Quantum nonlocality, tensor norms and operator spaces

International Graduate Student Summer School: Quantum Correlations and Group C^*-Algebra Theory

University of Zhejiang, Hangzhou (China).

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A comment on these notes

These notes are based on a series of lectures that I delivered in July 2016 as a part of the *International Graduate Student Summer School: Quantum Correlations and Group C^*-Algebra Theory* at the University of Zhejiang in Hangzhou (China).

The aim of this course was to present an introduction to the use of operator spaces in the field of quantum information theory. Since this course was expected to last no more than 30 hours, in order to give a self-contained and deep enough introduction, I saw myself forced to select only one topic among many different possibilities. As a consequence of this, the course was focused on the connections between *quantum nonlocality*, a major topic in quantum information theory, and the theory of *tensor norms of Banach/operator spaces*. In particular, the course was devoted to explaining why the theory of operator spaces is very suitable to study *violations of Bell inequalities*.

The course was thought to be self-contained. To this end, all basic concepts and results that are used (except for well known theorems, for which references are provided) are introduced in detail. At the same time, one of the main goals of the course was to explain some of the recent results in the field so that, at the end, the interested students were able to handle some of the current techniques in the topic.

These notes faithfully follow the recent survey, written in a joint work with T. Vidick,

- *Survey on Nonlocal Games and Operator Space Theory*, C. Palazuelos, T. Vidick. Journal of Mathematical Physics 57, 015220 (2016)(Special Issue: Operator Algebras and Quantum Information Theory).

However, in the present notes fewer topics than in the previous survey are considered, with the advantage that they are explained in complete detail.

In Chapter 1 the basic concepts of quantum nonlocality are introduced. Here, I give the basic definitions about quantum nonlocality and also explain the connections between quantum nonlocality and a famous theorem due to Grothendieck which links the topic of quantum nonlocality with the theory of Banach spaces. In fact, I found it interesting to look at the topic of quantum nonlocality from three different perspective: Physical point of view (or at least, the original one considered by Bell), mathematical point of view (Banach spaces and tensor norms) and computer science point of view (nonlocal games).

In Chapter 2 the general bipartite scenario of Bell functionals with many possible inputs and outputs for Alice and Bob is considered. First, I introduce the setting in detail and explain the motivation to consider tensor products to describe Bell functionals. Later, a brief introduction about operator spaces is given, by explaining those concepts and results that I will need in the rest of the notes. Finally, in the last part of Chapter 2 I explain the precise connections between the classical and the quantum value of Bell functionals (or two-prover games) and Banach spaces/operator spaces.

Chapter 3 is devoted to explaining how to use the connections shown in Chapter 2 between nonlocality and operator spaces to study Bell inequality violations. First, I show how to obtain upper bounds for such violations by using basic results on operator spaces. In addition, it is shown how to construct examples of Bell functions with large violations. This is in hard contrast with the results studied in Chapter 1 for correlation Bell functionals, for which Bell violations are uniformly upper bounded by Grothendieck's constant. Although the construction of Bell functionals with large violations is probably the most technical part of the course, the section is (expected to be) self-contained. In the last part of Chapter 3 some other interesting constructions leading to large Bell violations are briefly mentioned.

Finally, I would like to thank Professor Junde Wu for his invitation to this very nice summer school and also for his hospitality, which definitely made our stay in Hangzhou memorable!

Capítulo 1

Introduction to Quantum nonlocality

Since the birth of quantum mechanics many scientists have questioned this theory. Indeed, though quantum mechanics has been shown to be very *useful* to explain the behavior of matter and energy on the atomic and subatomic level, some of the most important scientists of the 20th century were skeptic about it owing to its nondeterministic nature. The study of theories based on *hidden variable models* is partially motivated by the attempt to avoid the intrinsic uncertainty in Nature, assumed by quantum mechanics.

Quantum mechanics assumes that a physical system is completely described by an "object" (vector state) which somehow contains all the information that we can obtain about the system. The impossibility of obtaining a more accurate information about it does not depend on the precision of our devices, but it is intrinsic in Nature. Contrary to this assumption, models based on hidden variables propose that such a lack of knowledge about Nature is because our own restrictions. These models assume that there exists a classical probability over the "states of the world" to which we do not have a complete access and which models our uncertainly. However, for one such fixed state, we are in a completely deterministic situation. Consequently, any action on a physical system can be understood as a classical average over deterministic processes.

In 1935 Einstein, Podolsky and Rosen proposed an experiment [**6**] whose aim was to show the incompleteness of quantum mechanics as a model of Nature. The possibility of using two spatially separated particles to produce an immediate effect in one of them by just acting on the other one (a consequence of quantum entanglement referred as "spooky action at a distance") was considered impossible by the authors, as it violated the local realist view of causality. It took almost 30 years to understand that the apparent dilemma presented in [**6**] could be formulated in terms of assumptions which naturally lead to a refutable prediction. Bell showed that the assumption of a local hidden variable model implies some inequalities on the set of correlations obtained in a certain measurement scenario and that these inequalities are violated by certain quantum correlations produced with an entangled state [**2**]. Those inequalities are since then called *Bell inequalities*. Bell's work led to different experimental verifications of a such counterintuitive phenomenon [**1, 8**], which provide the strongest evidence that Nature does not obey the laws of classical mechanics.

Though initially discovered in the context of foundations of quantum mechanics, violations of Bell inequalities, commonly known as *quantum nonlocality* (because they show the existence of quantum correlation which cannot be explained by a local hidden variable model), are nowadays a key point in a wide range of branches of quantum information science. In particular, nonlocal correlations provide advantages in communication complexity and information theoretical protocols as well as in the security of quantum cryptography protocols.

1. Bell's result: Correlations in EPR

We forget for a moment about quantum mechanics and we perform the following mental experiment ([**14**, Section 2.6]). Charlie prepares two particles, in whatever way he wants, and he sends one of these particles to Alice and the other to Bob. Upon receiving her particle, Alice measures either property Q or property R of the particle, and assume that these measurements can only take the two values ± 1. We want to impose that Alice does not know in advance what measurement she will choose to perform, so we can assume that after getting her particle, Alice flips a coin to take her decision. Bob does the same with his particle, and let us call S, T to the properties he measures, again with the possible outcomes ± 1. In addition, we assume that Alice and Bob can perform their measurements in a casually disconnected manner. That is, sufficiently simultaneously and far apart that Alice's measurement can not influence in Bob's measurement and vice versa. Let us also assume that Charlie can prepare similar pair of particles once and again and we can repeat the experiment as many times as we want.

Let us consider the number

$$QS + RS + RT - QT = (Q+R)S + (R-Q)T.$$

From a local and deterministic point of view, it is clear that either $(Q+R)$ or $(R-Q)$ is 0 and

$$QS + RS + RT - QT = \pm 2.$$

Let us first assume that Nature can be explained by a Local Hidden Variable Model. The locality hypothesis means exactly what we have just explained about the possibility of Alice and Bob to perform their measurements in a casually disconnected manner. In particular, the special relativity theory implies this hypothesis. Note that the randomness in Alice's and Bob's choice of measurements is needed, since the fact that their measurements were a priori determined would affect the locality. On the other hand, a hidden variable model is based on the hypothesis of the existence of a hidden probability on the space of "all possible states of the world" and such that each of these possible states is deterministic.

Going back to our experiment, call $p(q, r, s, t)$ to the (hidden) probability that, for a given preparation of the pair of particles, $Q = q$, $R = r$, etc. Then, it is trivial to calculate

$$\begin{aligned} |\mathbb{E}(QS) + \mathbb{E}(RS) + \mathbb{E}(RT) - \mathbb{E}(QT)| &= |\mathbb{E}(QS + RS + RT - QT)| \\ = \Big| \sum_{q,r,s,t} p(q,r,s,t)(qs + rs + rt - qt)\Big| &\leq 2 \sum_{q,r,s,t} p(q,r,s,t) = 2. \end{aligned}$$

This defines an inequality on the set of measurement correlations obtained in the previous experiment,

$$|\mathbb{E}(QS) + \mathbb{E}(RS) + \mathbb{E}(RT) - \mathbb{E}(QT)| \leq 2, \tag{1.1}$$

which is known as *CHSH-inequality*.

Let us assume now that Nature is explained by quantum mechanics and assume that the state formed by both particles is described by

$$|\varphi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The first qubit goes to Alice and the second to Bob. Alice then measures with the observables $Q = \sigma_z$, $R = \sigma_x$ and Bob measures with observables $S = \frac{-\sigma_z - \sigma_x}{\sqrt{2}}$, $T = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$, where here we are using the standard notation for *the Pauli matrices*:

$$\sigma_0 = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, easy calculations allow to obtain the following estimates.

$$\langle QS\rangle = -\frac{1}{\sqrt{2}},\ \langle RS\rangle = -\frac{1}{\sqrt{2}},\ \langle RT\rangle = -\frac{1}{\sqrt{2}},\ \langle QT\rangle = \frac{1}{\sqrt{2}}.$$

Hence,

$$|\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle| = 2\sqrt{2}. \tag{1.2}$$

Equation (1.1) and Equation (1.2) show that quantum mechanics can not be explained by a local hidden variable model. That is, quantum mechanics predicts that we can obtain certain correlations in the previous measurement-experiment which can not be explained by a local hidden variable model. The very recent free-loophole experimental verification of this phenomenon [**8**] can be consider as an irrefutable evidence of the nonlocality of Nature.

2. Tsirelson's theorem and Grothendieck's theorem

The previous Alice-Bob scenario can be naturally generalized to the case of more than two measurements per site. In this case Alice can perform N different measurements $A_1 \cdots, A_N$, each with possible outputs ± 1 and similarly to Bob with measurements $B_1 \cdots, B_N$.[1] Let us denote

$$\gamma_{x,y} = \mathbb{E}[A_x B_y], \quad \text{for every} \quad x, y = 1, \cdots, N.$$

Here, $\mathbb{E}[A_x B_y]$ denotes the expected value of the product of the outputs of A_x and B_y for every x, y. $\gamma := (\gamma_{x,y})_{x,y=1}^N$ is usually called *correlation matrix*.

The correlation matrices obtainable if we assumed a local hidden variable model of Nature are those of the form

$$\gamma_{x,y} = \int_\Omega A_x(\omega) B_y(\omega) d\mathbb{P}(\omega), \tag{2.1}$$

where $(\Omega, \mathbb{P})$ is the hidden probability space and, fixed one of these states ω, $A_x(\omega) = +1$ or -1 and similarly for $B_y(\omega)$, for every x, y. We call these matrices *classical correlation matrices* and we denote by $\mathcal{L}_N$ the set of classical correlation matrices of size N. Note that the elements in $\mathcal{L}_N$ are those matrices with entries $\mathbb{E}[A_x B_y]$ given by the expected value of the product of the outcomes of the binary measurements A_x and B_y when we describe the corresponding measurement procedure by using a local hidden variable model in the same way we did in Section 1.

It is very easy to check that $\mathcal{L}_N$ is the convex hull of the elements of the form $(t_x s_y)_{x,y=1}^N$ with $t_x = \pm 1, s_y = \pm 1$ for every $x, y = 1, \cdots, N$ and these are precisely the extreme points of the set:

$$\mathcal{L}_N = \operatorname{conv}\Big\{ (t_x s_y)_{x,y=1}^N : t_x = \pm 1, s_y = \pm 1 \ \text{ for every } \ x, y = 1, \cdots, N \Big\}. \tag{2.2}$$

That is, $\mathcal{L}_N$ is a polytope in $\mathbb{R}^{N^2}$ with 2^{2N} vertices.

According to the postulates of quantum mechanics, in order to define the bipartite system we are measuring on, we must specify a quantum state $\rho \in S_1(\mathcal{H} \otimes \mathcal{H})$[2]. On the other hand, each of Alice's two outputs measurements A_x will be described by a POVM $\{E_x, \mathrm{Id} - E_x\}$, where E_x is a positive operator acting on $\mathcal{H}$ associated to the output 1 and $\mathrm{Id} - E_x$ is a positive operator acting on $\mathcal{H}$ associated to the output -1. Similarly, we will have to consider the corresponding POVMs to describe Bob's measurements $\{F_y, \mathrm{Id} - F_y\}$ for every y. Then, if Alice and Bob perform the measurements A_x and B_y respectively, we know that the corresponding table of probabilities is given by

$$P(x,y) = \begin{cases} tr\big((E_x \otimes F_y)\rho\big) & \text{is the probability of outputs 1 and 1 respectively} \\ tr\big((E_x \otimes (\mathrm{Id} - F_y))\rho\big) & \text{is the probability of outputs 1 and -1 respectively} \\ tr\big(((\mathrm{Id} - E_x) \otimes F_y))\rho\big) & \text{is the probability of outputs -1 and 1 respectively} \\ tr\big(((\mathrm{Id} - E_x) \otimes (\mathrm{Id} - F_y))\rho\big) & \text{is the probability of outputs -1 and -1 respectively.} \end{cases}$$

[1]We assume the same number of measurements N for each of them to simply notation, but the case where $x = 1, \cdots, \mathrm{X}$, and $y = 1, \cdots, \mathrm{Y}$, is completely analogous.

[2]We assume that Alice's and Bob's systems are described by the same Hilbert space $\mathcal{H}$ just for simplicity.

Then,

$$\begin{aligned}\gamma_{x,y} = \mathbb{E}[A_x B_y] =& \big[P(1,1|x,y) + P(-1,-1|x,y)\big] - \big[P(-1,1|x,y) + P(1,-1|x,y)\big]\\ =& tr\Big(\big(E_x \otimes F_y + (\mathrm{Id} - E_x) \otimes (\mathrm{Id} - F_y) - E_x \otimes (\mathrm{Id} - F_y) - (\mathrm{Id} - E_x) \otimes F_y\big)\rho\Big)\\ =& tr\Big(\big((\mathrm{Id} - 2E_x) \otimes (\mathrm{Id} - 2F_y)\big)\rho\Big).\end{aligned}$$

Note that if we denote $A_x = \mathrm{Id} - 2E_x$, this is a self-adjoint operators acting on $\mathcal{H}$ verifying $\|A_x\| \leq 1$ for every x. Here, for a given linear map $T : \mathcal{H} \to \mathcal{H}$ we denote

$$\|T\| = \sup_{\|h\|=1} \|T(h)\|.$$

Reciprocally, every self-adjoint operator $\|A_x\| \leq 1$ can be written as $\mathrm{Id} - 2E_x$, where E_x is a positive operator smaller than the identity. We can reason in a similar way for $B_y = \mathrm{Id} - 2F_y$, for every y. We say that $\gamma := (\gamma_{x,y})_{x,y=1}^N$ is a *quantum correlation matrix* if there exit self-adjoint operators $A_1, \cdots, A_N, B_1, \cdots, B_N$ acting on a Hilbert space $\mathcal{H}$[3] with $\text{máx}_{x,y}\{\|A_x\|, \|B_y\|\} \leq 1$ and a density operator ρ acting on $\mathcal{H} \otimes \mathcal{H}$ such that

$$\gamma_{x,y} = tr(A_x \otimes B_y \rho), \quad \text{for every} \quad x, y = 1, \cdots, N. \tag{2.3}$$

We denote by $\mathcal{Q}_N$ the set of quantum correlation matrices of order N.

It is well known that every density operator ρ acting on $\mathcal{H}$ can be purified. That is, there exists a unit vector $|\psi\rangle$ in $\mathcal{H} \otimes \mathcal{H}$ (whose singular values are precisely the square root of the singular values of ρ) such that

$$\rho = (\mathrm{Id}_{\mathcal{H}} \otimes tr_{\mathcal{H}})(|\psi\rangle\langle\psi|).$$

It can be easily deduced from this fact that we can restrict to pure states $\rho = |\psi\rangle\langle\psi|$ in the definition of a quantum correlation matrix (2.3). Note, however, that in this process we must increase the dimension of the corresponding Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

It is easy to see that $\mathcal{L}_N \subseteq \mathcal{Q}_N$. Indeed, to see this inclusion let us consider a general element $\gamma \in \mathcal{L}_N$. Note that for a fixed N we can always assume that the integral in Equation (2.1) is a finite sum since, according to (2.2), $\mathcal{L}_N$ is a convex hull of a finite number of points. Let us assume that our probability space is of size K. Then,

$$\gamma_{x,y} = \sum_{k=1}^{K} p(k) A_x(k) B_y(k),$$

where $A_x(k)$ and $B_y(k)$ are as explained before. Then, considering the $K \times K$ matrices

$$A_x = \begin{pmatrix} A_x(1) & 0 & \cdots & 0 \\ 0 & A_x(2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_x(K) \end{pmatrix} \quad \text{and} \quad B_y = \begin{pmatrix} B_y(1) & 0 & \cdots & 0 \\ 0 & B_y(2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_y(K) \end{pmatrix}$$

and the K-dimensional state $\rho = \sum_{k=1}^{K} p(k)|kk\rangle\langle kk|$, it is trivial to check that

$$tr(A_x \otimes B_y \rho) = \sum_{k=1}^{K} p(k) A_x(k) B_y(k) = \gamma_{x,y} \tag{2.4}$$

for every x, y. Since A_x and B_y are clearly self-adjoint matrices with norm ≤ 1, the inclusion $\mathcal{L}_N \subseteq \mathcal{Q}_N$ is proved.

As it is the case of $\mathcal{L}_N$, the set $\mathcal{Q}_N$ is also convex (although $\mathcal{Q}_N$ is not a polytope anymore since it can be seen that it has infinitely many extreme points). In order to show the convexity

[3] In this definition we allow for infinite dimensional Hilbert spaces $\mathcal{H}$. However, a direct consequence of Theorem 2.2 below is that we can always assume that $\mathcal{H}$ has finite dimension.

of this set, let us consider two elements γ, γ' in $\mathcal{Q}_N$ and $\lambda \in [0,1]$. Then, $\gamma_{x,y} = tr(A_x \otimes B_y \rho)$ and $\gamma'_{x,y} = tr(A'_x \otimes B'_y \rho')$ for every $x, y = 1, \cdots, N$, where A_x, A'_x, B_y, B'_y, ρ and ρ' are as in (2.3) and we must show that $\lambda\gamma + (1-\lambda)\gamma' \in \mathcal{Q}_N$. To this end, we define the elements

$$\tilde{A}_x = \begin{pmatrix} A_x & 0 \\ 0 & A'_x \end{pmatrix}, \quad \tilde{B}_y = \begin{pmatrix} B_y & 0 \\ 0 & B'_y \end{pmatrix} \text{ and } \tilde{\rho} = \begin{pmatrix} \lambda\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\lambda)\rho \end{pmatrix}.$$

Then, $\tilde{A}_i$, $\tilde{B}_j$ are self-adjoint operators in M_{2N} of norm lower than or equal to one, $\tilde{\rho}$ is a density matrix in M_{4N^2} and one can easily check that

$$tr(\tilde{A}_i \otimes \tilde{B}_j \tilde{\rho}) = \lambda tr(A_x \otimes B_y \rho) + (1-\lambda) tr(A'_x \otimes B'_y \rho') \text{ for every } x, y = 1, \cdots N,$$

which finishes the proof.

In the previous proof we needed to increase the Hilbert space dimension of the corresponding quantum correlation matrices. It is interesting to note that if we fix a given dimension d and denote by $\mathcal{Q}_N^d$ the set of quantum correlation matrices of orden N which can be written by using Hilbert spaces $\mathcal{H}$ of dimension lower than or equal to d, then the set $\mathcal{Q}_N^d$ is not convex in general.

Since $\mathcal{L}_N$ is a polytope, it is described by its facets. The inequalities which describe these facets are usually called *(correlation) Bell inequalities*. Note that one of these inequalities will be of the form

$$\sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} \leq C \text{ for every } \gamma := (\gamma_{x,y})_{x,y=1}^{N} \in \mathcal{L}_N, \tag{2.5}$$

where $M = (M_{x,y})_{x,y=1}^N$ are the (real) coefficients of the corresponding inequality and C is the independent term. Actually, we have already studied one of these inequalities. Indeed, in the case $N = 2$ we have defined the CHSH-inequality in Section 1 as the one given by

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } C = 2.$$

It can be seen that in the case $N = 2$ this is the only Bell inequality up to certain symmetries.

As we showed before there exist certain quantum correlation matrices $\gamma \in \mathcal{Q}_N$ for which

$$\sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} = 2\sqrt{2}.$$

In this case, we say that the correlation γ violates the corresponding correlation Bell inequality or that we have a *Bell inequality violation*. By convexity, this is equivalent to say that we have a proper content

$$\mathcal{L}_N \subsetneq \mathcal{Q}_N.$$

For every matrix $M = (M_{x,y})_{x,y=1}^N$ of real numbers we can associate an inequality

$$\left| \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} \right| \leq \omega(M), \tag{2.6}$$

where

$$\omega(M) := \sup\Big\{\Big|\sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y}\Big| : (\gamma_{x,y})_{x,y=1}^{N} \in \mathcal{L}_N\Big\}$$
$$= \sup\Big\{\Big|\sum_{x,y=1}^{N} M_{x,y}t_x s_y\Big| : t_x = \pm 1, s_y = \pm 1, \text{ for every } x, y\Big\}.$$

Here, the last equality is straightforward from (2.2).

To be more precise, every matrix $M = (M_{x,y})_{x,y=1}^{N}$ defines a functional M acting on the set of correlation matrices by means of the dual action:

$$\langle M, \gamma\rangle = \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y}.$$

Hence, from now on we will call *correlation Bell functional* to any matrix $M = (M_{x,y})_{x,y=1}^{N}$ of real numbers [4] and the value $\omega(M)$ will be called *classical value of* M.

On the other hand, we will define the *quantum value of* M by

$$\omega^*(M) := \sup\left\{\left|\sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y}\right| : (\gamma_{x,y})_{x,y=1}^{N} \in \mathcal{Q}_N\right\}.$$

The value

$$LV(M) := \frac{\omega^*(M)}{\omega(M)}$$

is usually called *the largest violation of* M.

The previously proved inclusion $\mathcal{L}_N \subseteq \mathcal{Q}_N$ means that $\omega^*(M) \geq \omega(M)$ or, equivalently, $LV(M) \geq 1$ for every correlation Bell functional M. Then, M gives rise to a Bell violation whenever $LV(M) > 1$. As a particular example, we have seen that the CHSH-inequality M_{CHSH} verifies

$$LV(M_{CHSH}) \geq \sqrt{2}.$$

In fact, it is not very difficult to see that $LV(M_{CHSH}) = \sqrt{2}$. This is a direct consequence of the so called *Tsirelson's bound*, which shows that

$$\omega^*(M_{CHSH}) \leq 2\sqrt{2}.$$

Here, we present the original proof of Tsirelson [**22**].

PROPOSITION 2.1. *Let A_1, A_2, B_1, B_2 be self-adjoint operators of norm lower than or equal to 1 such that $[A_x, B_y] = 0$ for every $x, y = 1, 2$. Then,*

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \leq 2\sqrt{2}\,\mathrm{Id}\,.$$

[4] The corresponding inequality defined by this functional (2.5) is not properly a correlation Bell inequality since it does not necessarily describe a facet of $\mathcal{L}_N$. However, for the purpose of our study this is completely irrelevant.

Demostración. Let A_1, A_2, B_1, B_2 be as in the statement. Then,

$$\begin{aligned} A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 &= \frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) \\ &- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_1 - B_1) + A_2 - B_2\Big)^2 \\ &- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_1 - B_2) - A_2 - B_1\Big)^2 \\ &- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_2 - B_1) + A_1 + B_2\Big)^2 \\ &- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_2 + B_2) - A_1 - B_1\Big)^2 \\ &\leq \frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) \\ &\leq 2\sqrt{2}\,\mathrm{Id}\,. \end{aligned}$$

□

Given self-adjoint operators A_1, A_2, B_1, B_2 of norm lower than or equal to 1, by applying the previous proposition to the operators $A_1 \otimes \mathrm{Id}$, $A_2 \otimes \mathrm{Id}$, $\mathrm{Id} \otimes B_1$, $\mathrm{Id} \otimes B_2$ and for every state ρ we have

$$\begin{aligned} &|tr(A_1 \otimes B_1\rho) + tr(A_1 \otimes B_2\rho) + tr(A_2 \otimes B_1\rho) - tr(A_2 \otimes B_2\rho)| \\ &\leq \|A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2\| \leq 2\sqrt{2}. \end{aligned}$$

This shows Tsireslon's bound.

Surprisingly, the value $LV(M_{CHSH}) = \sqrt{2}$ is not far from being optimal, even if we consider matrices of order N as large as we want. This is a consequence of a deep theorem due to Grothendieck in the context of functional analysis. Before explaining this point, we need to see that the value $\omega^*(M)$ defined above can be expressed in a simpler way.

2.1. Tsirelson's theorem. Tsirelson's theorem tells us that, in the same way as the classical value of a correlation Bell inequality $\omega(M)$ can be written as a "combinatorial quantity", the quantum value $\omega^*(M)$ can be understood as a "geometrical quantity".

