Perturbation of elliptic operators in 1-sided NTA domains satisfying the capacity density condition

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ABSTRACT. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided non-tangentially accessible domain (aka uniform domain), that is, a set which satisfies the interior Corkscrew and Harnack chain conditions, which are respectively scale-invariant/quantitative versions of openness and path-connectedness. Assume also that Ω satisfies the so-called capacity density condition, a quantitative version of the fact that all boundary points are Wiener regular. Let $L_0 u = -\operatorname{div}(A_0 \nabla u)$, $Lu = -\operatorname{div}(A \nabla u)$ be two real (non-necessarily symmetric) uniformly elliptic operators, and write ω_{L_0}, ω_L for the respective associated elliptic measures. The goal of this paper is to find sufficient conditions guaranteeing that ω_L satisfies an A_∞-condition or a RH_q-condition with respect to ω_{L_0}. We show that if the discrepancy of the two matrices satisfies a natural Carleson measure condition with respect to ω_{L_0} then $\omega_L \in A_\infty(\omega_{L_0})$. Moreover, we obtain that $\omega_L \in RH_q(\omega_{L_0})$ for any given $1 < q < \infty$ if the Carleson measure condition is assumed to hold with a sufficiently small constant. This "small constant" case extends previous work of Fefferman-Kenig-Pipher and Milakis-Pipher together with the last author of the present paper who considered symmetric operators in Lipschitz and bounded chord-arc domains respectively. Here we go beyond those settings, our domains satisfy a capacity density condition which is much weaker than the existence of exterior Corkscrew balls. Moreover, the boundaries of our domains need not to be Ahlfors regular and the restriction of the n-dimensional Hausdorff measure to the boundary could be even locally infinite. The "large constant" case, that is, the one on which we just assume that the discrepancy of the two matrices satisfies a Carleson measure condition, is new even in the case of nice domains (such as the unit ball, the upper-half space, or non-tangentially accessible domains) and in the case of symmetric operators. We emphasize that our results hold in the absence of a nice surface measure: all the analysis is done with the underlying measure ω_{L_0}, which behaves well in the settings we are considering. When particularized to the setting of Lipschitz, chord-arc, or 1-sided chord-arc domains, our methods allow us to immediately recover a number of existing perturbation results as well as extend some of them. Of independent interest, is the extension of the Dahlberg-Jerison-Kenig result concerning conical square function and non-tangential maximal functions to 1-sided non-tangentially accessible domains satisfying the capacity density condition.

Contents

CHAPTER 1

Introduction and Main results

The purpose of this article is to study some perturbation problems for second order divergence form real elliptic operators with bounded measurable coefficients in domains with rough boundaries. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and let $Lu = -\operatorname{div}(A\nabla u)$ be a second order divergence form real elliptic operator defined in Ω. Here the coefficient matrix $A = (a_{i,j}(\cdot))_{i,j=1}^{n+1}$ is real (non-necessarily symmetric) with $a_{i,j} \in L^\infty(\Omega)$ and is uniformly elliptic, that is, there exists a constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \eta| \leq \Lambda|\xi|\,|\eta| \tag{1.1}$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$. Associated with L one can construct a family of positive Borel measures $\{\omega_L^X\}_{X\in\Omega}$, defined on $\partial\Omega$ with $\omega^X(\partial\Omega) \leq 1$ for every $X \in \Omega$, so that for each $f \in C_c(\partial\Omega)$ one can define its associated weak-solution

$$u(X) = \int_{\partial\Omega} f(z)\,d\omega_L^X(z), \quad \text{whenever } X \in \Omega, \tag{1.2}$$

which satisfies $Lu = 0$ in Ω in the weak sense. In principle, unless we assume some further condition, u needs not be continuous all the way to the boundary but still we think of u as the solution to the continuous Dirichlet problem with boundary data f. We call ω_L^X the elliptic measure of Ω associated with the operator L with pole at $X \in \Omega$. For convenience, we will sometimes write ω_L and call it simply the elliptic measure, dropping the dependence on the pole.

Given two such operators $L_0 u = -\operatorname{div}(A_0\nabla u)$ and $Lu = -\operatorname{div}(A\nabla u)$, one may wonder whether one can find conditions on the matrices A_0 and A so that some "good estimates" for the Dirichlet problem or for the elliptic measure for L_0 might be transferred to the operator L. Similarly, one may try to see whether A being "close" to A_0 in some sense gives some relationship between ω_L and ω_{L_0}. In this direction, a celebrated result of Littman, Stampacchia, and Weinberger in [**LSW63**] states that the continuous Dirichlet problem for the Laplace operator $L_0 = \Delta$, (i.e., A_0 is the identity) is solvable if and only if it is solvable for any real elliptic operator L. By solvability here we mean that the elliptic measure solutions as in (1.2) are indeed continuous in $\overline{\Omega}$. It is well known that solvability in this sense is in fact equivalent to the fact that all boundary points are regular in the sense of Wiener, a condition which entails some capacitary thickness of the complement of Ω. Note that, for this result, one does not need to know that L is "close" to the Laplacian in any sense (other than the fact that both operators are uniformly elliptic).

On the other hand, if $\Omega = \mathbb{R}^2_+$ is the upper-half plane and $L_0 = \Delta$, then the harmonic measure associated with Δ is the Poisson-kernel which is mutually absolutely continuous with respect to the surface measure on the boundary. However, Caffarelli, Fabes, and Kenig in [CFK81] constructed a uniformly real elliptic operator L in the plane – the pullback of the Laplacian via a quasiconformal mapping of the upper half plane to itself – for which the associated elliptic measure ω_L is not even absolutely continuous with respect to the surface measure (see also [MM81] for another example). Hence, in principle the "good behavior" of harmonic measure does not always transfer to any elliptic measure even in a nice domain such as the upper-half plane. Consequently, it is natural to see if those good properties can be transferred by assuming some conditions reflecting the fact that L is "close" to L_0 or, in other words, imposing some conditions on the disagreement of A and A_0.

To describe positive results in this direction, with L_0 and L as above, we define the disagreement of A and A_0 as

$$\varrho(A, A_0)(X) := \sup_{Y \in B(X,\delta(X)/2)} |A(Y) - A_0(Y)|, \qquad X \in \Omega,$$

where $\delta(X) = \operatorname{dist}(X, \partial\Omega)$ (thus, the supremum is taken over a Whitney ball). Define, for every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$,

$$h(x,r) = \left(\frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r)\cap\Omega} \frac{\varrho(A, A_0)(X)^2}{\delta(X)} dX \right)^{\frac{1}{2}},$$

where $\sigma = \mathcal{H}^n|_{\partial\Omega}$ (i.e, the n-dimensional Hausdorff measure restricted to the boundary). The study of perturbation of elliptic operators was initiated by Fabes, Jerison, and Kenig in [FJK84] and later studied by Dahlberg [Dah86] for symmetric operators. Dahlberg in the case of $\Omega = B(0,1)$ observed that if

$$\lim_{r\to 0} \sup_{|x|=1} h(x,r) = 0$$

and if $\omega_{L_0} \ll \sigma$ with $d\omega_0/d\sigma \in RH_q(\sigma)$ (the classical reverse Hölder condition with respect to the surface measure) for some $1 < q < \infty$ then $\omega_L \ll \sigma$ and $d\omega_L/d\sigma \in RH_q(\sigma)$. The importance of these reverse Hölder conditions comes from the fact that $d\omega_L/d\sigma \in RH_q(\sigma)$ is equivalent to the $L^{q'}$-solvability of the Dirichlet problem, that is, the non tangential maximal function for the solution u given in (1.2) is controlled by f in the $L^{q'}(\sigma)$-norm. Dahlberg's approach was to define $A_t = (1-t)A_0 + tA$ for $0 \le t \le 1$, obtaining a differential inequality for the best constant in the reverse Hölder inequality for $d\omega_{L_t}/d\sigma$. Later, Fefferman in [Fef89] made the first attempt to remove the smallness of the function h. Working again in the domain $\Omega = B(0,1)$ and with symmetric operators, he showed than an $A_\infty(\sigma)$ condition is still inherited from the first measure (that is, $\omega_0 \in A_\infty(\sigma)$ implies $\omega_L \in A_\infty(\sigma)$) provided that $\mathcal{A}(\varrho(A, A_0)) \in L^\infty(\partial B(0,1))$ (and the bound needs not to be small). Here,

$$\mathcal{A}(\varrho(A, A_0))(x) := \left(\iint_{\Gamma(x)} \frac{\varrho(A, A_0)(X)^2}{\delta(X)^{n+1}} dX \right)^{\frac{1}{2}} \tag{1.3}$$

and $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$ with angular aperture $\theta < \pi/2$. Using Fubini's theorem one can easily see the connection between $h(x,r)$ and $\mathcal{A}(\varrho(A, A_0))(x)$:

$$h(x,r) \lesssim \left(\frac{1}{\sigma(B(x,Cr) \cap \partial\Omega)} \int_{B(x,Cr)\cap\partial\Omega} \mathcal{A}(\varrho(A, A_0))(x)^2 d\sigma \right)^{\frac{1}{2}}.$$

It was also noted in [**FKP91**] that finiteness of $\|\mathcal{A}(\varrho(A,A_0))\|_{L^\infty(\partial B(0,1))}$ does not allow one to preserve the reverse Hölder exponent. Indeed it was shown that for a given $1<p<\infty$, there exist uniformly elliptic symmetric matrices A_0 and A with the property that $\mathcal{A}(\varrho(A,A_0)) \in L^\infty(\partial B(0,1))$, $\omega_{L_0} \in RH_p(\sigma)$ but $\omega_L \notin RH_p(\sigma)$. On the other hand, one of the main results in the pioneering perturbation article by Fefferman, Kenig, and Pipher [**FKP91**] established that if the Carleson norm $\sup_{0<r<1,\ |x|=1} h(x,r)$ is merely assumed to be finite (not necessarily going to zero as $r \to 0$) then $\omega_{L_0} \in A_\infty(\sigma)$ implies $\omega_L \in A_\infty(\sigma)$ in the symmetric case. In the same article, it was shown that the assumption that the previous Carleson norm $\sup_{0<r<1,\ |x|=1} h(r,x)$ being finite is also necessary and cannot be weakened. One of the ingredients in [**FKP91**] was to see that if Ω is a Lipschitz domain and if

(1.4)
$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \left(\frac{1}{\omega_{L_0}(B(x,r)\cap\partial\Omega)}\iint_{B(x,r)\cap\Omega} \varrho(A,A_0)^2(X)\frac{G_{L_0}(X)}{\delta(X)^2}dX\right)^{\frac12} < \varepsilon_0$$

for ε_0 sufficiently small then $\omega_L \in RH_2(\omega_{L_0})$, where $G_{L_0}(X)=G_{L_0}(X_0,X)$ is the Green function for L_0 in Ω with a pole at some fixed $X_0\in\Omega$. We also remark that in [**FKP91**] the authors also considered L^r-averages of the disagreement function $\varrho(A,A_0)$ as opposed to the supremum. Using that approach it was shown that there exists r (depending on ellipticity) such that for each $q>1$ there exists ε_q so that $\omega_L \in RH_q(\omega_{L_0})$ provided that L^r-average of the disagreement function $\varrho(A,A_0)$ satisfies (1.4) with ε_q.

Milakis, Pipher, and the fourth author of this article in [**MPT13**] made the first attempt to study perturbation problems for symmetric operators beyond the Lipschitz setting. To describe their results we need more notions which will be described briefly here and made precise later. A domain is called non-tangentially accessible (NTA for short) if it satisfies quantitative interior and exterior openness as well as quantitative (interior) path-connectedness. A boundary of a domain is called Ahlfors regular if the surface measure of balls with center on the boundary and radius r behaves like r^n (in ambient dimension $n+1$). Note that NTA domains with Ahlfors regular boundaries (called chord-arc domains) are not necessarily Lipschitz domains and in general they cannot be locally represented as graphs. The first result of Fefferman, Kenig, and Pipher discussed above was generalized in [**MPT13**] to bounded chord-arc domains. That is, if Ω is a chord-arc domain and if (1.4) is satisfied for some $\varepsilon_0>0$ small then $\omega_L \in RH_2(\omega_{L_0})$ (see also [**MT10**]). Also, [**MPT13**] established that if $h(x,r)$ is small enough (uniformly in $x\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$) and $w_{L_0}\in RH_q(\sigma)$ for some $1<q<\infty$ then $w_{L_0}\in RH_q(\sigma)$. Also, assuming that $h(x,r)$ is merely bounded (uniformly in $x\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$) then $w_{L_0}\in RH_q(\sigma)$ for some $1<q<\infty$ implies that $w_L\in RH_p(\sigma)$ for some $1<p<\infty$. We also mention that Escauriaza in [**Esc96**] showed that if Ω is a Lipschitz domain and $h(x,r)$ is uniformly bounded for every $x\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$ then $\log(d\omega_L/d\sigma)\in \mathrm{VMO}(\sigma)$ if $\log(d\omega_{L_0}/d\sigma)\in\mathrm{VMO}(\sigma)$, here VMO stands for the space of vanish mean oscillation introduced by Sarason. This result was further generalized to bounded chord-arc domains in [**MPT14**].

In [**CHM19**], Cavero, the second and the third authors of this article studied the “small”and “large” perturbation for symmetric operators when the domain is 1-sided NTA domain with Ahlfors regular boundary (called 1-sided chord-arc domains). Here 1-sided NTA domains (aka uniform domains) satisfy only quantitative interior openness and path-connectedness. In [**CHM19**], the perturbation results of [**FKP91, MPT13**] were generalized to 1-sided chord-arc domains. Again, smallness of $h(x,r)$ allowed the authors to preserve the exponent in the reverse Hölder condition while finiteness yields only that the A_∞ condition is transferred from one operator to the other. It is relevant to mention that the approach in [**CHM19**], which is different from [**FKP91, MPT13**], uses the extrapolation of Carleson measure, originally introduced by Lewis and Murray in [**LM91**] (but based on the Corona construction of [**Car62, CG75**]) and later developed in [**HL01, HM14, HM12**], as well as good properties of sawtooth domains (following the sawtooth construction in [**DJK84**]). The bottom line is that the large perturbation case can be reduced to the small perturbation in some sawtooth subdomains. We would like to note that the arguments of [**FKP91, MPT13, CHM19**] are written explicitly only in the case of real symmetric coefficients, but we would expect that similar arguments could be carried over to the non-symmetric case as well. We also mention [**CHMT20**] where the non-symmetric case is also considered by using a different method, as well as [**MP20**], where perturbation theory for certain degenerate elliptic operators is developed in the setting of domains with lower dimensional boundaries.

One common feature in the previous perturbation results is that the surface measures of the boundary of the domains always have good properties, since in all cases the boundary is Ahlfors regular. For those results on which one is perturbing $RH_q(\sigma)$ or $A_\infty(\sigma)$, this is natural as one implicitly needs to make sense of σ and to that extent the Ahlfors regularity is natural. However, if one carefully looks at (1.4) and the conclusion derived from it, that is, $\omega_L \in RH_2(\omega_{L_0})$, there is no appearance of the surface measure, and these conditions make sense whether or not the surface measure is a well-behaved object. Another natural question that arises from (1.4) is whether one can target some other reverse Hölder conditions by allowing ε_0 to be larger, or ultimately to investigate what are the conclusions that can be obtained assuming that ε_0 is just an arbitrary large finite constant.

The goal of this monograph is to answer these questions. Our setting is that of 1-sided NTA domains satisfying the so called capacity density condition (CDC for short), see Chapter 2 for the precise definitions. The latter is a quantitative version of the well-known Wiener criterion and it is weaker than the Ahlfors regularity of the boundary or the existence of exterior Corkscrews. This setting guarantees among other things that any elliptic measure is doubling in some appropriate sense, hence one can see that a suitable portion of the boundary of the domain endowed with the Euclidean distance and with a given elliptic measure ω_{L_0} is a space of homogeneous type. In particular, classes like $A_\infty(\omega_{L_0})$ or $RH_p(\omega_{L_0})$ have the same good features of the corresponding ones in the Euclidean setting. However, our assumptions do not guarantee that the surface measure σ has any good behavior and could even be locally infinite. One of our main results considers both the case in which (1.4) holds with small or large ε_0. The small constant case can be seen as an extension of [**FKP91, MPT13**] to a setting on which surface measure is not a good object. Furthermore, the large constant case is new even in nice domains such as balls, upper-half spaces, Lipschitz domains or chord-arc domains. In this line, our main result is, to the best of our knowledge, the first one being able to establish perturbation results on sets with bad surface measures. Also, for the first time we are able to consider large perturbations in the sense of (1.4). Finally, we do not require the operators to be symmetric. Our main result is formulated as follows:

THEOREM 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain (cf. Definition 2.5) satisfying the capacity density condition (cf. Definition 2.10). Let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0\nabla u)$ be real (non-necessarily symmetric) elliptic operators. Define the disagreement between A and A_0 in Ω by*

$$\varrho(A, A_0)(X) := \|A - A_0\|_{L^\infty(B(X,\delta(X)/2))}, \qquad X \in \Omega, \tag{1.6}$$

where $\delta(X) := \operatorname{dist}(X, \partial\Omega)$, and

$$|||\varrho(A, A_0)||| := \sup_B \sup_{B'} \frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B'\cap\Omega} \varrho(A, A_0)(X)^2 \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2}\, dX, \tag{1.7}$$

where $\Delta = B \cap \partial\Omega$, $\Delta' = B' \cap \partial\Omega$, and the sups are taken respectively over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $B' = B(x', r')$ with $x' \in 2\Delta$ and $0 < r' < rc_0/4$, and c_0 is the Corkscrew constant.

(a) *If $|||\varrho(A, A_0)||| < \infty$, then $\omega_L \in A_\infty(\partial\Omega, \omega_{L_0})$ (cf. Definition 2.58). More precisely, there exists $1 < q < \infty$ such that $\omega_L \in RH_q(\partial\Omega, \omega_{L_0})$ (cf.Definition 2.58). Here, q and $[\omega_L]_{RH_q(\partial\Omega,\omega_0)}$ (cf. Definition 2.58) depend only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, and $|||\varrho(A, A_0)|||$.*

(b) *Given $1 < p < \infty$, there exists $\varepsilon_p > 0$ (depending only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, and p) such that if one has $|||\varrho(A, A_0)||| \leq \varepsilon_p$, then $\omega_L \in RH_p(\partial\Omega, \omega_{L_0})$ (cf. Definition 2.58). Here, $[\omega_L]_{RH_p(\partial\Omega,\omega_0)}$ (cf. Definition 2.58) depends only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, and p.*

REMARK 1.8. Let us make a few remarks regarding the expression in (1.6). First, the collection of B' in the second sup is chosen so that $X_\Delta \notin 4B'$, hence the Green function is not singular in the domain of integration. But even if the domain of integration contained X_Δ this would not cause any problem, since the corresponding estimate near X_Δ becomes easy by invoking Lemma 2.59 below:

$$\begin{aligned}
&\frac{1}{\omega_{L_0}^{X_\Delta}(\Delta')} \iint_{B(X_\Delta,\delta(X_\Delta)/2)} \varrho(A, A_0)(X)^2 \frac{G_{L_0}(X_\Delta, X)}{\delta(X)^2}\, dX \\
&\qquad\lesssim \left(\|A - A_0\|_{L^\infty(B(X_\Delta,\delta(X_\Delta)/2))}\right)^2 \frac{1}{\delta(X_\Delta)^2} \iint_{B(X_\Delta,\delta(X_\Delta)/2)\cap\Omega} |X - X_\Delta|^{1-n}\, dX \\
&\qquad\lesssim \left(\|A - A_0\|_{L^\infty(B(X_\Delta,\delta(X_\Delta)/2))}\right)^2.
\end{aligned}$$

Second, at a first glance (1.6) seems different than (1.4), the condition imposed by Fefferman, Kenig, and Pipher in [**FKP91**], which in the current case and if Ω is **bounded** (avoiding the pole as just mentioned) would read as

$$|||\varrho(A, A_0)|||_* := \sup_{B'} \frac{1}{\omega^{X_\Omega}(\Delta')} \iint_{B'\cap\Omega} \varrho(A, A_0)(X)^2 \frac{G_{L_0}(X_\Omega, X)}{\delta(X)^2}\, dX, \tag{1.9}$$

where $X_\Omega \in \Omega$ is a "center" of Ω (say, X_Ω is the Corkscrew point associated with the surface ball $\Delta(x_0, \operatorname{diam}(\partial\Omega)/2)$ for some fixed $x_0 \in \Omega$) so that $\delta(X_\Omega) \approx$

$\operatorname{diam}(\partial\Omega)$; $\Delta' = B' \cap \partial\Omega$ and the sup is taken over all balls $B' = B(x', r')$ with $x' \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)c_0/4$. We can easily see that $|||\varrho(A, A_0)||| \approx |||\varrho(A, A_0)|||_*$. First, using Lemma 2.69 below and possibly Harnack's inequality, one can see that for $B = B(x, r)$ and $B' = B(x', r')$ as in (1.7) if $X \in B' \cap \Omega$ then $\frac{G_{L_0}(X_\Delta, X)}{\omega_{L_0}^{X_\Delta}(\Delta')} \approx \frac{G_{L_0}(X_\Omega, X)}{\omega_{L_0}^{X_\Omega}(\Delta')}$. Thus, $|||\varrho(A, A_0)||| \lesssim |||\varrho(A, A_0)|||_*$. To obtain the converse inequality, let $B' = B(x', r')$ with $x' \in \partial\Omega$ and $0 < r' < \operatorname{diam}(\partial\Omega)c_0/4$. Pick $\max\{\frac{1}{2}, 4\,r'/(\operatorname{diam}(\partial\Omega)c_0)\} < \theta < 1$ and write $r = \theta \operatorname{diam}(\partial\Omega)$ so that $\operatorname{diam}(\partial\Omega)/2 < r < \operatorname{diam}(\partial\Omega)$ and $r' < rc_0/4$. Set $B = B(x', r)$ and note that the Harnack chain condition and Harnack's inequality easily yield $\omega^{X_\Omega}(\Delta') \approx \omega^{X_\Delta}(\Delta')$, and also $G_{L_0}(X_\Omega, X) \approx G_{L_0}(X_\Delta, X)$ for every $X \in B' \cap \Omega$, where $\Delta = B \cap \partial\Omega$ and $\Delta' = B' \cap \partial\Omega$. All these give at once that $|||\varrho(A, A_0)|||_* \lesssim |||\varrho(A, A_0)|||$. Hence, $|||\varrho(A, A_0)||| \approx |||\varrho(A, A_0)|||_*$ when Ω is **bounded**.

In the **unbounded** case, one could use a similar argument working with a pole at infinity, which would require to normalize appropriately ω_{L_0} and G_{L_0}; here we will simply work with the scale-invariant expression (1.7) to avoid that issue.

Finally, we also have a generalization of a result [**Fef89, FKP91, MPT13**]:

THEOREM 1.10. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain (cf. Definition 2.5) satisfying the capacity density condition (cf. Definition 2.10), and let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be real (non-necessarily symmetric) elliptic operators. Given $\alpha > 0$, set*

$$\mathcal{A}_\alpha(\varrho(A, A_0))(x) := \left(\iint_{\Gamma_\alpha(x)} \frac{\varrho(A, A_0)(X)^2}{\delta(X)^{n+1}} dX \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega, \tag{1.11}$$

where $\Gamma_\alpha(x) = \{Y \in \Omega : |Y - x| < (1+\alpha)\delta(Y)\}$.

(a) *If $\mathcal{A}_\alpha(\varrho(A, A_0)) \in L^\infty(\omega_{L_0})$, then $\omega_L \in A_\infty(\partial\Omega, \omega_{L_0})$ (cf. Definition 2.58). More precisely, there exists $1 < q < \infty$ such that $\omega_L \in RH_q(\partial\Omega, \omega_{L_0})$ (cf. Definition 2.58). Here, q and $[\omega_L]_{RH_q(\partial\Omega,\omega_0)}$ (cf. Definition 2.58) depend only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, α, and $\|\mathcal{A}_\alpha(\varrho(A, A_0))\|_{L^\infty(\omega_{L_0})}$.*

(b) *Given p, $1 < p < \infty$, there exists $\varepsilon_p > 0$ (depending only on p, dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, and α) such that if $\mathcal{A}_\alpha(\varrho(A, A_0)) \in L^\infty(\omega_{L_0})$ with $\|\mathcal{A}_\alpha(\varrho(A, A_0))\|_{L^\infty(\omega_{L_0})} \leq \varepsilon_p$, then $\omega_L \in RH_p(\partial\Omega, \omega_{L_0})$ (cf. Definition 2.58). Here $[\omega_L]_{RH_p(\partial\Omega,\omega_0)}$ (cf. Definition 2.58) depends only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, α, and p.*

REMARK 1.12. Note that in the previous result we are not specifying the pole for the elliptic measure ω_{L_0}. However there is no ambiguity since, as a matter of fact, for any given X, $Y \in \Omega$ one has that $\omega_{L_0}^X$ and $\omega_{L_0}^Y$ are mutually absolutely continuous, thus $L^\infty(\partial\Omega, \omega_{L_0}^X) = L^\infty(\partial\Omega, \omega_{L_0}^Y)$ with $\|\cdot\|_{L^\infty(\partial\Omega,\omega_{L_0}^X)} = \|\cdot\|_{L^\infty(\partial\Omega,\omega_{L_0}^Y)}$.

The plan of this paper is as follows. Next chapter contains some of the preliminaries, definitions, and tools which will be used throughout the paper. Chapter 3 is devoted to proving our main results. As a matter of fact Theorem 1.5 follows from a local version, interesting on its own right, which is valid on bounded domains, see Proposition 3.1. The proof of Theorem 1.10 is also in Chapter 3.

The proof of Proposition 3.1 is in Sections 3.2 and 3.3 which respectively handle the large and small constant cases. The proof of the large constant case is based on the extrapolation of Carleson measure technique mentioned above. Chapter 4 contains some dyadic version of the main lemma of [**DJK84**] which is needed in our arguments. In Chapter 5 we present a couple of results which are interesting on its own right. In Theorem 5.1 we show that bounded weak-solutions satisfy Carleson measure estimates adapted to the elliptic measure and in Theorem 5.3 we establish that the conical square function can be controlled by the non-tangential maximal function in norm with respect to the elliptic measure, extending [**DJK84**] to our general setting. Finally, in Appendix A we apply our main results to consider the case of 1-sided CAD (cf. Definition 2.9) —hence the domain is 1-sided NTA and satisfies the CDC condition— and show in Corollaries A.2 and A.5 that one can immediately recover some results from [**CHM19**, **CHMT20**] (see also [**Dah86**, **Fef89**, **FKP91**, **MPT13**]) as well as give new extensions.

CHAPTER 2

Preliminaries

2.1. Notation and conventions

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. Unless otherwise specified upper case constants are greater than 1 and lower case constants are smaller than 1. In some occasions it is important to keep track of the dependence on a given parameter γ, in that case we write $a \lesssim_\gamma b$ or $a \approx_\gamma b$ to emphasize that the implicit constants in the inequalities depend on γ.
- Our ambient space is $\mathbb{R}^{n+1}$, $n \geq 2$.
- Given $E \subset \mathbb{R}^{n+1}$ we write $\operatorname{diam}(E) = \sup_{x,y \in E} |x - y|$ to denote its diameter.
- Given a domain $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus \partial\Omega$).
- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$. A *surface ball* is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and unless otherwise specified it is implicitly assumed that $x \in \partial\Omega$.
- If $\partial\Omega$ is bounded, it is always understood (unless otherwise specified) that all surface balls have radii controlled by the diameter of $\partial\Omega$, that is, if $\Delta = \Delta(x, r)$ then $r \lesssim \operatorname{diam}(\partial\Omega)$. Note that in this way $\Delta = \partial\Omega$ if $\operatorname{diam}(\partial\Omega) < r \lesssim \operatorname{diam}(\partial\Omega)$.
- For $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.
- We let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\mathbf{1}_A$ denote the usual indicator function of A, i.e. $\mathbf{1}_A(X) = 1$ if $X \in A$, and $\mathbf{1}_A(X) = 0$ if $X \notin A$.
- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean cube with sides parallel to the coordinate axes, and we let $\ell(I)$ denote the side length of I. We use Q to denote dyadic "cubes" on E or $\partial\Omega$. The latter exist as a consequence of Lemma 2.33 below.

2.2. Some definitions

Definition 2.1 (**Corkscrew condition**). Following [**JK82**], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Corkscrew condition* if for some uniform constant $0 < c_0 < 1$ and for every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, if we write $\Delta := \Delta(x,r)$, there is a ball $B(X_\Delta, c_0 r) \subset B(x,r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *Corkscrew point relative to* Δ (or, relative to B). We note that we may allow $r < C \operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c_0.

Definition 2.2 (**Harnack Chain condition**). Again following [**JK82**], we say that Ω satisfies the *Harnack Chain condition* if there are uniform constants $C_1, C_2 > 1$ such that for every pair of points $X, X' \in \Omega$ there is a chain of balls $B_1, B_2, \dots, B_N \subset \Omega$ with $N \le C_1(2 + \log_2^+ \Pi)$ where

$$\Pi := \frac{|X - X'|}{\min\{\delta(X), \delta(X')\}}. \tag{2.3}$$

such that $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and for every $1 \le k \le N$

$$C_2^{-1} \operatorname{diam}(B_k) \le \operatorname{dist}(B_k, \partial\Omega) \le C_2 \operatorname{diam}(B_k). \tag{2.4}$$

The chain of balls is called a *Harnack Chain*.

We note that in the context of the previous definition if $\Pi \le 1$ we can trivially form the Harnack chain $B_1 = B(X, 3\delta(X)/5)$ and $B_2 = B(X', 3\delta(X')/5)$ where (2.4) holds with $C_2 = 3$. Hence the Harnack chain condition is non-trivial only when $\Pi > 1$.

Definition 2.5 (**1-sided NTA and NTA**). We say that a domain Ω is a *1-sided non-tangentially accessible domain* (1-sided NTA) if it satisfies both the Corkscrew and Harnack Chain conditions. Furthermore, we say that Ω is a *non-tangentially accessible domain* (NTA domain) if it is a 1-sided NTA domain and if, in addition, $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ also satisfies the Corkscrew condition.

Remark 2.6. In the literature, 1-sided NTA domains are also called *uniform domains*. We remark that the 1-sided NTA condition is a quantitative form of path connectedness.

Definition 2.7 (**Ahlfors regular**). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is *n-dimensional Ahlfors regular* (AR for short) if there is some uniform constant $C_1 > 1$ such that

$$C_1^{-1} r^n \le \mathcal{H}^n(E \cap B(x,r)) \le C\, r^n, \qquad x \in E, \quad 0 < r < \operatorname{diam}(E). \tag{2.8}$$

Definition 2.9 (**1-sided CAD and CAD**). A *1-sided chord-arc domain* (1-sided CAD) is a 1-sided NTA domain with AR boundary. A *chord-arc domain* (CAD) is an NTA domain with AR boundary.

We next recall the definition of the capacity of a set. Given an open set $D \subset \mathbb{R}^{n+1}$ (where we recall that we always assume that $n \ge 2$) and a compact set $K \subset D$ we define the capacity of K relative to D as

$$\operatorname{Cap}_2(K, D) = \inf\left\{ \iint_D |\nabla v(X)|^2 dX : \; v \in C_0^\infty(D),\, v(x) \ge 1 \text{ in } K \right\}.$$

DEFINITION 2.10 (**Capacity density condition**). An open set Ω is said to satisfy *capacity density condition* (CDC for short) if there exists a uniform constant $c_1 > 0$ such that

$$\frac{\operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega, B(x,2r))}{\operatorname{Cap}_2(\overline{B(x,r)}, B(x,2r))} \geq c_1 \tag{2.11}$$

for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$.

The CDC is also known as the uniform 2-fatness as studied by Lewis in [**Lew88**]. Using [**HKM06**, Example 2.12] one has that

$$\operatorname{Cap}_2(\overline{B(x,r)}, B(x,2r)) \approx r^{n-1}, \qquad \text{for all } x \in \mathbb{R}^{n+1} \text{ and } r > 0, \tag{2.12}$$

and hence the CDC is a quantitative version of the Wiener regularity, in particular every $x \in \partial\Omega$ is Wiener regular. It is easy to see that the exterior Corkscrew condition implies CDC. Also, it was proved in [**Zha18**, Section 3] and [**HLMN17**, Lemma 3.27] that a set with Ahlfors regular boundary satisfies the capacity density condition with constant c_1 depending only on n and the Ahlfors regular constant.

2.3. Dyadic analysis

Throughout this section we will work with $E \subset \mathbb{R}^{n+1}$ and a countable collection of Borel sets $\mathbb{D} = \{Q\}_{Q \in \mathbb{D}}$ which is a dyadic grid on E, whose elements will be called "cubes". This means that $\mathbb{D} = \bigcup_{k \in \mathbb{Z}} \mathbb{D}_k$ (with $\mathbb{D}_k \neq \emptyset$ for each $k \in \mathbb{Z}$) and the following properties hold:

- $E = \bigcup_{Q \in \mathbb{D}_k} Q$ for every $k \in \mathbb{Z}$ with the union comprising pairwise disjoint sets.
- If $Q \in \mathbb{D}_k$ and $Q' \in \mathbb{D}_j$ with $k \geq j$ then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$.
- If for every $k > j$ and $Q \in \mathbb{D}_k$ there exists (a unique) $Q' \in \mathbb{D}_j$ such that $Q \subset Q'$.

See Section 2.4 below (and the references [**Chr90**], and [**HK12, HK13**]) for a discussion of the existence of such a dyadic system, as well as its additional properties.

Note that by assumption, within the same generation (that is, within each $\mathbb{D}_k$) the cubes are pairwise disjoint (hence, there are no repetitions). On the other hand, we allow repetitions in the different generations, that is, we could have that $Q \in \mathbb{D}_k$ and $Q' \in \mathbb{D}_{k-1}$ agree. Then, although Q and Q' are the same set, as cubes we understand that they are different. In short, it is then understood that $\mathbb{D}$ is an indexed collection of sets where repetitions of sets are allowed in the different generations but not within the same generation. With this in mind, we can give a proper definition of the "length" of a cube (this concept has no geometric meaning in this context). For every $Q \in \mathbb{D}_k$, we set $\ell(Q) = 2^{-k}$, which is called the "length" of Q. Note that the "length" is well defined when considered on $\mathbb{D}$, but it is not well-defined on the family of sets induced by $\mathbb{D}$. It is important to observe that the "length" refers to the way the cubes are organized in the dyadic grid and in general may not have a geometrical meaning (see the examples below).

REMARK 2.13. We would like to observe that in our notion of dyadic grid the generations run for all $k \in \mathbb{Z}$. However, as we are about to see, sometimes it is natural to truncate the generations (from above or from below). For instance, it could be that $E = Q_0$ for some $Q_0 \in \mathbb{D}_{k_0}$ and $k_0 \in \mathbb{Z}$, hence $\mathbb{D}_k = \{Q_0\}$ for all $k \leq k_0$. In that scenario it is convenient to ignore those $k \in \mathbb{Z}$ with $k < k_0$ and work with $\mathbb{D} = \bigcup_{k \geq k_0} \mathbb{D}_k$. We will actually use this convention throughout this paper and, more specifically, when E is bounded we will be working with the generations $k \in \mathbb{Z}$ so that $2^{-k} \lesssim \operatorname{diam}(E)$. In any case, the results and proofs in this section remain valid with or without the truncation of generations.

It is interesting to introduce some examples. In $\mathbb{R}^n$ we can consider the collection of classical dyadic cubes. Note that here there are no repetitions at all, $E = \mathbb{R}^n$, and that if we let $\mathbb{D}_k$ be the collection of those dyadic cubes with side length 2^{-k}, then the "length" is indeed the side length. Analogously, with $E = \mathbb{R}^n$ we can let $\mathbb{D}_{2k}$ be the collection of those dyadic cubes with side length 2^{-k} and $\mathbb{D}_{2k+1} = \mathbb{D}_{2k}$. Hence there are repetitions of cubes in $\mathbb{D}$ and "length" is comparable to the square root of the side length.

Another example is the collection of dyadic subcubes of the unit cube $Q_0 = [0,1)^n$. To frame this in the previous definition (without truncating the generations), we let $\mathbb{D}_k$ be the collection of dyadic subcubes of Q_0 if $k \geq 0$ and $\mathbb{D}_k = \{Q_0\}$ for $k \leq 0$. In this scenario $E = Q_0$ and we have that all the dyadic ancestors of Q_0 are indeed Q_0, hence there are repetitions in $\mathbb{D}$. Observe that the "length" agrees with the side length in $\mathbb{D}_k$ for $k \geq 0$. On the other hand, for $Q_k \in \mathbb{D}_k$ with $k \leq 0$ we have that $\ell(Q_k) = 2^{-k}$ (note that Q_k and Q_0 are the same set but as dyadic cubes they are distinct). In this case, it may be convenient and more natural to truncate the generations and just work with $\mathbb{D}_k$, $k \geq 0$, in which case the "length" agrees with the side length.

We can also consider all classical dyadic cubes with side length at least 1. In this scenario, let $\mathbb{D}_k$ be the set of classical dyadic cubes with side length 2^{-k} for $k \leq 0$, and $\mathbb{D}_k$ the collection of classical dyadic cubes with side length 1 for $k \geq 0$. In this scenario, $E = \mathbb{R}^n$ and he have that all the dyadic descendants of any cube Q with length side equal 1 are indeed Q, hence there are repetitions in $\mathbb{D}$. Note that "length" agrees with the side length in $\mathbb{D}_k$ for $k \leq 0$, however in $\mathbb{D}_k$ for $k \geq 0$ the "length" is 2^{-k} although the cubes comprising that family have side length 1. Again, in this example, it may be more natural to truncate the generations and work with $\mathbb{D}_k$, $k \leq 0$, so that "length" and side length agree.

Our last example is that of dyadic subcubes of the unit cube $Q_0 = [0,1)^n$ with side length at least 2^{-N} with $N \in \mathbb{N}$ fixed. We let $\mathbb{D}_k$ be the collection of dyadic subcubes of Q_0 if $0 \leq k \leq N$, $\mathbb{D}_k = \{Q_0\}$ for $k \leq 0$, and $\mathbb{D}_k$, $k \geq N$, is the collection of all dyadic subcubes of Q_0 of side length 2^{-N}. In this case, $E = Q_0$, all the dyadic ancestors of Q_0 are indeed Q_0, and all the dyadic descendants of any cube Q with length side equal 2^{-N} are indeed Q. We have infinitely many cubes but only a finite number of different sets. Here the reasonable thing is to truncate the generations and just work with $\mathbb{D}_k$, $0 \leq k \leq N$.

We next introduce the "discretized Carleson region" relative to Q, $\mathbb{D}_Q = \{Q' \in \mathbb{D} : Q' \subset Q\}$. Let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}$ be a family of pairwise disjoint cubes. The "global discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes $Q \in \mathbb{D}$ that are not contained in any $Q_i \in \mathcal{F}$, that is,

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D} \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i}.$$

For a given $Q \in \mathbb{D}$, the "local discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes in $\mathbb{D}_Q$ that are not contained in any $Q_i \in \mathcal{F}$ or, equivalently,

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q.$$

We also allow $\mathcal{F}$ to be the null set in which case $\mathbb{D}_{\emptyset} = \mathbb{D}$ and $\mathbb{D}_{\emptyset,Q} = \mathbb{D}_Q$.

With a slight abuse of notation, let Q^0 be either E, and in that case $\mathbb{D}_{Q^0} := \mathbb{D}$, or a fixed cube in $\mathbb{D}$, hence $\mathbb{D}_{Q^0}$ is the family of dyadic subcubes of Q^0. Let μ be a non-negative Borel measure on Q^0 so that $0 < \mu(Q) < \infty$ for every $Q \in \mathbb{D}_{Q^0}$. Consider the operators $\mathcal{A}_{Q^0}$, $\mathcal{B}_{Q^0}$ defined by
(2.14)

$$\mathcal{A}^{\mu}_{Q^0}\alpha(x) := \Bigg(\sum_{x \in Q \in \mathbb{D}_{Q^0}} \frac{1}{\mu(Q)}\,\alpha_Q^2\Bigg)^{\frac12}, \quad \mathcal{B}^{\mu}_{Q^0}\alpha(x) := \sup_{x \in Q \in \mathbb{D}_{Q^0}} \Bigg(\frac{1}{\mu(Q)} \sum_{Q' \in \mathbb{D}_Q} \alpha_{Q'}^2\Bigg)^{\frac12},$$

where $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q^0}}$ is a sequence of real numbers. Note that these operators are discrete analogues of those used in [**CMS85**] to develop the theory of tent spaces. Sometimes, we use a truncated version of $\mathcal{A}^{\mu}_{Q^0}$, defined for each $k \geq 0$ by

$$\mathcal{A}^{\mu,k}_{Q^0}\alpha(x) := \Bigg(\sum_{x \in Q \in \mathbb{D}^k_{Q^0}} \frac{1}{\mu(Q)}\,\alpha_Q^2\Bigg)^{\frac12},$$

where $\mathbb{D}^k_Q := \{Q' \in \mathbb{D}_Q : \ell(Q') \leq 2^{-k}\ell(Q)\}$.

The following lemma is a discrete version of [**CMS85**, Theorem 1] and extends [**CHM19**, Lemma 3.8]:

Lemma 2.15. *Under the previous considerations, given Q^0 as above, and $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q^0}}$, $\beta = \{\beta_Q\}_{Q \in \mathbb{D}_{Q^0}}$ sequences of real numbers, we have that*

$$\sum_{Q \in \mathbb{D}_{Q^0}} |\alpha_Q \beta_Q| \leq 4 \int_{Q^0} \mathcal{A}^{\mu}_{Q^0}\alpha(x)\mathcal{B}^{\mu}_{Q^0}\beta(x)\,d\mu(x). \tag{2.16}$$

Proof. The proof follows the argument in [**CHM19**, Lemma 3.8] which in turn is based on [**CMS85**, Theorem 1]. We first claim that it suffices to assume that $Q^0 \in \mathbb{D}$. Indeed, if $Q^0 = E$ we have

$$\begin{aligned}\sum_{Q \in \mathbb{D}_{Q^0}} |\alpha_Q\beta_Q| &= \sum_{Q \in \mathbb{D}} |\alpha_Q\beta_Q| = \sup_N \sum_{Q \in \mathbb{D}_{-N}} \sum_{Q' \in \mathbb{D}_Q} |\alpha_{Q'}\beta_{Q'}| \\ &\leq 4 \sup_N \sum_{Q \in \mathbb{D}_{-N}} \int_Q \mathcal{A}^{\mu}_{Q}\alpha(x)\mathcal{B}^{\mu}_{Q}\beta(x)\,d\mu(x) \leq 4\int_E \mathcal{A}^{\mu}_{Q^0}\alpha(x)\mathcal{B}^{\mu}_{Q^0}\beta(x)\,d\mu(x),\end{aligned}$$

where in the first estimate we have used our claim for Q, which has finite length, and in the second one the fact that the cubes in $\mathbb{D}_{-N}$ are pairwise disjoint.

From now on we assume $Q^0 \in \mathbb{D}$, hence $\ell(Q^0) < \infty$. Recall $\mathbb{D}$ that is countable collection of cubes and then we can find $\mathbb{D}^1 \subset \mathbb{D}^2 \subset \cdots \subset \mathbb{D}^N \subset \cdots \subset \mathbb{D}$ with $\mathbb{D} = \bigcup_{N \geq 1} \mathbb{D}^N$ and $\#\mathbb{D}^N \leq N$. Given $N \geq 1$, let $\beta^N = \{\beta^N_Q\}_{Q \in \mathbb{D}_{Q^0}}$ where $\beta^N_Q = \beta_Q$ if $Q \in \mathbb{D}^N$ and $\beta^N_Q = 0$ otherwise. With this notation in mind, if we show (2.16) for β^N then observing that $\mathcal{B}^{\mu}_{Q^0}\beta^N \leq \mathcal{B}^{\mu}_{Q^0}\beta$ we just need to let $N \to \infty$ and the desired estimate follows at once.

Thus from now on we work with β^N. To simplify the presentation we drop the exponent and keep in mind that $\beta_Q = 0$ for every $Q \notin \mathbb{D}^N$. For $Q \in \mathbb{D}_{Q^0}$, let $k_Q \geq 0$ be so that $\ell(Q) = 2^{-k_Q}\ell(Q^0)$. Suppose that $Q' \in \mathbb{D}_{Q^0}$ satisfies $\ell(Q') \leq 2^{-k_Q}\ell(Q^0) = \ell(Q)$ and $Q' \cap Q \neq \emptyset$, then necessarily $Q' \in \mathbb{D}_Q$ and for every $x \in Q$

$$\xi_Q := \fint_Q \big(\mathcal{A}^{\mu,k_Q}_{Q^0}\beta(y)\big)^2\,d\mu(y) = \fint_Q \sum_{Q'\in\mathbb{D}_Q} \mathbf{1}_{Q'}(y)\frac{1}{\mu(Q')}\,\beta^2_{Q'}\,d\mu(y) = \frac{1}{\mu(Q)}\sum_{Q'\in\mathbb{D}_Q}\beta^2_{Q'} \leq \big(\mathcal{B}^{\mu}_{Q^0}\beta(x)\big)^2. \tag{2.17}$$

Since $\beta_Q = 0$ for $Q \notin \mathbb{D}^N$ and $\#\mathbb{D}^N \leq N$, we have $\mathcal{A}^{\mu}_{Q^0}\beta(x) \leq C_N < \infty$ for every $x \in Q^0$ and hence $\xi_Q \leq C_N^2 < \infty$. Now, define

$$F_0 := \big\{x \in Q^0 :\ \mathcal{A}^{\mu,k}_{Q^0}\beta(x) > 2\,\mathcal{B}^{\mu}_{Q^0}\beta(x),\ \forall k \geq 0\big\}.$$

In particular, using (2.17), we have $\mathcal{A}^{\mu,k_Q}_{Q^0}\beta(x) > 2\,\xi_Q^{\frac12}$ for each $x \in Q\cap F_0$. We claim that $4\mu(Q\cap F_0) \leq \mu(Q)$. Indeed, if $\xi_Q = 0$ then one can see that $\mathcal{A}^{\mu,k_Q}_{Q^0}\beta(y) = 0$ for every $y \in Q$ and hence $Q \cap F_0 = \emptyset$, which trivially gives that $4\mu(Q \cap F_0) \leq \mu(Q)$. On the other hand, if $\xi_Q > 0$, we have

$$4\,\xi_Q\,\mu(Q\cap F_0) \leq \int_{Q\cap F_0}\big(\mathcal{A}^{\mu,k_Q}_{Q^0}\beta(y)\big)^2\,d\mu(y) \leq \xi_Q\,\mu(Q),$$

and the desired estimate follows since $0 < \xi_Q < \infty$. Let us now consider

$$k(x) := \min\big\{k \geq 0 :\ \mathcal{A}^{\mu,k}_{Q^0}\beta(x) \leq 2\,\mathcal{B}^{\mu}_{Q^0}\beta(x)\big\}, \qquad x \in Q^0 \setminus F_0. \tag{2.18}$$

Setting $F_{1,Q} := \{x \in Q\setminus F_0 :\ k(x) > k_Q\}$ and using (2.17) we obtain

$$F_{1,Q} \subset \{x \in Q \setminus F_0 :\ \mathcal{A}^{\mu,k_Q}_{Q^0}\beta(x) > 2\,\xi_Q^{\frac12}\}.$$

Applying Chebyshev's inequality, it follows that

$$\mu(F_{1,Q}) \leq \frac{1}{4\,\xi_Q}\int_{Q\setminus F_0}\big(\mathcal{A}^{\mu,k_Q}_{Q^0}\beta(y)\big)^2\,d\mu(y) \leq \frac14\mu(Q).$$

Setting $F_{2,Q} := \{x \in Q\setminus F_0 :\ k(x) \leq k_Q\}$, and gathering the above estimates, we have

$$\mu(F_{2,Q}) = \mu(Q) - \mu(Q\cap F_0) - \mu(F_{1,Q}) \geq \frac12\mu(Q).$$

Hence, Cauchy-Schwarz's inequality and (2.18) yield

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{Q^0}}|\alpha_Q\beta_Q| &\leq 2\sum_{Q\in\mathbb{D}_{Q^0}}\mu(F_{2,Q})\frac{|\alpha_Q\beta_Q|}{\mu(Q)} \leq 2\int_{Q^0\setminus F_0}\sum_{Q\in\mathbb{D}_{Q^0}}\frac{|\alpha_Q\beta_Q|}{\mu(Q)}\mathbf{1}_{F_{2,Q}}(x)\,d\mu(x)\\
&\leq 2\int_{Q^0\setminus F_0}\mathcal{A}^{\mu}_{Q^0}\alpha(x)\Big(\sum_{Q\in\mathbb{D}_{Q^0}}\frac{1}{\mu(Q)}\,\beta_Q^2\mathbf{1}_{F_{2,Q}}(x)\Big)^{\frac12}d\mu(x)\\
&\leq 2\int_{Q^0\setminus F_0}\mathcal{A}^{\mu}_{Q^0}\alpha(x)\mathcal{A}^{\mu,k(x)}_{Q^0}\beta(x)\,d\mu(x)\\
&\leq 4\int_{Q^0}\mathcal{A}^{\mu}_{Q^0}\alpha(x)\mathcal{B}^{\mu}_{Q^0}\beta(x)\,d\mu(x),
\end{aligned}$$

where we have used that $Q \in \mathbb{D}_{Q^0}^{k(x)}$ for each $x \in F_{2,Q}$. This completes the proof of (2.16). ☐

LEMMA 2.19. *Under the previous considerations, given Q^0 as above, let μ and ν be two non-negative Borel measures on Q^0 so that $0 < \mu(Q), \nu(Q) < \infty$ for every $Q \in \mathbb{D}_{Q^0}$. Assume that there exist $\alpha, \beta \in (0,1)$ such that*

$$F \subset Q \in \mathbb{D}_{Q^0}, \ \frac{\mu(F)}{\mu(Q)} > \alpha \qquad \Longrightarrow \qquad \frac{\nu(F)}{\nu(Q)} \geq \beta. \tag{2.20}$$

Given $\gamma = \{\gamma_Q\}_{Q\in\mathbb{D}_{Q^0}}$, a sequence of non-negative real numbers, if we set

$$|||\gamma|||_\nu := \sup_{Q\in\mathbb{D}_{Q^0}} \frac{1}{\nu(Q)} \sum_{Q'\in\mathbb{D}_Q} \gamma_{Q'}\, \nu(Q'), \qquad |||\gamma|||_\mu := \sup_{Q\in\mathbb{D}_{Q^0}} \frac{1}{\mu(Q)} \sum_{Q'\in\mathbb{D}_Q} \gamma_{Q'}\, \mu(Q').$$

then,

$$(1-\alpha)\,\beta\, |||\gamma|||_\mu \leq |||\gamma|||_\nu \leq \frac{1}{(1-\alpha)\,\beta}\, |||\gamma|||_\mu. \tag{2.21}$$

Let us observe that when μ is dyadically doubling (that is, there exists C_μ such that $\mu(Q) \leq C_\mu \mu(Q')$ for every $Q, Q' \in \mathbb{D}_{Q^0}$ with $\ell(Q) = 2\ell(Q')$), the assumption (2.20) means exactly $\nu \in A_\infty^{\rm dyadic}(Q^0, \mu)$ (see Definition 2.24 below).

