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ANILLOS DE CHOW ARITMETICOS

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Indice

INTRODUCCION

Desde la Matemática griega, Geometría y Teoría de Números han estado estrechamente vinculadas, proponiéndose mutuamente problemas y soluciones. Ejemplos clásicos de esta interrelación son la irracionalidad de $\sqrt{2}$ o la trascendencia de π.

El tema de la presente tesis es la Teoría de Arakelov, que refuerza también la relación entre Geometría y Teoría de Números, y que abre amplias perspectivas en lo que modernamente se ha llamado Geometría Aritmética.

Esta memoria se centra, más concretamente, en el estudio de los anillos de Chow aritméticos de Gillet y Soulé, con especial atención en la componente arquimediana de los mismos. Nuestro punto de partida es el estudio de las propiedades cohomológicas de las formas diferenciales C^{∞} con singularidades logarítmicas. Utilizando estas formas diferenciales, damos una definición cohomológica de las formas de Green y del producto $*$, a partir de la cual definimos unos anillos de Chow aritméticos cohomológicos. En el caso de variedades proyectivas, demostramos que esta definición coincide con la definición de anillos de Chow aritméticos debida a Gillet y Soulé. Pero, en el caso de variedades quasi-proyectivas los anillos de Chow aritméticos definidos aquí tienen mejores propiedades en relación con la teoría de Hodge.

El hecho de que la definición de formas de Green sea puramente cohomológica nos permite construir variantes de los grupos de Chow aritméticos, con buenas propiedades en función de los problemas considerados. En particular, usando corrientes, damos una definición de grupos de Chow aritméticos homológicos, que son covariantes para morfismos propios no necesariamente lisos en la fibra genérica. Estos grupos de Chow aritméticos homológicos son un módulo sobre el anillo de Chow aritmético cohomológico.

Gracias a los grupos de Chow aritméticos homológicos, damos una variante aritmética de la fórmula de Riemann-Hurwitz. Esta fórmula puede considerarse como el caso más sencillo de una posible extensión del Teorema de Riemann-Roch aritmético de Gillet y Soulé, a morfismos que no son lisos en la fibra genérica. Sin embargo, los resultados aquí presentados quedan muy lejos todavía de esta generalización.

Lo que queda de esta introducción está dividida en dos partes. En la primera de ellas haremos un resumen de la Teoría de Arakelov, centrándonos, principalmente, en la parte de esta teoría que tiene relación directa con esta memoria. Mientras que en la segunda parte, haremos una descripción más detallada del contenido de esta tesis.

§1. Una Introducción a la Teoría de Arakelov.

La analogía entre cuerpos de números y cuerpos de funciones.

Una de las ideas fundamentales de la Teoría de Arakelov es la analogía entre las variedades algebraicas proyectivas complejas, y las variedades definidas sobre un anillo de enteros algebraicos, “completadas” con alguna estructura adicional en el “infinito”.

El punto de partida de esta analogía es la similitud, observada por A. Weil ([We]), entre la fórmula de los resíduos de Cauchy y la fórmula del producto de Artin. Recordemos a continuación esta similitud. Sea X una curva proyectiva compleja y sea $\operatorname{Div}(X)$ el grupo de divisores de X. Un elemento $D \in \operatorname{Div}(X)$ es una suma finita

$$D = \sum_{p \in X(\mathbb{C})} n_p p,$$

donde los n_p son números enteros. Como la suma $\sum n_p$ es finita, el entero $\deg D = \sum n_p$ está bien definido y se denomina el grado de D.

Dada una función racional $f \in \mathbb{C}(X)$ y un punto $p \in X(\mathbb{C})$, se define la valoración de f en p, $\nu_p(f)$, como el orden de anulación de f en p. Si f presenta un polo en p, entonces $\nu_p(f)$ es el orden de anulación de f^{-1} con signo negativo. Dado que el número de ceros y polos de una función racional, así como sus órdenes, es finito, tenemos un divisor bien definido:

$$\operatorname{div}(f) = \sum_{p \in X(\mathbb{C})} \nu_p(f) p.$$

Como la valoración $\nu_p(f)$ es el resíduo de la forma diferencial df/f en p, la fórmula de los resíduos de Cauchy implica que

$$\text{(FR)} \qquad \deg(\operatorname{div} f) = \sum_{p \in X(\mathbb{C})} \nu_p(f) = \sum_{p \in X(\mathbb{C})} \operatorname{Res}_p \left(\frac{df}{f} \right) = 0.$$