Theorem 2.2 (Tsirelson). *Let $\gamma = (\gamma_{x,y})_{x,y=1}^N$ be a matrix with real entries. Then, the following statements are equivalent:*

1. $\gamma = (\gamma_{x,y})_{x,y=1}^N \in \mathcal{Q}_N$.
2. *There exist norm one elements $u_1, \cdots, u_N, v_1, \cdots, v_N$ in a real Hilbert $\mathcal{K}_{\mathbb{R}}$ space such that*

$$\gamma_{x,y} = \langle u_x, v_y\rangle \ \ \textit{for every} \ \ x, y = 1, \cdots N.$$

In particular,

$$\omega^*(M) := \sup\left\{\left|\sum_{x,y=1}^N M_{x,y}\langle u_x, v_y\rangle\right|\right\},$$

where the sup *is taken over elements $u_1, \cdots, u_N, v_1, \cdots, v_N$ in the unit sphere of a real Hilbert space $\mathcal{K}_{\mathbb{R}}$.*

Note that, the dimension of the Hilbert space $\mathcal{K}_{\mathbb{R}}$ can always be assumed lower than or equal to $2N$ by just considering the span of the vectors $u_1, \cdots, u_N, v_1, \cdots, v_N$. In fact, this dimension can be taken lower than or equal to $N + 1$ by considering the projection of the vectors v's to the span of the normalized vectors $u_1, \cdots, u_N$ and then adding an extra dimension to make the new vectors $\tilde{v}$'s unit.

In order to prove Theorem 2.2 we need to introduce the *Canonical Anti-commutation Relations* (CAR)-algebra: Given $N \geq 2$, we consider a set of operators $X_1, \cdots, X_N$, such that they verify the following properties:

1. $X_i^* = X_i$ for every $i = 1, \cdots, N$.
2. $X_iX_j + X_jX_i = 2\delta_{i,j}\,\mathrm{Id}$ for every $i, j = 1, \cdots, N$..

The proof of the existence of such operators is completely constructive. Indeed, we can realize them as elements of $\bigotimes_{[\frac{N}{2}]} M_2 = M_{2^{[\frac{N}{2}]}}$ by using tensor products of Pauli matrices. Here, for every positive real number r, $[r]$ denotes the least natural number z such that $z \leq r$.

Let us first assume that $N = 2k$ for some k. Then, we define the following operators in M_N:

$$\begin{cases} X_1 = \sigma_x \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id}, & X_2 = \sigma_y \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id} \\ X_3 = \sigma_z \otimes \sigma_x \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id}, & X_4 = \sigma_z \otimes \sigma_y \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id} \\ \qquad\vdots & \qquad\vdots \\ X_{2k-3} = \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_x \otimes \mathrm{Id}, & X_{2k-2} = \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_y \otimes \mathrm{Id} \\ X_{2k-1} = \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_x, & X_{2k} = \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_y. \end{cases}$$

In the case $N = 2k + 1$, we add the element

$$X_{2k+1} = \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_z.$$

Proving that the operators $X_1, \cdots, X_N$ verify the CAR-relations is straightforward. We leave it as an exercise for the reader.

PROOF OF THEOREM 2.2. In order to prove the implication 1. ⇒ 2., recall that we can assume, without loss of generality, that the state ρ is pure ($\rho = |\varphi\rangle\langle\varphi|$) so that

$$\gamma_{x,y} = \langle\varphi|A_x \otimes B_y|\varphi\rangle \text{ for every } x, y. \tag{2.7}$$

Let us then define

$$|w_x\rangle = (A_x \otimes \mathrm{Id}_H)|\varphi\rangle \in H \otimes H \text{ and } |z_y\rangle = (\mathrm{Id}_H \otimes B_y)|\varphi\rangle \in H \otimes H$$

for every $x, y = 1, \cdots, N$.

It is clear that these vectors are in the unit ball of $H \otimes H$, $\mathrm{Ball}(H \otimes H)$. However, they could be complex vectors. The key point here is that we know that $\gamma_{x,y} = \langle w_x|z_y\rangle \in \mathbb{R}$ for all x, y. Indeed, this is due to the fact that A_x and B_y are self-adjoint operators in (2.7).

Hence, if we define

$$|\tilde{w}_x\rangle = \mathrm{Re}(|w_x\rangle) \oplus \mathrm{Im}(|w_x\rangle) \text{ and } |\tilde{z}_y\rangle = \mathrm{Re}(|z_y\rangle) \oplus \mathrm{Im}(|z_y\rangle),$$

we obtain new real vectors with norm lower than or equal to one and such that

$$\langle\tilde{w}_x|\tilde{z}_y\rangle = \mathrm{Re}\left(\langle w_x|z_y\rangle\right) = \gamma_{x,y} \text{ for every } x, y.$$

Finally, note that we can modify these vectors so that they have norm exactly one, by replacing our Hilbert space from H to $H \oplus \mathbb{R} \oplus \mathbb{R}$. Indeed, let us just define

$$u_x = \tilde{w}_x \oplus \sqrt{1 - \|\tilde{w}_x\|^2} \oplus 0 \text{ and } v_y = \tilde{z}_y \oplus 0 \oplus \sqrt{1 - \|z_y\|^2}$$

for every $x, y = 1, \cdots, N$.

To show implication 2. ⇒ 1., let M be the dimension of the real Hilbert space $\mathcal{K}_{\mathbb{R}}$. Without lost of generality we can assume that $\mathcal{K}_{\mathbb{R}} = (\mathbb{R}^M, \langle\cdot,\cdot\rangle)$. Recall that $u_1, \cdots, u_N, v_1, \cdots, v_N \in \mathcal{K}_{\mathbb{R}}$ are unit vectors. We have seen that we can realize the Clifford operators $X_1, \cdots, X_M$ (of order M) as elements in M_{2^n} with $n = [\frac{M}{2}]$. Let us consider the linear map

$$J : \mathbb{R}^M \to CL_M = \mathrm{span}\{X_1, \cdots, X_M\} \subset M_{2^n},$$

defined by

$$J\left(\sum_{k=1}^{M}\alpha_k e_k\right)=\sum_{k=1}^{M}\alpha_k X_k \quad \text{for every} \quad \sum_{k=1}^{M}\alpha_k e_k\in\mathbb{R}^M.$$

It can be deduced from the CAR-relations that $J:\mathbb{R}^M\to M_{2^n}$ defines a (not surjective) isometry, when we consider in M_{2^n} the standard operator norm. Indeed, in order to show this we write, for a given $x=\sum_{k=1}^M\alpha_k e_k\in\mathbb{R}^M$,

$$\|J(x)\|_{M_{2^n}}=\|J(x)J(x)^*\|_{M_{2^n}}^{\frac{1}{2}}=\left\|\left(\sum_{k=1}^{M}\alpha_k X_k\right)\left(\sum_{k=1}^{M}\alpha_k X_k\right)^*\right\|_{M_{2^n}}^{\frac{1}{2}}=\left\|\sum_{k,k'=1}^{M}\alpha_k\bar{\alpha}_{k'}X_kX_{k'}^*\right\|_{M_{2^n}}^{\frac{1}{2}}$$

$$=\left\|\sum_{k=1}^{M}|\alpha_k|^2\,\mathrm{Id}_{M_{2^n}}+\sum_{k\neq k'}^{M}\alpha_k\alpha_{k'}X_kX_{k'}\right\|_{M_{2^n}}^{\frac{1}{2}}=\left\|\sum_{k=1}^{M}|\alpha_k|^2\,\mathrm{Id}_{M_{2^n}}\right\|_{M_{2^n}}^{\frac{1}{2}}=\left(\sum_{k=1}^{M}|\alpha_k|^2\right)^{\frac{1}{2}}.$$

Here, we have used that

$$\sum_{k\neq k'}^{M}\alpha_k\alpha_{k'}X_kX_{k'}=0,$$

which can be easily deduced from the CAR-relations.

The CAR-relations also imply that

$$\frac{1}{2^n}tr\left(J(x)J(y)\right)=\langle x,y\rangle \text{ for every } \ x,y\in\mathbb{R}^M,$$

since for $x=\sum_{k=1}^M\alpha_k e_k$ and $y=\sum_{k=1}^M\beta_k e_k$ in $\mathbb{R}^M$ we have

$$\frac{1}{2^n}tr\left(J(x)J(y)\right)=\frac{1}{2^n}tr\left(\sum_{k,k'=1}^{M}\alpha_k\beta_{k'}X_kX_{k'}\right)=\frac{1}{2^n}\sum_{k,k'=1}^{M}\alpha_k\beta_{k'}tr\left(X_kX_{k'}\right)=\sum_{k=1}^{M}\alpha_k\beta_k.$$

Here, we have used that $tr(\mathrm{Id}_{M_{2^n}})=2^n$ and for every $k\neq k'$, we have $tr(X_kX_{k'})=0$.

If we consider now the state $|\psi\rangle=\frac{1}{2^{\frac{N}{2}}}\sum_{i=1}^{2^n}|ii\rangle\in\mathbb{C}^{2^n}\otimes_2\mathbb{C}^{2^n}$, it is easy to check that for every $A,B\in M_{2^n}$ we have

$$\frac{1}{2^n}tr(AB^{tr})=tr(A\otimes B|\psi\rangle\langle\psi|)=\langle\psi|A\otimes B|\psi\rangle.$$

Therefore, if we define the operators $A_x=J(u_x)\in M_{2^n}, B_y=\overline{J(v_y)}\in M_{2^n}$ for every x,y, where the bar denotes the conjugate operator, we obtain a family of self-adjoint operators with norm one, and such that

$$\langle\psi|A_x\otimes B_y|\psi\rangle=\frac{1}{2^n}tr(A_xB_y^{tr})=\frac{1}{2^n}tr(J(u_x)J(v_y)^*)=\langle x,y\rangle=\gamma_{x,y}$$

for every $x,y=1,\cdots,N$, where in the last equality we have used that the operators $J(v_y)$ are self-adjoint. This concludes the proof. □

Remark 2.3 (Maximally entangled states suffice in correaltions Bell inequalities). *An important consequence of the proof of Theorem 2.2 is that, for a fixed N, every correlation matrix $\gamma\in\mathcal{Q}_N$ can be written by using the maximally entangled state in dimension $d=2^{[\frac{N}{2}]}$:*

$$|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle\in\mathbb{C}^d\otimes\mathbb{C}^d.$$

One can define the *(commuting) quantum correlation matrices* as those $\gamma := (\gamma_{x,y})_{x,y=1}^N$ such that there exist self-adjoint operators $A_1, \cdots, A_N, B_1, \cdots, B_N$ acting on a Hilbert space $\mathcal{H}$ with $\max_{x,y=1,\cdots,N}\{\|A_x\|, \|B_y\|\} \leq 1$ verifying $[A_x, B_y] = 0$ for every $x, y = 1, \cdots, N$ and a density operator ρ acting on $\mathcal{H}$ such that

$$\gamma_{x,y} = tr(A_x B_y \rho), \quad \text{for every} \quad x, y = 1, \cdots, N. \tag{2.8}$$

One can easily deduce from the proof of Theorem 2.2 that this definition is equivalent to the previous one. Indeed, if we add the item

1'. $\gamma = (\gamma_{x,y})_{x,y=1}^N \in \mathcal{Q}_N^c$

in the statement of Theorem 2.2, where $\mathcal{Q}_N^c$ denotes the set of correlation matrices which can be written as in (2.8) it is trivial to check that [1] implies [1'] and a modification of the proof [1] $\rightarrow$ [2] allows to show [1′] $\rightarrow$ [2].

2.2. Grothendieck's theorem. Theorem 2.2 allows us to bring Grothendieck's *fundamental theorem in the metric theory of tensor products* **[7]** to the context of Bell inequalities.

Theorem 2.4 (Grothendieck's inequality). *There exits a positive universal constant $K_G^{\mathbb{R}}$ such that for every natural number N and for every matrix of real coefficients $(M_{x,y})_{x,y=1}^N$ the following inequality holds:*

$$\sup\Big\{\Big|\sum_{x,y=1}^N M_{x,y}\langle u_x, v_y\rangle\Big| : \|u_x\|, \|v_y\| = 1 \ \forall x, y\Big\} \leq K_G^{\mathbb{R}} \cdot \sup\Big\{\Big|\sum_{x,y=1}^N M_{x,y} t_x s_y\Big| : t_x, s_y = \pm 1 \ \forall x, y\Big\}.$$

Here, the first supremum is taken over families of vectors $u_1, \cdots, u_N, v_1, \cdots, v_N$ in an arbitrary real Hilbert space.

The constant $K_G^{\mathbb{R}}$, known as *the (real) Grothendieck's constant*, verifies

$$1{,}67696... \leq K_G^{\mathbb{R}} < \frac{\pi}{2\log((1+\sqrt{2})} = 1{,}7822139781...$$

Howevere, the exact value of this constant is still an open question.

Remark 2.5. *There exists a complex version of Grothendieck's inequality for complex matrices. Then, the optimization in the left hand side is over unit vectors in a complex Hilbert spaces and the optimization in the right hand side is over elements $(t_x)_{x=1}^N, (s_y)_{y=1}^N$ such that $|t_x| \leq 1$ and $|s_y| \leq 1$ for every $x, y = 1, \cdots, N$. The corresponding constant is denoted by $K_G^{\mathbb{C}}$.*
Although the exact value of both constants real and complex are unknown, it is known that

$$K_G^{\mathbb{C}} < K_G^{\mathbb{R}}.$$

According to our previous description of the values $\omega(M)$ and $\omega^*(M)$, Theorem 2.4 can be reformulated as follows.

Theorem 2.6. *There exists a positive universal constant $K_G^{\mathbb{R}}$ such that for every natural number N and for every correlation Bell functional $(M_{x,y})_{x,y=1}^N$ the following inequality holds:*

$$\omega^*(M) \leq K_G^{\mathbb{R}} \cdot \omega(M) \quad \textit{or, equivalently,} \ LV(M) \leq K_G^{\mathbb{R}}.$$

Hence, the basic example defined by the CHSH-inequality, which provides a Bell violation $LV(M_{CHSH}) \geq \sqrt{2}$, is "close to be optimal".

3. Banach space point of view

3.1. Basic definitions. We will start this section by fixing some notation. We recall that a Banach space X is a normed space which is complete under that norm (that is, every Cauchy sequence converges to an element of the space). In these notes, we will mostly consider finite dimensional normed spaces and they are automatically complete. Here, will use notation X, Y, Z, $\cdots$ to denote generic Banach spaces. We will use notation $\|\cdot\|_X$ for the norm on X and $\mathrm{Ball}(X)$ for the closed unit ball of the normed space X. Given a vector space $\mathcal{V}$ we will use $\{e_x\}$ to denote an arbitrary fixed orthonormal basis of $\mathcal{V}$, identified as a "canonical basis" for $\mathcal{V}$.

Notation ℓ_p^n will be used to denote the n-dimensional complex ℓ_p-space, and $\ell_p^n(\mathbb{R})$ will be used in the real case. Unless specified otherwise, the space $\mathbb{C}^n$ will always be endowed with the Hilbertian norm and identified with ℓ_2^n. The inner product between two elements $x, y \in \ell_2^n$ is denoted by $\langle x, y\rangle$.

Given Banach spaces X, Y and Z, $\mathrm{L}(X, Y)$ will be the set of linear maps from X to Y. We will also write $\mathrm{L}(X)$ for $\mathrm{L}(X, X)$, and $\mathrm{Id}_X \in \mathrm{L}(X)$ for the identity map on X. $\mathrm{B}(X, Y; Z)$ will be the set of bilinear maps from $X \times Y$ to Z. If $Z = \mathbb{C}$ we also write $\mathrm{B}(X, Y)$.

Given a linear map $T : X \to Y$ between Banach spaces, T is *bounded* if its norm is finite:

$$\|T\| := \sup\{\|T(x)\|_Y : \|x\|_X \leq 1\} < \infty.$$

We will denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear maps from X to Y, and write $\mathcal{B}(X)$ for $\mathcal{L}(X, X)$. Similarly, $B \in \mathrm{B}(X, Y; Z)$ is said bounded if

$$\|B\| := \sup\{\|B(x, y)\|_Z : \|x\|_X, \|y\|_Y \leq 1\} < \infty,$$

and we will write $\mathcal{B}(X, Y; Z)$ for the set of such maps. If X is a Banach space, we write $X^\star = \mathrm{L}(X, \mathbb{K})$ and $X^* = \mathcal{L}(X, \mathbb{K})$ for their algebraic and topological dual respectively, where $\mathbb{K}$ will be always $\mathbb{R}$ or $\mathbb{C}$.

The natural correspondence between liar maps, bilinear forms and tensor products will play an important role in these notes. It is obtained through the identifications

$$L(X, Y^\star) = \mathrm{B}(X, Y) = (X \otimes Y)^\star. \tag{3.1}$$

Here the first equality is obtained by identifying a linear map $T : X \to Y^\star$, with the bilinear form $B_T : X \times Y \to \mathbb{K}$ defined by $B_T(x, y) = \langle T(x), y\rangle$ for every $(x, y) \in X \times Y$ and, reciprocally, any bilinear form $B : X \times Y \to \mathbb{K}$ with the linear map $T_B : X \to Y^\star$ defined as $\langle T_B(x), y\rangle = B(x, y)$. For the second identification, given a bilinear form $B : X \times Y \to \mathbb{C}$ we associate the linear form $S_B : X \otimes Y \to \mathbb{K}$ defined by $S_B(x \otimes y) = B(x, y)$ for every x, y and, reciprocally, for every linear map $S : X \otimes Y \to \mathbb{K}$ we associate the bilinear form $B_S : X \times Y \to \mathbb{K}$ defined as $B_S(x, y) = S(x \otimes y)$ for every $(x, y) \in X \times Y$.

In the particular case of $\mathbb{R}^N \otimes \mathbb{R}^N$, we can identify this space with the space of $N \times N$ matrices M_N by the canonical identification

$$\sum_{i,j=1}^N \alpha_{i,j} e_i \otimes e_j \to (\alpha_{i,j})_{i,j=1}^N,$$

and the identifications (3.1) can be equivalently explained by using matrices.

Another interesting identification is

$$X^\star \otimes Y = F(X, Y) \tag{3.2}$$

where $F(X, Y)$ denotes the linear maps from X to Y with finite rank. To understand this identification, let us consider an element $Q = \sum_{s,t=1}^N x_s^\star \otimes y_t \in X^\star \otimes Y$. Then, we associate the linear map $T_Q : X \to Y$ defined by $T_Q(x) = \sum_{s,t=1}^N \langle x_s^\star, x\rangle y_t$ for every $x \in X$. Reciprocally, if $T \in F(X, Y)$ has finite rank, then it can be written (using for instance the singular value decompostion) as $T = \sum_s \langle x_s^\star, \cdot\rangle y_s$ for certain elements $x_s^\star \in X^\star$, $y_s \in Y$. Then, we associate the tensor $Q_T = \sum_s x_s^\star \otimes y_s \in X^\star \otimes Y$.

Hence, if X or Y is a finite dimensional spaces (so that $F(X,Y) = L(X,Y)$) we naturally obtain the identification

$$(X \otimes Y)^\star = L(X, Y^*) = X^\star \otimes Y^\star. \tag{3.3}$$

3.2. Three tensor norms. The identifications (3.1) can be made isometric. Given two Banach spaces X, Y, define the π-*norm* of $z \in X \otimes Y$ as

$$\|z\|_{X\otimes_\pi Y} = \text{ínf}\left\{\sum_{i=1}^n \|u_i\|\|v_i\| : z = \sum_{i=1}^n u_i \otimes v_i\right\}, \tag{3.4}$$

where the infimum runs over all representations $z = \sum_{i=1}^n u_i \otimes v_i$. We will denote by $X \otimes_\pi Y$ the completion of the space $X \otimes Y$ under the norm π.[5]

It is very simple to show that the π-norm on the tensor product $X \otimes Y$ is the Minkowski functional of the set

$$\Gamma = \text{conv}\left(\text{Ball}(X) \otimes \text{Ball}(Y)\right).$$

In other words, we can re-write the π-norm of z as

$$\|z\|_{X\otimes_\pi Y} = \text{ínf}\left\{\sum_{i=1}^m |\lambda_i| : z = \sum_{i=1}^m \lambda_i f_i \otimes g_i,\ \ f_i \in \text{Ball}(X), g_i \in \text{Ball}(Y)\right\}. \tag{3.5}$$

Indeed, it is clear that for every representation $z = \sum_{i=1}^n u_i \otimes v_i$, we can write $z = \sum_{i=1}^n \lambda_i \frac{u_i}{\|u_i\|} \otimes \frac{v_i}{\|v_i\|}$ with $\lambda_i = \|u_i\|\|v_i\|$ for every i, to conclude that the expression (3.5) is lower than or equal to (3.4). On the other hand, for every representation of the form $z = \sum_{i=1}^n \lambda_i f_i \otimes g_i$ we can denote $\tilde{f}_i = \lambda_i f_i$ for every i to obtain a representation $z = \sum_{i=1}^n \tilde{f}_i \otimes g_i$ which leads to lower bound (3.5) by (3.4).

With the expression (3.5) at hand it is very easy to show that the following identifications are isometric (details are left to the reader):

$$\mathcal{L}(X, Y^*) = \mathcal{B}(X, Y) = (X \otimes_\pi Y)^*. \tag{3.6}$$

In addition the π-norm verifies de so called *metric mapping property*: For all Banach spaces X, Y, Z, W and all linear maps $T : X \to Z$ and $S : Y \to W$, the following estimate holds:

$$\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\| \leq \|T\|\|S\|. \tag{3.7}$$

Here, the linear map $T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W$ is defined by canonically extending (to the completion space) the linear map $T \otimes S : X \otimes Y \to Z \otimes W$ defined by

$$(T \otimes S)\left(\sum_i u_i \otimes y_i\right) = \sum_i T(u_i) \otimes S(y_i).$$

Estimate (3.7) follows easily from the definition of the π-norm.

Given two Banach spaces X, Y, define the ϵ-*norm* of $z \in X \otimes Y$ as

$$\|z\|_{X\otimes_\epsilon Y} = \sup_{x^* \in \text{Ball}(X^*),\, y^* \in \text{Ball}(Y^*)} \left|\langle z, x^* \otimes y^*\rangle\right|. \tag{3.8}$$

[5]We note that this space is usually denoted as $X\hat{\otimes}_\pi Y$ but since these notes will mostly deal with finite dimensional spaces (where no completion is required), we prefer using this notation.

It is very easy to show that restricting the elements x^* and y^* to the set of extreme points of the corresponding unit balls $\mathrm{ext}(\mathrm{Ball}(X^*))$ and $\mathrm{ext}(\mathrm{Ball}(Y^*))$ respectively does not change the definition of the norm. Moreover, it is also straightforward to check that

$$\|z\|_{X\otimes_\epsilon Y} = \|T_z : X^* \to Y\| = \sup_{x^*\in\mathrm{Ball}(X^*)} \|T_z(x^*)\|_Y\,, \tag{3.9}$$

where for a given $z = \sum_i x_i \otimes y_i$ and a given $x^* \in \mathrm{Ball}(X^*)$ we denote $T_z(x^*) = \sum_i x^*(x_i) y_i \in Y$.

We will denote by $X \otimes_\epsilon Y$ the completion of the space $X \otimes Y$ under the ϵ-norm. It follows from (3.9) that the identification (3.2) becomes isometric in the form:

$$\mathcal{K}(X, Y) = X^* \otimes_\epsilon Y, \tag{3.10}$$

where $\mathcal{K}(X, Y)$ denotes the compact linear maps from X to Y endowed with the operator norm. In particular, if X or Y are finite dimensional spaces (so that $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$ isometrically) we naturally obtain the isometric identifications

$$(X \otimes_\pi Y)^* = X^* \otimes_\epsilon Y^* \quad\text{and}\quad (X \otimes_\epsilon Y)^* = X^* \otimes_\pi Y^* \tag{3.11}$$

It is again very easy to check that the ϵ-norm verifies the metric mapping property: For all Banach spaces X, Y, Z, W and all linear maps $T : X \to Z$ and $S : Y \to W$, the following estimate holds:

$$\|T \otimes S : X \otimes_\epsilon Y \to Z \otimes_\epsilon W\| \leq \|T\|\|S\|. \tag{3.12}$$

In addition, this norm verifies a very important property; namely, it is injective. This means that if $j_1 : X \to Z$ and $j_2 : Y \to W$ are linear isometries (resp. linear isomorphisms), then

$$j_1 \otimes j_2 : X \otimes_\epsilon Y \to Z \otimes_\epsilon W$$

is a linear isometry (resp. linear isomorphim).

In general a *tensor norm* α on the class BANACH of all Banach spaces[6] assigns to each pair (X, Y) of Banach spaces X and Y a norm $\|\cdot\|_{X\otimes_\alpha Y}$ on the algebraic tensor product $X \otimes Y$ so that $X \otimes_\alpha Y$ is a Banach space and such that the following two conditions are satisfied:

1. α is *reasonable*: $\epsilon \leq \alpha \leq \pi$.
2. α satisfies the metric mapping property.