PROOF. We first consider the case on which $\#\{Q \in \mathbb{D}_{Q^0} : \gamma_Q \neq 0\} < \infty$ so that $|||\gamma|||_\nu$, $|||\gamma|||_\mu < \infty$ (albeit with constants depending on the set $\{Q : \gamma_Q \neq 0\}$), condition which will be used qualitatively. We will eventually see how to pass to the general case.

Fix $Q_0 \in \mathbb{D}_{Q^0}$. Let $\mathcal{F} = \{Q_j\}_j$ be the collection of dyadic cubes contained in Q_0 that are maximal with respect to the inclusion, and therefore pairwise disjoint, with respect to the property that

$$\frac{\nu(Q)}{\mu(Q)} > \frac{1}{1-\alpha}\, \frac{\nu(Q_0)}{\mu(Q_0)} \tag{2.22}$$

Note that $\mathcal{F} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ since $(1-\alpha)^{-1} > 1$. Also, the maximality of the cubes in $\mathcal{F}$ immediately gives

$$\frac{\nu(Q)}{\mu(Q)} \leq \frac{1}{1-\alpha}\, \frac{\nu(Q_0)}{\mu(Q_0)}, \qquad \forall\, Q \in \mathbb{D}_{\mathcal{F},Q_0}. \tag{2.23}$$

Set

$$E_0 = \bigcup_{Q_j\in\mathcal{F}} Q_j$$

and note that if $\mathcal{F}$ is the null set then we understand that E_0 is also empty. The definition of the family $\mathcal{F}$ gives

$$\frac{\mu(E_0)}{\mu(Q_0)} = \sum_{Q_j\in\mathcal{F}} \frac{\mu(Q_j)}{\mu(Q_0)} < (1-\alpha) \sum_{Q_j\in\mathcal{F}} \frac{\nu(Q_j)}{\nu(Q_0)} = (1-\alpha)\, \frac{\nu(E_0)}{\nu(Q_0)} \leq 1-\alpha.$$

Applying (2.20) to $F = Q_0 \setminus E_0$ which satisfies $\mu(Q_0 \setminus E_0) > \alpha\, \mu(Q_0)$ we obtain $\nu(Q_0 \setminus E_0) \geq \beta\, \nu(Q_0)$ and eventually $\nu(E_0) \leq (1-\beta)\, \nu(Q_0)$. Therefore,

$$\sum_{Q\in\mathbb{D}_{Q_0}\setminus\mathbb{D}_{\mathcal{F},Q_0}} \gamma_Q\, \nu(Q) = \sum_{Q_j\in\mathcal{F}} \sum_{Q\in\mathbb{D}_{Q_j}} \gamma_Q\, \nu(Q) \leq |||\gamma|||_\nu \sum_{Q_j\in\mathcal{F}} \nu(Q_j)$$

$$= |||\gamma|||_\nu \, \nu\Big(\bigcup_{Q_j \in \mathcal{F}} Q_j \Big) = |||\gamma|||_\nu \, \nu(E_0) \le (1-\beta)\, |||\gamma|||_\nu \, \nu(Q_0).$$

On the other hand, invoking (2.23),

$$\frac{1}{\nu(Q_0)} \sum_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \gamma_Q \, \nu(Q) \le \frac{1}{1-\alpha} \frac{1}{\mu(Q_0)} \sum_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \gamma_Q \, \mu(Q)$$
$$\le \frac{1}{1-\alpha} \frac{1}{\mu(Q_0)} \sum_{Q \in \mathbb{D}_{Q_0}} \gamma_Q \, \mu(Q) \le \frac{1}{1-\alpha} \, |||\gamma|||_\mu.$$

Combining the previous estimates we arrive that

$$\frac{1}{\nu(Q_0)} \sum_{Q \in \mathbb{D}_{Q_0}} \gamma_Q \, \nu(Q) = \frac{1}{\nu(Q_0)} \Big(\sum_{Q \in \mathbb{D}_{Q_0} \setminus \mathbb{D}_{\mathcal{F},Q_0}} \gamma_Q \, \nu(Q) + \sum_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \gamma_Q \, \nu(Q) \Big)$$
$$\le (1-\beta)\, |||\gamma|||_\nu + \frac{1}{1-\alpha} \, |||\gamma|||_\mu.$$

We next take the supremum over all $Q_0 \in \mathbb{D}_{Q^0}$ to conclude

$$|||\gamma|||_\nu \le (1-\beta)\, |||\gamma|||_\nu + \frac{1}{1-\alpha} \, |||\gamma|||_\mu.$$

Recalling that in the current case $|||\gamma|||_\nu < \infty$ (and this is used qualitatively) the first term in the right hand side can be absorbed and we eventually see that

$$|||\gamma|||_\nu \le \frac{1}{(1-\alpha)\,\beta} \, |||\gamma|||_\mu.$$

Let us now remove the assumption $\#\{Q : \gamma_Q \neq 0\} < \infty$. Much as in the proof of Lemma 2.15 we can find $\mathbb{D}^1 \subset \mathbb{D}^2 \subset \cdots \subset \mathbb{D}^N \subset \cdots \subset \mathbb{D}$ with $\mathbb{D} = \bigcup_{N \ge 1} \mathbb{D}^N$ and $\#\mathbb{D}^N \le N$. Given $N \ge 1$, let $\gamma^N = \{\gamma_Q^N\}_{Q \in \mathbb{D}_{Q^0}}$ where $\gamma_Q^N = \beta_Q$ if $Q \in \mathbb{D}^N$ and $\gamma_Q^N = 0$ otherwise. Note that $\#\{Q : \gamma_Q^N \neq 0\} \le N < \infty$ hence the previous estimate applies to γ^N. Thus, for every $Q_0 \in \mathbb{D}_{Q^0}$

$$\frac{1}{\nu(Q_0)} \sum_{Q \in \mathbb{D}_{Q_0}} \gamma_Q \, \nu(Q) = \sup_{N \ge 1} \frac{1}{\nu(Q_0)} \sum_{Q \in \mathbb{D}_{Q_0} \cap \mathbb{D}_N} \gamma_Q \, \nu(Q)$$
$$= \sup_{N \ge 1} \frac{1}{\nu(Q_0)} \sum_{Q \in \mathbb{D}_{Q_0}} \gamma_Q^N \, \nu(Q) \le \sup_{N \ge 1} \frac{1}{(1-\alpha)\,\beta} \, |||\gamma^N|||_\mu \le \frac{1}{(1-\alpha)\,\beta} \, |||\gamma|||_\mu.$$

Taking now the supremum over all $Q_0 \in \mathbb{D}_{Q^0}$ we conclude the second estimate in (2.21).

Obtaining the first estimate in (2.21) is now easy. Set $\widetilde{\alpha} = 1-\beta$ and $\widetilde{\beta} = 1-\alpha$, and note that for any $F \subset Q \in \mathbb{D}_{Q^0}$, applying the contrapositive of (2.20) to $Q \setminus F$ we obtain

$$\frac{\nu(F)}{\nu(Q)} > \widetilde{\alpha} \implies \frac{\nu(Q \setminus F)}{\nu(Q)} < \beta \implies \frac{\mu(Q \setminus F)}{\mu(Q)} \le \alpha \implies \frac{\mu(F)}{\mu(Q)} \ge \widetilde{\beta}.$$

That is, in (2.20) holds with μ and ν swapped, and with $\widetilde{\alpha}$, and $\widetilde{\beta}$. Hence, the second estimate in (2.21) with μ and ν swapped yields

$$|||\gamma|||_\mu \le \frac{1}{(1-\widetilde{\alpha})\,\widetilde{\beta}} \, |||\gamma|||_\nu = \frac{1}{(1-\alpha)\,\beta} \, |||\gamma|||_\nu,$$

which is the first estimate in (2.21). This completes the proof. □

As above, Q^0 is either E, and in which case $\mathbb{D}_{Q^0} := \mathbb{D}$, or a fixed cube in $\mathbb{D}$, hence $\mathbb{D}_{Q^0}$ is the family of dyadic subcubes of Q^0. For the rest of the section we will be working with μ which is dyadically doubling in Q^0. This means that there exists C_μ such that $\mu(Q) \le C_\mu \mu(Q')$ for every $Q, Q' \in \mathbb{D}_{Q^0}$ with $\ell(Q) = 2\ell(Q')$.

DEFINITION 2.24 ($A_\infty^{\rm dyadic}$). Given Q^0 and μ, a non-negative dyadically doubling measure in Q^0, a non-negative Borel measure ν defined on Q^0 is said to belong to $A_\infty^{\rm dyadic}(Q^0, \mu)$ if there exist constants $0 < \alpha, \beta < 1$ such that for every $Q \in \mathbb{D}_{Q^0}$ and for every Borel set $F \subset Q$, we have that

$$\frac{\mu(F)}{\mu(Q)} > \alpha \qquad \Longrightarrow \qquad \frac{\nu(F)}{\nu(Q)} > \beta. \tag{2.25}$$

It is well known (see [**CF74**, **GCRdF85**]) that since μ is a dyadically doubling measure in Q^0, $\nu \in A_\infty^{\rm dyadic}(Q^0, \mu)$ if and only if $\nu \ll \mu$ in Q^0 and there exists $1 < p < \infty$ such that $\nu \in RH_p^{\rm dyadic}(Q^0, \mu)$, that is, there is a constant $C \ge 1$ such that

$$\left(\fint_Q k(x)^p \, d\mu(x)\right)^{\frac{1}{p}} \le C \fint_Q k(x) \, d\mu(x) = C\, \frac{\nu(Q)}{\mu(Q)},$$

for every $Q \in \mathbb{D}_{Q^0}$, and where $k = d\nu/d\mu$ is the Radon-Nikodym derivative.

For each $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q^0}$, a family of pairwise disjoint dyadic cubes, and each $f \in L^1_{\rm loc}(\mu)$, we define the projection operator

$$\mathcal{P}_{\mathcal{F}}^\mu f(x) = f(x) \mathbf{1}_{E \setminus (\bigcup_{Q_i \in \mathcal{F}} Q_i)}(x) + \sum_{Q_i \in \mathcal{F}} \Big(\fint_{Q_i} f(y) \, d\mu(y)\Big) \mathbf{1}_{Q_i}(x).$$

If ν is a non-negative Borel measure on Q^0, we may naturally then define the measure $\mathcal{P}_{\mathcal{F}}^\mu \nu$ as $\mathcal{P}_{\mathcal{F}}^\mu \nu(F) = \int_E \mathcal{P}_{\mathcal{F}}^\mu \mathbf{1}_F \, d\nu$, that is,

$$\mathcal{P}_{\mathcal{F}}^\mu \nu(F) = \nu\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \nu(Q_i), \tag{2.26}$$

for each Borel set $F \subset Q^0$.

The next result follows easily by adapting the arguments in [**HM14**, Lemma B.1] and [**HM10**, Lemma 4.1] to the current scenario.

LEMMA 2.27. *Given Q^0, let μ be a non-negative dyadically doubling measure in Q^0, and let ν be a non-negative Borel measure in Q^0.*

(a) *If ν is dyadically doubling on Q^0 then $\mathcal{P}_{\mathcal{F}}^\mu \nu$ is dyadically doubling on Q^0.*
(b) *If $\nu \in A_\infty^{\rm dyadic}(Q^0, \mu)$ then $\mathcal{P}_{\mathcal{F}}^\mu \nu \in A_\infty^{\rm dyadic}(Q^0, \mu)$.*

Let $\gamma = \{\gamma_Q\}_{Q \in \mathbb{D}_{Q^0}}$ be a sequence of non-negative numbers. For any collection $\mathbb{D}' \subset \mathbb{D}_{Q^0}$, we define an associated "discrete measure"

$$\mathfrak{m}_\gamma(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \gamma_Q. \tag{2.28}$$

We say that $\mathfrak{m}_\gamma$ is a "discrete Carleson measure" (with respect to μ) in Q^0, if

$$\|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q^0, \mu)} := \sup_{Q \in \mathbb{D}_{Q^0}} \frac{\mathfrak{m}_\gamma(\mathbb{D}_Q)}{\mu(Q)} = \sup_{Q \in \mathbb{D}_{Q^0}} \frac{1}{\mu(Q)} \sum_{Q' \in \mathbb{D}_Q} \gamma_{Q'} < \infty. \tag{2.29}$$

For simplicity, when $Q^0 = E$ we simply write $\|\mathfrak{m}_\gamma\|_{\mathcal{C}(\mu)}$.

Given $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q^0}$, a (possibly empty) family of pairwise disjoint dyadic cubes, we define $\mathfrak{m}_{\gamma,\mathcal{F}}$ by

$$\mathfrak{m}_{\gamma,\mathcal{F}}(\mathbb{D}') = \mathfrak{m}_\gamma(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q \in \mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}} \gamma_Q, \qquad \mathbb{D}' \subset \mathbb{D}_{Q^0}. \tag{2.30}$$

Equivalently, $\mathfrak{m}_{\gamma,\mathcal{F}} = \mathfrak{m}_{\gamma_{\mathcal{F}}}$ where $\gamma_{\mathcal{F}} = \{\gamma_{\mathcal{F},Q}\}_{Q \in \mathbb{D}_{Q^0}}$ is given by

$$\gamma_{\mathcal{F},Q} = \begin{cases} \gamma_Q & \text{if } Q \in \mathbb{D}_{\mathcal{F},Q^0}, \\ 0 & \text{if } Q \in \mathbb{D}_{Q_0} \setminus \mathbb{D}_{\mathcal{F},Q^0}. \end{cases} \tag{2.31}$$

Note that if $\mathcal{F} = \emptyset$, then $\gamma_{\mathcal{F}} = \gamma$ and hence $\mathfrak{m}_{\gamma,\emptyset} = \mathfrak{m}_\gamma$.

The following result was proved in [**HM14**, Lemma 8.5] under the additional assumption that $\partial\Omega$ is AR, however a careful inspection of the proof shows that the same argument can be carried out under the current assumption. We note that [**HM14**, Lemma 8.5] was formulated and proved in the case that $Q^0 \in \mathbb{D}$, but clearly that implies the case $Q^0 = E$. We caution the reader to beware of the distinction between sub- and super-script, Q_0 vs. Q^0, in the statement of the following lemma.

Lemma 2.32 ([**HM14**, Lemma 8.5]). *Given Q^0, let μ, ν be a pair of non-negative dyadically doubling Borel measures on Q^0, and let $\mathfrak{m}_\gamma$ be a discrete Carleson measure with respect to μ, with*

$$\|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q^0,\mu)} \le M_0.$$

Suppose that there exists ε such that for every $Q_0 \in \mathbb{D}_{Q^0}$ and every family of pairwise disjoint dyadic cubes $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ verifying

$$\|\mathfrak{m}_{\gamma,\mathcal{F}}\|_{\mathcal{C}(Q_0,\mu)} = \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}_\gamma(\mathbb{D}_{\mathcal{F},Q})}{\mu(Q)} \le \varepsilon,$$

we have that $\mathcal{P}^\mu_{\mathcal{F}}\nu$ satisfies the following property:

$$\forall \zeta \in (0,1), \quad \exists\, C_\zeta > 1 \text{ such that } \Big(F \subset Q_0, \quad \frac{\mu(F)}{\mu(Q_0)} \ge \zeta \implies \frac{\mathcal{P}^\mu_{\mathcal{F}}\nu(F)}{\mathcal{P}^\mu_{\mathcal{F}}\nu(Q_0)} \ge \frac{1}{C_\zeta}\Big).$$

Then, there exist $\eta_0 \in (0,1)$ and $1 < C_0 < \infty$ such that, for every $Q_0 \in \mathbb{D}_{Q^0}$

$$F \subset Q_0, \quad \frac{\mu(F)}{\mu(Q_0)} \ge 1 - \eta_0 \quad \implies \quad \frac{\nu(F)}{\nu(Q_0)} \ge \frac{1}{C_0}.$$

In other words, $\nu \in A_\infty^{\text{dyadic}}(Q^0, \mu)$.

2.4. Existence of a dyadic grid

In this section we introduce a dyadic grid along the lines of that obtained in [**Chr90**]. More precisely, we will use the dyadic structure from [**HK12, HK13**], with a modification from [**HMMM17**, Proof of Proposition 2.12]:

Lemma 2.33 (**Existence and properties of the "dyadic grid"**). *Let $E \subset \mathbb{R}^{n+1}$ be a closed set. Then there exists a constant $C \ge 1$ depending just on n such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets (called "cubes")*

$$\mathbb{D}_k := \big\{Q_j^k \subset E : \ j \in \mathfrak{J}_k\big\},$$

where $\mathfrak{J}_k$ denotes some (possibly finite) index set depending on k satisfying:

(a) $E = \bigcup_{j\in\mathfrak{J}_k} Q_j^k$ *for each* $k \in \mathbb{Z}$.

(b) *If* $m \le k$ *then either* $Q_j^k \subset Q_i^m$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(c) *For each* $k \in \mathbb{Z}$, $j \in \mathfrak{J}_k$, *and* $m < k$, *there is a unique* $i \in \mathfrak{J}_m$ *such that* $Q_j^k \subset Q_i^m$.

(d) *For each* $k \in \mathbb{Z}$, $j \in \mathfrak{J}_k$ *there is* $x_j^k \in E$ *such that*

$$B(x_j^k, C^{-1}2^{-k}) \cap E \subset Q_j^k \subset B(x_j^k, C2^{-k}) \cap E.$$

Proof. We first note that E is geometric doubling. That is, there exists N depending just on n such that for every $x \in E$ and $r > 0$ one can cover the surface ball $B(x,r) \cap E$ with at most N surface balls of the form $B(x_i, r/2) \cap E$ with $x_i \in E$ —observe that geometric doubling for E is inherited from the corresponding property on $\mathbb{R}^{n+1}$ and that is why N depends only on n and it is independent of E. Besides, letting $\eta = \frac{1}{16}$, for every $k \in \mathbb{Z}$ it is easy to find a countable collection $\{x_j^k\}_{j\in\mathfrak{J}_k} \subset E$ such that

$$|x_j^k - x_{j'}^k| \ge \eta^k, \qquad j, j' \in \mathfrak{J}_k,\ j \ne j'; \qquad \min_{j\in\mathfrak{J}_k} |x - x_j| < \eta^k, \qquad \forall\, x \in E.$$

Invoking then [**HK12, HK13**] on E with the Euclidean distance and $c_0 = C_0 = 1$ one can construct a family of dyadic cubes associated with these families of points, say $\mathfrak{D}_k$ for $k \in \mathbb{Z}$. These satisfy (a)–(d) in the statement with the only difference that we have to replace 2^{-k} by η^k in (d).

At this point we follow the argument in [**HMMM17**, Proof of Proposition 2.12] with $\eta = \frac{1}{16}$. For any $k \in \mathbb{Z}$ we set $\mathbb{D}_j = \mathfrak{D}_k$ for every $4k \le j < 4(k+1)$. It is straightforward to show that properties (a), (b) and (c) for the families $\mathbb{D}_k$ follow at once from those for the families $\mathfrak{D}_k$. Regarding (d), let $Q^i \in \mathbb{D}_j$ and let $k \in \mathbb{Z}$ such that $4k \le j < 4(k+1)$ so that $Q^i \in \mathbb{D}_j = \mathfrak{D}_k$. Writing $x^i \in E$ for the corresponding point associated with $Q^i \in \mathfrak{D}_k$ and invoking (d) for $\mathfrak{D}_k$ we conclude

$$B(x^i, C^{-1}2^{-j}) \cap E \subset B(x^i, C^{-1}\eta^k) \cap E \subset Q^i \subset B(x^i, C\eta^k) \cap E \subset B(x^i, 16C2^{-j}) \cap E,$$

hence (d) holds. □

In what follows given $B = B(x,r)$ with $x \in E$ we will denote $\Delta = \Delta(x,r) = B \cap E$. A few remarks are in order concerning this lemma. We first observe that if E is bounded and $k \in \mathbb{Z}$ is such that $\operatorname{diam}(E) < C^{-1}2^{-k}$, then there cannot be two distinct cubes in $\mathbb{D}_k$. Thus, $\mathbb{D}_k = \{Q^k\}$ with $Q^k = E$. Therefore, as explained in Remark 2.13 we are going to ignore those $k \in \mathbb{Z}$ such that $2^{-k} \lesssim \operatorname{diam}(E)$. Hence, we shall denote by $\mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D}(E) := \bigcup_k \mathbb{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those $k \in \mathbb{Z}$ such that $2^{-k} \lesssim \operatorname{diam}(E)$. For a dyadic cube $Q \in \mathbb{D}_k$, as explained above we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. It is clear from (d) that $\operatorname{diam}(Q) \lesssim \ell(Q)$ (we will see below that in our setting the converse hold, see Remark 2.73). We write $\Xi = 2C^2$, with C being the constant in Lemma 2.33, which is a purely dimensional. For $Q \in \mathbb{D}(E)$ we will set $k(Q) = k$ if $Q \in \mathbb{D}_k$.

Property (d) implies that for each cube $Q \in \mathbb{D}$, there exist $x_Q \in E$ and r_Q, with $\Xi^{-1}\ell(Q) \le r_Q \le \ell(Q)$ (indeed $r_Q = (2C)^{-1}\ell(Q)$), such that

$$\Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, \Xi r_Q). \tag{2.34}$$

We shall denote these balls and surface balls by

$$B_Q := B(x_Q, r_Q), \qquad \Delta_Q := \Delta(x_Q, r_Q), \tag{2.35}$$

$$\widetilde{B}_Q := B(x_Q, \Xi r_Q), \qquad \widetilde{\Delta}_Q := \Delta(x_Q, \Xi r_Q), \tag{2.36}$$

and we shall refer to the point x_Q as the "center" of Q.

Let $Q \in \mathbb{D}_k$ and consider the family of its dyadic children $\{Q' \in \mathbb{D}_{k+1} : Q' \subset Q\}$. Note that for any two distinct children Q', Q'', one has $|x_{Q'} - x_{Q''}| \ge r_{Q'} = r_{Q''} = r_Q/2$, otherwise $x_{Q''} \in Q'' \cap \Delta_{Q'} \subset Q'' \cap Q'$, contradicting the fact that Q' and Q'' are disjoint. Also $x_{Q'}, x_{Q''} \in Q \subset \Delta(x_Q, r_Q)$, hence by the geometric doubling property we have a purely dimensional bound for the number of such $x_{Q'}$ and hence the number of dyadic children of a given dyadic cube is uniformly bounded.

Lemma 2.37. *Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $\mathbb{D}(E)$ be the dyadic grid as in Lemma 2.33. Assume that there is a Borel measure μ which is doubling, that is, there exists $\mathcal{C}_\mu \ge 1$ such that $\mu(\Delta(x, 2r)) \le C_\mu \mu(\Delta(x, r))$ for every $x \in E$ and $r > 0$. Then $\mu(\partial Q) = 0$ for every $Q \in \mathbb{D}(E)$. Moreover, there exist $0 < \tau_0 < 1$, C and $\eta > 0$ depending only on dimension and C_μ such that for every $\tau \in (0, \tau_0)$ and $Q \in \mathbb{D}(E)$*

$$\mu\big(\{x \in Q : \operatorname{dist}(x, E \setminus Q) \le \tau \ell(Q)\}\big) \le C_1 \tau^\eta \mu(Q). \tag{2.38}$$

Proof. The argument is a refinement of that in [**HM14**, Proposition 6.3] (see also [**GCRdF85**, p. 403] where the Euclidean case was treated). Fix an integer k, a cube $Q \in \mathbb{D}_k$, and a positive integer m to be chosen. Fix $\tau > 0$ small enough to be chosen and write

$$\Sigma_\tau = \big\{x \in \overline{Q} : \operatorname{dist}(x, E \setminus Q) < \tau \ell(Q)\big\}.$$

We set

$$\{Q_i^1\} := \mathbb{D}^1 := \mathbb{D}_Q \cap \mathbb{D}_{k+m}\,,$$

and make the disjoint decomposition $Q = \bigcup Q_i^1$. We then split $\mathbb{D}^1 = \mathbb{D}^{1,1} \cup \mathbb{D}^{1,2}$, where $Q_i^1 \in \mathbb{D}^{1,1}$ if $\overline{Q_i^1}$ meets Σ_τ, and $Q_i^1 \in \mathbb{D}^{1,2}$ otherwise. We then write $\overline{Q} = R^{1,1} \cup R^{1,2}$, where

$$R^{1,1} := \bigcup_{\mathbb{D}^{1,1}} \widehat{Q}_i^1\,, \qquad R^{1,2} := \bigcup_{\mathbb{D}^{1,2}} Q_i^1,$$

and for each cube $Q_i^1 \in \mathbb{D}^{1,1}$, we construct $\widehat{Q}_i^1$ as follows. We enumerate the elements in $\mathbb{D}^{1,1}$ as $Q_{i_1}^1, Q_{i_2}^1, \dots, Q_{i_N}^1$, and then set $(Q_i^1)^* = Q_i^1 \cup (\partial Q_i^1 \cap \partial Q)$ and

$$\widehat{Q}_{i_1}^1 := (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_2}^1 := (Q_{i_2}^1)^* \setminus (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_3}^1 := (Q_{i_3}^1)^* \setminus ((Q_{i_1}^1)^* \cup (Q_{i_2}^1)^*), \ \dots$$

so that $R^{1,1}$ covers Σ_τ and the modified cubes $\widehat{Q}_i^1$ are pairwise disjoint.

We also note from (2.34) that if $2^{-m} < \Xi^{-2}/4$ then

$$\operatorname{dist}\big(\Delta_Q, E \setminus Q\big) \ge r_Q \ge \Xi^{-1}\ell(Q), \qquad \operatorname{diam}(Q_i^1) \le 2\Xi r_{Q_i^1} \le 2\Xi \ell(Q_i^1) < \frac{\Xi^{-1}}{2}\ell(Q).$$

Then $R^{1,1}$ misses Δ_Q provided $\tau < \Xi^{-1}/2$. Otherwise, we can find $x \in \overline{Q_i^1} \cap \Delta_Q$ with $Q_i^1 \in \mathbb{D}^{1,1}$. The latter implies that there is $y \in \overline{Q_i^1} \cap \Sigma_\tau$. All these yield a contradiction:

$$\Xi^{-1}\ell(Q) \leq \operatorname{dist}\big(\Delta_Q, E\setminus Q\big) \leq |x-y| + \operatorname{dist}\big(y, E\setminus Q\big) \\ \leq \operatorname{diam}(\overline{Q_i^1}) + \tau\ell(Q) < \Xi^{-1}\ell(Q).$$

Consequently, by the doubling property,

$$\mu(\overline{Q}) \leq \mu(2\widetilde{\Delta}_Q) \leq C'_\mu\, \mu(\Delta_Q) \leq C'_\mu\, \mu(R^{1,2}).$$

Since $R^{1,1}$ and $R^{1,2}$ are disjoint, the latter estimate yields

$$\mu(R^{1,1}) \leq \Big(1 - \frac{1}{C'_\mu}\Big)\,\mu(\overline{Q}) =: \theta\,\mu(\overline{Q}),$$

where we note that $0 < \theta < 1$.

Let us now repeat this procedure, decomposing $\widehat{Q}_i^1$ for each $Q_i^1 \in \mathbb{D}^{1,1}$. We set $\mathbb{D}^2(Q_i^1) = \mathbb{D}_{Q_i^1} \cap \mathbb{D}_{k+2m}$ and split it into $\mathbb{D}^{2,1}(Q_i^1)$ and $\mathbb{D}^{2,2}(Q_i^1)$ where $Q' \in \mathbb{D}^{2,1}(Q_i^1)$ if $\overline{Q'}$ meets Σ_τ. Associated to any $Q' \in \mathbb{D}^{2,1}(Q_i^1)$ we set $(Q')^* = (Q' \cap \widehat{Q}_i^1) \cup (\partial Q' \cap (\partial Q \cap \widehat{Q}_i^1))$. Then we make these sets disjoint as before and we have that $R^{2,1}(Q_i^1)$ is defined as the disjoint union of the corresponding $\widehat{Q'}$. Note that $\widehat{Q}_i^1 = R^{2,1}(Q_i^1) \cup R^{2,2}(Q_i^1)$ and this is a disjoint union. As before, $R^{2,1}(Q_i^1)$ misses $\Delta_{Q_i^1}$ provided $\tau < 2^{-m}\Xi^{-1}/2$ so that by the doubling property

$$\mu(\widehat{Q}_i^1) \leq \mu(2\widetilde{\Delta}_{Q_i^1}) \leq C'_\mu\, \mu(\Delta_{Q_i^1}) \leq C'_\mu\, \mu(R^{2,2}(Q_i^1))$$

and then $\mu(R^{2,1}(Q_i^1)) \leq \theta\,\mu(\widehat{Q}_i^1)$. Next we set $R^{2,1}$ and $R^{2,2}$ as the union of the corresponding $R^{2,1}(Q_i^1)$ and $R^{2,2}(Q_i^1)$ with $Q_i^1 \in \mathbb{D}^{1,1}$. Then,

$$\mu(R^{2,1}) := \mu\Big(\bigcup_{Q_i^1 \in \mathbb{D}^{1,1}} R^{2,1}(Q_i^1)\Big) = \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu\big(R^{2,1}(Q_i^1)\big) \\ \leq \theta \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu(\widehat{Q}_i^1) = \theta\,\mu(R^{1,1}) \leq \theta^2\,\mu(\overline{Q}).$$

Iterating this procedure we obtain that for every $k = 0, 1, \dots$, if $\tau < 2^{-km}\Xi^{-1}/2$ then $\mu(R^{k+1,1}) \leq \theta^{k+1}\mu(\overline{Q})$. Let us see that this leads to the desired estimates. Fix $\tau < \Xi^{-1}/2$ and find $k \geq 0$ such that $2^{-(k+1)m}\Xi^{-1}/2 \leq \tau < 2^{-km}\Xi^{-1}/2$. By construction $\Sigma_\tau \subset R^{k+1,1}$ and then

$$\mu(\Sigma_\tau) \leq \mu(R^{k+1,1}) \leq \theta^{k+1}\mu(\overline{Q}) \leq (2\Xi)^{\frac{\log_2\theta^{-1}}{m}}\,\tau^{\frac{\log_2\theta^{-1}}{m}}\,\mu(\overline{Q}),$$

which easily gives (2.38) with $C_1 = (2\Xi)^{\frac{\log_2\theta^{-1}}{m}}$ and $\eta = \frac{\log_2\theta^{-1}}{m}$. On the other hand, note that

$$\partial Q \subset \bigcap_{j: 2^{-j} < \Xi^{-1}/2} \Sigma_{2^{-j}},$$

also $\Sigma_{2^{-(j+1)}} \subset \Sigma_{2^{-j}}$. Thus clearly,

$$0 \leq \mu(\partial Q) \leq \lim_{j\to\infty} \mu(\Sigma_{2^{-j}}) \leq \lim_{j\to\infty} C_1 2^{-j\eta}\mu(Q) = 0,$$

yielding that $\mu(\partial Q) = 0$. □

REMARK 2.39. Note that the previous argument is local in the sense that if we just want to obtain the desired estimates for a fixed Q_0 we would only need to assume that μ is doubling in $2\widetilde{\Delta}_{Q_0}$. Indeed we would just need to know that $\mu(\Delta(x,2r)) \le C\,\mu(\Delta(x,r))$ for every $x \in Q_0$ and $0 < r < \Xi\ell(Q_0)$, and the involved constants in the resulting estimates will depend only on dimension and C_μ. Further details are left to the interested reader.

2.5. Sawtooth domains

In the sequel, $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, will be a 1-sided NTA domain satisfying the CDC. Write $\mathbb{D} = \mathbb{D}(\partial\Omega)$ for the dyadic grid obtained from Lemma 2.33 with $E = \partial\Omega$. In Remark 2.73 below we shall show that under the present assumptions one has that $\operatorname{diam}(\Delta) \approx r_\Delta$ for every surface ball Δ. In particular $\operatorname{diam}(Q) \approx \ell(Q)$ for every $Q \in \mathbb{D}$ in view of (2.34). Given $Q \in \mathbb{D}$ we define the "Corkscrew point relative to Q" as $X_Q := X_{\Delta_Q}$. We note that

$$\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q).$$

Much as we did in Section 2.3 of, given $Q \in \mathbb{D}$ and $\mathcal{F}$ a possibly empty family of pairwise disjoint dyadic cubes, we can define $\mathbb{D}_Q$, the "discretized Carleson region"; $\mathbb{D}_{\mathcal{F}}$, the "global discretized sawtooth" relative to $\mathcal{F}$; and $\mathbb{D}_{\mathcal{F},Q}$, the "local discretized sawtooth" relative to $\mathcal{F}$. Note that if $\mathcal{F}$ to be the null set in which case $\mathbb{D}_{\emptyset} = \mathbb{D}$ and $\mathbb{D}_{\emptyset,Q} = \mathbb{D}_Q$.

We also introduce the "geometric" Carleson regions and sawtooths. Given $Q \in \mathbb{D}$ we want to define some associated regions which inherit the good properties of Ω. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of $\Omega \subset \mathbb{R}^{n+1}$, so that the cubes in $\mathcal{W}$ form a covering of Ω with non-overlapping interiors, and satisfy

$$4\operatorname{diam}(I) \le \operatorname{dist}(4I, \partial\Omega) \le \operatorname{dist}(I, \partial\Omega) \le 40\operatorname{diam}(I), \qquad \forall I \in \mathcal{W}, \tag{2.40}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2), \text{ whenever } I_1 \text{ and } I_2 \text{ touch.}$$

Let $X(I)$ denote the center of I, let $\ell(I)$ denote the side length of I, and write $k = k_I$ if $\ell(I) = 2^{-k}$.

Given $0 < \lambda < 1$ and $I \in \mathcal{W}$ we write $I^* = (1+\lambda)I$ for the "fattening" of I. By taking λ small enough, we can arrange matters, so that, first, $\operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J)$ for every $I, J \in \mathcal{W}$. Secondly, I^* meets J^* if and only if ∂I meets ∂J (the fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathcal{W}$ whose boundaries touch, so that the Harnack Chain property then holds locally in $I^* \cup J^*$, with constants depending upon λ). By picking λ sufficiently small, say $0 < \lambda < \lambda_0$, we may also suppose that there is $\tau \in (\frac{1}{2}, 1)$ such that for distinct $I, J \in \mathcal{W}$, we have that $\tau J \cap I^* = \emptyset$. In what follows we will need to work with dilations $I^{**} = (1+2\lambda)I$ or $I^{***} = (1+4\lambda)I$, and in order to ensure that the same properties hold we further assume that $0 < \lambda < \lambda_0/4$.

For every $Q \in \mathbb{D}$ we can construct a family $\mathcal{W}_Q^* \subset \mathcal{W}(\Omega)$, and define

$$U_Q := \bigcup_{I \in \mathcal{W}_Q^*} I^*,$$

satisfying the following properties: $X_Q \in U_Q$ and there are uniform constants k^* and K_0 such that

$$\begin{aligned} k(Q) - k^* \le k_I \le k(Q) + k^*, &\quad \forall I \in \mathcal{W}_Q^*, \\ X(I) \to_{U_Q} X_Q, &\quad \forall I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I, Q) \le K_0 2^{-k(Q)}, &\quad \forall I \in \mathcal{W}_Q^*. \end{aligned} \tag{2.41}$$

Here, $X(I) \to_{U_Q} X_Q$ means that the interior of U_Q contains all balls in a Harnack Chain (in Ω) connecting $X(I)$ to X_Q, and moreover, for any point Z contained in any ball in the Harnack Chain, we have $\operatorname{dist}(Z, \partial\Omega) \approx \operatorname{dist}(Z, \Omega \setminus U_Q)$ with uniform control of the implicit constants. The constants k^*, K_0 and the implicit constants in the condition $X(I) \to_{U_Q} X_Q$, depend on at most allowable parameters and on λ. Moreover, given $I \in \mathcal{W}(\Omega)$ we have that $I \in \mathcal{W}_{Q_I}^*$, where $Q_I \in \mathbb{D}$ satisfies $\ell(Q_I) = \ell(I)$, and contains any fixed $\widehat{y} \in \partial\Omega$ such that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, \widehat{y})$. The reader is referred to [**HM14, HMT14**] for full details.

For a given $Q \in \mathbb{D}$, the "Carleson box" relative to Q is defined by

$$T_Q := \operatorname{int}\Bigg(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}\Bigg).$$

For a given family $\mathcal{F} = \{Q_i\} \subset \mathbb{D}$ of pairwise disjoint cubes and a given $Q \in \mathbb{D}$, we define the "local sawtooth region" relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} = \operatorname{int}\Bigg(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'}\Bigg) = \operatorname{int}\Bigg(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^*\Bigg), \tag{2.42}$$

where $\mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \mathcal{W}_Q^*$. Note that in the previous definition we may allow $\mathcal{F}$ to be empty in which case clearly $\Omega_{\varnothing,Q} = T_Q$. Similarly, the "global sawtooth region" relative to $\mathcal{F}$ is defined as

$$\Omega_{\mathcal{F}} = \operatorname{int}\Bigg(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} U_{Q'}\Bigg) = \operatorname{int}\Bigg(\bigcup_{I \in \mathcal{W}_{\mathcal{F}}} I^*\Bigg), \tag{2.43}$$

where $\mathcal{W}_{\mathcal{F}} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} \mathcal{W}_Q^*$. If $\mathcal{F}$ is the empty set clearly $\Omega_\varnothing = \Omega$. For a given $Q \in \mathbb{D}$ and $x \in \partial\Omega$ let us introduce the "truncated dyadic cone"

$$\Gamma_Q(x) := \bigcup_{x \in Q' \in \mathbb{D}_Q} U_{Q'},$$

where it is understood that $\Gamma_Q(x) = \varnothing$ if $x \notin Q$. Analogously, we can slightly fatten the Whitney boxes and use I^{**} to define new fattened Whitney regions and sawtooth domains. More precisely, for every $Q \in \mathbb{D}$,

$$T_Q^* := \operatorname{int}\Bigg(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}^*\Bigg), \quad \Omega_{\mathcal{F},Q}^* := \operatorname{int}\Bigg(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'}^*\Bigg), \quad \Gamma_Q^*(x) := \bigcup_{x \in Q' \in \mathbb{D}_{Q_0}} U_{Q'}^*$$

where $U_Q^* := \bigcup_{I \in \mathcal{W}_Q^*} I^{**}$. Similarly, we can define T_Q^{**}, $\Omega_{\mathcal{F},Q}^{**}$, $\Gamma_Q^{**}(x)$, and U_Q^{**} by using I^{***} in place of I^{**}.

Given Q we next define the "localized dyadic non-tangential maximal function"

$$\mathcal{N}_Q u(x) := \sup_{Y \in \Gamma_Q^*(x)} |u(Y)|, \qquad x \in \partial\Omega, \tag{2.44}$$

for every $u \in C(T_Q^*)$, where it is understood that $\mathcal{N}_Q u(x) = 0$ for every $x \in \partial\Omega \setminus Q$ (since $\Gamma_Q^*(x) = \emptyset$ in such a case). Finally, let us introduce the "localized dyadic conical square function"

$$\mathcal{S}_Q u(x) := \left(\iint_{\Gamma_Q(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega, \tag{2.45}$$

for every $u \in W^{1,2}_{\mathrm{loc}}(T_{Q_0})$. Note that again $\mathcal{S}_Q u(x) = 0$ for every $x \in \partial\Omega \setminus Q$.

To define the "Carleson box" T_Δ associated with a surface ball $\Delta = \Delta(x,r)$, let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200r \le 2^{-k}$, and set

$$\mathbb{D}^\Delta := \big\{ Q \in \mathbb{D}_{k(\Delta)} :\ Q \cap 2\Delta \neq \emptyset \big\}. \tag{2.46}$$

We then define

$$T_\Delta := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q} \Big). \tag{2.47}$$

We can also consider fattened versions of T_Δ given by

$$T_\Delta^* := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q^*} \Big), \qquad T_\Delta^{**} := \operatorname{int}\Big(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q^{**}} \Big).$$

Following [**HM14, HMT14**], one can easily see that there exist constants $0 < \kappa_1 < 1$ and $\kappa_0 \ge 16\Xi$ (with Ξ the constant in (2.34)), depending only on the allowable parameters, so that

$$\kappa_1 B_Q \cap \Omega \subset T_Q \subset T_Q^* \subset T_Q^{**} \subset \overline{T_Q^{**}} \subset \kappa_0 B_Q \cap \overline{\Omega} =: \tfrac{1}{2} B_Q^* \cap \overline{\Omega}, \tag{2.48}$$

$$\tfrac{5}{4} B_\Delta \cap \Omega \subset T_\Delta \subset T_\Delta^* \subset T_\Delta^{**} \subset \overline{T_\Delta^{**}} \subset \kappa_0 B_\Delta \cap \overline{\Omega} =: \tfrac{1}{2} B_\Delta^* \cap \overline{\Omega}, \tag{2.49}$$

and also

$$Q \subset \kappa_0 B_\Delta \cap \partial\Omega = \tfrac{1}{2} B_\Delta^* \cap \partial\Omega =: \tfrac{1}{2}\Delta^*, \qquad \forall\, Q \in \mathbb{D}^\Delta, \tag{2.50}$$

where B_Q is defined as in (2.35), $\Delta = \Delta(x,r)$ with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and $B_\Delta = B(x,r)$ is so that $\Delta = B_\Delta \cap \partial\Omega$. From our choice of the parameters one also has that $B_Q^* \subset B_{Q'}^*$ whenever $Q \subset Q'$.

In the remainder of this section we show that if Ω is a 1-sided NTA domain satisfying the CDC then Carleson boxes and local and global sawtooth domains are also 1-sided NTA domains satisfying the CDC. We next present some of the properties of the capacity which will be used in our proofs. From the definition of capacity one can easily see that given a ball B and compact sets $F_1 \subset F_2 \subset \overline{B}$ then

$$\operatorname{Cap}_2(F_1, 2B) \le \operatorname{Cap}_2(F_2, 2B). \tag{2.51}$$

Also, given two balls $B_1 \subset B_2$ and a compact set $F \subset \overline{B_1}$ then

$$\operatorname{Cap}_2(F, 2B_2) \le \operatorname{Cap}_2(F, 2B_1). \tag{2.52}$$

On the other hand, [**HKM06**, Lemma 2.16] gives that if F is a compact with $F \subset \overline{B}$ then there is a dimensional constant C_n such that

$$C_n^{-1} \operatorname{Cap}_2(F, 2B) \le \operatorname{Cap}_2(F, 4B) \le \operatorname{Cap}_2(F, 2B). \tag{2.53}$$

LEMMA 2.54. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be a 1-sided NTA domain satisfying the CDC. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$, and $\Omega_{\mathcal{F},Q}$ are 1-sided NTA domains and satisfy the CDC with uniform implicit constants depending only on dimension and on the corresponding constants for Ω.*

PROOF. A careful examination of the proofs in [**HM14**, Appendices A.1-A.2] reveals that if Ω is a 1-sided NTA domain then all Carleson boxes T_Q and T_Δ, and local and global sawtooth domains $\Omega_{\mathcal{F},Q}$ and $\Omega_{\mathcal{F}}$ inherit the interior Corkscrew and Harnack chain conditions, hence they are also 1-sided NTA domains. Therefore, we only need to prove the CDC. We are going to consider only the case $\Omega_{\mathcal{F},Q}$ (which in particular gives the desired property for T_Q by allowing $\mathcal{F}$ to be the null set). The other proofs require minimal changes which are left to the interested reader. To this end, fix $Q \in \mathbb{D}$ and $\mathcal{F} \subset \mathbb{D}_Q$ a (possibly empty) family of pairwise disjoint dyadic cubes. Let $x \in \partial\Omega_{\mathcal{F},Q}$ and $0 < r < \operatorname{diam}(\Omega_{\mathcal{F},Q}) \approx \ell(Q)$.

Case 1: $\delta(x) = 0$. In that case we have that $x \in \partial\Omega$ and we can use that Ω satisfies the CDC with constant c_1, (2.51) and the fact that $\Omega_{\mathcal{F},Q} \subset \Omega$ to obtain the desired estimate

$$c_1 r^{n-1} \lesssim \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega, B(x,2r)) \leq \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x,2r)).$$

Case 2: $0 < \delta(x) < r/M$ with M large enough to be chosen. In this case $x \in \Omega \cap \partial\Omega_{\mathcal{F},Q}$ and hence there exist $Q' \in \mathbb{D}_{\mathcal{F},Q}$ and $I \in \mathcal{W}^*_{Q'}$ such that $x \in \partial I^*$. Note that by (2.41)

$$|x - x_{Q'}| \leq \operatorname{diam}(I^*) + \operatorname{dist}(I,Q') + \operatorname{diam}(Q') \lesssim \ell(Q') \approx \ell(I) \approx \delta(x) \lesssim \frac{r}{M}.$$

Let $Q'' \in \mathbb{D}_Q$ be such that $x_{Q'} \in Q''$ and $\frac{r}{2M} \leq \ell(Q'') < \frac{r}{M} < \ell(Q)$ provided that M is taken large enough. If $Z \in B_{Q''}$ then taking M large enough

$$|Z - x| \leq |Z - x_{Q''}| + |x_{Q''} - x_{Q'}| + |x_{Q'} - x| \lesssim \ell(Q'') + \frac{r}{M} \lesssim \frac{r}{M} < r$$

and $B_{Q''} \subset B(x,r)$. On the other hand, if $Z \in B(x,2r)$, we analogously have provided M is large enough

$$|Z - x_{Q''}| \leq |Z - x| + |x - x_{Q'}| + |x_{Q'} - x_{Q''}| < 2r + C\frac{r}{M} + \Xi r_{Q''} < 6M\Xi r_{Q''}$$

and thus $B(x,2r) \subset 6M\Xi B_{Q''}$. Once M has been fixed so that the previous estimates hold, we use them in conjunction with the fact that Ω satisfies the CDC with constant c_1, (2.51)–(2.53), and that $\Omega_{\mathcal{F},Q} \subset \Omega$ to obtain

$$\begin{aligned} \frac{c_1}{(2M\Xi)^{n-1}} r^{n-1} &\leq c_1 r_{Q''}^{n-1} \lesssim \operatorname{Cap}_2(\overline{B_{Q''}} \setminus \Omega, 2B_{Q''}) \lesssim \operatorname{Cap}_2(\overline{B_{Q''}} \setminus \Omega, 6M\Xi B_{Q''}) \\ &\leq \operatorname{Cap}_2(\overline{B_{Q''}} \setminus \Omega, B(x,2r)) \leq \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x,2r)), \end{aligned}$$

which gives us the desired lower bound in the present case.

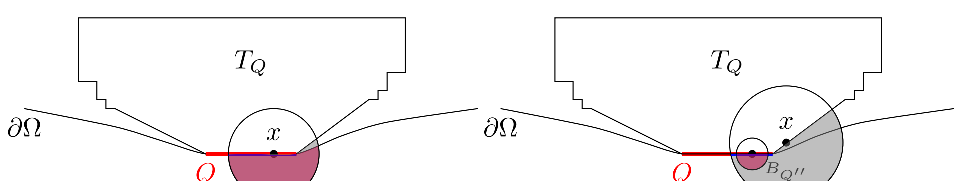

FIGURE 1. **Case 1** and **Case 2** for T_Q.

Case 3: $\delta(x) > r/M$. In this case $x \in \Omega \cap \partial\Omega_{\mathcal{F},Q}$ and hence there exists $Q' \in \mathbb{D}_{\mathcal{F},Q}$ and $I \in \mathcal{W}^*_{Q'}$ such that $x \in \partial I^*$ and $\operatorname{int}(I^*) \subset \Omega_{\mathcal{F},Q}$. Also there exists $J \in \mathcal{W}$, with $J \ni x$ such that $J \notin \mathcal{W}^*_{Q''}$ for any $Q'' \in \mathbb{D}_{\mathcal{F},Q}$ which implies that $\tau J \subset \Omega \setminus \Omega_{\mathcal{F},Q}$

for some $\tau \in (\frac{1}{2}, 1)$ (see Section 2.5). Note that $\ell(I) \approx \ell(J) \approx \delta(x) \gtrsim r$, and more precisely $r/M < \delta(x) < 41 \operatorname{diam}(J)$ by (2.40).

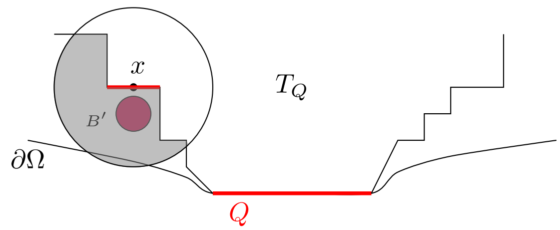

FIGURE 2. **Case 3** for T_Q.