Por tanto, si denotamos por $\operatorname{Rat}(X)$ el subgrupo de $\operatorname{Div}(X)$ generado por los elementos de la forma $\operatorname{div}(f)$, y por $\operatorname{CH}^1(X) = \operatorname{Div}(X)/\operatorname{Rat}(X)$ el grupo de clases de divisores de X, el grado nos determina un morfismo

$$\deg : \operatorname{CH}^1(X) \longrightarrow \mathbb{Z}.$$

Veamos ahora el análogo aritmético. Sea K un cuerpo de números y sea $\mathcal{O}_K$ su anillo de enteros. Denotaremos por $X = \operatorname{Spec} \mathcal{O}_K$ el conjunto de ideales primos de $\mathcal{O}_K$. Si $\mathfrak{p} \in X$ es un ideal primo no nulo, entonces el anillo localizado $(\mathcal{O}_K)_\mathfrak{p}$ es un anillo de valoración discreta. Esto implica que, para cada elemento $f \in K^*$, podemos definir su orden en $\mathfrak{p}$. Denotaremos este orden por $\nu_\mathfrak{p}(f)$ y diremos que $\nu_\mathfrak{p}$ es la valoración $\mathfrak{p}$-ádica.

Si consideramos únicamente estas valoraciones no obtenemos ninguna fórmula similar a la fórmula de los resíduos (FR). Esto es debido a que $X = \operatorname{Spec} \mathcal{O}_K$ no es una variedad "completa" sino que es el análogo de una curva afín.

A partir de una valoración $\mathfrak{p}$-ádica podemos obtener un valor absoluto escribiendo

$$\|x\|_\mathfrak{p} = (\sharp \mathcal{O}_K/\mathfrak{p})^{-\nu(x)}.$$

De esta forma obtenemos, salvo equivalencia, los valores absolutos no arquimedianos de K. Sea M_K^0 el conjunto de todos estos valores absolutos no arquimedianos.

Por otro lado, podemos obtener otro tipo de valores absolutos mediante las distintas inmersiones de K en $\mathbb{C}$. Sea $\sigma : K \longrightarrow \mathbb{C}$ una tal inmersión. Entonces escribiremos

$$\|x\|_\sigma = \|\sigma(x)\|.$$

Por este procedimiento obtenemos, también salvo equivalencia, los valores absolutos arquimedianos. Denotaremos por M_K^∞ el conjunto de todas las inmersiones complejas de K. Nótese que dos inmersiones complejas conjugadas dan lugar al mismo valor absoluto.

Sea ahora $M_K = M_K^0 \cup M_K^\infty$. Como hemos visto, cada elemento $v \in M_K$ define un valor absoluto $\|\cdot\|_v$. Si $v \in M_K$ y $x \in K^*$, escribiremos

$$v(x) = \log \|x\|_v,$$

así, en el caso de una valoración $\mathfrak{p}$-ádica se tiene

$$v(x) = -\nu(x) \log(\sharp \mathcal{O}_K/\mathfrak{p}).$$

Con estas notaciones, el análogo de la fórmula de los resíduos (FR) es la fórmula del producto de Artin que, en su versión aditiva, afirma que para todo $x \in K^*$,

$$\text{(FP)} \qquad \sum_{v \in M_K} v(x) = 0.$$

Gracias a esta fórmula, se puede definir un grupo de clases de divisores, $\widehat{\mathrm{CH}}^1(X)$, que posee una morfismo con valores reales:

$$\widehat{\deg} : \widehat{\mathrm{CH}}^1(X) \longrightarrow \mathbb{R},$$

que denominaremos grado aritmético.

La analogía entre los cuerpos de números y las curvas proyectivas complejas se puede llevar mucho más lejos. Así, uno de los teoremas fundamentales de la teoría de curvas, el Teorema de Riemann-Roch, tiene un análogo aritmético que denominaremos el Teorema de Riemann-Roch Aritmético y, por ejemplo, el Teorema de Dirichlet, sobre la finitud del grupo de clases de ideales del anillo de enteros de un cuerpo de números, se puede interpretar en términos de este Teorema de Riemann-Roch Aritmético (véase [Sz]).