In particular, π and ϵ are (the extreme) tensor norms.

Given a tensor norm α, we define its *dual tensor norm* α^* by defining, for every pair of finite dimensional Banach spaces X, Y,

$$\|u\|_{X\otimes_{\alpha^*} Y} = \sup\left\{|\langle u, v\rangle| : \|v\|_{X^*\otimes_\alpha Y^*} \leq 1\right\},$$

where the dual action is the one used in the algebraic identification $(X \otimes Y)^\star = X^\star \otimes Y^\star$.

Formally, we should define α^* on every pair of not necessarily finite dimensional Banach spaces. However, it suffices to define it on finite dimensional spaces and one can extend it to the class BANACH in a canonical way. Note that we only use the definition of α on finite dimensional Banach spaces to define α^*. In fact, the norms we are using: π-norm, ϵ-norm and the ones defined below, are *finitely generated* (as well as all the so called *usual tensor norms*); that is, they are determined by their restrictions to finite dimensional spaces (see [**4**, Section 12.4] for a formal definition).

Note that (3.11) expresses that the π-norm and the ϵ-norm are dual of each other.

Let us also mention that the π tensor norm and the ϵ tensor norm can be analogously defined on the tensor product of k Banach spaces

$$X_1 \otimes \cdots \otimes X_k.$$

[6] In general, a tensor norm is usually defined on the class NORM of normed spaces but it can also be defined on subclasses, such as BANACH, and this will be enough of our purposes.

It is easy to check that these norms are both associative and commutative. Moreover, all previously completely isometric identifications also hold in the general context.

We will need to consider another tensor norm which is very important for the purpose of these notes. Given two Banach spaces X, Y, define the γ_2*-tensor norm* of $z \in X \otimes Y$ as

$$\|z\|_{X\otimes_{\gamma_2} Y} = \text{ínf}\left\{\omega_2\left((u_x)_i; X\right)\omega_2\left((y_i)_i; Y\right) : z = \sum_{i=1}^n u_i \otimes v_i\right\}, \tag{3.13}$$

where for a given sequence $(z_i)_{i=1}^n \subset Z$, we denote

$$\omega_2\left((z_i)_i; Z\right) = \sup_{z^*\in \text{Ball}(Z^*)} \left(\sum_{i=1}^n |z^*(z_i)|^2\right)^{\frac{1}{2}}.$$

The reader can easily check that, given a sequence $(z_i)_{i=1}^n \subset Z$, the quantity $\omega_2\left((z_i)_i; Z\right)$ coincides with the norm $\|T\|$ of the linear map $T : \ell_2^n \to Z$ defined by $T(e_i) = z_i$ for every $i = 1, \cdots, n$.

One can check that γ_2 defines indeed a tensor norm. Its dual norm will be denoted by γ_2^*.

3.3. Connections with quantum nonlocality. Tensor norms perfectly fit in the study of quantum nonlocality. To see this, let us recall the notation $\ell_1^n(\mathbb{R})$, $\ell_\infty^n(\mathbb{R})$ and $\ell_2^n(\mathbb{R})$ for the corresponding real spaces.

We start by relating the classical value of a correlation Bell functional with the ϵ-norm:

Proposition 3.1 (Classical value of a Bell inequality). *Let $M = (M_{x,y})_{x,y=1}^N$ be a matrix with real entries and let us denote by $M = \sum_{x,y=1}^N M_{x,y} e_x \otimes e_y \in \mathbb{R}^N \otimes \mathbb{R}^N$ the corresponding tensor. Then,*

$$\omega(M) = \|M\|_{\ell_1^N(\mathbb{R})\otimes_\epsilon \ell_1^N(\mathbb{R})}.$$

Equivalently, we have

$$\mathcal{L}_N = B_{\ell_\infty^N(\mathbb{R})\otimes_\pi \ell_\infty^N(\mathbb{R})}.$$

Demostración. Given $(M_{x,y})_{x,y=1}^N \in \mathbb{R}^{N^2}$ the following equalities hold:

$$\omega(M) = \sup\left\{|\langle M, \gamma\rangle| : \gamma \in \mathcal{L}_N\right\} = \sup\left\{\left|\sum_{x,y=1}^N M_{x,y} t_x s_y\right| : t_x = \pm 1, s_y = \pm 1, \quad \text{for every} \quad i,j\right\}$$

$$= \sup\left\{\left|\sum_{x,y=1}^N M_{x,y}\langle e_x, \hat{t}\rangle\langle e_y, \hat{s}\rangle\right| : \hat{t}, \hat{s} \in \text{ext}(\text{Ball}(\ell_\infty^N(\mathbb{R})))\right\} = \left\|\sum_{x,y=1}^N M_{x,y} e_x \otimes e_y\right\|_{\ell_1^N(\mathbb{R})\otimes_\epsilon \ell_1^N(\mathbb{R})}.$$

The second statement of the proposition is indeed equivalent to the first one by duality. In fact, according to (2.2) and (3.5) a correlation matrix γ verifies

$$\gamma \in \mathcal{L}_N \Leftrightarrow \|\gamma\|_{\ell_\infty^N(\mathbb{R})\otimes_\pi \ell_\infty^N(\mathbb{R})} \leq 1.$$

□

In order to study the quantum value of a correlation Bell inequality $\omega(M)$, we need to analyze the γ_2-norm on the space $\ell_\infty^N(\mathbb{R}) \otimes \ell_\infty^N(\mathbb{R})$. The following lemma will be proved for completeness, but the reader could skip it in a first reading of the notes and accept expression (3.14) as the definition of the γ_2-norm on $\ell_\infty^N(\mathbb{R}) \otimes \ell_\infty^N(\mathbb{R})$.

Lemma 3.2. *Given an element $\tau \in \mathbb{R}^N \otimes \mathbb{R}^N$, we have that*

$$\|\tau\|_{\ell_\infty^N(\mathbb{R})\otimes_{\gamma_2}\ell_\infty^N(\mathbb{R})} = \text{ínf}\left\{\max_{1\leq i,j\leq N} \|u_i\|_2\|v_j\|_2 : \ \tau_{i,j} = \langle u_i, v_j\rangle \quad \textit{for every} \quad i,j\right\}, \tag{3.14}$$

where u_i and v_j are unit vectors in a real Hilbert space for every $i,j = 1, \cdots, N$.

DEMOSTRACIÓN. Let us first assume that $\|\tau\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2}\ell_1^N(\mathbb{R})} = d$. By definition of this norm, for every $\epsilon > 0$ there exist sequences $(x_k)_{k=1}^r$ and $(y_k)_{k=1}^r$ in $\ell_\infty^N(\mathbb{R})$ such that $\tau = \sum_{k=1}^r x_k \otimes y_k$ and $\omega_2\left((x_k)_k; \ell_\infty^N(\mathbb{R})\right)\omega_2\left((y_k)_k; \ell_\infty^N(\mathbb{R})\right) \leq d + \epsilon$. Then,

$$\tau = \sum_{k=1}^r \left(\sum_{i=1}^N x_k(i)e_i\right) \otimes \left(\sum_{j=1}^N y_k(j)e_j\right) = \sum_{i,j=1}^N \left(\sum_{k=1}^r x_k(i)y_k(j)\right) e_i \otimes e_j.$$

Let us then define the real vectors $u_i = (x_k(i))_{k=1}^r$ and $v_j = (y_k(j))_{k=1}^r$ in $\mathbb{R}^r$ so that we have

$$\tau_{i,j} = \langle u_i, v_j\rangle \text{ for every } i, j = 1, \cdots, N.$$

We must show that $\max_{1\leq i,j\leq N} \|u_i\|_2\|v_j\|_2 \leq d + \epsilon$. To this end, fix i and write

$$\|u_i\|_2 = \left(\sum_{k=1}^r |u_i(k)|^2\right) = \left(\sum_{k=1}^r |\langle x_k, e_i\rangle|^2\right) \leq \sup_{e\in \mathrm{Ball}(\ell_1^N(\mathbb{R}))} \left(\sum_{k=1}^r |\langle x_k, e\rangle|^2\right) = \omega_2\left((x_k)_k; \ell_\infty^N(\mathbb{R})\right).$$

Here, we have used that $(\ell_\infty^N(\mathbb{R}))^* = \ell_1^N(\mathbb{R})$ and that $\mathrm{ext}(\mathrm{Ball}(\ell_1^N(\mathbb{R}))) = \{\pm e_i : i = 1, \cdots, N\}$.

Applying the same argument to upper bound $\|v_j\|_2$ by $\omega_2\left((y_k)_k; \ell_\infty^N(\mathbb{R})\right)$ we can conclude that

$$\max_{1\leq i,j\leq N} \|u_i\|_2\|v_j\|_2 \leq \omega_2\left((x_k)_k; \ell_\infty^N(\mathbb{R})\right)\omega_2\left((y_k)_k; \ell_\infty^N(\mathbb{R})\right) \leq d + \epsilon.$$

In order to prove the converse inequality, let us assume that $\tau_{i,j} = \langle u_i, v_j\rangle$ for every $i, j = 1, \cdots, N$, where u_i and v_j are vectors in $\mathbb{R}^r$ for a certain r verifying $\max_{1\leq i,j\leq N} \|u_i\|_2\|v_j\|_2 \leq d + \epsilon$.

Then, we can write

$$\tau = \sum_{i,j=1}^N \tau_{i,j} e_i \otimes e_j = \sum_{i,j=1}^N \left(\sum_{k=1}^r u_i(k)v_j(k)\right) e_i \otimes e_j = \sum_{k=1}^r \left(\sum_{i=1}^N u_i(k)e_i\right) \otimes \left(\sum_{j=1}^N v_j(k)e_j\right).$$

By defining $x_k = \sum_{i=1}^r u_i(k)e_i$ and $y_k = \sum_{j=1}^r v_j(k)e_j$ for every $k = 1, \cdots, r$, we trivially obtain

$$\tau = \sum_{k=1}^r x_k \otimes y_k.$$

In addition, we note that

$$\omega_2\left((x_k)_k; \ell_\infty^N(\mathbb{R})\right) = \sup_{e\in \mathrm{Ball}(\ell_1^N(\mathbb{R}))} \left(\sum_{k=1}^r |\langle x_k, e\rangle|^2\right) = \max_{1\leq i\leq N} \left(\sum_{k=1}^r |\langle x_k, e_i\rangle|^2\right)$$
$$= \max_{1\leq i\leq N} \left(\sum_{k=1}^r |u_i(k)|^2\right) = \max_{1\leq i\leq N} \|u_i\|_2.$$

Applying the same argument to upper bound $\omega_2\left((y_k)_k; \ell_\infty^N(\mathbb{R})\right)$ by $\max_{1\leq j\leq N} \|v_j\|_2$ we conclude that

$$\omega_2\left((x_k)_k; \ell_\infty^N(\mathbb{R})\right)\omega_2\left((y_k)_k; \ell_\infty^N(\mathbb{R})\right) \leq \max_{1\leq i,j\leq N} \|u_i\|_2\|v_j\|_2 \leq d + \epsilon.$$

This finishes the proof. □

Another way of writing (3.14), buy using a matrix formulation, is

$$\|\tau\|_{\ell_\infty^N(\mathbb{R})\otimes_{\gamma_2}\ell_\infty^N(\mathbb{R})} := \inf\left\{ \|X\|_{\ell_2^K(\mathbb{R})\to\ell_\infty^N(\mathbb{R})}\|Y\|_{\ell_1^N(\mathbb{R})\to\ell_2^K(\mathbb{R})} : \tau = XY \right\},$$

where, denoting by $R_i(X)$ the i^{th} row of a matrix X and by $C_j(Y)$ the j^{th} column of a matrix Y, we have

$$\|X\|_{\ell_2^K(\mathbb{R})\to\ell_\infty^N(\mathbb{R})} = \max_{1\leq i\leq N}\|R_i(X)\|_2 \text{ and } \|Y\|_{\ell_1^N(\mathbb{R})\to\ell_2^K(\mathbb{R})} = \max_{1\leq j\leq N}\|C_j(Y)\|_2.$$

With Lemma 3.2 at hand, it is very easy to connect the γ_2-norm with the set of quantum correlation matrices

PROPOSITION 3.3. *Let $M = (M_{x,y})_{x,y=1}^N$ be a matrix with real entries and let us denote by $M = \sum_{x,y=1}^N M_{x,y} e_x \otimes e_y \in \mathbb{R}^N \otimes \mathbb{R}^N$. Then,*

$$\omega^*(M) = \|M\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}.$$

Equivalently, we have

$$\mathcal{Q}_N = \mathrm{Ball}(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})).$$

DEMOSTRACIÓN. According to Theorem 2.2 and Lemma 3.2 we trivially conclude that

$$\mathcal{Q}_N \subseteq \mathrm{Ball}(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})).$$

Furthermore, according to Lemma 3.2, if $\|\gamma\|_{\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})} < 1$ we know that there exist vectors u_x, v_y in a certain space $\mathbb{R}^r$ such that

$$\gamma_{x,y} = \langle u_x, v_y \rangle \ \text{ for every } \ x, y = 1, \cdots, N,$$

verifying $\mathrm{máx}_{1 \leq x,y \leq N} \|u_x\| \|v_y\| < 1$. Without loose of generality we can assume that $\|u_x\| \leq 1$, $\|v_y\| \leq 1$ for every x, y. Indeed, this can be done by considering

$$u_x' = \frac{u_x}{\mathrm{máx}_x \|u_x\|} \quad \text{and} \quad v_y' = \mathrm{máx}_x \|u_x\| v_y$$

for every x, y.

Now, by increasing the dimension r (to $r+1$ is enough) we can obtain new vectors $\tilde{u}_x$ and $\tilde{v}_y$ so that $\|\tilde{u}_x\| = \|\tilde{v}_y\| = 1$ and $\gamma_{x,y} = \langle \tilde{u}_x, \tilde{v}_y \rangle$ for every x, y. Hence, we conclude that

$$\left\{ \gamma \in \mathbb{R}^N \otimes \mathbb{R}^N : \|\gamma\|_{\ell_\infty^N \otimes_{\gamma_2} \ell_\infty^N} < 1 \right\} \subseteq \mathcal{Q}_N.$$

Using that $\mathcal{Q}_N$ is a closed (so compact) set, which can be easily deduced from the continuity of the bilinear form

$$B : \ell_\infty^N(\ell_2^r(\mathbb{R})) \times \ell_\infty^N(\ell_2^r(\mathbb{R})) \to M_N,$$

defined as

$$B((u_x)_x, (v_y)_y) = (\langle u_x, v_y \rangle)_{x,y=1}^N,$$

we immediately conclude that

$$\mathcal{Q}_N = \mathrm{Ball}(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})),$$

which proves the second assertion of the statement.

As a trivial consequence of this equality we obtain the desired description for $\omega^*(M)$:

$$\begin{aligned} \omega^*(M) &= \sup \{ |\langle M, \gamma \rangle| : \gamma \in \mathcal{Q}_N \} = \sup \left\{ |\langle M, \gamma \rangle| : \gamma \in \mathrm{Ball}(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})) \right\} \\ &= \|M\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}. \end{aligned}$$

□

Finally, let us reformulate Grothedieck's inequality in the language of tensor norms:

THEOREM 3.4 (Grothendieck's inequality). *There exits a positive universal constant $K_G^{\mathbb{R}}$ such that for every natural number N and for every tensor $\tau \in \mathbb{R}^N \otimes \mathbb{R}^N$ the following inequality holds:*

$$\|\tau\|_{\ell_\infty^N(\mathbb{R}) \otimes_\pi \ell_\infty^N(\mathbb{R})} \leq K_G^{\mathbb{R}} \|\tau\|_{\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})}.$$

We see that Theorem 2.6 on the upper bound for the largest violation of correlation Bell functionals can be recover from Theorem 3.4 by invoking Propositions 3.1 and Proposition 3.3. From a geometrical point of view, Grothendieck's inequality can be understood as the second inclusion in:

$$\mathcal{L}_N \subset \mathcal{Q}_N \subset K_G^{\mathbb{R}} \mathcal{L}_N, \quad \text{for every} \quad N.$$

4. Computer science point of view: XOR games

A two-player one-round game $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$ is specified by finite sets $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}$, a probability distribution $\pi : \mathbf{X} \times \mathbf{Y} \to [0,1]$ and a payoff function $V : \mathbf{A} \times \mathbf{B} \times \mathbf{X} \times \mathbf{Y} \to [0,1]$.[7] These are games between two players, Alice and Bob, and a referee. A referee asks Alice a question $x \in \mathbf{X}$ and asks Bob a question $y \in \mathbf{y}$ according to the distribution π. He receives back an answer $a \in \mathbf{A}$ from Alice and an answer $b \in \mathbf{B}$ from Bob. Finally the referee declares that the players win the game with probability precisely $V(a, b, x, y)$; alternatively one may say that the players are attributed a payoff $V(a, b, x, y)$ for their answers. During the game Alice and Bob are space-like separated: they are put so far away that information, which travels at finite speed, cannot be exchanged between them until they produce the answers. That is, they cannot communicate with each other as a part of their strategy. The value of the game is the largest probability with which the players can win the game, where the probability is taken over the referee's choice of questions, the players' strategy (to be defined below), and the randomness in the referee's final decision; alternatively the value can be interpreted as the maximum expected payoff that can be achieved by the players.

XOR games are arguably the simplest and best understood class of two-player one-round games that are interesting from the point of view of quantum nonlocality. Two-player XOR games correspond to the restricted family of games for which the answer alphabets are binary, $\mathbf{A} = \mathbf{B} = \{0, 1\}$, and the payoff function $V(a, b, x, y)$ depends only on x, y, and the parity of a and b. We further restrict our attention to functions that take the form $V(a, b, x, y) = \frac{1}{2}(1 + (-1)^{a \oplus b \oplus c_{xy}})$ for some $c_{xy} \in \{0, 1\}$. This corresponds to deterministic predicates such that for every pair of questions there is a unique parity $a \oplus b$ that leads to a win for the players. [8] We will further restrict to the case where $\mathbf{X} = \mathbf{Y} = \{1, \cdots, N\}$, but the general situation is completely analogous.

In general, a strategy for the players is specified by an element $P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY})$ which gives the probability that Alice and Bob answer a and b when they are asked questions x and y respectively. Given one such strategy the value achieved by P in G can be expressed as

$$\begin{aligned}
\omega(G; P) &= \sum_{x,y=1}^{N} \sum_{a,b=0,1} \pi(x, y) V(a, b, x, y) P(a, b|x, y) \\
&= \sum_{x,y=1}^{N} \sum_{a,b=0,1} \pi(x, y) \frac{1 + (-1)^{a \oplus b \oplus c_{xy}}}{2} P(a, b|x, y) \\
&= \frac{1}{2} + \frac{1}{2} \sum_{x,y=1}^{N} \pi(x, y)(-1)^{c_{xy}} \left[P(0, 0|x, y) + P(1, 1|x, y) - P(0, 1|x, y) - P(1, 0|x, y) \right].
\end{aligned}$$

This last expression motivates the introduction of the *bias* $\beta(G; P) = 2\omega(G; P) - 1 \in [-1, 1]$ of an XOR game, a quantity that will prove more convenient to work with than the value of the

[7] Sometimes the function V is required to take values in $\{0, 1\}$. Our slightly more general definition can be interpreted as allowing for randomized predicates.

[8] This additional restriction is not essential, and all results discussed in this section extend to general $V(a, b, x, y) = V(a \oplus b, x, y) \in [0, 1]$.

game. Hence,

$$\beta(G;P) = \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} \left[P(0,0|x,y) + P(1,1|x,y) - P(0,1|x,y) - P(1,0|x,y) \right]. \tag{4.1}$$

We see that in order to compute the bias of an XOR games (4.1) we only use the correlations of the strategy $P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY})$. The *classical bias* of an XOR game is the largest value in (4.1) when optimizing over all classical correlations (matrices) as in (2.1). That is, we denote

$$\beta(G) = \sup_{\gamma \in \mathcal{L}_N} \left| \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} \gamma_{x,y} \right| = \|G\|_{\ell_1^{\mathrm{N}}(\mathbb{R}) \otimes_\epsilon \ell_1^{\mathrm{N}}(\mathbb{R})},$$

where we identify the game G with the tensor $G = \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} e_x \otimes e_y$. This value corresponds to the largest bias achievable by the players if they use a local and determinist strategy implemented with the help of shared randomness.

Analogously, the *quantum bias* of an XOR game is the largest value in (4.1) when optimizing over all quantum correlations (matrices) as in (2.3). That is,

$$\beta^*(G) = \sup_{\gamma \in \mathcal{Q}_N} \left| \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} \gamma_{x,y} \right| = \|G\|_{\ell_1^{\mathrm{N}}(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^{\mathrm{N}}(\mathbb{R})} \tag{4.2}$$

where G is as before.

The quantity $\beta^*(G)$ corresponds to the largest bias of the game when the players are allowed to perform a quantum strategy to play the game. That is, the probability of answering the pair of outputs (a,b) when the players are asked questions x and y respectively is defined by the application of some POVMs $\{E_x^a\}_{a=0,1}$ for Alice and $\{F_y^b\}_{b=0,1}$ for Bob, on a shared quantum state ρ.

We see that XOR games are particular examples of correlation Bell functional corrections. More precisely, computing the classical and quantum bias of an XOR games can be seen as the computation of the classical and the quantum value respectively of certain correlation Bell functional. It is interesting to observe that the correspondence goes both ways. To any tensor $G \in \ell_1^{\mathrm{N}}(\mathbb{R}) \otimes \ell_1^{\mathrm{N}}(\mathbb{R})$ with real coefficients that satisfies the mild normalization condition $\sum_{x,y} |G_{x,y}| = 1$ we may associate an XOR game by defining $\pi(x,y) = |G_{xy}|$ and $(-1)^{c_{xy}} = \mathrm{sign}(G_{xy})$. In particular any correlation Bell functional M can, up to normalization, be made into an equivalent XOR game and in this setting there is no difference between the viewpoints of Bell inequalities and of games.

To conclude this section we observe that the CHSH inequality can be written (by adding a normalization factor 1/4) as

$$G = G_{CHSH} = \sum_{x,y=0,1} \frac{1}{4} (-1)^{x \wedge y} e_x \otimes e_y,$$

which can be read very easily in terms of an XOR game: Here, the inputs $x \in \{0,1\}$ and $y \in \{0,1\}$ are uniformly distributed, and Alice and Bob win the game if their respective outputs $a \in \{0,1\}$ and $b \in \{0,1\}$ satisfy $a \oplus b = x \wedge y$; in other words, a must equal b unless $x = y = 1$. According to the computations made in Section 1, we have that classical players can achieve a bias of at most $1/2$, while entangled players can achieve a bias of $1/\sqrt{2}$.

Capítulo 2

Quantum nonlocality: The general case

5. Bell functionals and multiplayer games

5.1. Bell functionals. Bell functionals are linear forms acting on multipartite conditional probability distributions. For clarity we focus on the bipartite case, but the framework extends easily. Given finite sets $\mathbf{X}$ and $\mathbf{A}$ denote by $\mathcal{P}(\mathbf{A}|\mathbf{X})$ the set

$$\mathcal{P}(\mathbf{A}|\mathbf{X}) = \Big\{ P = \big(P(a|x)\big)_{x,a} \in \mathbb{R}_+^{\mathrm{AX}} : \forall x \in \mathbf{X}, \sum_{a \in \mathbf{A}} P(a|x) = 1 \Big\}.$$

In the case of bipartite conditional distributions we will use notation $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ instead of $\mathcal{P}(\mathbf{A} \times \mathbf{B}|\mathbf{X} \times \mathbf{Y})$.

A *Bell functional* M is simply a linear form on $\mathbb{R}^{\mathrm{ABXY}}$. Any such functional is specified by a family of real coefficients $M = (M_{x,y}^{a,b})_{x,y;a,b} \in \mathbb{R}^{\mathbf{A}\times\mathbf{B}\times\mathbf{X}\times\mathbf{Y}}$, and its action on $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ is given by

$$P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY}) \mapsto \omega(M;P) := \langle M, P \rangle = \sum_{x,y;a,b} M_{x,y}^{a,b} P(a,b|x,y) \in \mathbb{R}. \tag{5.1}$$

We refer to x and y as the *inputs* of the game and to a and b as the *outputs* of the system acted on by the Bell functional.