Let $B' = B(x', s)$ with $s = r/(300M)$ and x' being the point in the segment joining x and the center of J at distance $2s$ from x. It is easy to see that $B' \subset B(x, r) \subset B(x, 2r) \subset 1000MB'$ and also $\overline{B'} \subset \operatorname{int}(J) \setminus \Omega_{\mathcal{F},Q}$. We can then use (2.12) and (2.51)–(2.53) to obtain the desired estimate:

$$\frac{1}{(300M)^{n-1}} r^{n-1} = s^{n-1} \approx \operatorname{Cap}_2(\overline{B'}, 2B') \lesssim \operatorname{Cap}_2(\overline{B'}, 1000MB')$$
$$\leq \operatorname{Cap}_2(\overline{B'}, B(x, 2r)) \leq \operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x, 2r)).$$

Collecting the 3 cases and using (2.12) we have been able to show that

(2.55)
$$\frac{\operatorname{Cap}_2(\overline{B(x,r)} \setminus \Omega_{\mathcal{F},Q}, B(x, 2r))}{\operatorname{Cap}_2(\overline{B(x,r)}, B(x, 2r))} \gtrsim 1, \qquad \forall\, x \in \partial\Omega_{\mathcal{F},Q},\ 0 < r < \operatorname{diam}(\Omega_{\mathcal{F},Q}),$$

which eventually gives that $\Omega_{\mathcal{F},Q}$ satisfies the CDC. This completes the proof. □

2.6. Uniformly elliptic operators, elliptic measure and the Green function

Next, we recall several facts concerning elliptic measure and the Green functions. To set the stage let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Throughout we consider elliptic operators L of the form $Lu = -\operatorname{div}(A\nabla u)$ with $A(X) = (a_{i,j}(X))_{i,j=1}^{n+1}$ being a real (non-necessarily symmetric) matrix such that $a_{i,j} \in L^\infty(\Omega)$ and there exists $\Lambda \geq 1$ such that the following uniform ellipticity condition holds

$$\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \eta| \leq \Lambda |\xi|\,|\eta| \tag{2.56}$$

for all $\xi, \eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$. We write $L^\top$ to denote the transpose of L, or, in other words, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A.

We say that u is a weak solution to $Lu = 0$ in Ω provided that $u \in W^{1,2}_{\text{loc}}(\Omega)$ satisfies

$$\iint A(X)\nabla u(X) \cdot \nabla \phi(X) dX = 0 \quad \text{whenever } \phi \in C_0^\infty(\Omega).$$

Associated with L one can construct an elliptic measure $\{\omega_L^X\}_{X\in\Omega}$ and a Green function G_L (see [**HMT14**] for full details). Sometimes, in order to emphasize the dependence on Ω, we will write $\omega_{L,\Omega}$ and $G_{L,\Omega}$. If Ω satisfies the CDC then it follows that all boundary points are Wiener regular and hence for a given $f \in C_c(\partial\Omega)$ we can define

$$u(X) = \int_{\partial\Omega} f(z) d\omega_L^X(z), \quad \text{whenever } X \in \Omega,$$

so that $u \in W^{1,2}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ satisfying $u = f$ on $\partial\Omega$ and $Lu = 0$ in the weak sense. Moreover, if $f \in \mathrm{Lip}(\Omega)$ then $u \in W^{1,2}(\Omega)$.

We first define the reverse Hölder class and the A_∞ classes with respect to fixed elliptic measure in Ω. One reason we take this approach is that we do not know whether $\mathcal{H}^n|_{\partial\Omega}$ is well-defined since we do not assume any Ahlfors regularity in Theorem 1.5. Hence we have to develop these notions in terms of elliptic measures. To this end, let Ω satisfy the CDC and let L_0 and L be two real (non-necessarily symmetric) elliptic operators associated with $L_0 u = -\operatorname{div}(A_0 \nabla u)$ and $Lu = -\operatorname{div}(A\nabla u)$ where A and A_0 satisfy (2.56). Let ω_0^X and ω_L^X be the elliptic measures of Ω associated with the operators L_0 and L respectively with pole at $X \in \Omega$. Note that if we further assume that Ω is connected then $\omega_L^X \ll \omega_L^Y$ on $\partial\Omega$ for every $X, Y \in \Omega$. Hence if $\omega_L^{X_0} \ll \omega_{L_0}^{Y_0}$ on $\partial\Omega$ for some $X_0, Y_0 \in \Omega$ then $\omega_L^X \ll \omega_{L_0}^Y$ on $\partial\Omega$ for every $X, Y \in \Omega$ and thus we can simply write $\omega_L \ll \omega_{L_0}$ on $\partial\Omega$. In the latter case we will use the notation

$$h(\cdot\,; L, L_0, X) = \frac{d\omega_L^X}{d\omega_{L_0}^X} \tag{2.57}$$

to denote the Radon-Nikodym derivative of ω_L^X with respect to $\omega_{L_0}^X$, which is a well-defined function $\omega_{L_0}^X$-almost everywhere on $\partial\Omega$.

DEFINITION 2.58 (Reverse Hölder and A_∞ classes). Fix $\Delta_0 = B_0 \cap \partial\Omega$ where $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$. Given p, $1 < p < \infty$, we say that $\omega_L \in RH_p(\Delta_0, \omega_{L_0})$, provided that $\omega_L \ll \omega_{L_0}$ on Δ_0, and there exists $C \geq 1$ such that

$$\begin{aligned}\left(\fint_\Delta h(y; L, L_0, X_{\Delta_0})^p d\omega_{L_0}^{X_{\Delta_0}}(y)\right)^{\frac{1}{p}} &\leq C \fint_\Delta h(y; L, L_0, X_{\Delta_0}) d\omega_{L_0}^{X_{\Delta_0}}(y) \\ &= C \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)},\end{aligned}$$

for every $\Delta = B \cap \partial\Omega$ where $B \subset B(x_0, r_0)$, $B = B(x, r)$ with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$. The infimum of the constants C as above is denoted by $[\omega_L]_{RH_p(\Delta_0, \omega_{L_0})}$.

Similarly, we say that $\omega_L \in RH_p(\partial\Omega, \omega_{L_0})$ provided that for every $\Delta_0 = \Delta(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ one has $\omega_L \in RH_p(\Delta_0, \omega_{L_0})$ uniformly on Δ_0, that is,

$$[\omega_L]_{RH_p(\partial\Omega, \omega_{L_0})} := \sup_{\Delta_0} [\omega_L]_{RH_p(\Delta_0, \omega_{L_0})} < \infty.$$

Finally,

$$A_\infty(\Delta_0, \omega_{L_0}) = \bigcup_{p>1} RH_p(\Delta_0, \omega_{L_0}) \quad \text{and} \quad A_\infty(\partial\Omega, \omega_{L_0}) = \bigcup_{p>1} RH_p(\partial\Omega, \omega_{L_0}).$$

The following lemmas state some properties for the Green functions and elliptic measures, proofs may be found in [**HMT14**].

LEMMA 2.59. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is an open set satisfying the CDC. Given a real (non-necessarily symmetric) elliptic operator $L = -\operatorname{div}(A\nabla)$, there exist $C > 1$ (depending only on dimension and on the ellipticity constant of L) and $c_\theta > 0$ (depending on the above parameters and on $\theta \in (0,1)$) such that G_L, the Green function associated with L, satisfies*

$$G_L(X,Y) \leq C|X-Y|^{1-n}; \tag{2.60}$$

$$c_\theta |X-Y|^{1-n} \leq G_L(X,Y), \quad \text{if } |X-Y| \leq \theta\delta(X), \quad \theta \in (0,1); \tag{2.61}$$

$$G_L(\cdot,Y) \in C\big(\overline{\Omega}\setminus\{Y\}\big) \quad \text{and} \quad G_L(\cdot,Y)|_{\partial\Omega} \equiv 0 \quad \forall Y \in \Omega; \tag{2.62}$$

$$G_L(X,Y) \geq 0, \quad \forall X,Y \in \Omega, \quad X \neq Y; \tag{2.63}$$

$$G_L(X,Y) = G_{L^\top}(Y,X), \quad \forall X,Y \in \Omega, \quad X \neq Y. \tag{2.64}$$

Moreover, $G_L(\cdot,Y) \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{Y\})$ for any $Y \in \Omega$ and satisfies $LG_L(\cdot,Y) = \delta_Y$ in the sense of distributions, that is,

$$\iint_\Omega A(X)\nabla_X G_L(X,Y)\cdot\nabla\varphi(X)\,dX = \varphi(Y), \qquad \forall\,\varphi \in C_c^\infty(\Omega). \tag{2.65}$$

In particular, $G_L(\cdot,Y)$ is a weak solution to $LG_L(\cdot,Y) = 0$ in the open set $\Omega\setminus\{Y\}$.

Finally, the following Riesz formula holds:

$$\iint_\Omega A^\top(X)\nabla_X G_{L^\top}(X,Y)\cdot\nabla\varphi(X)\,dX = \varphi(Y) - \int_{\partial\Omega}\varphi\,d\omega_L^Y, \quad \text{for a.e. } Y \in \Omega,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$.

REMARK 2.66. If we also assume that Ω is bounded, following [HMT14] we know that the Green function G_L coincides with the one constructed in [GW82]. Consequently, for each $Y \in \Omega$ and $0 < r < \delta(Y)$, there holds

$$G_L(\cdot,Y) \in W^{1,2}(\Omega\setminus B(Y,r)) \cap W_0^{1,1}(\Omega). \tag{2.67}$$

Moreover, for every $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $B(Y,r)$ with $0 < r < \delta(Y)$, we have that

$$(1-\varphi)G_L(\cdot,Y) \in W_0^{1,2}(\Omega). \tag{2.68}$$

The following result lists a number of properties which will be used throughout the paper:

LEMMA 2.69. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a 1-sided NTA domain satisfying the CDC. Let $L_0 = -\operatorname{div}(A_0\nabla)$ and $L = -\operatorname{div}(A\nabla)$ be two real (non-necessarily symmetric) elliptic operators, there exist $C_1 \geq 1$, $\rho \in (0,1)$ (depending only on dimension, the 1-sided NTA constants, the CDC constant, and the ellipticity of L) and $C_2 \geq 1$ (depending on the same parameters and on the ellipticity of L_0), such that for every $B_0 = B(x_0,r_0)$ with $x_0 \in \partial\Omega$, $0 < r_0 < \operatorname{diam}(\partial\Omega)$, and $\Delta_0 = B_0 \cap \partial\Omega$ we have the following properties:*

(a) *$\omega_L^Y(\Delta_0) \geq C_1^{-1}$ for every $Y \in C_1^{-1}B_0 \cap \Omega$ and $\omega_L^{X_{\Delta_0}}(\Delta_0) \geq C_1^{-1}$.*

(b) *If $B = B(x,r)$ with $x \in \partial\Omega$ and $\Delta = B \cap \partial\Omega$ is such that $2B \subset B_0$, then for all $X \in \Omega \setminus B_0$ we have that*

$$\frac{1}{C_1}\omega_L^X(\Delta) \le r^{n-1} G_L(X, X_\Delta) \le C_1 \omega_L^X(\Delta).$$

(c) *If $X \in \Omega \setminus 4B_0$, then*

$$\omega_L^X(2\Delta_0) \le C_1 \omega_L^X(\Delta_0).$$

(d) *If $B = B(x,r)$ with $x \in \partial\Omega$ and $\Delta := B \cap \partial\Omega$ is such that $B \subset B_0$, then for every $X \in \Omega \setminus 2\kappa_0 B_0$ with κ_0 as in (2.49), we have that*

$$\frac{1}{C_1}\omega_L^{X_{\Delta_0}}(\Delta) \le \frac{\omega_L^X(\Delta)}{\omega_L^X(\Delta_0)} \le C_1 \omega_L^{X_{\Delta_0}}(\Delta).$$

As a consequence,

$$\frac{1}{C_1}\frac{1}{\omega_L^X(\Delta_0)} \le \frac{d\omega_L^{X_{\Delta_0}}}{d\omega_L^X}(y) \le C_1 \frac{1}{\omega_L^X(\Delta_0)}, \qquad \text{for } \omega_L^X\text{-a.e. } y \in \Delta_0.$$

(e) *If $B = B(x,r)$ with $x \in \Delta_0$, $0 < r < r_0/4$ and $\Delta = B \cap \partial\Omega$, then we have that*

$$\frac{1}{C_1}\omega_{L,\Omega}^{X_\Delta}(F) \le \omega_{L,T_{\Delta_0}}^{X_\Delta}(F) \le C_1 \omega_{L,\Omega}^{X_\Delta}(F), \qquad \text{for every Borel set } F \subset \Delta.$$

(f) *If $L \equiv L_0$ in $B(x_0, 2\kappa_0 r_0) \cap \Omega$ with κ_0 as in (2.49), then*

$$\frac{1}{C_2}\omega_{L_0}^{X_{\Delta_0}}(F) \le \omega_L^{X_{\Delta_0}}(F) \le C_2 \omega_{L_0}^{X_{\Delta_0}}(F), \qquad \text{for every Borel set } F \subset \Delta_0.$$

(g) *For every $X \in B_0 \cap \Omega$ and for any $j \ge 1$*

$$\frac{d\omega_L^X}{d\omega_L^{X_{2^j\Delta_0}}}(y) \le C_1 \left(\frac{\delta(X)}{2^j\, r_0}\right)^\rho, \qquad \text{for } \omega_L^X\text{-a.e. } y \in \partial\Omega \setminus 2^j\,\Delta_0.$$

REMARK 2.70. We note that from (*d*) in the previous result and Harnack's inequality one can easily see that given $Q, Q', Q'' \in \mathbb{D}$

$$\frac{\omega_L^{X_{Q''}}(Q)}{\omega_L^{X_{Q''}}(Q')} \approx \omega_L^{X_{Q'}}(Q), \qquad \text{whenever } Q \subset Q' \subset Q''. \tag{2.71}$$

Also, (*d*), Harnack's inequality, and (2.34) give

$$\frac{d\omega_L^{X_{Q'}}}{d\omega_L^{X_{Q''}}}(y) \approx \frac{1}{\omega_L^{X_{Q''}}(Q')}, \qquad \text{for } \omega_L^{X_{Q''}}\text{-a.e. } y \in Q', \text{whenever } Q' \subset Q''. \tag{2.72}$$

Observe that since $\omega_L^{X_{Q''}} \ll \omega_L^{X_{Q'}}$ we can easily get an analogous inequality for the reciprocal of the Radon-Nikodym derivative.

REMARK 2.73. Given Ω, a 1-sided NTA domain satisfying the CDC, we claim that if $\Delta = \Delta(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ then $\operatorname{diam}(\Delta) \approx r$. To see this we first observe that $\operatorname{diam}(\Delta) \le 2r$. If $\operatorname{diam}(\Delta) \ge c_0 r/4$ —c_0 is the Corkscrew constant— then clearly $\operatorname{diam}(\Delta) \approx r$. Hence, we may assume that $\operatorname{diam}(\Delta) < c_0 r/4$. Let $s = 2\operatorname{diam}(\Delta)$ so that $\operatorname{diam}(\Delta) < s < r$ and note that one can easily see that $\Delta = \Delta' := \Delta(x,s)$. Associated with Δ and Δ' we can consider

X_Δ and $X_{\Delta'}$ the corresponding Corkscrew points. These are different, despite the fact that $\Delta = \Delta(x, r)$. Indeed,

$$c_0 r \le \delta(X_\Delta) \le |X_\Delta - X_{\Delta'}| + |X_{\Delta'} - x| \le |X_\Delta - X_{\Delta'}| + s < |X_\Delta - X_{\Delta'}| + \frac{c_0}{2} r$$

which yields that $|X_\Delta - X_{\Delta'}| \ge \frac{c_0}{2} r$. Note that $X_\Delta \notin 2B' := B(x, 2s)$ since otherwise we would get a contradiction: $c_0 r \le \delta(X_\Delta) \le |X_\Delta - x| < 2s < c_0 r$. Hence we can invoke Lemma 2.69 parts (a) and (b) and (2.60) to see that

$$1 \approx \omega_L^{X_\Delta}(\Delta) = \omega_L^{X_\Delta}(\Delta') \approx s^{n-1} G_L(X_\Delta, X_{\Delta'}) \lesssim s^{n-1} |X_\Delta - X_{\Delta'}|^{1-n} \lesssim (s/r)^{n-1}.$$

This and the fact that $n \ge 2$ easily yields that $r \lesssim s$ as desired.

CHAPTER 3

Proofs of the main results

In order to prove Theorem 1.5 we are going to obtain a local version valid for bounded domains, interesting on its own right, which in turn will imply the desired results.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a* ***bounded*** *1-sided NTA domain satisfying the CDC. Let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0\nabla u)$ be two real (non-necessarily symmetric) elliptic operators. Fix $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ and let $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$. Set*

$$\|\|\varrho(A, A_0)\|\|_{B_0} := \sup_B \frac{1}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)} \iint_{B\cap\Omega} \varrho(A, A_0)(X)^2 \frac{G_{L_0}(X_{\Delta_0}, X)}{\delta(X)^2}\, dX, \tag{3.2}$$

where $\varrho(A, A_0)$ was defined in (1.6), $\Delta = B \cap \partial\Omega$, and the sup is taken over all balls $B = B(x, r)$ with $x \in 2\Delta_0$ and $0 < r < r_0 c_0/4$ (c_0 is the Corkscrew constant).

- (a) *If $\|\|\varrho(A, A_0)\|\|_{B_0} < \infty$, then $\omega_L \in A_\infty(\Delta_0, \omega_{L_0})$, that is, there exists $1 < q < \infty$ such that $\omega_L \in RH_q(\Delta_0, \omega_{L_0})$. Here, q and the implicit constant depend only on dimension, the 1-sided NTA and CDC constants, the ellipticity constants of L_0 and L, and $\|\|\varrho(A, A_0)\|\|_{B_0}$.*
- (b) *Given $1 < p < \infty$, there exists $\varepsilon_p > 0$ (depending only on p, dimension, the 1-sided NTA and CDC constants and the ellipticity constants of L_0 and L) such that if one has $\|\|\varrho(A, A_0)\|\|_{B_0} \leq \varepsilon_p$, then $\omega_L \in RH_p(\Delta_0, \omega_{L_0})$, with the implicit constant depending only on p, dimension, the 1-sided NTA and CDC constants, and the ellipticity constant of L_0 and L.*

Assuming this result momentarily we can prove Theorem 1.5:

PROOF OF THEOREM 1.5, PART (a).
Case 1: Ω bounded.
For every $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$, we clearly have $\|\|\varrho(A, A_0)\|\|_{B_0} \leq \|\|\varrho(A, A_0)\|\| < \infty$. We can then invoke Proposition 3.1 part (a) to find q, $1 < q < \infty$, such that $\omega_L \in RH_q(\Delta_0, \omega_{L_0})$. Moreover, since $\sup_{B_0} \|\|\varrho(A, A_0)\|\|_{B_0} \leq \|\|\varrho(A, A_0)\|\|$ then the same q is valid for every B_0 and also $\sup_{\Delta_0} [\omega_L]_{RH_q(\Delta_0, \omega_{L_0})} < \infty$. This means that $\omega_L \in RH_q(\partial\Omega, \omega_{L_0})$ and hence $\omega_L \in A_\infty(\partial\Omega, \omega_{L_0})$.

Case 2: Ω unbounded.
Fix $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$. From Lemma 2.54, we know that every T_Δ is a 1-sided NTA domain satisfying the CDC and moreover all the implicit constants depend on the corresponding ones for Ω. Write $c_0^\star$ for the associated Corkscrew constant (which is independent of Δ), set $K = \max\{1, c_0^\star/c_0\}$ and fix $M > 16K \geq 16$. We have two sub-cases:

Case 2a: $0 < r_0 < \operatorname{diam}(\partial\Omega)/(2M)$.

Set $\widehat{B}_0 = MB_0$, so that $r_{\widehat{B}_0} < \operatorname{diam}(\partial\Omega)/2$, and let $\widehat{\Delta}_0 = \widehat{B}_0 \cap \partial\Omega$. Define $\Omega_\star = T_{\widehat{\Delta}_0} \subset \Omega$, and our goal is to apply Proposition 3.1 in this bounded domain. From Lemma 2.54, it follows that $\Omega_\star$ is a 1-sided NTA domain satisfying the CDC and moreover all the implicit constants depend on the corresponding ones for Ω but are uniform on M. In particular, the interior Corkscrew condition holds with $c_0^\star$ (which does not depend on M).

Write $\widetilde{B}_0 = B(x_0, \widetilde{r}_0) = B(x_0, Kr_0)$ so that $8B_0 \subset 8\widetilde{B}_0 \subset \widehat{B}_0$, and set $\widetilde{\Delta}_0 = \widetilde{B}_0 \cap \partial\Omega$, $\widetilde{\Delta}_0^\star = \widetilde{B}_0 \cap \partial\Omega_\star$, and $\Delta_0^\star := B_0 \cap \partial\Omega_\star$. Note that by (2.49) we have $8\widetilde{B}_0 \cap \Omega \subset \widehat{B}_0 \cap \Omega \subset T_{\widehat{\Delta}_0} = \Omega_\star$ and hence $8\widetilde{\Delta}_0 = 8\widetilde{\Delta}_0^\star$. Moreover, one can also see that for every $X \in 4\widetilde{B}_0 \cap \Omega = 4\widetilde{B}_0 \cap \Omega_\star$ then $\delta(X) = \operatorname{dist}(X, \partial\Omega_\star) =: \delta_\star(X)$. Consequently, if $X_{\Delta_0^\star}$ denotes the Corkscrew point relative to $\Delta_0^\star$ for the domain $\Omega_\star$ and $X_{\widetilde{\Delta}_0}$ denotes the Corkscrew point relative to $\widetilde{\Delta}_0$ for the domain Ω we have

$$c_0^\star r_0 \le \delta_\star(X_{\Delta_0^\star}) = \delta(X_{\Delta_0^\star}) \le r_0, \quad c_0 r_0 \le \delta(X_{\widetilde{\Delta}_0}) = \delta_\star(X_{\widetilde{\Delta}_0}) \le r_0,$$

and $|X_{\Delta_0^\star} - X_{\widetilde{\Delta}_0}| \le (1+K)r_0$.

Fix $x \in 2\Delta_0$, $0 < r < r_0 c_0^\star/4$, write $B = B(x,r)$, $\Delta = B \cap \partial\Omega$, $\Delta^\star = B \cap \partial\Omega_\star$, and note that from the above observations $\Delta = \Delta^\star$. Invoking Lemma 2.69 part (e), the Harnack chain condition for $\Omega_\star$ allows us to obtain

$$\omega_{L_0,\Omega_\star}^{X_{\Delta_0^\star}}(\Delta^\star) \approx \omega_{L_0,\Omega_\star}^{X_{\widetilde{\Delta}_0}}(\Delta) \approx \omega_{L_0,\Omega}^{X_{\widetilde{\Delta}_0}}(\Delta).$$

On the other hand if $Y \in B \cap \Omega_\star = B \cap \Omega$ and we pick $y \in \partial\Omega$ so that $|Y - y| = \delta(Y) = \delta_\star(Y) < r_0$. Write $B_Y = B(y, 2\delta(Y))$ which satisfies $B_Y \subset 5B_0$ and hence $\Delta_Y := B_Y \cap \partial\Omega = B_Y \cap \partial\Omega_\star =: \Delta_Y^\star$. Then if X_{Δ_Y} (respectively $X_{\Delta_Y^\star}$) stands for the Corkscrew point relative to Δ_Y (respectively $\Delta_Y^\star$) with respect to Ω (respectively $\Omega_\star$) we observe that

$$\begin{aligned} G_{L_0,\Omega_\star}(X_{\Delta_0^\star}, Y) &\approx G_{L_0,\Omega_\star}(X_{\Delta_0^\star}, X_{\Delta_Y^\star}) \approx \delta(Y)^{1-n} \omega_{L_0,\Omega_\star}^{X_{\Delta_0^\star}}(\Delta_Y^\star) \\ &\approx \delta(Y)^{1-n} \omega_{L_0,\Omega_\star}^{X_{\widetilde{\Delta}_0}}(\Delta_Y) \approx \delta(Y)^{1-n} \omega_{L_0,\Omega}^{X_{\widetilde{\Delta}_0}}(\Delta_Y) \\ &\approx G_{L_0,\Omega}(X_{\widetilde{\Delta}_0}, X_{\Delta_Y}) \approx G_{L_0,\Omega}(X_{\widetilde{\Delta}_0}, Y), \end{aligned}$$

where we have used the Harnack chain condition in both Ω and $\Omega_\star$, Harnack's inequality, and Lemma 2.69 parts (b) and (e). Finally,

$$\varrho_\star(A, A_0)(Y) := \|A - A_0\|_{B(Y, \delta_\star(Y)/2)} = \|A - A_0\|_{B(Y,\delta(Y)/2)} = \varrho(A, A_0)(Y)$$

since $Y \in B \cap \Omega \subset 4\widetilde{B}_0 \cap \Omega = 4\widetilde{B}_0 \cap \Omega_\star$ and hence $\delta(Y) = \delta_\star(Y)$.

At this point we collect the previous estimates to obtain that

$$\begin{aligned} &|||\varrho(A, A_0)|||_{B_0,\Omega_\star} \\ &\quad := \sup_{\substack{B=B(x,r) \\ x \in \Delta_0^\star, 0<r<r_0 c_0^\star/4}} \frac{1}{\omega_{L_0,\Omega_\star}^{X_{\Delta_0^\star}}(\Delta^\star)} \iint_{B\cap\Omega_\star} \varrho_\star(A, A_0)(X)^2 \frac{G_{L_0,\Omega_\star}(X_{\Delta_0^\star}, X)}{\delta_\star(X)^2}\, dX \\ &\quad \lesssim \sup_{\substack{B=B(x,r) \\ x \in \widetilde{\Delta}_0, 0<r<\widetilde{r}_0 c_0/4}} \frac{1}{\omega_{L_0,\Omega}^{X_{\widetilde{\Delta}_0}}(\Delta)} \iint_{B\cap\Omega} \varrho(A, A_0)(X)^2 \frac{G_{L_0,\Omega}(X_{\widetilde{\Delta}_0}, Y)}{\delta(X)^2}\, dX \\ &\quad \le |||\varrho(A, A_0)||| < \infty, \end{aligned}$$

where all the implicit constants are independent of M and uniform in B_0. We can then invoke Proposition 3.1 part (a) (since $\Omega_\star$ is bounded) to find q, $1 < q < \infty$, such that $\omega_{L,\Omega_\star} \in RH_q(\Delta_0, \omega_{L_0,\Omega_\star})$. On the other hand, by Lemma 2.69 part (e) we have that $\omega_{L,\Omega_\star}$ and $\omega_{L,\Omega}$ are comparable in Δ_0 and so are $\omega_{L_0,\Omega_\star}$ and $\omega_{L_0,\Omega}$. Thus eventually, $\omega_{L,\Omega} \in RH_q(\Delta_0, \omega_{L_0,\Omega})$. Moreover, the previous estimate is independent of B_0 and the same q is valid for every B_0 as in the present case.

Case 2b: $\operatorname{diam}(\partial\Omega)/(2M) < r_0 < \operatorname{diam}(\partial\Omega)$.

Note first that this case is vacuous if $\partial\Omega$ is unbounded. Hence we may assume that $\partial\Omega$ is bounded. We first find a finite maximal collection of points $\{x_j\}_{j=1}^J \in \Delta_0$ with $1 \le J \le (1+20M)^{n+1}$ such that $|x_j - x_k| \ge \operatorname{diam}(\partial\Omega)/(10M)$ for $1 \le j < k \le J$. For any of the balls $B_j = B(x_j, \operatorname{diam}(\partial\Omega)/(10M))$ by **Case 2a** we have that $\omega_L \in RH_q(3\Delta_j, \omega_{L_0})$ where the implicit constants do not depend on j, and we have written $\omega_{L_0} = \omega_{L_0,\Omega}$ and $\omega_L = \omega_{L,\Omega}$.

To show that $\omega_L \in RH_q(\Delta_0, \omega_{L_0})$, let $B = B(x,r) \subset B_0$ with $x \in \partial\Omega$ and $\Delta = B \cap \partial\Omega$. If $\Delta \cap \Delta_j \neq \emptyset$ and $0 < r < \operatorname{diam}(\partial\Omega)/(10M)$ we note that $\Delta \cap \Delta_j \subset \Delta \subset 3\Delta_j$ and thus

$$\left(\frac{1}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)} \int_{\Delta\cap\Delta_j} h(y; L, L_0, X_{\Delta_0})^q d\omega_{L_0}^{X_{\Delta_0}}(y)\right)^{\frac{1}{q}}$$
$$\lesssim \left(\fint_{\Delta} h(y; L, L_0, X_{3\Delta_j})^q d\omega_{L_0}^{X_{3\Delta_j}}(y)\right)^{\frac{1}{q}} \lesssim \frac{\omega_L^{X_{3\Delta_j}}(\Delta)}{\omega_{L_0}^{X_{3\Delta_j}}(\Delta)} \approx \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)},$$

where we have used Harnack's inequality and that $\omega_L \in RH_q(3\Delta_j, \omega_{L_0})$. On the other hand, if $\Delta \cap \Delta_j \neq \emptyset$ and $\operatorname{diam}(\partial\Omega)/(10M) < r < r_0$ we have that $r \approx r_0 \approx \operatorname{diam}(\partial\Omega)$. Thus, by Lemma 2.69 parts (a), (b), and (c), $\omega_{L_0}^{X_{\Delta_0}}(\Delta) \approx \omega_{L_0}^{X_{\Delta_j}}(\Delta_j) \approx 1$ and the same occurs for ω_L. These yield

$$\left(\frac{1}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)} \int_{\Delta\cap\Delta_j} h(y; L, L_0, X_{\Delta_0})^q d\omega_{L_0}^{X_{\Delta_0}}(y)\right)^{\frac{1}{q}}$$
$$\lesssim \left(\fint_{\Delta_j} h(y; L, L_0, X_{\Delta_j})^q d\omega_{L_0}^{X_{\Delta_j}}(y)\right)^{\frac{1}{q}} \lesssim \frac{\omega_L^{X_{\Delta_j}}(\Delta_j)}{\omega_{L_0}^{X_{\Delta_j}}(\Delta_j)} \approx 1 \approx \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)},$$

where we have used Harnack's inequality and the fact that $\omega_L \in RH_q(3\Delta_j, \omega_{L_0})$. All these, the fact $\Delta \subset \bigcup_j \Delta_j \cap \Delta$, and the bound $J \le (1+2M)^{n+1}$ imply

$$\left(\fint_{\Delta} h(y; L, L_0, X_{\Delta_0})^q d\omega_{L_0}^{X_{\Delta_0}}(y)\right)^{\frac{1}{q}}$$
$$\le \left(\sum_{j=1}^{J} \frac{1}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)} \int_{\Delta\cap\Delta_j} h(y; L, L_0, X_{\Delta_0})^q d\omega_{L_0}^{X_{\Delta_0}}(y)\right)^{\frac{1}{q}} \lesssim \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\omega_{L_0}^{X_{\Delta_0}}(\Delta)},$$

which eventually shows $\omega_{L,\Omega} \in RH_q(\Delta_0, \omega_{L_0,\Omega})$ in the current case.

Collecting **Case 2a** and **Case 2b** we have shown that $\omega_{L,\Omega} \in RH_q(\Delta_0, \omega_{L_0,\Omega})$ uniformly on Δ_0 which eventually means that $\omega_{L,\Omega} \in RH_q(\partial\Omega, \omega_{L_0,\Omega})$ and hence $\omega_{L,\Omega} \in A_\infty(\partial\Omega, \omega_{L_0,\Omega})$. This completes the proof.

□

Proof of Theorem 1.5, part (b). We follow the same argument as in the previous proof using part (b) in place of part (a) in Proposition 3.1. Further details are left to the interested reader. □

Proof of Theorem 1.10. Fix $\alpha > 0$. It is immediate to see that parts (a) and (b) follow respectively from parts (a) and (b) in Theorem 1.5 and the following estimate:

$$\|\!|\varrho(A, A_0)|\!\| \lesssim_\alpha \|\mathcal{A}_\alpha(\varrho(A, A_0))\|^2_{L^\infty(\omega_{L_0})}, \tag{3.3}$$

where, as explained in Remark 1.12, the pole for ω_{L_0} needs not to be specified. Hence everything reduces to obtaining such estimate. With this goal in mind, fix $\Delta_0 = B_0 \cap \partial\Omega$ with $B_0 = B(x_0, r_0)$, $x_0 \in \partial\Omega$, and $0 < r_0 < \operatorname{diam}(\partial\Omega)$. Let $\Delta = B \cap \partial\Omega$ with $B = B(x, r)$, $x \in 2\Delta_0$, and $0 < r < r_0 c_0/4$, here c_0 is the Corkscrew constant. Write $X_0 = X_{\Delta_0}$ and $\omega_0 = \omega_{L_0}^{X_0}$. Note that this choice guarantees that $X_0 \notin 4B$. Define

$$\mathcal{W}_B = \{I \in \mathcal{W} : I \cap B \neq \emptyset\}$$

and for every $I \in \mathcal{W}_B$ let $X_I \in I \cap B$ so that $4\operatorname{diam}(I) \leq \operatorname{dist}(I, \partial\Omega) \leq \delta(X_I) < r$ and hence $I \subset \frac{5}{4}B$. Pick $x_I \in \partial\Omega$ such that $|X_I - x_I| = \delta(X_I) \leq \operatorname{diam}(I) + \operatorname{dist}(I, \partial\Omega)$ and let $Q_I \in \mathbb{D}$ be such that $x_I \in Q_I$ and $\ell(I) = \ell(Q_I)$. By Lemma 2.69 parts (a)–(c) and Harnack's inequality one can show that

$$\omega_0(Q_I) \approx \ell(I)^{n-1} G_{L_0}(X_0, X_I) \approx \delta(Y)^{n-1} G_{L_0}(X_0, Y), \qquad \text{for every } Y \in I.$$

Then,

$$\begin{aligned}
\mathcal{I}_B :&= \iint_{B\cap\Omega} \varrho(A, A_0)(Y)^2 \frac{G_{L_0}(X_0, Y)}{\delta(Y)^2}\, dY \\
&\lesssim \sum_{I \in \mathcal{W}_B} \iint_{B \cap I} \frac{\varrho(A, A_0)(Y)^2}{\delta(Y)^{n+1}}\, dY\, \omega_0(Q_I) \\
&= \iint_{B\cap\Omega} \frac{\varrho(A, A_0)(Y)^2}{\delta(Y)^{n+1}} \sum_{I \in \mathcal{W}_B} \mathbf{1}_I(Y)\, \omega_0(Q_I)\, dY.
\end{aligned}$$

Fix $Y \in B$ and note that by the nature of the Whitney cubes one has $\#\{I \in \mathcal{W}_B : I \ni Y\} \leq C_n$ for some dimensional constant (indeed the I's have non-overlapping interiors and hence for a.e. $Y \in \Omega$, there is just one I_Y containing Y). Pick $y \in \partial\Omega$ such that $|Y - y| = \delta(Y)$. Let $z \in Q_I$, then by (2.34) and (2.40)

$$\begin{aligned}
|z - y| &\leq |z - x_I| + |x_I - X_I| + |X_I - Y| + |Y - y| \\
&\leq \Xi\ell(Q_I) + \delta(X_I) + \operatorname{diam}(I) + \delta(Y) < 3\Xi\delta(Y)
\end{aligned}$$

and therefore $Q_I \subset \Delta(y, 3\Xi\delta(Y))$. Note also that

$$\Delta(y, \alpha\delta(Y)) \subset B(Y, (1+\alpha)\delta(Y)) \cap \partial\Omega \subset (2+\alpha)\Delta.$$

Then using Lemma 2.69 parts (a) and (c)

$$\begin{aligned}
\sum_{I \in \mathcal{W}_B} \mathbf{1}_I(Y)\, \omega_0(Q_I) &\leq C_n \omega_0\big(\Delta(y, 3\Xi\delta(Y))\big) \\
&\lesssim_\alpha \omega_0\big(\Delta(y, \alpha\delta(Y))\big) \leq \omega_0\big(B(Y, (1+\alpha)\delta(Y)) \cap \partial\Omega\big).
\end{aligned}$$

Hence, using again Lemma 2.69 parts (a) and (c), and Harnack's inequality we conclude:

$$
\begin{aligned}
\mathcal{I}_B &\lesssim_\alpha \iint_{B\cap\Omega} \frac{\varrho(A,A_0)(Y)^2}{\delta(Y)^{n+1}} \omega_0\big(B(Y,(1+\alpha)\delta(Y))\cap\partial\Omega\big)\,dY \\
&= \int_{(2+\alpha)\Delta}\iint_{B\cap\Omega} \frac{\varrho(A,A_0)(Y)^2}{\delta(Y)^{n+1}} \mathbf{1}_{B(Y,(1+\alpha)\delta(Y))\cap\partial\Omega}(z)\,dY\,d\omega_0(z) \\
&\le \int_{(2+\alpha)\Delta}\iint_{\Gamma_\alpha(z)} \frac{\varrho(A,A_0)(Y)^2}{\delta(Y)^{n+1}}\,d\omega_0(z) \\
&= \int_{(2+\alpha)\Delta} \mathcal{A}_\alpha(\varrho(A,A_0))(z)^2\,d\omega_0(z) \\
&\lesssim \|\mathcal{A}_\alpha(\varrho(A,A_0))\|^2_{L^\infty(\omega_0)}\omega_0((2+\alpha)\Delta) \\
&\lesssim_\alpha \|\mathcal{A}_\alpha(\varrho(A,A_0))\|^2_{L^\infty(\omega_0)}\omega_0(\Delta).
\end{aligned}
$$

This eventually shows (3.3) and this completes the proof of Theorem 1.10. □

3.1. Auxiliary results

We next state some auxiliary lemmas which will be needed for our arguments.

LEMMA 3.4. *Let Ω be a* ***bounded*** *1-sided NTA domain satisfying the CDC. Consider $L_0 = -\operatorname{div}(A_0\nabla)$ and $L = -\operatorname{div}(A\nabla)$ two real (non-necessarily symmetric) elliptic operators, and let $u_0 \in W^{1,2}(\Omega)$ be a weak solution to $L_0u_0 = 0$ in Ω. Then,*

$$
\iint_\Omega A_0^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla u_0(Y)\,dY = 0, \qquad \text{for a.e. } X\in\Omega. \tag{3.5}
$$

PROOF. We follow the argument in [**CHM19**, Lemma 3.12] where it was assumed that $\partial\Omega$ is AR and the operators were symmetric. Pick $\varphi \in C_0^\infty(\mathbb{R})$ with $\mathbf{1}_{(0,1)} \le \varphi \le \mathbf{1}_{(0,2)}$. Fix $X_0\in\Omega$, for each $0<\varepsilon<\delta(X_0)/16$ we set $\varphi_\varepsilon(X) = \varphi(|X-X_0|/\varepsilon)$ and $\psi_\varepsilon = 1-\varphi_\varepsilon$. By (2.68), one has that $G_{L^\top}(\cdot,X_0)\psi_\varepsilon \in W_0^{1,2}(\Omega)$, which together with the assumption that $u_0\in W^{1,2}(\Omega)$ is a weak solution to $L_0u_0 = 0$ in Ω, allows us to see that

$$
\iint_\Omega A_0^\top(Y)\nabla\big(G_{L^\top}(\cdot,X_0)\psi_\varepsilon\big)(Y)\cdot\nabla u_0(Y)\,dY = 0.
$$

As a consequence,

$$
\begin{aligned}
\iint_\Omega A_0^\top\nabla G_{L^\top}(\cdot,X_0)\cdot\nabla u_0\,dY &= \iint_\Omega A_0^\top\nabla\big(G_{L^\top}(\cdot,X_0)\varphi_\varepsilon\big)\cdot\nabla u_0\,dY \\
&= \iint_\Omega A_0^\top\nabla G_{L^\top}(\cdot,X_0)\cdot\nabla u_0\,\varphi_\varepsilon\,dY + \iint_\Omega A_0^\top\nabla\varphi_\varepsilon\cdot\nabla u_0\,G_{L^\top}(\cdot,X_0)\,dY \\
&=: \mathcal{I}_\varepsilon + \mathcal{II}_\varepsilon.
\end{aligned}
$$

For the first term, we use (1.1), Cauchy-Schwarz's inequality, Caccioppoli's inequality for $G_{L^\top}(\cdot,X_0)$ (which satisfies $L^\top G_{L^\top}(\cdot,X_0) = 0$ in the weak sense in $\Omega\setminus\{X_0\}$), and (2.60)

$$
|\mathcal{I}_\varepsilon| \lesssim \iint_{B(X_0,2\varepsilon)} |\nabla G_{L^\top}(\cdot,X_0)|\,|\nabla u_0|\,dY
$$

$$\lesssim \sum_{j=0}^{\infty}\left(\iint_{2^{-j}\varepsilon\le|Y-X_0|<2^{-j+1}\varepsilon}|\nabla_Y G_{L^\top}(Y,X_0)|^2\,dY\right)^{\frac12}$$
$$\times\left(\iint_{B(X_0,2^{-j+1}\varepsilon)}|\nabla u_0|^2\,dY\right)^{\frac12}$$
$$\lesssim M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0)$$
$$\times\sum_{j=1}^{\infty}(2^{-j}\varepsilon)^{\frac{n-1}{2}}\left(\iint_{2^{-j-1}\varepsilon\le|Y-X_0|<2^{-j+2}\varepsilon}|G_{L^\top}(Y,X_0)|^2\,dY\right)^{\frac12}$$
$$\lesssim \varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0),$$

where $M_2 f := M(|f|^2)^{\frac12}$, with M being the Hardy-Littlewood maximal operator on $\mathbb{R}^{n+1}$. For the second term, we invoke again (2.60) and Jensen's inequality:

$$|\mathcal{II}_\varepsilon|\lesssim\varepsilon^{-1}\iint_{\varepsilon\le|Y-X_0|<2\varepsilon}|G_{L^\top}(Y,X_0)|\,|\nabla u_0(Y)|\,dY$$
$$\lesssim\varepsilon^{-n}\iint_{B(X_0,2\varepsilon)}|\nabla u_0(Y)|\,dY\lesssim\varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0).$$

Combining the obtained estimates we have shown that, for every $X_0\in\Omega$ and for every $0<\varepsilon<\delta(X_0)/16$,

$$\left|\iint_\Omega A_0^\top\nabla G_{L^\top}(\cdot,X_0)\cdot\nabla u_0\,dY\right|\lesssim\varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0). \tag{3.6}$$

Since $u_0\in W^{1,2}(\Omega)$ it the follows that $M_2(|\nabla u_0|\mathbf{1}_\Omega)\in L^{2,\infty}(\mathbb{R}^{n+1})$, and as a result $M_2(|\nabla u_0|\mathbf{1}_\Omega)$ is finite almost everywhere in $\mathbb{R}^{n+1}$. Thus, we can let $\varepsilon\to 0^+$ in (3.6) to obtain the desired equality. □

LEMMA 3.7. *Let Ω be a* ***bounded*** *1-sided NTA domain satisfying the CDC. Let $L_0=-\operatorname{div}(A_0\nabla)$ and $L=-\operatorname{div}(A\nabla)$ be two real (non-necessarily symmetric) elliptic operators. Given $g\in\operatorname{Lip}(\partial\Omega)$, consider the solutions u_0 and u given by*

$$u_0(X)=\int_{\partial\Omega}g(y)\,d\omega_{L_0}^X(y),\qquad u(X)=\int_{\partial\Omega}g(y)\,d\omega_L^X(y),\qquad X\in\Omega.$$

Then,

$$u(X)-u_0(X)=\iint_\Omega(A_0-A)^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla u_0(Y)\,dY \tag{3.8}$$

for almost every $X\in\Omega$.

PROOF. We again follow the argument in [**CHM19**, Lemma 3.18] with some appropriate changes. Following [**HMT14**] we know that $u_0=\widetilde{g}-v_0$ and $u=\widetilde{g}-v$, where $\widetilde{g}\in\operatorname{Lip}_c(\mathbb{R}^{n+1})$ is a Lipschitz extension of g, and $v_0,v\in W_0^{1,2}(\Omega)$ are the Lax-Milgram solutions of $L_0v_0=L_0\widetilde{g}$ and $Lv=L\widetilde{g}$ respectively. Hence, we have that $u-u_0=v_0-v\in W_0^{1,2}(\Omega)$, and following again [**HMT14**] one can extend (2.65) so that

$$(u-u_0)(X)=\iint_\Omega A^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla(u-u_0)(Y)\,dY,\quad\text{for a.e. } X\in\Omega.$$

For almost every $X\in\Omega$ we then have that

$$(u-u_0)(X) - \iint_\Omega (A_0-A)^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla u_0(Y)\,dY$$
$$= \iint_\Omega A^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla u(Y)\,dY - \iint_\Omega A_0^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla u_0(Y)\,dY.$$

Using Lemma 3.4 for both terms the right side of the above equality vanishes almost everywhere, and this proves (3.8). □

For the following result, we recall the definition of the localized dyadic conical square function in (2.45). Also, if μ is a non-negative Borel measure on Q_0 so that $0<\mu(Q)<\infty$ for every $Q\in\mathbb{D}_{Q_0}$, we define the localized dyadic maximal function with respect to μ as

$$M^{\mathbf{d}}_{Q_0,\mu}\nu(x) := \sup_{x\in Q\in\mathbb{D}_{Q_0}} \frac{\nu(Q)}{\mu(Q)},$$

where ν is a non-negative Borel measure on Q_0.

LEMMA 3.9. *Let Ω be a 1-sided NTA domain satisfying the CDC and let $L_0 = -\operatorname{div}(A_0\nabla)$ and $L=-\operatorname{div}(A\nabla)$ be two real (non-necessarily symmetric) elliptic operators. Let $Q_0\in\mathbb{D}$ and let $\mathcal{F}=\{Q_j\}_j\subset\mathbb{D}_{Q_0}$ be a (possibly empty) family of pairwise disjoint dyadic cubes. Let $u_0\in W^{1,2}_{\rm loc}(\Omega)$, and let $0\le H\in L^\infty(\Omega)$. Let $Y_0\in\Omega\setminus B^*_{Q_0}$ (see (2.48)) and define $\gamma_{Y_0}=\{\gamma_{Y_0,Q}\}_{Q\in\mathbb{D}_{Q_0}}$ where*

$$\gamma_{Y_0,Q} := \omega^{Y_0}_{L_0}(Q)\sum_{I\in\mathcal{W}^*_Q}\|H\|^2_{L^\infty(I^*)}, \qquad Q\in\mathbb{D}_{Q_0}.$$

Then,

$$\iint_{\Omega_{\mathcal{F},Q_0}} H(Y)|\nabla_Y G_{L^\top}(Y,Y_0)|\,|\nabla u_0(Y)|dY \lesssim \|\mathfrak{m}_{\gamma_{Y_0},\mathcal{F}}\|^{\frac12}_{\mathcal{C}(Q_0,\omega^{Y_0}_{L_0})}\int_{Q_0} M^{\mathbf{d}}_{Q_0,\omega^{Y_0}_{L_0}}(\omega^{Y_0}_L)(x)\mathcal{S}_{Q_0}u_0(x)d\omega^{Y_0}_{L_0}(x). \tag{3.10}$$

PROOF. To ease the notation let us write $\omega_0:=\omega^{Y_0}_{L_0}$, $\omega:=\omega^{Y_0}_L$, $\gamma_{Y_0,Q}=\gamma_Q$, and $\gamma_{Y_0}=\gamma$. From the definition of $\Omega_{\mathcal{F},Q_0}$; Cauchy-Schwarz's, Caccioppoli's and Harnack's inequalities (applied to $G_{L^\top}(\cdot,Y_0)$ which satisfies $L^\top G_{L^\top}(\cdot,Y_0)=0$ in the weak sense in $\Omega\setminus\{Y_0\}$); the fact that $\ell(I)\approx\ell(Q)\approx\delta(Y)$ for every $Y\in I^*\in\mathcal{W}^*_Q$; (2.64); and Lemma 2.69 part (b) in conjunction with (2.48), we clearly have

$$\begin{aligned}
\mathcal{I}_0 &:= \iint_{\Omega_{\mathcal{F},Q_0}} H(Y)|\nabla_Y G_{L^\top}(Y,Y_0)|\,|\nabla u_0(Y)|dY\\
&\le \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\sum_{I\in\mathcal{W}^*_Q}\|H\|_{L^\infty(I^*)}\iint_{I^*}|\nabla_Y G_{L^\top}(Y,Y_0)|\,|\nabla u_0(Y)|dY\\
&\le \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\sum_{I\in\mathcal{W}^*_Q}\|H\|_{L^\infty(I^*)}\left(\iint_{I^*}|\nabla_Y G_{L^\top}(Y,Y_0)|^2dY\right)^{\frac12}\left(\iint_{I^*}|\nabla u_0(Y)|^2dY\right)^{\frac12}\\
&\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\sum_{I\in\mathcal{W}^*_Q}\|H\|_{L^\infty(I^*)}\ell(I)^n\frac{G_{L^\top}(X_Q,Y_0)}{\delta(X_Q)}\left(\iint_{I^*}|\nabla u_0(Y)|^2\delta(Y)^{1-n}dY\right)^{\frac12}
\end{aligned}$$

$$\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \sum_{I\in\mathcal{W}_Q^*} \|H\|_{L^\infty(I^*)}\omega(Q) \left(\iint_{I^*} |\nabla u_0(Y)|^2\delta(Y)^{1-n}dY\right)^{\frac12}$$

$$\le \sum_{Q\in\mathbb{D}_{Q_0}} \left(\omega_0(Q)\left(\frac{\omega(Q)}{\omega_0(Q)}\right)^2 \iint_{U_Q} |\nabla u_0(Y)|^2\delta(Y)^{1-n}dY\right)^{\frac12} \gamma_{\mathcal{F},Q}^{\frac12},$$

where in the last estimate we have used that the family $\{I^*\}_{I\in\mathcal{W}_Q^*}$ has bounded overlap. If we now set $\alpha = \{\alpha_Q\}_{Q\in\mathbb{D}_{Q_0}}$ with

$$\alpha_Q := \left(\omega_0(Q)\left(\frac{\omega(Q)}{\omega_0(Q)}\right)^2 \iint_{U_Q} |\nabla u_0(Y)|^2\delta(Y)^{1-n}dY\right)^{\frac12}, \qquad Q\in\mathbb{D}_{Q_0},$$

we obtain by invoking Lemma 2.15 with $\mu=\omega_0$

$$\mathcal{I}_0 \lesssim \sum_{Q\in\mathbb{D}_{Q_0}} \alpha_Q\gamma_{\mathcal{F},Q}^{\frac12} \le 4\int_{Q_0} \mathcal{A}_{Q_0}^{\omega_0}\alpha(x)\mathcal{B}_{Q_0}^{\omega_0}\big(\{\gamma_{\mathcal{F},Q}^{\frac12}\}_{Q\in\mathbb{D}_{Q_0}}\big)(x)\,d\omega_0(x).$$

Note that for every $x\in Q_0$

$$\mathcal{A}_{Q_0}^{\omega_0}\alpha(x) = \left(\sum_{x\in Q\in\mathbb{D}_{Q_0}} \left(\frac{\omega(Q)}{\omega_0(Q)}\right)^2 \iint_{U_Q} |\nabla u_0(Y)|^2\delta(Y)^{1-n}dY\right)^{\frac12}$$

$$\lesssim M_{Q_0,\omega_0}^{\mathbf{d}}\omega(x)\,\mathcal{S}_{Q_0}u_0(x)$$

where we have used that the family $\{U_Q\}_{Q\in\mathbb{D}_{Q_0}}$ has finite overlap. Besides, if $x\in Q_0$

$$\mathcal{B}_{Q_0}^{\omega_0}\big(\{\gamma_{\mathcal{F},Q}^{\frac12}\}_{Q\in\mathbb{D}_{Q_0}}\big)(x) = \sup_{x\in Q\in\mathbb{D}_{Q_0}} \left(\frac{1}{\omega_0(Q)}\sum_{Q'\in\mathbb{D}_Q}\gamma_{\mathcal{F},Q'}\right)^{\frac12} \le \|\mathfrak{m}_{\gamma,\mathcal{F}}\|_{\mathcal{C}(Q_0,\omega_0)}^{\frac12}.$$

Collecting all the obtained estimates completes the proof of (3.10). □

Our next auxiliary result adapts [**HMT17**, Lemma 4.44] to our current setting:

LEMMA 3.11. *Let $\Omega\subset\mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Given $Q_0\in\mathbb{D}$ and $N\ge 4$ consider the family of pairwise disjoint cubes $\mathcal{F}_N = \{Q\in\mathbb{D}_{Q_0} : \ell(Q) = 2^{-N}\ell(Q_0)\}$ and let $\Omega_N := \Omega_{\mathcal{F}_N,Q_0}$ and $\Omega_N^* := \Omega_{\mathcal{F}_N,Q_0}^*$. There exists $\Psi_N\in C_c^\infty(\mathbb{R}^{n+1})$ and a constant $C\ge 1$ depending only on dimension n, the 1-sided NTA constants, the CDC constant, and independent of N and Q_0 such that the following hold:*

(*i*) $C^{-1}\,\mathbf{1}_{\Omega_N}\le\Psi_N\le\mathbf{1}_{\Omega_N^*}$.

