition 3.4, for all $p \in \mathcal{J}_w(L)$, S_ε is bounded on $L^p(w)$ (the norm must depend on ε) and for all $p \in \mathcal{K}_w(L)$, ∇S_ε is bounded on $L^p(w)$ with no control yet on the norm with respect to ε.

Theorem 5.2. *Let $w \in A_\infty$ be such that $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$. For all $p \in \mathrm{Int}\,\mathcal{K}_w(L)$ and $f \in L_c^\infty$,*

$$\|\nabla L^{-1/2} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}. \tag{5.2}$$

Hence, $\nabla L^{-1/2}$ has a bounded extension to $L^p(w)$.

We note that for a given $p \in \mathrm{Int}\,\mathcal{K}_w(L)$, once (5.2) is established, similar arguments using Theorem 4.2 imply convergence in $L^p(w)$ of $\nabla S_\varepsilon f$ to $\nabla L^{-1/2} f$ for $f \in L_c^\infty$.

Proof. We split the argument in three cases: $p \in (\widetilde{q}_-, \widetilde{q}_+)$, $p \in (\widetilde{q}_-, \widehat{q}_+)$, $p \in (\widehat{q}_-, \widetilde{q}_+)$.

Case $p \in (\widetilde{q}_-, \widetilde{q}_+)$: By (iii) and (iv) in Proposition 2.1, there exist p_0, q_0 such that

$$q_- < p_0 < p < q_0 < q_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

The desired estimate (5.2) is obtained by applying Theorem 2.2 with underlying measure dx and weight w. Hence, it suffices to verify (2.1) and (2.2) on $\mathcal{D} = L_c^\infty$ for

$T = \nabla L^{-1/2}$, $S = I$ and $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, with m a large enough integer. These conditions will be proved as in [Aus], but here we use the whole range of exponents for which the Riesz transforms are bounded on unweighted L^p spaces, that is, (q_-, q_+).

Lemma 5.3. *Fix a ball B. For $f \in L_c^\infty$ and m large enough,*

$$\left(\fint_B |\nabla L^{-1/2}(I - e^{-r^2 L})^m f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq \sum_{j \geq 1} g_1(j) \left(\fint_{C_j(B)} |f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \qquad (5.3)$$

and for $f \in L^{p_0}$ such that $\nabla f \in L^{p_0}$ and $1 \leq k \leq m$,

$$\left(\fint_B |\nabla e^{-k \, r^2 \, L} f|^{q_0} \, dx \right)^{\frac{1}{q_0}} \leq \sum_{j \geq 1} g_2(j) \left(\fint_{2^{j+1} \, B} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}}, \qquad (5.4)$$

where $g_1(j) = C_m \, 2^{j \, \theta} \, 4^{-m \, j}$ and $g_2(j) = C_m \, 2^j \sum_{l \geq j} 2^{l \, \theta} \, e^{-\alpha \, 4^l}$ for some $\theta > 0$.

Assume this is proved. Note that if $2m > \theta$ then $\sum_{j \geq 1} g_1(j) < \infty$ and the first estimate is (2.1).

Next, expanding $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, the latter estimate applied to $S_\varepsilon f$ in place of f (since $S_\varepsilon f \in L^{p_0}$ and $\nabla S_\varepsilon f \in L^{p_0}$) and the commuting rule $\mathcal{A}_r S_\varepsilon = S_\varepsilon \mathcal{A}_r$ give us

$$\left(\fint_B |\nabla S_\varepsilon \mathcal{A}_r f|^{q_0} \, dx \right)^{\frac{1}{q_0}} \leq \sum_{j \geq 1} g_2(j) \left(\fint_{2^{j+1} \, B} |\nabla S_\varepsilon f|^{p_0} \, dx \right)^{\frac{1}{p_0}}.$$

By letting ε go to 0 (the justification uses the observations made at the beginning of this section and is left to the reader), we obtain (2.2) using $\sum_{j \geq 1} g_2(j) < \infty$. Therefore, by Theorem 2.2, (5.2) holds for $f \in L_c^\infty$.

Proof of Lemma 5.3. We begin with the first estimate. Decomposing f as in (4.7),

$$\left(\fint_B |\nabla L^{-1/2}(I - e^{-r^2 L})^m f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq \sum_{j \geq 1} \left(\fint_B |\nabla L^{-1/2}(I - e^{-r^2 L})^m f_j|^{p_0} \, dx \right)^{\frac{1}{p_0}}.$$

For $j = 1$, since $q_- < p_0 < q_+$, $\nabla L^{-1/2}$ and $e^{-r^2 L}$ are bounded on L^{p_0} by Theorem 5.1 and Proposition 3.3. Hence,

$$\left(\fint_B |\nabla L^{-1/2}(I - e^{-r^2 L})^m f_1|^{p_0} \, dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4 \, B} |f|^{p_0} \, dx \right)^{\frac{1}{p_0}}.$$

For $j \geq 2$, we use a different approach. If $h \in L^2$, by (5.1)

$$\nabla L^{-1/2}(I - e^{-r^2 L})^m h = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \, \nabla \varphi(L, t) h \, \frac{dt}{t},$$

where $\varphi(z, t) = e^{-t z} (1 - e^{-r^2 z})^m$. The functions $\eta_\pm(\cdot, t)$ associated with $\varphi(\cdot, t)$ by (4.3) satisfy

$$|\eta_\pm(z, t)| \lesssim \frac{r^{2 m}}{(|z| + t)^{m+1}}, \qquad z \in \Gamma_\pm, t > 0.$$

Since $\sqrt{z} \, \nabla e^{-z L} \in \mathcal{O}(L^{p_0} - L^{p_0})$ (note that $p_0 \in \mathcal{K}(L)$ and we are using the equivalence between the two notions of off-diagonal estimates for the Lebesgue measure),

$$\left(\fint_B \left| \int_{\Gamma_+} \eta_+(z) \sqrt{t} \, \nabla e^{-z L} f_j \, dz \right|^{p_0} \, dx \right)^{\frac{1}{p_0}}$$

$$\leq \int_{\Gamma_+} \Big(\fint_B |\sqrt{z}\,\nabla e^{-z\,L} f_j|^{p_0}\,dx\Big)^{\frac{1}{p_0}} \frac{\sqrt{t}}{\sqrt{|z|}}\,|\eta_+(z)|\,|dz|$$

$$\lesssim 2^{j\,\theta_1} \int_{\Gamma_+} \Upsilon\Big(\frac{2^j\,r}{\sqrt{|z|}}\Big)^{\theta_2} e^{-\frac{\alpha\,4^j\,r^2}{|z|}} \frac{\sqrt{t}}{\sqrt{|z|}}\,|\eta_+(z)|\,|dz| \Big(\fint_{C_j(B)} |f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$

$$\lesssim 2^{j\,\theta_1} \int_0^\infty \Upsilon\Big(\frac{2^j\,r}{\sqrt{s}}\Big)^{\theta_2} e^{-\frac{\alpha\,4^j\,r^2}{s}} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2\,m}}{(s+t)^{m+1}}\,ds \Big(\fint_{C_j(B)} |f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}.$$

Observing that when $2\,m > \theta_2$

$$\int_0^\infty \int_0^\infty \Upsilon\Big(\frac{2^j\,r}{\sqrt{s}}\Big)^{\theta_2} e^{-\frac{\alpha\,4^j\,r^2}{s}} \frac{\sqrt{t}}{\sqrt{s}} \frac{r^{2\,m}}{(s+t)^{m+1}}\,ds\frac{dt}{t} = C\,4^{-j\,m},$$

and plugging this, plus the corresponding term for Γ_-, into the representation (4.2), we obtain

$$\Big(\fint_B |\nabla e^{-t\,L}(I - e^{-r^2\,L})^m f_j|^{p_0}\,dx\Big)^{\frac{1}{p_0}} \lesssim \int_0^\infty \Big(\fint_B |\sqrt{t}\,\nabla\varphi(L,t) f_j|^{p_0}\,dx\Big)^{\frac{1}{p_0}} \frac{dt}{t}$$

$$\lesssim 2^{j\,(\theta_1 - 2\,m)} \Big(\fint_{C_j(B)} |f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}. \qquad (5.5)$$

This readily yields the first estimate in the lemma.

Let us get the second one. Fix $1 \leq k \leq m$. Let $f \in L^{p_0}$ such that $\nabla f \in L^{p_0}$. We write $h = f - f_{4\,B}$ where $f_{\lambda\,B}$ is the dx-average of f on $\lambda\,B$. Then by the conservation property (see [Aus]) $e^{-t\,L}1 = 1$ for all $t > 0$, we have

$$\nabla e^{-k\,r^2\,L} f = \nabla e^{-k\,r^2\,L}(f - f_{4\,B}) = \nabla e^{-k\,r^2\,L} h = \sum_{j\geq 1} \nabla e^{-k\,r^2\,L} h_j,$$

with $h_j = h\,\chi_{C_j(B)}$. Hence,

$$\Big(\fint_B |\nabla e^{-k\,r^2\,L} f|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \leq \sum_{j\geq 1} \Big(\fint_B |\nabla e^{-k\,r^2\,L} h_j|^{q_0}\,dx\Big)^{\frac{1}{q_0}}.$$

Since $p_0 \leq q_0$ and $p_0, q_0 \in \mathcal{K}(L)$, $\sqrt{t}\,\nabla e^{-t\,L} \in \mathcal{O}(L^{p_0} - L^{q_0})$. This and the L^{p_0}-Poincaré inequality for dx yield

$$\Big(\fint_B |\nabla e^{-k\,r^2\,L} h_j|^{q_0}\,dx\Big)^{\frac{1}{q_0}} \lesssim \frac{2^{j\,(\theta_1 + \theta_2)}\,e^{-\alpha\,4^j}}{r} \Big(\fint_{C_j(B)} |h_j|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$

$$\leq \frac{2^{j\,(\theta_1 + \theta_2)}\,e^{-\alpha\,4^j}}{r} \Big(\fint_{2^{j+1}\,B} |f - f_{4\,B}|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$

$$\leq \frac{2^{j\,(\theta_1 + \theta_2)}\,e^{-\alpha\,4^j}}{r} \Big(\Big(\fint_{2^{j+1}\,B} |f - f_{2^{j+1}\,B}|^{p_0}\,dx\Big)^{\frac{1}{p_0}} + \sum_{l=2}^j |f_{2^l\,B} - f_{2^{l+1}\,B}|\Big)$$

$$\lesssim \frac{2^{j\,(\theta_1 + \theta_2)}\,e^{-\alpha\,4^j}}{r} \sum_{l=1}^j \Big(\fint_{2^{l+1}\,B} |f - f_{2^{l+1}\,B}|^{p_0}\,dx\Big)^{\frac{1}{p_0}}$$

$$\lesssim 2^{j\,(\theta_1 + \theta_2)}\,e^{-\alpha\,4^j} \sum_{l=1}^j 2^l \Big(\fint_{2^{l+1}\,B} |\nabla f|^{p_0}\,dx\Big)^{\frac{1}{p_0}}, \qquad (5.6)$$

which is the desired estimate with $\theta = \theta_1 + \theta_2$. $\qquad\qquad\qquad\qquad\square$

Case $p \in (\widetilde{q}_-, \widehat{q}_+)$: Take p_0, q_0 such that $\widetilde{q}_- < p_0 < \widetilde{q}_+$ and $p_0 < p < q_0 < \widehat{q}_+$. Let $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some $m \geq 1$ to be chosen later. As $p_0 \in (\widetilde{q}_-, \widetilde{q}_+)$, both $\nabla L^{-1/2}$ and $\mathcal{A}_r$ are bounded on $L^{p_0}(w)$ (we have just shown it for the Riesz transforms and $\mathcal{A}_r$ are bounded uniformly in r by Proposition 3.4). By Theorem 2.2 with underlying doubling measure dw and no weight, it is enough to verify (2.1) and (2.2) on $\mathcal{D} = L_c^\infty$ for $T = \nabla L^{-1/2}$, $S = I$ and $\mathcal{A}_r$. To do so, it suffices to copy the proof of Lemma 5.3 in the weighted case by changing systematically dx to dw, off-diagonal estimates with respect to dx by those with respect to dw given the choice of p_0, q_0. Also in the argument with dx we used a Poincaré inequality. Here, since $p_0 \in \mathcal{W}_w(q_-, q_+)$, $w \in A_{p_0/q_-}$ and in particular $w \in A_{p_0}$ (since $q_- \geq 1$). Therefore we can use the $L^{p_0}(w)$-Poincaré inequality (see [FPW]):

$$\left(\fint_B |f - f_{B,w}|^{p_0} \, dw \right)^{\frac{1}{p_0}} \lesssim r(B) \left(\fint_B |\nabla f|^{p_0} \, dw \right)^{\frac{1}{p_0}}$$

for all $f \in L^1_{\mathrm{loc}}(w)$ such that $\nabla f \in L^{p_0}_{\mathrm{loc}}(w)$, where $f_{B,w}$ is the dw-average of f over B. We leave further details to the reader.

Case $p \in (\widehat{q}_-, \widetilde{q}_+)$: Take p_0, q_0 such that $\widetilde{q}_- < q_0 < \widetilde{q}_+$ and $\widehat{q}_- < p_0 < p < q_0$. Set $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some integer $m \geq 1$ to be chosen later. Since $q_0 \in (\widetilde{q}_-, \widetilde{q}_+)$, it follows that $\nabla L^{-1/2}$ is already bounded on $L^{q_0}(w)$ and so is $\mathcal{A}_r$. That $\nabla L^{-1/2}$ is bounded on $L^p(w)$ will follow on applying Theorem 2.4 with underlying measure w. Hence it is enough to check both (2.4) and (2.5).