De esta analogía se desprende que los valores absolutos arquimedianos de K o, lo que es lo mismo, las inmersiones complejas de K, juegan el papel de los puntos del infinito de la variedad afín $\operatorname{Spec} \mathcal{O}_K$, y nos permiten "completarla". Así, una de las posibles definiciones que se puede dar de variedad aritmética es: un esquema X regular, de tipo finito y plano sobre el anillo de enteros de un cuerpo de números $\mathcal{O}_K$, junto con la variedad algebraica compleja X_∞, obtenida mediante las distintas inmersiones de K en $\mathbb{C}$. Siempre que definamos un objeto sobre una variedad aritmética tendremos dos partes, una definida sobre $\mathcal{O}_K$, que denominaremos la componente no arquimediana y otra definida sobre $\mathbb{C}$, que denominaremos la componente arquimediana.

Las alturas de Weil.

Otra manifestación de la misma idea puede encontrarse en la teoría de alturas. Recordemos que, en su tesis, A. Weil generalizó el Teorema de Mordell, sobre la generación finita del grupo de puntos racionales de una curva elíptica, a variedades abelianas de dimensión superior. Para ello, introdujo unos conceptos que, modernamente, se han desarrollado en lo que se conoce como teoría de alturas, y que vamos a describir a continuación (véase [Si]).

Sea K, como antes, un cuerpo de números y $M_K = M_K^0 \cup M_K^\infty$ el conjunto formado por las valoraciones $\mathfrak{p}$-ádicas y las inmersiones complejas de K. La altura (logarítmica) de un punto $p = (x_0 : \cdots : x_n)$ del espacio proyectivo $\mathbb{P}^n(K)$ se define por

$$h(p) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \max_{x_i \neq 0} \{v(x_i)\}.$$

Gracias a la fórmula del producto, la altura de un punto es independiente de las coordenadas homogéneas que lo representan. Además, si L es una extensión finita de K, la altura de p, considerado como un punto de $\mathbb{P}^n(L)$, coincide con la anterior. Así, se obtiene que la aplicación altura esta definida sobre $\mathbb{P}^n(\overline{\mathbb{Q}})$:

$$h : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

Obsérvese que la definición de altura tiene también una componente arquimediana y una componente no arquimediana.

La altura de un punto mide su "complejidad aritmética". Por ejemplo, si $p \in \mathbb{P}^n(\mathbb{Q})$, podemos encontrar unas coordenadas homogéneas para p, $(x_0 : \cdots : x_n)$, con los x_i enteros y tal que $\operatorname{mcd}(x_0, \ldots, x_n) = 1$. Entonces

$$h(p) = \log(\max_i \{\|x_i\|\}).$$

Sea ahora V una variedad proyectiva definida sobre $\overline{\mathbb{Q}}$. Para definir la altura de un punto $p \in V$, podemos elegir un morfismo $F : V \longrightarrow \mathbb{P}^n$ y poner

$$h_F(p) = h(F(p)).$$

Naturalmente, esta altura depende del morfismo F y es trivial si F es constante.

Examinemos la relación entre alturas y haces inversibles. Un morfismo $F : V \longrightarrow \mathbb{P}^n$ determina un haz inversible $\mathcal{L} = F^*\mathcal{O}(1)$, pero el mismo haz inversible puede provenir de distintos morfismos. Sea $G : V \longrightarrow \mathbb{P}^m$ otro morfismo, con $G^*\mathcal{O}(1)$

isomorfo a $\mathcal{L}$. En general, las alturas obtenidas mediante F y G no coinciden. Sin embargo, un teorema de Weil afirma que su diferencia es una función acotada. Es decir, la altura $h_{\mathcal{L}}$, asociada a un haz inversible $\mathcal{L}$, está bien definida módulo funciones acotadas. Además, si notamos

$$\mathcal{H}(V) = \left\{ \begin{array}{c} \text{funciones} \\ f : V \longrightarrow \mathbb{R} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{funciones} \\ \text{acotadas} \end{array} \right\},$$

se obtiene un morfismo de grupos

$$\begin{array}{ccc} \operatorname{Pic} V & \longrightarrow & \mathcal{H}(V) \\ \mathcal{L} & \longmapsto & h_{\mathcal{L}}. \end{array}$$