(*ii*) $\sup_{X\in\Omega}|\nabla\Psi_N(X)|\,\delta(X)\le C$.

(*iii*) *Setting*

(3.12)
$$\mathcal{W}_N := \bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}} \mathcal{W}_Q^*, \quad \mathcal{W}_N^\Sigma := \big\{I\in\mathcal{W}_N : \exists\, J\in\mathcal{W}\setminus\mathcal{W}_N \text{ with } \partial I\cap\partial J\neq\emptyset\big\}.$$

one has

$$\nabla\Psi_N\equiv 0 \quad in \quad \bigcup_{I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma} I^{**} \tag{3.13}$$

and there exists a family $\{\widehat{Q}_I\}_{I\in\mathcal{W}_N^\Sigma}$ *so that*

$$C^{-1}\,\ell(I)\le \ell(\widehat{Q}_I)\le C\,\ell(I),\quad \operatorname{dist}(I,\widehat{Q}_I)\le C\,\ell(I),\quad \sum_{I\in\mathcal{W}_N^\Sigma}\mathbf{1}_{\widehat{Q}_I}\le C. \tag{3.14}$$

Proof. We proceed as in [**HMT17**, Lemma 4.44]. Recall that given I, any closed dyadic cube in $\mathbb{R}^{n+1}$, we set $I^*=(1+\lambda)I$ and $I^{**}=(1+2\,\lambda)I$. Let us introduce $\widetilde{I^*}=(1+\frac{3}{2}\,\lambda)I$ so that

$$I^*\subsetneq \operatorname{int}(\widetilde{I^*})\subsetneq \widetilde{I^*}\subset \operatorname{int}(I^{**}). \tag{3.15}$$

Given $I_0:=[-\frac12,\frac12]^{n+1}\subset\mathbb{R}^{n+1}$, fix $\phi_0\in C_c^\infty(\mathbb{R}^{n+1})$ such that $1_{I_0^*}\le\phi_0\le 1_{\widetilde{I_0^*}}$ and $|\nabla\phi_0|\lesssim 1$ (the implicit constant depends on the parameter λ). For every $I\in\mathcal{W}=\mathcal{W}(\Omega)$ we set $\phi_I(\cdot)=\phi_0\big(\frac{\cdot\,-X(I)}{\ell(I)}\big)$ so that $\phi_I\in C^\infty(\mathbb{R}^{n+1})$, $1_{I^*}\le\phi_I\le 1_{\widetilde{I^*}}$ and $|\nabla\phi_I|\lesssim \ell(I)^{-1}$ (with implicit constant depending only on n and λ).

For every $X\in\Omega$, we let $\Phi(X):=\sum_{I\in\mathcal{W}}\phi_I(X)$. It then follows that $\Phi\in C^\infty_{\rm loc}(\Omega)$ since for every compact subset of Ω, the previous sum has finitely many non-vanishing terms. Also, $1\le\Phi(X)\lesssim C_\lambda$ for every $X\in\Omega$ since the family $\{\widetilde{I^*}\}_{I\in\mathcal{W}}$ has bounded overlap by our choice of λ. Hence we can set $\Phi_I=\phi_I/\Phi$ and one can easily see that $\Phi_I\in C_c^\infty(\mathbb{R}^{n+1})$, $C_\lambda^{-1}1_{I^*}\le\Phi_I\le 1_{\widetilde{I^*}}$ and $|\nabla\Phi_I|\lesssim\ell(I)^{-1}$. With this in hand set

$$\Psi_N(X):=\sum_{I\in\mathcal{W}_N}\Phi_I(X)=\frac{\sum\limits_{I\in\mathcal{W}_N}\phi_I(X)}{\sum\limits_{I\in\mathcal{W}}\phi_I(X)},\qquad X\in\Omega.$$

We first note that the number of terms in the sum defining Ψ_N is bounded depending on N. Indeed, if $Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}$ then $Q\in\mathbb{D}_{Q_0}$ and $2^{-N}\ell(Q_0)<\ell(Q)\le\ell(Q_0)$ which implies that $\mathbb{D}_{\mathcal{F}_N,Q_0}$ has finite cardinality with bounds depending only on the AR property and N (here we recall that the number of dyadic children of a given cube is uniformly controlled). Also, by construction $\mathcal{W}_Q^*$ has cardinality depending only on the allowable parameters. Hence, $\#\mathcal{W}_N\lesssim C_N<\infty$. This and the fact that each $\Phi_I\in C_c^\infty(\mathbb{R}^{n+1})$ yield that $\Psi_N\in C_c^\infty(\mathbb{R}^{n+1})$. Note also that (3.15) and the definition of $\mathcal{W}_N$ give

$$\begin{aligned}\operatorname{supp}\Psi_I\subset\bigcup_{I\in\mathcal{W}_N}\widetilde{I^*}=\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}}\bigcup_{I\in\mathcal{W}_Q^*}\widetilde{I^*}\subset\operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}}\bigcup_{I\in\mathcal{W}_Q^*}I^{**}\Big)\\ =\operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_N,Q_0}}U_Q^*\Big)=\Omega_N^*\end{aligned}$$

This, the fact that $\mathcal{W}_N\subset\mathcal{W}$, and the definition of Ψ_N immediately give that $\Psi_N\le\mathbf{1}_{\Omega_N^*}$. On the other hand if $X\in\Omega_N=\Omega_{\mathcal{F}_N,\widetilde{Q}_0}$ then the exists $I\in\mathcal{W}_N$ such that $X\in I^*$ in which case $\Psi_N(X)\ge\Phi_I(X)\ge C_\lambda^{-1}$. All these imply (i). Note that (ii) follows by observing that for every $X\in\Omega$

$$|\nabla\Psi_N(X)|\le\sum_{I\in\mathcal{W}_N}|\nabla\Phi_I(X)|\lesssim\sum_{I\in\mathcal{W}}\ell(I)^{-1}\,1_{\widetilde{I^*}}(X)\lesssim\delta(X)^{-1}$$

where we have used that if $X\in\widetilde{I^*}$ then $\delta(X)\approx\ell(I)$ and also that the family $\{\widetilde{I^*}\}_{I\in\mathcal{W}}$ has bounded overlap.

To see (iii) fix $I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma$ and $X\in I^{**}$, and set $\mathcal{W}_X:=\{J\in\mathcal{W}:\phi_J(X)\neq0\}$ so that $I\in\mathcal{W}_X$. We first note that $\mathcal{W}_X\subset\mathcal{W}_N$. Indeed, if $\phi_J(X)\neq0$ then $X\in\widetilde{J^*}$.

Hence $X \in I^{**} \cap J^{**}$ and our choice of λ gives that ∂I meets ∂J, this in turn implies that $J \in \mathcal{W}_N$ since $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. All these yield

$$\Psi_N(X) = \frac{\sum_{J \in \mathcal{W}_N} \phi_J(X)}{\sum_{J \in \mathcal{W}} \phi_J(X)} = \frac{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)}{\sum_{J \in \mathcal{W}_X} \phi_J(X)} = \frac{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)}{\sum_{J \in \mathcal{W}_N \cap \mathcal{W}_X} \phi_J(X)} = 1.$$

Hence $\Psi_N\big|_{I^{**}} \equiv 1$ for every $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. This and the fact that $\Psi_N \in C_c^\infty(\mathbb{R}^{n+1})$ immediately give that $\nabla \Psi_N \equiv 0$ in $\bigcup_{I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma} I^{**}$.

We are left with showing the last part of (iv) and for that we borrow some ideas from [**HMM16**, Appendix A.2]. Fix $I \in \mathcal{W}_N^\Sigma$ and let J be so that $J \in \mathcal{W} \setminus \mathcal{W}_N$ with $\partial I \cap \partial J \neq \emptyset$, in particular $\ell(I) \approx \ell(J)$. Since $I \in \mathcal{W}_N^\Sigma$ there exists $Q_I \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ (that is, $Q_I \subset Q_0$ with $2^{-N}\ell(Q_0) < \ell(Q_I) \leq \ell(Q_0)$ so that $I \in \mathcal{W}_{Q_I}^*$. Pick $Q_J \in \mathbb{D}$ so that $\ell(Q_J) = \ell(J)$ and it contains any fixed $\widehat{y} \in \partial\Omega$ such that $\operatorname{dist}(J, \partial\Omega) = \operatorname{dist}(J, \widehat{y})$. Then, as observed in Section 2.5, one has $J \in \mathcal{W}_{Q_J}^*$. But, since $J \in \mathcal{W} \setminus \mathcal{W}_N$, we necessarily have $Q_J \notin \mathbb{D}_{\mathcal{F}_N, Q_0} = \mathbb{D}_{\mathcal{F}_N} \cap \mathbb{D}_{Q_0}$. Hence, $\mathcal{W}_N^\Sigma = \mathcal{W}_N^{\Sigma,1} \cup \mathcal{W}_N^{\Sigma,2} \cup \mathcal{W}_N^{\Sigma,3}$ where

$$\begin{aligned}
\mathcal{W}_N^{\Sigma,1} &:= \{I \in \mathcal{W}_N^\Sigma : Q_0 \subset Q_J\}, \\
\mathcal{W}_N^{\Sigma,2} &:= \{I \in \mathcal{W}_N^\Sigma : Q_J \subset Q_0,\ \ell(Q_J) \leq 2^{-N}\ell(Q_0)\}, \\
\mathcal{W}_N^{\Sigma,3} &:= \{I \in \mathcal{W}_N^\Sigma : Q_J \cap Q_0 = \emptyset\}.
\end{aligned}$$

For later use it is convenient to observe that

$$\operatorname{dist}(Q_J, I) \leq \operatorname{dist}(Q_J, J) + \operatorname{diam}(J) + \operatorname{diam}(I) \approx \ell(J) + \ell(I) \approx \ell(I). \tag{3.16}$$

Let us first consider $\mathcal{W}_N^{\Sigma,1}$. If $I \in \mathcal{W}_N^{\Sigma,1}$ we clearly have

$$\ell(Q_0) \leq \ell(Q_J) = \ell(J) \approx \ell(I) \approx \ell(Q_I) \leq \ell(Q_0)$$

and since $Q_I \in \mathbb{D}_{Q_0}$

$$\operatorname{dist}(I, x_{Q_0}) \lesssim \operatorname{diam}(I) + \operatorname{dist}(I, Q_I) + \operatorname{diam}(Q_I) \approx \ell(I).$$

In particular, $\#\mathcal{W}_N^{\Sigma,1} \lesssim 1$. Thus if we set $\widehat{Q}_I := Q_J$ it follows from (3.16) that the two first conditions in (3.14) hold and also $\sum_{I \in \mathcal{W}_N^{\Sigma,1}} \mathbf{1}_{\widehat{Q}_I} \leq \#\mathcal{W}_N^{\Sigma,1} \lesssim 1$.

Consider next $\mathcal{W}_N^{\Sigma,2}$. For any $I \in \mathcal{W}_N^{\Sigma,2}$ we also set $\widehat{Q}_I := Q_J$ so that from (3.16) we clearly see that the two first conditions in (3.14) hold. It then remains to estimate the overlap. With this goal in mind we first note that if $I \in \mathcal{W}_N^{\Sigma,2}$, the fact that $Q_I \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ yields

$$2^{-N}\ell(Q_0) < \ell(Q_I) \approx \ell(I) \approx \ell(J) \approx \ell(Q_J) \leq 2^{-N}\ell(Q_0),$$

hence $\ell(I) \approx 2^{-N}\ell(Q_0)$. Suppose next that $Q_J \cap Q_J' = \widehat{Q}_I \cap \widehat{Q}_I \neq \emptyset$ for $I, I' \in \mathcal{W}_N^{\Sigma,2}$. Then since I touches J and I' touches J'

$$\begin{aligned}
\operatorname{dist}(I, I') \leq \operatorname{diam}(J) + \operatorname{dist}(J, Q_J) + \operatorname{diam}(Q_J) + \operatorname{diam}(Q_J') + \operatorname{diam}(J') \\
\approx \ell(J) + \ell(J') \approx 2^{-N}\ell(Q_0).
\end{aligned}$$

Hence fixed $I \in \mathcal{W}_N^{\Sigma,2}$ there is a uniformly bounded number of $I' \in \mathcal{W}_N^{\Sigma,2}$ with $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$, and, in particular, $\sum_{I \in \mathcal{W}_N^{\Sigma,2}} \mathbf{1}_{\widehat{Q}_I} \lesssim 1$.

We finally take into consideration the most delicate collection $\mathcal{W}_N^{\Sigma,3}$. In this case for every $I \in \mathcal{W}_N^{\Sigma,3}$ we pick $\widehat{Q}_I \in \mathbb{D}$ so that $\widehat{Q}_I \ni x_{Q_J}$ and $\ell(\widehat{Q}_I) = 2^{-M'}\ell(Q_J)$

with $M \geq 3$ large enough so that $2^{M'} \geq 2\Xi^2$ (cf. (2.34)). Note that since $M \geq 3$ we have that $\widehat{Q}_I \subset Q_J$ which, together with (3.16), implies

$$\operatorname{dist}(I, \widehat{Q}_I) \leq \operatorname{dist}(I, Q_J) + \operatorname{diam}(Q_J) \lesssim \ell(I).$$

Hence the first two conditions in (3.14) hold in the current situation.

On the other hand, the choice of M and (2.34) guarantee that

$$\operatorname{diam}(\widehat{Q}_I) \leq 2\,\Xi\, r_{\widehat{Q}_I} \leq 2\,\Xi\,\ell(\widehat{Q}_I) = 2^{-M'+1}\,\Xi\,\ell(Q_J) \leq \Xi^{-1}\,\ell(Q_J). \tag{3.17}$$

Also, since $\Delta_{Q_J} \subset Q_J$, it follows that $Q_0 \cap \Delta_{Q_J} = \emptyset$ and therefore $2\Xi^{-1}\,\ell(Q_J) \leq \operatorname{dist}(x_{Q_J}, Q_0)$. Besides, since $Q_I \subset Q_0$

$$\begin{aligned}\operatorname{dist}(x_{Q_J}, Q_0) &\leq \operatorname{diam}(Q_J) + \operatorname{dist}(Q_J, J) + \operatorname{diam}(J)\\ &\qquad + \operatorname{diam}(I) + \operatorname{dist}(I, Q_I) + \operatorname{diam}(Q_I) \approx \ell(J) \approx \ell(I).\end{aligned}$$

Thus, $2\,\Xi^{-1}\,\ell(Q_J) \leq \operatorname{dist}(x_{Q_J}, Q_0) \leq C\,\ell(J)$. Suppose next that $I, I' \in \mathcal{W}_N^{\Sigma,3}$ are so that $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$ and assume without loss of generality that $\widehat{Q}_{I'} \subset \widehat{Q}_I$, hence $\ell(J') \leq \ell(J)$. Then, since $x_{Q_J} \in \widehat{Q}_I$ and $x_{Q_{J'}} \in \widehat{Q}_{I'} \subset \widehat{Q}_I$ we get from (3.17)

$$\begin{aligned}2\,\Xi^{-1}\,\ell(Q_J) &\leq \operatorname{dist}(x_{Q_J}, Q_0) \leq |x_{Q_J} - x_{Q_{J'}}| + \operatorname{dist}(x_{Q_{J'}}, Q_0)\\ &\leq \operatorname{diam}(\widehat{Q}_I) + C\ell(J') \leq \Xi^{-1}\,\ell(Q_J) + C\ell(J')\end{aligned}$$

and therefore $\Xi^{-1}\,\ell(Q_J) \leq C\,\ell(J)$ which in turn gives $\ell(I) \approx \ell(J) \approx \ell(J') \approx \ell(I')$. Note also that since I touches J, I' touches J', and $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$ we obtain

$$\begin{aligned}\operatorname{dist}(I, I') &\leq \operatorname{diam}(J) + \operatorname{dist}(J, Q_J) + \operatorname{diam}(Q_J) + \operatorname{diam}(Q_{J'})\\ &\qquad + \operatorname{dist}(Q_{J'}, J') + \operatorname{diam}(J') \approx \ell(J) + \ell(J') \approx \ell(I).\end{aligned}$$

Consequently, fixed $I \in \mathcal{W}_N^{\Sigma,3}$ there is a uniformly bounded number of $I' \in \mathcal{W}_N^{\Sigma,3}$ with $\widehat{Q}_I \cap \widehat{Q}_{I'} \neq \emptyset$. As a result, $\sum_{I \in \mathcal{W}_N^{\Sigma,3}} \mathbf{1}_{\widehat{Q}_I} \lesssim 1$. This clearly completes the proof of (iii) and hence that of Lemma 3.11. □

LEMMA 3.18. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Given $Q_0 \in \mathbb{D}$ and let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint cubes. There exists $Y_{Q_0} \in \Omega \cap \Omega_{\mathcal{F},Q_0} \cap \Omega^*_{\mathcal{F},Q_0}$ so that*

$$\operatorname{dist}(Y_{Q_0}, \partial\Omega) \approx \operatorname{dist}(Y_{Q_0}, \partial\Omega_{\mathcal{F},Q_0}) \approx \operatorname{dist}(Y_{Q_0}, \partial\Omega^*_{\mathcal{F},Q_0}) \approx \ell(Q_0), \tag{3.19}$$

where the implicit constants depend only on dimension, the 1-sided NTA constants, the CDC constant, and is independent of Q_0 and $\mathcal{F}$.

PROOF. Note first that $\Omega_{\mathcal{F},Q_0}$ is a 1-sided NTA domain satisfying the CDC (see Lemma 2.54). Pick an arbitrary $x_0 \in \partial\Omega_{\mathcal{F},Q_0}$ and let Y_0 be a Corkscrew point relative to $B(x_0, \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0})/2) \cap \partial\Omega_{\mathcal{F},Q_0}$ for the bounded domain $\Omega_{\mathcal{F},Q_0}$ (recall that one has $\operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0}) \approx \ell(Q_0) < \infty$ by (2.48)). Note that $Y_0 \in \Omega_{\mathcal{F},Q_0} \subset \Omega$, which is comprised of fattened Whitney boxes, then $Y_0 \in I^{**}$ for some $I \in \mathcal{W}$, with $\operatorname{int}(I^{**}) \subset \Omega_{\mathcal{F},Q_0}$. Let $Y_{Q_0} = X(I)$ be the center of I so that $\delta(Y_0) \approx \ell(I) \approx \delta(Y_{Q_0})$. Then

$$\begin{aligned}\ell(Q_0) &\approx \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0}) \approx \operatorname{dist}(Y_0, \partial\Omega_{\mathcal{F},Q_0}) \leq \operatorname{dist}(Y_0, \partial\Omega_{\mathcal{F},Q_0^*}) \leq \delta(Y_0)\\ &\approx \delta(Y_{Q_0}) \approx \ell(I) \leq \operatorname{diam}(\Omega_{\mathcal{F},Q_0}) = \operatorname{diam}(\partial\Omega_{\mathcal{F},Q_0}) \approx \ell(Q_0).\end{aligned}$$

This completes the proof. □

LEMMA 3.20. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Given $Q_0 \in \mathbb{D}$ and $f \in C(\partial\Omega)$ with $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$ let*

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \partial\Omega.$$

Then for every $x \in Q_0$,

$$\mathcal{N}_{Q_0} u(x) \lesssim \sup_{\substack{\Delta \ni x \\ 0<r_\Delta<4\Xi r_{Q_0}}} \fint_\Delta |f(y)|\, d\omega_L^{X_{Q_0}}(y), \tag{3.21}$$

and, as a consequence, for every $1 < q \leq \infty$

$$\|\mathcal{N}_{Q_0} u\|_{L^q(Q_0, \omega_L^{X_{Q_0}})} \lesssim \|f\|_{L^q(2\widetilde{\Delta}_{Q_0}, \omega_L^{X_{Q_0}})}. \tag{3.22}$$

Moreover, the implicit constants depend just on dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constant of L and on q in (3.22).

PROOF. By decomposing f into its positive and negative parts we may assume that f is non-negative with $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$ and construct the associated u as in the statement which is non-negative. Fix $x \in Q_0$ and let $X \in \Gamma^*_{Q_0}(x)$. Then, by definition there are $Q \in \mathbb{D}_{Q_0}$ and $I \in \mathcal{W}^*_Q$ such that $x \in Q$ and $X \in I^{**}$. Hence using Harnack's inequality and the notation introduced in (2.34)–(2.36)

$$\begin{aligned} u(X) &= \int_{\partial\Omega} f(y)\, d\omega_L^X(y) \approx \int_{\partial\Omega} f(y)\, d\omega_L^{X_Q}(y) \\ &\leq \int_{4\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_Q}(y) + \sum_{j=3}^\infty \int_{2^j\,\widetilde{\Delta}_Q \setminus 2^{j-1}\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_Q}(y) =: \sum_{j=2}^\infty \mathcal{I}_j. \end{aligned}$$

Let $k_0 \geq 0$ be such that $\ell(Q) = 2^{-k_0}\ell(Q_0)$. Observe that for every $j \geq k_0 + 3$ one has that $2\widetilde{\Delta}_{Q_0} \setminus 2^{j-1}\widetilde{\Delta}_Q = \emptyset$. Otherwise there is $z \in 2\widetilde{\Delta}_{Q_0} \setminus 2^{j-1}\widetilde{\Delta}_Q$ and hence we get a contradiction:

$$4\,\Xi\, r_{Q_0} \leq 2^{j-1-k_0}\,\Xi\, r_{Q_0} = 2^{j-1}\,\Xi\, r_Q \leq |z - x_Q| \leq |z - x_{Q_0}| + |x_Q - x_{Q_0}| \leq 3\,\Xi\, r_{Q_0}.$$

With this in hand, and since $\operatorname{supp} f \subset 2\widetilde{\Delta}_{Q_0}$, we clearly see that $\mathcal{I}_j = 0$ for $j \geq k_0 + 3$.

In order to estimate the $\mathcal{I}_j$'s we need some preparatives. Note that for every $2 \leq j \leq k_0 + 2$ one has $2^j\widetilde{B}_Q \subset 5\widetilde{B}_{Q_0}$. We claim that

$$\frac{d\omega_L^{X_{2^j\widetilde{\Delta}_Q}}}{d\omega_L^{X_{Q_0}}}(y) \lesssim \frac{1}{\omega_L^{X_{Q_0}}(2^j\,\widetilde{\Delta}_Q)}, \qquad \text{for } \omega_L^{X_{Q_0}}\text{-a.e. } y \in 2^j\,\widetilde{\Delta}_Q,\ 2 \leq j \leq k_0 + 2. \tag{3.23}$$

Indeed, this estimate follows from Harnack's inequality and Lemma 2.69 part (a) when $j \approx k_0$ since $2^j\,\ell(Q) \approx \ell(Q_0)$, and from Lemma 2.69 part (d) whenever $j \ll k_0$. We also observe that Lemma 2.69 part (a) and Harnack's inequality readily give that

$$\omega_L^{X_{2^j\widetilde{\Delta}_Q}}(2^j\,\widetilde{\Delta}_Q) \approx 1, \qquad \text{for every } 2 \leq j \leq k_0 + 2. \tag{3.24}$$

Finally, by Lemma 2.69 part (g) and Harnack's inequality it follows that

$$\frac{d\omega_L^{X_Q}}{d\omega_L^{X_{2^{j-1}\widetilde{\Delta}_Q}}}(y) \lesssim 2^{-j\,\rho}, \qquad \text{for } \omega_L^{X_Q}\text{-a.e. } y \in \partial\Omega \setminus 2^{j-1}\,\widetilde{\Delta}_Q,\ j \geq 3. \tag{3.25}$$

Let us start estimating $\mathcal{I}_2$. Use Harnack's inequality and (3.24), (3.23) with $j=2$, to conclude that

$$\mathcal{I}_2 \approx \fint_{4\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{\widetilde{\Delta}_Q}}(y) \approx \fint_{4\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y).$$

On the other hand, for $3 \le j \le k_0+2$, we employ (3.25), Harnack's inequality, (3.24), and (3.23)

$$\begin{aligned}\mathcal{I}_j \lesssim 2^{-j\,\rho} \int_{2^j\,\widetilde{\Delta}_Q \setminus 2^{j-1}\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{2^{j-1}\,\widetilde{\Delta}_Q}}(y) &\lesssim 2^{-j\,\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{2^j\,\widetilde{\Delta}_Q}}(y) \\ &\approx 2^{-j\,\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y).\end{aligned}$$

If we now collect all the obtained estimates we conclude as desired (3.21):

$$\begin{aligned} u(X) &\lesssim \sum_{j=2}^{k_0+2} \mathcal{I}_j \lesssim \sum_{j=2}^{k_0+2} 2^{-j\,\rho} \fint_{2^j\,\widetilde{\Delta}_Q} f(y)\, d\omega_L^{X_{Q_0}}(y) \\ &\le \sup_{\substack{\Delta\ni x \\ 0<r_\Delta<8\Xi r_{Q_0}}} \fint_{\Delta} |f(y)|\, d\omega_L^{X_{Q_0}}(y) \sum_{j=2}^{\infty} 2^{-j\,\rho} \lesssim \sup_{\substack{\Delta\ni x \\ 0<r_\Delta<4\Xi r_{Q_0}}} \fint_{\Delta} |f(y)|\, d\omega_L^{X_{Q_0}}(y).\end{aligned}$$

To complete the proof we just need to obtain (3.22) but this follows at once upon using (3.21) and observing that the local Hardy-Littlewood maximal function on its right hand side is bounded on $L^q(20\,\widetilde{\Delta}_{Q_0}, \omega_L^{X_{Q_0}})$ since $\omega_L^{X_{Q_0}}$ is a doubling measure in $20\,\widetilde{\Delta}_{Q_0}$ by Lemma 2.69 parts (a) and (c). □

Throughout the rest of this section we will always assume that Ω is a 1-sided NTA domain satisfying the CDC, hence $\partial\Omega$ is also bounded. We fix $\mathbb{D} = \mathbb{D}(\partial\Omega)$ the dyadic grid for Lemma 2.33 with $E = \partial\Omega$. Let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be two real (non-necessarily symmetric) elliptic operators. Fix $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ and let $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$. From now on $X_0 := X_{\Delta_0}$, $\omega_0 := \omega_{L_0}^{X_0}$ and $\omega := \omega_L^{X_0}$.

We further assume that $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$. In particular $r_{2\Delta_0} < \operatorname{diam}(\partial\Omega)$. We introduce the following notation (which should not be confused with the one introduced in (2.46)):

$$\mathbb{D}_*^{\Delta_0} = \Big\{ Q \in \mathbb{D} :\ Q \cap \tfrac{3}{2}\Delta_0 \neq \emptyset,\ \tfrac{c_0}{16\kappa_0} r_0 \le \ell(Q) < \tfrac{c_0}{8\kappa_0} r_0 \Big\}. \tag{3.26}$$

Fixed $\varphi \in C^\infty(0,\infty)$ with $\mathbf{1}_{(0,1)} \le \varphi \le \mathbf{1}_{(0,2)}$, we define

$$P_t g(x) := \int_{\partial\Omega} \varphi_t(x,y) g(y)\, d\omega_0(y) \qquad \text{whenever } x \in \partial\Omega, \tag{3.27}$$

where

$$\varphi_t(x,y) := \frac{\varphi\left(\frac{|x-y|}{t}\right)}{\int_{\partial\Omega} \varphi\left(\frac{|x-z|}{t}\right) d\omega_0(z)} \qquad \text{whenever } x, y \in \partial\Omega. \tag{3.28}$$

A variant of the following lemma was shown in [**CHM19**, Lemma 3.5].

LEMMA 3.29. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. Let $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be a real (non-necessarily symmetric) elliptic operator. Fix $\varphi \in C^\infty(0,\infty)$ with $\mathbf{1}_{(0,1)} \le \varphi \le \mathbf{1}_{(0,2)}$. There exists C depending only on dimension n, the 1-sided NTA constants, the CDC constant, the ellipticity constant of L_0, and φ (and independent of Δ_0), such that for every $Q \in \mathbb{D}_{Q^0}$ with $Q^0 \in \mathbb{D}_*^{\Delta_0}$, and with P_t as above then the following statements are true:*

(a) *If $g \in L^q(\partial\Omega, \omega_0)$, $1 \le q \le \infty$, then*

$$\sup_{0<t<\ell(Q)} \|P_t g\|_{L^q(2\widetilde{\Delta}_Q, \omega_0)} \le C \|g\|_{L^q(3\widetilde{\Delta}_Q, \omega_0)}.$$

(b) *If $g \in L^q(\partial\Omega, \omega_0)$, $1 \le q \le \infty$, and $0 < t < \ell(Q)$ then $P_t(g\mathbf{1}_Q) \in \operatorname{Lip}(\partial\Omega) \cap L^\infty(\partial\Omega, \omega_0)$.*

(c) *If $g \in L^q(\partial\Omega, \omega_0)$, $1 \le q < \infty$, then $P_t g \to g$ in $L^q(2\widetilde{\Delta}_Q, \omega_0)$ as $t \to 0^+$.*

(d) *If $g \in C(\partial\Omega)$ then $P_t g(x) \to g(x)$ as $t \to 0^+$ for every $x \in 2\widetilde{\Delta}_Q$.*

(e) *If $\operatorname{supp}(g) \subset \overline{\Delta(x,r)}$ then $\operatorname{supp}(P_t g) \subset \overline{\Delta(x, r+2t)}$.*

PROOF. We start with some preliminaries. Fix $Q \in \mathbb{D}_{Q^0}$ with $Q^0 \in \mathbb{D}_*^{\Delta_0}$. Set

$$H(x) := \int_{\partial\Omega} \varphi\left(\frac{|x-z|}{t}\right) d\omega_0(z), \qquad x \in \partial\Omega$$

and observe that $\omega_0(\Delta(x,t)) \le H(x) \le \omega_0(\Delta(x,2t))$. Hence if $x, y \in \partial\Omega$

$$\frac{\mathbf{1}_{\Delta(x,t)}(y)}{\omega_0(\Delta(x,2t))} \le \varphi_t(x,y) \le \frac{\mathbf{1}_{\Delta(x,2t)}(y)}{\omega_0(\Delta(x,t))}. \tag{3.30}$$

This easily implies (e) and also, recalling the notation in (2.34)–(2.36),

$$\frac{\mathbf{1}_{\Delta(x,t)}(y)}{\omega_0(\Delta(x,t))} \lesssim \varphi_t(x,y) \lesssim \frac{\mathbf{1}_{\Delta(x,2t)}(y)}{\omega_0(\Delta(x,2t))}, \qquad 0 < t < \ell(Q^0), \quad x \in 4\widetilde{\Delta}_{Q^0}, \tag{3.31}$$

by Lemma 2.69 part (c), and the implicit constant does not depend on t. Moreover, for every $x \in 4\widetilde{\Delta}_Q$

$$\sup_{0<t<\ell(Q)} |P_t g(x)| \le C \sup_{0<t<2\ell(Q)} \fint_{\Delta(x,t)} |g(y)|\, d\omega_0(y). \tag{3.32}$$

Note also that fixed $0 < t < \ell(Q) \le \ell(Q^0) < r_0$ for every $x \in 4\widetilde{\Delta}_Q$ we have $\delta(X_{\Delta(x,2t)}) \ge c_0 2t$ and since $Q^0 \in \mathbb{D}_*^{\Delta_0}$

$$\begin{aligned}|X_{\Delta(x,2t)} - X_{\Delta_0}| &\le |X_{\Delta(x,2t)} - x| + |x - x_Q| + |x_Q - x_{Q^0}| + |x_{Q^0} - x_0| + |x_0 - X_{\Delta_0}| \\ &\le 2t + 6\Xi\ell(Q^0) + 3r_0 \lesssim r_0.\end{aligned}$$

Hence, the Harnack Chain condition and Harnack's inequality yield

$$\omega_0(\Delta(x,2t)) \approx_t \omega_{L_0}^{X_{\Delta(x,2t)}}(\Delta(x,2t)) \approx 1 \tag{3.33}$$

where the last estimate follows from Lemma 2.69 part (a) and the implicit constants depend on t but are uniform in $x \in 4\widetilde{\Delta}_Q$.

To show (a), note first $(P_t g)\mathbf{1}_{2\widetilde{\Delta}_Q} = (P_t(g\mathbf{1}_{3\widetilde{\Delta}_Q}))\mathbf{1}_{2\widetilde{\Delta}_Q}$ whenever $0 < t < \ell(Q)$. This, Fubini's theorem and (3.32) yield

$$\|P_t g\|_{L^1(2\widetilde{\Delta}_Q, \omega_0)} \le \|g\|_{L^1(3\widetilde{\Delta}_Q, \omega_0)} \qquad \text{and} \qquad \|P_t g\|_{L^\infty(2\widetilde{\Delta}_Q, \omega_0)} \le C \|g\|_{L^\infty(3\widetilde{\Delta}_Q, \omega_0)}.$$

Thus, (a) follows easily from Marcinkiewicz's interpolation theorem.

To obtain (b) we first observe that (e) yields $\operatorname{supp}(P_t(g\mathbf{1}_Q)) \subset 3\widetilde{\Delta}_Q$. This, (3.31), Hölder's inequality, and (3.33) give for every $x \in 3\widetilde{\Delta}_Q$

$$|P_t(g\mathbf{1}_Q)(x)| \lesssim \fint_{\Delta(x,2t)} |g(y)|\mathbf{1}_Q(y)\, d\omega_0(y) \lesssim_t \|g\|_{L^q(Q,\omega_0)}.$$

Thus, $P_t(g\mathbf{1}_Q) \in L^\infty(\partial\Omega, \omega_0)$.

We next see that $P_t(g\mathbf{1}_Q) \in \mathrm{Lip}(\partial\Omega)$. Using what we have proved so far it is trivial to see that it suffices to consider the case on which $|x - x'| < \ell(Q)$ and both $x, x' \in 4\widetilde{\Delta}_Q$. Taking such points we note that

$$|P_t(g\mathbf{1}_Q)(x) - P_t(g\mathbf{1}_Q)(x')| \le \int_{\partial\Omega} |\varphi_t(x,y) - \varphi_t(x',y)|\, |g(y)|\, \mathbf{1}_Q(y)\, d\omega_0(y).$$

Note that for every $y \in Q$ we have by the mean value theorem and easy calculations

$$\begin{aligned}
|\varphi_t(x,y) - \varphi_t(x',y)| &\le \frac{1}{H(x)} \left| \varphi\left(\frac{|x-y|}{t}\right) - \varphi\left(\frac{|x'-y|}{t}\right) \right| \\
&\qquad + \varphi\left(\frac{|x'-y|}{t}\right) \left| \frac{1}{H(x)} - \frac{1}{H(x')} \right| \\
&\lesssim \frac{\|\nabla\varphi\|_{L^\infty}}{t\,\omega_0(\Delta(x,t))} \left(1 + \frac{1}{\omega_0(\Delta(x',t))}\right) |x - x'| \\
&\lesssim_t \|\nabla\varphi\|_{L^\infty} |x - x'|,
\end{aligned}$$

where in the last estimate we have used (3.33). Consequently,

$$\begin{aligned}
|P_t(g\mathbf{1}_Q)(x) - P_t(g\mathbf{1}_Q)(x')| &\lesssim_t \|\nabla\varphi\|_{L^\infty} |x - x'| \int_{\partial\Omega} |g(y)|\mathbf{1}_Q(y)\, d\omega_0(y) \\
&\lesssim \|\nabla\varphi\|_{L^\infty} \|g\|_{L^q(\omega_0, Q)} |x - x'|,
\end{aligned}$$

and this completes the proof of (b).

Let us now establish (d). Since $g \in C(\partial\Omega)$ and $\partial\Omega$ is bounded, g is uniformly continuous and hence given $\varepsilon > 0$ there exists $\eta > 0$ such that $|g(y) - g(x)| < \varepsilon$ whenever $|x - y| < \min\{\eta, \ell(Q)\}$. Hence, if $0 < t < \eta/2$ and $x \in 4\widetilde{\Delta}_Q$ by (3.31)

$$|P_t g(x) - g(x)| \lesssim \fint_{\Delta(x,2t)} |g(y) - g(x)| d\omega_0 < \varepsilon$$

and therefore $P_t g(x) \to g(x)$ for every $x \in 4\widetilde{\Delta}_Q$ (which is indeed stronger than what stated in (d)).

Finally, we show (c). To set the stage, fix $\varepsilon > 0$ and $g \in L^q(\omega_0, \partial\Omega)$, $1 \le q < \infty$. Pick $h \in C(\partial\Omega)$ such that $\|g - h\|_{L^q(\partial\Omega,\omega_0)} < \varepsilon$. Proceeding as in the proof of (d) there exists $\eta > 0$ such that $|h(y) - h(x)| < \varepsilon$ whenever $|x - y| < \min\{\eta, \ell(Q)\}$. Hence, if $0 < t < \eta/2$ and $x \in 2\widetilde{\Delta}_Q$ by (3.31)

$$|P_t h(x) - h(x)| \lesssim \fint_{\Delta(x,2t)} |h(y) - h(x)| d\omega_0 \le \varepsilon.$$

Using all these we obtain for all $0 < t < \eta/2$

$$\begin{aligned}
\|P_t g - g\|_{L^q(2\widetilde{\Delta}_Q,\omega_0)} &\le \|P_t(g - h)\|_{L^q(2\widetilde{\Delta}_Q,\omega_0)} \\
&\qquad + \|P_t h - h\|_{L^q(2\widetilde{\Delta}_Q,\omega_0)} + \|h - g\|_{L^q(2\widetilde{\Delta}_Q,\omega_0)} \lesssim \varepsilon
\end{aligned}$$

where we have used item (a) and the fact that $\omega_0(\partial\Omega) \le 1$. This completes the proof. $\square$

LEMMA 3.34. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC and adopt the notation introduced above. There exists $\kappa > 0$ depending only on dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constant of L_0 (and independent of Δ_0) such that if $Q^0 \in \mathbb{D}_*^{\Delta_0}$ and we set*

$$\gamma_Q = \gamma_{X_0,Q} := \omega_0(Q) \sum_{I \in \mathcal{W}_Q^*} \|A - A_0\|_{L^\infty(I^*)}^2, \qquad Q \in \mathbb{D}_{Q^0}, \tag{3.35}$$

then $\|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q^0,\omega_0)} \le \kappa \|\!|\!|\varrho(A, A_0)|\!|\!|_{B_0}$.

PROOF. Fix $Q^0 \in \mathbb{D}_*^{\Delta_0}$ and pick $y_0 \in Q^0 \cap \Delta_0$. Let $Q \in \mathbb{D}_{Q^0}$ and note that by (2.34) and the fact that $\kappa_0 \ge 16\Xi$

$$|x_Q - x_0| \le |x_Q - y_0| + |y_0 - x_0| < 2\Xi r_{Q^0} + r_0 \le 2\Xi\ell(Q^0) + r_0 \le \left(\frac{\Xi c_0}{4\kappa_0} + 1\right) r_0 < 2r_0.$$

Hence $x_Q \in 2\Delta_0$. Note also that $r_{B_Q^*} = 2\kappa_0 r_Q \le 2\kappa_0 \ell(Q^0) < r_0 c_0/4$. This means that B_Q^* is one of the balls in the sup in (3.2). Also, $X_0 \notin 4B_Q^*$ hence if $Q' \in \mathbb{D}_Q$ and $Y \in I^* \in \mathcal{W}_{Q'}^*$ we have by Harnack's inequality and Lemma 2.69 parts (a)–(c),

$$\omega_0(Q') \approx \omega_0(\Delta_{Q'}) \approx \ell(Q')^{n-1} G_{L_0}(X_0, X_{Q'}) \approx \delta(Y)^{n-1} G_{L_0}(X_0, Y).$$

On the other hand, by (2.40) and recalling that $I^* = (1+\lambda)I$ with $0 < \lambda < 1$, it follows that $I^* \subset B(Y, \delta(Y)/2)$ and thus $\|A - A_0\|_{L^\infty(I^*)} \le \varrho(A, A_0)(Y)$. All these imply

$$\begin{aligned}
\mathfrak{m}_\gamma(\mathbb{D}_Q) &= \sum_{Q' \in \mathbb{D}_Q} \omega_0(Q') \sum_{I \in \mathcal{W}_{Q'}^*} \|A - A_0\|_{L^\infty(I^*)}^2 \\
&\le \sum_{Q' \in \mathbb{D}_Q} \omega_0(Q') \sum_{I \in \mathcal{W}_{Q'}^*} \iint_{I^*} \frac{\varrho(A, A_0)(Y)^2}{\ell(I)^{n+1}} dY \\
&\approx \sum_{Q' \in \mathbb{D}_Q} \iint_{U_{Q'}} \varrho(A, A_0)(Y)^2 \, \frac{\omega_0(Q')}{\delta(Y)^{n+1}} dY \\
&\approx \sum_{Q' \in \mathbb{D}_Q} \iint_{U_{Q'}} \varrho(A, A_0)(Y)^2 \, \frac{G_{L_0}(X_0, Y)}{\delta(Y)^2} dY \\
&\lesssim \iint_{T_Q} \varrho(A, A_0)(Y)^2 \frac{G_{L_0}(X_0, Y)}{\delta(Y)^2} dY \\
&\lesssim \iint_{B_Q^*} \varrho(A, A_0)(Y)^2 \frac{G_{L_0}(X_0, Y)}{\delta(Y)^2} dY \\
&\lesssim \|\!|\!|\varrho(A, A_0)|\!|\!|_{B_0} \, \omega_0(\Delta_Q^*) \\
&\lesssim \|\!|\!|\varrho(A, A_0)|\!|\!|_{B_0} \, \omega_0(Q),
\end{aligned} \tag{3.36}$$

where have used that the families $\{I^*\}_{I \in \mathcal{W}}$ and $\{U_{Q'}\}_{Q' \in \mathbb{D}_Q}$ have bounded overlap, (2.48), and Lemma 2.69, parts (b) and (c). This leads to the desired estimate. $\square$

For each $j \in \mathbb{N}$ (large enough), let (see Figure 1)

$$A^j(Y) = \begin{cases} A(Y) & \text{if } Y \in \Omega \text{ and } \delta(Y) \geq 2^{-j}; \\ A_0(Y) & \text{if } Y \in \Omega \text{ and } \delta(Y) < 2^{-j}, \end{cases} \tag{3.37}$$

and define $L^j u = -\operatorname{div}(A^j \nabla u)$. Note that the matrix A^j is uniformly elliptic with constant $\Lambda_0 = \max\{\Lambda_A, \Lambda_{A_0}\}$, where Λ_A and Λ_{A_0} are the ellipticity constants of A and A_0 respectively. Let ω_{L^j} be elliptic measure of Ω associated to the operator L^j with pole at X_0.

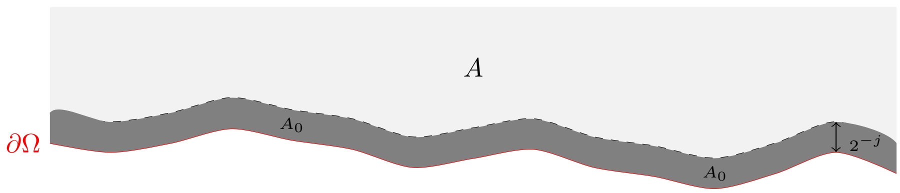

FIGURE 1. Definition of the matrix A^j in Ω.

The following result is a version of [CHM19, Proposition 4.28] adapted to our setting.

LEMMA 3.38. *Let $\Omega \subset \mathbb{R}^{n+1}$ be* ***bounded*** *1-sided NTA domain satisfying the CDC. Assume that there exists q, $1 < q < \infty$, such that $\omega_{L^j} \in RH_q(\frac{5}{4}\Delta_0, \omega_0)$ for every $j \geq j_0$ and with implicit constants which are uniform in j and in Δ_0. Then $\omega_L \in RH_q(\Delta_0, \omega_0)$ with $[\omega_L]_{RH_q(\Delta_0,\omega_0)} \lesssim \sup_{j\geq j_0}[\omega_{L^j}]_{RH_q(\frac{5}{4}\Delta_0,\omega_0)}$, with an implicit constant depending on dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constants of L_0 and L (and independent of Δ_0).*

PROOF. Set $\Upsilon := \sup_{j\geq j_0}[\omega_{L^j}]_{RH_q(\frac{5}{4}\Delta_0,\omega_0)}$. Consider an arbitrary $\Delta_0' = B_0' \cap \partial\Omega$ with $B_0' = B(x_0', r_0') \subset B_0$. Write $X_0' = X_{\Delta_0'}$, $\omega' = \omega_L^{X_0'}$, $\omega_0' = \omega_{L_0}^{X_0'}$ (and note that $\omega_0 = \omega_{L_0}^{X_0}$ since $X_0 = X_{\Delta_0}$). Write $\Delta_1 = \frac{5}{4}\Delta_0'$, let $r_1 = \frac{5}{4}r_0'$ be its radius and set $X_1 = X_{\Delta_1}$. By hypotheses $\omega_{L^j} \ll \omega_0$ in $\frac{5}{4}\Delta_0$, hence $h(\cdot\,; L^j, L_0, X)$ is defined ω_0-a.e. in $\frac{5}{4}\Delta_0$.

If $r_0' < c_0 r_0/(3\kappa_0)$ so that $X_0 \in \Omega \setminus 2\kappa_0 B_1$, by Lemma 2.69 part ($d$) applied to L_j and L_0 we have

$$h(\cdot\,; L^j, L_0, X_0) = \frac{d\omega_{L^j}^{X_0}}{d\omega_{L_0}^{X_0}} = \frac{d\omega_{L^j}^{X_0}}{d\omega_{L^j}^{X_1}} \frac{d\omega_{L^j}^{X_1}}{d\omega_{L_0}^{X_1}} \frac{d\omega_{L_0}^{X_1}}{d\omega_{L_0}^{X_0}} \approx \frac{\omega_{L^j}^{X_0}(\Delta_1)}{\omega_{L_0}^{X_0}(\Delta_1)} h(\cdot\,; L^j, L_0, X_1), \tag{3.39}$$

ω_0-a.e. in Δ_1. This and Lemma 2.69 part (d) give

$$\begin{aligned} \|h(\cdot\,; L^j, L_0, X_1)\|_{L^q(\Delta_1, \omega_{L_0}^{X_1})} &\approx \frac{1}{\omega_{L^j}^{X_0}(\Delta_1)} \|h(\cdot\,; L^j, L_0, X_0)\|_{L^q(\Delta_1,\omega_0)} \\ &\leq [\omega_{L^j}]_{RH_q(\frac{5}{4}\Delta_0,\omega_0)} \omega_0(\Delta_1)^{-\frac{1}{q'}} \leq \Upsilon \omega_0(\Delta_0')^{-\frac{1}{q'}}, \end{aligned} \tag{3.40}$$

where the implicit constants are independent of j.

For any $f \in C(\partial\Omega)$, we define

$$\Phi(f) := \int_{\partial\Omega} f(y)\, d\omega'(y).$$

Let $f \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(f) \subset \Delta_1$ and consider the following solutions to the Dirichlet problems associated with the operators L and L^j in Ω:

$$u(X) = \int_{\partial\Omega} f(y) d\omega_L^X(y) \quad \text{and} \quad u_j(X) = \int_{\partial\Omega} f(y) d\omega_{L^j}^X(y), \qquad X \in \Omega.$$

Implicit in the way that ω_{L^j} is defined and since Ω is bounded one has that $u_j = F - v_j$ where F is a compactly supported Lipschitz extension (e.g., [**EG92**, p. 80] multiplied some cut-off function) of f such that $\|F\|_{\mathrm{Lip}(\mathbb{R}^{n+1})} \le \|f\|_{\mathrm{Lip}(\partial\Omega)} + \|f\|_{L^\infty(\partial\Omega)}$ and $v_j \in W_0^{1,2}(\Omega)$ is the unique Lax-Milgram solution to the problem $L^j v_j = L^j F$ in Ω. Also, one has

$$\sup_j \|u_j\|_{W^{1,2}(\Omega)} \le C_\Omega \|F\|_{W^{1,2}(\Omega)} < \infty \tag{3.41}$$

where the implicit constants depend on $\mathrm{diam}(\partial\Omega)$ and Λ_0.