We begin with (2.5). By Proposition 3.4, $\inf \mathcal{J}_w(L) = \inf \mathcal{K}_w(L) = \widehat{q}_-$. Since $p_0 > \widehat{q}_-$ and $p_0 \leq q_0 \in \mathcal{W}_w(q_-, q_+) \subset \mathcal{W}_w(p_-, p_+) \subset \mathcal{J}_w(L)$, we have $p_0, q_0 \in \mathcal{J}_w(L)$ and so $e^{-tL} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$. This yields (4.11), hence (2.5) with $g(j) = C \, 2^{j\,(\theta_1+\theta_2)} \, e^{-c\,4^j}$ which clearly satisfies $\sum_j g(j)\,2^{D\,j} < \infty$, with D the doubling order of dw.

We next show (2.4). Let $f \in L_c^\infty$ be supported on a ball B and $j \geq 2$. The argument is the same as the one for (5.5) by reversing the roles of $C_j(B)$ and B, and using dw and $\sqrt{z}\,\nabla e^{-z\,L} \in \mathcal{O}(L^{p_0}(w) - L^{p_0}(w))$ (since $p_0 \in \mathcal{K}_w(L)$) instead of dx and $\sqrt{z}\,\nabla e^{-z\,L} \in \mathcal{O}(L^{p_0} - L^{p_0})$. Hence, we obtain

$$\left(\fint_{C_j(B)} |\nabla L^{-1/2}(I - e^{-r^2 L})^m f|^{p_0} \, dw \right)^{\frac{1}{p_0}} \lesssim 2^{j\,(\theta_1 - 2\,m)} \left(\fint_B |f|^{p_0} \, dw \right)^{\frac{1}{p_0}}$$

provided $2\,m > \theta_2$ and it remains to impose further $2\,m > \theta_1 + D$ to conclude. $\quad\square$

Remark 5.4. If $\mathcal{W}_w(q_-, q_+) \neq \varnothing$, the last part of the proof yields weighted weak-type $(\widehat{q}_-, \widehat{q}_-)$ of $\nabla L^{-1/2}$ provided $\widehat{q}_- \in \mathcal{K}_w(L)$: one only needs to take $p_0 = \widehat{q}_-$.

Remark 5.5. Theorem 5.1 asserts that $\mathrm{Int}\,\mathcal{K}(L)$ is the *exact* range of L^p boundedness for the Riesz transforms when $w = 1$. When $w \neq 1$, we cannot repeat the same argument as it used Sobolev embedding which has no simple counterpart in the weighted situation. However, if we insert in the integral of (5.1) a function $m(t)$ with $m \in L^\infty(0, \infty)$, then (5.2) holds with a constant proportional to $\|m\|_\infty$. Indeed, Let $\varphi_m(z) = \int_0^\infty z^{1/2} e^{-t\,z} m(t) \frac{dt}{\sqrt{t}}$ for $z \in \Sigma_\mu$, $\vartheta < \mu < \pi/2$. Then, φ_m is holomorphic in

Σ_μ and bounded with $\|\varphi_m\|_\infty \le c_\mu \|m\|_\infty$. Now for $f \in L_c^\infty$,

$$\int_0^\infty \nabla e^{-tL} f \, m(t) \, \frac{dt}{\sqrt{t}} = \nabla L^{-1/2} \varphi_m(L) f$$

hence, combining Theorems 4.2 and 5.2, we obtain for $p \in \text{Int} \, \mathcal{K}_w(L)$,

$$\Big\| \int_0^\infty \sqrt{t} \, \nabla e^{-tL} f \, m(t) \, \frac{dt}{t} \Big\|_{L^p(w)} \lesssim \|m\|_\infty \, \|f\|_{L^p(w)}.$$

Conversely, given an exponent $p \in (1, \infty)$, assume that this $L^p(w)$ estimate holds for all $m \in L^\infty$. Using randomization techniques which we skip (see Section 8 for some account on such techniques), this implies

$$\Big\| \Big(\int_0^\infty |\sqrt{t} \, \nabla e^{-tL} f|^2 \, \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

This square function estimate is proved directly in Section 7 and we indicate at the end of that section why this inequality implies $p \in \overline{\widetilde{\mathcal{K}}_w(L)}$. Thus, the range in p is sharp up to endpoints (see Proposition 3.4).

6. REVERSE INEQUALITIES FOR SQUARE ROOTS

We continue on square roots by studying when the inequality opposite to (5.2) hold. First we recall the unweighted case.

Theorem 6.1. [Aus] *If* $\max\big\{1, \frac{n\,p_-(L)}{n+p_-(L)}\big\} < p < p_+(L)$ *then for* $f \in \mathcal{S}$,

$$\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p. \tag{6.1}$$

To state our result, we need a new exponent. For $p > 0$, define

$$p_{w,*} = \frac{n \, r_w \, p}{n \, r_w + p},$$

where $r_w = \inf\{r \ge 1 \ : \ w \in A_r\}$. Set also $p_w^* = \frac{n \, r_w \, p}{n \, r_w - p}$ for $p < n \, r_w$ and $p_w^* = \infty$ otherwise. Note that $(p_{w,*})_w^* = p$.

Theorem 6.2. *Let* $w \in A_\infty$ *with* $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \ne \emptyset$. *If* $\max\big\{r_w, (\widehat{p}_-)_{w,*}\big\} < p < \widehat{p}_+$ *then for* $f \in \mathcal{S}$,

$$\|L^{1/2} f\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}. \tag{6.2}$$

Remark 6.3. Recall that $\widetilde{p}_- = p_-(L) \, r_w$ and we have $(\widehat{p}_-)_{w,*} < \widehat{p}_- \le \widetilde{p}_-$. If $p_-(L) = 1$, then $(\widehat{p}_-)_{w,*} \le r_w$, so $\max\big\{r_w, (\widehat{p}_-)_{w,*}\big\} = r_w = \widetilde{p}_-$. This happens for example when L is real or when $n = 1, 2$.

Define $\dot{W}^{1,p}(w)$ as the completion of $\mathcal{S}$ under the semi-norm $\|\nabla f\|_{L^p(w)}$. Arguing as in [AT1] (see [Aus]) combining Theorems 5.2 and 6.2, we obtain the following consequence.

Corollary 6.4. *Assume* $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \ne \emptyset$. *If* $p \in \text{Int} \, \mathcal{K}_w(L)$ *with* $p > r_w$, *then* $L^{1/2}$ *extends to an isomorphism from* $\dot{W}^{1,p}(w)$ *into* $L^p(w)$.

Proof of Theorem 6.2. We split the argument in three cases: $p \in (\widetilde{p}_-, \widetilde{p}_+)$, $p \in (\widetilde{p}_-, \widehat{p}_+)$, $p \in (\max\big\{r_w, (\widehat{p}_-)_{w,*}\big\}, \widetilde{p}_+)$.

Case $p \in (\widetilde{p}_-, \widetilde{p}_+)$: It relies on the following lemma.

Lemma 6.5. *Let $p_0 \in \operatorname{Int} \mathcal{J}(L)$ and $q_0 \in \mathcal{J}(L)$ with $p_0 < q_0$. Let B be a ball and $m \geq 1$ an integer. For all $f \in \mathcal{S}$, we have*

$$\left(\fint_B |L^{1/2}(I - e^{-r^2 L})^m f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq \sum_{j \geq 1} g_1(j) \left(\fint_{2^{j+1} B} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \qquad (6.3)$$

for m large enough depending on p_0 and q_0, and

$$\left(\fint_B |L^{1/2}(I - (I - e^{-r^2 L})^m) f|^{q_0} \, dx \right)^{\frac{1}{q_0}} \leq \sum_{j \geq 1} g_2(j) \left(\fint_{2^{j+1} B} |L^{1/2} f|^{p_0} \, dx \right)^{\frac{1}{p_0}}, \quad (6.4)$$

where $g_1(j) = C_m \, 2^{j\theta} \, 4^{-mj}$ and $g_2(j) = C_m \, 2^{j\theta} \, e^{-\alpha 4^j}$ for some $\theta > 0$, and the implicit constants are independent of B and f.

Admit this lemma for a moment. Since $p \in (\widetilde{p}_-, \widetilde{p}_+) = \mathcal{W}_w(p_-, p_+)$, by (iii) and (iv) in Proposition 2.1, there exist p_0, q_0 such that

$$p_- < p_0 < p < q_0 < p_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

Note that (6.3) and (6.4) are respectively the conditions (2.1) and (2.2) of Theorem 2.2 with underlying measure dx and weight w, $T = L^{1/2}$, $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, with m large enough, and $Sf = \nabla f$. Hence we obtain (6.2).

Proof of Lemma 6.5. We first show (6.4). Using the commutation rule and expanding $(I - e^{-r^2 L})^m$ it suffices to apply (4.8) as $p_0, q_0 \in \mathcal{J}(L)$ to $h = L^{1/2} f$.

We turn to (6.3). If $\varphi(z) = z^{1/2}(1 - e^{-r^2 z})^m$, then $\varphi(L) f = L^{1/2}(I - e^{-r^2 L})^m f$. By the conservation property

$$\varphi(L) f = \varphi(L) (f - f_{4B}) = \sum_{j \geq 1} \varphi(L) h_j, \qquad (6.5)$$

where $h_j = (f - f_{4B}) \phi_j$. Here, $\phi_j = \chi_{C_j(B)}$ for $j \geq 3$, ϕ_1 is a smooth function with support in $4B$, $0 \leq \phi_1 \leq 1$, $\phi_1 = 1$ in $2B$ and $\|\nabla \phi_1\|_\infty \leq C/r$ and, eventually, ϕ_2 is taken so that $\sum_{j \geq 1} \phi_j = 1$. We estimate each term in turn. For $j = 1$, since $p_- < p_0 < p_+$, by the bounded holomorphic functional calculus on L^{p_0} (Theorem 4.1) and $\varphi(L) h_1 = (I - e^{-r^2 L})^m L^{1/2} h_1$, one has uniformly in r,

$$\|\varphi(L) h_1\|_{p_0} \lesssim \|L^{1/2} h_1\|_{p_0}.$$

Next, Theorem 6.1, L^{p_0}-Poincaré inequality and the definition of h_1 imply

$$\|L^{1/2} h_1\|_{p_0} \lesssim \|\nabla h_1\|_{p_0} \lesssim \|\nabla f\|_{L^{p_0}(4B)}.$$

Therefore,

$$\left(\fint_B |\varphi(L) h_1|^{p_0} \, dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4B} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}}.$$

For $j \geq 3$, the functions $\eta_\pm$ associated with φ by (4.3) satisfy

$$|\eta_\pm(z)| \lesssim \frac{r^{2m}}{|z|^{m+3/2}}, \qquad z \in \Gamma_\pm.$$

Since $p_0 \in \mathcal{J}(L)$, $\{e^{-zL}\}_{z \in \Gamma_\pm} \in \mathcal{O}(L^{p_0} - L^{p_0})$ and so

$$\left(\fint_B \left| \int_{\Gamma_+} \eta_+(z) \, e^{-zL} h_j \, dz \right|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq \int_{\Gamma_+} \left(\fint_B |e^{-zL} h_j|^{p_0} \, dx \right)^{\frac{1}{p_0}} |\eta_+(z)| \, |dz|$$

$$\lesssim \; 2^{j\,\theta_1} \int_{\Gamma_+} \Upsilon\left(\frac{2^j\,r}{\sqrt{|z|}}\right)^{\theta_2} e^{-\frac{\alpha\,4^j\,r^2}{|z|}} \frac{r^{2\,m}}{|z|^{m+3/2}}\,|dz|\left(\fint_{C_j(B)}|h_j|^{p_0}\,dx\right)^{\frac{1}{p_0}}$$

$$\lesssim \; 2^{j\,(\theta_1-2\,m-1)}\sum_{l=1}^{j}2^l\left(\fint_{2^{l+1}B}|\nabla f|^{p_0}\,dx\right)^{\frac{1}{p_0}},$$

provided $2\,m+1>\theta_2$, where the last inequality follows by repeating the calculations made to derive (5.6). The term corresponding to Γ_- is controlled similarly. Plugging both estimates into the representation of $\varphi(L)$ given by (4.2) one obtains

$$\left(\fint_B|\varphi(L)h_j|^{p_0}\,dx\right)^{\frac{1}{p_0}}\lesssim 2^{j\,(\theta_1-2\,m-1)}\sum_{l=1}^{j}2^l\left(\fint_{2^{l+1}B}|\nabla f|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

The treatment for the term $j=2$ is similar using

$$|h_2|\le|f-f_{4\,B}|\,\chi_{8\,B\setminus2\,B}\le|f-f_{2\,B}|\,\chi_{8\,B\setminus2\,B}+|f_{4\,B}-f_{2\,B}|\,\chi_{8\,B\setminus2\,B}.$$

Applying Minkowski's inequality and (6.5), we obtain (6.3). The lemma is proved. $\square$

Case $p\in(\widetilde{p}_-,\widehat{p}_+)$: Take p_0,q_0 such that $\widetilde{p}_-<p_0<\widetilde{p}_+$ and $p_0<p<q_0<\widehat{p}_+$. Observe that $p_0\in\mathcal{W}_w(p_-,p_+)\subset\mathrm{Int}\,\mathcal{J}_w(L)$ and $q_0\in\mathcal{J}_w(L)$. The proof of Lemma 6.5 extends *mutatis mutandis* with dw replacing dx since there is an L^{p_0}-Poincaré inequality for dw (see Section 5). It suffices to apply Theorem 2.2 with underlying measure dw and no weight. We leave further details to the reader.

Case $p\in(\max\{r_w,(\widehat{p}_-)_{w,*}\},\widetilde{p}_+)$: It follows a method in the unweighted case by [Aus] using an adapted Calderón-Zygmund decomposition.