En este punto, cabe preguntarse si dado un haz inversible $\mathcal{L}$, existe una función altura canónica $\widehat{h}_{\mathcal{L}}$, de tal forma que, si extendemos por linealidad $\widehat{h}_{\mathcal{L}}$ a $\mathrm{Z}_0(V)$, el grupo de cero-ciclos de V, se obtenga una aplicación bilineal

$$\begin{array}{ccc} \operatorname{Pic}(V) \otimes \mathrm{Z}_0(V) & \longrightarrow & \mathbb{R} \\ \mathcal{L} \otimes \alpha & \longmapsto & \langle \mathcal{L}, \alpha \rangle = \widehat{h}_{\mathcal{L}}(\alpha). \end{array}$$

Por otra parte, en el caso de que exista una aplicación bilineal como la anterior, cabe preguntarse si desciende a una apareamiento bilineal entre $\mathrm{CH}^1(V)$, el grupo de Chow de divisores de V y $\mathrm{CH}_0(V)$, el grupo de Chow de cero-ciclos de V.

El apareamiento de Néron y Tate.

En el caso de variedades abelianas, y por tanto en el caso de curvas vía la jacobiana de la curva, Néron ([Ne]) y Tate dan una respuesta a las preguntas anteriores. Vamos a explicar brevemente sus resultados (véase también [La 1], [La 2] y [Gro]).

Sea A una variedad abeliana definida sobre $\overline{\mathbb{Q}}$. En este caso, si $\mathcal{L}$ es un haz inversible, cualquier altura $h_{\mathcal{L}}$ asociada a $\mathcal{L}$ se puede descomponer, de forma única, como

$$h_{\mathcal{L}} = q_{\mathcal{L}} + l_{\mathcal{L}} + O(1),$$

donde $q_{\mathcal{L}}$ es una función cuadrática, $l_{\mathcal{L}}$ una función lineal y $O(1)$ una función acotada. Por tanto, la altura $\widehat{h}_{\mathcal{L}} = q_{\mathcal{L}} + l_{\mathcal{L}}$ está unívocamente determinada por $\mathcal{L}$ y se denomina la altura canónica o de Néron-Tate. Esta función sólo depende de la clase de isomorfismo de $\mathcal{L}$. Además, la asignación $\mathcal{L} \longmapsto \widehat{h}_{\mathcal{L}}$ es un morfismo de grupos entre $\operatorname{Pic}(A)$ y el conjunto de funciones reales sobre A. Por tanto, tenemos una respuesta afirmativa a la primera de las preguntas anteriores.

El carácter cuadrático de $\widehat{h}_{\mathcal{L}}$ indica que la segunda pregunta, tal como está planteada, tiene respuesta negativa. Sin embargo, si $\mathcal{L}$ es algebraicamente equivalente a cero, entonces $q_{\mathcal{L}} = 0$, es decir, $\widehat{h}_{\mathcal{L}}$ es una función lineal. Por tanto, si denotamos por $\operatorname{Pic}^0(A)$ el grupo de haces inversibles de A algebraicamente equivalentes a cero, se obtiene una aplicación bilineal:

$$\begin{array}{ccc} \operatorname{Pic}^0(A) \times A & \longrightarrow & \mathbb{R} \\ (\mathcal{L}, x) & \longmapsto & \langle \mathcal{L}, x \rangle = \widehat{h}_{\mathcal{L}}(x), \end{array}$$

que se denomina el apareamiento por las alturas global (global height pairing).

Si ahora denotamos por $\mathrm{CH}^1(A)_0$ el grupo de Chow de divisores algebraicamente equivalentes a cero, y por $\mathrm{CH}_0(A)_0$ el grupo de Chow de cero-ciclos de A de grado cero, el apareamiento por las alturas induce una aplicación bilineal

$$\mathrm{CH}^1(A)_0 \otimes \mathrm{CH}_0(A)_0 \longrightarrow \mathbb{R}.$$

Veremos más adelante que, con ciertas condiciones, este apareamiento se puede extender a variedades no necesariamente abelianas. Para llegar a esta extensión, es importante recordar la expresión de Néron del apareamiento por las alturas, pues es aquí donde se realiza la conexion entre el apareamiento global y sus componentes locales, tanto arquimedianas como no arquimedianas.