Since $f \in \mathrm{Lip}(\partial\Omega)$ it follows that we can use Lemma 3.7 (slightly moving X_0' if needed) to obtain

$$u(X_0') - u_j(X_0') = \int_\Omega (A^j - A)^\top(Y) \nabla_Y G_{L^\top}(Y, X_0') \cdot \nabla u_j(Y) dY.$$

We want to estimate the right hand-side of this identity. To this end, if $j > j_0$ is large enough so that $2^{-j} < \delta(X_{\Delta_0'})/2$ then

$$\Sigma_j := \{Y \in \Omega : \ \delta(Y) < 2^{-j}\} \cap B(X_0', \delta(X_0')/2) = \emptyset.$$

Then using (1.1) and Hölder's inequality we have

$$\begin{aligned} |u(X_0') - u_j(X_0')| &\lesssim \int_{\Omega \cap \Sigma_j} |\nabla_Y G_{L^\top}(Y, X_0')|\, |\nabla u_j(Y)| dY \\ &\lesssim \|\nabla_Y G_{L^\top}(\cdot, X_0')\, \mathbf{1}_{\Sigma_j}\|_{L^2(\Omega)} \sup_j \|u_j\|_{W^{1,2}(\Omega)}. \end{aligned} \tag{3.42}$$

By Remark 2.66 and (3.41) the dominated convergence theorem gives that $u_j(X_0') \to u(X_0')$ as $j \to \infty$. Using this observation, the definitions of u, u_j, Φ, and the fact that $\mathrm{supp}(f) \subset \Delta_1$, we get that for every $f \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(f) \subset \Delta_1$

$$\begin{aligned} |\Phi(f)| &= |u(X_0')| = \lim_{j\to\infty} |u_j(X_0')| \\ &\lesssim \|f\|_{L^{q'}(\Delta_1, \omega_0')} \sup_{j \ge j_1} \|h(\cdot\,; L^j, L_0, X_{\Delta_1})\|_{L^q(\Delta_1, \omega_{L_0}^{X_1})} \lesssim \|f\|_{L^{q'}(\Delta_1, \omega_0)} \Upsilon\, \omega_0(\Delta_0')^{-\frac{1}{q'}}. \end{aligned} \tag{3.43}$$

Note that in the previous inequalities we have employed that $\Delta_0' \subset \Delta_1$ have comparable radii, Harnack's inequality, and (3.40).

We next write $\Delta_2 = \frac{9}{8}\Delta_0'$ so that $\Delta_0' \subset \overline{\Delta_0'} \subset \Delta_2 \subset \overline{\Delta_2} \subset \Delta_1$ and let $f \in L^{q'}(\Delta_2, \omega_0')$ (where we recall that $\omega_0' = \omega_{L_0}^{X_{\Delta_0'}}$). Abusing the notation we extend f by 0 in $\partial\Omega \setminus \Delta_2$ so that $\mathrm{supp}(f) \subset \overline{\Delta_2}$. By definition of $\mathbb{D}_*^{\Delta_0'}$, see (3.26), we have that $\Delta_0' \subset \Delta_1 \subset \bigcup_{Q \in \mathbb{D}_*^{\Delta_0'}} Q$ where the cubes in $\mathbb{D}_*^{\Delta_0'}$ are pairwise disjoint. Also, by Harnack's inequality and Lemma 2.69 parts (a) and (c)

$$\#\mathbb{D}_*^{\Delta_0'} \approx \#\mathbb{D}_*^{\Delta_0'} \omega_0'(\Delta_0') \le \sum_{Q \in \mathbb{D}_*^{\Delta_0'}} \omega_0'(Q) \le \omega_0\Big(\bigcup_{Q \in \mathbb{D}_*^{\Delta_0'}} Q\Big) \le 1,$$

hence $\#\mathbb{D}_*^{\Delta_0}$ is uniformly bounded. This means that by Lemma 3.29 applied with ω_0' in place of ω_0

$$P_t f = \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} P_t(f\mathbf{1}_Q) \in L^\infty(\partial\Omega,\omega_0')\cap \mathrm{Lip}(\partial\Omega)$$

provided $0<t<c_0 r_0'/(32\kappa_0) =: t_0$. Note that $t_0 \le \ell(Q)$ for every $Q\in\mathbb{D}_*^{\Delta_0'}$. Also Lemma 3.29 applied with ω_0' in place of ω_0 implies that

$$\mathrm{supp}(P_t f) \subset \overline{\Delta(x_0', \frac{9}{8}r_0' + 2t)} \subset \Delta_1,$$

provided $0<t<r_0'/16$. Consequently, if $0<t<t_0$ we have shown that $P_t f \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(P_t f)\subset \Delta_1$. We can then invoke (3.43) to see that

$$\begin{aligned}
\Upsilon^{-1}\,\omega_0(\Delta_0')^{\frac{1}{q'}} \sup_{0<t<t_0} |\Phi(P_t f)| &\lesssim \sup_{0<t<t_0} \|P_t f\|_{L^{q'}(\Delta_1',\omega_0)} \\
&\le \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} \sup_{0<t<\ell(Q)} \|P_t(f\mathbf{1}_Q)\|_{L^{q'}(2\widetilde{\Delta}_Q,\omega_0')} \\
&\lesssim \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} \|f\mathbf{1}_Q\|_{L^{q'}(3\widetilde{\Delta}_Q,\omega_0')} \lesssim \|f\|_{L^{q'}(\Delta_2,\omega_0')},
\end{aligned}$$

where we have used that $\mathrm{supp}(P_t(f\mathbf{1}_Q)) \subset \overline{\Delta(x_Q, Cr_Q+2t)} \subset 2\widetilde{\Delta}_Q$ for every $Q\in\mathbb{D}_*^{\Delta_0}$, Lemma 3.29 applied with ω_0' in place of ω_0, and that $\#\mathbb{D}_*^{\Delta_0}$ is uniformly bounded.

On the other hand, if $0<t,s<t_0$ we have that $P_t f - P_s f \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(P_t f - P_s f)\subset\Delta_1$ and again we can invoke (3.43) to see that a similar computation lead us to

$$\begin{aligned}
\Upsilon^{-1}\omega_0(\Delta_0')^{\frac{1}{q'}}|\Phi(P_t f)-\Phi(P_s f)| &= \Upsilon^{-1}\omega_0(\Delta_0')^{\frac{1}{q'}}|\Phi(P_t f - P_s f)| \\
&\lesssim \|P_t f - P_s f\|_{L^{q'}(\Delta_1,\omega_0')} \\
&\le \|P_t f - f\|_{L^{q'}(\Delta_1,\omega_0')} + \|P_s f - f\|_{L^{q'}(\Delta_1,\omega_0')} \\
&\le \sum_{Q\in\mathbb{D}_*^{\Delta_0}} \|P_t(f\mathbf{1}_Q) - f\mathbf{1}_Q\|_{L^{q'}(2\widetilde{\Delta}_Q,\omega_0')} + \|P_s(f\mathbf{1}_Q)-f\mathbf{1}_Q\|_{L^{q'}(2\widetilde{\Delta}_Q,\omega_0')}.
\end{aligned}$$

This and Lemma 3.29 applied with ω_0' in place of ω_0 yield that $\{\Phi(P_t f)\}_{0<t<t_0}$ is a Cauchy sequence and we can define $\widetilde{\Phi}(f) := \lim_{t\to 0^+}\Phi(P_t f)$. Clearly, $\widetilde{\Phi}$ is a well-defined linear operator and satisfies

$$|\widetilde{\Phi}(f)| = \lim_{t\to0^+}|\Phi(P_t)| \le \sup_{0<t<t_0}|\Phi(P_t f)| \lesssim \Upsilon\omega_0(\Delta_0')^{-\frac{1}{q'}}\|f\|_{L^{q'}(\Delta_2,\omega_0')}.$$

Consequently, there exists $g\in L^q(\Delta_2,\omega_0')$ with $\|g\|_{L^q(\Delta_2,\omega_0')} \lesssim \Upsilon\omega_0(\Delta_0')^{-\frac{1}{q'}}$ such that

$$\widetilde{\Phi}(f) = \int_{\Delta_2} f(y)g(y)\,d\omega_0'(y), \qquad \forall\, f\in L^{q'}(\Delta_2,\omega_0'). \tag{3.44}$$

We now assume that $f\in C(\partial\Omega)$ with $\mathrm{supp}(f)\subset\Delta_2$, thus $f\in L^{q'}(\Delta_2,\omega_0')$ and hence $P_t f\in\mathrm{Lip}(\partial\Omega)$. Also, proceeding as above

$$\sup_{0<t<t_0} |\Phi(P_t f)| \le \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} \sup_{0<t<\ell(Q)} \|P_t(f\mathbf{1}_Q)\|_{L^\infty(2\widetilde{\Delta}_Q,\omega_0')}$$
$$\lesssim \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} \|f\mathbf{1}_Q\|_{L^\infty(3\widetilde{\Delta}_Q,\omega_0')} \lesssim \|f\|_{L^\infty(\partial\Omega,\omega_0')}.$$

Note also that, as mentioned above, for t small enough one has $\operatorname{supp}(P_t f) \subset \Delta_1$ and the cubes in $\mathbb{D}_*^{\Delta_0'}$ cover Δ_1. Hence by Lemma 3.29 applied with ω_0' in place of ω_0 it follows that $P_t f(x) \to f(x)$ as $t \to 0^+$ for every $y \in \Delta_1$. These, the definitions of Φ, $\widetilde{\Phi}$, and the dominated convergence theorem yield for every $f \in C(\partial\Omega)$ with $\operatorname{supp}(f) \subset \Delta_2$

$$(3.45)\quad \widetilde{\Phi}(f) = \lim_{t\to 0^+} \Phi(P_t f) = \lim_{t\to 0^+} \int_{\partial\Omega} P_t f(y) d\omega'(y) = \lim_{t\to 0^+} \int_{\Delta_1} P_t f(y) d\omega'(y)$$
$$= \int_{\Delta_1} f(y) d\omega'(y) = \int_{\partial\Omega} f(y) d\omega'(y) = \Phi(f).$$

Our next goal is to show that $\omega' = \omega_L^{X_0'} \ll \omega_{L_0}^{X_0'} = \omega_0'$ in $\Delta_3 = \frac{17}{16}\Delta_0'$. Let $E \subset \Delta_3$ a Borel set. Since both measures are Borel regular, given $\varepsilon > 0$ we can find a compact set K and open set U such that $K \subset E \subset U \subset \Delta_2$ satisfying

$$\omega(U \setminus K) + \omega_0(U \setminus K) < \varepsilon.$$

Using Urysohn's lemma we construct $f \in C_c(\partial\Omega)$ such that $\mathbf{1}_K \le f \le \mathbf{1}_U$ and $\operatorname{supp}(f) \subset \Delta_2$. Thus, combining (3.44) and (3.45), and using definition of Φ and $\widetilde{\Phi}$ we have

$$\omega'(E) \le \varepsilon + \omega'(K) \le \varepsilon + \int_{\partial\Omega} f(y) d\omega'(y) = \varepsilon + \Phi(f) = \varepsilon + \widetilde{\Phi}(f)$$
$$\le \varepsilon + \|f\|_{L^{q'}(\Delta_2,\omega_0')} \|g\|_{L^q(\Delta_2,\omega_0')} \lesssim \varepsilon + [(\varepsilon + \omega_0'(E))^{\frac{1}{q'}} \Upsilon \omega_0(\Delta_0')^{-\frac{1}{q'}}].$$

By letting $\varepsilon \to 0$ we see that $\omega'(E) \lesssim \omega_0'(E)^{\frac{1}{q'}} \Upsilon \omega_0(\Delta_0')^{-\frac{1}{q'}}$ and consequently $\omega' \ll \omega_0'$ in Δ_3. Thus we can write $h(\cdot) := h(\cdot\,; L, L_0, X_0') = \frac{d\omega_L^{X_0'}}{d\omega_{L_0}^{X_0'}} = \frac{d\omega'}{d\omega_0'} \in L^1(\Delta_3, \omega_0')$ which is well-defined for ω_0'-a.e. point in Δ_3 and if $f \in C(\partial\Omega)$ with $\operatorname{supp} f \subset \Delta_3 \subset \Delta_2$

(3.46)
$$\int_{\Delta_3} f(y) g(y) d\omega_0'(y) = \widetilde{\Phi}(f) = \Phi(f) = \int_{\partial\Omega} f(y) d\omega'(y) = \int_{\Delta_3} f(y) h(y) d\omega_0'(y).$$

Note that $\widetilde{h} = (g - h)\mathbf{1}_{\Delta_3} \in L^1(\partial\Omega, \omega_0')$ hence proceeding as above if $0 < t < t_0$ Lemma 3.29 applied with ω_0' in place of ω_0 gives

$$\|P_t\widetilde{h} - \widetilde{h}\|_{L^1(\Delta_3,\omega_0')} \le \sum_{Q\in\mathbb{D}_*^{\Delta_0'}} \|P_t(\widetilde{h}\mathbf{1}_Q) - \widetilde{h}\mathbf{1}_Q\|_{L^1(2\widetilde{\Delta}_Q,\omega_0')} \to 0, \qquad \text{as } t \to 0^+.$$

On the other hand, for any $x \in \Delta_0'$ and $0 < t < r_0'/32$ if we consider φ_t as in (3.28) with ω_0' in place of ω_0 we have $\operatorname{supp}(\varphi_t(x,\cdot)) \subset \overline{\Delta(x, 2t)} \subset \Delta_3$. Thus, we can invoke (3.46) with $f = \varphi_t(x,\cdot)$ to get $P_t\widetilde{h}(x) = 0$ for every $x \in \Delta_0'$. Thus, Lemma 3.29

part (c) applied with ω_0' allows us to conclude that $\widetilde{h} = 0$ ω_0'-a.e. in Δ_0'. Hence $g = h \geq 0$ ω_0'-a.e. in Δ_0' and using that $\|g\|_{L^q(\Delta_2,\omega_0')} \lesssim \Upsilon \omega_0(\Delta_0')^{-\frac{1}{q'}}$

$$(3.47) \quad \left(\fint_{\Delta_0'} h(y; L, L_0, X_0')^q d\omega_0'(y) \right)^{\frac{1}{q}} = \left(\fint_{\Delta_0'} h(y)^q d\omega_0'(y) \right)^{\frac{1}{q}}$$
$$= \left(\fint_{\Delta_0'} g(y)^q d\omega_0'(y) \right)^{\frac{1}{q}} \lesssim \Upsilon \frac{\omega_0(\Delta_0')^{-\frac{1}{q'}}}{\omega_0'(\Delta_0')^{\frac{1}{q}}} \approx \Upsilon \omega_0(\Delta_0')^{-\frac{1}{q'}},$$

where the last estimate follows from Lemma 2.69 part (a). At this point we can repeat the computations we have done in (3.39) replacing L^j by L and Δ_1 by Δ_3 —we already know that $\omega' \ll \omega_0'$ in $\Delta_3 = \frac{17}{16}\Delta_0'$ where B_0' was arbitrary chosen so that $B_0' \subset B_0$, hence taking $B_0' = B_0$ we conclude that $\omega \ll \omega_0$ in Δ_3— to obtain that

$$h(z; L, L_0, X_0) \approx \frac{\omega_L^{X_0}(\Delta_3)}{\omega_{L_0}^{X_0}(\Delta_3)} h(z; L, L_0, X_{\Delta_3}) \approx \frac{\omega(\Delta_0')}{\omega_0(\Delta_0')} h(z; L, L_0, X_0'),$$

for ω_0-a.e. $z \in \Delta_3$, and where we have used Harnack's inequality to pass from X_0' to X_{Δ_3}. This, Lemma 2.69 part (d), and (3.47) give

$$\left(\fint_{\Delta_0'} h(y; L, L_0, X_0)^q d\omega_0(y) \right)^{\frac{1}{q}} \approx \frac{\omega(\Delta_0')}{\omega_0(\Delta_0')^{\frac{1}{q}}} \left(\fint_{\Delta_0'} h(y; L, L_0, X_0')^q d\omega_0'(y) \right)^{\frac{1}{q}}$$
$$\lesssim \Upsilon \frac{\omega(\Delta_0')}{\omega_0(\Delta_0')} = \Upsilon \fint_{\Delta_0'} h(y; L, L_0, X_0) d\omega_0(y).$$

Since $\Delta_0' = B_0' \cap \partial\Omega$ was arbitrary with $B_0' = B(x_0', r_0') \subset B_0$ we therefore conclude that $\omega_L \in RH_q(\Delta_0, \omega_0)$ with $[\omega_L]_{RH_q(\Delta_0,\omega_{L_0})} \lesssim \Upsilon$ and this completes the proof. $\square$

3.2. Proof Proposition 3.1, part (a)

We start assuming that Ω is a **bounded** 1-sided NTA domain satisfying the CDC and whose boundary $\partial\Omega$ is bounded. We fix $\mathbb{D} = \mathbb{D}(\partial\Omega)$ the dyadic grid from Lemma 2.33 with $E = \partial\Omega$. As in the statement of Proposition 3.1 let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be two real (non-necessarily symmetric) elliptic operators. Fix $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ and let $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$. From now on $X_0 := X_{\Delta_0}$, $\omega_0 := \omega_{L_0}^{X_0}$ and $\omega := \omega_L^{X_0}$.

We first observe that we can reduce the proof to the case $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$. Assuming that this has been already proved we now explain how to consider the general case. Let $B_0 = B(x_0, r_0)$ with $\operatorname{diam}(\partial\Omega)/2 \leq r_0 < \operatorname{diam}(\partial\Omega)$. We proceed as **Case 2b** in the proof of Theorem 1.5 part (a) with $M = 1$ to find the corresponding collection $\{x_j\}_{j=1}^J$ with $J \leq 21^{n+1}$. Let $B_j = B(x_j, \operatorname{diam}(\partial\Omega)/10)$ for $1 \leq j \leq J$. Then we can easily see that Harnack's inequality yields $\sup_{1 \leq j \leq J} |||\varrho(A, A_0)|||_{B_j,\Omega_\star} \lesssim |||\varrho(A, A_0)|||_{B_0}$ and since $r_{B_j} < \operatorname{diam}(\partial\Omega)/2$ we can apply the claimed case to conclude that $\omega_L \in RH_q(3\Delta_j, \omega_{L_0})$ (for part (b), $q = p$). At this point we carry out the same argument *mutatis mutandis* to conclude that $\omega_L \in RH_q(\Delta_0, \omega_{L_0})$ which completes the proof.

We split the proof in several steps.

3.2.1. Step 0. We first make a reduction which will allow us to use some qualitative properties of the elliptic measure. By Lemma 3.38 it suffices to show that there exists $1 < q < \infty$ such that for every j large enough $\omega_{L^j} \in RH_q(\frac{5}{4}\Delta_0, \omega_0)$ uniformly in j and in Δ_0. Thus we fix $j \in \mathbb{N}$ and let $\widetilde{L} = L^j$ be the operator defined by $\widetilde{L}u = -\operatorname{div}(\widetilde{A}\nabla u)$, with $\widetilde{A} = A^j$ (see (3.37)). As mentioned above $\widetilde{A}$ is uniformly elliptic with constant $\Lambda_0 = \max\{\Lambda_A, \Lambda_{A_0}\}$. Also, since $\widetilde{L} \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$, by Lemma 2.69 part (f) and Harnack's inequality give that $\omega_{L_0} \ll \omega_{\widetilde{L}} \ll \omega_{L_0}$, hence recalling (2.57) we have that $h(\cdot; \widetilde{L}, L_0, X)$ exists ω_0^X-a.e. for every $X \in \Omega$. Moreover, fixed $\Delta_1 = \Delta(x_1, r_1)$ with $x_1 \in \partial\Omega$ and $0 < r_1 < 2^{-j-2}/\kappa_0$ for every $\Delta = B \cap \partial\Omega$ with $B = B(x, r) \subset B_1$, $x \in \partial\Omega$, and $0 < r < \operatorname{diam}(\partial\Omega)$, we have by Lemma 2.69 part (f)

$$1 \approx \frac{\omega_{\widetilde{L}}^{X_{\Delta_1}}(\Delta(x,r))}{\omega_{L_0}^{X_{\Delta_1}}(\Delta(x,r))} = \fint_{\Delta(x,r)} h(y; \widetilde{L}, L_0, X_{\Delta_1})\, d\omega_{L_0}^{X_{\Delta_1}}(y).$$

Letting $r \to 0^+$ the Lebesgue differentiation theorem (whose applicability is ensured by the fact that $\omega_{L_0}^{X_{\Delta_1}}$ is doubling in Δ_1) yields

$$h(y; \widetilde{L}, L_0, X_{\Delta_1}) \approx 1, \qquad \text{for } \omega_{L_0}^{X_{\Delta_1}}\text{-a.e. } x \in \Delta_1.$$

Thus, by Harnack's inequality $h(\cdot\,; \widetilde{L}, L_0, X) \in L^\infty_{\rm loc}(\partial\Omega, \omega_{L_0}^Y)$ for every $X, Y \in \Omega$ — the actual norm will depend on X, Y and j, but we will use this fact in a qualitative fashion. This qualitative control will be essential in the following steps. At the end of **Step 3** we will have obtained the desired conclusion for the operator $\widetilde{L} = L^j$, with constants independent of $j \in \mathbb{N}$, which as observed above will allow us to complete the proof by Lemma 3.38.

3.2.2. Step 1. Let us recall that we have fixed already $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$ and let $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$, $X_0 = X_{\Delta_0}$, and $\omega_0 = \omega_{L_0}^{X_0}$. Set $\widetilde{\omega} := \omega_{\widetilde{L}}^{X_0}$. Fix $Q^0 \in \mathbb{D}_*^{\Delta_0}$ (see (3.26)), so that by (2.48),

$$X_0 \in \Omega \setminus B_{Q^0}^* \subset \Omega \setminus \tfrac{1}{2}B_{Q^0}^* \subset \Omega \setminus T_{Q^0}^{**}. \tag{3.48}$$

Set $\mathcal{E}(Y) := A(Y) - A_0(Y)$, $Y \in \Omega$, and consider $\gamma = \{\gamma_Q\}_{Q \in \mathbb{D}_{Q^0}}$

$$\gamma_Q = \gamma_{X_0, Q} := \omega_0(Q) \sum_{I \in \mathcal{W}_Q^*} \sup_{Y \in I^*} \|\mathcal{E}\|_{L^\infty(I^*)}^2 \qquad \text{whenever } Q \in \mathbb{D}_{Q^0}. \tag{3.49}$$

Lemma 3.34 yields that $\|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q^0, \omega_0)} \lesssim \|\!|\!|\varrho(A, A_0)|\!|\!|_{B_0} < \infty$, hence $\mathfrak{m}_\gamma$ is a discrete Carleson measure with respect to ω_0 in Q^0. Our goal is to show that $\widetilde{\omega} \in A_\infty^{\rm dyadic}(Q^0, \omega_0)$ and we will use Lemma 2.32 with $\mu = \omega_0$. To this aim we fix $Q_0 \in \mathbb{D}_{Q^0}$ and a family of pairwise disjoint dyadic cubes $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ such that

$$\|\mathfrak{m}_{\gamma, \mathcal{F}}\|_{\mathcal{C}(Q_0, \omega_0)} = \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}_\gamma(\mathbb{D}_{\mathcal{F}, Q})}{\omega_0(Q)} \leq \varepsilon_0, \tag{3.50}$$

with $\varepsilon_0 > 0$ sufficiently small to be chosen and where we have used the notation introduced in (2.30) and (2.31).

We modify the operator $\widetilde{L}$ inside the region $\Omega_{\mathcal{F},Q_0}$ (see (2.42)), by defining $L_1 = L_1^{\mathcal{F},Q_0}$ as $L_1 u = -\operatorname{div}(A_1 \nabla u)$, where

$$A_1(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in \Omega_{\mathcal{F},Q_0}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus \Omega_{\mathcal{F},Q_0}. \end{cases}$$

See Figure 2. Recalling that $\widetilde{A} = A^j$ (see (3.37)), it is clear that $\mathcal{E}_1 := A_1 - A_0$ verifies $|\mathcal{E}_1| \le |\mathcal{E}| \mathbf{1}_{\Omega_{\mathcal{F},Q_0}}$ and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$ (this latter condition will be used qualitatively). Hence much as before if we write $\omega_1^X = \omega_{L_1}^X$ for every $X \in \Omega$ and $\omega_1 = \omega_1^{X_0}$ we have that $\omega_1 \ll \omega_0$ and hence we can write $h(\cdot\,; L_1, L_0, X_0) = d\omega_1/d\omega_0$ which is well-defined ω_0-a.e. Also, as shown in **Step 0** we have that $h(\cdot\,; L_1, L_0, X_0) \in L^\infty_{\rm loc}(\partial\Omega, \omega_0)$ (the bound depends on X_0 and the fixed j but we will use this qualitatively).

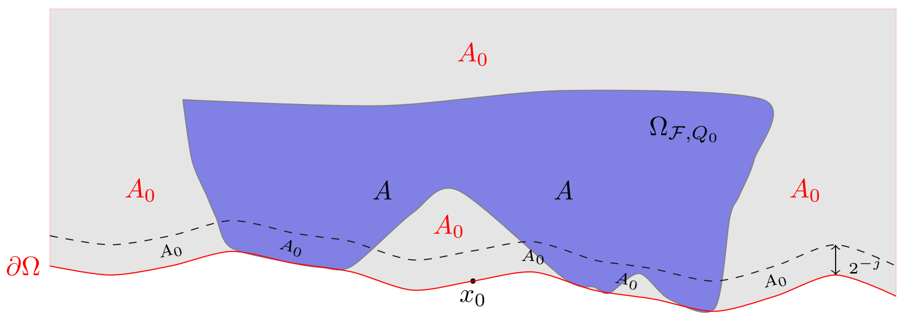

FIGURE 2. Definition of the matrix A_1 in Ω.

We next fix $Q_0^\star \in \mathbb{D}_{Q_0}$ an define $L_1^\star = L_1^{\mathcal{F},Q_0^\star}$ as $L_1^\star u = -\operatorname{div}(A_1^\star \nabla u)$ where

$$A_1^\star(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in \Omega_{\mathcal{F},Q_0^\star}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus \Omega_{\mathcal{F},Q_0^\star}. \end{cases}$$

Note that if $Q_0^\star = Q_0$ then $L_1^\star \equiv L_1$. Again $\mathcal{E}_1^\star := A_1^\star - A_0$ verifies $|\mathcal{E}_1^\star| \le |\mathcal{E}| \mathbf{1}_{\Omega_{\mathcal{F},Q}}$ and also $\mathcal{E}_1^\star(Y) = 0$ if $\delta(Y) < 2^{-j}$ (this latter condition will be used qualitatively). Hence if write $\omega_\star^X = \omega_{L_1^\star}^X$ for every $X \in \Omega$ we have that $\omega_\star^X \ll \omega_0^X$ for every $X \in \Omega$ and hence we can write $h(\cdot\,; L_1^\star, L_0, X) = d\omega_\star^X/d\omega_0^X$ which is well-defined ω_0^X-a.e. Also, as shown in **Step 0** we have $h(\cdot\,; L_1^\star, L_0, X) \in L^\infty_{\rm loc}(\partial\Omega, \omega_0^Y)$ for every $X, Y \in \Omega$ (the bound depends on X, Y and the fixed j but we will use this qualitatively).

Set $X_\star := X_{c_0^{-1}\Delta^*_{Q_0^\star}}$ which satisfies $2\kappa_0 r_{Q_0^\star} \le \delta(X_\star) < r_0$ since $\ell(Q_0^\star) \le \ell(Q_0) \le \ell(Q^0) \le \frac{c_0}{8\kappa_0} r_0$. Moreover, $X_\star \in \Omega \setminus B^*_{Q_0^\star}$. To simplify the notation set $\omega_\star = \omega_\star^{X_\star}$ and $\omega_0^\star = \omega_0^{X_\star}$.

We have two cases:

Case 1: $Q_0^\star \notin \mathbb{D}_{\mathcal{F},Q_0}$, that is, $Q_0^\star \subset Q_j \in \mathcal{F}$ for some j. Clearly, $\Omega_{\mathcal{F},Q_0^\star} = \emptyset$ and hence $L_1^\star \equiv L_0$ in Ω. As a consequence, $\omega_\star^X \equiv \omega_0^X$ for every $X \in \Omega$ and $h(\cdot\,; L_1^\star, L_0, X_\star) \equiv 1$ in $\partial\Omega$. In turn we obtain

$$\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star, \omega_0)} = \omega_0^\star(Q_0^\star)^{\frac{1}{q'}}. \tag{3.51}$$

Case 2: $Q_0^\star \in \mathbb{D}_{\mathcal{F},Q_0}$. In this case it is easy to see that

$$\mathcal{F}_\star := \{Q_j \in \mathcal{F} : Q_j \cap Q_0^\star \neq \emptyset\} = \{Q_j \in \mathcal{F} : Q_j \subset Q_0^\star\} \subset \mathbb{D}_{Q_0^\star}.$$

Thus, $\mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_{Q_0^\star} = \mathbb{D}_{\mathcal{F}_\star} \cap \mathbb{D}_{Q_0^\star}$ and $\Omega_{\mathcal{F},Q_0^\star} = \Omega_{\mathcal{F}_\star,Q_0^\star}$. On the other hand, we set $\gamma^\star = \{\gamma_Q^\star\}_{Q\in\mathbb{D}_{Q_0^\star}}$ where

$$\gamma_Q^\star := \omega_0^{X_\star}(Q) \sum_{I\in\mathcal{W}_Q^*} \sup_{Y\in I^*} \|\mathcal{E}\|^2_{L^\infty(I^*)}, \qquad \text{whenever } Q \in \mathbb{D}_{Q_0^\star}.$$

Using (2.71) and Harnack's inequality we have that $\omega_0^\star(Q) \approx \omega_0(Q)/\omega_0(Q_0^\star)$ for $Q \in \mathbb{D}_{Q_0^\star}$ where $\omega_0^\star = \omega_0^{X_\star}$. Hence, by (3.49),

$$\gamma_Q^\star \approx \frac{\omega_0(Q)}{\omega_0(Q_0^\star)} \sum_{I\in\mathcal{W}_Q^*} \sup_{Y\in I^*} \|\mathcal{E}\|^2_{L^\infty(I^*)} = \frac{\gamma_Q}{\omega_0(Q_0^\star)}, \qquad Q \in \mathbb{D}_{Q_0^\star}.$$

and, by (3.50),

$$(3.52)\quad \|\mathbf{m}_{\gamma^\star,\mathcal{F}_\star}\|_{\mathcal{C}(Q_0^\star,\omega_0^\star)} = \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_{\gamma^\star}(\mathbb{D}_Q \cap \mathbb{D}_{\mathcal{F}_\star})}{\omega_0^\star(Q)} = \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_{\gamma^\star}(\mathbb{D}_Q \cap \mathbb{D}_{\mathcal{F}})}{\omega_0^\star(Q)}$$

$$\approx \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_\gamma(\mathbb{D}_{\mathcal{F},Q})}{\omega_0^\star(Q)\omega_0(Q_0^\star)} \approx \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_\gamma(\mathbb{D}_{\mathcal{F},Q})}{\omega_0(Q)} \le \|\mathbf{m}_{\gamma,\mathcal{F}}\|_{\mathcal{C}(Q_0,\omega_0)} \le \varepsilon_0.$$

We next fix $1 < q < \infty$ and $0 \le g \in L^q(Q_0^\star, \omega_0^\star)$ with $\|g\|_{L^q(Q_0^\star,\omega_0^\star)} = 1$. Extend g by 0 in $\partial\Omega \setminus Q_0^\star$. Set $g_t = P_t g$ with $0 < t < \ell(Q_0^\star)/3$ (see (3.27)) and note that Lemma 3.29 gives that $g_t \in \operatorname{Lip}(\partial\Omega)$ with $\operatorname{supp}(g_t) \subset 2\widetilde{\Delta}_{Q_0^\star}$. We then consider

$$u_0^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_0^X(y) \quad \text{and} \quad u_\star^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_\star^X(y), \qquad X \in \Omega.$$

Since Ω is bounded, we can use Lemma 3.7 (slightly moving $X_\star$ if needed). This, Lemma 3.9, (3.52), and Hölder's inequality yield

$$\begin{aligned}
|u_\star^t(X_\star) - u_0^t(X_\star)| &= \left| \iint_\Omega (A_0 - A_1^\star)^\top(Y) \nabla_Y G_{(L_1^\star)^\top}(Y, X_\star) \cdot \nabla u_0^t(Y)\, dY \right| \\
&\le \iint_{\Omega_{\mathcal{F}_\star,Q_0^\star}} |\mathcal{E}(Y)|\, |\nabla_Y G_{(L_1^\star)^\top}(Y, X_\star)|\, |\nabla u_0^t(Y)|\, dY \\
&\lesssim \|\mathfrak{m}_{\gamma^\star,\mathcal{F}_\star}\|^{\frac12}_{\mathcal{C}(Q_0^\star,\omega_0^\star)} \int_{Q_0^\star} M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}(\omega_1^\star)(x) \mathcal{S}_{Q_0^\star} u_0^t(x)\, d\omega_0^\star(x) \\
&\lesssim \varepsilon_0^{\frac12} \int_{Q_0^\star} M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}(\omega_\star)(x) \mathcal{S}_{Q_0^\star} u_0^t(x)\, d\omega_0^\star(x) \\
&\le \varepsilon_0^{\frac12} \|M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}(\omega_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} \|\mathcal{S}_{Q_0^\star} u_0^t(x)\|_{L^q(Q_0^\star,\omega_0^\star)}.
\end{aligned}$$

Using the well-known fact that $M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}$ is bounded on $L^{q'}(Q_0^\star, \omega_0^\star)$ and that, as mentioned before $\omega_\star \ll \omega_0^\star$ with $h(\cdot\,; L_1^\star, L_0, X_\star) = d\omega_\star/d\omega_0^\star$, it readily follows that

$$\|M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}(\omega_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} \lesssim \|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)}.$$

On the other hand, Theorem 5.3, Lemmas 3.20 and 3.29, Remark 2.70, and Harnack's inequality to pass from $X_\star$ to $X_{Q_0^\star}$, and the fact that $\operatorname{supp} g \subset Q_0^\star$, yield

$$\|\mathcal{S}_{Q_0^\star} u_0^t(x)\|_{L^q(Q_0^\star,\omega_0^\star)} \lesssim \|\mathcal{N}_{Q_0^\star} u_0^t\|_{L^q(Q_0^\star,\omega_0^\star)} \lesssim \|g_t\|_{L^q(Q_0^\star,\omega_0^\star)}$$

$$\approx \frac{1}{\omega_0(Q_0^\star)^{\frac{1}{q}}}\|g_t\|_{L^q(Q_0^\star,\omega_0)} \lesssim \frac{1}{\omega_0(Q_0^\star)^{\frac{1}{q}}}\|g\|_{L^q(Q_0^\star,\omega_0)} \approx \|g\|_{L^q(Q_0^\star,\omega_0^\star)} = 1.$$

Thus we conclude that

$$|u_\star^t(X_\star) - u_0^t(X_\star)| \lesssim \varepsilon_0^{\frac{1}{2}}\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)},$$

and hence using the definitions of u_0^t and $u_\star^t$ we conclude that

$$(3.53)\quad \begin{aligned} &\Big|\int_{\partial\Omega} g(y)d\omega_\star(y) - \int_{\partial\Omega} g(y)d\omega_0^\star(y)\Big| \\ &\quad\le |u_\star^t(X_\star) - u_0^t(X_\star)| + \|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)} + \|g-g_t\|_{L^1(\partial\Omega,\omega_\star)} \\ &\quad\lesssim \varepsilon_0^{\frac{1}{2}}\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} + \|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)} + \|g-g_t\|_{L^1(\partial\Omega,\omega_\star)}. \end{aligned}$$

Since $g \in L^q(Q_0,\omega_0)$ with $\operatorname{supp}(g) \subset Q_0^\star$, it follows that $\operatorname{supp}(g), \operatorname{supp}(g_t) \subset 2\widetilde{\Delta}_{Q_0^\star}$. Hence, Lemma 3.29, Harnack's inequality and (2.72) give

$$(3.54)\quad \begin{aligned} \|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)} &= \|g-P_t g\|_{L^1(2\widetilde{\Delta}_{Q_0^\star},\omega_0^\star)} \\ &\approx \frac{1}{\omega_0(Q_0^\star)}\|g-P_t g\|_{L^1(2\widetilde{\Delta}_{Q_0^\star},\omega_0)} \to 0, \quad \text{as } t\to 0^+. \end{aligned}$$

Similarly, using also that as mentioned above $\omega_\star \ll \omega_0$ with $h(\cdot\,; L_1^\star, L_0, X_\star) \in L^\infty_{\rm loc}(\partial\Omega,\omega_0)$

$$(3.55)\quad \begin{aligned} &\|g-g_t\|_{L^1(\partial\Omega,\omega_\star)} = \|g-P_t g\|_{L^1(2\widetilde{\Delta}_{Q_0^\star},\omega_\star)} \\ &\quad\le \|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^\infty(2\widetilde{\Delta}_{Q_0^\star},\omega_0^\star)}\|g-P_t g\|_{L^1(2\widetilde{\Delta}_{Q_0^\star},\omega_0^\star)} \to 0, \qquad \text{as } t\to 0^+. \end{aligned}$$

Combining (3.53), (3.54), (3.55) and letting $t\to 0^+$ we conclude that

$$\begin{aligned} 0 \le \int_{Q_0^\star} h(y; L_1^\star, L_0, X_\star)\, g(y) d\omega_0^\star(y) &= \int_{\partial\Omega} h(y; L_1^\star, L_0, X_\star)\, g(y) d\omega_0^\star(y) \\ &= \int_{\partial\Omega} g(y) d\omega_\star(y) \\ &\lesssim \varepsilon_0^{\frac{1}{2}}\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0)} + \int_{\partial\Omega} g(y)d\omega_0^\star(y) \\ &\le \varepsilon_0^{\frac{1}{2}}\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} + \omega_0^\star(Q_0^\star)^{\frac{1}{q'}}. \end{aligned}$$

Taking now the sup over all $0 \le g \in L^q(Q_0^\star,\omega_0^\star)$ with $\|g\|_{L^q(Q_0^\star,\omega_0^\star)} = 1$ we eventually get

$$(3.56)\quad \|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} \lesssim \varepsilon_0^{\frac{1}{2}}\|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} + \omega_0^\star(Q_0^\star)^{\frac{1}{q'}}.$$

Since $h(\cdot\,; L_1, L_0^\star, X_\star) \in L^\infty_{\rm loc}(\partial\Omega,\omega_0^\star)$ (albeit with bounds which may depend on $X_\star$ or j) we can hide the first term on the right hand side and eventually obtain fixing ε_0 small enough (depending on n, the 1-sided NTA constants, the CDC constant, the ellipticity constants of L_0 and L_2, and on q),

$$(3.57)\quad \|h(\cdot\,; L_1^\star, L_0, X_\star)\|_{L^{q'}(Q_0^\star,\omega_0^\star)} \lesssim \omega_0^\star(Q_0^\star)^{\frac{1}{q'}}.$$

Note then that by (3.51) we conclude that (3.57) holds for any $Q_0^\star \in \mathbb{D}_{Q_0}$. On the other hand, using [**HM14**, Lemma 3.55] (which holds as well in our scenario), there exists $0 < \widehat{\kappa}_1 < \kappa_1$ (see (2.48)), depending only on the allowable parameters, such that $\widehat{\kappa}_1 B_{Q_0^\star} \cap \Omega_{\mathcal{F},Q_0} = \widehat{\kappa}_1 B_{Q_0^\star} \cap \Omega_{\mathcal{F},Q_0^\star}$, Hence $L_1^\star \equiv L_1$ in $\widehat{\kappa}_1 B_{Q_0^\star} \cap \Omega$ which, by Lemma 2.69 part (f) and Harnack's inequality, gives that $\omega_\star$ and $\omega_0^\star$ are comparable in $\eta\Delta_{Q_0^\star}$ with $\eta = \widehat{\kappa}_1/(2\kappa_0)$, thus $h(\cdot\,; L_1^\star, L_0, X_\star) \approx h(\cdot\,; L_1, L_0, X_\star)$ for $\omega_0^\star$-a.e. in $\eta\Delta_{Q_0^\star}$ (hence, also ω_0-a.e.). This, Remark 2.70, Harnack's inequality, and Lemma 2.69 part (c) yield

$$h(\cdot\,; L_1, L_0, X_0) = \frac{d\omega_{L_1}^{X_0}}{d\omega_{L_0}^{X_0}} = \frac{d\omega_{L_1}^{X_0}}{d\omega_{L_1}^{X_\star}} \frac{d\omega_{L_1}^{X_\star}}{d\omega_{L_0}^{X_\star}} \frac{d\omega_{L_0}^{X_\star}}{d\omega_{L_0}^{X_0}}$$
$$\approx \frac{\omega_1(Q_0^\star)}{\omega_0(Q_0^\star)} h(\cdot\,; L_1, L_0, X_\star) \approx \frac{\omega_1(\eta\Delta_{Q_0^\star})}{\omega_0(\eta\Delta_{Q_0^\star})} h(\cdot\,; L_1^\star, L_0, X_\star),$$

and these hold ω_0-a.e. in $\eta\Delta_{Q_0^\star}$ an $\forall\, Q_0^\star \in \mathbb{D}_{Q_0}$ (recall that ω_1 and ω_0 are mutually absolutely continuous). Eventually, (3.57), Remark 2.70 and Harnack's inequality allow us to conclude that for all $Q_0^\star \in \mathbb{D}_{Q_0}$

$$(3.58)\quad \left(\fint_{\eta\Delta_{Q_0^\star}} h(y; L_1, L_0, X_0)^{q'} d\omega_0(y) \right)^{\frac{1}{q'}}$$
$$\approx \frac{\omega_1(\eta\Delta_{Q_0^\star})}{\omega_0(\eta\Delta_{Q_0^\star})} \left(\fint_{\eta\Delta_{Q_0^\star}} h(y; L_1^\star, L_0, X_\star)^{q'} d\omega_0^\star(y) \right)^{\frac{1}{q'}}$$
$$\lesssim \frac{\omega_1(\eta\Delta_{Q_0^\star})}{\omega_0(\eta\Delta_{Q_0^\star})} = \fint_{\eta\Delta_{Q_0^\star}} h(y; L_1, L_0, X_0) d\omega_0(y).$$

Our next goal is to show that the latter implies that $\omega_1 \in A_\infty^{\text{dyadic}}(Q_0, \omega_0)$ and to show that we use an argument similar to [**CHM19**, Lemma 3.1]. Let $Q \in \mathbb{D}_{Q_0}$ and a Borel set $F \subset \eta\Delta_Q$ and note that by (3.58) applied to Q

$$\frac{\omega_1(F)}{\omega_0(\eta\Delta_Q)} = \fint_{\eta\Delta_Q} \mathbf{1}_F(y) h(y; L_1, L_0, X_0) d\omega_0$$
$$\leq \left(\frac{\omega_0(F)}{\omega_0(\eta\Delta_Q)} \right)^{\frac{1}{q}} \left(\fint_{\eta\Delta_Q} h(y; L_1, L_0, X_0)^{q'} d\omega_0(y) \right)^{\frac{1}{q'}}$$
$$\leq C_1 \left(\frac{\omega_0(F)}{\omega_0(\eta\Delta_Q)} \right)^{\frac{1}{q}} \left(\fint_{\eta\Delta_Q} h(y; L_1, L_0, X_0) d\omega_0(y) \right)$$
$$= C_1 \left(\frac{\omega_0(F)}{\omega_0(\eta\Delta_Q)} \right)^{\frac{1}{q}} \frac{\omega_1(\eta\Delta_Q)}{\omega_0(\eta\Delta_Q)}$$

and hence

$$(3.59)\quad \frac{\omega_1(F)}{\omega_1(\eta\Delta_Q)} \leq C_1 \left(\frac{\omega_0(F)}{\omega_0(\eta\Delta_Q)} \right)^{\frac{1}{q}}, \qquad \forall\, F \subset \eta\Delta_Q,\ Q \in \mathbb{D}_{Q_0}.$$

On the other hand, by Lemma 2.69 part (c), $\omega_0(Q) \leq C_2 \omega_0(\eta\Delta_Q)$ for all $Q \in \mathbb{D}_{Q_0}$. Fix then α, $0 < \alpha < (C_2 C_1^q)^{-1}$, and take $F \subset Q$ such that $\omega_0(F) > (1-\alpha)\omega_0(Q)$. Writing $F_0 = \eta\Delta_Q \cap F$ and $F_1 = \eta\Delta_Q \setminus F$, it is clear that

$$(1-\alpha)\frac{\omega_0(Q)}{\omega_0(\eta\Delta_Q)} < \frac{\omega_0(F)}{\omega_0(\eta\Delta_Q)} \leq \frac{\omega_0(F_0)}{\omega_0(\eta\Delta_Q)} + \frac{\omega_0(Q \setminus \eta\Delta_Q)}{\omega_0(\eta\Delta_Q)}$$

$$= \frac{\omega_0(F_0)}{\omega_0(\eta\Delta_Q)} + \frac{\omega_0(Q)}{\omega_0(\eta\Delta_Q)} - 1,$$

and hence

$$\frac{\omega_0(F_1)}{\omega_0(\eta\Delta_Q)} = 1 - \frac{\omega_0(F_0)}{\omega_0(\eta\Delta_Q)} < \alpha\, \frac{\omega_0(Q)}{\omega_0(\eta\Delta_Q)} \le C_2\,\alpha. \tag{3.60}$$

Combining (3.59) and (3.60) applied to F_1 we obtain $\omega_1(F_1)/\omega_1(\eta\Delta_Q) < C_1\big(C_2\alpha\big)^{\frac{1}{q}}$. This and the fact that $\omega_1(Q) \le C_3\omega_1(\eta\Delta_Q)$, by Lemma 2.69 part (c), yield

$$\frac{\omega_1(F)}{\omega_1(Q)} \ge \frac{\omega_1(\eta\Delta_Q)}{\omega_1(Q)}\, \frac{\omega_1(F_0)}{\omega_1(\eta\Delta_Q)} \ge C_3^{-1}\left(1 - \frac{\omega_1(F_1)}{\omega_1(\eta\Delta_Q)}\right)$$
$$> C_3^{-1}\big(1 - C_1(C_2\alpha)^{\frac{1}{q}}\big) =: 1-\beta,$$

with $0<\beta<1$ by our choice of α. This eventually shows that $\omega_1 \in A_\infty^{\rm dyadic}(Q_0,\omega_0)$ (see Definition (2.24)) as desired. This with the help of Lemma 2.27 allows us to obtain that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1 \in A_\infty^{\rm dyadic}(Q_0,\omega_0)$, which is the conclusion of **Step 1**.

3.2.3. Step 2. We next define a new operator $L_2 u = -\operatorname{div}(A_2\nabla u)$ where (see Figure 3):

$$A_2(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in T_{Q_0}\setminus \Omega_{\mathcal{F},Q_0}, \\ A_1(Y) & \text{if } Y \in \Omega \setminus (T_{Q_0}\setminus \Omega_{\mathcal{F},Q_0}). \end{cases}$$

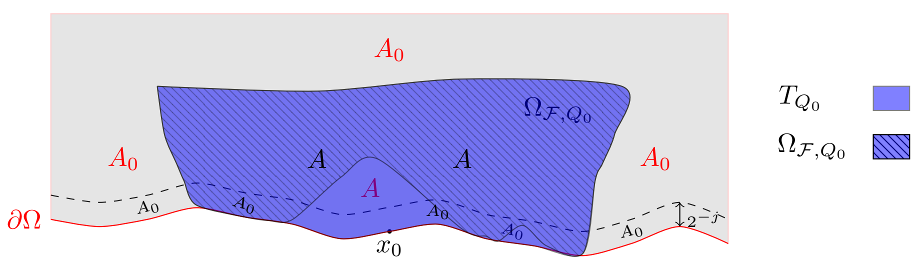

FIGURE 3. Definition of matrix A_2 in Ω.

The goal of this step is to show that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2 \in A_\infty^{\rm dyadic}(Q_0,\omega_0)$, where much as before let $\omega_2 = \omega_{L_2}^{X_0}$.

We apply Lemma 3.18 to obtain $Y_{Q_0} \in \Omega\cap\Omega_{\mathcal{F},Q_0}$ satisfying (3.19). For $k=1,2$ we write $\omega_k^{Y_{Q_0}} = \omega_{L_k,\Omega}^{Y_{Q_0}}$ a for the elliptic measures associated with L_k for the domain Ω and with pole at Y_{Q_0}. Likewise, let $\omega_{k,*}^{Y_{Q_0}} = \omega_{L_k,\Omega_{\mathcal{F},Q_0}}^{Y_{Q_0}}$ be the elliptic measures associated with L_k for the domain $\Omega_{\mathcal{F},Q_0}$ and with pole at Y_{Q_0}. By definition $A_2 = \tilde{A}$ in T_{Q_0}, $A_2 = A_0$ in $\Omega\setminus T_{Q_0}$, and $A_2 = A_1$ in $\Omega_{\mathcal{F},Q_0}$. Hence $L_2 \equiv L_1$ in $\Omega_{\mathcal{F},Q_0}$, and thus $\omega_{2,*}^{Y_{Q_0}} \equiv \omega_{1,*}^{Y_{Q_0}}$. If we now consider the associated measures $\nu_{L_1}^{Y_{Q_0}}$ and $\nu_{L_2}^{Y_{Q_0}}$ in (4.5) from Lemma 4.4 it follows from (4.6) (with $\mu=\omega_0$ which is clearly (dyadically) doubling in Q_0 by Lemma 2.69 part (c)) that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_1}^{Y_{Q_0}} = \mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_2}^{Y_{Q_0}}$ as measures on Q_0.