Lemma 6.6. *Let* $n\ge1$, $w\in A_\infty$ *and* $1\le p<\infty$ *such that* $w\in A_p$. *Assume that* $f\in\mathcal{S}$ *is such that* $\|\nabla f\|_{L^p(w)}<\infty$. *Let* $\alpha>0$. *Then, one can find a collection of balls* $\{B_i\}_i$, *smooth functions* $\{b_i\}_i$ *and a function* $g\in L^1_{\mathrm{loc}}(w)$ *such that*

$$f=g+\sum_i b_i \tag{6.6}$$

and the following properties hold:

$$|\nabla g(x)|\le C\alpha,\quad for\ \mu\text{-}a.e.\ x \tag{6.7}$$

$$\mathrm{supp}\,b_i\subset B_i\quad and\quad \int_{B_i}|\nabla b_i|^p\,dw\le C\alpha^p w(B_i), \tag{6.8}$$

$$\sum_i w(B_i)\le\frac{C}{\alpha^p}\int_{\mathbb{R}^n}|\nabla f|^p\,dw, \tag{6.9}$$

$$\sum_i\chi_{B_i}\le N, \tag{6.10}$$

where C *and* N *depends only on the dimension, the doubling constant of* μ *and* p. *In addition, for* $1\le q<p_w^*$, *we have*

$$\left(\fint_{B_i}|b_i|^q\,dw\right)^{\frac{1}{q}}\lesssim\alpha\,r(B_i). \tag{6.11}$$

Proof. Since $w \in A_p$, we have an $L^p(w)$ Poincaré inequality (see [FPW]). On the other hand, as $w \in A_p$ and $1 \leq q < p_w^*$ (if $p = 1$, i.e. $w \in A_1$, it holds also at $q = 1_w^* = \frac{n}{n-1}$ when $n \geq 2$) we can apply [FPW, Corollary 3.2] (when checking the "balance condition" in that reference we have used that $w \in A_r$ implies $(|E|/|B|)^r \lesssim w(E)/w(B)$ for any ball B and any $E \subset B$). Thus there is an $L^p(w) - L^q(w)$ Poincaré inequality:

$$\left(\fint_B |f - f_{B,w}|^q \, dw \right)^{\frac{1}{q}} \lesssim r(B) \left(\fint_B |\nabla f|^p \, dw \right)^{\frac{1}{p}} \tag{6.12}$$

for all locally Lipschitz functions f and all balls B. These are all the ingredients needed to invoke [AM1, Proposition 9.1]. $\square$

We use the following resolution of $L^{1/2}$:

$$L^{1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty L e^{-t L} f \, \frac{dt}{\sqrt{t}}.$$

It suffices to work with $\int_\varepsilon^R \ldots$, to obtain bounds independent of ε, R, and then to let $\varepsilon \downarrow 0$ and $R \uparrow \infty$: indeed, the truncated integrals converge to $L^{1/2} f$ in L^2 when $f \in \mathcal{S}$ and a use of Fatou's lemma concludes the proof. For the truncated integrals, all the calculations are justified. We write $L^{1/2}$ where it is understood that it should be replaced by its approximation at all places.

Take q_0 so that $\widetilde{p}_- < q_0 < \widetilde{p}_+$. By the first case of the proof,

$$\|L^{1/2} f\|_{L^{q_0}(w)} \lesssim \|\nabla f\|_{L^{q_0}(w)}. \tag{6.13}$$

We may assume that $\max\{ r_w, (\widehat{p}_-)_{w,*} \} < p < \widetilde{p}_-$, otherwise there is nothing to prove. We claim that it is enough to show that

$$\|L^{1/2} f\|_{L^{p,\infty}(w)} \lesssim \|\nabla f\|_{L^p(w)}. \tag{6.14}$$

Assuming this estimate we want to interpolate. To this end, we use the following lemma.

Lemma 6.7. *Assume $r > r_w$. Then $\mathcal{D} = \{(-\Delta)^{1/2} f \, : \, f \in \mathcal{S}, \operatorname{supp} \widehat{f} \subset \mathbb{R}^n \setminus \{0\}\}$ is dense in $L^r(w)$, where $\widehat{f}$ denotes the Fourier transform of f.*

Proof. It is easy to see that $\mathcal{D} \subset \mathcal{S}$ hence $\mathcal{D} \subset L^r(w)$. As in [Gra, p. 353], using that the classical Littlewood-Paley series converges in $L^r(w)$ since $w \in A_r$, it follows that the set

$$\widetilde{\mathcal{D}} = \{g \in \mathcal{S} \, : \, \operatorname{supp} \widehat{g} \subset \mathbb{R}^n \setminus \{0\}, \ \operatorname{supp} \widehat{g} \ \text{is compact}\}$$

is dense in $L^r(w)$. We see that $\widetilde{\mathcal{D}} \subset \mathcal{D}$ and so $\mathcal{D}$ is dense in $L^r(w)$. For $g \in \widetilde{\mathcal{D}}$, $f = (-\Delta)^{-1/2} g$ is well-defined in $\mathcal{S}$ as $\widehat{f}(\xi) = c \, |\xi|^{-1/2} \widehat{g}(\xi)$ and $\operatorname{supp} \widehat{f} \subset \mathbb{R}^n \setminus \{0\}$. Hence, $g = (-\Delta)^{1/2} f \in \mathcal{D}$. $\square$

If $r > r_w$, the usual Riesz transforms, $\nabla(-\Delta)^{-1/2}$, are bounded on $L^r(w)$ (this can be reobtained from the results in Section 5). Also, for $g \in L^r(w)$, one has

$$\|g\|_{L^r(w)} \sim \|\nabla(-\Delta)^{-1/2} g\|_{L^r(w)}$$

using the identity $-I = R_1^2 + \cdots + R_n^2$ where $R_j = \partial_j(-\Delta)^{-1/2}$. Thus, for $g \in \mathcal{D}$, $L^{1/2}(-\Delta)^{-1/2} g = L^{1/2} f$ if $f = (-\Delta)^{-1/2} g$ and $\|\nabla f\|_{L^r(w)} \sim \|g\|_{L^r(w)}$ for $r > r_w$. As $r_w < p < q_0$, (6.13) and (6.14) reformulate into weighted strong type (q_0, q_0) and weak type (p, p) of $T = L^{1/2}(-\Delta)^{-1/2}$ *a priori* defined on $\mathcal{D}$. Since $\mathcal{D}$ is dense in all $L^r(w)$

when $r > r_w$ by the above lemma, we can extend T by density in both cases and their restrictions to the space of simple functions agree. Hence, we can apply Marcinkiewicz interpolation and conclude again by density that (6.13) holds for all q with $p < q < q_0$ which leads to the desired estimate.

Our goal is thus to establish (6.14), more precisely: for $f \in \mathcal{S}$ and $\alpha > 0$,

$$w\{|L^{1/2}f| > \alpha\} = w\{x \in \mathbb{R}^n : |L^{1/2}f(x)| > \alpha\} \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw. \qquad (6.15)$$

Since $p > r_w$, we have $w \in A_p$. From the condition $(\widehat{p}_-)_{w,*} < p$, we have $\widehat{p}_- < p_w^*$. Therefore, there exists $q \in (\widehat{p}_-, \widehat{p}_+) = \operatorname{Int} \mathcal{J}_w(L)$ such that $\widehat{p}_- < q < p_w^*$. Thus, we can apply the Calderón-Zygmund decomposition of Lemma 6.6 to f at height α for the measure dw and write $f = g + \sum_i b_i$. Using (6.13), (6.7) and $q_0 > p$, we have

$$w\left\{|L^{1/2}g| > \frac{\alpha}{3}\right\} \lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} |L^{1/2}g|^{q_0} \, dw \lesssim \frac{1}{\alpha^{q_0}} \int_{\mathbb{R}^n} |\nabla g|^{q_0} \, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla g|^p \, dw$$

$$\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + \frac{1}{\alpha^p} \int_{\mathbb{R}^n} \left|\sum_i \nabla b_i\right|^p \, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw,$$

where the last estimate follows by applying (6.10), (6.8), (6.9).

To compute $L^{1/2}(\sum_i b_i)$, let $r_i = 2^k$ if $2^k \leq r(B_i) < 2^{k+1}$, hence $r_i \sim r(B_i)$ for all i. Write

$$L^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^{r_i^2} L e^{-tL} \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_{r_i^2}^\infty L e^{-tL} \frac{dt}{\sqrt{t}} = T_i + U_i,$$

and then

$$w\left\{\left|\sum_i L^{1/2}b_i\right| > \frac{2\alpha}{3}\right\} \leq w\left(\bigcup_i 4B_i\right) + w\left\{\left|\sum_i U_i b_i\right| > \frac{\alpha}{3}\right\}$$

$$+ w\left(\left(\mathbb{R}^n \setminus \bigcup_i 4B_i\right) \bigcap \left\{\left|\sum_i T_i b_i\right| > \frac{\alpha}{3}\right\}\right)$$

$$\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw + I + II,$$

where we have used (6.9). We estimate II. Since $q \in \mathcal{J}_w(L)$, it follows that $t L e^{-tL} \in \mathcal{O}(L^q(w) - L^q(w))$ by Proposition 3.4, hence

$$II \lesssim \frac{1}{\alpha} \sum_i \sum_{j \geq 2} \int_{C_j(B_i)} |T_i b_i| \, dw \lesssim \frac{1}{\alpha} \sum_i \sum_{j \geq 2} w(2^j B_i) \int_0^{r_i^2} \fint_{C_j(B_i)} |t L e^{-tL} b_i| \, dw \frac{dt}{t^{3/2}}$$

$$\lesssim \frac{1}{\alpha} \sum_i \sum_{j \geq 2} 2^{jD} w(B_i) \int_0^{r_i^2} 2^{j\theta_1} \Upsilon\left(\frac{2^j r_i}{\sqrt{t}}\right)^{\theta_2} e^{-\frac{c 4^j r_i^2}{t}} \frac{dt}{t^{3/2}} \left(\fint_{B_i} |b_i|^q \, dw\right)^{\frac{1}{q}}$$

$$\lesssim \sum_i \sum_{j \geq 2} 2^{jD} e^{-c 4^j} w(B_i) \lesssim \sum_i w(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, dw,$$

where we have used (6.11) and (6.9), and D is the doubling order of dw.

It remains to handling the term I. Using functional calculus for L one can compute U_i as $r_i^{-1}\psi(r_i^2 L)$ with ψ the holomorphic function on the sector $\Sigma_{\pi/2}$ given by

$$\psi(z) = c \int_1^\infty z\, e^{-tz}\, \frac{dt}{\sqrt{t}}. \tag{6.16}$$

It is easy to show that $|\psi(z)| \le C|z|^{1/2}e^{-c|z|}$, uniformly on subsectors Σ_μ, $0 \le \mu < \frac{\pi}{2}$. We claim that, since $q \in \mathrm{Int}\,\mathcal{J}_w(L)$,

$$\Big\| \sum_{k\in\mathbb{Z}} \psi(4^k L)\,\beta_k \Big\|_{L^q(w)} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(w)}. \tag{6.17}$$

The proof of this inequality is postponed until the end of Section 7. We set $\beta_k = \sum_{i\,:\,r_i=2^k} \frac{b_i}{r_i}$. Then,

$$\sum_i U_i\, b_i = \sum_{k\in\mathbb{Z}} \psi(4^k L)\Big(\sum_{i\,:\,r_i=2^k} \frac{b_i}{r_i} \Big) = \sum_{k\in\mathbb{Z}} \psi(4^k L)\beta_k.$$

Using (6.17), the bounded overlap property (6.10), (6.11), $r_i \sim r(B_i)$ and (6.9), one has

$$I \lesssim \frac{1}{\alpha^q} \Big\| \sum_i U_i b_i \Big\|_{L^q(w)}^q \lesssim \frac{1}{\alpha^q} \Big\| \Big(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(w)}^q \lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n} \sum_i \frac{|b_i|^q}{r_i^q}\, dw$$

$$\lesssim \sum_i w(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p\, dw.$$

Collecting the obtained estimates, we conclude (6.14) as desired. $\qquad\square$

Remark 6.8. If $w \in A_1$, $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \varnothing$ and $(\widehat{p}_-)_{w,*} < 1$ then for all $f \in \mathcal{S}$

$$\|L^{1/2} f\|_{L^{1,\infty}(w)} \lesssim \|\nabla f\|_{L^1(w)}.$$

This (that is (6.15) with $p = 1$) uses a similar argument (left to the reader) once we have chosen an appropriate q for which $L^1(w) - L^q(w)$ Poincaré inequality holds: since $w \in A_1$, one needs $q \le \frac{n}{n-1}$. As $r_w = 1$, the assumption $(\widehat{p}_-)_{w,*} < 1$ means that $\widehat{p}_- < \frac{n}{n-1}$ and so we pick $q \in \mathrm{Int}\,\mathcal{J}_w(L)$ with $\widehat{p}_- < q < \frac{n}{n-1}$.

7. SQUARE FUNCTIONS

We define the square functions for $x \in \mathbb{R}^n$ and $f \in L^2$,

$$g_L f(x) = \Big(\int_0^\infty |(t L)^{1/2} e^{-t L} f(x)|^2\, \frac{dt}{t} \Big)^{\frac{1}{2}},$$

$$G_L f(x) = \Big(\int_0^\infty |\nabla e^{-t L} f(x)|^2\, dt \Big)^{\frac{1}{2}}.$$

They are representative of a larger class of square functions and we restrict our discussion to them to show the applicability of our methods. They satisfy the following L^p estimates.