We recall that, motivated by the previous chapter, we can identify M with the tensor

$$\sum_{x,y;a,b} M_{x,y}^{a,b} e_x \otimes e_a \otimes e_y \otimes e_b \in \mathbb{R}^{\mathrm{X}} \otimes \mathbb{R}^{\mathrm{A}} \otimes \mathbb{R}^{\mathrm{Y}} \otimes \mathbb{R}^{\mathrm{B}}.$$

In order to talk about *Bell inequalities* we must consider the subset of $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ corresponding to *classical* conditional distributions. Informally, classical distributions are those that can be implemented locally with the help of shared randomness. Formally, they correspond to the convex hull of product distributions (so they form a polytope),

$$\mathcal{P}_{\mathcal{C}}(\mathbf{AB}|\mathbf{XY}) = \mathrm{Conv}\Big\{ (P(a|x)Q(b|y))_{x,y;a,b} : \ P \in \mathcal{P}(\mathbf{A}|\mathbf{X}),\ Q \in \mathcal{P}(\mathbf{B}|\mathbf{Y}) \Big\}. \tag{5.2}$$

It follows by definition that $\mathcal{P}_{\mathcal{C}}(\mathbf{AB}|\mathbf{XY})$ is a convex and closed set (in fact, it is a polytope) of $\mathbb{R}^{\mathrm{XYAB}}$.

Bell inequalities are those inequalities which describe the facets of the set $\mathcal{P}_{\mathcal{C}}(\mathbf{AB}|\mathbf{XY})$. As in the case of correlation Bell inequalities, we will be actually interested in the study of the *classical value* of a general Bell functional M:

$$\omega(M) := \sup_{P \in \mathcal{P}_{\mathcal{C}}(\mathbf{AB}|\mathbf{XY})} |\omega(M;P)|. \tag{5.3}$$

The second value associated to a Bell functional is its *quantum value* (or *entangled value*), which corresponds to its supremum over the subset of $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ consisting of those distributions that can be implemented locally using measurements on a bipartite quantum state:

$$\mathcal{P}_{\mathcal{Q}}(\mathbf{AB}|\mathbf{XY}) = \Big\{ \big(\langle\psi| A_x^a \otimes B_y^b |\psi\rangle\big)_{x,y,a,b} : \ d \in \mathbb{N},\ |\psi\rangle \in \mathrm{Ball}(\mathbb{C}^d \otimes \mathbb{C}^d),\ A_x^a, B_y^b \in \mathrm{Pos}(\mathbb{C}^d),$$
$$\sum_a A_x^a = \sum_b B_y^b = \mathrm{Id} \ \ \forall (x,y) \in \mathbf{X} \times \mathbf{Y} \Big\}.$$

Here the constraints $A_x^a \in \mathrm{Pos}(\mathbb{C}^d)$ and $\sum_a A_x^a = \mathrm{Id}$ for every x correspond to the general notion of measurement called *positive operator-valued measurement* (POVM) in quantum information. As we explained before, considering pure states in the previous definition is equivalent to considering general density matrices. As in the case of correlation matrices, it is very easy to see that $\mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$ is a convex set. Interestingly, it is not known whether it is closed.

With this definition of the set $\mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$, the entangled value of M is defined as

$$\omega^*(M) := \sup_{P \in \mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})} |\omega(M;P)|. \tag{5.4}$$

Since $\mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY}) \subseteq \mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$, which can be easily proved as in (2.4), it is clear that $\omega(M) \leq \omega^*(M)$ in general. A *Bell inequality violation* corresponds to the case when the inequality is strict: those elements M such that $\omega(M) < \omega^*(M)$ serve as witnesses that the set of quantum conditional distributions is strictly larger than the classical set.

5.1.1. Connection with correlation matrices. According to the definitions of $\mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY})$ and $\mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$, we can understand the setting of correlation Bell functionals studied in the previous chapter as a particular case of the setting considered in this section. To be more precise, let us consider the situation where $\mathbf{A} = \mathbf{B} = \{-1, 1\}$ and $\mathbf{X} = \mathbf{Y} = \{1, \cdots, N\}$, and compute the correlations of a given conditional distributions $P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY})$. That is, for every x and y we compute

$$\gamma_{x,y} = \mathbb{E}[a_x b_y] = P(1,1|x,y) + P(-1,-1|x,y) - P(1,-1|x,y) - P(-1,1|x,y).$$

It is very easy to check that the set of correlation matrices $(\gamma_{x,y})_{x,y=1}^N$ written of this form coincides with the set of classical correlation matrices $\mathcal{L}_N$ (2.1) if we restrict to $P \in \mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY})$ and coincides with the set of quantum correlation matrices $\mathcal{Q}_N$ (2.3) if we restrict to $P \in \mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$.

5.2. Two-prover one-round games. We have already introduced two-player one-round games $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$ in Section 4 of Chapter 1. We see that two-prover one-round games are the sub-class of Bell functionals M such that all coefficients are of the form $M_{x,y}^{a,b} = \pi(x,y)V(x,y,a,b)$; in particular, they are all non-negative. Conversely, any Bell functional with non-negative coefficients and verifying $\sum_{x,y;a,b} M_{x,y}^{a,b} = 1$[1] can be made into a game by setting $\pi(x,y) = \sum_{a,b} M_{x,y}^{a,b}$ and $V(a,b,x,y) = M_{x,y}^{a,b}/\pi(x,y)$. For this reason we will sometimes refer to games as "Bell functionals with non-negative coefficients". Beyond a mere change of language (inputs and outputs to the systems will be referred to as *questions* and *answers* to the *players*), the fact that such functionals can be interpreted as games allows for fruitful connections with the extensive literature on this topic developed in computer science and leads to many interesting constructions.

We can extend the definitions of classical and quantum values given earlier for Bell functionals to corresponding quantities for games. Precisely, given a game G we can define

$$\begin{aligned} \omega(G) &:= \sup_{P \in \mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY})} \Big| \sum_{x,y;a,b} \pi(x,y)V(a,b,x,y)P(a,b|x,y) \Big|, \\ \omega^*(G) &:= \sup_{P \in \mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})} \Big| \sum_{x,y;a,b} \pi(x,y)V(a,b,x,y)P(a,b|x,y) \Big|. \end{aligned} \tag{5.5}$$

The values in (5.5) are precisely the probability of winning the game G when the players are allowed to use classical resources (so local strategies and shared randomness) in the first case and quantum resources (so strategies using entangled states) in the second case. Note that the absolute value is redundant in (5.5) since all coefficients are non-negative in the corresponding

[1]Note that this normalization condition is irrelevant if we want to study the quantity $\omega^*(M)/\omega(M)$.

sums. This is not the case when considering general Bell functionals (5.2), where the absolute value plays an important role.

6. Operator spaces

We have seen in Chapter 1 that the classical and the quantum value of a correlation Bell functional M can be exactly described by means of tensor norms (the ϵ-norm and the γ_2-norm respectively) on the tensor product $\ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$. Instead of invoking Tsirelon's theorem (Theorem 2.2) to describe the quantum value of M by means of the γ_2-*norm*, let us look again at the definition of quantum correlation matrices (2.3) and ask: is there any natural norm to describe the value

$$\omega^*(M) = \sup_{\substack{A_1,\dots,A_N\\ B_1,\dots,B_N\\ \rho}} \left| \sum_{x,y=1}^{N} M_{x,y} tr\left(A_x \otimes B_y \rho\right) \right| = \sup_{\substack{A_1,\dots,A_N\\ B_1,\dots,B_N}} \left\| \sum_{x,y=1}^{N} M_{x,y} A_x \otimes B_y \right\|_{\mathcal{B}(\mathcal{H}\otimes\mathcal{H})} \quad ? \tag{6.1}$$

Here, the first supremum runs over all Hilbert spaces $\mathcal{H}$, all families of self-adjoint operators $A_1,\dots,A_N,B_1,\dots,B_N$ in $\mathcal{B}(\mathcal{H})$ of norm lower than or equal to one and all density matrices ρ in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$. The equality between the supremums follows from the fact that $\sum_{x,y=1}^{N} M_{x,y}A_x\otimes B_y$ is a self-adjoint operator in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$.

Colloquially, quantum mechanics is often thought of as a "non-commutative" extension of classical mechanics, and it is perhaps not so surprising that the answer to the previous question comes out of a non-commutative extension of Banach space theory, the theory of *operator spaces*. Initiated in Ruan's thesis (1988), operator space theory is a "quantized extension" of Banach space theory, whose basic objects are no longer the elements of a Banach space X but sequences of matrices $M_d(X)$, $d \geq 1$, with entries in X. This very recent theory provides us with the right tools: we will see that in operator space terminology (6.1) is precisely the *minimal* norm of the tensor M, which plays the same role in the operator space category as the ϵ-norm in the Banach space category.

One could still wonder why we should consider this new point of view if we already have a nice description of the value $\omega^*(M)$ by means of the γ_2-norm. A possible answer to this question is that the previous description is only suitable in the particular case of bipartite correlation Bell inequalities. As soon as we want to study the general context of Bell inequalities (including two-prover one-round games) and the k- partite scenario with $k \geq 3$, one "must" consider operator spaces!

6.1. Basic definitions about operator spaces.

We introduce all notions from operator space theory needed to understand the results presented in these notes. For the reader interested in a deeper treatment of the theory we refer to the standard books **[5]**, **[17]** and **[18]**.

An *operator space* X is a closed subspace of the space of all bounded operators on a complex Hilbert space $\mathcal{H}$. For any such subspace the operator norm on $\mathcal{B}(\mathcal{H})$ automatically induces a sequence of *matrix norms* $\|\cdot\|_d$ on $M_d(X)$ via the inclusions

$$M_d(X) \subseteq M_d(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{\oplus d}) = \mathcal{B}(\ell_2^d \otimes_2 \mathcal{H}).$$

That is, a matrix A with entries $A_{ij} \in X$ is regarded as a bounded operator on $\mathcal{H}^{\oplus d}$ through

$$A\left(\sum_j x_j \otimes e_j\right) = \sum_{ij} A_{ij}(x_j) \otimes e_i,$$

and equipped with the corresponding operator norm.

Ruan's Theorem **[20]** characterizes those sequences of norms which can be obtained in this way. It provides an alternative definition of an operator space as a complex vector space X

equipped with a sequence of matrix norms $(M_d(X), \|\cdot\|_d)$ satisfying the following two conditions for every pair of integers (c, d):

- (P1) $\|v \oplus w\|_{c+d} = \text{máx}\{\|v\|_c, \|w\|_d\}$ for every $v \in M_c(X)$ and $w \in M_d(X)$,
- (P2) $\|\alpha v \beta\|_d \leq \|\alpha\| \, \|v\|_d \, \|\beta\|$ for every $\alpha, \beta \in M_d$ and $v \in M_d(X)$.

Let us state Ruan's theorem in a precise form (see [**18**, Section 2.2] for the proof).

THEOREM 6.1 (Ruan). *Let X be a complex vector space. Let $\|\cdot\|_d$ be a sequence of norms on the spaces $M_d(X)$ and let $\|\cdot\|$ be the corresponding norm on $\mathcal{K}_0 \otimes X$, where*

$$\mathcal{K}_0 = \bigcup_{d \geq 1} M_d.$$

The following assertions are equivalent:

1. *The sequence of norms $\|\cdot\|_d$ verifies properties (P1) and (P2).*
2. *For a suitable Hilbert space $\mathcal{H}$, there is a linear embedding $\mathcal{J} : X \hookrightarrow \mathcal{B}(\mathcal{H})$ such that for any d, $\text{Id}_{M_d} \otimes \mathcal{J}$ is an isometry between $(M_d(X), \|\cdot\|_d)$ and $M_d(\mathcal{J}(X)) \subset \mathcal{B}(\ell_2^d \otimes_2 \mathcal{H})$.*
3. *For a suitable Hilbert space $\mathcal{H}$, there is a linear embedding $\mathcal{J} : X \hookrightarrow \mathcal{B}(\mathcal{H})$ such that $\text{Id}_{\mathcal{K}_0} \otimes \mathcal{J}$ is an isometry between $(\mathcal{K}_0 \otimes X, \|\cdot\|)$ and $\mathcal{K}_0 \otimes_{min} \mathcal{J}(X) \subset \mathcal{B}(\ell_2 \otimes_2 \mathcal{H})$.*
4. *For a suitable Hilbert space $\mathcal{H}$, there is a linear embedding $\mathcal{J} : X \hookrightarrow \mathcal{B}(\mathcal{H})$ such that $\text{Id}_{\mathcal{K}} \otimes \mathcal{J}$ is an isometry between $(\mathcal{K} \otimes X, \|\cdot\|)$ and $\mathcal{K} \otimes_{min} \mathcal{J}(X) \subset \mathcal{B}(\ell_2 \otimes_2 \mathcal{H})$. Here, $\mathcal{K}$ denotes the space of compact operators of ℓ_2 so that $\bar{\mathcal{K}}_0 = \mathcal{K}$.*

A sequence of matrix norms satisfying both conditions, or alternatively an explicit inclusion of X into $\mathcal{B}(\mathcal{H})$, which automatically yields such a sequence, is called an *operator space structure* (o.s.s.) on X.

A given Banach space X can be endowed with different o.s.s. Important examples are the *row* and *column* o.s.s. on ℓ_2^N. The space ℓ_2^N can be viewed as a subspace R_N of M_N via the map $e_i \mapsto E_{1i} = |1\rangle\langle i|$, $i = 1, \cdots, N$, where we use the Dirac notation $|i\rangle = e_i$ and $\langle i| = e_i^*$ that is standard in quantum information theory. Thus each vector $u \in \ell_2^N$ is identified with the matrix A_u which has that vector as its first row and zero elsewhere; clearly $\|u\|_2 = \|A_u\|$ and the embedding is norm-preserving. We can also use the map $e_i \mapsto E_{i1} = |i\rangle\langle 1|$, $i = 1, \cdots, N$, identifying ℓ_2^N with the subspace C_N of M_N of matrices that have all but their first column set to zero. An element $A \in M_d(R_N)$ is a $d \times d$ matrix whose each entry is an $N \times N$ matrix that is 0 except in its first row. Alternatively, A can be seen as an $N \times N$ matrix whose first row is made of $d \times d$ matrices $A_1, \ldots, A_N$ and all other entries are zero. The operator A acts on $M_d(M_N) \simeq M_{dN}$ by block-wise matrix multiplication, and one immediately verifies that the corresponding sequence of norms can be expressed as

$$\left\| \sum_{i=1}^N A_i \otimes e_i \right\|_{M_d(R_N)} = \left\| \sum_{i=1}^N A_i A_i^* \right\|_{M_d}^{\frac{1}{2}}. \tag{6.2}$$

Similarly, for $M_d(C_N)$ we obtain

$$\left\| \sum_{i=1}^N A_i \otimes e_i \right\|_{M_d(C_N)} = \left\| \sum_{i=1}^N A_i^* A_i \right\|_{M_d}^{\frac{1}{2}}. \tag{6.3}$$

These two expressions make it clear that the two o.s.s. they define on ℓ_2^N can be very different; consider for instance the norms of the element $A = \sum_{i=1}^N |i\rangle\langle 1| \otimes e_i \in M_N(\ell_2^N)$.

It is easy to deduce from the previous definition that these structures do not depend on the orthonormal basis $(e_i)_{i=1}^N$ chosen. Indeed, let us assume that $(f_j)_{j=1}^N$ is another orthonormal basis of ℓ_2^N. Let us write $f_j = \sum_{i=1}^N a_{i,j} e_i$ for every $j = 1, \cdots, N$ and note that $U = (a_{i,j})_{i,j=1}^N$ is a unitary matrix. Then,

$$
\begin{aligned}
(6.4) \qquad \left\| \sum_{j=1}^N A_j \otimes f_j \right\|_{M_d(R_N)} &= \left\| \sum_{j=1}^N A_j \otimes \left(\sum_{i=1}^N a_{i,j} e_i \right) \right\| = \left\| \sum_{i=1}^N \left(\sum_{j=1}^N a_{i,j} A_j \right) \otimes e_i \right\|_{M_d(R_N)} \\
&= \left\| \sum_{i=1}^N \left(\sum_{j=1}^N a_{i,j} A_j \right) \left(\sum_{j=1}^N a_{i,j} A_j \right)^* \right\|_{M_d} \\
&= \left\| \sum_{i=1}^N \sum_{j,j'=1}^N a_{i,j} \bar{a}_{i,j'} A_j A_{j'}^* \right\|_{M_d} \\
&= \left\| \sum_{j,j'=1}^N \left(\sum_{i=1}^N U(i,j) U^\dagger(j',i) \right) A_j A_{j'}^* \right\|_{M_d} \\
&= \left\| \sum_{i=1}^N A_i A_i^* \right\|_{M_d} .
\end{aligned}
$$

Certain Banach spaces have a natural o.s.s. This happens for the case of C*-algebras which, by the GNS representation, have a canonical inclusion in $\mathcal{B}(\mathcal{H})$ for a certain Hilbert space $\mathcal{H}$ obtained from the GNS construction. A first canonical example is the space $\mathcal{B}(\mathcal{H})$ itself, for which the natural inclusion is the identity. Note that when $\mathcal{H} = \mathbb{C}^N$, $\mathcal{B}(\mathbb{C}^N)$ is identified with M_N and $M_d(M_N)$ with M_{dN}. A second example is $\ell_\infty^N = (\mathbb{C}^N, \|\cdot\|_\infty)$, for which the natural inclusion is given by the map $e_i \mapsto E_{ii} = |i\rangle\langle i|$, $i = 1, \cdots, N$ embedding an element of ℓ_∞^N as the diagonal of an N-dimensional matrix. This inclusion yields the sequence of norms

$$
(6.5) \qquad \left\| \sum_{i=1}^N A_i \otimes e_i \right\|_{M_d(\ell_\infty^N)} = \sup \left\{ \|A_i\|_{M_d} : i = 1, \ldots, N \right\}.
$$

Bounded linear maps are the natural morphisms in the category of Banach spaces in the sense that two Banach spaces X, Y can be identified whenever there exists an isomorphism $T : X \to Y$ such that $\|T\| \|T^{-1}\| = 1$; in this case we say that X and Y are *isometric*. When considering operator spaces the norm on linear operators should take into account the sequence of matrix norms defined by the o.s.s. Given a linear map between operator spaces $T : X \to Y$, let T_d denote the linear map

$$
T_d : \; v = (v_{ij}) \in M_d(X) \; \mapsto \; (\mathrm{Id}_{M_d} \otimes T)(v) = (T(v_{ij}))_{i,j} \in M_d(Y).
$$

The map T is said to be *completely bounded* if its completely bounded norm if finite:

$$
\|T\|_{cb} := \sup_{d \in \mathbb{N}} \|T_d\| \; < \; \infty.
$$

It is very easy to see that given two operator spaces X and Y, the space of completely bounded maps from the first to the second becomes a Banach space when this space is endowed with the cb-norm. Let us denote this space by

$$
CB(X,Y).
$$

The definition of R_N and C_N joint with property (6.4) make the computation of the completely bounded norm of a linear map $T : C_N \to R_N$ particularly easy, in contrast to the general case. Indeed, the following well-known lemma tells us how to compute such a norm.

LEMMA 6.2. *Let $T : \ell_2^N \to \ell_2^N$ be a linear map. Then,*

$$\|T : C_N \to R_N\|_{cb} = \left(\sum_{i=1}^{N} |\lambda_i|^2\right)^{\frac{1}{2}} = \|T : R_N \to C_N\|_{cb}, \tag{6.6}$$

where $T = \sum_{i=1}^{N} \lambda_i |e_i\rangle\langle f_i|$ is the Schmidt decomposition of T.

DEMOSTRACIÓN. We will only prove the first equality since the second one can be proved analogously. By the comments above, we can assume that $T = \sum_{i=1}^{N} \lambda_i |i\rangle\langle i|$.

Let us denote $A_i = |1\rangle\langle i| \in M_N$ for every $i = 1, \cdots, N$. Then, according to Equation (6.3) we trivially have $\left\|\sum_{i=1}^{N} A_i \otimes e_i\right\|_{M_N(C_N)} = 1$. Therefore,

$$\left\|(id_N \otimes T)\left(\sum_{i=1}^{N} A_i \otimes e_i\right)\right\|_{M_N(R_N)} = \left\|\sum_{i=1}^{N} \lambda_i A_i \otimes e_i\right\|_{M_N(R_N)} \leq \|T\|_{cb}.$$

Now, note that $\left\|\sum_{i=1}^{N} \lambda_i A_i \otimes e_i\right\|_{M_N(R_N)} = \left(\sum_{i=1}^{N} |\lambda_i|^2\right)^{\frac{1}{2}}$. Therefore, $\|T\|_{cb} \geq \left(\sum_{i=1}^{N} |\lambda_i|^2\right)^{\frac{1}{2}}$.

To see the converse inequality, let us consider a sequence of matrices $(A_i)_{i=1}^N$ in M_d such that $\left\|\sum_{i=1}^{N} A_i \otimes e_i\right\|_{M_d(C_N)} = \left\|\sum_{i=1}^{N} A_i^* A_i\right\|_{M_d}^{\frac{1}{2}} \leq 1$. Then,

$$\begin{aligned}\left\|(id_N \otimes T)\left(\sum_{i=1}^{N} A_i \otimes e_i\right)\right\|_{M_d(R_N)} &= \left\|\sum_{i=1}^{N} \lambda_i A_i \otimes e_i\right\|_{M_d(R_N)} \\ &= \left\|\sum_{i=1}^{N} |\lambda_i|^2 A_i A_i^*\right\|_{M_d}^{\frac{1}{2}} \leq \left(\sum_{i=1}^{N} |\lambda_i|^2\right)^{\frac{1}{2}},\end{aligned}$$

where we have used that $\|A_i A_i^*\| \leq 1$ for every i. □

Two operator spaces are said *completely isometric* whenever there exists an isomorphism $T : X \to Y$ such that $\|T\|_{cb}\|T^{-1}\|_{cb} = 1$. The spaces R_N and C_N introduced earlier are isometric Banach spaces: if id denotes the identity map,

$$\|id : R_N \to C_N\| = \|id : C_N \to R_N\| = 1.$$

But they are *not* completely isometric. According to Lemma 6.2 we have that

$$\|id : R_N \to C_N\|_{cb} = \|id : C_N \to R_N\|_{cb} = \sqrt{N}.$$

In fact, any isomorphism $u : R_N \to C_N$ verifies $\|u\|_{cb}\|u^{-1}\|_{cb} \geq N$. Indeed, in this case

$$\begin{aligned}N = tr(id_N) = tr(u \circ u^{-1}) &\leq \left(\sum_{i=1}^{N} |\beta_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} |\lambda_i|^2\right)^{\frac{1}{2}} \\ &= \|u^{-1} : R_N \to C_N\|_{cb}\|u : C_N \to R_N\|_{cb},\end{aligned}$$

where we have used Cauchy-Schwarz inequality, the Schmidt decompositions $u = \sum_{i=1}^{N} \lambda_i |e_i\rangle\langle f_i|$, $u^{-1} = \sum_{i=1}^{N} \beta_i |e_i\rangle\langle f_i|$ and Lemma 6.2 in the last equality.

We leave the proof of the following lemma, which can be easily obtained as a consequence of (6.4), as an exercise for the reader.

LEMMA 6.3. *Let n and k be two natural numbers and let $T : \ell_2^n \to \ell_2^k$ be a linear map. Then,*

$$\|T : R_n \to R_k\|_{cb} = \|T : \ell_2^n \to \ell_2^k\| = \|T : C_n \to C_k\|_{cb}.$$

For C*-algebras A, B with unit a linear map $T : A \to B$ is *completely positive* if $T_d(x) = (\mathrm{Id}_{M_d} \otimes T)(x)$ is a positive element in $M_d(B)$ for every d and every positive element $x \in M_d(A)$, and it is *unital* if $T(\mathrm{Id}_A) = \mathrm{Id}_B$. Although every completely positive map is trivially positive, the converse is not true. An easy counterexample can be given by the transpose map $T : M_2 \to M_2$ defined as $T(x) = x^t$ for every $x \in M_2$. Indeed, it is trivial that this map is positive; while if we consider the positive element

$$A = |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 2| \otimes |1\rangle\langle 2| + |2\rangle\langle 1| \otimes |2\rangle\langle 1| + |2\rangle\langle 2| \otimes |2\rangle\langle 2|$$
$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

and apply $\mathrm{Id}_{M_2} \otimes T$, we obtain the non-positive element

$$(\mathrm{Id}_{M_2} \otimes T)(A) = |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 2| \otimes |2\rangle\langle 1| + |2\rangle\langle 1| \otimes |1\rangle\langle 2| + |2\rangle\langle 2| \otimes |2\rangle\langle 2|$$
$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

However, if A is a commutative C*-algebra (such as ℓ_∞^N) it is straightforward to check that positivity implies complete positivity. The following very standard lemma will be used often, and we include a short proof.