In **Step 1** we showed that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1 \in A_\infty^{\rm dyadic}(Q_0,\omega_0)$, then there is $1<\widetilde{q}<\infty$ such that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1 \in RH_{\widetilde{q}}^{\rm dyadic}(Q_0,\omega_0)$. Note that by Remark 2.70 and Harnack's inequality we have that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_k^{Y_{Q_0}} \approx \mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_k/\omega_1(Q_0)$ for $k = 1, 2$. Then given $Q \in \mathbb{D}_{Q_0}$ and a Borel set $F \subset Q$ we have that all these yield

$$\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(Q)} \approx \frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_{L_2}^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_{L_2}^{Y_{Q_0}}(Q)} \lesssim \left(\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_2}^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_2}^{Y_{Q_0}}(Q)}\right)^{\frac{1}{\theta_2}} = \left(\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_1}^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\nu_{L_1}^{Y_{Q_0}}(Q)}\right)^{\frac{1}{\theta_2}}$$
$$\lesssim \left(\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_{L_1}^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_{L_1}^{Y_{Q_0}}(Q)}\right)^{\frac{1}{\theta_2}} \lesssim \left(\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1(Q)}\right)^{\frac{1}{\theta_2}} \lesssim \left(\frac{\omega_0(F)}{\omega_0(Q)}\right)^{\frac{1}{\theta_2 \widetilde{q}'}}$$

where in the second and third estimates we have invoked Lemma 4.4 respectively for L_2 (with parameter θ_2) and L_1, and the last estimate follows easily from the fact that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_1 \in RH_{\widetilde{q}}^{\text{dyadic}}(Q_0, \omega_0)$ and Hölder's inequality. This, the fact that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2$ is dyadic doubling in Q_0 by Lemma 2.27 part (a) since ω_2 is indeed doubling in $4\widetilde{\Delta}_{Q_0}$ by Lemma 2.69 part (c), and [**HM14**, Lemma B.7] (which is a purely dyadic result and hence applies in our setting) gives that there exists $\theta, \theta' > 0$ such that

$$\left(\frac{\omega_0(F)}{\omega_0(Q)}\right)^{\theta} \lesssim \frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(Q)} \lesssim \left(\frac{\omega_0(F)}{\omega_0(Q)}\right)^{\theta'}, \qquad \forall F \subset Q,\ Q \in \mathbb{D}_{Q_0}. \tag{3.61}$$

3.2.4. Step 3. In this part, we change the operator outside of T_{Q_0} to complete the process. To this end, let $L_3 u = -\operatorname{div}(A_3 \nabla u)$, where

$$A_3(Y) := \begin{cases} A_2(Y) & \text{if } Y \in T_{Q_0}, \\ \widetilde{A}(Y) & \text{if } Y \in \Omega \setminus T_{Q_0}, \end{cases}$$

and note that $L_3 \equiv \widetilde{L}$ in Ω (see Figure 4). Let $w_3^{X_0} := \omega_{L_3}^{X_0}$ be the elliptic measure of Ω associated with the operator $L_3 \equiv \widetilde{L}$ with pole at X_0.

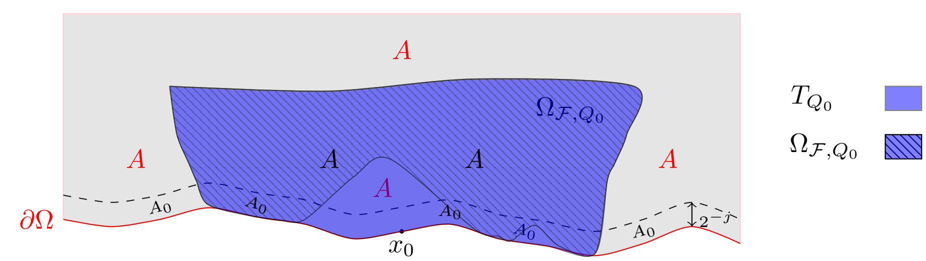

FIGURE 4. Definition of the matrix A_3 in Ω.

In this step we are going to need the following property: if $\tau > 0$ is small enough, there exists $C_\tau > 1$ such that

$$C_\tau^{-1}\frac{\omega_3(E)}{\omega_3(Q_0)} \le \frac{\omega_2(E)}{\omega_2(Q_0)} \le C_\tau \frac{\omega_3(E)}{\omega_3(Q_0)}, \qquad \forall E \subset Q_0 \setminus \Sigma_\tau, \tag{3.62}$$

where $\Sigma_\tau := \{x \in Q_0 : \operatorname{dist}(x, \partial\Omega \setminus Q_0) < \tau \ell(Q_0)\}$.

Assuming this momentarily, our final goal is to prove that for every ζ, $0 < \zeta < 1$, there exists $C_\zeta > 1$ such that

$$F \subset Q_0, \quad \frac{\omega_0(F)}{\omega_0(Q_0)} \ge \zeta \quad \Longrightarrow \quad \frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(Q_0)} \ge \frac{1}{C_\zeta}. \tag{3.63}$$

Fix then $\zeta \in (0,1)$, and $F \subset Q_0$ with $\omega_0(F) \geq \zeta\omega_0(Q_0)$. Consider first the case on which $\mathcal{F} = \{Q_0\}$, in which case

$$\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(Q_0)} = \frac{\frac{\omega_0(F)}{\omega_0(Q_0)}\omega_3(Q_0)}{\frac{\omega_0(Q_0)}{\omega_0(Q_0)}\omega_3(Q_0)} = \frac{\omega_0(F)}{\omega_0(Q_0)} \geq \zeta,$$

which is the desired estimate with $C_\zeta = \zeta$. Thus we may assume that $\mathcal{F} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$. Let $\tau \ll 1$ small enough to be chosen and let $Q_0^\tau := Q_0 \setminus \bigcup_{Q' \in \mathcal{I}_\tau} Q'$, where

$$\mathcal{I}_\tau = \big\{Q' \in \mathbb{D}_{Q_0} : \tau\ell(Q_0) < \ell(Q') \leq 2\tau\ell(Q_0),\ Q' \cap \Sigma_\tau \neq \emptyset\big\}.$$

By construction, $\Sigma_\tau \subset \bigcup_{Q' \in \mathcal{I}_\tau} Q'$, and by (2.34) every $Q' \in \mathcal{I}_\tau$ satisfies $Q' \subset \Sigma_{(1+4\Xi)\tau}$. Using Lemma 2.37 and Remark 2.39, along with the fact that ω_0 is doubling in $4\Delta_0$ with a constant which does not depend on Δ_0 (see Lemma 2.69 part (c)), if $\tau = \tau(\zeta) > 0$ is sufficiently small then

$$\omega_0(Q_0 \setminus Q_0^\tau) \leq \omega_0(\Sigma_{(1+4\Xi)\tau}) \lesssim \tau^\eta \omega_0(Q_0) \leq \frac{\zeta}{2}\,\omega_0(Q_0).$$

Letting $F' = F \cap Q_0^\tau$, it follows that

$$\zeta\omega_0(Q_0) \leq \omega_0(F) \leq \omega_0(F') + \omega_0(Q_0 \setminus Q_0^\tau) \leq \omega_0(F') + \frac{\zeta}{2}\omega_0(Q_0).$$

Hence $\omega_0(F')/\omega_0(Q_0) \geq \zeta/2$ and by (3.61), we conclude that

$$\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(F')}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(Q_0)} \gtrsim \left(\frac{\omega_0(F')}{\omega_0(Q_0)}\right)^\theta \geq \left(\frac{\zeta}{2}\right)^\theta. \tag{3.64}$$

Our next goal is to show that there exists $c_\zeta > 0$ such that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(F') \geq c_\zeta \mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(F')$. To see this let $Q_k \in \mathcal{F}$ be such that $F' \cap Q_k \neq \emptyset$. We consider two cases. If $Q_k \subset Q_0^\tau$, we can invoke (3.62) since $Q_0^\tau \subset Q_0 \setminus \Sigma_\tau$, to conclude that

$$\frac{\omega_2(Q_k)}{\omega_2(Q_0)} \approx_\tau \frac{\omega_3(Q_k)}{\omega_3(Q_0)}. \tag{3.65}$$

Otherwise, $Q_k \setminus Q_0^\tau \neq \emptyset$, and there exists $Q' \in \mathcal{I}_\tau$ such that $Q_k \cap Q' \neq \emptyset$. Then necessarily $Q' \subsetneq Q_k$ —if $Q_k \subset Q'$ then $Q_k \subset Q_0 \setminus Q_0^\tau$, contradicting that $F' \cap Q_k \neq \emptyset$ and $F' \subset Q_0^\tau$— and, in particular, $\ell(Q_k) > \tau\ell(Q_0)$. Take $\widehat{Q}_k \in \mathbb{D}_{Q_k}$ with $x_{Q_k} \in \widehat{Q}_k$, $\ell(\widehat{Q}_k) = 2^{-M}\ell(Q_k)$ and $M > 1$ to be chosen. Note that $\operatorname{diam}(\widehat{Q}_k) \approx 2^{-M}\ell(Q_k)$ (see Remark 2.73) and clearly

$$\begin{aligned}\ell(Q_k) \approx r_{Q_k} \leq \operatorname{dist}(x_{Q_k}, \partial\Omega \setminus \Delta_{Q_k}) &\leq \operatorname{diam}(\widehat{Q}_k) + \operatorname{dist}(\widehat{Q}_k, \partial\Omega \setminus \Delta_{Q_k}) \\ &\approx 2^{-M}\ell(Q_k) + \operatorname{dist}(\widehat{Q}_k, \partial\Omega \setminus \Delta_{Q_k}).\end{aligned}$$

Taking $M \gg 1$ large enough, we conclude that

$$c\tau\ell(Q_0) < c\ell(Q_k) \leq \operatorname{dist}(\widehat{Q}_k, \partial\Omega \setminus \Delta_{Q_k}) \leq \operatorname{dist}(\widehat{Q}_k, \partial\Omega \setminus Q_0)$$

and hence $\widehat{Q}_k \subset Q_0 \setminus \Sigma_{c\tau}$. Using again (3.62) (with $c\tau$ in place of τ) and Lemma 2.69 part (c) we obtain

$$\frac{\omega_3(Q_k)}{\omega_3(Q_0)} \geq \frac{\omega_3(\widehat{Q}_k)}{\omega_3(Q_0)} \approx_\tau \frac{\omega_2(\widehat{Q}_k)}{\omega_2(Q_0)} \gtrsim \frac{\omega_2(Q_k)}{\omega_2(Q_0)}. \tag{3.66}$$

Combining (3.65), (3.66) and invoking (3.62), since $F' \subset Q_0^\tau \subset Q_0 \setminus \Sigma_\tau$, we conclude that

$$\frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(F)}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(Q_0)} \geq \frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(F')}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_3(Q_0)} = \frac{\omega_3(F' \setminus \bigcup_{Q_k \in \mathcal{F}} Q_k)}{\omega_3(Q_0)} + \sum_{Q_k \in \mathcal{F}} \frac{\omega_0(Q_k \cap F')}{\omega_0(Q_k)} \frac{\omega_3(Q_k)}{\omega_3(Q_0)}$$
$$\gtrsim_\zeta \frac{\omega_2(F' \setminus \bigcup_{Q_k \in \mathcal{F}} Q_k)}{\omega_2(Q_0)} + \sum_{Q_k \in \mathcal{F}} \frac{\omega_0(Q_k \cap F')}{\omega_0(Q_k)} \frac{\omega_2(Q_k)}{\omega_2(Q_0)} = \frac{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(F')}{\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_2(Q_0)} \gtrsim \Big(\frac{\zeta}{2}\Big)^\theta,$$

where we have used that $\tau = \tau(\zeta)$, that $\mathcal{P}_{\mathcal{F}}^{\omega_0}\omega_i(Q_0) = \omega_i(Q_0)$ for $i = 2, 3$, and the last estimate follows from (3.64). This eventually proves (3.63) in the present case and it remains to establish our claim (3.62).

To show (3.62) write $r = \tau\ell(Q_0)/(8\kappa_0)$ (see (2.49)) and find a maximal collection of points $\{x_k\}_{k\in\mathcal{K}} \subset Q_0 \setminus \Sigma_\tau$ with respect to the property that $|x_k - x_{k'}| > 2r/3$ for every $k, k' \in \mathcal{K}$ with $k \neq k'$. Write $\Delta_k = \Delta(x_k, r)$ and observe that $\{\frac{1}{3}\Delta_k\}_{k\in\mathcal{K}}$ is a family of pairwise disjoint surface balls such that $Q_0 \setminus \Sigma_\tau \subset \bigcup_{k\in\mathcal{K}} \Delta_k$. Note that by (2.34), we have $\frac{1}{3}\Delta_k \subset 2\widetilde{\Delta}_{Q_0} \subset \Delta(x_k, 3\Xi\ell(Q_0))$, for every $k \in \mathcal{K}$, hence Lemma 2.69 part (c) yields

$$\#\mathcal{K} C_\tau^{-1} \omega_0(2\widetilde{\Delta}_{Q_0}) \leq \sum_{k\in\mathcal{K}} \omega_0(\tfrac{1}{3}\Delta_k) = \omega_0\Big(\bigcup_{k\in\mathcal{K}} \tfrac{1}{3}\Delta_k\Big) \leq \omega_0(2\widetilde{\Delta}_{Q_0}),$$

which eventually gives $\#\mathcal{K} \leq C_\tau$.

We claim that $B_k^* \cap \Omega \subset T_{Q_0}$, with $B_k^* := B_{\Delta_k}^* = B(x_k, 2\kappa_0 r)$ and κ_0 as in (2.49). To see this let $Y \in B_k^* \cap \Omega$ and take $I \in \mathcal{W}$ such that $Y \in I$. Pick $y_k \in \partial\Omega$ verifying $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, y_k)$ and let $R_k \in \mathbb{D}$ be the unique dyadic cube such that $y_k \in R_k$ and $\ell(R_k) = \ell(I)$, thus $I \in \mathcal{W}_{R_k}^*$. Let us see that $R_k \in \mathbb{D}_{Q_0}$. First, by (2.40) and our choice of M

$$\ell(R_k) = \ell(I) \leq \operatorname{dist}(I, \partial\Omega) \leq |x_k - Y| < 2\kappa_0 r = \frac{1}{4}\tau\ell(Q_0) < \frac{1}{4}\ell(Q_0).$$

Also, since $x_k \in Q_0 \setminus \Sigma_\tau$, we can write by (2.40)

$$\tau\ell(Q_0) \leq \operatorname{dist}(x_k, \partial\Omega \setminus Q_0) \leq |x_k - Y| + \operatorname{diam}(I) + \operatorname{dist}(I, y_k) + \operatorname{dist}(y_k, \partial\Omega \setminus Q_0)$$
$$< \frac{1}{4}\tau\ell(Q_0) + \frac{5}{4}\operatorname{dist}(I, \partial\Omega) + \operatorname{dist}(y_k, \partial\Omega \setminus Q_0) \leq \frac{9}{16}\tau\ell(Q_0) + \operatorname{dist}(y_k, \partial\Omega \setminus Q_0),$$

and hence $y_k \in \operatorname{int}(Q_0)$. Since $y_k \in Q_0 \cap R_k$ and $\ell(R_k) < \ell(Q_0)/4$ it follows that $R_k \in \mathbb{D}_{Q_0}$. This and the fact that $Y \in I \in \mathcal{W}_{R_k}^*$ allow us to conclude that $Y \in T_{Q_0}$. Consequently, we have shown that $B_k^* \cap \Omega \subset T_{Q_0}$ and thus $L_2 \equiv L_3$ in $B_k^* \cap \Omega$ for every $k \in \mathcal{K}$.

Next, we observe that $\delta(X_{Q_0}) \approx \ell(Q_0)$, $\delta(X_{\Delta_k}) \approx \tau\ell(Q_0)$, and $|X_{Q_0} - X_{\Delta_k}| \lesssim \ell(Q_0)$. Hence, we can use Harnack's inequality to move from X_{Q_0} to X_{Δ_k} with constants depending on τ, Lemma 2.69 part (f) and Remark 2.70 to obtain that if $F_j \subset \Delta_j \cap Q_j$

$$\frac{\omega_2(F_k)}{\omega_2(Q_0)} \approx \omega_2^{X_{Q_0}}(F_k) \approx_\tau \omega_2^{X_{\Delta_j}}(F_k) \approx \omega_3^{X_{\Delta_j}}(F_k) \approx_\tau \omega_3^{X_{Q_0}}(F_k) \approx \frac{\omega_3(F_k)}{\omega_3(Q_0)}.$$

This and the fact $Q_0 \setminus \Sigma_\tau \subset \bigcup_{k\in\mathcal{K}} \Delta_k$ readily give (3.62) and we finish **Step 3**.

3.2.5. Step 4. Let us recap what we have obtained so far. Fixed $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$, we set $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$, $X_0 = X_{\Delta_0}$, and $\omega_0 = \omega_{L_0}^{X_0}$, in **Step 0** we took an arbitrary j and wrote $\widetilde{L} = L^j$, (see (3.37)) and $\widetilde{\omega} = \omega_{\widetilde{L}}^{X_0}$. For an arbitrary $Q^0 \in \mathbb{D}_*^{\Delta_0}$ (see (3.26)), and for any given $Q_0 \in \mathbb{D}_{Q^0}$ we let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes such that (3.50) holds with ε_0 small enough to be chosen. Combining **Step 1**–**Step 3** we have shown that if ε_0 is small enough (depending only in the allowable parameters) then (3.63) is satisfied. Note that keeping track of the constants one can easily see that C_ζ does not depend on j, x_0, r_0, Q^0 and Q_0 —the fact that $\widetilde{L} = L^j$, which agrees with L_0 in small boundary strip, was mainly used, and only in a qualitative fashion, in (3.56) in **Step 1** to *a priori* know that some term is finite so that it can be hidden. We can then invoke Lemma 2.32 with the dyadically doubling measures (see Lemma 2.69 part (c)) $\mu = \omega_0$ and $\nu = \widetilde{\omega}$ to eventually show that (3.63) (recalling that $L_3 \equiv \widetilde{L}$ as mentioned in **Step 3**) yields $\widetilde{\omega} \in A_\infty^{\rm dyadic}(Q^0, \omega_0)$ (uniformly on the implicit j and Q^0), that is, there exist $1 < q < \infty$ and C (independent of j an Q^0) such that for every $Q \in \mathbb{D}_{Q^0}$ with $Q^0 \in \mathbb{D}_*^{\Delta_0}$

$$(3.67)\qquad \left(\fint_Q h(y; \widetilde{L}, L_0, X_0)^q d\omega_0(y)\right)^{\frac{1}{q}} \le C \fint_Q h(y; \widetilde{L}, L_0, X_0) d\omega_0(y) = C\frac{\widetilde{\omega}(Q)}{\omega_0(Q)}.$$

Our next goal is to see that $\widetilde{\omega} \in RH_q(\frac{5}{4}\Delta_0, \omega_0)$ (uniformly in j). To do this let $\Delta = B \cap \partial\Omega$ with $B = B(x, r) \subset \frac{5}{4}B_0$ such that $x \in \partial\Omega$. Write $\widetilde{r} = \min\{\frac{r}{4\Xi}, \frac{c_0 r_0}{32\kappa_0}\}$, where Ξ is the constant in (2.34), and let

$$\widetilde{\mathbb{D}}^\Delta = \left\{Q \in \mathbb{D} :\ Q \cap \Delta \neq \emptyset,\ \widetilde{r} \le \ell(Q) < 2\widetilde{r}\right\}.$$

Clearly, $\widetilde{\mathbb{D}}^\Delta$ is a family of pairwise disjoint cubes such that $\Delta \subset \bigcup_{Q\in\widetilde{\mathbb{D}}^\Delta} Q \subset 2\Delta$. Note that if $Q \in \widetilde{\mathbb{D}}^\Delta$ then $\emptyset \neq Q \cap \Delta \subset Q \cap \frac{5}{4}\Delta_0 \subset Q \cap \frac{3}{2}\Delta_0$, thus $Q \cap Q^0 \neq \emptyset$ for some $Q^0 \in \mathbb{D}_*^{\Delta_0}$. Besides, $\ell(Q) < 2\widetilde{r} < c_0 r_0/(16\kappa_0) \le \ell(Q^0)$. Consequently, $Q \in \mathbb{D}_{Q^0}$ and (3.67) applies to each $Q \in \widetilde{\mathbb{D}}^\Delta$. All in one we have

$$\begin{aligned}\left(\fint_\Delta h(y; \widetilde{L}, L_0, X_0)^q d\omega_0(y)\right)^{\frac{1}{q}} &\lesssim \sum_{Q\in\widetilde{\mathbb{D}}^\Delta} \left(\fint_Q h(y; \widetilde{L}, L_0, X_0)^q d\omega_0(y)\right)^{\frac{1}{q}} \\ &\lesssim \sum_{Q\in\widetilde{\mathbb{D}}^\Delta} \frac{\widetilde{\omega}(Q)}{\omega_0(Q)} \lesssim \frac{1}{\omega_0(\Delta)}\widetilde{\omega}\Big(\bigcup_{Q\in\widetilde{\mathbb{D}}^\Delta} Q\Big) \lesssim \frac{\widetilde{\omega}(2\Delta)}{\omega_0(\Delta)} \lesssim \frac{\widetilde{\omega}(\Delta)}{\omega_0(\Delta)},\end{aligned}$$

where we have used that $\omega_0(\Delta) \approx \omega_0(Q)$ for every $Q \in \widetilde{\mathbb{D}}^\Delta$, and also that $\widetilde{\omega}(2\Delta) \approx \widetilde{\omega}(\Delta)$. These in turn follow from Lemma 2.69 part (c) and the facts that Q meets Δ and $\ell(Q) \approx \widetilde{r} \approx r$ since $0 < r < r_0$. This eventually establishes that $\omega_{L^j}^{X_0} = \widetilde{\omega} \in RH_q(\frac{5}{4}\Delta_0, \omega_0)$ with a constant that depends only on the allowable parameters and which is ultimately independent of j and Δ_0. This, as explained in **Step 0**, allows us to conclude that $\omega_L \in RH_q(\Delta_0, \omega_0)$ with the help of Lemma 3.38, completing the proof of Proposition 3.1, part (a). □

3.3. Proof Proposition 3.1, part (b)

We start assuming that Ω is a **bounded** 1-sided NTA domain satisfying the CDC and whose boundary $\partial\Omega$ is bounded. We fix $\mathbb{D} = \mathbb{D}(\partial\Omega)$ the dyadic grid from Lemma 2.33 with $E = \partial\Omega$. As in the statement of Proposition 3.1 let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0\nabla u)$ be two real (non-necessarily symmetric) elliptic operators. Fix $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$ and let $B_0 = B(x_0, r_0)$,

$\Delta_0 = B_0 \cap \partial\Omega$. From now on $X_0 := X_{\Delta_0}$, $\omega_0 := \omega_{L_0}^{X_0}$ and $\omega := \omega_L^{X_0}$. As observed in the proof of part (a), without loss of generality we may assume that $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$.

We fix $1 < p < \infty$ and assume that $|||\varrho(A, A_0)|||_{B_0} < \varepsilon$, where ε is a small enough parameter to be chosen. Our goal is to obtain that $\omega \in RH_p(\Delta_0, \omega_0)$.

We split the proof in several steps.

3.3.1. Step 0. Much as before Lemma 3.38 guarantee that just need to see that for every j large enough $\omega_{L^j} \in RH_p(\frac{5}{4}\Delta_0, \omega_0)$ uniformly in j and in Δ_0. Thus we fix $j \in \mathbb{N}$ and let $\widetilde{L} = L^j$ be the operator defined by $\widetilde{L}u = -\operatorname{div}(\widetilde{A}\nabla u)$, with $\widetilde{A} = A^j$ (see (3.37)), and set $\widetilde{\omega} := \omega_{\widetilde{L}}^{X_0}$. As mentioned above $\widetilde{A}$ is uniformly elliptic with constant $\Lambda_0 = \max\{\Lambda_A, \Lambda_{A_0}\}$. Also, since $\widetilde{L} \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$, the analogous step in part (a) showed, $\omega_0 \ll \omega_{\widetilde{L}} \ll \omega_0$ and $h(\cdot\,; \widetilde{L}, L_0, X) \in L^\infty_{\rm loc}(\partial\Omega, \omega_{L_0}^Y)$ for every $X, Y \in \Omega$ —the actual norm will depend on X, Y and j, but we will use this fact in a qualitative fashion. This qualitative control will be essential in the following steps. At the end of **Step 3** we will have obtained the desired conclusion for the operator $\widetilde{L} = L^j$, with constants independent of $j \in \mathbb{N}$, which as observed above will allow us to complete the proof by Lemma 3.38.

3.3.2. Step 1. Consider an arbitrary surface ball $\Delta_1 = \Delta(x_1, r_1)$ with $x_1 \in \frac{5}{4}\Delta_0$ and $0 < r_1 \le \frac{c_0}{10^5\kappa_0^3} r_0$, and let $B_1 = B(x_1, r_1)$. Set $\Delta_\star := B_\star \cap \partial\Omega$ with $B_\star := B(x_\star, r_\star)$ where $x_\star = x_1$ and $r_\star = 2\kappa_0 r_1$ (hence $\Delta_\star = 2\kappa_0\Delta_1$) satisfy $x_\star \in \frac{5}{4}\Delta_0$ and $0 < r_\star \le \frac{2c_0}{10^5\kappa_0^2} r_0$. By (2.49), (2.50) we have

$$X_\star = X_{c_0^{-1}\Delta_\star^*} \in \Omega \setminus B_{\Delta_\star}^* \subset \Omega \setminus \tfrac{1}{2}B_{\Delta_\star}^* \subset \Omega \setminus T_{\Delta_\star}^{**}. \tag{3.68}$$

Note also that $2\kappa_0 r_\star \le \delta(X_\star) < r_0$. We claim that $\mathbb{D}^{\Delta_\star} \subset \mathbb{D}^{\Delta_0}_{**} := \bigcup_{Q^0 \in \mathbb{D}^{\Delta_0}_*} \mathbb{D}_{Q^0}$ (see (2.46) and (3.26)). To see this, let $Q_0 \in \mathbb{D}^{\Delta_\star}$ and pick $y_\star \in Q_0 \cap 2\Delta_\star$. Then

$$|y_\star - x_0| \le |y_\star - x_\star| + |x_\star - x_0| < 2r_\star + \frac{5}{4}r_0 \le \Big(\frac{4c_0}{10^5\kappa_0^2} + \frac{5}{4}\Big)r_0 < \frac{3}{2}r_0,$$

hence $y_\star \in \frac{3}{2}\Delta_0$ and there exists a unique $Q^0 \in \mathbb{D}^{\Delta_0}_*$ such that $y_\star \in Q^0$. Moreover, by construction

$$\ell(Q_0) = 2^{-k(\Delta_\star)} < 400 r_\star \le \frac{c_0}{125\kappa_0^2} r_0 < \frac{c_0}{16\kappa_0} r_0 < \ell(Q^0),$$

and therefore $Q_0 \in \mathbb{D}_{Q^0}$ as desired.

Set $\mathcal{E}(Y) := A(Y) - A_0(Y)$, $Y \in \Omega$, and consider $\gamma = \{\gamma_Q\}_{Q \in \mathbb{D}^{\Delta_0}_{**}}$

$$\gamma_Q = \gamma_{X_0, Q} := \omega_0(Q) \sum_{I \in \mathcal{W}_Q^*} \sup_{Y \in I^*} \|\mathcal{E}\|^2_{L^\infty(I^*)}, \qquad \text{whenever } Q \in \mathbb{D}^{\Delta_0}_{**}. \tag{3.69}$$

Lemma 3.34 yields that for every $Q_0 \in \mathbb{D}^{\Delta_\star}$, if $Q^0 \in \mathbb{D}^{\Delta_0}_*$ is selected so that $Q_0 \in \mathbb{D}_{Q^0}$

$$\|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q_0, \omega_0)} \le \|\mathfrak{m}_\gamma\|_{\mathcal{C}(Q^0, \omega_0)} \lesssim |||\varrho(A, A_0)|||_{B_0} < \varepsilon, \tag{3.70}$$

where the last inequality is our main assumption in the current scenario and ε is to be chosen.

We also set $\omega_0^\star = \omega_0^{X_\star}$ and $\gamma^\star = \{\gamma_Q^\star\}_{Q\in\mathbb{D}^{\Delta_\star}}$ where

$$\gamma_Q^\star := \omega_0^\star(Q) \sum_{I\in\mathcal{W}_Q^*} \sup_{Y\in I^*} \|\mathcal{E}\|^2_{L^\infty(I^*)}, \qquad \text{whenever } Q \in \mathbb{D}^{\Delta_\star}.$$

Using (2.71) and Harnack's inequality we have that $\omega_0^\star(Q) \approx \omega_0(Q)/\omega_0(Q_0^\star)$. Hence, by (3.69)

$$\gamma_Q^\star \approx \frac{\omega_0(Q)}{\omega_0(Q_0^\star)} \sum_{I\in\mathcal{W}_Q^*} \sup_{Y\in I^*} \|\mathcal{E}\|^2_{L^\infty(I^*)} = \frac{\gamma_Q}{\omega_0(Q_0^\star)}, \qquad Q \in \mathbb{D}^{\Delta_\star}.$$

and, by (3.70),

$$\|\mathbf{m}_{\gamma^\star}\|_{\mathcal{C}(Q_0^\star,\omega_0^\star)} = \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_{\gamma^\star}(\mathbb{D}_Q)}{\omega_0^\star(Q)} \approx \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_{\gamma}(\mathbb{D}_Q)}{\omega_0^\star(Q)\omega_0(Q_0^\star)} \approx \sup_{Q\in\mathbb{D}_{Q_0^\star}} \frac{\mathbf{m}_{\gamma}(\mathbb{D}_Q)}{\omega_0(Q)} \le \|\mathbf{m}_\gamma\|_{\mathcal{C}(Q_0,\omega_0)} \lesssim \varepsilon. \tag{3.71}$$

We modify the operator $\widetilde{L}$ inside the region $T_{\Delta_\star}$ (see (2.47)), by defining $L_1 = L_1^{\Delta_\star}$ as $L_1 u = -\operatorname{div}(A_1\nabla u)$, where

$$A_1(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in T_{\Delta_\star}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus T_{\Delta_\star}. \end{cases}$$

See Figure 5. Write $\omega_1^X = \omega_{L_1}^X$ for every $X\in\Omega$ and $\omega_\star = \omega_{L_1}^{X_\star}$.

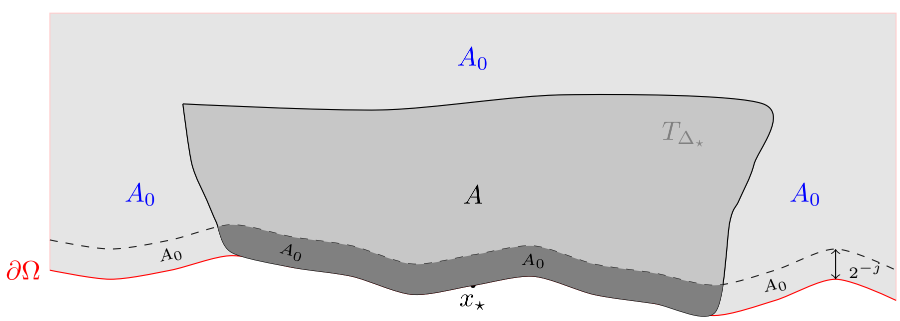

FIGURE 5. Definition of A_1 in Ω.

Recalling that $\widetilde{A} = A^j$ (see (3.37)), it is clear that $\mathcal{E}_1 := A_1 - A_0$ verifies $|\mathcal{E}_1| \le |\mathcal{E}|\mathbf{1}_{T_{\Delta_\star}}$ and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$ (this latter condition will be used qualitatively). Hence much as before if write $\omega_1^X = \omega_{L_1}^X$ for every $X\in\Omega$ we have that $\omega_1^X \ll \omega_0^X$ for every $X\in\Omega$ and hence we can write $h(\cdot\,; L_1, L_0, X) = d\omega_1^X/d\omega_0^X$ which is well-defined ω_0^X-a.e. Also, as shown in **Step 0** we have that $h(\cdot\,; L_1, L_0, X) \in L^\infty_{\rm loc}(\partial\Omega, \omega_0^Y)$ for every $X, Y\in\Omega$ (the bound depends on X, Y and the fixed j but we will use this qualitatively).

In order to simplify the notation, we recall (2.49), (2.50), and set $\widehat{\Delta}_\star := \frac{1}{2}\Delta_\star^* = \Delta(x_\star, \kappa_0 r_\star)$ and let $0 \le g \in L^{p'}(\widehat{\Delta}_\star, \omega_0^\star)$ with $\|g\|_{L^{p'}(\widehat{\Delta}_\star,\omega_0^\star)} = 1$. Extend g by 0 in $\partial\Omega\setminus\widehat{\Delta}_\star$. Set $g_t = P_t g$ with $0 < t < \kappa_0 r_1/3$ (see (3.27)). It is easy to see that $\widehat{\Delta}_\star \subset \frac{3}{2}\Delta_0$, hence $\widehat{\Delta}_\star$ can be covered by the cubes in $\mathbb{D}_*^{\Delta_0}$. This and the fact that

$r_\star/3 < c_0 r_0/(16\kappa_0)$ guarantee that Lemma 3.29 applies to give $g_t \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(g_t) \subset \Delta_\star^*$. We then consider

$$u_0^t(X) = \int_{\partial\Omega} g_t(y) d\omega_0^X(y) \quad \text{and} \quad u_1^t(X) = \int_{\partial\Omega} g_t(y) d\omega_1^X(y), \qquad X \in \Omega.$$

Since Ω is bounded, we can use Lemma 3.7 (slightly moving $X_\star$ if needed). This, Lemma 3.9 with $\mathcal{F} = \emptyset$, (3.71), and Hölder's inequality yield

(3.72)

$$\begin{aligned}
|u_1^t(X_\star) - u_0^t(X_\star)| &= \left| \iint_\Omega (A_0 - A_1)^\top(Y) \nabla_Y G_{L_1^\top}(Y, X_\star) \cdot \nabla u_0^t(Y) dY \right| \\
&\le \iint_{T_{\Delta_\star}} |\mathcal{E}(Y)| \, |\nabla_Y G_{L_1^\top}(Y, X_\star)| \, |\nabla u_0^t(Y)| \, dY \\
&\le \sum_{Q_0 \in \mathbb{D}^{\Delta_\star}} \iint_{T_{Q_0}} |\mathcal{E}(Y)| \, |\nabla_Y G_{L_1^\top}(Y, X_\star)| \, |\nabla u_0^t(Y)| \, dY \\
&\le \sum_{Q_0 \in \mathbb{D}^{\Delta_1}} \|\mathfrak{m}_{\gamma^\star}\|_{\mathcal{C}(Q_0,\omega_0^\star)}^{\frac12} \int_{Q_0} M^{\mathbf{d}}_{Q_0^\star,\omega_0^\star}(\omega_\star)(x) \mathcal{S}_{Q_0^\star} u_0^t(x) d\omega_0^\star(x) \\
&\le \varepsilon^{\frac12} \sum_{Q_0 \in \mathbb{D}^{\Delta_\star}} \int_{Q_0} M^{\mathbf{d}}_{Q_0,\omega_0^\star}(\omega_\star)(x) \mathcal{S}_{Q_0} u_0^t(x) d\omega_0^\star(x) \\
&\lesssim \varepsilon^{\frac12} \sum_{Q_0 \in \mathbb{D}^{\Delta_\star}} \|M^{\mathbf{d}}_{Q_0,\omega_0^\star}(\omega_\star)\|_{L^p(Q_0,\omega_0^\star)} \|\mathcal{S}_{Q_0} u_0^t(x)\|_{L^{p'}(Q_0,\omega_0^\star)}.
\end{aligned}$$

Using the well-known fact that $M^{\mathbf{d}}_{Q_0,\omega_0^\star}$ is bounded on $L^p(Q_0,\omega_0^\star)$ and that, as mentioned before $\omega_\star \ll \omega_0^\star$ with $h(\cdot\,; L_1^\star, L_0, X_\star) = d\omega_\star/d\omega_0^\star$, it readily follows that

$$\|M^{\mathbf{d}}_{Q_0,\omega_0^\star}(\omega_\star)\|_{L^p(Q_0,\omega_0^\star)} \lesssim \|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(Q_0,\omega_0^\star)}.$$

On the other hand, given $Q_0 \in \mathbb{D}^{\Delta_\star}$, let $Q^0 \in \mathbb{D}_*^{\Delta_0}$ be such that $Q_0 \subset Q^0$. We claim that $\Delta_\star^* \subset 2\widetilde{\Delta}_{Q^0}$ and hence $\mathrm{supp}\, g_t \subset 2\widetilde{\Delta}_{Q^0}$. Indeed, if $y \in \Delta_\star^*$ and we recall that $y_\star \in Q_0 \cap 2\Delta_\star$ we obtain

$$\begin{aligned}
|y - x_{Q^0}| \le |y - x_\star| + |x_\star - y_\star| + |y_\star - x_{Q^0}| &< 2(\kappa_0+1) r_\star + C r_{Q^0} \\
&\le \frac{8c_0}{10^5 \kappa_0} r_0 + \Xi r_{Q^0} < \frac{128}{10^5} \ell(Q^0) + \Xi r_{Q^0} < 2\Xi r_{Q^0},
\end{aligned}$$

thus $y \in 2\widetilde{\Delta}_{Q^0}$ as desired. On the other hand, observe that $X_0 \in \Omega \setminus 2\kappa_0 B_{\Delta_\star}^* = B(x_\star, 2\kappa_0^2 r_\star)$, for otherwise we would get a contradiction:

$$c_0 r_0 \le \delta(X_0) \le |X_0 - x_\star| < 2\kappa_0^2 r_\star \le \frac{4c_0}{10^5} r_0.$$

Hence Lemma 2.69 part (d) and Harnack's inequality to pass from $X_\star$ to $X_{\Delta_\star^*}$

$$\frac{d\omega_0^\star}{d\omega_0} \approx \frac{1}{\omega_0(\Delta_\star^*)}, \qquad \omega_0\text{-a.e. in } \Delta_\star^*. \tag{3.73}$$

After all these observations we use Harnack's inequality to pass from $X_\star$ to X_{Q^0} and from X_{Q^0} to X_0, Remark 2.70, Theorem 5.3, and Lemmas 3.20, 3.29, and 2.69 to conclude

$$\|\mathcal{S}_{Q_0} u_0^t(x)\|_{L^{p'}(Q_0,\omega_0^\star)} \lesssim \frac{1}{\omega_0(Q_0)^{\frac{1}{p'}}} \|\mathcal{S}_{Q^0} u_0^t(x)\|_{L^{p'}(Q^0,\omega_0^{X_{Q^0}})}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\omega_0(Q_0)^{\frac{1}{p'}}}\|g_t\|_{L^{p'}(Q^0,\omega_0^{X_{Q^0}})} \\
&\approx \frac{1}{\omega_0(Q_0)^{\frac{1}{p'}}}\|g_t\|_{L^{p'}(Q^0,\omega_0)} \\
&\lesssim \frac{1}{\omega_0(Q_0)^{\frac{1}{p'}}}\|g\|_{L^{p'}(3\widetilde{\Delta}_{Q^0},\omega_0)} \\
&= \frac{1}{\omega_0(Q_0)^{\frac{1}{p'}}}\|g\|_{L^{p'}(\widehat{\Delta}_\star,\omega_0)} \\
&\approx \frac{\omega_0(\Delta_\star^*)^{\frac{1}{p'}}}{\omega_0(Q_0)^{\frac{1}{p'}}}\|g\|_{L^{p'}(\widehat{\Delta}_\star,\omega_0^\star)} \\
&\approx 1.
\end{aligned}
$$

Plugging the obtained estimates into (3.72) we conclude that

$$
\begin{aligned}
|u_1^t(X_\star)-u_0^t(X_\star)| &\lesssim \varepsilon^{\frac12}\sum_{Q_0\in\mathbb{D}^{\Delta_\star}}\|h(\cdot\,;L_1,L_0,X_\star)\|_{L^p(Q_0,\omega_0^\star)} \\
&\lesssim \varepsilon^{\frac12}\|h(\cdot\,;L_1,L_0,X_\star)\|_{L^p(\widehat{\Delta}_\star,\omega_0^\star)},
\end{aligned}
$$

where we have used (2.50) and that $\mathbb{D}^{\Delta_\star}$ has bounded cardinality, which follows from $\omega_0(Q_0)\approx\omega_0(\widehat{\Delta}_\star)$ for every $Q_0\in\mathbb{D}^{\Delta_\star}$ and (2.50). Using then the definitions of u_0^t and u_1^t we conclude that

(3.74)
$$
\begin{aligned}
&\Big|\int_{\partial\Omega} g(y)d\omega_\star(y)-\int_{\partial\Omega} g(y)d\omega_0^\star(y)\Big| \\
&\qquad\le |u_1^t(X_\star)-u_0^t(X_\star)|+\|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)}+\|g-g_t\|_{L^1(\partial\Omega,\omega_\star)} \\
&\qquad\lesssim \varepsilon^{\frac12}\|h(\cdot\,;L_1,L_0,X_\star)\|_{L^p(\widehat{\Delta}_\star,\omega_0^\star)}+\|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)}+\|g-g_t\|_{L^1(\partial\Omega,\omega_\star)}.
\end{aligned}
$$

Fix $Q_0\in\mathbb{D}^{\Delta_\star}$, we showed before that if we pick $Q^0\in\mathbb{D}_*^{\Delta_0}$ so that $Q_0\subset Q^0$, then $\Delta_\star^*\subset 2\widetilde{\Delta}_{Q^0}$. Recalling that $0\le g\in L^{p'}(\widehat{\Delta}_\star,\omega_0^\star)$, with $\operatorname{supp}(g),\operatorname{supp}(g_t)\subset\Delta_\star^*$, then (3.73) and Lemma 3.29 give

$$
\begin{aligned}
\text{(3.75)}\quad \|g-g_t\|_{L^1(\partial\Omega,\omega_0^\star)} &= \|g-g_t\|_{L^1(\Delta_\star^*,\omega_0^\star)}\approx\frac{1}{\omega_0(\Delta_\star^*)}\|g-g_t\|_{L^1(\Delta_\star^*,\omega_0)} \\
&\le \frac{1}{\omega_0(\Delta_\star^*)}\|g-P_tg\|_{L^1(2\widetilde{\Delta}_{Q^0},\omega_0)}\to 0,\quad \text{as } t\to 0^+.
\end{aligned}
$$

Similarly, using also that as mentioned above $\omega_1\ll\omega_0$ with $h(\cdot\,;L_1,L_0,X_\star)\in L^\infty_{\rm loc}(\partial\Omega,\omega_0)$

$$
\begin{aligned}
\text{(3.76)}\quad \|g-g_t\|_{L^1(\partial\Omega,\omega_\star)} &= \|g-P_tg\|_{L^1(\Delta_\star^*,\omega_\star)} \\
&\le \|h(\cdot\,;L_1,L_0,X_\star)\|_{L^\infty(\Delta_\star^*,\omega_0^\star)}\|g-P_tg\|_{L^1(\Delta_\star^*,\omega_0^\star)}\to 0,\qquad \text{as } t\to 0^+.
\end{aligned}
$$

Combining (3.74), (3.75), (3.76) and letting $t\to 0^+$ we conclude that

$$
\begin{aligned}
0\le\int_{\widehat{\Delta}_\star} g(y)d\omega_\star(y) &= \int_{\partial\Omega} g(y)d\omega_\star(y) \\
&= \int_{\partial\Omega} g(y)\,h(y;L_1,L_0,X_\star)\,d\omega_0^\star(y)
\end{aligned}
$$

$$\le \varepsilon^{\frac{1}{2}} \|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(\widehat{\Delta}_\star, \omega_0^\star)} + \int_{\partial\Omega} g(y) d\omega_0^\star(y)$$
$$\le \varepsilon^{\frac{1}{2}} \|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(\widehat{\Delta}_\star, \omega_0^\star)} + \omega_0^\star(\widehat{\Delta}_\star)^{\frac{1}{p}}.$$

Taking now the sup over all $0 \le g \in L^{p'}(\widehat{\Delta}_\star, \omega_0^\star)$ with $\|g\|_{L^{p'}(\widehat{\Delta}_\star, \omega_0^\star)} = 1$ we eventually get

$$\|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(\widehat{\Delta}_\star, \omega_0^\star)} \lesssim \varepsilon^{\frac{1}{2}} \|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(\widehat{\Delta}_\star, \omega_0^\star)} + \omega_0^\star(\widehat{\Delta}_\star)^{\frac{1}{p}}. \tag{3.77}$$

Since $h(\cdot\,; L_1, L_0, X_\star) \in L^\infty_{\rm loc}(\partial\Omega, \omega_0^\star)$ (albeit with bounds which may depend on $X_\star$ or j) we can hide the first term on the right hand side and eventually obtain fixing ε small enough (depending on n, the 1-sided NTA constants, the CDC constant, the ellipticity constants of L_0 and L_2, and on p)

$$\|h(\cdot\,; L_1, L_0, X_\star)\|_{L^p(\widehat{\Delta}_\star, \omega_0^\star)} \lesssim \omega_0^\star(\widehat{\Delta}_\star)^{\frac{1}{p}}. \tag{3.78}$$

3.3.3. Step 2. Let us next define

$$A_2(Y) := \begin{cases} A_1(Y) & \text{if } Y \in T_{\Delta_\star}, \\ \widetilde{A}(Y) & \text{if } Y \in \Omega \setminus T_{\Delta_\star}, \end{cases}$$

and set $L_2 u := -\operatorname{div}(A_2 \nabla u)$. Note that $L_2 \equiv \widetilde{L}$ in Ω (see Figure 6). Since $\widetilde{L} \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$ we have already mentioned in **Step 0** that $\omega_{L_2} = \omega_{\widetilde{L}}$ and ω_{L_0} are mutually absolutely continuous with $h(\cdot\,; \widetilde{L}, L_0, X) \in L^\infty_{\rm loc}(\partial\Omega, \omega^Y_{L_0})$ for every $X, Y \in \Omega$.

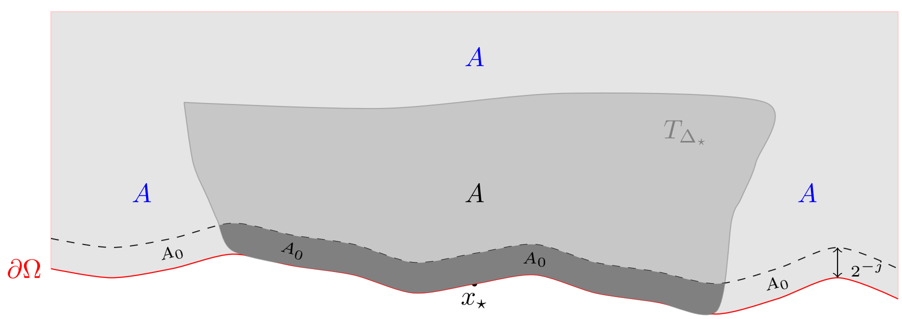

FIGURE 6. Definition of A_2 in Ω.

Note that by construction $B_1 = \frac{1}{2\kappa_0} B_\star$. Besides, by (2.49), $2\kappa_0 B_1 \cap \Omega \subset \frac{5}{4} B_\star \cap \Omega \subset T_{\Delta_\star}$ and since $\widetilde{L} \equiv L_2 \equiv L_1$ in T_{Δ_0}, Lemma 2.69 part (f) and Harnack's inequality give that $\omega_{\widetilde{L}}^{X_\star}$ and $\omega_{L_1}^{X_\star} = \omega_\star$ are comparable in Δ_1, thus $h(\cdot\,; L_1, L_0, X_\star) \approx h(\cdot\,; \widetilde{L}, L_0, X_\star)$ for $\omega_0^\star$-a.e. $y \in \Delta_1$ (and also ω_0-a.e.). On the other hand using that as shown above $X_0 \in \Omega \setminus 2\kappa_0 B^\star_{\Delta_\star} \subset \Omega \setminus 2\kappa_0 B_1$ we can invoke Lemma 2.69 part (d) and Harnack's inequality to see that

$$h(\cdot\,; \widetilde{L}, L_0, X_0) = \frac{d\omega_{\widetilde{L}}^{X_0}}{d\omega_{L_0}^{X_0}} = \frac{d\omega_{\widetilde{L}}^{X_0}}{d\omega_{\widetilde{L}}^{X_\star}} \frac{d\omega_{\widetilde{L}}^{X_\star}}{d\omega_{L_0}^{X_\star}} \frac{d\omega_{L_0}^{X_\star}}{d\omega_{L_0}^{X_0}}$$
$$\approx \frac{\omega_1(\Delta_1)}{\omega_0(\Delta_1)} h(\cdot\,; \widetilde{L}, L_0, X_\star) \approx \frac{\widetilde{\omega}(\Delta_1)}{\omega_0(\Delta_1)} h(\cdot\,; L_1, L_0, X_\star),$$

for ω_0-a.e. $y \in \Delta_1$ (recall that $\omega_{\widetilde{L}}$ and ω_0 are mutually absolutely continuous). This, the fact that $\Delta_1 \subset \widehat{\Delta}_\star$, (3.78) and Lemma 2.69 part (d) yield

$$\left(\fint_{\Delta_1} h(y;\widetilde{L},L_0,X_0)^p d\omega_0(y)\right)^{\frac{1}{p}} \approx \frac{\widetilde{\omega}(\Delta_1)}{\omega_0(\Delta_1)}\left(\fint_{\Delta_1} h(y;\widetilde{L},L_0,X_\star)^p d\omega_0^\star(y)\right)^{\frac{1}{p}} \lesssim \frac{\widetilde{\omega}(\Delta_1)}{\omega_0(\Delta_1)}. \tag{3.79}$$

3.3.4. Step 3. Let us summarize what we have obtained up to this point. We fixed $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)/2$, we set $B_0 = B(x_0, r_0)$, $\Delta_0 = B_0 \cap \partial\Omega$, $X_0 = X_{\Delta_0}$, and $\omega_0 = \omega_{L_0}^{X_0}$. We also fix $1 < p < \infty$ and assumed that $\|\varrho(A, A_0)\|_{B_0} < \varepsilon$ with ε small enough at our disposal. In **Step 0** we took an arbitrary j and wrote $\widetilde{L} = L^j$, (see (3.37)) and $\widetilde{\omega} = \omega_{\widetilde{L}}^{X_0}$. For an arbitrary surface ball $\Delta_1 = \Delta(x_1, r_1)$ with $x_1 \in \frac{5}{4}\Delta_0$ and $0 < r_1 \le \frac{c_0}{10^5\kappa_0^3} r_0$ we have obtained, combining **Step 1** and **Step 2**, that provided ε is small enough (independently of j and Δ_1) then (3.79) holds.