Theorem 7.1 ([Aus]).

$$\mathrm{Int}\,\big\{ 1 < p < \infty : \|g_L f\|_p \sim \|f\|_p, \forall f \in L^p \cap L^2 \big\} = \big(p_-(L), p_+(L)\big)$$

and

$$\text{Int}\left\{1 < p < \infty : \|G_L f\|_p \sim \|f\|_p, \forall\, f \in L^p \cap L^2\right\} = \big(q_-(L), q_+(L)\big).$$

In this statement, $\sim$ can be replaced by $\lesssim$: the square function estimates for L (with $\lesssim$) automatically imply the reverse ones for L^*. The part concerning g_L can be obtained using an abstract result of Le Merdy [LeM] as a consequence of the bounded holomorphic functional calculus on L^p. The method in [Aus] is direct. We remind the reader that in [Ste], these inequalities for $L = -\Delta$ were proved differently and the boundedness of $G_{-\Delta}$ follows from that of $g_{-\Delta}$ and of the Riesz transforms $\partial_j(-\Delta)^{-1/2}$ (or vice-versa) using the commutation between ∂_j and $e^{-t\Delta}$. Here, no such thing is possible.

We have the following weighted estimates for square functions.

Theorem 7.2. *Let $w \in A_\infty$.*

(a) *If $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$ and $p \in \text{Int}\, \mathcal{J}_w(L)$ then for all $f \in L_c^\infty$ we have*
$$\|g_L f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

(b) *If $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$ and $p \in \text{Int}\, \mathcal{K}_w(L)$ then for all $f \in L_c^\infty$ we have*
$$\|G_L f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Note that the operators $(t\,L)^{1/2}\, e^{-t\,L}$ and $\nabla e^{-t\,L}$ extend to $L^p(w)$ when $p \in \text{Int}\, \mathcal{J}_w(L)$ and $p \in \text{Int}\, \mathcal{K}_w(L)$ respectively. By seeing g_L and G_L as linear operators from scalar functions to $\mathbb{H}$-valued functions (see below for definitions), the above inequalities extend to all $f \in L^p(w)$ by density (see the proof).

We also get reverse weighted square function estimates as follows.

Theorem 7.3. *Let $w \in A_\infty$.*

(a) *If $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$ and $p \in \text{Int}\, \mathcal{J}_w(L)$ then*
$$\|f\|_{L^p(w)} \lesssim \|g_L f\|_{L^p(w)}, \qquad f \in L^p(w) \cap L^2.$$

(b) *If $r_w < p < \infty$,*
$$\|f\|_{L^p(w)} \lesssim \|G_L f\|_{L^p(w)}, \qquad f \in L^p(w) \cap L^2.$$

The restriction that $f \in L^2$ can be removed provided g_L and G_L are appropriately interpreted: see the proofs. We add a comment about sharpness of the ranges of p at the end of the section.

As a corollary, g_L (resp. G_L) defines a new norm on $L^p(w)$ when $p \in \text{Int}\, \mathcal{J}_w(L)$ (resp. $p \in \text{Int}\, \mathcal{K}_w(L)$ and $p > r_w$). Again, Le Merdy's result cited above [LeM] also gives such a result for g_L, but not for G_L. The restriction $p > r_w$ in part (b) comes from the argument. We do not know whether it is necessary for a given weight non identically 1.

Before we begin the arguments, we recall some basic facts about Hilbert-valued extensions of scalar inequalities. To do so we introduce some notation: by $\mathbb{H}$ we mean $L^2\big((0, \infty), \frac{dt}{t}\big)$ and $\|\cdot\|$ denotes the norm in $\mathbb{H}$. Hence, for a function $h \colon \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$, we have for $x \in \mathbb{R}^n$
$$\|h(x, \cdot)\| = \left(\int_0^\infty |h(x, t)|^2 \,\frac{dt}{t}\right)^{1/2}.$$

In particular,
$$g_L f(x) = \|\varphi(L, \cdot) f(x)\|$$
with $\varphi(z, t) = (t z)^{1/2} e^{-t z}$ and
$$G_L f(x) = \|\nabla \varphi(L, \cdot) f(x)\|$$
with $\varphi(z, t) = \sqrt{t} \, e^{-t z}$. Let $L_{\mathbb{H}}^p(w)$ be the space of $\mathbb{H}$-valued $L^p(w)$-functions equipped with the norm
$$\|h\|_{L_{\mathbb{H}}^p(w)} = \left(\int_{\mathbb{R}^n} \|h(x, \cdot)\|^p \, dw(x) \right)^{\frac{1}{p}}.$$

Lemma 7.4. *Let μ be a Borel measure on $\mathbb{R}^n$ (for instance, given by an A_∞ weight). Let $1 \le p \le q < \infty$. Let $\mathcal{D}$ be a subspace of $\mathcal{M}$, the space of measurable functions in $\mathbb{R}^n$. Let S, T be linear operators from $\mathcal{D}$ into $\mathcal{M}$. Assume there exists $C_0 > 0$ such that for all $f \in \mathcal{D}$, we have*
$$\|Tf\|_{L^q(\mu)} \le C_0 \sum_{j \ge 1} \alpha_j \|Sf\|_{L^p(F_j, \mu)},$$
where F_j are subsets of $\mathbb{R}^n$ and $\alpha_j \ge 0$. Then, there is an $\mathbb{H}$-valued extension with the same constant: for all $f \colon \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ such that for (almost) all $t > 0$, $f(\cdot, t) \in \mathcal{D}$,
$$\|Tf\|_{L_{\mathbb{H}}^q(\mu)} \le C_0 \sum_{j \ge 1} \alpha_j \|Sf\|_{L_{\mathbb{H}}^p(F_j, \mu)}.$$

The extension of a linear operator T on $\mathbb{C}$-valued functions to $\mathbb{H}$-valued functions is defined for $x \in \mathbb{R}^n$ and $t > 0$ by $(Th)(x, t) = T\big(h(\cdot, t)\big)(x)$, that is, t can be considered as a parameter and T acts only on the variable in $\mathbb{R}^n$. This result is essentially the same as the Marcinkiewicz-Zygmund theorem and the fact that $\mathbb{H}$ is isometric to ℓ^2. That the norm decreases uses $p \le q$. We refer to, for instance, [Gra, Theorem 4.5.1] for an argument that extends straightforwardly to our setting.

Proof of Theorem 7.2. Part (a). We split the argument in three cases: $p \in (\widetilde{p}_-, \widetilde{p}_+)$, $p \in (\widetilde{p}_-, \widehat{p}_+)$, $p \in (\widehat{p}_-, \widetilde{p}_+)$.

Case $p \in (\widetilde{p}_-, \widetilde{p}_+)$: By Proposition 2.1, there exist p_0, q_0 such that
$$p_- < p_0 < p < q_0 < p_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

We are going to apply Theorem 2.2 with $T = g_L$, $S = I$, $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, m large enough, underlying measure dx and weight w. We first see that (2.2) holds for all $f \in L_c^\infty$. Here, we could have used the approach in [Aus], but the one below adapts to the other two cases with minor changes.

As $p_0, q_0 \in \mathcal{J}(L)$ and $p_0 \le q_0$, we know that $e^{-t L} \in \mathcal{O}(L^{p_0} - L^{q_0})$. If B is a ball, $j \ge 1$ and $g \in L^{p_0}$ with $\operatorname{supp} g \subset C_j(B)$ we have
$$\left(\fint_B |e^{-k \, r^2 L} g|^{q_0} \, dx \right)^{\frac{1}{q_0}} \le C_0 \, 2^{j \, (\theta_1 + \theta_2)} \, e^{-\alpha \, 4^j} \left(\fint_{C_j(B)} |g|^{p_0} \, dx \right)^{\frac{1}{p_0}}. \tag{7.1}$$

Lemma 7.4 applied to $S = I$, $T \colon L^{p_0} = L^{p_0}(\mathbb{R}^n, dx) \longrightarrow L^{q_0} = L^{q_0}(\mathbb{R}^n, dx)$ given by
$$Tg = \left(C_0 \, 2^{j \, (\theta_1 + \theta_2)} \, e^{-\alpha \, 4^j} \right)^{-1} \frac{|2^{j+1} B|^{\frac{1}{p_0}}}{|B|^{\frac{1}{q_0}}} \, \chi_B \, e^{-k \, r^2 L} (\chi_{C_j(B)} \, g)$$

yields

$$\left(\fint_B \|e^{-k\,r^2\,L} g(x,\cdot)\|^{q_0}\,dx \right)^{\frac{1}{q_0}} \leq C_0\, 2^{j\,(\theta_1+\theta_2)}\, e^{-\alpha\,4^j} \left(\fint_{C_j(B)} \|g(x,\cdot)\|^{p_0}\,dx \right)^{\frac{1}{p_0}} \qquad (7.2)$$

for all $g \in L_{\mathbb{H}}^{p_0}$ with $\operatorname{supp} g(\cdot, t) \subset C_j(B)$ for each $t > 0$.

As in (4.7), for $h \in L_{\mathbb{H}}^{p_0}$ write

$$h(x,t) = \sum_{j \geq 1} h_j(x,t), \quad x \in \mathbb{R}^n,\ t > 0,$$

where $h_j(x,t) = h(x,t)\,\chi_{C_j(B)}(x)$. Using (7.2), we have for $1 \leq k \leq m$,

$$\left(\fint_B \|e^{-k\,r^2\,L} h(x,\cdot)\|^{q_0}\,dx \right)^{\frac{1}{q_0}} \leq \sum_j \left(\fint_B \|e^{-k\,r^2\,L} h_j(x,\cdot)\|^{q_0}\,dx \right)^{\frac{1}{q_0}}$$

$$\lesssim \sum_{j \geq 1} 2^{j\,(\theta_1+\theta_2)}\, e^{-\alpha\,4^j} \left(\fint_{2^{j+1}\,B} \|h(x,\cdot)\|^{p_0}\,dx \right)^{\frac{1}{p_0}}. \qquad (7.3)$$

Take $h(x,t) = (t\,L)^{1/2}\,e^{-t\,L} f(x)$. Since $g_L f(x) = \|h(x,\cdot)\|$ and $f \in L_c^\infty$, $h \in L_{\mathbb{H}}^{p_0}$ by Theorem 7.1 and

$$g_L(e^{-k\,r^2\,L} f)(x) = \left(\int_0^\infty |(t\,L)^{1/2}\,e^{-t\,L}\,e^{-k\,r^2\,L} f(x)|^2\,\frac{dt}{t} \right)^{\frac{1}{2}} = \|e^{-k\,r^2\,L} h(x,\cdot)\|.$$

Thus (7.3) implies

$$\left(\fint_B |g_L(e^{-k\,r^2\,L} f)|^{q_0}\,dx \right)^{\frac{1}{q_0}} \lesssim \sum_{j \geq 1} 2^{j\,(\theta_1+\theta_2)}\, e^{-\alpha\,4^j} \left(\fint_{2^{j+1}\,B} |g_L f|^{p_0}\,dx \right)^{\frac{1}{p_0}}$$

and it follows that g_L satisfies (2.2).

It remains to show that (2.1) with $Sf = f$ holds for all $f \in L_c^\infty$. Write $f = \sum_{j \geq 1} f_j$ as before. If $j = 1$ we use that both g_L and $(I - e^{-r^2\,L})^m$ are bounded on L^{p_0} (see Theorem 7.1 and Proposition 3.3):

$$\left(\fint_B |g_L(I - e^{-r^2\,L})^m f_1|^{p_0}\,dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4B} |f|^{p_0}\,dx \right)^{\frac{1}{p_0}}. \qquad (7.4)$$

For $j \geq 2$, we observe that

$$g_L(I - e^{-r^2\,L})^m f_j(x) = \left(\int_0^\infty |(t\,L)^{1/2} e^{-t\,L}\,(I - e^{-r^2\,L})^m f_j(x)|^2\,\frac{dt}{t} \right)^{\frac{1}{2}} = \|\varphi(L,\cdot) f_j(x)\|$$

where $\varphi(z,t) = (t\,z)^{1/2}\,e^{-t\,z}\,(1 - e^{-r^2\,z})^m$. As in [Aus], the functions $\eta_\pm(\cdot, t)$ associated with $\varphi(\cdot, t)$ by (4.3) verify

$$|\eta_\pm(z,t)| \lesssim \frac{t^{1/2}}{(|z| + t)^{3/2}}\,\frac{r^{2\,m}}{(|z| + t)^m}, \qquad z \in \Gamma_\pm,\ t > 0.$$

Thus,

$$\|\eta_\pm(z,\cdot)\| \leq \left(\int_0^\infty \frac{t}{(|z| + t)^3}\,\frac{r^{4\,m}}{(|z| + t)^{2\,m}}\,\frac{dt}{t} \right)^{\frac{1}{2}} \lesssim \frac{r^{2\,m}}{|z|^{m+1}}. \qquad (7.5)$$

Next, applying Minkowski's inequality and $e^{-zL} \in \mathcal{O}(L^{p_0} - L^{p_0})$, since $p_0 \in \mathcal{J}(L)$, we have

$$\left(\fint_B \left\| \int_{\Gamma_+} e^{-zL} f_j(x) \, \eta_+(z, \cdot) dz \right\|^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\leq \ \left(\fint_B \left(\int_{\Gamma_+} |e^{-zL} f_j(x)| \, \|\eta_+(z, \cdot)\| \, |dz| \right)^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\leq \ \int_{\Gamma_+} \left(\fint_B |e^{-zL} f_j|^{p_0} dx \right)^{\frac{1}{p_0}} \frac{r^{2m}}{|z|^{m+1}} \, |dz|$$

$$\lesssim \ 2^{j\theta_1} \int_0^\infty \Upsilon\left(\frac{2^j r}{\sqrt{s}} \right)^{\theta_2} e^{-\frac{\alpha \, 4^j \, r^2}{s}} \frac{r^{2m}}{s^m} \frac{ds}{s} \left(\fint_{C_j(B)} |f|^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\lesssim \ 2^{j(\theta_1 - 2m)} \left(\fint_{C_j(B)} |f|^{p_0} dx \right)^{\frac{1}{p_0}}$$

provided $2m > \theta_2$. This plus the corresponding term for Γ_- yield

$$\left(\fint_B |g_L(I - e^{-r^2 L})^m f_j|^{p_0} dx \right)^{\frac{1}{p_0}} \lesssim 2^{j(\theta_1 - 2m)} \left(\fint_{C_j(B)} |f|^{p_0} dx \right)^{\frac{1}{p_0}}. \qquad (7.6)$$

Collecting the latter estimate and (7.4), we obtain that (2.1) holds whenever $2m > \max\{\theta_1, \theta_2\}$.