LEMMA 6.4. *Let $T : A \to B$ be a completely positive and unital map between C^*-algebras. Then*

$$\|T\| = \|T\|_{cb} = 1.$$

DEMOSTRACIÓN. The inequalities $1 \leq \|T\| \leq \|T\|_{cb}$ hold trivially for any unital map. To show the converse inequalities, first recall that an element z in a unital C^*-algebra D verifies

$$\|z\| \leq 1 \text{ if and only if } \begin{pmatrix} \mathrm{Id}_D & z \\ z^* & \mathrm{Id}_D \end{pmatrix} \text{ is a positive element in } M_2(D). \tag{6.7}$$

This can be easily seen by considering the canonical inclusion of the C*-algebra $M_2(D)$ in $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Now given an element $x \in M_d(A)$ such that $\|x\| \leq 1$ we use the positivity of

$$\begin{pmatrix} \mathrm{Id}_{M_d(A)} & x \\ x^* & \mathrm{Id}_{M_d(A)} \end{pmatrix} \in M_2(M_d(A))$$

and the fact that T is completely positive and unital to conclude that

$$T_{2d} \begin{pmatrix} \mathrm{Id}_{M_d(A)} & x \\ x^* & \mathrm{Id}_{M_d(A)} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{M_d(B)} & T_d(x) \\ T_d(x)^* & \mathrm{Id}_{M_d(B)} \end{pmatrix}$$

is a positive element in $M_2(M_d(B))$. Using (6.7) again we conclude that $\|T_d(x)\| \leq 1$ for every d, thus $\|T\|_{cb} \leq 1$. □

The following remark will be very important to understand the connections between quantum nonlocality and operator spaces.

REMARK 6.5. *Given a POVM $\{E_a\}$ the linear map $T : \ell_\infty^K \to M_d$, $T : e_a \mapsto E_a$ is completely positive and unital, thus by Lemma 6.4 $\|T\|_{cb} = 1$; conversely any completely positive and unital map $T : \ell_\infty^K \to M_d$ defines a POVM.*

Completely bounded maps can be understood as a generalization of completely positive maps. In particular, some fundamental theorems for completely positive maps generalize to the setting of completely bounded maps. The following factorization/extension theorem for completely bounded maps is a generalization of Stinespring theorem (see [**18**, Theorem 1.6] and the references therein for the proof).

THEOREM 6.6. *Let X and Y be operator spaces given by the embeddings $\mathcal{J}_1 : X \hookrightarrow \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 : Y \hookrightarrow \mathcal{B}(\mathcal{H}_2)$ respectivelly. Given a completely bounded map $T : X \to Y$, there exist a Hilbert space $\hat{\mathcal{H}}$, a $*$-homomorphism $\pi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\hat{\mathcal{H}})$ and operators $V_1 : \mathcal{H}_2 \to \hat{\mathcal{H}}$, $V_2 : \hat{\mathcal{H}} \to \mathcal{H}_2$ such that $\|V_1\|\|V_2\| = \|T\|_{cb}$ and*

$$\mathcal{J}_2(T(x)) = V_2\pi(\mathcal{J}_1(x))V_1 \quad \text{for every} \quad x \in X.$$

As an immediate consequence of the previous theorem we obtain the following version of the *Hahn-Banach theorem* for operator spaces: For every operator space X with $\mathcal{J} : X \hookrightarrow \mathcal{B}(\mathcal{H})$ and every completely bounded map $T : X \to \mathcal{B}(\hat{\mathcal{H}})$, there exists a completely bounded map $\tilde{T} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\hat{\mathcal{H}})$ which extends the map T (that is, $\tilde{T} \circ \mathcal{J} = T$) and such that $\|\tilde{T}\|_{cb} = \|T\|_{cb}$.

One can also prove (see for instance [**17**, Proposition 8.1]) that for every linear map $T : X \to M_d$ it holds that $\|T\|_{cb} = \|T_d\|$, so in this case the completely bounded norm is attained by considering amplifications up to dimension d.

Given an operator space X it is possible to define an o.s.s. on X^*, the dual space of X. The norms on $M_d(X^*)$ are specified through the natural identification with the space of linear maps from X to M_d, according to which to an element $z = \sum_i A_i \otimes x_i^* \in M_d(X^*)$ is associated the map

$$T^z : x \in X \mapsto \sum_i x_i^*(x) A_i \in M_d.$$

This leads to the sequence of norms

$$\|z\|_{M_d(X^*)} = \|T^z : X \to M_d\|_{cb}, \qquad d \geq 1. \tag{6.8}$$

We encourage the reader to check the (completely isometric) identifications $C_N^* = R_N$ and $R_N^* = C_N$.

Duality allows us to introduce a natural o.s.s. on the spaces ℓ_1^N, dual space of ℓ_∞^N. From (6.5) and (6.8) one immediately obtains that[2]

$$\left\| \sum_{i=1}^N A_i \otimes e_i \right\|_{M_d(\ell_1^N)} = \sup \left\| \sum_{i=1}^N A_i \otimes B_i \right\|_{M_{d^2}}, \tag{6.9}$$

where the supremum runs over all families of operators $\{B_i\}_i$ in M_d such that $\sup_i \|B_i\| \leq 1$. Note that, by convexity, the supremum in (6.9) can be restricted to families of unitaries $\{U_i\}_i$ in M_d.

6.2. Tensor norms in the operator spaces category. As in the case of linear maps, working with bilinear maps on operator spaces requires the introduction of a norm on such maps which captures the o.s.s. Given a bilinear form on operator spaces $B : X \times Y \to \mathbb{C}$, for every d define a bilinear operator

$$B_d = B \otimes \mathrm{Id}_{M_d} \otimes \mathrm{Id}_{M_d} : M_d(X) \times M_d(Y) \to M_{d^2}$$

[2]Note that here the o.s.s. is defined without explicitly specifying an embedding of ℓ_1^N in some $\mathcal{B}(\mathcal{H})$. Although Ruan's theorem assures that such an embedding must exist that leads to the sequence of norms (6.9), finding that embedding explicitly can be a difficult problem. In particular, for ℓ_1^n the simplest embedding is based on the universal C*-algebra associated to the free group with n generators $C^*(\mathbb{F}_n)$.

by $B(a\otimes x, b\otimes y) = B(x,y)a\otimes b$ for every $a, b \in M_d$, $x \in X$, $y \in Y$. We say that B is *completely bounded* if its completely bounded norm is finite:

$$\|B\|_{cb} := \sup_d \|B_d\| < \infty.$$

Given two operator spaces X, Y the space of completely bounded bilinear forms $B : X\times Y \to \mathbb{C}$ becomes a Banach spaces when it is inherit with the norm $\|\cdot\|_{cb}$. We will denote this space by $\mathcal{B}il_{cb}(X,Y)$. It is not difficult to see that the previous definition makes the identity (3.1) an isometry when we consider completely bounded norms:

$$CB(X,Y^*) = \mathcal{B}il_{cb}(X,Y).$$

This is the equivalent identification to (3.6) in the category of operator spaces.

As in the case of Banach spaces, there exists a tensor norm (in the category of operator spaces) such that the identification (3.1) becomes isometric when we consider the $\|\cdot\|_{cb}$-norms. Given $z \in X\otimes Y$ we define its *projective tensor norm* as

$$\|z\|_{X\otimes^\wedge Y} := \text{ínf}\left\{\|\alpha\|_{HS}\|x\|_{M_l(X)}\|y\|_{M_m(Y)}\|\beta\|_{HS}\right\},$$

where the infimum runs over all possible factorization of the form

$$z = \alpha\cdot(x\otimes y)\cdot\beta = \sum_{\substack{1\le i,j\le l\\ 1\le p,q\le m}} \alpha_{i,p}x_{i,j}\otimes y_{p,q}\beta_{j,q}.$$

We denote by $X\otimes^\wedge Y$ the completion of $X\otimes Y$ under the previous norm. It is now an exercise to verify that the identification

$$CB(X,Y^*) = \mathcal{B}il_{cb}(X,Y) = (X\otimes^\wedge Y)^* \tag{6.10}$$

is indeed isometric.

More generally, one can define an operator space structure on $X\otimes^\wedge Y$ by considering the following sequence of norms in $M_d(X\otimes Y)$:

$$\|z\|_{M_d(X\otimes^\wedge Y)} := \text{ínf}\left\{\|\alpha\|_{M_{d,lm}}\|x\|_{M_l(X)}\|y\|_{M_m(Y)}\|\beta\|_{M_{lm,d}}\right\},$$

where the infimum runs over all possible factorization of the form

$$z_{r,s} = \alpha\cdot(x\otimes y)\cdot\beta = \sum_{\substack{1\le i,j\le l\\ 1\le p,q\le m}} \alpha_{r,ip}x_{i,j}\otimes y_{p,q}\beta_{jq,s} \quad \text{for every} \quad 1\le r,s\le d.$$

The identification $\mathcal{B}il_{cb}(X,Y) = CB(X,Y^*) = (X\otimes^\wedge Y)^*$ allows to define operator space structures on $CB(X,Y^*)$ and $\mathcal{B}il_{cb}(X,Y)$ so that these spaces are, by definition, completely isometric.

As in the Banach space setting, it can be shown that the projective tensor norm verifies the metric mapping property (in the category of operator spaces): For all operator spaces X, Y, Z, W and all linear maps $T : X\to Z$ and $S : Y\to W$, the following estimate holds:

$$\left\|T\otimes S : X\otimes^\wedge Y \to X\otimes^\wedge Y\right\|_{cb} \le \|T\|_{cb}\|S\|_{cb}. \tag{6.11}$$

Let us now introduce the analogous norm to the ϵ-norm in the category of operator spaces. Given any $z\in X\otimes Y$, let us define the *minimal tensor norm* (or *min*-norm) of $z\in X\otimes Y$ by

$$\|z\|_{X\otimes_{min}Y} = \|(\mathcal{J}_1\otimes\mathcal{J}_2)(z)\|_{\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)},$$

where $\mathcal{J}_1 : X\hookrightarrow \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 : Y\hookrightarrow \mathcal{B}(\mathcal{H}_2)$ are embeddings defining the operator space structure of X and Y respectively. Here, we are using the algebraic inclusion $\mathcal{B}(\mathcal{H}_1)\otimes\mathcal{B}(\mathcal{H}_2)\subset \mathcal{B}(\mathcal{H}_1\otimes_2\mathcal{H}_2)$. We will denote by $X\otimes_{min} Y$ the completion of $X\otimes Y$ under the minimal norm.

Notice that with this definition at hand, it is clear that $M_d\otimes_{min} X = M_d(X)$ for every d and, in particular, $M_d\otimes_{min} M_d = M_{d^2}$.

The Banach space $X \otimes_{min} Y$ has a natural operator spaces structure given by the embedding

$$\mathcal{J}_1 \otimes \mathcal{J}_2 : X \otimes Y \hookrightarrow \mathcal{B}(\mathcal{H}_1 \otimes_2 \mathcal{H}_2).$$

As in the Banach space setting, it can be shown that the minimal tensor norm verifies the metric mapping property in the category of operator spaces: For all operator spaces X, Y, Z, W and all linear maps $T : X \to Z$ and $S : Y \to W$, the following estimate holds:

$$\|T \otimes S : X \otimes_{min} Y \to Z \otimes_{min} W\|_{cb} \leq \|T\|_{cb} \|S\|_{cb}. \tag{6.12}$$

This estimate can be easily shown by using Theorem 6.6. Indeed, this theorem allows to reduce the proof of (6.12) to the case where all the operator spaces are of the form $\mathcal{B}(\mathcal{H})$. The proof of this case follows by definition of the minimal norm.

In addition, an easy consequence of its definition is that the minimal norm, analogously to the ϵ-norm in the category of Banach spaces, is injective in the category of operator spaces; that is, if $\mathcal{J}_1 : X \to Z$ and $\mathcal{J}_2 : Y \to W$ are complete isometries (resp. complete isomorphisms), then

$$\mathcal{J}_1 \otimes \mathcal{J}_2 : X \otimes_{min} Y \to Z \otimes_{min} W$$

is a complete isometry (resp. complete isomorphim).

In fact, there is an alternative way to define the minimal norm which stresses the idea that considering finite dimensional Hilbert spaces suffices to compute it. Given two operator spaces X and Y, define the $\|\cdot\|_\alpha$-norm of $z \in X \otimes Y$ as $\|z\|_\alpha = \sup_d \|z\|_{X \otimes_{min_d} Y}$, where

$$\|z\|_{X \otimes_{min_d} Y} = \sup_{T \in \mathrm{L}(X, M_d), S \in \mathrm{L}(Y, M_d) \colon \|T\|_{cb}, \|S\|_{cb} \leq 1} \big\|(T \otimes S)(z)\big\|_{M_{d^2}}. \tag{6.13}$$

Lemma 6.7. *Given any $z \in X \otimes Y$, the following identity holds:*

$$\|z\|_{X \otimes_{min} Y} = \|z\|_\alpha.$$

Demostración. Let us assume that $\mathcal{J}_1 : X \to \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 : Y \to \mathcal{B}(\mathcal{H}_2)$ are embeddings defining the operator spaces structures of X and Y respectively. In order to simplify notation, let us identify the element $z = \sum_{i=1}^N x_i \otimes y_i \in X \otimes Y$ with th element $(\mathcal{J}_1 \otimes \mathcal{J}_2)(z) = \sum_{i=1}^N \mathcal{J}_1(x_i) \otimes \mathcal{J}_2(y_i) \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. By definition we have that

$$\|z\|_{min} = \sup\{|\langle zs, t\rangle|\}$$

where this supremum runs over all elements in the unit ball $s, t \in \mathcal{H}_1 \otimes_2 \mathcal{H}_2$. By density, we can restrict the supremum to elements in the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then, it is clear that for some finite dimensional subspecies $\mathcal{H}_n \subset \mathcal{H}_1$ and $\mathcal{H}'_n \subset \mathcal{H}_2$ we have $s, t \in \mathcal{H}_n \otimes \mathcal{H}'_n$. We can identify the spaces $\mathcal{H}_n$ and $\mathcal{H}'_n$ with ℓ_2^n and so, $\mathcal{B}(\mathcal{H}_n)$ and $\mathcal{B}(\mathcal{H}'_n)$ with M_n. Let $u : \mathcal{B}(\mathcal{H}_1) \to M_n$ be the mapping taking x to $P_{\mathcal{H}_n} x|_{\mathcal{H}_n}$. Let $\mathcal{A}_n$ be the collection of all such mappings with $\mathcal{H}_n$ arbitrary n-dimensional and $\mathcal{B}_n$ the corresponding set for $\mathcal{H}_2$. Then, we have that

$$\begin{aligned} \|z\|_\alpha &= \sup\Big\{\Big|\Big\langle\Big(\sum_{i=1}^N x_i \otimes y_i\Big)s, t\Big\rangle\Big|\Big\} = \sup\Big\{\Big|\Big\langle\Big(\sum_{i=1}^N u(x_i) \otimes v(y_i)\Big)s, t\Big\rangle\Big|\Big\} \\ &\leq \sup_{n \in \mathbb{N}, u \in \mathcal{A}_n, v \in \mathcal{B}_n} \Big\|\sum_{i=1}^N u(a_i) \otimes v(b_i)\Big\|_{M_{n^2}}. \end{aligned}$$

This shows that

$$\|z\|_{X \otimes_{min} Y} \leq \|z\|_\alpha.$$

In order to prove the converse inequality let us consider maps $u : X \to M_n$ and $v : Y \to M_n$ whose completely bounded norm are lower than or equal to one. According to Theorem 6.6 we can find some extensions $\tilde{u}:\mathcal{B}(\mathcal{H}_1)\to M_n$ and $\tilde{v}:\mathcal{B}(\mathcal{H}_2)\to M_n$ of u and v respectively such that $\|u\|_{cb}=\|\tilde{u}\|_{cb}$ and $\|v\|_{cb}=\|\tilde{v}\|_{cb}$. Then, we easily deduce that

$$\begin{aligned}\|(u\otimes v)(z)\|_{M_{n^2}} &= \|(\tilde{u}\otimes\tilde{v})\left((\mathcal{J}_1\otimes\mathcal{J}_2)(z)\right)\|_{M_{n^2}} \leq \|\tilde{u}\|_{cb}\|\tilde{v}\|_{cb}\|(\mathcal{J}_1\otimes\mathcal{J}_2)(z)\|_{\mathcal{B}(\mathcal{H}_1\otimes_2\mathcal{H}_2)}\\ &\leq \|(\mathcal{J}_1\otimes\mathcal{J}_2)(z)\|_{\mathcal{B}(\mathcal{H}_1\otimes_2\mathcal{H}_2)} = \|z\|_{min},\end{aligned}$$

where in the first inequality we have used the metric mapping property of the minimal norm (6.12).

Since this happens for every n we obtain the desired inequality. □

Following the analogy with the Banach space setting, it is easy to check that if X and Y are finite dimensional operator spaces, we have the following (completely) isometric identification

$$X^*\otimes_{min} Y = CB(X,Y). \tag{6.14}$$

In addition, this immediately implies that, for finite dimensional operator spaces, the identifications

$$(X\otimes^\wedge Y)^* = X^*\otimes_{min} Y^* \quad\text{and}\quad (X\otimes_{min} Y)^* = X^*\otimes^\wedge Y^* \tag{6.15}$$

also hold (completely isometrically).

The following simple lemma will be used very often in these notes.

Lemma 6.8. *Let X be any operator spaces. Then, for ever natural number N we have isometric identifications:*

$$\ell_\infty^N(X) = \ell_\infty^N\otimes_\epsilon X = \ell_\infty^N\otimes_{min} X. \tag{6.16}$$

Here, $\ell_\infty^N(X)$ is the algebraic space $\mathbb{C}^N\otimes X$ joint with the norm

$$\left\|\sum_{i=1}^N e_i\otimes x_i\right\| = \sup_{i=1,\cdots,N}\|x_i\|_X.$$

Demostración. The first identification follows easily from the definition of the ϵ- norm. Indeed, according to (3.9), for a given element $z=\sum_{i=1}^N e_i\otimes x_i\in\ell_\infty^N\otimes X$ we have

$$\begin{aligned}\left\|\sum_{i=1}^N e_i\otimes x_i\right\|_{\ell_\infty^N\otimes_\epsilon X} &= \sup_{e^*\in\mathrm{Ball}((\ell_\infty^N)^*)}\left\|\sum_{i=1}^N e^*(e_i)x_i\right\|_X\\ &= \sup_{e^*\in\mathrm{Ball}(\ell_1^N)}\left\|\sum_{i=1}^N e^*(i)x_i\right\|_X = \sup_{i=1,\cdots,N}\|x_i\|_X,\end{aligned}$$

where here we denote by $e^*(i)$ the i-th coefficient of the vector e^* and the last equality follows easily from the definition of the 1-norm.

To show the identification $\ell_\infty^N(X)=\ell_\infty^N\otimes_{min} X$ note that by the very definition of the min-norm we have, for a given element $z=\sum_{i=1}^N e_i\otimes x_i\in\ell_\infty^N\otimes X$,

$$\begin{aligned}\left\|\sum_{i=1}^N e_i\otimes x_i\right\|_{\ell_\infty^N\otimes_{min}X} &= \left\|\sum_{i=1}^N |i\rangle\langle i|\otimes\mathcal{J}(x_i)\right\|_{M_N\otimes_{min}\mathcal{B}(\mathcal{H})} = \left\|\sum_{i=1}^N |i\rangle\langle i|\otimes\mathcal{J}(x_i)\right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))}\\ &= \sup_{i=1,\cdots,N}\|\mathcal{J}(x_i)\|_{\mathcal{B}(\mathcal{H})} = \sup_{i=1,\cdots,N}\|x_i\|_X\,.\end{aligned}$$

Here, the first two equalities are by definition of the minimal norm, the third one follows from the fact that the element $\sum_{i=1}^N |i\rangle\langle i|\otimes\mathcal{J}(x_i)$ is diagonal in the first component, and the last one is trivial from the fact that $\mathcal{J}$ is an isometry. □

The dual formulation of the previous lemma says that for every operator space X and any natural number N we have isometric identifications:

$$\ell_1^N(X) = \ell_1^N \otimes_\pi X = \ell_1^N \otimes^\frown X, \tag{6.17}$$

where $\ell_1^N(X)$ is the algebraic space $\mathbb{C}^N \otimes X$ joint with the norm

$$\left\| \sum_{i=1}^N e_i \otimes x_i \right\| = \sum_{i=1}^N \|x_i\|_X.$$

Let us also mention that the projective and the minimal tensor norms can be analogously defined on the tensor product of k operator spaces

$$X_1 \otimes \cdots \otimes X_k.$$

It is easy to check that the norms are associative and commutative. Moreover, all previously completely isometric identifications also hold in the general context.

6.2.1. Coming back to quantum nonlocality. Before finishing this section, let us explain why operator spaces are relevant in the context of quantum nonlocality, even in the correlation case. Let $M = (M_{x,y})_{x,y=1}^N$ be a matrix with real entries and let us denote $M = \sum_{x,y=1}^N M_{x,y} e_x \otimes e_y \in \mathbb{R}^N \otimes \mathbb{R}^N$.[3] In the first chapter, we consider the γ_2-norm in order to define the quantum value of M when it is regarded as a correlation Bell functional. Let us, instead, consider the minimal norm.

$$\|M\|_{\ell_1^N(\mathbb{C}) \otimes_{min} \ell_1^N(\mathbb{C})} = \sup_{d;\, U_x, V_y \in \mathrm{Ball}(M_d(\mathbb{C}))} \left\| \sum_{x,y=1}^N M_{xy}\, U_x \otimes V_y \right\|. \tag{6.18}$$

Note that the only difference with the value $\omega^*(G)$ is that in (6.1) the supremum is taken over all self-adjoints operators A_x and B_y, while in (6.18) we consider unitary matrices. Nevertheless taking advantage of the unrestricted dimension d the mapping

$$A \to \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

shows that restricting the supremum in (6.18) to Hermitian operators leaves it unchanged. Then, we conclude that for a given correlation Bell functional we have

$$\omega^*(M) = \|M\|_{\ell_1^N(\mathbb{C}) \otimes_{min} \ell_1^N(\mathbb{C})}.$$

7. Description of two-player games with operator spaces

Given a game $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$, a classical strategy for the players is described by an element $P \in \mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY})$, defined in (5.2) as the convex hull of the set of product strategies $\mathcal{P}(\mathbf{A}|\mathbf{X}) \times \mathcal{P}(\mathbf{B}|\mathbf{X})$. In order to simplify notation we will assume that $\mathbf{X} = \mathbf{Y} = \{1, \cdots, N\}$ and $\mathbf{A} = \mathbf{B} = \{1, \cdots, K\}$. However, the analysis of the general case can be done completely analogously. According to our simplification, we will simply write $\mathcal{P}_\mathcal{C}(K|N)$ and $\mathcal{P}_\mathcal{Q}(K|N)$ to denote $\mathcal{P}_\mathcal{C}(\mathbf{AB}|\mathbf{XY})$ and $\mathcal{P}_\mathcal{Q}(\mathbf{AB}|\mathbf{XY})$ respectively.

The normalization condition $\sup_x \sum_a |P(a|x)| \leq 1$ suggests that the space $\ell_\infty^N(\ell_1^K)$ should play a role here, where this space is defined as $\mathbb{C}^{NK}$ equipped with the norm

$$\|(R(x,a))_{x,a}\|_{\infty,1} = \sup_{x=1,\cdots,N} \sum_{a=1}^K |R(x,a)|. \tag{7.1}$$

[3]In order to use operator spaces, we must realize our element in the complex space, so we can always understand M as an element in $\mathbb{C}^N \otimes \mathbb{C}^N$.

In fact, regarding Proposition 3.1 and Proposition 3.3, in order to describe the classical and the quantum value of a game G the natural space to work with is the dual space $(\ell_\infty^N(\ell_1^K))^* = \ell_1^N(\ell_\infty^K) = (\mathbb{C}^{NK}, \|\cdot\|_{1,\infty})$, where

$$\big\|(R(x,a))_{x,a}\big\|_{1,\infty} = \sum_{x=1}^{N} \sup_{a=1,\cdots,K} |R(x,a)|.$$

Let us consider the tensor

$$G = \sum_{x,y;a,b=1}^{N,K} G_{x,y}^{a,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b) \in \ell_1^N(\ell_\infty^K) \otimes \ell_1^N(\ell_\infty^K),$$

where $G_{x,y}^{a,b} = \pi(x,y)V(a,b,x,y)$ for every x, y, a, b. [4]

Then, according to the definition of the ϵ-norm (3.8) we have

$$\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)} = \sup\Big\{\Big|\sum_{x,y;a,b} G_{x,y}^{a,b}P(x,a)Q(y,b)\Big| : \|P\|_{\ell_\infty^N(\ell_1^K)}, \|Q\|_{\ell_\infty^N(\ell_1^K)} \leq 1\Big\}. \tag{7.2}$$

While (7.1) and (7.2) make it clear that $\omega(G) \leq \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}$, since the space $\ell_\infty^N(\ell_1^K)$ allows for elements with complex coefficients there could a priori be cases where the inequality is strict. As will be seen in Section 8 this can indeed happen for general Bell functionals M; however for the case of a game G both quantities coincide.