Our next goal is to see that (3.79) holds as well with $\frac{5}{4}\Delta_0$ replacing Δ_1. To do this $r = \frac{c_0}{10^5\kappa_0^3} r_0$ and find a maximal collection of points $\{x_k\}_{k\in\mathcal{K}} \subset \frac{5}{4}\Delta_0$ with respect to the property that $|x_k - x_{k'}| > 2r/3$ for every $k, k' \in \mathcal{K}$ with $k \neq k'$. Write $\Delta_k = \Delta(x_k, r)$ and note that $\{\frac{1}{3}\Delta_k\}_{k\in\mathcal{K}}$ is a family of pairwise disjoint surface balls such that $\frac{5}{4}\Delta_0 \subset \bigcup_{k\in\mathcal{K}} \Delta_k \subset \frac{3}{2}\Delta_0$. Note that since $r \approx r_0$ and $x_k \in \frac{5}{4}\Delta_0$ it follows from Lemma 2.69 part (c) that $\omega_0(\frac{5}{4}\Delta_0) \approx \omega_0(\Delta_k)$ and $\widetilde{\omega}(\frac{3}{2}\Delta_0) \approx \widetilde{\omega}(\frac{5}{4}\Delta_0) \approx \widetilde{\omega}(\Delta_k) \approx \widetilde{\omega}(\frac{1}{3}\Delta_k)$ for every $k \in \mathcal{K}$. Thus using (3.79) for every Δ_k (whose applicability is ensure by the facts that $x_k \in \frac{5}{4}\Delta_0$ and $r_{\Delta_k} = r = \frac{c_0}{10^5\kappa_0^3} r_0$) it follows that

$$\begin{aligned}\left(\fint_{\frac{5}{4}\Delta_0} h(y;\widetilde{L},L_0,X_0)^p d\omega_0(y)\right)^{\frac{1}{p}} &\lesssim \sum_{k\in\mathcal{K}}\left(\fint_{\Delta_k} h(y;\widetilde{L},L_0,X_0)^p d\omega_0(y)\right)^{\frac{1}{p}} \\ \lesssim \sum_{k\in\mathcal{K}} \frac{\widetilde{\omega}(\Delta_k)}{\omega_0(\Delta_k)} &\approx \frac{1}{\omega_0(\frac{5}{4}\Delta_0)}\sum_{k\in\mathcal{K}} \widetilde{\omega}(\tfrac{1}{3}\Delta_k) = \frac{1}{\omega_0(\frac{5}{4}\Delta_0)}\widetilde{\omega}\Big(\bigcup_{k\in\mathcal{K}} \tfrac{1}{3}\Delta_k\Big) \\ &\le \frac{\widetilde{\omega}(\frac{3}{2}\Delta_0)}{\omega_0(\frac{5}{4}\Delta_0)} \approx \frac{\widetilde{\omega}(\frac{5}{4}\Delta_0)}{\omega_0(\frac{5}{4}\Delta_0)}.\end{aligned} \tag{3.80}$$

We now have all the ingredients to show that $\widetilde{\omega} \in RH_p(\frac{5}{4}\Delta_0, \omega_{L_0})$ (uniformly in j) and to do this we let $\Delta = B \cap \partial\Omega$ with $B = B(x, r) \subset \frac{5}{4}B_0$ and $x \in \partial\Omega$. If $r_\Delta < 1 < \frac{c_0}{10^5\kappa_0^3} r_0$ then we can invoke (3.79) with $\Delta_1 = \Delta$ and this gives us the desired estimate. Assume otherwise that $r_\Delta \ge 1\frac{c_0}{10^5\kappa_0^3} r_0$, hence $r_\Delta \approx r_0$ since $B \subset \frac{5}{4}B_0$ implies that $r_\Delta < \frac{5}{4}r_0$. In that scenario using that $\Delta \subset \frac{5}{4}\Delta_0$ and that $\omega_0(\Delta) \approx \omega_0(\frac{5}{4}\Delta_0)$, $\widetilde{\omega}(\Delta) \approx \widetilde{\omega}(\frac{5}{4}\Delta_0)$ by Lemma 2.69 part (c) we obtain that (3.80) gives as desired

$$\left(\fint_{\Delta} h(y;\widetilde{L},L_0,X_0)^p d\omega_0(y)\right)^{\frac{1}{p}} \lesssim \left(\fint_{\frac{5}{4}\Delta_0} h(y;\widetilde{L},L_0,X_0)^p d\omega_0(y)\right)^{\frac{1}{p}} \lesssim \frac{\widetilde{\omega}(\frac{5}{4}\Delta_0)}{\omega_0(\frac{5}{4}\Delta_0)} \approx \frac{\widetilde{\omega}(\Delta)}{\omega_0(\Delta)}.$$

All in one, we have shown that $\widetilde{\omega} \in RH_p(\frac{5}{4}\Delta_0, \omega_{L_0})$, where the implicit constant depends only on the allowable parameters and which is ultimately independent of j and Δ_0. This, as argued in **Step 0**, permits us to show that $\omega_L \in RH_p(\Delta_0, \omega_{L_0})$ with the help of Lemma 3.38. The proof of Proposition 3.1, part (b) is then complete. $\square$

CHAPTER 4

Dyadic sawtooth lemma for projections

In this section, we shall prove two dyadic versions of the main lemma in [**DJK84**]. To set the stage we quote a proposition from [**HM14**, Proposition 6.7] which is proved under the further assumption that $\partial\Omega$ is Ahlfors regular. However, a careful examination of the proof shows that the same argument applies in our scenario.

We present some versions of the main lemma in [DJK] which are valid for discrete sawtooth regions based on dyadic cubes. The first result involves the projection operators and was used in Step 2 above. The second result (cf. Lemma A.2) is interesting in its own right and is a dyadic analog of the main lemma in [DJK]. For both lemmas, the proofs follow the idea of the argument in [DJK], but are technically much simpler, owing to the dyadic setting in which we work here

Proposition 4.1 ([**HM14**, Proposition 6.7]). *Let Ω be a 1-sided NTA domain satisfying the CDC. Fix $Q_0 \in \mathbb{D}$ and let $\mathcal{F} = \{Q_k\}_k \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes. Then for each $Q_j \in \mathcal{F}$, there is an n-dimensional cube $P_j \subset \partial\Omega_{\mathcal{F},Q_0}$, which is contained in a face of I^* for some $I \in \mathcal{W}$, and which satisfies*

$$\ell(P_j) \approx \operatorname{dist}(P_j, Q_j) \approx \operatorname{dist}(P_j, \partial\Omega) \approx \ell(I) \approx \ell(Q_j), \tag{4.2}$$

where the constants depend on allowable parameters.

Next we claim that

$$\sum_j 1_{P_j} \leq C, \tag{4.3}$$

with C depending on the allowable parameters.

To see this, observe that as in [**HM14**, Remark 6.9] if $P_j \cap P_k \neq \emptyset$ then $\ell(Q_j) \approx \ell(Q_k)$. Indeed from the previous result $P_j \subset I_j^*$ and $P_k \subset I_k^*$ for some $I_j, I_k \in \mathcal{W}$. Thus I_j^* meets I_k^* and by construction I_j and I_k meet. Using (4.2) and the nature of the Whitney cubes we see that $\ell(Q_j) \approx \ell(I_j) \approx \ell(I_k) \approx \ell(Q_k)$. Using this and (4.2) one can also see that $\operatorname{dist}(Q_j, Q_k) \lesssim \ell(Q_j) \approx \ell(Q_k)$. Hence, fixing P_{j_0} and $x \in P_{j_0}$ we have some constant $k_0 \geq 1$ (depending on the allowable parameters) such that

$$\begin{aligned}\sum_j 1_{P_j}(x) &\leq \#\{P_k : \ P_k \cap P_{j_0} \neq \emptyset\} \\ &\leq \#\big\{Q_k : \ 2^{-k_0} \leq \tfrac{\ell(Q_k)}{\ell(Q_{j_0})} \leq 2^{k_0}, \ \operatorname{dist}(Q_k, Q_{j_0}) \leq 2^{k_0}\ell(Q_{j_0})\big\} \\ &= \sum_{k=-k_0}^{k_0} \#\big\{Q_k : \ \ell(Q_k) = 2^k \ell(Q_{j_0}), \ \operatorname{dist}(Q_k, Q_{j_0}) \leq 2^{k_0}\ell(Q_{j_0})\big\} =: \sum_{k=-k_0}^{k_0} N_k.\end{aligned}$$

To estimate each of the terms in the last sum fix k and note that since the cubes belong to the same generation then Q_k's involved are disjoint and hence so they are the corresponding Δ_{Q_k}'s which all have radius $(2C)^{-1}2^k\ell(Q_{j_0})$. In particular, $|x_{Q_k} - x_{Q'_k}| \gtrsim 2^k\ell(Q_{j_0}) \geq 2^{-k_0}\ell(Q_{j_0})$ for any such cubes Q_k and $Q_{k'}$. Moreover,

$$|x_{Q_k} - x_{Q_{j_0}}| \leq \operatorname{diam}(Q_k) + \operatorname{dist}(Q_k, Q_{j_0}) + \operatorname{diam}(Q_{j_0}) \lesssim 2^{k_0}\ell(Q_{j_0}).$$

Thus it is easy to see (since $\mathbb{R}^{n+1}$ is geometric doubling) that $N_k \lesssim 2^{2k_0(n+1)}$. All these together gives us desired (4.3) —we note in passing that the argument in [**HM14**, Remark 6.9] used the fact there $\partial\Omega$ is AR to estimate each N_k, while here we are invoking the geometric doubling property of the ambient space $\mathbb{R}^{n+1}$.

We are now ready to state the first main result of this chapter which is a version of [**HM14**, Lemma 6.15] (see also [**DJK84**]) valid in our setting:

LEMMA 4.4 (Discrete sawtooth lemma for projections). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a* ***bounded*** *1-sided NTA domain satisfying the CDC. Let $Q_0 \in \mathbb{D}$, let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes, and let μ be a dyadically doubling measure in Q_0. Given two real (non-necessarily symmetric) elliptic L_0, L, we write $\omega_0^{Y_{Q_0}} = \omega_{L_0,\Omega}^{Y_{Q_0}}$, $\omega_L^{Y_{Q_0}} = \omega_{L,\Omega}^{Y_{Q_0}}$ for the elliptic measures associated with L_0 and L for the domain Ω with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$ (cf. Lemma 3.18). Let $\omega_{L,*}^{Y_{Q_0}} = \omega_{L,\Omega_{\mathcal{F},Q_0}}^{Y_{Q_0}}$ be the elliptic measure associated with L for the domain $\Omega_{\mathcal{F},Q_0}$ with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$. Consider $\nu_L^{Y_{Q_0}}$ the measure defined by*

$$\nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}\Big(F \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) + \sum_{Q_i\in\mathcal{F}} \frac{\omega_L^{Y_{Q_0}}(F\cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i), \qquad F \subset Q_0, \tag{4.5}$$

where P_i is the cube produced in Proposition 4.1. Then $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$ (see (2.26)) depends only on $\omega_0^{Y_{Q_0}}$ and $\omega_{L,*}^{Y_{Q_0}}$, but not on $\omega_L^{Y_{Q_0}}$. More precisely,

$$\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}\Big(F \setminus \bigcup_{Q_i\in\mathcal{F}} Q_i\Big) + \sum_{Q_i\in\mathcal{F}} \frac{\mu(F\cap Q_i)}{\mu(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i), \qquad F \subset Q_0. \tag{4.6}$$

Moreover, there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and all $F \subset Q$, we have

$$\left(\frac{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(Q)}\right)^{\theta} \lesssim \frac{\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(Q)} \lesssim \frac{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(Q)}. \tag{4.7}$$

PROOF. Our argument follows the ideas from [**HM14**, Lemma 6.15] and we use several auxiliary technical results from [**HM14**, Section 6] which were proved under the additional assumption that $\partial\Omega$ is AR. However, as we will indicate along the proof, most of them can be adapted to our setting. Those arguments that require new ideas will be explained in detail.

We first observe that (4.6) readily follows from the definitions of $\mathcal{P}_{\mathcal{F}}^{\mu}$ and $\nu_L^{Y_{Q_0}}$. We first establish the second estimate in (4.7). With this goal in mind let us fix $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q_0$.

Case 1: There exists $Q_i \in \mathcal{F}$ such that $Q \subset Q_i$. By (4.6) we have

$$\frac{\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(Q)} = \frac{\frac{\mu(F\cap Q_i)}{\mu(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i)}{\frac{\mu(Q\cap Q_i)}{\mu(Q_i)}\omega_{L,*}^{Y_{Q_0}}(P_i)} = \frac{\mu(F)}{\mu(Q)} = \frac{\frac{\mu(F\cap Q_i)}{\mu(Q_i)}\omega_L^{Y_{Q_0}}(Q_i)}{\frac{\mu(Q\cap Q_i)}{\mu(Q_i)}\omega_L^{Y_{Q_0}}(Q_i)} = \frac{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^{\mu}\omega_L^{Y_{Q_0}}(Q)}.$$

Case 2: $Q \not\subset Q_i$ for any $Q_i \in \mathcal{F}$, that is, $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. In particular if $Q \cap Q_i \neq \emptyset$ with $Q_i \in \mathcal{F}$ then necessarily $Q_i \subsetneq Q$. Let $x_i^\star$ denote the center of P_i and pick $r_i \approx \ell(Q_i) \approx \ell(P_i)$ so that $P_i \subset \Delta_\star(x_i^\star, r_i) := B(x_i^\star, r_i) \cap \partial\Omega_{\mathcal{F},Q_0}$. Note that by Lemma 2.54, Harnack's inequality and Lemma 2.69 parts (a) and (c) we have that $\omega_{L,*}^{Y_{Q_0}}(P_i) \approx \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i))$. On the other hand as in [**HM14**, Proposition 6.12] one can see that

$$\Delta_\star^Q := B(x_Q^\star, t_Q) \cap \partial\Omega_{\mathcal{F},Q_0} \subset \Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big) \tag{4.8}$$

with $t_Q \approx \ell(Q)$, $x_Q^\star \in \partial\Omega_{\mathcal{F},Q_0}$ and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$ with implicit constants depending on the allowable parameters. We note that the last expression is slightly different to that in [**HM14**, Proposition 6.2], nonetheless the one stated here follows from the proof in account of [**HM14**, (6.14) and Proposition 6.1] as ∂Q_i is contained in $\overline{T_{Q_i}}$. Besides, Proposition 4.1 easily yields

$$\begin{aligned} &\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} P_i\Big) \\ &\qquad \subset \Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big) \subset C\Delta_\star^Q, \end{aligned} \tag{4.9}$$

hence

$$\omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \Delta_\star(x_i^\star, r_i)\Big)\Big) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q). \tag{4.10}$$

Writing $E_0 = Q_0 \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i \subset \partial\Omega \cap \partial\Omega_{\mathcal{F},Q}$ (see [**HM14**, Proposition 6.1]) we have

$$\begin{aligned} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q) &\leq \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i)) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &= \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(Q \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &= \mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q) \end{aligned} \tag{4.11}$$

and, by (4.3),

$$\begin{aligned} \mathcal{P}_{\mathcal{F}}^{\mu} \nu_L^{Y_{Q_0}}(Q) &= \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(Q \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &= \omega_{L,*}^{Y_{Q_0}}(Q \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \omega_{L,*}^{Y_{Q_0}}(P_i) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}\Big((Q \cap E_0) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} P_i\Big)\Big) \\ &\lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q). \end{aligned} \tag{4.12}$$

Since $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ we can invoke [**HM14**, Proposition 6.4] (which also holds in the current setting) to find $Y_Q \in \Omega_{\mathcal{F},Q_0}$ which serves as a Corkscrew point simultaneously for $\Omega_{\mathcal{F},Q_0}$ with respect to the surface ball $\Delta_\star(y_Q, s_Q)$ for some $y_Q \in$

$\Omega_{\mathcal{F},Q}$ and some $s_Q \approx \ell(Q)$, and for Ω with respect to each surface ball $\Delta(x, s_Q)$, for every $x \in Q$. Applying (2.72) and Harnack's inequality to join Y_Q with X_Q and Y_{Q_0} with Y_Q we have

$$\frac{d\omega_L^{Y_Q}}{d\omega_L^{Y_{Q_0}}} \approx \frac{1}{\omega_L^{Y_{Q_0}}(Q)}, \qquad \omega_L^{Y_{Q_0}}\text{-a.e. in } Q. \tag{4.13}$$

On the other hand one can see that

$$\widetilde{B}_Q \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} B(x_i^\star, r_i) \Big) \subset B(y_Q, \widehat{s}_Q), \tag{4.14}$$

for some $\widehat{s}_Q \approx s_Q$. Invoking then Lemma 2.54, and Lemma 2.69 parts (c) and (e) in the domain $\Omega_{\mathcal{F},Q_0}$ we can analogously see

$$\frac{d\omega_{L,*}^{Y_Q}}{d\omega_{L,*}^{Y_{Q_0}}} \approx \frac{1}{\omega_{L,*}^{Y_{Q_0}}(\Delta(y_Q, \widehat{s}_Q))} \approx \frac{1}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)}, \qquad \omega_{L,*}^{Y_{Q_0}}\text{-a.e. in } \Delta(y_Q, \widehat{s}_Q). \tag{4.15}$$

Next we invoke (4.11), (4.14), and (4.13) to obtain

$$\begin{aligned} \frac{\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}(Q)} &\approx \frac{\omega_{L,*}^{Y_{Q_0}}(F \cap E_0)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \frac{\omega_{L,*}^{Y_{Q_0}}(P_i)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} \\ &\approx \omega_{L,*}^{Y_Q}(F \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_{L,*}^{Y_Q}(P_i). \end{aligned} \tag{4.16}$$

We claim the following estimates hold

$$\omega_{L,*}^{Y_Q}(F \cap E_0) \lesssim \omega_L^{Y_Q}(F \cap E_0), \qquad \omega_{L,*}^{Y_Q}(P_i) \lesssim \omega_L^{Y_Q}(Q_i). \tag{4.17}$$

The first estimate follows easily from the maximum principle since $\Omega_{\mathcal{F},Q_0} \subset \Omega$ and $F \cap E_0 \subset \partial\Omega \cap \partial\Omega_{\mathcal{F},Q_0}$. For the second one, by the maximum principle we just need to see that $\omega_L^X(Q_i) \gtrsim 1$ for $X \in P_i$, but this follows from Lemma 2.69 part (a), (2.34), Harnack's inequality, and (4.2).

With the previous estimates at our disposal we can the continue with our estimate (4.16):

$$\begin{aligned} \frac{\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}(Q)} &\lesssim \omega_L^{Y_Q}(F \cap E_0) + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \omega_L^{Y_Q}(Q_i) \\ &\approx \frac{\omega_L^{Y_{Q_0}}(F \cap E_0)}{\omega_L^{Y_{Q_0}}(Q)} + \sum_{Q_i \in \mathcal{F}: Q_i \subsetneq Q} \frac{\mu(F \cap Q_i)}{\mu(Q_i)} \frac{\omega_L^{Y_{Q_0}}(Q_i)}{\omega_L^{Y_{Q_0}}(Q)} \\ &= \frac{\mathcal{P}_{\mathcal{F}}^\mu \omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)} = \frac{\mathcal{P}_{\mathcal{F}}^\mu \omega_L^{Y_{Q_0}}(F)}{\mathcal{P}_{\mathcal{F}}^\mu \omega_L^{Y_{Q_0}}(Q)}, \end{aligned}$$

where we have used (4.14) and that $\mathcal{P}_{\mathcal{F}}^\mu \omega_L^{Y_{Q_0}}(Q) = \omega_L^{Y_{Q_0}}(Q)$. This proves the second estimate in (4.7) in the current case.

Once we have shown the second estimate in (4.7) we can invoke [**HM14**, Lemma B.7] (which is a purely dyadic result and hence applies in our setting) along with Lemma 4.21 below to eventually obtain the first estimate in (4.7). □

As a consequence of the previous result we can easily obtain a dyadic analog of the main lemma in [**DJK84**].

Lemma 4.18 (Discrete sawtooth lemma). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a* ***bounded*** *1-sided NTA domain satisfying the CDC. Let $Q_0 \in \mathbb{D}$ and let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes. Given two real (non-necessarily symmetric) elliptic L_0, L, we write $\omega_0^{Y_{Q_0}} = \omega_{L_0,\Omega}^{Y_{Q_0}}$, $\omega_L^{Y_{Q_0}} = \omega_{L,\Omega}^{Y_{Q_0}}$ for the elliptic measures associated with L_0 and L for the domain Ω with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$ (cf. Lemma 3.18). Let $\omega_{L,*}^{Y_{Q_0}} = \omega_{L,\Omega_{\mathcal{F},Q_0}}^{Y_{Q_0}}$ be the elliptic measure associated with L for the domain $\Omega_{\mathcal{F},Q_0}$ with fixed pole at $Y_{Q_0} \in \Omega_{\mathcal{F},Q_0} \cap \Omega$. Consider $\nu_L^{Y_{Q_0}}$ the measure defined by (4.5). Then, there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and all $F \subset Q$, we have*

$$\left(\frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}\right)^\theta \lesssim \frac{\nu_L^{Y_{Q_0}}(F)}{\nu_L^{Y_{Q_0}}(Q)} \lesssim \frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}. \tag{4.19}$$

In particular, if $F \subset Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i$, we have

$$\left(\frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}\right)^\theta \lesssim \frac{\omega_{L,*}^{Y_{Q_0}}(F)}{\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^Q)} \lesssim \frac{\omega_L^{Y_{Q_0}}(F)}{\omega_L^{Y_{Q_0}}(Q)}, \tag{4.20}$$

*where $\Delta_\star^Q := B(x_Q^\star, t_Q) \cap \partial\Omega_{\mathcal{F},Q_0}$ with $t_Q \approx \ell(Q)$, $x_Q^\star \in \partial\Omega_{\mathcal{F},Q_0}$ and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$ with implicit constants depending on the allowable parameters (see [**HM14**, Proposition 6.12]).*

Proof. Letting $\mu = \omega_L^{Y_{Q_0}}$, which is dyadically doubling in Q_0, one easily has $\mathcal{P}_{\mathcal{F}}^\mu \omega_L^{Y_{Q_0}} = \omega_L^{Y_{Q_0}}$ and $\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}} = \nu_L^{Y_{Q_0}}$. Thus (4.7) in Lemma 4.4 readily yields (4.19). Next, to obtain (4.20) we may assume that F is non-empty. Observe that if $F \subset Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i$, then $\nu_L^{Y_{Q_0}}(F) = \omega_{L,*}^{Y_{Q_0}}(F)$. On the other hand, if $F \subset Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i$ we must be in **Case 2** of the proof of Lemma 4.4, hence (4.11) and (4.12) hold. With all these we readily obtain (4.20). $\square$

Our last result in this section establishes that both $\nu_L^{Y_{Q_0}}$ and $\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}$ are dyadically doubling on Q_0.

Lemma 4.21. *Under the assumptions of Lemma 4.4, $\nu_L^{Y_{Q_0}}$ and $\mathcal{P}_{\mathcal{F}}^\mu \nu_L^{Y_{Q_0}}$ are dyadically doubling on Q_0.*

Proof. We follow the ideas in [**HM14**, Lemma B.2]. We shall first see $\nu_L^{Y_{Q_0}}$ is dyadically doubling. To this end, let $Q \in \mathbb{D}_{Q_0}$ be fixed and let Q' be one of its dyadic children. We consider three cases:

Case 1: There exists $Q_i \in \mathcal{F}$ such that $Q \subset Q_i$. In this case we have

$$\nu_L^{Y_{Q_0}}(Q) = \frac{\omega_L^{Y_{Q_0}}(Q)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \lesssim \frac{\omega_L^{Y_{Q_0}}(Q')}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) = \nu_L^{Y_{Q_0}}(Q')$$

where we have used Harnack's inequality and Lemma 2.69 parts and (a) and (c).

Case 2: $Q' \in \mathcal{F}$. For simplicity say $Q' = Q_1 \in \mathcal{F}$ and in this case $\nu_L^{Y_{Q_0}}(Q') = \omega_{L,*}^{Y_{Q_0}}(P_1)$. Note that then $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and we let $\mathcal{F}_1$ be the family of cubes $Q_i \in \mathcal{F}$ with $Q_i \cap Q \neq \emptyset$ and observe that if $Q_i \in \mathcal{F}_1$ then $Q_i \subsetneq Q$. Then by (4.3)

$$\nu_L^{Y_{Q_0}}(Q) = \omega_{L,*}^{Y_{Q_0}}\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}_1} \frac{\omega_L^{Y_{Q_0}}(Q \cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \tag{4.22}$$

$$= \omega_{L,*}^{Y_{Q_0}}\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}_1} \omega_{L,*}^{Y_{Q_0}}(P_i)$$
$$\lesssim \omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big)\Big).$$

Recall that in **Case 2** in the proof of Lemma 4.4 we mentioned that $P_1 \subset \Delta_\star(x_1^\star, r_1)$ with $x_1^\star$ being the center of P_1 and $r_1 \approx \ell(P_1) \approx \ell(Q_1) \approx \ell(Q)$ since Q is the dyadic parent of Q_1. Note that since $Q_i \in \mathcal{F}_1$ by (4.2)

$$\ell(P_i) \approx \operatorname{dist}(P_i, Q) \approx \ell(Q_i) \lesssim \ell(Q) = 2\ell(Q_1) \approx \ell(P_1) \approx \operatorname{dist}(Q_1, P_1) \approx r_1.$$

Thus

$$\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big) \subset \Delta_\star(x_1^\star, Cr_1),$$

where we here and below we use the notation $\Delta_\star$ for the surface balls with respect to $\partial\Omega_{\mathcal{F},Q_0}$. Using this, (4.22), and Lemma 2.69 parts (a) and (c) and Harnack's inequality we derive

$$\nu_L^{Y_{Q_0}}(Q) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_1^\star, Cr_1)) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_1^\star, r_1)) \lesssim \omega_{L,*}^{Y_{Q_0}}(P_1) = \nu_L^{Y_{Q_0}}(Q').$$

Case 3: None of the conditions in the previous cases happen, and necessarily $Q, Q' \in \mathbb{D}_{\mathcal{F},Q_0}$. We take the same set $\mathcal{F}_1$ as in the previous case and again if $Q_i \in \mathcal{F}_1$ then $Q_i \subsetneq Q$ (otherwise we are driven to **Case 1**). Introduce $\mathcal{F}_2$, the family of cubes $Q_i \in \mathcal{F}$ with $Q_i \cap Q' \neq \emptyset$. Again, if $Q_i \in \mathcal{F}_2$ we have $Q_i \subsetneq Q'$; otherwise either $Q' = Q_i$ which is **Case 2**, or $Q' \subsetneq Q_i$ which implies $Q \subset Q_i$ and we are back to **Case 1**.

Note that since Q is the dyadic parent of Q', using the same notation as in (4.8) applied to $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$ we have that

$$\operatorname{dist}(x_{Q'}^\star, Q) \le \operatorname{dist}(x_{Q'}^\star, Q') \lesssim \ell(Q') \approx \ell(Q) \approx t_{Q'}.$$

Also by (4.2)

$$\operatorname{dist}(x_{Q'}^\star, P_i) \lesssim \operatorname{dist}(x_{Q'}^\star, Q) + \ell(Q) + \operatorname{dist}(Q, P_i) \lesssim \ell(Q) + \operatorname{dist}(Q_i, P_i) \lesssim \ell(Q) \approx t_{Q'}.$$

These readily give

$$\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big) \subset \Delta_\star(x_{Q'}^\star, Ct_{Q'}).$$

We can then proceed as in the previous case (see (4.22)) to obtain

$$\nu_L^{Y_{Q_0}}(Q) \lesssim \omega_{L,*}^{Y_{Q_0}}\Big(\Big(Q \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) \bigcup \Big(\bigcup_{Q_i \in \mathcal{F}_1} P_i\Big)\Big)$$
$$\lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_{Q'}^\star, Ct_{Q'})) \lesssim \omega_{L,*}^{Y_{Q_0}}(\Delta_\star^{Q'}),$$

where $\Delta_\star^{Q'} = B(x_{Q'}^\star, t_{Q'}) \cap \partial\Omega_{\mathcal{F},Q_0}$ (see (4.8)) and we have used Lemma 2.69 parts (a) and (c) and Harnack's inequality. On the other hand, proceeding as in (4.11) with Q' in place of Q since $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$:

$$\omega_{L,*}^{Y_{Q_0}}(\Delta_\star^{Q'}) \le \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \omega_{L,*}^{Y_{Q_0}}(\Delta_\star(x_i^\star, r_i))$$
$$\lesssim \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \omega_{L,*}^{Y_{Q_0}}(P_i)$$

$$= \omega_{L,*}^{Y_{Q_0}}(Q' \cap E_0) + \sum_{Q_i \in \mathcal{F}_2} \frac{\omega_L^{Y_{Q_0}}(Q' \cap Q_i)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i)$$
$$= \nu_L^{Y_{Q_0}}(Q').$$

Eventually we obtain that $\nu_L^{Y_{Q_0}}(Q) \lesssim \nu_L^{Y_{Q_0}}(Q')$, completing the proof of the dyadic doubling property of $\nu_L^{Y_{Q_0}}$.

We next deal with $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$, and we could use Lemma 2.27 in which case the doubling constant would depend on μ and $\nu_L^{Y_{Q_0}}$, and for the latter it was shown above that depends of $\omega_L^{Y_{Q_0}}$. However, as stated in Lemma 4.4, $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$ does not depend on $\omega_L^{Y_{Q_0}}$ and hence it is reasonable to expect that the doubling constant does not depend on that measure. As a matter of fact we can simply follow the previous argument replacing $\omega_L^{Y_{Q_0}}$ by $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}$ to see that in **Cases 2** and **3** we have that $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(Q) = \nu_L^{Y_{Q_0}}(Q)$ and $\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(Q') = \nu_L^{Y_{Q_0}}(Q')$, hence the doubling condition follows from the previous calculations and the constant depend on that of $\omega_{L,*}^{Y_{Q_0}}$. With regard to case one on which $Q \subset Q_i$ for some $Q_i \in \mathcal{F}$ one can easily see that

$$\mathcal{P}_{\mathcal{F}}^{\mu}\nu_L^{Y_{Q_0}}(Q) = \frac{\mu(Q)}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) \lesssim \frac{\mu(Q')}{\omega_L^{Y_{Q_0}}(Q_i)} \omega_{L,*}^{Y_{Q_0}}(P_i) = \nu_L^{Y_{Q_0}}(Q'),$$

which uses that μ is dyadically doubling in Q_0. Eventually we have seen that doubling constant depend on that of $\omega_{L,*}^{Y_{Q_0}}$ and μ as desired. This completes the proof. □

CHAPTER 5

Square function and Non-tangential maximal function estimates

In this section we first show that bounded weak-solutions satisfy Carleson measure estimates adapted to the elliptic measure. This is in turn the main ingredient to obtain that the conical square function can be locally controlled by the non-tangential maximal function in norm with respect to the elliptic measure.

THEOREM 5.1. *Let $\Omega\subset\mathbb{R}^{n+1}$, $n\geq 2$, be a 1-sided NTA domain satisfying the capacity density condition. Let $Lu=-\operatorname{div}(A\nabla u)$ be a real (non-necessarily symmetric) elliptic operator. There exists C depending only on dimension n, the 1-sided NTA constants, the CDC constant, and the ellipticity constant of L, such that for every $u\in W^{1,2}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ with $Lu=0$ in the weak-sense in Ω there holds*

$$\sup_{B}\sup_{B'}\frac{1}{\omega_L^{X_\Delta}(\Delta')}\iint_{B'\cap\Omega}|\nabla u(X)|^2\,G_L(X_\Delta,X)\,dX\leq C\,\|u\|^2_{L^\infty(\Omega)},\tag{5.2}$$

where $\Delta=B\cap\partial\Omega$, $\Delta'=B'\cap\partial\Omega$, and the sups are taken respectively over all balls $B=B(x,r)$ with $x\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$, and $B'=B(x',r')$ with $x'\in 2\Delta$ and $0<r'<rc_0/4$, and c_0 is the Corkscrew constant.

Using this result we are able to extend some estimates from [DJK84] to our general setting.

THEOREM 5.3. *Let $\Omega\subset\mathbb{R}^{n+1}$, $n\geq 2$, be a 1-sided NTA domain satisfying the capacity density condition. Let $Lu=-\operatorname{div}(A\nabla u)$ be a real (non-necessarily symmetric) elliptic operator. For every $1<q<\infty$, there exists C_q depending only on dimension n, the 1-sided NTA constants, the CDC constant, the ellipticity constant of L, and q, such that for every $u\in W^{1,2}_{\rm loc}(\Omega)$ with $Lu=0$ in the weak-sense in Ω, for every $Q_0\in\mathbb{D}$, there holds*

$$\|\mathcal{S}_{Q_0}u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}\leq C_q\,\|\mathcal{N}_{Q_0}u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}.\tag{5.4}$$

5.1. Proof of Theorem 5.1

By renormalization we may assume without loss of generality that $\|u\|_{L^\infty(\Omega)}=1$. We will first prove a dyadic version of (5.2). Let $\mathbb{D}=\mathbb{D}(\partial\Omega)$ the dyadic grid from Lemma 2.33 with $E=\partial\Omega$. Our goal is to show that

$$M_0:=\sup_{Q^0\in\mathbb{D}}\sup_{\substack{Q_0\in\mathbb{D}_{Q^0}\\ \ell(Q_0)\leq\frac{\ell(Q^0)}{M}}}\frac{1}{\omega_L^{X_{Q^0}}(Q_0)}\iint_{T_{Q_0}}|\nabla u(X)|^2\,G_L(X_{Q^0},X)\,dX\lesssim 1\tag{5.5}$$

with $M\geq 4$ large enough. Assuming this momentarily let us see how to derive (5.2). Fix B and B' as in the suprema in (5.2). Let $k,k'\in\mathbb{Z}$ be so that $2^{k-1}<r\leq 2^k$

and $2^{k'-1} < r' \le 2^{k'}$, and define $k'' := \min\{k', k - 10k_M\}$ where $k_M \ge 1$ is large enough to be chosen depending on M and the allowable parameters. Set

$$\mathcal{W}' := \{I \in \mathcal{W} : I \cap B' \neq \emptyset, \ell(I) < 2^{k''}\} \bigcup \{I \in \mathcal{W} : I \cap B' \neq \emptyset, \ell(I) \ge 2^{k''}\} =: \mathcal{W}_1' \cup \mathcal{W}_2'.$$

Note that for every $I \in \mathcal{W}$ with $I \cap B' \neq \emptyset$ we have

$$\ell(I) < \operatorname{diam}(I) \le \frac{\operatorname{dist}(I, \partial\Omega)}{4} < \frac{r'}{4} \le 2^{k'-2}.$$

As a consequence, if $\mathcal{W}_2' \neq \emptyset$, then $k'' = k - 10\,k_M$, and picking $I \in \mathcal{W}_2' \neq \emptyset$ one has

$$r \approx 2^k \approx_M 2^{k''} \le \ell(I) \le 2^{k'-2} \approx r' \lesssim r.$$

This gives $r' \approx_M r$ and $\#\mathcal{W}_2' \lesssim_M 1$.

To proceed, let us write

$$\iint_{B'\cap\Omega} |\nabla u(X)|^2 \, G_L(X_\Delta, X)\, dX \le \iint_{\bigcup_{I\in\mathcal{W}_1'} I} |\nabla u(X)|^2 \, G_L(X_\Delta, X)\, dX + \sum_{I\in\mathcal{W}_2'} \iint_I |\nabla u(X)|^2 \, G_L(X_\Delta, X)\, dX =: \mathcal{I} + \mathcal{II},$$

and we estimate each term in turn.

To estimate $\mathcal{II}$ we may assume that $\mathcal{W}_2' \neq \emptyset$, hence $k'' = k - 10\,k_M$, $r' \approx r$ and $\#\mathcal{W}_2' \lesssim 1$. Then Lemma 2.69, the fact that $\omega(\partial\Omega) \le 1$, Caccioppoli's inequality, the normalization $\|u\|_{L^\infty(\Omega)} = 1$, and Harnack's inequality give

$$\mathcal{II} = \sum_{I\in\mathcal{W}_2'} \iint_I |\nabla u(X)|^2 \, G_L(X_\Delta, X)\, dX \lesssim \sum_{I\in\mathcal{W}_2'} \ell(I)^{1-n} \iint_I |\nabla u(X)|^2 \, dX \lesssim \#\mathcal{W}_2' \lesssim 1 \approx \omega_L^{X_\Delta}(\Delta').$$

Next we deal with $\mathcal{I}$. Introduce the disjoint family $\mathcal{F}' = \{Q \in \mathbb{D} : \ell(Q) = 2^{k''-1}, Q \cap 3\,B' \neq \emptyset\}$. Given $I \in \mathcal{W}_1'$, let $X_I \in B' \cap I$, and $Q_I \in \mathbb{D}$ be so that $\ell(Q_I) = \ell(I)$ and it contains some fixed $y_I \in \partial\Omega$ such that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, y_I)$. Then, as observed in Section 2.5, one has $I \in \mathcal{W}_{Q_I}^*$. Note that

$$|y_I - x'| \le \operatorname{dist}(y_I, I) + \operatorname{diam}(I) + |X_I - x'| \le \frac{5}{4} \operatorname{dist}(I, \partial\Omega) + |X_I - x'| \le \frac{9}{4} |X_I - x'| < 3\,r',$$

hence $y_I \in Q_I \cap 3\,\Delta'$. This and the fact that, as observed before, $\ell(Q_I) = \ell(I) < 2^{k''}$ imply that $Q_I \subset Q$ for some $Q \in \mathcal{F}'$. Hence, $I \subset (1+\lambda)\,I \subset U_{Q_I} \subset \overline{T_Q}$ for some $Q \in \mathcal{F}'$. This eventually show that $\bigcup_{I\in\mathcal{W}_1'} I \subset \bigcup_{Q\in\mathcal{F}'} T_Q$ and therefore

$$\mathcal{I} \le \sum_{Q\in\mathcal{F}'} \iint_{T_Q} |\nabla u(X)|^2 \, G_L(X_\Delta, X)\, dX.$$

For any $Q \in \mathcal{F}'$ pick the unique (ancestor) $\widehat{Q} \in \mathbb{D}$ with $\ell(\widehat{Q}) = 2^{k-1}$ and $Q \subset \widehat{Q}$. Note that $\delta(X_\Delta) \approx r$, $\delta(X_{\widehat{Q}}) \approx \ell(\widehat{Q}) = 2^{k-1} \approx r$. Also,

$$|X_\Delta - X_{\widehat{Q}}| \le |X_\Delta - x| + |x - x'| + |x' - x_Q| + |x_Q - x_{\widehat{Q}}| + |x_{\widehat{Q}} - X_{\widehat{Q}}|$$
$$< 3\,r + 3\,r' + \operatorname{diam}(Q) + \operatorname{diam}(\widehat{Q}) + \ell(\widehat{Q}) \lesssim r + 2^{k''} + 2^k \lesssim r.$$

Hence by the Harnack chain condition one obtains that $G_L(X_\Delta, X) \approx G_L(X_{\widehat{Q}}, X)$ for every $X \in T_Q$ (in doing that we need to make sure that k_M is large enough so that the Harnack chain joining X_Δ and $X_{\widehat{Q}}$, which is $c\,r$-away from $\partial\Omega$, does not get near T_Q, which is $\kappa_0\,\ell(Q)$-close to $\partial\Omega$). Note also that $\frac{\ell(Q)}{\ell(\widehat{Q})} = 2^{k''-k} \le 2^{-k_M} < M^{-1}$, provided k_M is large enough depending on M. All in one we can eventually obtain from (5.5)

$$\mathcal{I} \lesssim \sum_{Q\in\mathcal{F}'} \iint_{T_Q} |\nabla u(X)|^2\, G_L(X_{\widehat{Q}}, X)\, dX \lesssim M_0 \sum_{Q\in\mathcal{F}'} \omega_L^{X_{\widehat{Q}}}(Q)$$
$$\le M_0 \sum_{Q\in\mathcal{F}'} \omega_L^{X_\Delta}(Q) \le M_0\, \omega_L^{X_\Delta}\Big(\bigcup_{Q\in\mathcal{F}'} Q\Big) \le M_0\, \omega_L^{X_\Delta}(C\Delta') \lesssim M_0\, \omega_L^{X_\Delta}(\Delta'),$$

where we have used Lemma 2.69. This completes the prof of the fact that (5.5) implies (5.2).

We next focus on showing (5.5). With this goal in mind we fix $Q^0 \in \mathbb{D} = \mathbb{D}(\partial\Omega)$ and let $Q_0 \in \mathbb{D}_{Q^0}$ with $\ell(Q_0) \le \ell(Q^0)/M$ with M large enough so that $X_{Q^0} \notin 4\,B_Q^*$ (cf. (2.48)). Write $\omega_L = \omega_L^{X_{Q^0}}$ and $\mathcal{G}_L = G_L(X_{Q^0}, \cdot)$ and note that our choice of M, (2.64), and (2.65) guarantee that $L^\top \mathcal{G}_L = L^\top G_{L^\top}(\cdot, X_{Q^0}) = 0$ in the weak sense in $4\,B_Q^*$.

Fix $N \gg 1$ and consider the family of pairwise disjoint cubes $\mathcal{F}_N = \{Q \in \mathbb{D}_{Q_0} : \ell(Q) = 2^{-N}\,\ell(Q_0)\}$ and let $\Omega_N = \Omega_{\mathcal{F}_N, Q_0}$ (cf. (2.42)). Note that by construction $\Omega_N \subset T_{Q_0}$ is an increasing sequence of sets converging to T_{Q_0}. Our goal is to show that for every $N \gg 1$ there holds

$$\iint_{\Omega_N} |\nabla u(X)|^2\, \mathcal{G}_L(X)\, dX \le M_0\, \omega_L(Q_0), \tag{5.6}$$

with M_0 independent of Q^0, Q_0, and N. Hence the monotone convergence theorem yields

$$\iint_{T_{Q_0}} |\nabla u(X)|^2\, \mathcal{G}_L(X)\, dX = \lim_{N\to\infty} \iint_{\Omega_N} |\nabla u(X)|^2\, \mathcal{G}_L(X)\, dX \le M_0\, \omega_L(Q_0),$$

which is (5.5).

Let us next start estimating (5.6). Using Ψ_N from Lemma 3.11 and the ellipticity of the matrix A we have

$$\iint_{\Omega_N} |\nabla u(X)|^2\, \mathcal{G}_L(X)\, dX \lesssim \iint_{\mathbb{R}^{n+1}} |\nabla u(X)|^2\, \mathcal{G}_L(X)\, \Psi_N(X)\, dX$$
$$\lesssim \iint_{\mathbb{R}^{n+1}} A(X)\nabla u(X) \cdot \nabla u(X)\, \mathcal{G}_L(X)\, \Psi_N(X)\, dX$$
$$= \iint_{\mathbb{R}^{n+1}} A(X)\nabla u(X) \cdot \nabla(u\, \mathcal{G}_L\, \Psi_N)(X)\, dX$$
$$- \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla(u^2\, \Psi_N)(X) \cdot \nabla \mathcal{G}_L(X)\, dX$$

$$
\begin{aligned}
&- \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla(u^2)(X)\cdot\nabla\Psi_N(X)\,\mathcal{G}_L(X)\,dX \\
&+ \frac{1}{2} \iint_{\mathbb{R}^{n+1}} A(X)\nabla\Psi_N(X)\cdot\nabla\mathcal{G}_L(X)\,u(X)^2\,dX \\
=:\,& \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned}
$$

We observe that $u\,\mathcal{G}_L\,\Psi_N$ and $u^2\,\Psi_N$ belong to $W^{1,2}(\Omega)$ since $u\in W^{1,2}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$, $\operatorname{supp}\Psi_N\subset\Omega_N^*$, $\delta(X)\gtrsim 2^{-N}\,\ell(Q_0)$ for every $X\in\Omega_N^*$, Lemma 2.59 and the fact that X_{Q^0} is away from Ω_N^* since $\delta(X_{Q^0})\approx\ell(Q^0)$ and by (2.48) one has $\delta(X)\lesssim\ell(Q_0)\le\ell(Q^0)/M\le\delta(X_{Q^0})/2$ for every $X\in\Omega_N^*$ and provided M is large enough. Using all these one can easily see via a limiting argument that the fact that $Lu=0$ in the weak sense in Ω implies that $\mathcal{I}_1=0$. Likewise, one can easily show that $\mathcal{I}_2=0$ by recalling that $\operatorname{supp}\Psi_N\subset\Omega_N^*\subset\frac12\,B_Q{}^*\cap\Omega$ (see (2.48)) and that as mentioned above $L^\top\mathcal{G}_L=0$ in the weak sense in $4\,B_Q^*$. Thus we are left with estimating the terms $\mathcal{I}_3$ and $\mathcal{I}_4$. By (iii) in Lemma 3.11 and the fact that $\|u\|_{L^\infty(\Omega)}=1$ we obtain

$$
\begin{aligned}
|\mathcal{I}_3|+|\mathcal{I}_4| &\lesssim \iint_{\bigcup_{I\in\mathcal{W}_N^\Sigma} I^{**}} \big(|\nabla u|\,\mathcal{G}_L+|\nabla\mathcal{G}_L|\big)\,\delta(\cdot)^{-1}\,dX \\
&\lesssim \sum_{I\in\mathcal{W}_N^\Sigma}\ell(I)^{\frac{n-1}{2}}\bigg(\Big(\iint_{I^{**}}|\nabla u|^2\,dX\Big)^{\frac12}\mathcal{G}_L(X(I))+\Big(\iint_{I^{**}}|\nabla\mathcal{G}_L|^2\,dX\Big)^{\frac12}\bigg) \\
&\lesssim \sum_{I\in\mathcal{W}_N^\Sigma}\ell(I)^{\frac{n-3}{2}}\bigg(\Big(\int_{I^{***}}|u|^2\,dX\Big)^{\frac12}+\ell(I)^{\frac{n+1}{2}}\bigg)\mathcal{G}_L(X(I)) \\
&\lesssim \sum_{I\in\mathcal{W}_N^\Sigma}\ell(I)^{n-1}\,\mathcal{G}_L(X(I)),
\end{aligned}
$$

where $X(I)$ denotes the center of I, and we have used Harnack's and Caccioppoli's inequalities, that $L^\top\mathcal{G}_L=0$ and $Lu=0$ in the weak sense in $I^{***}\subset\frac12\,B_Q^*\cap\Omega$ (see (2.48)). Invoking Lemmas 2.69 and Lemma 3.11 one can see that $\ell(I)^{n-1}\,\mathcal{G}_L(X(I))\lesssim\omega_L(\widehat{Q}_I)$ for every $I\in\mathcal{W}_N^\Sigma$. This together with Lemma 3.11 allows us to conclude

$$
|\mathcal{I}_3|+|\mathcal{I}_4|\lesssim\sum_{I\in\mathcal{W}_N^\Sigma}\omega_L(\widehat{Q}_I)\lesssim\omega_L\Big(\bigcup_{I\in\mathcal{W}_N^\Sigma}\widehat{Q}_I\Big).
$$

Note that if $y\in\widehat{Q}_I$ with $I\in\mathcal{W}_N^\Sigma$ one has

$$
|y-x_{Q_0}|\le\operatorname{diam}(\widehat{Q}_I)+\operatorname{dist}(\widehat{Q}_I,I)+\operatorname{diam}(I)+\operatorname{dist}(I,x_{Q_0})\lesssim\ell(I)+\ell(Q_0)\lesssim\ell(Q_0)
$$

where we have used (3.14) and (2.48). Thus, Lemma 2.69 gives

$$
|\mathcal{I}_3|+|\mathcal{I}_4|\lesssim\omega_L(C\,\Delta_{Q_0})\lesssim\omega_L(Q_0).
$$

This allows us to complete the proof of Theorem 5.1. □

5.2. Proof of Theorem 5.3

We borrow some ideas from [HMM19]. Given $k \in \mathbb{N}$ introduce the truncated localized conical square function: for every $Q \in \mathbb{D}_{Q_0}$ and $x \in Q$, let

$$\mathcal{S}_Q^k u(x) := \bigg(\iint_{\Gamma_Q^k(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \bigg)^{\frac{1}{2}}, \text{ where } \Gamma_Q^k(x) := \bigcup_{\substack{x \in Q' \in \mathbb{D}_Q \\ \ell(Q') \geq 2^{-k}\,\ell(Q_0)}} U_{Q'},$$

where if $\ell(Q) < 2^{-k}\,\ell(Q_0)$ it is understood that $\Gamma_Q^k(x) = \emptyset$ and $\mathcal{S}_Q^k u(x) = 0$. Note that by the monotone convergence theorem $\mathcal{S}_Q^k u(x) \nearrow \mathcal{S}_Q u(x)$ as $k \to \infty$ for every $x \in Q$.

Fixed k_0 large enough (eventually, $k_0 \to \infty$), our goal is to show that we can find $\vartheta > 0$ (independent of k_0) such that for every $\beta, \gamma, \lambda > 0$ we have

$$\begin{aligned}(5.7)\quad \omega_L^{X_{Q_0}}\big(\big\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\,\lambda,\ \mathcal{N}_{Q_0} u(x) \leq \gamma\,\lambda\big\}\big) \\ \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \omega_L^{X_{Q_0}}\big(\big\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > \beta\,\lambda\big\}\big),\end{aligned}$$

where the implicit constant depend on the allowable parameters and it is independent of k_0. To prove this we fix $\beta, \gamma, \lambda > 0$ and set

$$E_\lambda := \big\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > \lambda\big\}.$$

Consider first the case $E_\lambda \subsetneq Q_0$. Note that if $x \in E_\lambda$, by definition $\mathcal{S}_{Q_0}^{k_0} u(x) > \lambda$. Let $Q_x \in \mathbb{D}_{Q_0}$ be the unique dyadic cube such that $Q_x \ni x$ and $\ell(Q_x) = 2^{-k_0}\ell(Q_0)$. Then it is clear from construction that for every $y \in Q_x$ one has

$$\Gamma_{Q_0}^{k_0}(x) = \bigcup_{Q_x \subset Q \subset Q_0} U_Q = \Gamma_{Q_0}^{k_0}(y) \qquad \text{and} \qquad \lambda < \mathcal{S}_{Q_0}^{k_0} u(x) = \mathcal{S}_{Q_0}^{k_0} u(y).$$

Hence, $Q_x \subset E_\lambda$ and we have shown that for every $x \in E_\lambda$ there exists $Q_x \in \mathbb{D}_{Q_0}$ such that $Q_x \ni x$ and $Q_x \subset E_\lambda$. We then take the ancestors of Q_x, and look for the one with maximal side length $Q_x^{\max} \supset Q_x$ which is contained in E_λ. That is, $Q \subset E_\lambda$ for every $Q_x \subset Q \subset Q_x^{\max}$ and $\widehat{Q}_x^{\max} \cap Q_0 \setminus E_\lambda \neq \emptyset$ where $\widehat{Q}_x^{\max}$ is the dyadic parent of $Q_x^{\max}$ (during this proof we will use $\widehat{Q}$ to denote the dyadic parent of Q, that is, the only dyadic cube containing it with double side length). Note that the assumption $E_\lambda \subsetneq Q_0$ guarantees that $Q_x^{\max} \in \mathbb{D}_{Q_0} \setminus \{Q_0\}$. Let $\mathcal{F}_0 = \{Q_j\}_j$ be the collection of such maximal cubes as x runs in E_λ and we clearly have that the family is pairwise disjoint and also $E_\lambda = \bigcup_{Q_j \in \mathcal{F}_0} Q_j$. Also, by construction $\ell(Q_j) \geq 2^{-k_0}\ell(Q_0)$ and by the maximality of each Q_j we can select $x_j \in \widehat{Q}_j \setminus E_\lambda$.