Case $p \in (\widetilde{p}_-, \widehat{p}_+)$: Take p_0, q_0 such that $\widetilde{p}_- < p_0 < \widetilde{p}_+$ and $p_0 < p < q_0 < \widehat{p}_+$. Let $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some $m \geq 1$ to be chosen later. Remark that by the previous case, g_L is bounded in $L^{p_0}(w)$ and so does $\mathcal{A}_r$ by Proposition 3.4. We apply Theorem 2.2 to $T = g_L$ and $S = I$ with underlying measure dw and no weight: it is enough to see that g_L satisfies (2.1) and (2.2) on L_c^∞. But this follows by adapting the preceding argument replacing everywhere dx by dw and observing that $e^{-zL} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$. We skip details.

Case $p \in (\widehat{p}_-, \widetilde{p}_+)$: Take p_0, q_0 such that $\widetilde{p}_- < q_0 < \widetilde{p}_+$ and $\widehat{p}_- < p_0 < p < q_0$. Set $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ for some integer $m \geq 1$ to be chosen later. Since $q_0 \in (\widetilde{p}_-, \widetilde{p}_+)$, by the first case, g_L is bounded on $L^{q_0}(w)$ and so does $\mathcal{A}_r$ by Proposition 3.4. By Theorem 2.4 with underlying Borel doubling measure dw, it is enough to show (2.4) and (2.5). Fix a ball B, $f \in L_c^\infty$ supported on B.

Observe that (2.5) follows directly from (4.11) since $p_0, q_0 \in \mathcal{J}_w(L)$ and $p_0 \leq q_0$. We turn to (2.4). Assume $j \geq 2$. The argument is the same as the one for (7.6) by reversing the roles of $C_j(B)$ and B, and using dw and $e^{-zL} \in \mathcal{O}(L^{p_0}(w) - L^{p_0}(w))$ (since $p_0 \in \mathcal{J}_w(L)$) instead of dx and $e^{-zL} \in \mathcal{O}(L^{p_0} - L^{p_0})$. We obtain

$$\left(\fint_{C_j(B)} |g_L(I - e^{-r^2 L})^m f|^{p_0} dw \right)^{\frac{1}{p_0}} \lesssim 2^{j(\theta_1 - 2m)} \left(\fint_B |f|^{p_0} dw \right)^{\frac{1}{p_0}}$$

provided $2m > \theta_2$ and it remains to impose further $2m > \theta_1 + D$ to conclude, where D is the doubling order of w. $\qquad \square$

Proof of Theorem 7.2. Part (b). We split the argument in three cases: $p \in (\widetilde{q}_-, \widetilde{q}_+)$, $p \in (\widetilde{q}_-, \widehat{q}_+)$, $p \in (\widehat{q}_-, \widetilde{q}_+)$.

Case $p \in (\widetilde{q}_-, \widetilde{q}_+)$: By Proposition 2.1, there exist p_0, q_0 such that

$$q_- < p_0 < p < q_0 < q_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

We are going to apply Theorem 2.2 with underlying measure dx and weight w to $T = G_L$, $S = I$, $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, m large enough. We begin with (2.2). Fix $1 \le k \le m$ and B a ball. Combining (5.4) and Lemma 7.4 with $T = \nabla e^{-k\,r^2 L}$ and $S = \nabla$, we obtain

$$\left(\fint_B \|\!|\nabla e^{-k\,r^2 L} h(x, \cdot)\|\!|^{q_0}\, dx \right)^{\frac{1}{q_0}} \lesssim \sum_{j \ge 1} g_2(j) \left(\fint_{2^{j+1} B} \|\!|\nabla h(x, \cdot)\|\!|^{p_0}\, dx \right)^{\frac{1}{p_0}}$$

with $g_2(j) = C_m\, 2^j \sum_{l \ge j} 2^{l\,\theta}\, e^{-\alpha\, 4^l}$ for some $\theta > 0$ whenever $h \colon \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{C}$ is such that h and ∇h belong to L^{p_0} (our space $\mathcal{D}$). Setting $h(x, t) = \sqrt{t}\, e^{-t L} f(x)$ for $f \in L^\infty_c$, we note that $h(\cdot, t) \in L^{p_0}$ and $\nabla h(\cdot, t) \in L^{p_0}$ for each $t > 0$. Hence, the above estimate applies. Since $\|\!|\nabla h(x, \cdot)\|\!| = G_L f(x)$ and $\|\!|\nabla e^{-k\,r^2 L} h(x, \cdot)\|\!| = G_L(e^{-k\,r^2 L} f)(x)$, we obtain

$$\left(\fint_B |G_L(e^{-k\,r^2 L} f)|^{q_0}\, dx \right)^{\frac{1}{q_0}} \lesssim \sum_{j \ge 1} g_2(j) \left(\fint_{2^{j+1} B} |G_L f|^{p_0}\, dx \right)^{\frac{1}{p_0}},$$

which is (2.2) after expanding $\mathcal{A}_r$.

It remains to checking (2.1) for G_L and $S = I$ for $f \in L^\infty_c$. Fix a ball B. As before, write $f = \sum_{j \ge 1} f_j$ where $f_j = f\, \chi_{C_j(B)}$. Since $p_0 \in \operatorname{Int} \mathcal{K}(L)$, both G_L and $(I - e^{-r^2 L})^m$ are bounded on L^{p_0} by Theorem 7.1 and Proposition 3.3. Then for $j = 1$ we have

$$\left(\fint_B |G_L(I - e^{-r^2 L})^m f_1|^{p_0}\, dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4B} |f|^{p_0}\, dx \right)^{\frac{1}{p_0}}. \tag{7.7}$$

For $j \ge 2$, we observe that

$$G_L(I - e^{-r^2 L})^m f_j(x) = \left(\int_0^\infty |\sqrt{t}\,\nabla e^{-t L}\,(I - e^{-r^2 L})^m f_j(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} = \|\!|\nabla \varphi(L, \cdot) f_j(x)\|\!|$$

where $\varphi(z, t) = \sqrt{t}\, e^{-t z}\,(1 - e^{-r^2 z})^m$. As in [Aus], the functions $\eta_\pm(\cdot, t)$ associated for each $t > 0$ with $\varphi(\cdot, t)$ by (4.3) verify

$$|\eta_\pm(z, t)| \lesssim \frac{\sqrt{t}}{|z| + t} \frac{r^{2m}}{(|z| + t)^m}, \qquad z \in \Gamma_\pm,\ t > 0,$$

and so

$$\|\!|\eta_\pm(z, \cdot)\|\!| \le \left(\int_0^\infty \frac{t}{(|z| + t)^2} \frac{r^{4m}}{(|z| + t)^{2m}} \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim \frac{r^{2m}}{|z|^{m + 1/2}}. \tag{7.8}$$

Using Minkowski's inequality and $\sqrt{z}\,\nabla e^{-z L} \in \mathcal{O}\big(L^{p_0} - L^{p_0}\big)$ since $p_0 \in \mathcal{K}(L)$,

$$\left(\fint_B \Big\|\!\Big| \int_{\Gamma_+} \nabla e^{-z L} f_j(x)\, \eta_+(z, \cdot)\, dz \Big\|\!\Big|^{p_0}\, dx \right)^{\frac{1}{p_0}}$$

$$\le \left(\fint_B \left(\int_{\Gamma_+} |\sqrt{z}\,\nabla e^{-z L} f_j(x)|\, \|\!|\eta_+(z, \cdot)\|\!| \frac{|dz|}{|z|^{1/2}} \right)^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\le \int_{\Gamma_+} \left(\fint_B |\sqrt{z}\,\nabla e^{-z L} f_j(x)|^{p_0}\, dx \right)^{\frac{1}{p_0}} \frac{r^{2m}}{|z|^{m + 1/2}} \frac{|dz|}{|z|^{1/2}}$$

$$\lesssim\; 2^{j\,\theta_1}\int_0^\infty \Upsilon\left(\frac{2^j\,r}{\sqrt{s}}\right)^{\theta_2} e^{-\frac{\alpha\,4^j\,r^2}{s}}\,\frac{r^{2\,m}}{s^m}\,\frac{ds}{s}\left(\fint_{C_j(B)}|f|^{p_0}\,dx\right)^{\frac{1}{p_0}}$$

$$\lesssim\; 2^{j\,(\theta_1-2\,m)}\left(\fint_{C_j(B)}|f|^{p_0}\,dx\right)^{\frac{1}{p_0}}$$

provided $2\,m > \theta_2$. This, plus the corresponding term for Γ_-, yields

$$\left(\fint_B |G_L(I-e^{-r^2\,L})^m f_j|^{p_0}\,dx\right)^{\frac{1}{p_0}} \lesssim 2^{j\,(\theta_1-2\,m)}\left(\fint_{C_j(B)}|f|^{p_0}\,dx\right)^{\frac{1}{p_0}}. \qquad (7.9)$$

Collecting the latter estimate and (7.7), we obtain by Minkowski's inequality

$$\left(\fint_B |G_L(I-e^{-r^2\,L})^m f|^{p_0}\,dx\right)^{\frac{1}{p_0}} \lesssim \sum_{j\geq 1} 2^{j\,(\theta_1-2\,m)}\left(\fint_{C_j(B)}|f|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

Therefore, (2.1) holds on taking $2\,m > \sup(\theta_1,\theta_2)$.

Case $p \in (\widetilde{q}_-,\widehat{q}_+)$: Take p_0,q_0 such that $\widetilde{q}_- < p_0 < \widetilde{q}_+$ and $p_0 < p < q_0 < \widehat{q}_+$. Let $\mathcal{A}_r = I - (I - e^{-r^2\,L})^m$ for some $m \geq 1$ to be chosen later. As $p_0 \in (\widetilde{q}_-,\widetilde{q}_+)$, both G_L and $\mathcal{A}_r$ are bounded on $L^{p_0}(w)$ (we have just shown it for G_L and Proposition 3.4 yields it for $\mathcal{A}_r$ with a uniform norm in r). By Theorem 2.2 with underlying doubling measure dw and no weight, it is enough to verify (2.1) and (2.2) on $\mathcal{D} = L_c^\infty$ for $T = G_L$, $S = I$. It suffices to copy the preceding argument replacing everywhere dx by dw, observing that $p_0,q_0 \in \mathcal{K}_w(L)$ implies weighted off-diagonal estimates and an $L^{p_0}(w)$ Poincaré inequality, and applying Lemma 7.4 to obtain an $\mathbb{H}$-valued extension. We leave the details to the reader.

Case $p \in (\widehat{q}_-,\widetilde{q}_+)$: Take p_0,q_0 such that $\widetilde{q}_- < q_0 < \widetilde{q}_+$ and $\widehat{q}_- < p_0 < p < q_0$. Set $\mathcal{A}_r = I - (I - e^{-r^2\,L})^m$ for some $m \geq 1$ to be chosen later. Since $q_0 \in (\widetilde{q}_-,\widetilde{q}_+)$, it follows that G_L is bounded on $L^{q_0}(w)$ and so is $\mathcal{A}_r$ by Proposition 3.4. By Theorem 2.4 with underlying measure dw, it is enough to show (2.4) and (2.5).

Observe that (2.5) is nothing but (4.11) since $p_0,q_0 \in \mathcal{K}_w(L) \subset \mathcal{J}_w(L)$. The proof of (2.4) is again analogous to (7.9) in the weighted setting exchanging the roles of $C_j(B)$ and B. We skip details. $\qquad\square$

To prove Theorem 7.3, part (a), we introduce the following operator. Define for $f \in L_{\mathbb{H}}^2$ and $x \in \mathbb{R}^n$,

$$T_L f(x) = \int_0^\infty (t\,L)^{1/2}\,e^{-t\,L}f(x,t)\,\frac{dt}{t}.$$

Recall that $(t\,L)^{1/2}\,e^{-t\,L}f(x,t) = (t\,L)^{1/2}\,e^{-t\,L}(f(\cdot,t))(x)$. Hence, T_L maps $\mathbb{H}$-valued functions to $\mathbb{C}$-valued functions. We note that, for $f \in L_{\mathbb{H}}^2$ and $h \in L^2$, we have

$$\int_{\mathbb{R}^n} T_L f\,\overline{h}\,dx = \int_{\mathbb{R}^n}\int_0^\infty f(x,t)\,\overline{(t\,L^*)^{1/2}\,e^{-t\,L^*}h(x)}\,\frac{dt}{t}dx,$$

where L^* is the adjoint (on L^2) of L, hence,

$$\left|\int_{\mathbb{R}^n} T_L f\,\overline{h}\,dx\right| \leq \int_{\mathbb{R}^n}\|f(x,\cdot)\|\,g_{L^*}(h)(x)\,dx.$$

Let $p_-(L) < p < p_+(L)$. Since $p_-(L^*) = \big(p_+(L)\big)' < p' < \big(p_-(L)\big)' = p_+(L^*)$, g_{L^*} is bounded on $L^{p'}$. This and a density argument imply that T_L has a bounded extension from $L_{\mathbb{H}}^p$ to L^p. The weighted version is as follows.