Sea $\mathcal{L} \in \mathrm{Pic}^0(A)$, X un divisor en la clase de equivalencia lineal determinada por $\mathcal{L}$, y $\alpha \in \mathrm{Z}_0(A)_0$ un cero-ciclo de A de grado cero, cuyo soporte es disjunto con el de X. Si K es un cuerpo de números sobre el que A, α y X están definidos, entonces se tiene

$$\langle \mathcal{L}, \alpha \rangle = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \langle X, \alpha \rangle_v .$$

Los símbolos $\langle X, \alpha \rangle_v$ están definidos siempre que X y α sean disjuntos y están unívocamente determinados por las siguientes propiedades:

1) $\langle X, \alpha \rangle_v$ es bilineal.
2) Si $X = \operatorname{div} f$, es el divisor de una función racional entonces

$$\langle X, \alpha \rangle_v = v(f(\alpha)).$$

3) Es invariante por traslaciones.
4) Cumple una condición de acotación que no precisaremos.

Estos símbolos se construyen del siguiente modo. Si $v \in M_K^0$, K_v es el completado de K respecto de $\| \cdot \|_v$, y $\mathcal{O}_{K_v}$ es el anillo de valoración discreta de K_v, el símbolo $\langle X, \alpha \rangle_v$ se construye a partir del producto de intersección en el modelo de Néron de A sobre $\mathcal{O}_{K_v}$. Mientras que, si $\sigma \in M_K^\infty$, la construcción es la siguiente. A traves de la inmersión compleja σ de K, podemos suponer que A es una variedad compleja, que $\mathcal{L}$ es un haz inversible sobre A algebraicamente equivalente a cero, que $X = \operatorname{div} s$, con s una sección racional de $\mathcal{L}$ y que α es un cero-ciclo de A de grado cero. Escojamos una métrica hermítica $\| \cdot \|$ en $\mathcal{L}$ y consideremos la función $g'_X = -\log \|s\|^2$. La forma diferencial $\omega = \partial\overline{\partial} g'_X$ es C^∞ en todo A, es cerrada y representa la primera clase de Chern de $\mathcal{L}$. Al ser X algebraicamente equivalente a cero, la clase de cohomología de ω es cero y así, por el lema $\partial\overline{\partial}$, existe una función real f, C^∞ en todo A, tal que $\partial\overline{\partial} f = \omega$. Escribamos $g_X = g'_X - f$. Esta función esta unívocamente determinada por X salvo una constante. Por tanto, como α es de grado cero, el número real $g_X(\alpha)$ no depende de esta constante. La componente arquimediana correspondiente a σ es

$$\langle X, \alpha \rangle_\sigma = g_X(\alpha) \in \mathbb{R}.$$

A la función g_X se le denomina función de Green para el divisor X.

La teoría de Arakelov.

Supongamos ahora que V es una variedad aritmética proyectiva, es decir un esquema regular, de tipo finito, proyectivo y plano sobre $\mathcal{O}_K$. Denotaremos por

$V_K = V \otimes K$, la correspondiente variedad algebraica definida sobre K, y por V_∞ la variedad compleja definida por las inmersiones de K en $\mathbb{C}$. En este caso, un morfismo

$$F : V \longrightarrow \mathbb{P}^n_{\mathcal{O}_K}$$

determina un haz inversible $\mathcal{L} = F^*\mathcal{O}(1)$, que está definido, no sólo sobre V_K, sino en todo V. A su vez, $\mathcal{L}$ induce un haz inversible $\mathcal{L}_\infty$ sobre V_∞.

Otro dato que podemos extraer del morfismo F es una métrica hermítica $\| \cdot \|$ en $\mathcal{L}_\infty$, inducida por la métrica estandar de $\mathbb{C}^{n+1}$.

El interés de considerar el haz inversible $\mathcal{L}$ definido sobre toda la variedad aritmética V, junto con la métrica hermítica $\|\cdot\|$, es que la función altura h_F, asociada al morfismo F, está unívocamente determinada por el par $(\mathcal{L}, \| \cdot \|)$.