LEMMA 7.1. *Given a two-player game G,*

$$\omega(G) = \|G\|_{\ell_1^N(\ell_\infty^N)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}. \tag{7.3}$$

DEMOSTRACIÓN. We only need to prove that $\|G\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)} \leq \omega(G)$. To see this, for any P, Q such that $\|P\|_{\ell_\infty^N(\ell_1^K)}, \|Q\|_{\ell_\infty^N(\ell_1^K)} \leq 1$ write

$$\Big|\sum_{x,y;a,b} G_{x,y}^{a,b}P(x,a)Q(y,b)\Big| \leq \sum_{x,y;a,b} G_{x,y}^{a,b}|P(x,a)||Q(y,b)| \leq \omega(G),$$

where the first inequality follows from the triangle inequality and the non-negativity of G and the last inequality follows from $\||P|\|_{\ell_\infty^N(\ell_1^K)}, \||Q|\|_{\ell_\infty^N(\ell_1^K)} \leq 1$ and the fact that $|P|$, $|Q|$ as well as G have non-negative coefficients. $\square$

Let us also mention that according to the correspondence (3.6) and (3.11) between bilinear forms and tensor products, the classical value of a game G can be equivalently written as

$$\omega(G) = \|G : \ell_\infty^N(\ell_1^K) \times \ell_\infty^N(\ell_1^K) \to \mathbb{C}\|$$

where G is the bilinear form defined by

$$G(P,Q) = \sum_{x,y;a,b} G_{x,y}^{a,b}P(x,a)Q(y,b). \tag{7.4}$$

We proceed to analyze entangled strategies for the players, i.e. the set $\mathcal{P}_{\mathcal{Q}}(\mathbf{AB}|\mathbf{XY})$, and their relation to the minimal norm of $G \in \ell_1^N(\ell_\infty^K)\otimes \ell_1^N(\ell_\infty^K)$. Towards this end we need to define an o.s.s. on $\ell_\infty^N(\ell_1^K)$. Using the o.s.s. on ℓ_1^K introduced in (6.9), together with the natural o.s.s. on ℓ_∞^N, one can verify that the sequence of norms

$$\Big\|\sum_{x,a} T_x^a \otimes (e_x \otimes e_a)\Big\|_{M_d(\ell_\infty^N(\ell_1^K))} = \sup_x \Big\|\sum_a T_x^a \otimes e_a\Big\|_{M_d(\ell_1^K)}, \quad d \geq 1, \tag{7.5}$$

defines a suitable o.s.s. on $\ell_\infty^N(\ell_1^K)$. Moreover, a corresponding o.s.s. can be placed on $\ell_1^N(\ell_\infty^K) = (\ell_\infty^N(\ell_1^K))^*$ using duality.

[4]The only property of G that we will use in these notes is that the coefficients $G_{x,y}^{a,b}$ are non-negative.

According to the expression (6.13) for the minimal tensor norm, we conclude that

$$\|G\|_{\ell_1^N(\ell_\infty^N)\otimes_{min}\ell_1^N(\ell_\infty^K)} = \sup_{d\in\mathbb{N},\{T_x^a\},\{S_y^b\}} \Big\| \sum_{x,y;a,b} G_{x,y}^{a,b}\, T_x^a \otimes S_y^b \Big\|_{M_{d^2}} \tag{7.6}$$

Here, the supremum is taken over all $d \in \mathbb{N}$ and $T_x^a, S_y^b \in M_d$ such that

$$\text{máx}\left\{ \Big\| T : \ell_1^N(\ell_\infty^K) \to M_d \Big\|_{cb}, \Big\| S : \ell_1^N(\ell_\infty^K) \to M_d \Big\|_{cb} \right\} \le 1,$$

where the lineal map T (resp. S) is defined by $T(e_x \otimes e_a) = T_x^a$ for every x, a (resp. $S(e_y \otimes e_b) = S_y^b$ for every y, b). Note that, by definition of the operator space $\ell_1^N(\ell_\infty^K)$ as the dual operator space of $\ell_\infty^N(\ell_1^K)$, the previous "max-condition" is equivalent to

$$\text{máx}\left\{ \Big\| \sum_{x,a} T_x^a \otimes (e_x \otimes e_a) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))}, \Big\| \sum_{y,b} S_y^b \otimes (e_y \otimes e_b) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))} \right\} \le 1.$$

As for the case of the classical value it turns out that the entangled value of a two-player game equals the minimal tensor norm of the corresponding tensor. We will see in Section 8 that the corresponding result is false for general Bell functionals.

Lemma 7.2. *Given a two-player game G,*

$$\omega^*(G) = \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)}. \tag{7.7}$$

Demostración. Given a family of POVMs $\{E_x^a\}_a$ in M_d, for every $x = 1, \cdots, N$ we have

$$\Big\| \sum_{x,a} E_x^a \otimes (e_x \otimes e_a) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))} = 1.$$

According to (7.5) this follows from the fact that for every x, $\| \sum_a E_x^a \otimes e_a \|_{M_d(\ell_1^K)} = 1$. Indeed, since the map $T_x : \ell_\infty^K \to M_d$ defined by $T_x(e_a) = E_x^a$ for every a is completely positive and unital, by Lemma 6.4 $\|T_x\|_{cb} = 1$. Proceeding similarly with Bob's POVM, we deduce from (7.6) that

$$\omega^*(G) \le \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)}.$$

It remains to show the converse inequality. According to (7.6), given $\varepsilon > 0$ there exist an integer d, elements $T_x^a, S_y^b \in M_d$ satisfying

$$\Big\| \sum_{x,a} T_x^a \otimes (e_x \otimes e_a) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))} \le 1, \ \Big\| \sum_{y,b} S_y^b \otimes (e_y \otimes e_b) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))} \le 1,$$

and unit vectors $|u\rangle$, $|v\rangle$ in $\mathbb{C}^{d^2}$ such that

$$\langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} T_x^a \otimes S_y^b |v\rangle > \|G\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)} - \varepsilon.$$

By definition of the o.s.s. on ℓ_1 via duality (6.8), the condition

$$\Big\| \sum_{x,a} T_x^a \otimes (e_x \otimes e_a) \Big\|_{M_d(\ell_\infty^N(\ell_1^K))} = \sup_x \Big\| \sum_a T_x^a \otimes e_a \Big\|_{M_d(\ell_1^K)} \le 1$$

is equivalent to

$$\|T_x : \ell_\infty^K \to M_d\|_{cb} \le 1 \ \text{ for every } x, \tag{7.8}$$

where $T_x(e_a) = T_x^a$ for every a. The same bound applies to the operators S_y^b.

The main obstacle to conclude the proof is that the elements T_x^a, S_y^b are not necessarily positive, or even Hermitian. In order to recover a proper quantum strategy we appeal to the following.

THEOREM 7.3. *Let A be a C^*-algebra with unit and let $T : A \to \mathcal{B}(\mathcal{H})$ be completely bounded. Then there exist completely positive maps $\psi_i : A \to \mathcal{B}(\mathcal{H})$, with $\|\psi_i\|_{cb} = \|T\|_{cb}$ for $i = 1, 2$, such that the map $\Psi : A \to M_2(\mathcal{B}(\mathcal{H}))$ given by*

$$\Psi(a) = \begin{pmatrix} \psi_1(a) & T(a) \\ T^*(a) & \psi_2(a) \end{pmatrix}, \quad a \in A$$

is completely positive. Moreover, if $\|T\|_{cb} \leq 1$, then we may take ψ_1 and ψ_2 unital.

Theorem 7.3 is a direct consequence of [**17**, Theorem 8.3], where the same statement is proved with the map Ψ replaced by the map $\eta : M_2(A) \to M_2(\mathcal{B}(\mathcal{H}))$ given by

$$\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \psi_1(a) & T(b) \\ T^*(c) & \psi_2(d) \end{pmatrix}.$$

The complete positivity of η implies that the map Ψ defined in Theorem 7.3 is completely positive. In fact, it is an equivalence [**17**, Exercise 8.9]. (While in the moreover case η is unital, so $\|\eta\|_{cb} = 1$, in general one can only obtain $\|\Psi\|_{cb} \leq 2$.)

In our setting we take $A = \ell_\infty^K$. Since this is a commutative C*-algebra, a map $T : \ell_\infty^K \to \mathcal{B}(\mathcal{H})$ is completely positive if and only if it is positive; that is, $T(a) \in \mathcal{B}(\mathcal{H})$ is a positive element for every positive element $a \in \ell_\infty^K$. For every x, applying Theorem 7.3 to the map $T_x : \ell_\infty^K \to M_d$ defined in (7.8), we find completely positive and unital maps $\psi_x^i : \ell_\infty^K \to M_d$, $i = 1, 2$ such that the map $\Psi_x : \ell_\infty^K \to M_2(M_d)$ defined by

$$\Psi_x(a) = \begin{pmatrix} \psi_x^1(a) & T_x(a) \\ T_x^*(a) & \psi_x^2(a) \end{pmatrix}, \quad a \in \ell_\infty^K$$

is completely positive. Similarly, for every y we define $S_y : \ell_\infty^K \to M_d$ and find completely positive and unital maps $\varphi_y^i : \ell_\infty^K \to M_d$, $i = 1, 2$ and $\Phi_y : \ell_\infty^K \to M_2(M_d)$. Since these maps are positive, the element

$$\Gamma = \sum_{x,y;a,b} G_{x,y}^{a,b} \Psi_x(e_a) \otimes \Phi_y(e_b) \in M_2(M_d) \otimes M_2(M_d)$$

is positive. Consider the unit vectors $\tilde{u} = (u, 0, 0, 0) \in \mathbb{C}^{4d^2}$ and $\tilde{v} = (0, 0, 0, v) \in \mathbb{C}^{4d^2}$; we have

$$\begin{aligned} \Big|\langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} T_x^a \otimes S_y^b |v\rangle\Big| &= |\langle \tilde{u}|\Gamma|\tilde{v}\rangle| \leq |\langle \tilde{u}|\Gamma|\tilde{u}\rangle|^{\frac{1}{2}} |\langle \tilde{v}|\Gamma|\tilde{v}\rangle|^{\frac{1}{2}} \\ &= \Big|\langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} \psi_x^1(e_a) \otimes \varphi_y^1(e_b)|u\rangle\Big|^{\frac{1}{2}} \Big|\langle v| \sum_{x,y;a,b} G_{x,y}^{a,b} \psi_x^2(e_a) \otimes \varphi_y^2(e_b)|v\rangle\Big|^{\frac{1}{2}} \leq \omega^*(G). \end{aligned}$$

Here, the first inequality follows from the Cauchy-Schwarz inequality and the second inequality follows from the fact that the corresponding maps are completely positive and unital. □

Analogously to the classical case, the correspondences (6.10) and (6.15) allow us to write the quantum value of a game G as

$$\omega^*(G) = \|G : \ell_\infty^N(\ell_1^K) \times \ell_\infty^N(\ell_1^K) \to \mathbb{C}\|_{cb},$$

where G is the corresponding bilinear form defined as in (7.4).

8. Bell functionals with signed coefficients

It is sometimes interesting to consider arbitrary Bell functionals $M = \{M_{x,y}^{a,b}\}_{x,y;a,b}$, that may not directly correspond to games because of the presence of signed coefficients. As we have said before, the sets $\mathcal{P}_\mathcal{C}(K|N)$ are $\mathcal{P}_\mathcal{Q}(K|N)$ are convex and, so, their adherence (we do not know if the second set is closed) can be completely described by the hyperplanes supporting them. This means that, if our aim is to study the sets $\mathcal{P}_\mathcal{C}(K|N)$ are $\mathcal{P}_\mathcal{Q}(K|N)$, we must consider all possible hyperplanes (Bell functionals) and studying only those with non-negative coefficients is not enough. Interestingly, this additional freedom leads to phenomena with no equivalent in games. We will comment something about this point below.

The reason to introduce an absolute value in the definitions (5.3) and (5.4) of ω and ω^* respectively is precisely the possibility of considering signed coefficients. Indeed, while this absolute value is superfluous in the case of games in general it is needed for the quantity $\omega^*(M)/\omega(M)$ to be meaningful: without it this quantity could be made to take any value in $[-\infty,\infty]$ simply by shifting the coefficients $M_{x,y}^{a,b}\to M_{x,y}^{a,b}+c$. The presence of the absolute value in the definition allows one to show that the ratio $\omega^*(M)/\omega(M)$ can always be obtained as the ratio between the quantum and classical *biases*. Let us explain this point in more detail. Let M be a Bell functional given by some real arbitrary coefficients $M_{x,y}^{a,b}$ for $x,y=1,\cdots,N$; $a,b=1,\cdots,K$. Then, we can define an element G as

$$G_{x,y}^{a,b}=\frac{1}{2N^2}+\frac{1}{2N^2L}M_{x,y}^{a,b},\quad \text{for every } x,y,a,b;$$

where $L=\text{máx}_{x,y,a,b}|M_{x,y}^{a,b}|$. It is very easy to check that G has non-negative coefficients with values in $[0,1]$ (so it corresponds to a two-player game) and it verifies that

$$\frac{\beta^*(G)}{\beta(G)}=\frac{\omega^*(M)}{\omega(M)},$$

where $\beta(G)$ (resp. $\beta^*(G)$) is the classical (resp. quantum) bias of the game G defined as

$$\beta(G)=\sup\Big\{\Big|2\langle G,P\rangle-1\Big|:P\in\mathcal{P}_{\mathcal{C}}(K|N)\Big\}\ \text{ and }\ \beta^*(G)=\sup\Big\{\Big|2\langle G,P\rangle-1\Big|:P\in\mathcal{P}_{\mathcal{Q}}(K|N)\Big\}.$$

Unfortunately the use of signed coefficients makes the geometry of the problem more complicated. In particular, the correspondence between the classical and entangled values of a game and the ϵ and minimal norms of the associated tensor described in Section 7 no longer holds. As a simple example, consider $\{M_{x,y}^{a,b}\}_{x,y;a,b=1}^2$ such that $M_{1,1}^{a,b}=1=-M_{2,2}^{a,b}$ for every $a,b\in\{1,2\}$ and $M_{x,y}^{a,b}=0$ otherwise. Then, clearly $\omega(M)=\omega^*(M)=0$, but since $M\neq 0$ it must be that

$$\Big\|\sum_{x,y,a,b}M_{x,y}^{a,b}(e_x\otimes e_a)\otimes(e_y\otimes e_b)\Big\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon\ell_1^N(\ell_\infty^K)}\neq 0.$$

By considering a small perturbation of M it is possible to construct functionals $\tilde{M}$ such that $\omega(\tilde{M})\neq 0$ but the quotient $\|\tilde{M}\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon\ell_1^N(\ell_\infty^K)}/\omega(\tilde{M})$ is arbitrarily large; the same effect can be obtained for $\omega^*(M)$ with the minimal norm.

There are two different ways to circumvent this problem. The first, considered in [**11**], consists in a slight modification of the definition of the classical and entangled values $\omega(M)$ and $\omega^*(M)$. Intuitively, since payoffs may be negative it is natural to allow the players to avoid "losing" by giving them the possibility to refuse to provide an answer, or alternatively, to provide a "dummy" answer on which the payoff is always null. This leads to a notion of "incomplete" strategies which, aside from being mathematically convenient, is also natural to consider in the setting of Bell inequalities, for instance as a way to measure detector inefficiencies in experiments.[5]

Formally, a family of incomplete conditional distributions is specified by a vector $P_A=(P_A(a|x))_{x,a}\in\mathbb{R}^{NK}$ of non-negative reals such that $\sum_a P_A(a|x)\leq 1$ for every x. Define $\omega_{inc}(M)$ as in (5.3) where the supremum is extended to the convex hull of products of incomplete conditional distributions for each player. An analogous extension can be considered for the entangled bias, leading to a value $\omega_{inc}^*(M)$ obtained by taking a supremum over all distributions that can be obtained from a bipartite quantum state $|\psi\rangle\in\text{Ball}(\mathbb{C}^d\otimes\mathbb{C}^d)$ and families of positive

[5]We refer to [**11**, Section 5] for more on this.

operators $\{E_x^a\}_a$ and $\{F_y^b\}_b$ in M_d verifying $\sum_a E_x^a \leq \mathrm{Id}$ and $\sum_b F_y^b \leq \mathrm{Id}$. Note that in general it will always be the case that $\omega(M) \leq \omega_{inc}(M)$ and $\omega^*(M) \leq \omega_{inc}^*(M)$, and the example given at the beginning of the section can be used to show that the new quantities can be arbitrarily larger than the previous ones. The following lemma shows that considering incomplete distributions allows us to restore a connection with operator spaces, albeit only up to constant multiplicative factors.

LEMMA 8.1. *Let M be a Bell functional and write $M = \sum_{x,y;a,b} M_{x,y}^{a,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b) \in \mathbb{R}^{NK} \otimes \mathbb{R}^{NK}$ the associated tensor. Then*

$$\omega_{inc}(M) \leq \|M\|_{\ell_1^N(\ell_\infty^K(\mathbb{R}))\otimes_\epsilon \ell_1^N(\ell_\infty^K(\mathbb{R}))} \leq 4\,\omega_{inc}(M), \quad \text{and}$$
$$\omega_{inc}^*(M) \leq \|M\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)} \leq 4\,\omega_{inc}^*(M).$$

Here, in the first inequality the norm of M is taken over real spaces, while in the second the spaces are complex.

The first estimate in Lemma 8.1 is not hard to obtain (see [**11**, Proposition 4]), and readily extends to complex spaces with a factor of 16 instead of 4. The second estimate is proved in [**11**, Theorem 6] with a constant 16. The constant 4 stated in the theorem can be obtained by using the map Ψ introduced in Theorem 7.3 and the fact that $\|\Psi\|_{cb} \leq 2$.

Lemma 8.1 provides us with a method to translate constructions in operator space theory to Bell functionals for which the ratio of the entangled and classical values is large. Indeed, the correspondence established in the lemma implies that if M is a tensor such that $\|M\|_{min}/\|M\|_\varepsilon$ is large the associated functional M will be such that $\omega_{inc}^*(M)/\omega_{inc}(M)$ is correspondingly large (up to the loss of a factor 4). Increasing the number of possible outputs by 1, we may then add a "dummy" output for which the payoff is always zero. This results in a functional $\tilde{M}$ for which $\omega(\tilde{M}) = \omega_{inc}(M)$ and $\omega^*(\tilde{M}) = \omega_{inc}^*(M)$, so that any large violation for the incomplete values of M translates to a large violation for the values of $\tilde{M}$. Let us be more precise by stating a precise result.

PROPOSITION 8.2. [**11**, Corollary 4] *Let M be a Bell functional and $M = \sum_{x,y;a,b} M_{x,y}^{a,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b) \in \mathbb{R}^{NK} \otimes \mathbb{R}^{NK}$ the associated tensor. Let us assume that*

$$\frac{\|M\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)}}{\|M\|_{\ell_1^N(\ell_\infty^K(\mathbb{R}))\otimes_\epsilon \ell_1^N(\ell_\infty^K(\mathbb{R}))}} \geq \alpha.$$

Then, the Bell functional $\tilde{M} = (\tilde{M}_{x,y}^{a,b})_{x,y;a,b}$, where $x, y = 1, \cdots, N$ and $a, b = 1, \cdots, K+1$, obtained by adding extra zeros to the element M, verifies

$$\frac{\omega^*(\tilde{M})}{\omega(\tilde{M})} \geq C\alpha,$$

where C is a universal constant which can be taken equal to $1/4$.

DEMOSTRACIÓN. According to Lemma 8.1 we have that

$$\frac{\omega_{inc}^*(M)}{\omega_{inc}(M)} \geq C\alpha.$$

Then, the result follows by simply noting that $\omega_{inc}(M) = \omega(\tilde{M})$ and $\omega_{inc}^*(M) = \omega^*(\tilde{M})$. □

Unfortunately the lemma is not sufficient to obtain bounds in the other direction, upper bounds on the ratio $\omega^*(M)/\omega(M)$: as shown by the example described earlier, the values $\|M\|_\epsilon$

and $\omega(M)$, and $\|M\|_{min}$ and $\omega^*(M)$, are in general incomparable. In order to obtain such upper bounds a more sophisticated approach was introduced in [**10**]. Consider the space

$$\mathcal{N}_{\mathbb{K}}(K|N) = \Big\{\{R(a|x)\}_{x,a=1}^{N,K} \in \mathbb{K}^{NK} : \sum_{a=1}^{K} R(a|x) = \sum_{a=1}^{K} R(a|x') \;\forall x, x' = 1, \cdots, N\Big\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. This space is introduced to play the role of $\ell_\infty^N(\ell_1^K)$ above, while allowing a finer description of classical strategies. In particular, note that $\dim_{\mathbb{K}}(\mathcal{N}_{\mathbb{K}}(K|N)) = NK - N + 1$, while $\dim(\ell_\infty^N(\ell_1^K)) = NK$. The space $\mathcal{N}_{\mathbb{R}}(K|N)$ (resp. $\mathcal{N}_{\mathbb{C}}(K|N)$) can be endowed with a norm (resp. o.s.s.) such that the following holds.

Lemma 8.3 ([**10**]). *Let M be a Bell functional and $M = \sum_{x,y;a,b} M_{x,y}^{a,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b)$ the associated tensor, viewed as an element of $\mathcal{N}_{\mathbb{K}}(K|N)^* \otimes \mathcal{N}_{\mathbb{K}}(K|N)^*$. Then,*

$$\omega(M) = \|M\|_{\mathcal{N}_{\mathbb{R}}(K|N)^* \otimes_\epsilon \mathcal{N}_{\mathbb{R}}(K|N)^*} \qquad \textit{and} \qquad \omega^*(M) = \|M\|_{\mathcal{N}_{\mathbb{C}}(K|N)^* \otimes_{\min} \mathcal{N}_{\mathbb{C}}(K|N)^*}.$$

Lemma 8.3 shows that it is possible to describe a Banach space and an o.s.s. on it that precisely capture the classical and entangled values of arbitrary Bell functionals. Unfortunately there is a price to pay, which is that the relatively well-behaved space $\ell_1^N(\ell_\infty^K)$ is replaced by a more complex object, $\mathcal{N}_{\mathbb{K}}(K|N)^*$. We refer to [**10**, Section 5] for more details on the structure of these spaces and their use in placing upper bounds on the ratio $\omega^*(M)/\omega(M)$.

Capítulo 3

Large violations of Bell inequalities

In the previous two chapters we related the classical and entangled values of two-player games and Bell functionals to the ϵ-norm and the min-norm of the associated tensor respectively. This correspondence allows the application of tools developed for the study of these norms in operator space theory to quantify the ratio ω^*/ω, a quantity that can be interpreted as a measure of the nonlocality of quantum mechanics. For the case of two-player XOR games it was shown in Section 2 of Chapter 1 that this ratio is always bounded by a constant independent of the size of the game. As we will see in the present chapter chapter, the situation is different when we consider general two-player games. In Section 9 we discuss upper bounds on the ratio ω^*/ω that depend on the number of questions or answers in the game. We also analyze upper bounds as a function of the dimension of the Hilbert space used in the entangled strategy performed by Alice and Bob. In Section 10 we describe examples of games that come close to saturating these bounds, leading to violations that scale as the square root of the number of questions and answers. Finally, Section 11 is devoting to explain some other interesting constructions leading to large Bell violations.

9. Upper bounds on the ratio ω^*/ω in two-player games

As we mentioned previously, the results discussed in Section 7 and Section 8 can be analogously stated and proved in the general case where $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$ (without assuming the same number of inputs and outputs for Alice and Bob). In particular, given one such game we will have (see Lemma 7.1 and Lemma 7.2)

$$\omega(G) = \|G\|_{\ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}})\otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}})} \text{ and } \omega^*(G) = \|G\|_{\ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}})\otimes_{min} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}})}, \tag{9.1}$$

where G is regarded as

$$G = \sum_{x,y;a,b} G_{x,y}^{a,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b) \in \mathbb{R}^{\mathrm{X}} \otimes \mathbb{R}^{\mathrm{A}} \otimes \mathbb{R}^{\mathrm{Y}} \otimes \mathbb{R}^{\mathrm{B}}.$$

Considering here this general context is interesting because it will allow us to study the dependence of the upper bounds on the quantity ω^*/ω as a function of the parameters X, Y,A,B.

We will add another important parameter to our study. For any Bell functional M and integer d define

$$\omega_d^*(M) = \sup \Big| \sum_{x,y;a,b} M_{xy}^{ab} \langle\psi| A_x^a \otimes B_y^b |\psi\rangle \Big|, \tag{9.2}$$

where the supremum is taken over all $k \leq d$, $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^k \otimes \mathbb{C}^k)$ and families of POVM $\{A_x^a\}_a$ and $\{B_y^b\}_b$ in M_k. Clearly $(\omega_d^*(M))_d$ forms an increasing sequence that converges to $\omega^*(M)$ as $d \to \infty$. The quantity

$$\sup_M \frac{\omega_d^*(M)}{\omega(M)} \tag{9.3}$$

thus asks for the largest violation of a Bell functional achievable by states of Schmidt rank at most d.

The following result provides upper bounds for the ratio ω^*/ω as a function of the three parameters: inputs, outputs and Schmidt rank of the entangled state. Its proof is a good example of the application of estimates from the theory of operator spaces to bounds on the entangled and classical values of a multiplayer game.