Theorem 7.5. *Let $w \in A_\infty$. If $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \varnothing$ and $p \in \operatorname{Int} \mathcal{J}_w(L)$ then for all $f \in L_c^\infty(\mathbb{R}^n \times (0,\infty))$ we have*

$$\|T_L f\|_{L^p(w)} \lesssim \|f\|_{L_{\mathbb{H}}^p(w)}.$$

Hence, T_L has a bounded extension from $L_{\mathbb{H}}^p(w)$ to $L^p(w)$.

The duality argument above works for exponents in $\mathcal{W}_w\big(p_-(L), p_+(L)\big)$, but we do not know how to extend it to all of $\operatorname{Int} \mathcal{J}_w(L)$. Hence, we proceed via a direct proof where duality is used only when $w = 1$.

Proof. We split the argument in three cases: $p \in (\widetilde{p}_-, \widetilde{p}_+)$, $p \in (\widetilde{p}_-, \widehat{p}_+)$, $p \in (\widehat{p}_-, \widetilde{p}_+)$.

Case $p \in (\widetilde{p}_-, \widetilde{p}_+)$: By Proposition 2.1, there exist p_0, q_0 such that

$$p_- < p_0 < p < q_0 < p_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}.$$

We are going to apply Theorem 2.2 (in fact, its vector-valued extension) with underlying measure dx and weight w to the linear operator $T = T_L$ with $S = I$ and $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$, m large enough. Here, $\mathcal{A}_r$ denotes both the scalar operator and its $\mathbb{H}$-valued extension. We first see that T_L satisfies (2.2) with p_0, q_0 for $f \in L_c^\infty(\mathbb{R}^n \times (0,\infty))$. Let B be a ball. Note that $T_L \mathcal{A}_r f = \mathcal{A}_r T_L f$ with our confusion of notation. Hence (2.2) is a simple consequence of (7.1) applied to $g = T_L f$.

Next, it remains to check (2.1). Let $f \in L_c^\infty(\mathbb{R}^n \times (0,\infty))$ and let B be a ball. As in (4.7), we write

$$f(x,t) = \sum_{j \geq 1} f_j(x,t),$$

where $f_j(x,t) = f(x,t)\, \chi_{C_j(B)}(x)$. For $T_L(I - \mathcal{A}_r)f_1$, we use the boundedness of T_L from $L_{\mathbb{H}}^{p_0}$ to L^{p_0} noted above and the $L_{\mathbb{H}}^{p_0}$ boundedness of $\mathcal{A}_r$ to obtain

$$\left(\fint_B |T_L(I - \mathcal{A}_r)f_1|^{p_0}\, dx \right)^{\frac{1}{p_0}} \lesssim \left(\fint_{4B} \|f(x,\cdot)\|^{p_0}\, dx \right)^{\frac{1}{p_0}}.$$

For $j \geq 2$, the functions $\eta_\pm(z,t)$ associated with $\varphi(z,t) = (t\,z)^{1/2}\, e^{-t\,z}\,(1 - e^{-r^2 z})^m$ by (4.3) satisfy (7.5). Hence,

$$\left(\fint_B \left| \int_0^\infty \int_{\Gamma_+} e^{-z\,L} f_j(x,t) \eta_+(z,t)\, dz\, \frac{dt}{t} \right|^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\lesssim \left(\fint_B \left(\int_{\Gamma_+} \|e^{-z\,L} f_j(x,\cdot)\|\, \|\eta_+(z,\cdot)\|\, |dz| \right)^{p_0} dx \right)^{\frac{1}{p_0}}$$

$$\lesssim \int_{\Gamma_+} \left(\fint_B \|e^{-z\,L} f_j(x,\cdot)\|^{p_0}\, dx \right)^{\frac{1}{p_0}} \|\eta_+(z,\cdot)\|\, |dz|$$

$$\lesssim 2^{j\,\theta_1} \int_0^\infty \Upsilon\left(\frac{2^j\, r}{\sqrt{s}} \right)^{\theta_2} e^{-\frac{c\,4^j\,r^2}{s}}\, \frac{r^{2m}}{s^m}\, \frac{ds}{s} \left(\fint_{C_j(B)} \|f(x,\cdot)\|^{p_0}\, dx \right)^{\frac{1}{p_0}}$$

$$\lesssim \ 2^{j\,(\theta_1 - 2\,m)} \Big(\fint_{C_j(B)} \|f(x,\cdot)\|^{p_0} \, dx \Big)^{\frac{1}{p_0}}$$

where we used the $\mathbb{H}$-valued extension of $e^{-z\,L} \in \mathcal{O}\big(L^{p_0} - L^{p_0}\big)$ and assumed $2\,m > \theta_2$. This, plus the corresponding term for Γ_-, yields

$$\Big(\fint_B |T_L(I - \mathcal{A}_r)f_j|^{p_0} \, dx \Big)^{\frac{1}{p_0}} \lesssim 2^{j\,(\theta_1 - 2\,m)} \Big(\fint_{C_j(B)} \|f(x,\cdot)\|^{p_0} \, dx \Big)^{\frac{1}{p_0}} \tag{7.10}$$

and therefore (2.1) follows on taking also $2\,m > \theta_1$.

Case $p \in (\widetilde{p}_-, \widehat{p}_+)$: Take p_0, q_0 with $\widetilde{p}_- < p_0 < \widetilde{p}_+$ and $p_0 < p < q_0 < \widehat{p}_+$. It suffices to apply the $\mathbb{H}$-valued extension of Theorem 2.2 with underlying doubling measure dw and no weight to the linear operator $T = T_L$, $S = I$ and $\mathcal{A}_r = I - (I - e^{-r^2\,L})^m$, m large enough. This is done exactly as in the previous case. At some step we have to use that T_L is bounded from $L_{\mathbb{H}}^{p_0}(w)$ to $L^{p_0}(w)$ which follows by the previous case. We leave the details to the reader.

Case $p \in (\widehat{p}_-, \widetilde{p}_+)$: Take p_0, q_0 with $\widetilde{p}_- < q_0 < \widetilde{p}_+$ and $\widehat{p}_- < p_0 < p < q_0$. Since $q_0 \in (\widetilde{p}_-, \widetilde{p}_+)$, by the first case, T_L is bounded from $L_{\mathbb{H}}^{q_0}(w)$ to $L^{q_0}(w)$ and so does $\mathcal{A}_r = I - (I - e^{-r^2\,L})^m$, $m \geq 1$, on $L_{\mathbb{H}}^{q_0}(w)$ by Proposition 3.4 and Lemma 7.4. By Theorem 2.4 (in fact, its $\mathbb{H}$-valued extension) with underlying Borel doubling measure dw, it is enough to show (2.4) and (2.5) on $\mathcal{D} = L_c^{\infty}(\mathbb{R}^n \times (0, \infty))$ for $T = T_L$ and $\mathcal{A}_r$ with large enough m. As usual, the latter is a mere consequence of $e^{-t\,L} \in \mathcal{O}\big(L^{q_0}(w) - L^{q_0}(w)\big)$ and its $\mathbb{H}$-valued analog. The first condition is again a repetition of the argument for (7.10) in the weighted setting switching $C_j(B)$ and B. We skip details. $\square$

Proof of Theorem 7.3. We begin with part (a). Fix $p \in \operatorname{Int} \mathcal{J}_w(L)$ where $w \in A_\infty$ so that $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \varnothing$. Let $f \in L^2$ and define F by $F(x, t) = (t\,L)^{1/2}\,e^{-t\,L}f(x)$. Note that $F \in L_{\mathbb{H}}^2$ since $\|F\|_{L_{\mathbb{H}}^2} = \|g_L f\|_2$. By functional calculus on L^2, we have

$$f = 2 \int_0^\infty (t\,L)^{1/2}\,e^{-t\,L}F(\cdot, t)\,\frac{dt}{t} = 2\,T_L F \tag{7.11}$$

with convergence in L^2. Note that for $p \in \operatorname{Int} \mathcal{J}_w(L)$, $e^{-t\,L}$ has an infinitesimal generator on $L^p(w)$ as recalled in Remark 3.5. Let us call $L_{p,w}$ this generator. In particular $e^{-t\,L}$ and $e^{-t\,L_{p,w}}$ agree on $L^p(w) \cap L^2$. Our results assert that $L_{p,w}$ has a bounded holomorphic functional calculus on $L^p(w)$, hence replacing L by $L_{p,w}$ and $f \in L^2$ by $f \in L^p(w)$, we see that $F \in L_{\mathbb{H}}^p(w)$ with $\|F\|_{L^p(w)} = \|g_{L_{p,w}}f\|_{L^p(w)}$ and (7.11) is valid with convergence in $L^p(w)$ (this is standard fact from functional calculus and we skip details). Thus, by Theorem 7.5,

$$\|f\|_{L^p(w)} = 2\|T_{L_{p,w}}F\|_{L^p(w)} \lesssim \|F\|_{L_{\mathbb{H}}^p(w)} = \|g_{L_{p,w}}f\|_{L^p(w)}.$$

Noting that $g_L f = g_{L_{p,w}}f$ when $f \in L^2 \cap L^p(w)$ and $T_L F = T_{L_{p,w}}F$ when $F \in L_{\mathbb{H}}^2 \cap L_{\mathbb{H}}^p(w)$, part (a) is proved.

Let us show part (b), that is the corresponding inequality for G_L. Fix $w \in A_\infty$. We use the following estimate from [Aus]: for $f, h \in L^2$

$$\left| \int_{\mathbb{R}^n} f\,\overline{h} \, dx \right| \leq (1 + \|A\|_\infty) \int_{\mathbb{R}^n} G_L f\,G_{-\Delta}h \, dx,$$

where $G_{-\Delta}$ is the square function associated with the operator $-\Delta$. It is well known that $G_{-\Delta}$ is bounded on $L^q(u)$ for all $1 < q < \infty$ and all $u \in A_q$. Let us emphasize that, indeed, the results that we have proved can be applied to the operator $-\Delta$ and so $G_{-\Delta}$ is bounded on $L^q(u)$ for $u \in A_\infty$ and all $q \in \mathcal{W}_u\big(q_-(-\Delta), q_+(-\Delta)\big) = \mathcal{W}_u(1, \infty)$, that is, for all $1 < q < \infty$ and $u \in A_q$.

Coming back to the argument, let $p > r_w$, hence $w \in A_p$. Let $f \in L^2 \cap L^p(w)$. Then

$$\int_{\mathbb{R}^n} |f|^p \, dw = \lim_{N,k,R \to \infty} \int_{\mathbb{R}^n} f \, \overline{h} \, dw_N$$

with $w_N = \min\{w, N\}$ and $h = f|f|^{p-2} \chi_{B(0,R)} \chi_{\{0<|f|\leq k\}}$. Note that $\|h\|_{L^{p'}(w_N)} \leq \|f\|_{L^p(w)}^{p-1}$ and that hw_N is a bounded compactly supported function, hence in L^2.

Observe that $w_N \in A_p$ with A_p-constant smaller than the one for w. As observed, $G_{-\Delta}$ is bounded on $L^{p'}(w_N^{1-p'})$ since $w_N^{1-p'} \in A_{p'}$. Thus, we have

$$\left| \int_{\mathbb{R}^n} f \, \overline{h} \, dw_N \right| = \left| \int_{\mathbb{R}^n} f \, \overline{hw_N} \, dx \right| \leq (1 + \|A\|_\infty) \int_{\mathbb{R}^n} G_L f \, G_{-\Delta}(h \, w_N) \, dx$$

$$\leq (1 + \|A\|_\infty) \|G_L f\|_{L^p(w_N)} \|G_{-\Delta}(h \, w_N)\|_{L^{p'}(w_N^{1-p'})}$$

$$\leq C \|G_L f\|_{L^p(w_N)} \|h \, w_N\|_{L^{p'}(w_N^{1-p'})}$$

$$\leq C \|G_L f\|_{L^p(w)} \|f\|_{L^p(w)}^{p-1}$$

with C is independent of N, k, R and where we have used that $w_N \leq w$. Thus taking limits $N \to \infty$ first and then $k \to \infty$ and $R \to \infty$, we obtain

$$\|f\|_{L^p(w)}^p \leq C \|G_L f\|_{L^p(w)} \|f\|_{L^p(w)}^{p-1}.$$

$\square$

Proof of (6.17). The operator in (6.17) is similar to T_L, changing continuous times t to discrete times 4^k and $z^{1/2} e^{-z}$ to $\psi(z)$. Since $\psi(z)$ has the same quantitative properties as $z^{1/2} e^{-z}$ (decay at 0 and at infinity), the proof of Theorem 7.5 applies and furnishes (6.17). $\square$

Remark 7.6. Int $\mathcal{J}_w(L)$ is the sharp range up to endpoints for $\|g_L f\|_{L^p(w)} \sim \|f\|_{L^p(w)}$. Indeed, we have $g_L(e^{-tL}f) \leq g_L f$ for all $t > 0$. Hence, the equivalence implies the uniform $L^p(w)$ boundedness of e^{-tL}, which implies $p \in \widetilde{\mathcal{J}}_w(L)$ (see Proposition 3.4). Actually, Int $\mathcal{J}_w(L)$ is also the sharp range up to endpoints for the inequality $\|g_L f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$. It suffices to adapt the interpolation procedure in [Aus, Theorem 7.1, Step 7]. We skip details.

Similarly, this interpolation procedure also shows that Int $\mathcal{K}_w(L)$ is also sharp up to endpoints for $\|G_L f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$.

8. Some vector-valued estimates

In [AM1], we also obtained vector-valued inequalities.

Proposition 8.1. *Let $\mu, p_0, q_0, T, \mathcal{A}_r, \mathcal{D}$ be as in Theorem 2.2 and assume* (2.1) *and* (2.2) *with $S = I$. Let $p_0 < p, r < q_0$. Then, there is a constant C such that for all $f_k \in \mathcal{D}$*

$$\left\| \left(\sum_k |T f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mu)} \leq C \left\| \left(\sum_k |f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mu)}. \tag{8.1}$$

Let us see how it applies here.