Con este punto de vista, la función altura se puede interpretar del siguiente modo. Sea p un punto de $V(K)$. Este punto determina un morfismo $\varphi : \operatorname{Spec} \mathcal{O}_K \longrightarrow V$. Además, el haz inversible metrizado $\varphi^*(\mathcal{L}, \| \cdot \|)$ determina un elemento $x \in \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)$. Entonces la altura de p es, esencialmente, el grado aritmético de x. De este hecho se desprende que, si tuvieramos una noción de "divisor aritmético" X correspondiente al par $(\mathcal{L}, \| \cdot \|)$, entonces la altura de p se podría interpretar como una multiplicidad de intersección. En consecuencia, una teoría de intersección aritmética puede proporcionar interesantes resultados, sobre todo si tiene una fuerte analogía con la teoría de intersección geométrica.

En el caso de superficies aritméticas, S. J. Arakelov ([A]) ha desarrollado una teoría de intersección aritmética como la anterior. Para ello considera, junto a una superficie aritmética V, el dato de una forma de Kähler ω, anti-invariante por conjugación compleja, sobre las componentes de V_∞. El par $\overline{V} = (V, \omega)$ se denomina superficie de Arakelov. Una introducción a la Teoría de Intersección de Arakelov se puede encontrar en [La 3].

Para tener una buena noción de equivalencia racional, Arakelov tuvo la idea de definir el grupo de divisores de $\overline{V}$ como

$$\operatorname{Div}(\overline{V}) = \operatorname{Div}(V) \oplus \operatorname{Div}_\infty(V).$$

Siendo $\operatorname{Div}(V)$ el grupo de divisores de V, y $\operatorname{Div}_\infty(V)$ el grupo de divisores en el infinito, que está definido por

$$\operatorname{Div}_\infty(V) = \bigoplus_{\sigma \in M_K^\infty} \mathbb{R} V_\sigma.$$

Es decir, un divisor en el infinito es una suma formal, con coeficientes reales, de las variedades complejas $V_\sigma = V \underset{\sigma}{\otimes} \mathbb{C}$.

Sea $f \in k(V)^*$ una función racional. Esta función define un divisor $\operatorname{div} f \in \operatorname{Div}(V)$. Además, para cada $\sigma \in M_K^\infty$, f determina una función $f_\sigma \in k(V_\sigma)^*$. Designaremos por ω_σ la restricción de ω a X_σ, y escribiremos

$$\gamma_\sigma(f) = \int_{V_\sigma} - \log \|f_\sigma\|^2 \omega_\sigma.$$

Entonces el divisor de Arakelov de f está definido por

$$\overline{\operatorname{div}}(f) = \operatorname{div} f + \sum_{\sigma \in M_K^\infty} \gamma_\sigma(f) V_\sigma.$$

El subgrupo de $\operatorname{Div}(\overline{V})$ generado por los elementos de la forma $\overline{\operatorname{div}}(f)$ se denotará como $\operatorname{Rat}(\overline{V})$, y el grupo de clases de divisores de Arakelov se define por

$$\mathrm{CH}^1(\overline{V}) = \operatorname{Div}(\overline{V}) / \operatorname{Rat}(\overline{V}).$$

Vamos a dar una idea de como se define un producto de intersección en este grupo. El grupo $\operatorname{Div}(V)$ se puede descomponer en una suma directa de dos subgrupos. El primero, generado por las subvariedades que son planas sobre $\mathcal{O}_K$, se denomina el grupo de divisores horizontales. El segundo, generado por las subvariedades contenidas en alguna fibra del morfismo estructural $V \longrightarrow \operatorname{Spec} \mathcal{O}_K$, se denomina el grupo de divisores verticales. Tenemos, por tanto, tres tipos de divisores: divisores en el infinito, horizontales y verticales.

El producto de intersección de un divisor en el infinito por otro divisor en el infinito, o por un divisor vertical, es cero. Si α es un divisor horizontal, denotaremos por α_σ el divisor que determina α en V_σ. Si $\lambda V_\sigma \in \operatorname{Div}_\infty(V)$, con $\lambda \in \mathbb{R}$, y α es un divisor horizontal, entonces se define

$$\langle \alpha, \lambda V_\sigma \rangle = \lambda \deg(\alpha_\sigma).$$

Sean ahora $\alpha, \beta \in \operatorname{Div}(V)$ dos divisores horizontales sin componentes comunes. Se define su producto de intersección por

$$\langle \alpha, \beta \rangle = \deg(\alpha \cdot \beta) + \sum_{\sigma \in M_K^\infty} \langle \alpha_\sigma, \beta_\sigma \rangle_{\omega_\sigma},$$

donde la componente no arquimediana $\alpha \cdot \beta$ es un producto de intersección en teoría de esquemas y puede calcularse, por ejemplo, mediante la fórmula de los Tores de Serre.