On the other hand, for any $x \in Q_j$ we have, using that $x_j \in \widehat{Q}_j \setminus E_\lambda$,

$$\Gamma_{Q_0}^{k_0}(x) = \bigcup_{\substack{x \in Q \in \mathbb{D}_{Q_0} \\ \ell(Q) \geq 2^{-k_0}\,\ell(Q_0)}} U_Q = \Gamma_{Q_j}^{k_0}(x) \bigcup \Big(\bigcup_{Q_j \subsetneq Q \subset Q_0} U_Q \Big) \subset \Gamma_{Q_j}^{k_0}(x) \bigcup \Gamma_{Q_0}^{k_0}(x_j)$$

and therefore

$$\mathcal{S}_{Q_0}^{k_0} u(x) \leq \mathcal{S}_{Q_j}^{k_0} u(x) + \mathcal{S}_{Q_0}^{k_0} u(x_j) \leq \mathcal{S}_{Q_j}^{k_0} u(x) + \lambda.$$

As a consequence,

$$\big\{x \in Q_j : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\,\lambda\big\} \subset \big\{x \in Q_j : \mathcal{S}_{Q_j}^{k_0} u(x) > \beta\lambda\big\}$$

and

$$\begin{aligned}\big\{x\in Q_0:\mathcal{S}_{Q_0}^{k_0}u(x)>(1+\beta)\,\lambda\big\}&=\big\{x\in Q_0:\mathcal{S}_{Q_0}^{k_0}u(x)>(1+\beta)\,\lambda\big\}\cap E_\lambda\\&=\bigcup_{Q_j\in\mathcal{F}_0}\big\{x\in Q_j:\mathcal{S}_{Q_0}^{k_0}u(x)>(1+\beta)\,\lambda\big\}\subset\bigcup_{Q_j\in\mathcal{F}_0}\big\{x\in Q_j:\mathcal{S}_{Q_j}^{k_0}u(x)>\beta\lambda\big\}.\end{aligned}$$

This has been done under the assumption that $E_\lambda\subsetneq Q_0$. In the case $E_\lambda=Q_0$ we set $\mathcal{F}_0=\{Q_0\}$. Then in both cases we obtain

$$\big\{x\in Q_0:\mathcal{S}_{Q_0}^{k_0}u(x)>(1+\beta)\,\lambda\big\}\subset\bigcup_{Q_j\in\mathcal{F}_0}\big\{x\in Q_j:\mathcal{S}_{Q_j}^{k_0}u(x)>\beta\lambda\big\}.\tag{5.8}$$

Thus, to obtain (5.7) it suffices to see that for every $Q_j\in\mathcal{F}_0$

$$\omega_L^{X_{Q_0}}\big(\big\{x\in Q_j:\mathcal{S}_{Q_j}^{k_0}u(x)>\beta\,\lambda,\ \mathcal{N}_{Q_0}u(x)\le\gamma\,\lambda\big\}\big)\lesssim\Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\omega_L^{X_{Q_0}}(Q_j).\tag{5.9}$$

From this we just need to sum in $Q_j\in\mathcal{F}_0$ to see that (5.8) together with the previous facts yield the desired estimate (5.7):

$$\begin{aligned}&\omega_L^{X_{Q_0}}\big(\big\{x\in Q_0:\mathcal{S}_{Q_0}^{k_0}u(x)>(1+\beta)\,\lambda,\ \mathcal{N}_{Q_0}u(x)\le\gamma\,\lambda\big\}\big)\\&\qquad\le\sum_{Q_j\in\mathcal{F}_0}\omega_L^{X_{Q_0}}\big(\big\{x\in Q_j:\mathcal{S}_{Q_j}^{k_0}u(x)>\beta\,\lambda,\ \mathcal{N}_{Q_0}u(x)\le\gamma\,\lambda\big\}\big)\\&\qquad\lesssim\Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\sum_{Q_j\in\mathcal{F}_0}\omega_L^{X_{Q_0}}(Q_j)=\Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\omega_L^{X_{Q_0}}\Big(\bigcup_{Q_j\in\mathcal{F}_0}Q_j\Big)=\Big(\frac{\gamma}{\beta}\Big)^{\vartheta}\omega_L^{X_{Q_0}}(E_\lambda).\end{aligned}$$

Let us then obtain (5.9). Fix $Q_j\in\mathcal{F}_0$ and to ease the notation write $P_0=Q_j$. Set

$$\widetilde{E}_\lambda=\big\{x\in P_0:\mathcal{S}_{P_0}^{k_0}u(x)>\beta\,\lambda\big\},\qquad F_\lambda=\big\{x\in P_0:\mathcal{N}_{Q_0}u(x)\le\gamma\,\lambda\big\}.\tag{5.10}$$

If $\omega_L^{X_{Q_0}}(F_\lambda)=0$ then (5.9) is trivial, hence we may assume that $\omega_L^{X_{Q_0}}(F_\lambda)>0$ so that $P_0\cap F_\lambda=F_\lambda\neq\emptyset$. We subdivide P_0 dyadically and stop the first time that $Q\cap F_\lambda=\emptyset$. If one never stops we write $\mathcal{F}_{P_0}^*=\{\emptyset\}$, otherwise $\mathcal{F}_{P_0}^*=\{P_j\}_j\subset\mathbb{D}_{P_0}\setminus\{P_0\}$ is the family of stopping cubes which is maximal (hence pairwise disjoint) with respect to the property $F_\lambda\cap Q=\emptyset$. In particular, $F_\lambda\subset P_0\setminus(\cup_{\mathbb{D}_{\mathcal{F}_{P_0}^*,P_0}}P_j)$.

Next we claim that

$$\bigcup_{x\in F_\lambda}\Gamma_{P_0}^{k_0}(x)\subset\bigcup_{\substack{Q\in\mathbb{D}_{\mathcal{F}_{P_0}^*,P_0}\\ \ell(Q)\ge2^{-k_0}\,\ell(Q_0)}}U_Q\subset\operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F}_{P_0}^*,P_0}}U_Q^*\Big)=\Omega_{\mathcal{F}_{P_0}^*,P_0}^*=:\Omega_*\tag{5.11}$$

To verify the first inclusion, we fix $Y\in\Gamma_{P_0}^{k_0}(x)$ with $x\in F_\lambda$. Then, $Y\in U_Q$ where $x\in Q\in\mathbb{D}_{P_0}$. Since $x\in F_\lambda$ we must have $Q\in\mathbb{D}_{\mathcal{F}_{P_0}^*}$ (otherwise $Q\subset P_j$ for some $P_j\in\mathcal{F}_{P_0}^*$ and this would imply that $x\in P_j\cap F_\lambda=\emptyset$) and therefore $Q\in\mathbb{D}_{\mathcal{F}_{P_0}^*,P_0}$ which gives the first inclusion. The second inclusion in (5.11) is trivial (since $U_Q\subset\operatorname{int}(U_Q^*)$).

To continue we see that

$$|u(Y)|\le\gamma\,\lambda,\qquad\text{for all }Y\in\Omega_*.\tag{5.12}$$

Fix such a Y so that $Y\in U_Q^*$ for some $Q\in\mathbb{D}_{\mathcal{F}_{P_0}^*,P_0}$. If $Q\cap F_\lambda=\emptyset$, by maximality of the cubes in $\mathcal{F}_{P_0}^*$, it follows that $Q\subset P_j$ for some $P_j\in\mathcal{F}_{P_0}^*$, which contradicts the fact $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$. Thus, $Q \cap F_\lambda \neq \emptyset$ and we can select $x \in Q \cap F_\lambda$ so that by definition $|u(Y)| \le \mathcal{N}_{Q_0} u(x) \le \gamma\,\lambda$ since $Y \in U^*_Q \subset \Gamma^*_{Q_0}(x)$.

Apply Lemma 3.18 to find $X_* := Y_{P_0} \in \Omega_* \cap \Omega$ so that

$$\ell(P_0) \approx \operatorname{dist}(X_*, \partial\Omega_*) \approx \delta(X_*). \tag{5.13}$$

Let $\omega^*_L := \omega^{X_*}_{L,\Omega_*}$ be the elliptic measure associated with L relative to Ω_* with pole at X_* and write $\delta_* = \operatorname{dist}(\cdot, \partial\Omega_*)$. Given $Y \in \Omega_*$, we choose $y_Y \in \partial\Omega_*$ such that $|Y - y_Y| = \delta_*(Y)$. By definition, for $x \in F_\lambda$ and $Y \in \Gamma_{P_0}(x)$, there is a $Q \in \mathbb{D}_{P_0}$ such that $Y \in U_Q$ and $x \in Q$. Thus, by the triangle inequality, and the definition of U_Q, we have that for $Y \in \Gamma_{P_0}(x)$,

$$|x - y_Y| \le |x - Y| + \delta_*(Y) \approx \delta(Y) + \delta_*(Y) \approx \delta_*(Y) \tag{5.14}$$

where in the last step we have used that

$$\delta(Y) \approx \delta_*(Y) \qquad \text{for} \quad Y \in \bigcup_{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}} U_Q. \tag{5.15}$$

On the other hand, as observed above $F_\lambda \subset P_0 \setminus (\cup_{\mathcal{F}} Q_j) \subset \partial\Omega \cap \partial\Omega_*$, see [**HM14**, Proposition 6.1]. Using this and the fact that if $Q \cap F_\lambda \neq \emptyset$ then $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$ we have

(5.16)

$$\begin{aligned}
\int_{F_\lambda} \mathcal{S}^{k_0}_{P_0} u(x)^2 \, d\omega^*_L(x) &= \int_{F_\lambda} \iint_{\Gamma^{k_0}_{P_0}(x)} |\nabla u(Y)|^2 \, \delta(Y)^{1-n} \, dY \, d\omega^*_L(x) \\
&\le \int_{F_\lambda} \sum_{\substack{x \in Q \in \mathbb{D}_{P_0} \\ \ell(Q) \ge 2^{-k_0}\,\ell(Q_0)}} \iint_{U_Q} |\nabla u(Y)|^2 \, \delta(Y)^{1-n} \, dY \, d\omega^*_L(x) \\
&\lesssim \sum_{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}} \Big(\iint_{U_Q} |\nabla u(Y)|^2 \, dY \Big) \, \ell(Q)^{1-n} \, \omega^*_L(Q \cap F_\lambda) \\
&\lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} \\ \ell(Q) \ge M^{-1}\ell(P_0)}} \cdots + \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} \\ \ell(Q) < M^{-1}\ell(P_0)}} \cdots \\
&=: \Sigma_1 + \Sigma_2,
\end{aligned}$$

where M is a large constant to be chosen.

We start estimating Σ_1. Note first that $\#\{Q :\in \mathbb{D}_{P_0} : \ell(Q) \ge M^{-1}\ell(P_0)\} \le C_M$, thus

$$\begin{aligned}
\Sigma_1 &\lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} \\ \ell(Q) \ge M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I \in \mathcal{W}^*_Q} \iint_{I^*} |\nabla u(Y)|^2 \, dY \\
&\lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} \\ \ell(Q) \ge M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I \in \mathcal{W}^*_Q} \ell(I)^{-2} \iint_{I^{**}} |u(Y)|^2 \, dY \\
&\lesssim (\gamma\,\lambda)^2 \sum_{\substack{Q \in \mathbb{D}_{P_0} \\ \ell(Q) \ge M^{-1}\ell(P_0)}} \ell(Q)^{1-n} \sum_{I \in \mathcal{W}^*_Q} \ell(I)^{n-1}
\end{aligned}$$

$$\lesssim_M (\gamma\,\lambda)^2,$$

where we have used (5.12), along with the fact that $\text{int}(I^{**}) \subset \text{int}(U_Q^*) \subset \Omega_*$ for any $I \in \mathcal{W}_Q^*$ with $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$, and the fact that $\mathcal{W}_Q^*$ has uniformly bounded cardinality. To estimate Σ_2 we note that picking $y_Q \in Q \cap F_\lambda$ we have that $Q \cap F_\lambda \subset B(y_Q, 2\,\text{diam}(Q)) \cap \partial\Omega_* =: \Delta_Q^*$. Write X_Q^* for Corkscrew relative to Δ_Q^* with respect to Ω_* so that $\delta_*(X_Q^*) \approx \text{diam}(Q) \lesssim M^{-1}\ell(P_0)$. Note that by (5.13), we clearly have $X_* \in \Omega \setminus B(y_Q, 4\,\text{diam}(Q))$ provided M is sufficiently large. Hence, by Lemma 2.69 part (b) applied in Ω_*, which is a 1-sided NTA domain satisfying the CDC by Lemma 2.54, we obtain for every $Y \in U_Q$

(5.17)
$$\ell(Q)^{1-n}\,\omega_L^*(Q \cap F_\lambda) \lesssim \text{diam}(Q)^{1-n}\omega_L^*(\Delta_Q^*) \lesssim G_{L,*}(X_*, X_Q^*) \approx G_{L,*}(X_*, Y),$$

where $G_{L,*}$ is the Green function for the operator L relative to the domain Ω_*. Above the last estimate uses Harnack's inequality (we may need to tale M slightly larger) and the fact that by (5.15), one has $\delta_*(Y) \approx \ell(Q) \approx \text{diam}(Q) \approx \delta_*(X_Q^*)$ (see Remark 2.73) and that if $I \ni Y$ with $I \in \mathcal{W}_Q^*$

$$|Y - X_*| \le \text{diam}(I) + \text{dist}(I, Q) + \text{diam}(Q) + |y_Q - X^*| \lesssim \text{diam}(Q).$$

Write $\{P_0^i\}_i \subset \mathbb{D}_{P_0}$ for the collection of dyadic cubes with $M\,\ell(P_0) \le \ell(P_0^i) < 2\,M\ell(P_0)$ which has uniformly bounded cardinality depending on M. Note that

$$\{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} : \ell(Q) < M^{-1}\ell(P_0)\} \subset \bigcup_i \mathbb{D}_{\mathcal{F}^*_{P_0},P_0^i}.$$

For each i, if $\mathbb{D}_{\mathcal{F}^*_{P_0},P_0^i} \neq \emptyset$ then $P_0^i \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0}$ and hence $P_0^i \cap F_\lambda \neq \emptyset$. Pick then $y_i \in P_0^i \cap F_\lambda$ and note that for every $Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0^i}$ by (2.48) it follows that

$$U_Q \subset T_{P_0^i} \cap \Omega_* \subset B^*_{P_0^i} \cap \Omega_* \subset B(y_i, C\,\kappa_0\,\ell(P_0^i)) \cap \Omega_* =: B_i \cap \Omega_*.$$

Using then (5.17) we have

$$\begin{aligned}
\Sigma_2 &\lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0} \\ \ell(Q) < M^{-1}\ell(P_0)}} \iint_{U_Q} |\nabla u(Y)|^2\, G_{L,*}(X_*, Y)\, dY \\
&\lesssim \sum_i \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}^*_{P_0},P_0^i} \\ \ell(Q) < M^{-1}\ell(P_0)}} \iint_{U_Q} |\nabla u(Y)|^2\, G_{L,*}(X_*, Y)\, dY \\
&\lesssim \sum_i \iint_{B_i \cap \Omega_*} |\nabla u(Y)|^2\, G_{L,*}(X_*, Y)\, dY \\
&\lesssim \|u\|^2_{L^\infty(\Omega_*)} \sum_i \omega_L^*(B_i \cap \partial\Omega_*) \\
&\lesssim (\gamma\,\lambda)^2,
\end{aligned}$$

where we have invoked Theorem 5.1 applied in Ω_*, which is a 1-sided NTA domain satisfying the CDC by Lemma 2.54, and we may need to take M slightly larger and use Harnack's inequality; (5.12); and the fact that $\{P_0^i\}_i \subset \mathbb{D}_{P_0}$ has uniformly bounded cardinality.

Using Chebyshev's inequality, (5.16), and collecting the estimates for Σ_1 and Σ_2 we conclude that

$$\omega_L^*(\widetilde{E}_\lambda \cap F_\lambda) \le \frac{1}{(\beta\,\lambda)^2} \int_{\widetilde{E}_\lambda \cap F_\lambda} \mathcal{S}_{P_0}^{k_0} u(x)^2\, d\omega_L^*(x) \\ \le \frac{1}{(\beta\,\lambda)^2} \int_{F_\lambda} \mathcal{S}_{P_0}^{k_0} u(x)^2\, d\omega_L^*(x) \lesssim \Big(\frac{\gamma}{\beta}\Big)^2.$$

At this point we invoke Lemma 4.18 in P_0 with $\mathcal{F}_{P_0}^*$ —we warn the reader that P_0 and $\mathcal{F}_{P_0}^* = \{P_j\}_j$ play the role of Q_0 and $\{Q_j\}_j$ and that associated to each P_j one finds $\widetilde{P}_j$ as in Proposition 4.1, which now plays the role of P_j in that result, and $\mu = \omega_L^{X_*}$ (recall that $X_* = Y_{P_0}$) and observe that the fact that $F_\lambda \subset P_0 \setminus (\cup_{\mathbb{D}_{\mathcal{F}_{P_0}^*, P_0}} P_j)$ implies on account of (4.20) that for some $\vartheta > 0$ we have

$$\frac{\omega_L^{X_*}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_*}(P_0)} \lesssim \Big(\frac{\omega_L^*(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^*(\Delta_\star^{P_0})}\Big)^{\frac{\vartheta}{2}} \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta},$$

where we have used that $\omega_L^*(\Delta_\star^{P_0}) \approx 1$ since $\Delta_\star^{P_0} := B(x_{P_0}^\star, t_{P_0}) \cap \partial\Omega_*$ with $t_{P_0} \approx \ell(P_0) \approx \operatorname{diam}(\partial\Omega_*)$, $x_{P_0}^\star \in \partial\Omega_*$, (5.13), Harnack's inequality, and Lemma 2.69 part (a). We can then use Remark 2.70, Harnack's inequality, and (5.13), to conclude that

$$\frac{\omega_L^{X_{Q_0}}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_{Q_0}}(P_0)} \approx \frac{\omega_L^{X_{P_0}}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_{P_0}}(P_0)} \approx \frac{\omega_L^{X_*}(\widetilde{E}_\lambda \cap F_\lambda)}{\omega_L^{X_*}(P_0)} \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta}.$$

Recalling that $P_0 = Q_j \in \mathcal{F}_0$, and the definitions of $\widetilde{E}_\lambda$ and F_λ in (5.10) the previous estimates readily lead to (5.9).

To conclude we need to see how (5.7) yields (5.4). With this goal in mind we first observe that for every $x \in Q_0$ and $Y \in \Gamma_{Q_0}^{k_0}(x)$ one has that $Y \in \overline{B_{Q_0}^*} \cap \Omega$ (see (2.48)) and also $\delta(Y) \gtrsim 2^{-k_0}\,\ell(Q_0)$. Hence, since $u \in W^{1,2}_{\rm loc}(\Omega)$, one has

$$(5.18)\qquad \sup_{x\in Q_0} \mathcal{S}_{Q_0}^{k_0} u(x) = \sup_{x\in Q_0} \Big(\iint_{\Gamma_{Q_0}^{k_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY\Big)^{\frac12} \\ \lesssim (2^{-k_0}\,\ell(Q_0))^{\frac{1-n}{2}} \Big(\iint_{B_{Q_0}^* \cap \{Y\in\Omega:\delta(Y)\gtrsim 2^{-k_0}\,\ell(Q_0)\}} |\nabla u(Y)|^2\, dY\Big)^{\frac12} < \infty.$$

On the other hand, given $1 < q < \infty$, we can use (5.7)

$$(1+\beta)^{-q}\, \|\mathcal{S}_{Q_0}^{k_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q \\ = \int_0^\infty q\,\lambda^q\,\omega_L^{X_{Q_0}}\big(\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\ \le \int_0^\infty q\,\lambda^q\,\omega_L^{X_{Q_0}}\big(\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > (1+\beta)\,\lambda,\ \mathcal{N}_{Q_0} u(x) \le \gamma\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\ + \int_0^\infty q\,\lambda^q\,\omega_L^{X_{Q_0}}\big(\{x \in Q_0 : \mathcal{N}_{Q_0} u(x) > \gamma\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\ \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \int_0^\infty q\,\lambda^q\,\omega_L^{X_{Q_0}}\big(\{x \in Q_0 : \mathcal{S}_{Q_0}^{k_0} u(x) > \beta\,\lambda\}\big)\,\frac{d\lambda}{\lambda} \\ + \gamma^{-q}\, \|\mathcal{N}_{Q_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q \\ \lesssim \Big(\frac{\gamma}{\beta}\Big)^{\vartheta} \beta^{-q}\, \|\mathcal{S}_{Q_0}^{k_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q + \gamma^{-q}\, \|\mathcal{N}_{Q_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q.$$

We can then choose γ small enough so that we can hide the first term in the right hand side of the last quantity (which is finite by (5.18)) and eventually conclude that

$$\|\mathcal{S}_{Q_0}^{k_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q \lesssim \|\mathcal{N}_{Q_0} u\|_{L^q(Q_0,\omega_L^{X_{Q_0}})}^q.$$

Since the implicit constant does not depend on k_0 and $\mathcal{S}_{Q_0}^k u(x) \nearrow \mathcal{S}_{Q_0} u(x)$ as $k \to \infty$ for every $x \in Q$, the monotone convergence theorem yields at once (5.4) and the proof Theorem 5.3 is complete.

APPENDIX A

Domains with Ahlfors-regular boundary

Throughout this section we assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a 1-sided CAD (cf. Definition 2.9). This means that Ω is a 1-sided NTA domain (it satisfies the Corkscrew and Harnack Chain conditions) and $\partial\Omega$ is AR. As mentioned in Section 2.2, the latter condition implies that Ω satisfies the CDC, hence the theory we have developed in this paper applies to Ω. On the other hand, the fact that Ahlfors regularity condition says that the surface measure $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is a well-behaved object. The goal of this section is to show how some earlier perturbation results, valid in Lipschitz, NTA or 1-sided NTA settings, can be obtained easily from our results. Before giving the precise statements let us present some definition:

Definition A.1 (Reverse Hölder and A_∞ classes with respect to surface measure). Given p, $1 < p < \infty$, we say that $\omega_L \in RH_p(\partial\Omega, \sigma)$, provided that $\omega_L \ll \sigma$ on $\partial\Omega$, and there exists $C \geq 1$ such that, writing $k_L = \frac{d\omega_L}{d\sigma}$ for the associated Radon-Nikodym, for every $\Delta_0 = B_0 \cap \partial\Omega$ where $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$

$$\left(\fint_\Delta k_L^{X_{\Delta_0}}(y)^p \, d\sigma(y)\right)^{\frac{1}{p}} \leq C \fint_\Delta k_L^{X_{\Delta_0}} \, d\sigma(y) = C \, \frac{\omega_L^{X_{\Delta_0}}(\Delta)}{\sigma(\Delta)}$$

for every $\Delta = B \cap \partial\Omega$ where $B \subset B_0$, $B = B(x, r)$ with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$. The infimum of the constants C as above is denoted by $[\omega_L]_{RH_p(\partial\Omega,\sigma)}$.

We also define

$$A_\infty(\partial\Omega, \sigma) = \bigcup_{p>1} RH_p(\partial\Omega, \sigma).$$

These are the results that we can reprove with our methods:

Corollary A.2. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD. Consider $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ two real (non-necessarily symmetric) elliptic operators. Define the disagreement between A and A_0 in Ω by*

$$\varrho(A, A_0)(X) := \|A - A_0\|_{L^\infty(B(X,\delta(X)/2))}, \qquad X \in \Omega, \tag{A.3}$$

where $\delta(X) := \operatorname{dist}(X, \partial\Omega)$, and

$$|||\varrho(A, A_0)|||_\sigma := \sup_B \frac{1}{\sigma(\Delta)} \iint_{B\cap\Omega} \frac{\varrho(A, A_0)(X)^2}{\delta(X)} \, dX, \tag{A.4}$$

where $\Delta = B \cap \partial\Omega$, and the sup is taken over all balls $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$.

(a) *Assume that $|||\varrho(A, A_0)|||_\sigma < \infty$. If $\omega_{L_0} \in A_\infty(\partial\Omega, \sigma)$, then $\omega_L \in A_\infty(\partial\Omega, \sigma)$. More precisely, if $\omega_{L_0} \in RH_p(\partial\Omega, \sigma)$ for some p, $1 < p < \infty$, then $\omega_L \in RH_q(\partial\Omega, \sigma)$ for some q, $1 < q < \infty$. Here, q and $[\omega_L]_{RH_q(\partial\Omega,\sigma)}$ depend only* *on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, $\|\!|\varrho(A,A_0)|\!\|_\sigma$, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$.

(*b*) *If* $\omega_{L_0}\in RH_p(\partial\Omega,\sigma)$, *for some* p, $1<p<\infty$, *there exists* $\varepsilon_p>0$ *(depending only on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$*) such that if* $\|\!|\varrho(A,A_0)|\!\|_\sigma\le\varepsilon_p$, *then* $\omega_L\in RH_p(\partial\Omega,\sigma)$. *Here,* $[\omega_L]_{RH_q(\partial\Omega,\sigma)}$ *depends only on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$.

COROLLARY A.5. *Let* $\Omega\subset\mathbb{R}^{n+1}$, $n\ge 2$, *be a 1-sided CAD. Consider* $Lu=-\operatorname{div}(A\nabla u)$ *and* $L_0u=-\operatorname{div}(A_0\nabla u)$ *two real (non-necessarily symmetric) elliptic operators, and recall the definition of* $\mathcal{A}_\alpha(\varrho(A,A_0))$ *in* (1.11) *for any given* $\alpha>0$.

(*a*) *Assume that* $\mathcal{A}_\alpha(\varrho(A,A_0))\in L^\infty(\sigma)$. *If* $\omega_{L_0}\in A_\infty(\partial\Omega,\sigma)$, *then* $\omega_L\in A_\infty(\partial\Omega,\sigma)$. *More precisely, if* $\omega_{L_0}\in RH_p(\partial\Omega,\sigma)$ *for some* p, $1<p<\infty$, *then* $\omega_L\in RH_q(\partial\Omega,\sigma)$ *for some* q, $1<q<\infty$. *Here,* q *and* $[\omega_L]_{RH_q(\partial\Omega,\sigma)}$ *depend only on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, α, $\|\mathcal{A}_\alpha(\varrho(A,A_0))\|_{L^\infty(\sigma)}$, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$.

(*b*) *If* $\omega_{L_0}\in RH_p(\partial\Omega,\sigma)$, *for some* p, $1<p<\infty$, *there exists* $\varepsilon_p>0$ *(depending only on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$*), such that if* $\mathcal{A}_\alpha(\varrho(A,A_0))\in L^\infty(\sigma)$ *with* $\|\mathcal{A}_\alpha(\varrho(A,A_0))\|_{L^\infty(\sigma)}\le\varepsilon_p$, *then* $\omega_L\in RH_p(\partial\Omega,\sigma)$. *Here,* $[\omega_L]_{RH_p(\partial\Omega,\sigma)}$ *depends only on dimension, the 1-sided CAD constants, the ellipticity constants of* L_0 *and* L, α, p, *and* $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$.

In the case of symmetric operators, part (*b*) of Corollary A.2 has been proved for the unit ball in [**Dah86**], for bounded CAD in [**MPT13**], and for 1-sided CAD domains in [**CHM19**]. On the other hand, part (*a*) of Corollary A.2 can be found for Lipschitz domains in [**FKP91**] and for bounded CAD in [**MPT13**], both in the case of symmetric operators (but we would expect that similar arguments could be carried over to the non-symmetric case as well). The corresponding result in the setting of 1-sided CAD has been obtained in [**CHM19**] for symmetric operators and then extended to the general case in [**CHMT20**]. Note then that Corollary A.2 part (*b*) seems to be new in the case of non-symmetric operators in 1-sided CAD. Regarding Corollary A.5, part (*a*) for symmetric operators was proved in [**Fef89**] in the unit ball and in [**MPT13**] in the setting of bounded CAD.

Before proving the previous results we need the following auxiliary lemma:

LEMMA A.6. *Let* $\Omega\subset\mathbb{R}^{n+1}$ *be a 1-sided CAD and let* $Lu=-\operatorname{div}(A\nabla u)$ *and* $L_0u=-\operatorname{div}(A_0\nabla u)$ *be real (non-necessarily symmetric) elliptic operators. If* $\omega_{L_0}\in A_\infty(\partial\Omega,\sigma)$ *and* $\omega_L\in A_\infty(\partial\Omega,\omega_{L_0})$ *then* $\omega_L\in A_\infty(\partial\Omega,\sigma)$. *More precisely, if* $\omega_{L_0}\in RH_p(\partial\Omega,\sigma)$, $1<p<\infty$, *and* $\omega_L\in RH_q(\partial\Omega,\omega_{L_0})$, $1<q<\infty$, *then* $\omega_L\in RH_r(\partial\Omega,\sigma)$ *with* $r=\frac{p\,q}{p+q-1}\in(1,\min\{p,q\})$ *and, moreover,*

$$[\omega_L]_{RH_r(\partial\Omega,\sigma)}\le[\omega_L]_{RH_q(\partial\Omega,\omega_0)}\,[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}^{\frac{1}{q'}}.$$

PROOF. Fix $\Delta_0=B_0\cap\partial\Omega$ where $B_0=B(x_0,r_0)$ with $x_0\in\partial\Omega$ and $0<r_0<\operatorname{diam}(\partial\Omega)$. Write $\omega_0=\omega_{L_0}^{X_{\Delta_0}}$ and $\omega=\omega_L^{X_{\Delta_0}}$. By definition $\omega_0\ll\sigma$ and $\omega\ll\omega_0$, hence $\omega\ll\sigma$. Given $\Delta=B\cap\partial\Omega$ where $B\subset B(x_0,r_0)$, $B=B(x,r)$ with $x\in\partial\Omega$,

$0 < r < \operatorname{diam}(\partial\Omega)$, by Hölder's inequality with exponent $\frac{q}{r} > 1$ we obtain

$$\begin{aligned}
\left(\fint_\Delta \left(\frac{d\omega}{d\sigma}\right)^r d\sigma\right)^{\frac{1}{r}} &= \left(\fint_\Delta \left(\frac{d\omega}{d\omega_0}\,\frac{d\omega_0}{d\sigma}\right)^r d\sigma\right)^{\frac{1}{r}} \\
&= \left(\fint_\Delta \left(\frac{d\omega}{d\omega_0}\right)^r \left(\frac{d\omega_0}{d\sigma}\right)^{\frac{r}{q}} \left(\frac{d\omega_0}{d\sigma}\right)^{\frac{r}{q'}} d\sigma\right)^{\frac{1}{r}} \\
&\le \left(\frac{\omega_0(\Delta)}{\sigma(\Delta)}\right)^{\frac{1}{q}} \left(\fint_\Delta \left(\frac{d\omega}{d\omega_0}\right)^q d\omega_0\right)^{\frac{1}{q}} \left(\fint_\Delta \left(\frac{d\omega_0}{d\sigma}\right)^{\frac{r}{q'}\,(\frac{q}{r})'} d\sigma\right)^{\frac{1}{r\,(\frac{q}{r})'}} \\
&= \left(\frac{\omega_0(\Delta)}{\sigma(\Delta)}\right)^{\frac{1}{q}} \left(\fint_\Delta \left(\frac{d\omega}{d\omega_0}\right)^q d\omega_0\right)^{\frac{1}{q}} \left(\fint_\Delta \left(\frac{d\omega_0}{d\sigma}\right)^p d\sigma\right)^{\frac{1}{q'\,p}} \\
&\le [\omega_L]_{RH_q(\partial\Omega,\omega_0)}\, [\omega_{L_0}]^{\frac{1}{q'}}_{RH_p(\partial\Omega,\sigma)} \left(\frac{\omega_0(\Delta)}{\sigma(\Delta)}\right)^{\frac{1}{q}} \left(\fint_\Delta \frac{d\omega}{d\omega_0}\, d\omega_0\right) \left(\fint_\Delta \frac{d\omega_0}{d\sigma}\, d\sigma\right)^{\frac{1}{q'}} \\
&= [\omega_L]_{RH_q(\partial\Omega,\omega_0)}\, [\omega_{L_0}]^{\frac{1}{q'}}_{RH_p(\partial\Omega,\sigma)}\, \frac{\omega(\Delta)}{\sigma(\Delta)}.
\end{aligned}$$

Thus we conclude that $\omega_L \in RH_r(\partial\Omega,\sigma)$ with

$$[\omega_L]_{RH_r(\partial\Omega,\sigma)} \le [\omega_L]_{RH_q(\partial\Omega,\omega_0)}\, [\omega_{L_0}]^{\frac{1}{q'}}_{RH_p(\partial\Omega,\sigma)},$$

and the proof is complete. □

Proof of Corollary A.2. Assume that $\omega_{L_0} \in A_\infty(\partial\Omega,\sigma)$. Our first goal is to show that using the notation in (1.7) we have

$$\|\!|\varrho(A,A_0)|\!\| \lesssim \|\!|\varrho(A,A_0)|\!\|_\sigma. \tag{A.7}$$

To see this we take some ideas from the proof of Theorem 1.10. Let $\mathbb{D} = \mathbb{D}(\partial\Omega)$ be the dyadic grid from Lemma 2.33 with $E = \partial\Omega$. For any $Q \in \mathbb{D}$ we set

$$\gamma_Q = \frac{1}{\sigma(Q)} \iint_{U_Q} \frac{\varrho(A,A_0)(X)^2}{\delta(X)}\, dX.$$

Fix $B_0 = B(x_0,r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)$. Let $\Delta = B \cap \partial\Omega$ with $B = B(x,r)$, $x \in 2\Delta_0$, and $0 < r < r_0 c_0/4$, here c_0 is the Corkscrew constant. Write $X_0 = X_{\Delta_0}$ and $\omega_0 = \omega_{L_0}^{X_0}$. Note that this choice guarantees that $X_0 \notin 4B$. Define

$$\mathcal{W}_B = \{I \in \mathcal{W} : I \cap B \neq \emptyset\}$$

and for every $I \in \mathcal{W}_B$ let $X_I \in I \cap B$ so that $4\operatorname{diam}(I) \le \operatorname{dist}(I,\partial\Omega) \le \delta(X_I) < r$ and hence $I \subset \frac{5}{4}B$. Pick $x_I \in \partial\Omega$ such that $|X_I - x_I| = \delta(X_I) \le \operatorname{diam}(I) + \operatorname{dist}(I,\partial\Omega)$ and let $Q_I \in \mathbb{D}$ be such that $x_I \in Q_I$ and $\ell(I) = \ell(Q_I)$. By Lemma 2.69 parts (a)–(c), Harnack's inequality and the fact that $\partial\Omega$ is AR one has

$$\frac{G_{L_0}(X_0,Y)}{\delta(Y)} \approx \frac{G_{L_0}(X_0,X_I)}{\ell(I)} \approx \frac{\omega_0(Q_I)}{\ell(I)^n} \approx \frac{\omega_0(Q_I)}{\sigma(Q_I)}, \qquad \forall Y \in I.$$

Using this

$$\mathcal{I}_B := \iint_{B\cap\Omega} \varrho(A,A_0)(Y)^2 \frac{G_{L_0}(X_0,Y)}{\delta(Y)^2}\, dY$$

$$\begin{aligned}
&\lesssim \sum_{I\in\mathcal{W}_B} \iint_I \frac{\varrho(A,A_0)(Y)^2}{\delta(Y)}\,dY\,\frac{\omega_0(Q_I)}{\sigma(Q_I)}\\
&\le \sum_{I\in\mathcal{W}_B} \iint_{U_{Q_I}} \frac{\varrho(A,A_0)(Y)^2}{\delta(Y)}\,dY\,\frac{\omega_0(Q_I)}{\sigma(Q_I)}\\
&= \sum_{I\in\mathcal{W}_B} \gamma_{Q_I}\,\omega_0(Q_I),
\end{aligned}$$

where we have used that by construction $I\subset U_{Q_I}\in\mathcal{W}_{Q_I}$.

Note that $\ell(Q_I)=\ell(I)<\operatorname{diam}(Q_I)<r/4$. Also if $z\in Q_I$, then by (2.34) and (2.40)

$$\begin{aligned}
|z-x| &\le |z-x_I|+|x_I-X_I|+|X_I-x|\\
&\le \Xi\ell(Q_I)+\delta(X_I)+\frac{r}{4}<\Xi\ell(Q_I)+\operatorname{diam}(I)+\operatorname{dist}(I,\partial\Omega)+\frac{r}{4}<12\,\Xi\, r
\end{aligned}$$

and therefore $Q_I\subset 12\,\Xi\Delta$. Write then $\mathcal{F}_\Delta=\{Q\in\mathbb{D}:\frac{r}{4}\le\ell(Q)<\frac{r}{2}, Q\cap 12\,\Xi\Delta\neq\emptyset\}$, so that $\mathcal{F}_\Delta$ is a family of pairwise disjoint dyadic cubes with uniformly bounded cardinality and so that $12\,\Xi\Delta\subset\cup_{Q\in\mathcal{F}_\Delta}Q\subset 13\,\Xi\Delta$. By construction, if $I\in\mathcal{W}_B$, then $Q_I\subset Q$ for some $Q\in\mathcal{F}_\Delta$. Introducing the notation

$$|||\gamma|||_{\omega_0,\Delta}:=\sup_{Q\in\mathcal{F}_\Delta}\sup_{Q'\in\mathbb{D}_Q}\frac{1}{\omega_0(Q')}\sum_{Q''\in\mathbb{D}_{Q'}}\gamma_{Q''}\,\omega_0(Q''),$$

it follows that

$$\begin{aligned}
\text{(A.8)}\qquad \mathcal{I}_B &\le \sum_{Q\in\mathcal{F}_\Delta}\sum_{Q'\in\mathbb{D}_Q}\gamma_{Q'}\,\omega_0(Q')\le |||\gamma|||_{\omega_0,\Delta}\sum_{Q\in\mathcal{F}_\Delta}\omega_0(Q)\\
&\le |||\gamma|||_{\omega_0,\Delta}\,\omega_0(13\,\Xi\Delta)\lesssim |||\gamma|||_{\omega_0,\Delta}\,\omega_0(\Delta),
\end{aligned}$$

where we have used Lemma 2.69.

We next estimate $|||\gamma|||_{\omega_0,\Delta}$. Since we have assumed that $\omega_{L_0}\in A_\infty(\partial\Omega,\sigma)$, it follows that $\omega_{L_0}\in RH_p(\partial\Omega,\sigma)$ for some p, $1<p<\infty$, then it is straightforward to see using Lemma 2.69 that $\omega_{L_0}^{X_Q}\in RH_p^{\text{dyadic}}(Q,\sigma)$ for every $Q\in\mathbb{D}$ (cf. Definition 2.24). In particular, for every $Q'\in\mathbb{D}_Q$ with $Q\in\mathbb{D}$ and for every $F\subset Q'$ we have

$$\begin{aligned}
\frac{\omega_{L_0}^{X_Q}(F)}{\sigma(Q')} &= ⨍_{Q'}\mathbf{1}_F\,k_{L_0}^{X_Q}\,d\sigma(y)\le\left(\frac{\sigma(F)}{\sigma(Q')}\right)^{\frac{1}{p'}}\left(⨍_{Q'}k_{L_0}^{X_Q}(y)^p\,d\sigma(y)\right)^{\frac{1}{p}}\\
&\le [\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}\left(\frac{\sigma(F)}{\sigma(Q')}\right)^{\frac{1}{p'}}\frac{\omega_{L_0}^{X_Q}(Q')}{\sigma(Q')},
\end{aligned}$$

where $C>1$ is a uniform constant. Take then $\alpha=\frac12$, $\beta=(2\,C\,[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)})^{-p'}\in(0,1)$, and apply Lemma 2.19 with $\mu=\omega_{L_0}^{X_Q}$ and $\nu=\sigma$ to obtain

$$\begin{aligned}
\sup_{Q'\in\mathbb{D}_Q}\frac{1}{\omega_{L_0}^{X_Q}(Q')}\sum_{Q''\in\mathbb{D}_{Q'}}\gamma_{Q''}\,\omega_{L_0}^{X_Q}(Q'') &\lesssim \sup_{Q'\in\mathbb{D}_Q}\frac{1}{\sigma(Q')}\sum_{Q''\in\mathbb{D}_{Q'}}\gamma_{Q''}\,\sigma(Q'')\\
&= \sup_{Q'\in\mathbb{D}_Q}\frac{1}{\sigma(Q')}\sum_{Q''\in\mathbb{D}_{Q'}}\iint_{U_Q}\frac{\varrho(A,A_0)(X)^2}{\delta(X)}\,dX\\
&\lesssim \sup_{Q'\in\mathbb{D}_Q}\frac{1}{\sigma(Q')}\iint_{T_{Q'}}\frac{\varrho(A,A_0)(X)^2}{\delta(X)}\,dX
\end{aligned}$$

$$\lesssim \sup_{Q'\in\mathbb{D}_Q} \frac{1}{\sigma(\Delta_Q^*)} \iint_{B_Q^*} \frac{\varrho(A,A_0)(X)^2}{\delta(X)}\,dX$$
$$\le \|\!|\varrho(A,A_0)|\!\|_\sigma,$$

where we have used that the family $\{U_{Q'}\}_{Q'\in\mathbb{D}}$ has bounded overlap, (2.48), the AR property of σ and (A.4). Invoke once again Lemma 2.69 and Harnack's inequality to conclude that (A.8) along with the previous estimate readily yield

$$\mathcal{I}_B \lesssim \omega_0(\Delta) \sup_{Q\in\mathcal{F}_\Delta} \sup_{Q'\in\mathbb{D}_Q} \frac{1}{\omega_{L_0}^{X'_Q}(Q')} \sum_{Q''\in\mathbb{D}_{Q'}} \gamma_{Q''}\,\omega_{L_0}^{X'_Q}(Q'') \lesssim \omega_0(\Delta)\,\|\!|\varrho(A,A_0)|\!\|_\sigma,$$

Taking then the sup over all B and B_0 as above we have shown that (A.7) holds.

With (A.7) at hand we are now ready to prove (a) and (b) in the statement. To prove (a) note that by assumption $\|\!|\varrho(A,A_0)|\!\|_\sigma < \infty$ and $\omega_{L_0} \in A_\infty(\partial\Omega,\sigma)$. Hence, (A.7) says that $\|\!|\varrho(A,A_0)|\!\| < \infty$ and Theorem 1.5 part (a) yields $\omega_L \in A_\infty(\partial\Omega,\omega_{L_0})$. In turn, Lemma A.6 implies that $\omega_L \in A_\infty(\partial\Omega,\sigma)$ as desired.

To prove (b) we proceed as follows. Assume that $\omega_{L_0} \in RH_p(\partial\Omega,\sigma)$. By Gehring's lemma [**Geh73**] (see also [**CF74**]) there exists $s>1$ such that $\omega_{L_0} \in RH_{p\,s}(\partial\Omega,\sigma)$. Set $q := \frac{s\,p-1}{s-1} > 1$ and note that by (A.7) and Theorem 1.5 part (b) we can find $\varepsilon_p > 0$ sufficiently small (depending only on dimension, the 1-sided CAD constants, the ellipticity constants of L_0 and L, p, and $[\omega_{L_0}]_{RH_p(\partial\Omega,\sigma)}$) so that if $\|\!|\varrho(A,A_0)|\!\|_\sigma < \epsilon_p$ then $\omega_L \in RH_q(\partial\Omega,\omega_{L_0})$. If we apply Lemma A.6 with $p\,s$ and our choice of q we conclude that $\omega_L \in RH_r(\partial\Omega,\sigma)$ where $r = \frac{p\,s\,q}{p\,s+q-1} = p$. This completes the proof. □

Proof of Corollary A.5. Note first that in both cases (a) and (b), the fact that $\omega_{L_0} \in A_\infty(\partial\Omega,\sigma)$ implies $\omega_{L_0} \ll \sigma$. On the other hand, since the A_∞ property is symmetric we clearly have that $\sigma \ll \omega_{L_0}$. It is important to emphasize that by Harnack's inequality $\omega_L^X \ll \omega_L^Y$ for every $X, Y \in \Omega$, hence we do not need to specify the pole in ω_L. All these show that $\|\cdot\|_{L^\infty(\sigma)} = \|\cdot\|_{L^\infty(\omega_{L_0})}$.

To prove (a) we then observe that the assumption $\mathcal{A}_\alpha(\varrho(A,A_0)) \in L^\infty(\sigma)$ gives at once that $\mathcal{A}_\alpha(\varrho(A,A_0)) \in L^\infty(\omega_{L_0})$ and by Theorem 1.10 part (a) we conclude that $\omega_L \in A_\infty(\partial\Omega,\omega_{L_0})$. This, the fact that $\omega_{L_0} \in A_\infty(\partial\Omega,\sigma)$, and Lemma A.6 readily gives that $\omega_L \in A_\infty(\partial\Omega,\sigma)$ as desired.

To prove (b) we proceed much as in the corresponding case in the proof of Corollary A.2. Assume that $\omega_{L_0} \in RH_p(\partial\Omega,\sigma)$ and once again by Gehring's lemma find $s>1$ such that $\omega_{L_0} \in RH_{p\,s}(\partial\Omega,\sigma)$. Set $q := \frac{s\,p-1}{s-1} > 1$ and note that if $\|\mathcal{A}_\alpha(\varrho(A,A_0))\|_{L^\infty(\sigma)} = \|\mathcal{A}_\alpha(\varrho(A,A_0))\|_{L^\infty(\omega_{L_0})}$ is sufficiently small, Theorem 1.10 part (b) says that $\omega_L \in RH_q(\partial\Omega,\omega_{L_0})$. We next apply Lemma A.6 with $p\,s$ and our choice of q to conclude that $\omega_L \in RH_p(\partial\Omega,\sigma)$ much as we did before. □

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Symbol Index

$\mathcal{P}_{\mathcal{F}}^{\mu}$	projection operator, 17
Q	dyadic cube on E or $\partial\Omega$, 9
$RH_p(\partial\Omega, \omega_0)$	class of reverse Hölder weights with respect to ω_0, 27
$RH_p(\partial\Omega, \sigma)$	class of reverse Hölder weights with respect to σ, 87
RH_p^{dyadic}	class of dyadic reverse Hölder weights, 17
r_Q	radius of of the dyadic cube Q, 20
$\mathcal{S}_Q^k u(x)$	truncated localized dyadic conical square function, 81
$\mathcal{S}_Q u(x)$	localized dyadic conical square function, 24
T_Q	Carleson box relative to a dyadic cube, 23
T_Q^*, T_Q^{**}	fattened Carleson boxes relative to a dyadic cube, 23
T_Δ	Carleson box relative to a surface ball, 24
$T_\Delta^*, T_\Delta^{**}$	fattened Carleson boxes relative to a surface ball, 24
U_Q	Whitney region, 22
U_Q^*, U_Q^{**}	fattened Whitney regions, 23
$\mathcal{W}$	Whitney decomposition, 22
$\mathcal{W}(\Omega)$	Whitney decomposition of Ω, 22
$\mathcal{W}_Q^*$	family of Whitney cubes comprising the Whitney region above Q, 22
$X(I)$	center of I, 22
X_Q	Corkscrew point relative to Q, 22
X_Δ	Corkscrew point relative to Δ, 10
x_Q	center of the dyadic cube Q, 20
$\Gamma_Q(x)$	truncated dyadic cone, 23
$\Gamma_Q^*(x), \Gamma_Q^{**}(x)$	fattened truncated dyadic cones, 23
$\Gamma_Q^k(x)$	truncated dyadic cone from below, 81
$\delta(\cdot)$	distance to $\partial\Omega$, 9
$\Delta(x, r)$	$B(x, r) \cap \partial\Omega$ surface ball of radius r centered at $x \in \partial\Omega$, 9
Δ^*	large dilation of Δ, 24
κ_0, κ_1	relevant geometric constants, 24
λ_0	relevant geometric constant, 22
Ξ	relevant geometric constant, 19
$\varrho(A, A_0)$	disagreement between A and A_0, 5
$\vert\!\vert\!\vert \varrho(A, A_0) \vert\!\vert\!\vert$	Carleson measure norm of the disagreement between A and A_0, 5
$\vert\!\vert\!\vert \varrho(A, A_0) \vert\!\vert\!\vert_{B_0}$	Carleson measure norm of the disagreement between A and A_0 in B_0, 31
$\vert\!\vert\!\vert \varrho(A, A_0) \vert\!\vert\!\vert_{\sigma}$	Carleson measure norm of the disagreement between A and A_0 with respect to σ, 87
$\varphi_t(\cdot, \cdot)$	kernel of the approximation of the identity P_t, 43
Ω_{ext}	$\mathbb{R}^{n+1} \setminus \overline{\Omega}$ exterior of Ω, 10
$\Omega_{\mathcal{F}}$	global sawtooth region, 23
$\Omega_{\mathcal{F}}^*, \Omega_{\mathcal{F}}^{**}$	fattened global sawtooth regions, 23
$\Omega_{\mathcal{F},Q}$	local sawtooth region, 23
$\Omega_{\mathcal{F},Q}^*, \Omega_{\mathcal{F},Q}^{**}$	fattened local sawtooth regions, 23
$\omega_L, \omega_{L,\Omega}$	elliptic measure, 26
$\omega_L^X, \omega_{L,\Omega}^X,$	elliptic measure with pole at X, 26

Subject Index