First, let $T = \varphi(L)$ (φ bounded holomorphic in an appropriate sector). Theorem 4.2 says that T is bounded on $L^p(w)$ for all $p \in \operatorname{Int} \mathcal{J}_w(L)$. Also, for $p_0, q_0 \in \operatorname{Int} \mathcal{J}_w(L)$ with $p_0 < q_0$, we have $L^{p_0}(w) - L^{q_0}(w)$ off-diagonal estimates on balls for $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$. Hence, we can prove (2.1) and (2.2) with $S = I$ where dx is now replaced by $w(x)dx$ by mimicking the first case of the proof of Theorem 4.2 in the weighted context. Hence, one can apply the proposition above with $d\mu = w\,dx$ to above weighted vector-valued estimates for $\varphi(L)$ with all $p, r \in \operatorname{Int} \mathcal{J}_w(L)$.

The same weighted vector-valued estimates hold with all $p, r \in \operatorname{Int} \mathcal{J}_w(L)$ with $T = g_L$ starting from Theorem 7.2 and mimicking the proof of its first case with dx replaced with $w(x)dx$.

If $T = \nabla L^{-1/2}$ or $T = G_L$, then the same reasoning applies modulo the Poincaré inequality used towards obtaining (2.2). Hence, we conclude that for both $\nabla L^{-1/2}$ and G_L, one has (8.1) with $d\mu = wdx$ and $p, r \in \operatorname{Int} \mathcal{K}_w(L) \cap (r_w, \infty)$.

Other vector-valued inequalities of interest are

$$\left\| \left(\sum_{1 \le k \le N} |e^{-\zeta_k L} f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)} \le C \left\| \left(\sum_{1 \le k \le N} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)} \tag{8.2}$$

for $\zeta_k \in \Sigma_\alpha$ with $0 < \alpha < \pi/2 - \vartheta$ and $f_k \in L^p(w)$ with a constant C independent of N, the choice of the ζ_k's and the f_k's. We restrict to $1 < q < \infty$ and $w \in A_\infty$ (we keep working on $\mathbb{R}^n$). By a theorem of L. Weis [Wei, Theorem 4.2], we know that the existence of such a constant is equivalent to the maximal L^p-regularity of L on $L^q(w)$ with one/all $1 < p < \infty$, that is the existence of a constant C' such that for all $f \in L^p((0,\infty), L^q(w))$ there is a solution u of the parabolic problem on $\mathbb{R}^n \times (0, \infty)$,

$$u'(t) + Lu(t) = f(t), \ t > 0, \quad u(0) = 0,$$

with

$$\|u'\|_{L^p((0,\infty), L^q(w))} + \|Lu\|_{L^p((0,\infty), L^q(w))} \le C' \|f\|_{L^p((0,\infty), L^q(w))}.$$

Proposition 8.2. *Let $w \in A_\infty$ be such that $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \ne \emptyset$. Then for any $q \in \operatorname{Int} \mathcal{J}_w(L)$, (8.2) holds with $C = C_{q,w,L}$ independent of N, ζ_k, f_k.*

This result follows from an abstract result of Kalton-Weis [KW, Theorem 5.3] together with the bounded holomorphic functional calculus of L on those $L^q(w)$ that we established in Theorem 4.2. However, we wish to give a different proof using extrapolation and preceding ideas. Note that $q = 2$ may not be contained in $\operatorname{Int} \mathcal{J}_w(L)$ and the interpolation method of [BK2] may not work here.

Proof. There are three steps.

First step: Extrapolation. Letting N, ζ_k's and f_k's vary at will, we denote $\mathcal{F}$ the family of all ordered pairs (F, G) of the form

$$F = \left(\sum_{1 \le k \le N} |e^{-\zeta_k L} f_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad G = \left(\sum_{1 \le k \le N} |f_k|^2 \right)^{\frac{1}{2}}.$$

Then we have for all $(F, G) \in \mathcal{F}$,

$$\|F\|_{L^2(u)} \le C_u \|G\|_{L^2(u)}, \qquad \text{for all } u \in A_{2/p_-} \cap RH_{(p_+/2)'}. \tag{8.3}$$

Recall that $2 \in (p_-, p_+) = \operatorname{Int} \mathcal{J}(L)$ and $u \in A_{2/p_-} \cap RH_{(p_+/2)'}$ means $2 \in \mathcal{W}_u(p_-, p_+)$. In particular, $\{e^{-\zeta L} : \zeta \in \Sigma_\alpha\}$ is bounded in $\mathcal{L}(L^2(u))$. This inequality is trivially checked with C_u equal to the upper bound of this family. Applying our extrapolation result [AM1, Theorem 4.7], we deduce that, for all $p_- < q < p_+$ and $(F, G) \in \mathcal{F}$ we have

$$\|F\|_{L^q(u)} \leq C_{q,u} \|G\|_{L^q(u)}, \qquad \text{for all } u \in A_{q/p_-} \cap RH_{(p_+/q)'}. \tag{8.4}$$

In other words, for all $u \in A_\infty$ with $\mathcal{W}_u(p_-, p_+) \neq \varnothing$, (8.2) holds for $q \in \mathcal{W}_u(p_-, p_+)$ with C depending on q and w. This applies to our fixed weight w of the statement with $q \in \mathcal{W}_w(p_-, p_+)$. It remains to push the range of q's to all of $\operatorname{Int} \mathcal{J}_w(L)$.

Step 2: Pushing to the right. Take $p_0 \in \mathcal{W}_w(p_-, p_+)$, $q_0 \in \operatorname{Int} \mathcal{J}_w(L)$ with $p_0 < q < q_0$. Fix N and the ζ_k's. To prove (8.2) for that q, it suffices to apply the ℓ^2-valued version of Theorem 2.2 with underlying measure dw and no weight to $\mathbb{T}$ given by

$$\mathbb{T}f = (e^{-\zeta_1 L} f_1, \ldots, e^{-\zeta_N L} f_N), \qquad f = (f_1, \ldots, f_N),$$

with $S = I$. To check (2.1) and (2.2) we use $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ with m large enough (here, the ℓ^2-valued extension). Pick a ball B and $f \in (L_c^\infty)_{\ell^2}$. Using that $e^{-tL} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$ we can obtain (7.2), replacing $\mathbb{H}$ by ℓ^2 and with dw in place of dx. This and the fact that $\mathbb{T}\mathcal{A}_r = \mathcal{A}_r\mathbb{T}$ yield (2.2). We are left with checking (2.1). As usual we split f as $\sum_{j \geq 1} \chi_{C_j(B)} f$ (componentwise). The term with $\chi_{C_1(B)} f = \chi_{4B} f$ is treated using the $L_{\ell^2}^{p_0}(w)$ boundedness of $\mathbb{T}$ (first step) and of $I - \mathcal{A}_r$ (ℓ^2-valued extension of Proposition 3.4). The terms $\chi_{C_j(B)} f$, $j \geq 2$, are treated using off-diagonal estimates injecting the Khintchine inequality in the process: Let $F(x) = \|\mathbb{T}(I - \mathcal{A}_r)(\chi_{C_j(B)} f)(x)\|_{\ell^2}$. Let $r_1, \ldots, r_N$ be the N first Rademacher functions on $[0, 1]$. Then, by Khintchine's inequalities (see [Ste] for instance)

$$F(x) = \left(\int_0^1 \Big| \sum_{1 \leq k \leq N} r_k(t) \varphi_{\zeta_k}(L)(\chi_{C_j(B)} f_k)(x) \Big|^2 dt \right)^{\frac{1}{2}}$$

$$\sim \left(\int_0^1 \Big| \sum_{1 \leq k \leq N} r_k(t) \varphi_{\zeta_k}(L)(\chi_{C_j(B)} f_k)(x) \Big|^{p_0} dt \right)^{\frac{1}{p_0}}$$

where $z \longmapsto \varphi_\zeta(z) = e^{-\zeta z}(1 - e^{-r^2 z})^m$ for $\zeta \in \Sigma_\alpha$ is bounded on Σ_μ when $\vartheta < \mu < \pi/2 - \alpha$. Remark that the functions $\eta_{\pm,\zeta}$ associated to φ_ζ by (4.3) are easily shown to satisfy

$$|\eta_{\pm,\zeta}(z)| \lesssim \frac{1}{|z| + |\zeta|} \min\left(1, \left(\frac{r^2}{|z| + |\zeta|}\right)^m\right) \lesssim \frac{r^{2m}}{|z|^{m+1}}, \qquad z \in \Gamma_\pm,$$

where the implicit constant is independent of z, ζ, r. Thus, using the representation (4.2) for $\varphi_\zeta(L)$, integrating $F(x)^{p_0}$ against dw and using Minkowski's integral inequality we obtain

$$\left(\fint_B F(x)^{p_0} dw(x) \right)^{\frac{1}{p_0}} \lesssim \int_{\Gamma_+} \left(\int_0^1 \fint_B |e^{-zL}(\chi_{C_j(B)} h(\cdot, t, z))(x)|^{p_0} dw(x) dt \right)^{\frac{1}{p_0}} |dz|$$

where

$$h(x, t, z) = \sum_{1 \leq k \leq N} r_k(t) \eta_{+,\zeta_k}(z) f_k(x),$$

plus the similar term on Γ_-. Using $e^{-zL} \in \mathcal{O}\big(L^{p_0}(w) - L^{p_0}(w)\big)$ for $z \in \Gamma_+$, the right hand side in the above inequality is bounded by

$$2^{j\theta_1} \int_{\Gamma_+} \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-\frac{\alpha 4^j r^2}{|z|}} \left(\int_0^1 \fint_{C_j(B)} |h(x,t,z)|^{p_0} \, dw(x)\, dt\right)^{\frac{1}{p_0}} |dz|.$$

Using again Khintchine's inequality, this is comparable to

$$2^{j\theta_1} \int_{\Gamma_+} \Upsilon\left(\frac{2^j r}{\sqrt{|z|}}\right)^{\theta_2} e^{-\frac{\alpha 4^j r^2}{|z|}} \left(\fint_{C_j(B)} \Big(\sum_{1\le k\le N} |\eta_{+,\zeta_k}(z)\, f_k(x)|^2\Big)^{\frac{p_0}{2}} dw(x)\right)^{\frac{1}{p_0}} |dz|.$$

At this point, we use the upper bound on η_{ζ_k} and integrate in z if $2\,m > \theta_2$ to obtain that the latter is controlled by

$$2^{j(\theta_1 - 2\,m)} \left(\fint_{C_j(B)} \Big(\sum_{1\le k\le N} |f_k(x)|^2\Big)^{p_0/2} dw(x)\right)^{1/p_0}.$$

The condition (2.1) follows readily if $2\,m > \theta_1$ as well.

Step 3: Pushing to the left. This time, it suffices to use the ℓ^2-valued version of Theorem 2.4 with underlying measure dw and exponents p_0, q_0 such that $\widetilde{p}_- < q_0 < \widetilde{p}_+$ and $\widehat{p}_- < p_0 < q < q_0$. Then (2.5) follows from the ℓ^2-valued extension of $e^{-tL} \in \mathcal{O}\big(L^{p_0}(w) - L^{q_0}(w)\big)$, and (2.4) is obtained with a similar argument for the one just above to prove (2.1), by switching the role of B and $C_j(B)$ ($j \ge 2$), and using $e^{-zL} \in \mathcal{O}\big(L^{p_0}(w) - L^{p_0}(w)\big)$. $\qquad\square$

Remark 8.3. When $w = 1$, (8.2) holds for $p \in \text{Int}\, \mathcal{J}(L)$ and recall that this interval contains 2. Our proof contains two ways of seeing this. First, apply the extrapolation step and specialize to $u = 1$. Second, apply steps 2 and 3 with $w = 1$ and transition exponent 2 pushing to its right or to its left. Note that one could even reduce things to one of those two steps by using duality as, if we denote $\mathbb{T}$ by $\mathbb{T}_L$ then $\mathbb{T}^* = \mathbb{T}_{L^*}$. In [BK2], Step 3 and duality is used. However, duality does not seem to work when $w \ne 1$ on *all* of $\text{Int}\, \mathcal{J}_w(L)$.

9. COMMUTATORS WITH BOUNDED MEAN OSCILLATION FUNCTIONS

Let μ be a doubling measure in $\mathbb{R}^n$. Let $b \in \text{BMO}(\mu)$ (BMO is for bounded mean oscillation), that is,

$$\|b\|_{\text{BMO}(\mu)} = \sup_B \fint_B |b - b_B| d\mu = \sup_B \frac{1}{\mu(B)} \int_B |b(y) - b_B|\, d\mu < \infty$$

where the supremum is taken over balls and b_B stands for the μ-average of b on B. When $d\mu = dx$ we simply write BMO. If $w \in A_\infty$ (so dw is a doubling measure) then the reverse Hölder property yields that $\text{BMO}(w) = \text{BMO}$ with equivalent norms.

For T a sublinear operator, bounded in some $L^{p_0}(\mu)$, $1 \le p_0 \le \infty$, $b \in \text{BMO}$, $k \in \mathbb{N}$, we define the k-th order commutator

$$T_b^k f(x) = T\big((b(x) - b)^k\, f\big)(x), \qquad f \in L_c^\infty(\mu),\ x \in \mathbb{R}^n.$$

Note that $T_b^0 = T$. If T is linear they can be alternatively defined by recurrence: the first order commutator is

$$T_b^1 f(x) = [b, T]f(x) = b(x)\, Tf(x) - T(b\, f)(x)$$

and for $k \geq 2$, the k-th order commutator is given by $T_b^k = [b, T_b^{k-1}]$. As it is observed in [AM1], $T_b^k f(x)$ is well-defined almost everywhere when $f \in L_c^\infty(\mu)$ and it suffices to obtain boundedness with $b \in L^\infty$ with norm depending only on $\|b\|_{\mathrm{BMO}(\mu)}$. We state the results for commutators obtained in [AM1].