El dato de la forma de Kähler ω es el que permite definir las componentes arquimedianas. Sea $C = V_\sigma$ una superficie de Riemann provista de una métrica de Kähler $\omega = \omega_\sigma$. Sea p un punto de C. Se define la función de Green de p con respecto de ω como la función real, C^∞ sobre $C - \{p\}$, determinada unívocamente por las condiciones

(G1) $dd^c g_p = \lambda\omega$ en $C - \{p\}$, donde $\lambda \in \mathbb{R}$.

(G2) Si z es un parámetro local de C alrededor de p, entonces

$$g_p(z) = -\log z\overline{z} + \varphi,$$

donde φ es una función C^∞ en un entorno de p.

(G3) $\displaystyle\int_C g_p \omega = 0.$

Si $\alpha = \sum n_p p$ y $\beta = \sum m_q q$ son dos divisores disjuntos de C, se define el producto de α y β como

$$\langle \alpha, \beta \rangle_\omega = g_\alpha(\beta) = \sum_{p,q} n_p m_q g_p(q).$$

Se puede demostrar que este producto es conmutativo.

Por último, el producto de dos divisores verticales también queda determinado por el producto de intersección en teoría de esquemas.

De esta forma se construye un producto de intersección conmutativo

$$\mathrm{CH}^1(\overline{V}) \otimes \mathrm{CH}^1(\overline{V}) \dot{\to} \mathbb{R},$$

que tiene propiedades análogas al producto de intersección en superficies proyectivas complejas. Esta analogía se ha materializado en los trabajos de Arakelov, Faltings ([Fa 2]), Hriljac ([Hr]), etc., obteniéndose los análogos, en este contexto, de la Fórmula de Adjunción, del Teorema del Indice de Hodge y del Teorema de Riemann-Roch.

Veamos ahora que papel juegan los haces inversibles provistos de una métrica hermítica, en esta teoría. Sea $\mathcal{L}$ un haz inversible sobre V, y sea $\|\cdot\|$ una métrica hermítica sobre $\mathcal{L}_\infty$. Diremos que la métrica $\|\cdot\|$ es admisible si su forma de curvatura es un múltiplo de la forma de Kähler ω. Es decir, sea s una sección racional de $\mathcal{L}$, la métrica $\|\cdot\|$ es admisible si existe un número real λ tal que

$$dd^c(-\log \|s\|^2) = \lambda\omega.$$

Dado un haz inversible $\mathcal{L}$, con una métrica admisible $\|\cdot\|$, y una sección racional s, pondremos

$$\gamma_\sigma(s) = \int_{V_\sigma} -\log \|s\|^2 \omega_\sigma.$$

Entonces, el divisor de Arakelov asociado a la sección s es

$$\overline{\mathrm{div}} s = \mathrm{div}\, s + \sum_{\sigma \in M_K^\infty} \gamma_\sigma(s) V_\sigma.$$

La clase de este divisor en $\mathrm{CH}^1(\overline{V})$ no depende de la sección s. Esta construcción induce un isomorfismo entre el grupo de clases de isometría de haces inversibles, provistos de métricas hermíticas admisibles, y el grupo de clases de divisores de Arakelov. En particular, el producto de intersección de Arakelov se puede interpretar como un apareamiento entre haces inversibles provistos de métricas admisibles.

Las alturas se pueden recuperar a partir del producto de intersección de Arakelov. Sea $(\mathcal{L}, \|\cdot\|)$ un haz inversible sobre V provisto de una métrica admisible. Sea $x \in \mathrm{CH}^1(\overline{V})$ el divisor aritmético que determina y sea h la función altura asociada a $(\mathcal{L}, \|\cdot\|)$. Un punto $p \in V(K)$ determina, a su vez, un divisor $y \in \mathrm{CH}^1(\overline{V})$. Entonces, se tiene

$$h(p) = x \cdot y.$$

En [De 2] Deligne mostró que el apareamiento de Arakelov se podía extender a todo el grupo de clases de isometría de haces inversibles hermíticos sobre V, con lo que se podía eliminar la elección a priori de una forma de Kähler ω.