PROPOSITION 9.1. *The following inequalities hold for any two-player game G and any $d \geq 1$.*

1. $\omega^*(G) \leq \text{mín}\{\mathrm{X}, \mathrm{Y}\}\, \omega(G)$,
2. $\omega^*(G) \leq K_G^{\mathbb{C}} \sqrt{\mathrm{AB}}\, \omega(G)$, *where $K_G^{\mathbb{C}}$ is the complex Grothendieck's constant.*
3. $\omega_d^*(G) \leq d\, \omega(G)$.

DEMOSTRACIÓN. The proof of each item is based on a different way of bounding the norm of the identity map

$$id \otimes id: \ \ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_{\text{mín}} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}). \tag{9.4}$$

Using (7.3) and (7.7) any such bound immediately implies the same bound on the ratio ω^*/ω.

For the first item, assume without loss of generality that $\mathrm{X} \leq \mathrm{Y}$. The identity map (9.4) can be decomposed as a sequence

$$\ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_\infty^{\mathrm{XA}} \otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_\infty^{\mathrm{XA}} \otimes_{min} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_{min} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}), \tag{9.5}$$

where all arrows correspond to the identity. It follows directly from the definition of the ϵ and the *min* norms that the first and the third arrow in (9.5) have norm 1 and X respectively. In addition, according to Lemma 6.8, the second arrow has norm 1, so that the desired result is proved by composing the three norm estimates. Motivated by this decomposition, we give a self-contained proof relating the quantum and classical values. Start with the third arrow in (9.5). Given a family of POVMs $\{E_x^a\}_a$, $x \in \mathbf{X}$, for Alice, $\{\frac{1}{\mathrm{X}} E_x^a\}_{x,a}$ can be interpreted as a single POVM with XA outcomes. Thus,

$$\omega^*(G) \leq \mathrm{X} \sup \Big| \sum_{x,y;a,b} G_{x,y}^{a,b} \, \langle \psi | E^{x,a} \otimes F_y^a | \psi \rangle \Big|,$$

where the supremum is taken over all families of POVMs $\{F_y^b\}_b$ for Bob, a single POVM $\{E^{x,a}\}_{x,a}$ for Alice with XA outputs, and all bipartite states $|\psi\rangle$. Now that Alice is performing a single measurement, she may as well apply it before the game starts and her strategy can be assumed to be classical probabilistic. Thus,

$$\omega^*(G) \leq \mathrm{X} \sup \Big| \sum_{x,y;a,b} G_{x,y}^{a,b} \, P(x,a) Q(b|y) \Big|, \tag{9.6}$$

where the supremum is taken over all $P \in \mathcal{P}_{\mathcal{C}}(\mathbf{AX})$ and $Q \in \mathcal{P}_{\mathcal{C}}(\mathbf{B}|\mathbf{Y})$. This corresponds to the second arrow in (9.5). Finally, the fact that the first map in (9.5) has norm 1 corresponds, in this setting, to the observation that the distribution P can be transformed into an element $\tilde{P} \in \mathcal{P}_{\mathcal{C}}(\mathbf{A}|\mathbf{X})$ such that $\tilde{P}(a|x) \geq P(x,a)$ for every x, a. Since G has positive coefficients this can only increase the value. Thus the supremum on the right-hand side of (9.6) is at most $\omega(G)$, completing the proof of the first item in the proposition.

The proof of the second item makes use of the Fourier transform

$$\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N, \qquad \mathcal{F}_N : e_j \mapsto \sum_{k=1}^N e^{\frac{2\pi i jk}{N}} e_k \qquad \forall j \in \{1, \dots, N\}.$$

Note that the inverse of this map is

$$\mathcal{F}_N^{-1} : \mathbb{C}^N \to \mathbb{C}^N, \qquad \mathcal{F}_N^{-1} : e_j \mapsto \frac{1}{N} \sum_{k=1}^N e^{-\frac{2\pi i jk}{N}} e_k \qquad \forall j \in \{1, \dots, N\}.$$

Then, the identity map (9.4) can be decomposed as

$$\ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_1^{\mathrm{XA}} \otimes_\epsilon \ell_1^{\mathrm{YB}} \to \ell_1^{\mathrm{XA}} \otimes_{min} \ell_1^{\mathrm{YB}} \to \ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes_{min} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}), \tag{9.7}$$

where the first arrow is $(id_{\mathrm{X}} \otimes \mathcal{F}_{\mathrm{A}}) \otimes (id_{\mathrm{Y}} \otimes \mathcal{F}_{\mathrm{B}})$, the second is the identity, and the third arrow is $(id_{\mathrm{X}} \otimes \mathcal{F}_{\mathrm{A}}^{-1}) \otimes (id_{\mathrm{Y}} \otimes \mathcal{F}_{\mathrm{B}}^{-1})$. Here, it is not hard to verify that the norm of the first and the third maps are $(\mathrm{AB})^{3/2}$ and $\frac{1}{\mathrm{AB}}$ respectively. Indeed, this follows from the estimates $\|\mathcal{F}_N : \ell_\infty^N \to \ell_1^N\| \leq N^{3/2}$, $\|\mathcal{F}_N^{-1} : \ell_1^N \to \ell_\infty^N\| \leq N^{-1}$, equations (6.16), (6.17) and the metric mapping property of the ϵ-norm (3.12) and the π-norm (3.7). At the same time, Grothendieck's theorem can be used to show that the second map in (9.7) has norm at most $K_G^{\mathbb{C}}$. Composing the three estimates proves the second item. As for the first item, we give a self-contained proof directly relating the classical and entangled values. Start with the third arrow in (9.7). Given a family of POVMs $\{E_x^a\}_a$, $x \in \mathbf{X}$, for Alice, the (not necessarily self-adjoint) operators

$$A_{x,k} = \sum_{a\in\mathbf{A}} e^{-\frac{2\pi i a k}{\mathrm{A}}} E_x^a$$

verify that $\|A_{x,k}\| \leq 1$ for every $x \in \mathbf{X}$, $k \in \mathbf{A}$. To see this, for any unit vectors $|u\rangle, |v\rangle$ write

$$\begin{aligned} \big|\langle u|A_{x,k}|v\rangle\big| &= \Big|\sum_a e^{-\frac{2\pi i a k}{\mathrm{A}}} \langle u|E_x^a|v\rangle\Big| \\ &\leq \sum_a \big|\langle u|E_x^a|v\rangle\big| \\ &\leq \Big(\sum_a \langle u|E_x^a|u\rangle\Big)^{1/2}\Big(\sum_a \langle v|E_x^a|v\rangle\Big)^{1/2} \\ &= 1, \end{aligned}$$

where the second inequality uses $E_x^a \geq 0$ and the Cauchy-Schwarz inequality, and the last follows from $\sum_a E_x^a = \mathrm{Id}$. The same transformation can be applied to obtain operators $B_{y,k'}$ from Bob's POVM, thus

$$\omega^*(G) \leq \frac{1}{\mathrm{AB}} \sup\Big|\sum_{x,y;k,k'} \Big(\sum_{a,b} G_{x,y}^{a,b} e^{\frac{2\pi i a k}{\mathrm{A}}} e^{\frac{2\pi i b k'}{\mathrm{B}}}\Big) \langle\psi|A_{x,k} \otimes B_{y,k'}|\psi\rangle\Big|,$$

where the supremum on the right-hand side is taken over all d, states $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^d \otimes \mathbb{C}^d)$ and $A_{x,k}, B_{y,k'} \in M_d$ of norm at most 1. For the next step, interpret the coefficients

$$\Big(\sum_{a,b} G_{x,y}^{a,b}\, e^{\frac{2\pi i a k}{\mathrm{A}}} e^{\frac{2\pi i b k'}{\mathrm{B}}}\Big)_{x,k;y,k'}$$

as a complex $\mathrm{XA} \times \mathrm{YB}$ matrix and apply Grothendieck's inequality (see Remark 2.5) to obtain

$$\omega^*(G) \leq \frac{1}{\mathrm{AB}} K_G^{\mathbb{C}} \sup\Big|\sum_{x,y;k,k'} \Big(\sum_{a,b} G_{x,y}^{a,b}\, e^{\frac{2\pi i a k}{\mathrm{A}}} e^{\frac{2\pi i b k'}{\mathrm{B}}}\Big) t_{x,k} s_{y,k'}\Big|, \tag{9.8}$$

where the supremum is taken over all $(t_{x,k}) \in \ell_\infty^{\mathrm{XA}}$ and $(s_{y,k'}) \in \ell_\infty^{\mathrm{YB}}$ of norm at most 1. This gives the second arrow in (9.7). To obtain the first, observe that the expression appearing inside the supremum on the right-hand side of (9.8) may be rewritten as

$$\Big|\sum_{x,y;a,b} G_{x,y}^{a,b} \Big(\sum_k e^{\frac{2\pi i a k}{\mathrm{A}}} t_{x,k}\Big)\Big(\sum_{k'} e^{\frac{2\pi i b k'}{\mathrm{B}}} s_{y,k'}\Big)\Big|.$$

For any family of complex numbers $(t_{x,k})_{x,k}$ such that $\sup_{x\in\mathbf{X},k\in\mathrm{A}} |t_{x,k}| \leq 1$ it follows from Parseval's identity and the Cauchy-Schwarz inequality that the complex numbers $P(x,a) = \sum_k \exp(2\pi i a k/\mathrm{A}) t_{x,k}$, $x \in \mathbf{X}, a \in \mathbf{A}$ satisfy $\sum_a |P(x,a)| \leq \mathrm{A}^{3/2}$ for every $x \in \mathbf{X}$. Using that the coefficients $G_{x,y}^{a,b}$ are positive, the supremum in (9.8) is at most $(\mathrm{AB})^{\frac{3}{2}}\omega(G)$, concluding the proof of the second item in the proposition.

In order to prove the third item, consider families of POVMs $\{E_x^a\}_{x,a}$, $\{F_y^b\}_{y,b}$ in M_d for Alice and Bob respectively and a pure state $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^d \otimes \mathbb{C}^d)$. Absorbing local unitaries in the POVM elements, write the Schmidt decomposition as $|\psi\rangle = \sum_{i=1}^d \lambda_i |ii\rangle$, with $\sum_{i=1}^d |\lambda_i|^2 = 1$. Thus

$$\begin{aligned} \sum_{x,y;a,b} G_{x,y}^{a,b} \langle\psi| E_x^a \otimes F_y^b |\psi\rangle &= \sum_{i,j=1}^d \lambda_i\lambda_j \sum_{x,y;a,b} G_{x,y}^{a,b} \langle i|E_x^a|j\rangle\langle i|F_y^b|j\rangle \\ &\leq d \max_{i,j} \Big| \sum_{x,y;a,b} G_{x,y}^{a,b} \langle i|E_x^a|j\rangle\langle i|F_y^b|j\rangle \Big|, \end{aligned} \tag{9.9}$$

where for the inequality we used that $|\sum_{i,j=1}^n \lambda_i\lambda_j| = |\sum_{i=1}^d \lambda_i|^2 \leq d$ since $\sum_{i=1}^d |\lambda_i|^2 = 1$. For fixed i,j and any $x \in \mathbf{X}$ we have

$$\sum_a |\langle i|E_x^a|j\rangle| \leq \sum_a |\langle i|E_x^a|i\rangle|^{\frac{1}{2}} |\langle j|E_x^a|j\rangle|^{\frac{1}{2}} \leq \Big(\sum_a |\langle i|E_x^a|i\rangle|\Big)^{\frac{1}{2}} \Big(\sum_a |\langle j|E_x^a|j\rangle|\Big)^{\frac{1}{2}} \leq 1,$$

where the first two inequalities follow from the Cauchy-Schwarz inequality and the positivity of E_x^a, and the last uses $\sum_a E_x^a = \mathrm{Id}$. The same bound applies to Bob's POVM, hence for fixed i,j, using that G has non-negative coefficients

$$\Big| \sum_{x,y;a,b} G_{x,y}^{a,b} \langle i|E_x^a|j\rangle\langle i|F_y^b|j\rangle \Big| \leq \sum_{x,y;a,b} G_{x,y}^{a,b} |\langle i|E_x^a|j\rangle||\langle i|F_y^b|j\rangle| \leq \omega(G).$$

Together with (9.9) this proves the desired estimate. □

The bounds stated in Proposition 9.1 can be extended in several ways. First, the same estimates as stated in the proposition apply to the quantities $\|M\|_\varepsilon$ and $\|M\|_{min}$, for an arbitrary tensor $M \in \ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}})$, instead of $\omega(G)$ and $\omega^*(G)$ respectively. The proof of Proposition 9.1 goes through the complex spaces $\ell_1^{\mathrm{X}}(\ell_\infty^{\mathrm{A}}) \otimes \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}})$, thus using Lemma 8.1 and the comments that follow it similar bounds can be derived that relate the quantities $\omega_{inc}(M)$ and $\omega_{inc}^*(M)$ for arbitrary Bell functionals M. Bounds (slightly weakened by a constant factor) can then be obtained for the values $\omega(M)$ and $\omega^*(M)$ by using Lemma 8.3 and the fact that the space $\mathcal{N}_{\mathbb{K}}(K|N)^*$ is "very similar" to the space $\ell_1^N(\ell_\infty^{K-1})$; we refer to [**10**, Section 5] for details on this last point.

Second, we note that the proof of the first item in the proposition can be slightly modified to show that for any game G it holds that $\omega^*(G) \leq \min\{\mathrm{A},\mathrm{B}\}\omega(G)$. The key point to show this is that

$$\big\| id \otimes id : \ell_1^{\mathrm{XA}} \otimes_\epsilon \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \to \ell_1^{\mathrm{XA}} \otimes_{\min} \ell_1^{\mathrm{Y}}(\ell_\infty^{\mathrm{B}}) \big\|_+ = 1,$$

where $\|\cdot\|_+$ denotes the norm of a map when it is restricted to positive elements. Note that this bound is always stronger than the one provided in the second item in the proposition. However, as discussed above the latter also applies to general Bell functionals, while the above bound is no longer true. Indeed, in [**16**] the authors show the existence of a family of Bell functionals (M_n) with $\mathrm{A} = 2$, $\mathrm{B} = n$, $\mathrm{X} = \mathrm{Y} = 2^n$ such that $\omega^*(M_n)/\omega(M_n) \geq C\sqrt{n}/\log^2 n$, where C is a universal constant.

Third, a slight modification in the proof of the third item of Proposition 9.1 allows to prove that $\omega_d^*(M) \leq 2d\,\omega(M)$ for every Bell functional M.

Let us finally mention that, although the previous bounds are essentially optimal in terms of the asymptotic behavior in the number of outputs and in the dimensions d (we do not know if Proposition 9.1 is optimal as a function of the number of inputs), following a different approach one can improve some of the constants. In [**13**] the author shows the upper bound $\omega_d^*(M) \leq (2d-$

1) $\omega(M)$ which slightly improves the estimate mentioned in the previous paragraph (although it does not improve the third item in Proposition 9.1 for games).

10. Lower bounds on the largest violations in two-player games

In this section we will provide an example of a family of tensors $H_n \in \ell_1^n(\ell_\infty^n) \otimes \ell_1^n(\ell_\infty^n)$ for which

$$\frac{\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_{min}\ell_1^n(\ell_\infty^n)}}{\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_\epsilon\ell_1^n(\ell_\infty^n)}} \geq C\frac{\sqrt{n}}{\log n}, \tag{10.1}$$

where here C is a universal constant. Moreover, we will show that the dimension required to lower bound the minimal norm in (10.1) is $d = n$. According to Proposition 8.2 we can immediately obtain a Bell functional JP_n from H_n with n inputs and $n+1$ outputs verifying that

$$\frac{\omega_n^*(JP_n)}{\omega(JP_n)} \geq C\frac{\sqrt{n}}{\log n}.$$

This is therefore an example of a Bell functional for which the upper bounds shown in Proposition 9.1 (which, according to the comments below the proposition, also work for general Bell functionals) are only quadratically far for all the parameters of the problem (inputs, outputs and Schmidt rank of the entangled state) at the same time. We must note, however, that our Bell functional JP_n has signed coefficients, so the values $\omega(JP_n)$ and $\omega_n^*(JP_n)$ cannot be understood as the classical and the quantum value of a game respectively, but they should be understood as the classical bias and the quantum bias of a certain two-prover game (see Section 8). In Section 11 we will comment some other examples of Bell functionals leading to large Bell violations, some of them having non-negative coefficients.

We will first prove (10.1) by using a purely mathematical point of view and, afterward, we will look at our example in a more detailed way to provide a more explicit description of the corresponding game.

Description of the tensors H_n. Let us start by considering the linear map $T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)$ given by

$$T_n : e_k \to \frac{1}{n\sqrt{\log n}} \sum_{x,a=1}^n g_{x,a}^k e_x \otimes e_a,$$

where $(g_{x,a}^k)_{k,x,a=1}^n$ is a family of independent and normalized real Gaussian random variables. Our family of tensors will be given by the elements

$$H_n(x,y,a,b) := (T_n \otimes T_n)\left(\sum_{k=1}^n e_k \otimes e_k\right) = \frac{1}{n^2 \log n} \sum_{x,y,a,b=1}^n \left(\sum_{k=1}^n g_{x,a}^k g_{y,b}^k\right) (e_x \otimes e_a) \otimes (e_y \otimes e_b) \tag{10.2}$$

in $\ell_1^n(\ell_\infty^n) \otimes \ell_1^n(\ell_\infty^n)$. Note that this tensor is given by the coefficients

$$M_{x,y}^{a,b} = \frac{1}{n^2 \log n} \sum_{k=1}^n g_{x,a}^k g_{y,b}^k \quad \text{for every} \quad x,y,a,b = 1,\cdots,n.$$

It is clear that, since H_n is defined by means of random variables, it has random coefficients. We will prove that with probability strictly greater than 0, this coefficients verify equation (10.1). In fact, one can actually prove that this happens with probability tending to one exponential fast when n tends to infinity.

Upper bound for the ϵ-norm of H_n. Chevet's inequality [**21**, Theorem 43.1] provides good upper bounds for the average of the ϵ-tensor norm of random elements.

THEOREM 10.1 (Chevet's inequality). *There exists a universal constant K such that for every pair of Banach spaces X, Y, every sequences of elements $(x_i)_i$ and $(y_j)_j$ in X and Y respectively, and every sequence $(g_{i,j})_{i,j}$ of independent and normalized Gaussian random variables, we have*

$$\mathbb{E}\Big\| \sum_{i,j} g_{i,j} x_i \otimes y_j \Big\|_{X\otimes_\varepsilon Y} \leq K\omega_2((x_i)_i; X)\mathbb{E}\Big\| \sum_j g_j y_j \Big\|_Y + K\omega_2((y_i)_i; Y)\mathbb{E}\Big\| \sum_i g_i x_i \Big\|_X.$$

Here, $\omega_2((x_i)_i; X)$ and $\omega_2((y_i)_i; Y)$ denote the ω_2-norm already introduced in the definition of the γ_2-norm (3.13).

The constant K can be taken 1 for real Banach spaces and 4 for complex Banach spaces.

Our strategy to upper bound the value $\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_\epsilon \ell_1^n(\ell_\infty^n)}$ will consist in upper bounding the norm of the map T_n and then applying the metric mapping property for the ϵ-norm.

LEMMA 10.2. *Let n be a natural number and let $(g_{i,j}^k)_{i,j,k=1}^n$ be a family of independent and normalized real gaussian variables and G_n be the linear map defined by*

$$G_n(e_k) = \sum_{i,j=1}^n g_{i,j}^k e_i \otimes e_j \quad \textit{for every} \quad k = 1, \cdots, n.$$

Then,

$$\mathbb{E}\|G_n : \ell_2^n \to \ell_2^n(\ell_\infty^n)\| \leq C_1\sqrt{n\log n},$$

where $C_1 > 0$ is a universal constant. In particular, $\mathbb{E}\|G_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\| \leq C_1 n\sqrt{\log n}$.

DEMOSTRACIÓN. According to the identification (3.10) we have that

$$\|G_n : \ell_2^n \to \ell_2^n(\ell_\infty^n)\| = \Big\| \sum_{k,i,j=1}^n g_{i,j}^k e_k \otimes (e_i \otimes e_j) \Big\|_{\ell_2^n \otimes_\epsilon \ell_2^n(\ell_\infty^n)}.$$

Chevet's inequality implies then that

$$\mathbb{E}\,\|G_n : \ell_2^n \to \ell_2^n(\ell_\infty^n)\| \leq \omega_2((e_i)_i; \ell_2^n)\ \mathbb{E}\Big\| \sum_{i,j=1}^n g_{i,j} e_i \otimes e_j \Big\|_{\ell_2^n(\ell_\infty^n)}$$
$$+\, \omega_2((e_i \otimes e_j)_{i,j}; \ell_2^n(\ell_\infty^n))\ \mathbb{E}\Big\| \sum_{i=1}^n g_i e_i \Big\|_{\ell_2^n}.$$

It is very easy to see that $\omega_2((e_i)_i; \ell_2^n) = 1$ and $\mathbb{E}\,\|\sum_{i=1}^n g_i e_i\|_{\ell_2^n} \leq \sqrt{n}$. Also, the estimate $\omega_2((e_i \otimes e_j)_{i,j}; \ell_2^n(\ell_\infty^n)) = 1$ follows easily from the (equivalent) fact that

$$\|id \otimes id : \ell_2^n(\ell_2^n) \to \ell_2^n(\ell_\infty^n)\| \leq 1.$$

Hence, it suffices to show

$$\mathbb{E}\Big\| \sum_{i,j=1}^n g_{i,j} e_i \otimes e_j \Big\|_{\ell_2^n(\ell_\infty^n)} \leq C_1'\sqrt{n\log n}$$

for some constant C_1'. To this end, we use the well-known estimate $\mathbb{E}\|\sum_{i=1}^n g_i e_i\|_{\ell_\infty^n} \leq C_1''\sqrt{\log n}$ (see e.g. [**21**, Page 15]) to conlude

$$\mathbb{E}\Big\| \sum_{i,j=1}^n g_{i,j} e_i \otimes e_j \Big\|_{\ell_2^n(\ell_\infty^n)} \leq \sqrt{n}\,\mathbb{E}\Big\| \sum_{i,j=1}^n g_{i,j} e_i \otimes e_j \Big\|_{\ell_\infty^n(\ell_\infty^n)} \leq C_1''\sqrt{n}\sqrt{\log n^2} \leq C_1'\sqrt{n}\sqrt{\log n}\,.$$

The second assertion follows trivially from

$$\|id \otimes id : \ell_2^n(\ell_\infty^n) \to \ell_1^n(\ell_\infty^n)\| \leq \sqrt{n}.$$

□

By taking into account the coefficients in the definitions of the maps T_n and G_n, Lemma 10.2 implies that

$$\mathbb{E}\|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\| \leq C_1.$$

In addition, using Markov inequality we have that

$$\mathbb{P}\Big(\|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\| > 3\mathbb{E}\|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\|\Big) \leq \frac{1}{3}.$$

That is, for $\tilde{C}_1 := 3C_1$ we have

$$\mathbb{P}\left(\|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\| \leq \tilde{C}_1\right) \geq \frac{2}{3}. \tag{10.3}$$

A direct consequence of the bounds for the norm of T_n is that, with probability larger than 2/3, we have

$$\begin{aligned} \|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_\epsilon \ell_1^n(\ell_\infty^n)} &= \left\|(T_n \otimes T_n)\left(\sum_{k=1}^n e_k \otimes e_k\right)\right\|_{\ell_1^n(\ell_\infty^n)\otimes_\epsilon \ell_1^n(\ell_\infty^n)} \\ &\leq \|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\|^2 \left\|\sum_{k=1}^n e_k \otimes e_k\right\|_{\ell_2^n \otimes_\epsilon \ell_2^n} \\ &\leq \tilde{C}_1^2, \end{aligned} \tag{10.4}$$

where in the first inequality we have used the metric mapping property of the ϵ-norm and in the last inequality we have used the previous estimate on the norm of T_n and

$$\left\|\sum_{k=1}^n e_k \otimes e_k\right\|_{\ell_2^n \otimes_\epsilon \ell_2^n} = 1,$$

which can be easily deduced by identifying the element $\sum_{k=1}^n e_k \otimes e_k$ with the identity map on ℓ_2^n.

Lower bound the min-norm of H_n. In order to lower bound the minimal norm of the element H_n, we need a couple of lemmas.

LEMMA 10.3. *Let $(g_{i,j}^k)_{i,j,k=1}^n$ be a family of independent and normalized real gaussian variables and let G_n^* be the linear map defined by*

$$G_n^*(e_i \otimes e_j) = \sum_{k=1}^n g_{i,j}^k e_k \quad \textit{for every} \quad i,j = 1, \cdots, n.$$

Then,

$$\mathbb{E}\|G_n^* : \ell_1^n(\ell_2^n) \to \ell_2^n\| \leq C_2\sqrt{n},$$

where C_2 is a universal constant. In particular, $\mathbb{E}\|G_n^* : \ell_1^n(\ell_\infty^n) \to \ell_2^n\| \leq C_2 n$.