Theorem 9.1. *Let μ be a doubling measure on $\mathbb{R}^n$, $1 \leq p_0 < q_0 \leq \infty$ and $k \in \mathbb{N}$. Suppose that T is a sublinear operator bounded on $L^{p_0}(\mu)$, and let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $L_c^\infty(\mu)$ into $L^{p_0}(\mu)$. Assume that* (2.1) *and* (2.2) *hold with $S = I$. Let $p_0 < p < q_0$ and $w \in A_{\frac{p}{p_0}} \cap RH_{\left(\frac{q_0}{p}\right)'}$. If $\sum_j g(j)\, j^k < \infty$ then there is a constant C independent of f and $b \in \mathrm{BMO}(\mu)$ such that*

$$\|T_b^k f\|_{L^p(w)} \leq C\, \|b\|_{\mathrm{BMO}(\mu)}^k \|f\|_{L^p(w)}, \tag{9.1}$$

for all $f \in L_c^\infty(\mu)$.

Theorem 9.2. *Let $k \in \mathbb{N}$, μ be a doubling Borel measure on $\mathbb{R}^n$ with doubling order D and $1 \leq p_0 < q_0 \leq \infty$. Suppose that T is a sublinear operator and that T and T_b^m for $m = 1, \ldots, k$ are bounded on $L^{q_0}(\mu)$. Let $\{\mathcal{A}_r\}_{r>0}$ be a family of operators acting from $L_c^\infty(\mu)$ into $L^{q_0}(\mu)$. Assume that* (2.4) *and* (2.5) *hold. If $\sum_j g(j)\, 2^{Dj}\, j^k < \infty$, then for all $p_0 < p < q_0$, there exists a constant C (independent of b) such that for all $f \in L_c^\infty(\mu)$ and $b \in \mathrm{BMO}(\mu)$,*

$$\|T_b^k f\|_{L^p(\mu)} \leq C\, \|b\|_{\mathrm{BMO}(\mu)}^k \|f\|_{L^p(\mu)}.$$

With these results in hand, we have the following theorem.

Theorem 9.3. *Let $w \in A_\infty$, $k \in \mathbb{N}$ and $b \in \mathrm{BMO}$. Assume one of the following conditions:*

(a) $T = \varphi(L)$ *with φ bounded holomorphic on Σ_μ, $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$ and $p \in \mathrm{Int}\, \mathcal{J}_w(L)$.*

(b) $T = \nabla L^{-1/2}$, $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$ *and $p \in \mathrm{Int}\, \mathcal{K}_w(L)$.*

(c) $T = g_L$, $\mathcal{W}_w\big(p_-(L), p_+(L)\big) \neq \emptyset$ *and $p \in \mathrm{Int}\, \mathcal{J}_w(L)$.*

(d) $T = G_L$, $\mathcal{W}_w\big(q_-(L), q_+(L)\big) \neq \emptyset$ *and $p \in \mathrm{Int}\, \mathcal{K}_w(L)$.*

Then for $f \in L_c^\infty(\mathbb{R}^n)$,

$$\|T_b^k f\|_{L^p(w)} \leq C\, \|b\|_{\mathrm{BMO}}^k \|f\|_{L^p(w)},$$

where C does not depend on f, b, and is proportional to $\|\varphi\|_\infty$ in case (a).

Let us mention that, under kernel upper bounds assumptions, unweighted estimates for commutators in case (a) are obtained in [DY].

Proof of Theorem 9.3. Part (a). We fix $p \in \mathrm{Int}\, \mathcal{J}_w(L)$ and take $p_0, q_0 \in \mathrm{Int}\, \mathcal{J}_w(L)$ so that $p_0 < p < q_0$. We are going to apply Theorem 9.1 with $d\mu = dw$ and no weight to $T = \varphi(L)$ where φ satisfies (4.1).

First, as $p_0 \in \mathrm{Int}\, \mathcal{J}_w(L)$, Theorem 4.2 yields that $\varphi(L)$ is bounded on $L^{p_0}(w)$. Then, choosing $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ with $m \geq 1$ large enough, we proceed exactly as in the second case of the proof of Theorem 4.2. That is, we repeat the computations of the first case with dw replacing dx and using the corresponding weighted off-diagonal

estimates on balls. Applying (4.8) with dw in place of dx to $h = \varphi(L)$ we conclude (2.2). Besides, (4.9) and (4.10) with dw replacing dx lead us to (2.1) (with $S = I$). Therefore, Theorem 9.1 shows the boundedness of the commutators with BMO functions since $\|b\|_{\mathrm{BMO}}(w) \approx \|b\|_{\mathrm{BMO}}$ as noticed earlier.

It remains to remove the assumption on φ. This is done easily if one assumes that $b \in L^\infty$. Then the general case with $b \in \mathrm{BMO}$ follows as mentioned above. $\square$

Remark 9.4. The argument is the same as in the second case in Theorem 4.2 but for the whole range $\mathrm{Int}\,\mathcal{J}_w(L)$ (in place of working with $p \in (\widetilde{p}_-, \widehat{p}_+)$) since we already proved that $\varphi(L)$ is bounded in $\mathrm{Int}\,\mathcal{J}_w(L)$ by Theorem 4.2. That is, $T = \varphi(L)$ *a posteriori* satisfies (2.1) and (2.2) for $d\mu = dw$ and for *all* $p_0, q_0 \in \mathrm{Int}\,\mathcal{J}_w(L)$ with $p_0 < q_0$.

Proof of Theorem 9.3. Part (b). We write $T = \nabla L^{-1/2}$ and we already know that T is bounded on $L^p(w)$ for $p \in \mathrm{Int}\,\mathcal{K}_w(L)$ by Theorem 5.2.

First consider the case $p \in (\widetilde{q}_-, \widehat{q}_+)$. We take p_0, q_0 so that $\widetilde{q}_- < p_0 < \widetilde{q}_+$ and $p_0 < p < q_0 < \widehat{q}_+$. Let $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ where $m \geq 1$ is an integer to be chosen. As mentioned in the second case of the proof of Theorem 5.2, Lemma 5.3 holds with dw replacing dx. Thus, the hypotheses of Theorem 9.1 are fulfilled with $d\mu = dw$ and we can apply it with no weight.

Next we consider the case $p \in (\widehat{q}_-, \widetilde{q}_+)$. We take p_0, q_0 so that $\widetilde{q}_- < q_0 < \widetilde{q}_+$ and $\widehat{q}_- < p_0 < p < q_0$. Set $\mathcal{A}_r = I - (I - e^{-r^2 L})^m$ where $m \geq 1$ is an integer to be chosen. Notice that we have just proved that the operators T_b^l for $l = 0, \ldots, k$ are bounded on $L^{q_0}(w)$ as $q_0 \in (\widetilde{q}_-, \widehat{q}_+)$. We have already seen in the third case of the proof of Theorem 5.2

that T satisfies (2.4) and (2.5) with $d\mu = dw$. Choosing m large enough yields the needed condition for $g(j)$ to apply Theorem 9.2 with $d\mu = dw$. $\square$

Remark 9.5. In contrast with part (a), we do not know if Lemma 5.3 holds in the whole range $\mathrm{Int}\,\mathcal{K}_w(L)$ with dw replacing dx. Indeed, its proof relies on an $L^{p_0}(w)$-Poincaré inequality which is known only if $p_0 > r_w$. We get around this obstacle with Theorem 9.2 .

Proof of Theorem 9.3. Part (c). We proceed exactly as in part (a) using the arguments in Theorem 7.2, Part (a), in place of those in Theorem 4.2. Details are left to the reader. $\square$

Proof of Theorem 9.3. Part (d). We follow the same scheme as in part (b) using the arguments in Theorem 7.2, Part (b), in place of those in Theorem 5.2. Details are left to the reader. $\square$

Similar results can be proved for the multilinear commutators considered in [PT] (see also [AM1]) which are defined by replacing $(b(x) - b)^k$ in T_b^k by $\prod_{j=1}^k (b_j(x) - b_j)$ with $b_j \in \mathrm{BMO}$ for $1 \leq j \leq k$. Details are left to the reader.

10. REAL OPERATORS AND POWER WEIGHTS

Let us illustrate our results on Riesz transforms in a specific case and in particular discuss sharpness issues. Assume in this section that L has real coefficients. Then one knows that $q_-(L) = p_-(L) = 1$, $p_+(L) = \infty$.

If $n = 1$, one has also $q_+(L) = \infty$, so that we have obtained for all $1 < p < \infty$ and $w \in A_p$,

$$\|L^{1/2}f\|_{L^p(w)} \sim \|f'\|_{L^p(w)}.$$

For $p = 1$, there are two weak-type (1,1) estimates for A_1 weights. In fact, all this can be seen from [AT2] where it is shown that $L^{1/2} = R\frac{d}{dx}$ and $\frac{d}{dx} = M\tilde{R}L^{1/2}$ with R and $\tilde{R}$ being classical Calderón-Zygmund operators and M being the operator of pointwise multiplication by $1/a(x)$. Thus the usual weighted norm theory for Calderón-Zygmund operators applies.

Let us assume next that $n \geq 2$. In this case $q_+(L) > 2$. The next result will help us to study sharpness.

Proposition 10.1. *For each $q > 2$, there exists a real symmetric operator L on $\mathbb{R}^2$ for which $q_+(L) = q$.*

Proof. This is the example of Meyers-Kenig [AT1, p. 120]. Let $q > 2$ and set $\beta = -2/q \in (-1, 0)$. Consider the operator $L = -\operatorname{div} A\nabla$ obtained from $-\Delta$ by pulling back the associated quadratic form $\int \nabla u \cdot \nabla v$ by the quasi-conformal application $\varphi(x) = |x|^\beta x$, $x \in \mathbb{R}^2$. That is, A is obtained by writing out the change of variable in the relation

$$\int A(x)\nabla u(x) \cdot \nabla v(x)\, dx = \int \nabla(u \circ \varphi^{-1})(y) \cdot \nabla(v \circ \varphi^{-1})(y)\, dy,$$

with $u, v \in C_0^\infty(\mathbb{R}^n)$. It is easy to see that A is bounded and uniformly elliptic. Hence, u is a weak solution (in $W_{\mathrm{loc}}^{1,2}$) of L if and only if $u \circ \varphi^{-1}$ is a weak solution of $-\Delta$. In other words, weak solutions of L are harmonic functions composed with φ. Thus, the local L^p integrability of the gradient of such a solution is exactly that of $\nabla\varphi$. The latter is in L^p near 0 if and only if $p < -2/\beta$ and is bounded locally away from 0. Thus, for any weak solution u of L defined on a ball $2B$, $\nabla u \in L^p(B)$ for $p < -2/\beta$ and this is optimal if B is the unit ball. With this in hand, we can apply a result by Shen [She] which asserts that $q_+(L)$ is the supremum of those p for which all weak solutions of L defined on an arbitrary ball have ∇u in L^p locally inside that ball. In our case, $q_+(L) = -2/\beta = q$. $\square$

Remark 10.2. Let us also stress that if η is a smooth compactly supported function which is equal to 1 in a neighborhood of 0, then $v = \varphi\eta$ satisfies $|\nabla v(x)| \sim |x|^\beta$ near 0, whereas $|L^{1/2}v(x)| \leq c(1 + |x|)^{-1}$. See [AT1, p. 120], for this last fact.

Let us come back to a general situation and consider the power weights $w_\alpha(x) = |x|^\alpha$. Then, one has $p \in \mathcal{W}_{w_\alpha}\big(1, q_+(L)\big)$ if and only if

$$1 < p < q_+(L) \qquad \text{and} \qquad n\left(\frac{p}{q_+(L)} - 1\right) < \alpha < n(p-1). \tag{10.1}$$

For (p, α) tight with these relations Theorem 5.2 yields

$$\|\nabla f\|_{L^p(|x|^\alpha)} \lesssim \|L^{1/2}f\|_{L^p(|x|^\alpha)}.$$

In the latter inequality, we have in fact three parameters: $p \in (1, \infty)$, $\alpha \in (-n, \infty)$ (for $w_\alpha \in A_\infty$) and L in the family of real elliptic operators. One can study sharpness in various ways.

Fix L as in Proposition 10.1 with $n = 2$. The remark following this result implies that the L^p inequality can not hold for any (p, α) with $-2 < \alpha \leq 2\big(\frac{p}{q_+(L)} - 1\big)$ since

in this case, one can produce an $f(=v)$ where the left hand side is infinite and the right hand side finite.

If we fix $\alpha = 0$ and L, then the condition $1 < p < q_+(L)$ is necessary (and sufficient) to obtain the L^p estimate [Aus].

If we fix $p \in (1, \infty)$ and let L and α vary, then one can take $L = -\Delta$, in which case we are looking at the L^p power weight inequality for the usual Riesz transforms. In this case, it is known that this forces $w_\alpha \in A_p$, hence $\alpha < n(p-1)$.

Let us consider the reverse inequalities. For a given weight w, Theorem 6.2 says that the range of exponents for the L^p inequality contains $\mathcal{W}_w(1, \infty)$, which is the set of $p > 1$ for which $w \in A_p$. Hence, for w_α we have

$$\|L^{1/2}f\|_{L^p(|x|^\alpha)} \lesssim \|\nabla f\|_{L^p(|x|^\alpha)} \quad \text{if} \quad -n < \alpha < n(p-1).$$

This is the usual range for Calderón-Zygmund operators. This can also be seen from the fact proved in [AT1] that $L^{1/2} = T\nabla$ where T is a Calderón-Zygmund operator. Again for fixed $p \in (1, \infty)$, this range of α is best possible by taking $L = -\Delta$.

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