DEMOSTRACIÓN. According to the identification (3.10) we have that

$$\|G_n^* : \ell_1^n(\ell_2^n) \to \ell_2^n\| = \left\|\sum_{i,j,k=1}^n g_{i,j}^k (e_i \otimes e_j) \otimes e_k\right\|_{\ell_\infty^n(\ell_2^n)\otimes_\epsilon \ell_2^n}.$$

Chevet's inequality implies then that

$$\mathbb{E}\,\|G_n^* : \ell_1^n(\ell_2^n) \to \ell_2^n\| \leq \omega_2((e_i \otimes e_j)_{i,j}; \ell_\infty^n(\ell_2^n))\,\mathbb{E}\Big\|\sum_{i=1}^n g_i e_i\Big\|_{\ell_2^n}$$
$$+\,\omega_2((e_i)_i; \ell_2^n)\,\mathbb{E}\Big\|\sum_{i,j=1}^n g_{i,j} e_i \otimes e_j\Big\|_{\ell_\infty^n(\ell_2^n)}.$$

Using the simple estimates mentioned in the proof of the previous lemma, it suffices to see that $\omega_2((e_i \otimes e_j)_{i,j}; \ell_\infty^n(\ell_2^n)) \leq 1$ and $\mathbb{E}\|\sum_{i,j=1}^n g_{i,j} e_i \otimes e_j\|_{\ell_\infty^n(\ell_2^n)} \leq C_2'\sqrt{n}$. Indeed, for the first one just note that $\|id \otimes id : \ell_2^n(\ell_2^n) \to \ell_\infty^n(\ell_2^n)\| \leq 1$. The other inequality

$$\mathbb{E}\Big\|\sum_{i,j=1}^n g_{i,j} e_i \otimes e_j\Big\|_{\ell_\infty^n(\ell_2^n)} = \mathbb{E}\Big\|\sum_{i,j=1}^n g_{i,j} e_i \otimes e_j\Big\|_{\ell_\infty^n \otimes_\epsilon \ell_2^n} \leq C_2''\sqrt{n}$$

follows easily from a further application of Chevet's inequality.

For the last assertion note that

$$\|id \otimes id : \ell_1^n(\ell_\infty^n) \to \ell_1^n(\ell_2^n)\| \leq \sqrt{n}.$$

□

The following lemma is a consequence of (the complex version of) Grothendieck's theorem.

Lemma 10.4. *For every natural numbers $N, K, n \in \mathbb{N}$ and every operator $T : \ell_1^N(\ell_\infty^K) \to \ell_2^n$,*

$$\big\|T : \ell_1^N(\ell_\infty^K) \to R_n\big\|_{cb} \leq K_G^{\mathbb{C}}\|T\|.$$

In fact, this result can be obtained from a weaker version of Grothendieck's inequality; the so called *little Grothendieck theorem*, leading to a slightly better (and know!) constant. In order not to add new results, we will use here the complex version of Grothendieck's inequality, already introduced in Remark 2.5.

Demostración. According to Lemma 6.8, given an operator $T : \ell_1^N(\ell_\infty^K) \to \ell_2^n$, we have

$$\|T\| = \sup_i \|T_i\| \quad \text{and} \quad \|T\|_{cb} = \sup_i \|T_i\|_{cb}$$

where T_i is the associated operator $T_i : \ell_\infty^K \to \ell_2^n$ defined by $T_i(e_j) = T(e_i \otimes e_j)$ for every $j = 1, \cdots, K$ and $i = 1, .., N$. Hence, it suffices to show that for every operator $T : \ell_\infty^K \to \ell_2^n$ we have

$$\big\|T : \ell_\infty^K \to R_n\big\|_{cb} \leq K_G^{\mathbb{C}}\,\big\|T : \ell_\infty^K \to \ell_2^n\big\|.$$

Let us use notation $T(e_i) = \sum_{j=1}^n T_{i,j} e_j$ for every $i = 1, \cdots, n$, and let us consider an arbitrary element $\sum_{i=1}^N A_i \otimes e_i$ in the uni ball of $M_d \otimes_{min} \ell_\infty^N$. That is, the operators A_i verify $\sup_{i=1,\cdots,n} \|A_i\|_{M_d} \leq 1$. Then, we will have

$$(\mathrm{Id}_{M_d} \otimes T)\left(\sum_{i=1}^N A_i \otimes e_i\right) = \sum_{j=1}^n \left(\sum_{i=1}^N T_{i,j} A_i\right) \otimes e_j.$$

According to the definition of R_n (6.2) we have

$$\begin{aligned}\left\|(\mathrm{Id}_{M_d}\otimes T)\left(\sum_{i=1}^N A_i\otimes e_i\right)\right\|_{M_d(R_n)} &= \left\|\sum_{j=1}^n\left(\sum_{i=1}^N T_{i,j}A_i\right)\left(\sum_{i=1}^N T_{i,j}A_i\right)^*\right\|_{M_d}^{\frac{1}{2}}\\ &= \left\|\sum_{j=1}^n\sum_{i,i'=1}^N T_{i,j}\bar{T}_{i',j}A_iA_{i'}^*\right\|_{M_d}^{\frac{1}{2}}\\ &= \sup_{|\psi\rangle\in\mathrm{Ball}(\ell_2^d)}\left(\sum_{i,i'=1}^N\left(\sum_{j=1}^n T_{i,j}\bar{T}_{i',j}\right)\langle\psi|A_iA_{i'}^*|\psi\rangle\right)^{\frac{1}{2}}.\end{aligned}$$

Now, if we call $u_i = A_i^*|\psi\rangle$ for every $i = 1,\cdots,N$, they are clearly (possibly complex) vectors in the unit ball of ℓ_2^d. Hence, we can apply Grothendieck's inequality to conclude that

$$\left\|(\mathrm{Id}_{M_d}\otimes T)\left(\sum_{i=1}^N A_i\otimes e_i\right)\right\|_{M_d(R_n)} \le K_G^{\mathbb{C}}\sup_{\alpha,\beta\in\mathrm{Ball}(\ell_\infty^N)}\left(\sum_{i,i'=1}^N\left(\sum_{j=1}^n T_{i,j}\bar{T}_{i',j}\right)\alpha_i\beta_{i'}\right)^{\frac{1}{2}}.$$

Finally, note that for a fixed $\alpha = (\alpha_i)_{i=1}^N \in \mathrm{Ball}(\ell_\infty^N)$,

$$T\left(\sum_{i=1}^N \alpha_i e_i\right) = \sum_{j=1}^n\left(\sum_{i=1}^N \alpha_i T_{i,j}\right)e_j := |x_\alpha\rangle \in \ell_2^n$$

verifies

$$\||x_\alpha\rangle\|_{\ell_2^n} \le \|T\|,$$

and a similar estimate hold for every β and the analogous vectors $|x_\beta\rangle$

Hence, we can finish the proof by applying Cauchy Schwartz's inequality:

$$\left\|(\mathrm{Id}_{M_d}\otimes T)\left(\sum_{i=1}^N A_i\otimes e_i\right)\right\|_{M_d(R_n)} \le K_G^{\mathbb{C}}\sup_{\alpha,\beta\in\mathrm{Ball}(\ell_\infty^N)}\left(\langle x_\alpha|x_\beta\rangle\right)^{\frac{1}{2}} \le K_G^{\mathbb{C}}\|T\|.$$

□

Lemma 10.3 and Lemma 2.4 imply that $\mathbb{E}\|G_n^* : \ell_1^n(\ell_\infty^n)\to R_n\|_{cb} \le K_G^{\mathbb{C}}C_2 n$. As before, one can invoke Markov's inequality to conclude that for $\tilde{C}_2 := 3K_G C_2$ we have

$$\mathbb{P}\left(\|G_n^* : \ell_1^n(\ell_\infty^n)\to R_n\|_{cb} \le \tilde{C}_2 n\right) \ge 2/3. \tag{10.5}$$

In particular the probability that both estimates (10.3) and (10.5) happen is larger than 1/3. Let us consider some numbers $(g_{i,j}^k(\omega))_{i,j,k=1}^n$ for which both estimates are verified. In order to understand the lower bound for the minimal norm, let us assume for a moment that

$$\frac{1}{n^2}G_n^*G_n = \mathrm{Id}_{\ell_2^n}. \tag{10.6}$$

We will prove estimate (10.1) by assuming this (in general false) fact and later we will explain how the proof can be modified to obtain our result.

According to our definition of H_n in (10.2) and the metric mapping property for the minimal norm (6.12)

$$\begin{aligned}\|(G_n^*\otimes G_n^*)(H_n)\|_{R_n\otimes_{min}R_n} &\le \|G_n^* : \ell_1^n(\ell_\infty^n)\to R_n\|_{cb}^2\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_{min}\ell_1^n(\ell_\infty^n)}\\ &\le (\tilde{C}_2)^2n^2\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_{min}\ell_1^n(\ell_\infty^n)}.\end{aligned} \tag{10.7}$$

Now, according to (10.5) and (10.6),

(10.8)
$$\begin{aligned}\frac{1}{n^2}\|(G_n^*\otimes G_n^*)(H_n)\|_{R_n\otimes_{min}R_n} &= \frac{1}{n^2}\left\|(G_n^*\otimes G_n^*)\left(T_n\otimes T_n\right)\left(\sum_{k=1}^n e_k\otimes e_k\right)\right\|_{R_n\otimes_{min}R_n}\\ &= \frac{1}{\log n}\left\|\frac{1}{n^4}(G_n^*\otimes G_n^*)\left(G_n\otimes G_n\right)\left(\sum_{k=1}^n e_k\otimes e_k\right)\right\|_{R_n\otimes_{min}R_n}\\ &= \frac{1}{\log n}\left\|\sum_{k=1}^n e_k\otimes e_k\right\|_{R_n\otimes_{min}R_n} = \frac{\sqrt{n}}{\log n}.\end{aligned}$$

We immediately deduce from (10.7) and (10.8) that

$$\|H_n\|_{\ell_1^n(\ell_\infty^n)\otimes_{min}\ell_1^n(\ell_\infty^n)} \geq \frac{1}{(\tilde{C}_2)^2}\frac{\sqrt{n}}{\log n}.$$

This estimate, joint with the upper bound (10.4), leads to the desired result.

Slight modification of H_n. The following lemma says that the operator $\frac{1}{n^2}G_n^*G_n$ in (10.6) is invertible on a large subspace with high probability. This will allow to modify our proof to avoid the use of (10.6).

LEMMA 10.5. *There exists a constant $\delta\in(0,1/2)$ with the following property: Given natural numbers $n\leq m$ and a family of independent and normalized real Gaussian random variables $(g_{ij})_{i,j=1}^{n,m}$, consider the operator $\bar{G}:\ell_2^n\to\ell_2^m$ defined by*

$$\bar{G}(e_i)=\frac{1}{\sqrt{m}}\sum_{j=1}^m g_{i,j}e_j \quad \textit{for every} \quad i=1,\cdots,n.$$

Then, "with high probability" there exist operators $v_1:\ell_2^{\delta n}\to\ell_2^n$ and $v_2:\ell_2^n\to\ell_2^{\delta n}$ such that $\|v_1\|\|v_2\|\leq 2$ and $v_2\,\bar{G}^*\bar{G}\,v_1=\mathrm{Id}_{\ell_2^{\delta n}}$. Here, with high probability means that it happens with probability tending to one exponentially fast as n and m tend to infinity.

DEMOSTRACIÓN. It is very well known (and it can be deduced from Chevet's inequality) that

$$\mathbb{E}\|\bar{G}\|_{\ell_2^n\otimes_\epsilon\ell_2^m}\leq k_1\frac{1}{\sqrt{m}}(\sqrt{n}+\sqrt{m})\leq k_1' \tag{10.9}$$

for certain universal constant k_1. It is also very easy to check that

$$\mathbb{E}\|\bar{G}\|_{\ell_2^n\otimes_2\ell_2^m}=\mathbb{E}\|\bar{G}\|_{\ell_2^{nm}}\geq k_2\sqrt{n}, \tag{10.10}$$

for a certain constant $k_2>1/\sqrt{2}$.

Moreover, an easy application of Levi's lemma for gaussian random variables (see for instance [**12**, Chapter 1]) allows us to conclude that the probability that estimates (10.9) and (10.10) hold converges to 1 exponentially fast when n and m tend to infinity. Let us fix some numbers $(g_{i,j}(\omega))_{i,j=1}^{n,m}$ for which both estimates (10.9) and (10.10) are verified. Let us define $\delta=\frac{k_2^2}{2(k_1')^2}$ and let us denote by $s_i(\bar{G})$ the i-th singular value of $\bar{G}$. Then,

$$k_2^2 n\leq\sum_{i=1}^n s_i(\bar{G})^2\leq s_1(\bar{G})^2([\delta n]-1)+s_{[\delta n]}(\bar{G})^2 n,$$

where here $[x]$ denotes the smallest entire number z such that $x \leq z$. Using that $[\delta n] - 1 \leq \delta n$, we conclude that

$$k_2^2 n \leq (k_1')^2 \delta n + s_{[\delta n]}(\bar{G})^2 n,$$

We trivially deduce that

$$s_{[\delta n]}(\bar{G})^2 \geq k_2^2 - \delta (k_1')^2 = \frac{k_2^2}{2} \geq \frac{1}{4}.$$

By definition of singular values, the fact that $s_{[\delta n]}(\bar{G}) \geq 1/2$ implies that we can invert the operator $\bar{G}^* \bar{G}$ on a subspace of dimension δn. That is, we can define operators $v_1 : \ell_2^{\delta n} \to \ell_2^n$ and $v_2 : \ell_2^n \to \ell_2^{\delta n}$ as in the statement of the lemma. □

According to the previous lemma when we consider the particular case $m = n^2$, there must exist a choice of numbers $(g_{i,j}^k(\omega))_{i,j,k=1}^n$ such that they verify estimates (10.3), (10.5) and the property given by Lemma 10.5. Let us assume that $\|v_1\| \leq 2$ and $\|v_2\| \leq 1$. Then, we can follow a similar argument to the one above for H_n, by slightly replacing the maps T_n and G_n^*. Let us be more precise: We will consider $\tilde{T}_n = T_n \circ v_1 : \ell_2^{\delta n} \to \ell_1^n(\ell_\infty^n)$ so that our final tensor is

$$\tilde{H}_n := \left(\tilde{T}_n \otimes \tilde{T}_n\right) \left(\sum_{k=1}^{\delta n} e_k \otimes e_k\right) \in \ell_1^n(\ell_\infty^n) \otimes \ell_1^n(\ell_\infty^n).$$

It is clear that, since $\|\tilde{T}_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\| \leq 2\|T_n : \ell_2^n \to \ell_1^n(\ell_\infty^n)\|$, the same calculations done before lead us to

$$\|\tilde{H}_n\|_{\ell_1^n(\ell_\infty^n) \otimes_\epsilon \ell_1^n(\ell_\infty^n)} \leq 4\tilde{C}_1^2 := \bar{C}_1, \tag{10.11}$$

At the same time, if we define $\tilde{G}_n^* := v_2 \circ G_n^* : \ell_1^n(\ell_\infty^n) \to \ell_2^{\delta n}$, we will have, by the same calculations as before, that

$$\|\tilde{G}_n^* : v_2 \circ G_n^* : \ell_1^n(\ell_\infty^n) \to R_{\delta n}\|_{cb} \leq \tilde{C}_2 n.$$

This last estimate can be then used to lower bound the min-norm of $\tilde{H}_n$ exactly in the same way we lower bounded the min-norm of H_n above. Indeed, note that

$$\frac{1}{n}\tilde{G}_n^* \circ \tilde{T}_n = \frac{1}{n^2 \log n} v_2 \circ G_n^* \circ G_n \circ v_1 = \frac{1}{\log n} \mathrm{Id}_{\ell_2^{\delta n}}.$$

Then, one obtains

$$\begin{aligned}\|\tilde{H}_n\|_{\ell_1^n(\ell_\infty^n) \otimes_{min} \ell_1^n(\ell_\infty^n)} &\geq \frac{1}{n^2 (\tilde{C}_2)^2} \|(\tilde{G}_n^* \otimes \tilde{G}_n^*)(\tilde{H}_n)\|_{R_n \otimes_{min} R_{\delta n}} \\ &= \frac{1}{(\tilde{C}_2)^2 \log n} \left\| \sum_{k=1}^{\delta n} e_k \otimes e_k \right\|_{R_n \otimes_{min} R_{\delta n}} = \frac{\sqrt{\delta}}{(\tilde{C}_2)^2} \frac{\sqrt{n}}{\log n}.\end{aligned}$$

Some simplifications. The aim of Section 10 was to explain how, motivated by ideas from Banach spaces theory and operator space theory, one can find a Bell functional with a high quotient $\omega^*(M)/\omega(M)$. It turns out that, once the object is identified, some parts of the proof can be simplified.

It is not difficult to see that we can in fact consider the element H_n, so that we do not need to modify it to get a new tensor $\tilde{H}_n$. Indeed, our upper bound for the ϵ-norm holds, as we have proved for that element. At the same time, if one analyzes the estimate (10.5) carefully, one can define a very explicit quantum strategy in dimension $n+1$ which gives a good lower bound for the min-norm of H_n. Indeed, let us define $\{E_x^a\}_a$ in M_{n+1} given by

$$E_x^a = |u_x^a\rangle\langle u_x^a|,$$

where

$$|u_x^a\rangle = \frac{1}{K\sqrt{n}}(1, g_{x,a}^1, \cdots, g_{x,a}^n) \text{ for every } x, a = 1, \cdots, n, \tag{10.12}$$

where K is a universal constant. In fact, this constant can be chosen so that the operators $\{E_x^a\}_a$ verify $\sum_{a=1}^N E_x^a \leq \mathrm{Id}$ for every x. In addition, if we consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|11\rangle + \frac{1}{\sqrt{n}}\sum_{i=2}^{n+1}|ii\rangle\Big) \in \mathrm{Ball}\big(\mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}\big), \tag{10.13}$$

the estimate (10.8) can be adapted to show that

$$\sum_{x,y,a,b=1}^{n} H_n(x,y,a,b)\langle\psi|E_x^a\otimes E_y^b|\psi\rangle \geq K_2\frac{\sqrt{n}}{\log n}. \tag{10.14}$$

The state (10.13) is very different from a maximally entangled state, and it is not known whether the latter can be used to obtain a bound similar to (10.14).

Interpretation of the Bell functional JP_n as a game. As we explained in Section 8, Bell functionals M with large quotients $\omega^*(M)/\omega(M)$ are associated to games G with large quotients $\beta^*(G)/\beta(G)$, where $\beta(G) = 2\omega(G) - 1$ and $\beta(G)^* = 2\omega(G)^* - 1$ denote the classical and entangled biases respectively. We end this section by giving an interpretation of JP_n as a two-player game R_N due to Regev [**19**], verifying

$$\frac{\beta^*(R_N)}{\beta(R_N)} \geq C\frac{\sqrt{n}}{\log n}.$$

In fact, Regev suggested the use of independent coefficients $g_{i,x,a}$ and $g_{j,y,b}$ in the construction of JP_n which leads, in particular, to some slight improvements in the estimates presented above.

In the game R_N, each player is sent a uniformly random $x, y \in \{1, \dots, n\}$ respectively. Each player must return an answer $a, b \in \{1, \dots, n+1\}$. If any of the players returns the answer $n+1$ their payoff is 1/2. Otherwise, the payoff obtained from answers a, b on questions x, y is defined to be $\frac{1}{2}+\delta\langle u_x^a, u_y^b\rangle$, where the vectors u_x^a and v_y^b are defined from independent standard Gaussian random variables as in (10.12) and δ is a suitable factor chosen so that $\delta\langle u_x^a, u_y^b\rangle \in [-1/2, 1/2]$ with high probability. It is then straightforward to verify that both the classical and entangled biases of the game R_N are linearly related to the classical and entangled values of the Bell functional JP_n.

11. Some comments about different games

We will finish these notes by mentioning several constructions which complement in different senses the results we have studied in this course. Explaining these results in detail would make these notes too long. We will provide references where details can be found.

11.1. The role of maximally entangled states. One of the open questions implicitly posed in [**10**] is to study whether the maximally entangled state (in any dimension) can be used to obtain an estimate of the form

$$\frac{\omega_n^*(JP_n)}{\omega(JP_n)} \geq C\frac{\sqrt{n}}{\log n}. \tag{11.1}$$

Furthermore, it is not known if a maximally entangled state can be used to obtain *any* Bell violation by using the functional JP_n. However, in [**10**, Section 4] the authors gave an example of a (family of) Bell functional $\tilde{JP}_n$ such that it verifies (11.1) and it has the extra property

$$\omega^*_{max}(\tilde{JP}_n) \leq C, \tag{11.2}$$

where $\omega^*_{max}(\tilde{JP}_n)$ denotes the largest value of the Bell functional $\tilde{JP}_n$ acting on any quantum conditional distributions constructed with a maximally entangled state. In particular, this result implies the existence of quantum conditional distributions which cannot be obtained by using a maximally entangled state.

It is interesting to note that any Bell functional M verifying both properties (11.1) and (11.2) must have signed coefficients due to [**10**, Theorem 10] (see comments at the beginning of Section 8).

11.2. Khot-Visnoi game. In [**3**] the authors analyze a (family of) two-player game (so it has, in particular, non-negative coefficients) which shows that the bounds provided by the second and third items in Proposition 9.1 are essentially optimal. The game was originally introduced by Khot and Vishnoi to obtain the first integrality gap between the classical value of a unique game and the value returned by its "basic semidefinite relaxation". The game, known as the Khot-Vishnoi game KV_n, has $2^n/n$ questions and n answers per player. In [**3**] it is shown that for every n,[1]

$$\frac{\omega^*_n(KV_n)}{\omega(KV_n)} \geq C\,\frac{n}{\log^2 n}, \tag{11.3}$$

where C is a universal constant.

Estimate (11.3) follows from the bounds

$$\omega(KV_n) \leq C_1\frac{1}{n} \quad \text{and} \quad \omega^*_n(KV_n) \geq C_2\frac{1}{\log^2 n},$$

where the second one can be proved by using a maximally entangled state of dimension n. We refer to the paper [**3**] for more details.

A drawback of the Khot-Vishnoi game is that it requires exponentially many questions per player, so that the violation (11.3) is very far from the upper bound provided by item 1. in Proposition 9.1.

11.3. Reducing the number of inputs in Khot-Visnoi game. A natural question raised from item 1. in Proposition 9.1 and the number of questions required in the Khot-Visnoi game is: Does there exist a (family of) Bell functional M_n with n inputs per system such that $\frac{\omega^*(M_n)}{\omega(M_n)}$ is $\Omega(n)$?

Although this question is still unknown, in [**9**] the authors introduced a method to reduce the number of inputs in (some) Bell functionals while preserving the quotient $\omega^*(M)/\omega(M)$. Then, the author applied that procedure to the Khot-Visnoi game to obtain a new (family of) Bell functional KV_n^{red} (with possible signed coefficients) with $\simeq n^8$ inputs and n outputs per player an such that

$$\frac{\omega^*_n(KV_n^{red})}{\omega(KV_n^{red})} \geq C\,\frac{n}{\log^2 n}, \tag{11.4}$$

[1] In fact, the family of Khot-Visnoi games is parametrized by an integer ℓ and we should denote $n = 2^\ell$. See [**3**] for details.

Although this result is still far from the best upper bound $O(n)$ shown in item 1. in Proposition 9.1, (11.4) shows that the use of exponentially many inputs is not needed to obtain (almost) optimal estimates in the rest of the parameters, as it could be guessed from the example in [**3**].

11.4. Large bell violations with binary inputs in one party. In [**16**] the author defined an asymmetric version of the Khot-Visnoi game KV_n^{Asym} with $\mathrm{X} = 2^n/n$, $\mathrm{Y} = 2$, $\mathrm{A} = \mathrm{B} = n$ and proved that it verifies:

$$\frac{\omega_n^*(KV_n^{Asym})}{\omega(KV_n^{Asym})} \geq C \frac{\sqrt{n}}{\log^2 n}.$$

This example is very extreme in the following sense: According to item 1. in Proposition 9.1 applied to general Bell functionals (see comments below the proof of the proposition), for any Bell functional M with binary inputs, $\mathrm{X} = 2$, $\mathrm{Y} = 2$, the quantity $\frac{\omega^*(M)}{\omega(M)}$ is uniformly upper bounded by a constant, independently of the number of outputs A and B of Alice and Bob respectively. Hence, the setting considered in [**16**] is the simplest one where one can find large Bell violations. In addition, one can also prove that large Bell violations in this restricted setting cannot occur for Bell functionals with non-negative coefficients (see [**16**] for details). This is the reason why the asymmetric version KV_n^{Asym} of the Khot-Visnoi game KV_n cannot preserve the positivity of the latter.

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