Teoría de Arakelov en dimensión superior.

En [Be 2] y [Bl 2], Beilinson y Bloch generalizan el apareamiento por las alturas de Néron y Tate a variedades de dimensión superior. En concreto, sea X una variedad proyectiva y lisa de dimensión N definida sobre K y sea $\mathrm{CH}^*(X)_0$ el subgrupo del grupo de Chow formado por los ciclos que son homológicamente equivalentes a cero en X_∞. Entonces, con ciertas condiciones, existe una aplicación bilineal

$$\mathrm{CH}^p(V)_0 \otimes \mathrm{CH}^{N-p+1}(X)_0 \longrightarrow \mathbb{R}$$

que extiende el apareamiento por las alturas de Néron y Tate. Esta aplicación bilineal también se descompone en suma de componentes locales, arquimedianas y no arquimedianas.

Por otra parte, Gillet y Soulé generalizan, en [G-S 1], los trabajos de Arakelov a variedades de dimensión superior, con una construcción que también requiere la elección de una métrica de Kähler en X_∞.

En [G-S 2], los mismos autores, tomando el punto de vista de Deligne, desarrollan una teoría de intersección que no requiere la elección de una métrica de Kähler en V_∞. Vamos a describir brevemente esta teoría.

Sea, como antes, K un cuerpo de números y $\mathcal{O}_K$ su anillo de enteros. Sea X una variedad aritmética sobre $\operatorname{Spec}\mathcal{O}_K$.

Sea Z un ciclo algebraico de X de codimensión p y sea g una corriente real de tipo $(p-1,p-1)$ en X_∞. Se dice que g es una corriente de Green para el ciclo Z si satisface la ecuación

$$\text{(CG)} \qquad dd^c g + \delta_Z = \alpha,$$

donde α es una forma diferencial C^∞ en X_∞ y δ_Z es la corriente de integración a lo largo de Z_∞. Escribiremos $\widetilde{g}$ para denotar la clase de equivalencia de g módulo $\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}$. Se define un ciclo aritmético como un par $(Z,\widetilde{g})$, con Z un ciclo algebraico y g una corriente de Green para Z. Denotaremos por $\widehat{\mathrm{Z}}^p(X)$ al grupo de ciclos aritméticos de X.

Sea $j : W \hookrightarrow X$ una subvariedad de codimensión $p+1$ y sea $f \in k(W)^*$ una función racional. Entonces la corriente $j_*[-\log\|f\|^2]$ es una corriente de Green para el ciclo $\operatorname{div} f$, pues, por la fórmula de Poincaré-Lelong, se tiene:

$$dd^c j_*[-\log\|f\|^2] + \delta_{\operatorname{div} f} = 0.$$

Se define el divisor aritmético de f como

$$\widehat{\operatorname{div}}\, f = (\operatorname{div} f, j_*[-\log\|f\|^2]).$$

Sea $\widehat{\mathrm{Rat}}^p(X)$ el subgrupo de $\widehat{\mathrm{Z}}^p(X)$ generado por los elementos de la forma $\widehat{\operatorname{div}}\, f$. Los grupos de Chow aritméticos de Gillet-Soulé de X se definen como

$$\widehat{\mathrm{CH}}^p(X) = \widehat{\mathrm{Z}}^p(X)/\widehat{\mathrm{Rat}}^p(X).$$

Recordemos las primeras propiedades de estos grupos. Sea $f : X \longrightarrow Y$ un morfismo propio de variedades aritméticas. Si (Z,g) es un ciclo aritmético, entonces se tiene un ciclo bien definido $f_*(Z)$ y una corriente f_*g. Además, por la ecuación (CG), esta corriente satisface la ecuación de corrientes

$$dd^c f_*g + \delta_{f_*Z} = f_*\alpha.$$

Pero, a menos que el morfismo inducido $f_\infty : X_\infty \longrightarrow Y_\infty$ sea liso, no podemos asegurar que $f_*\alpha$ sea una forma C^∞ en todo Y_∞. En consecuencia, los grupos de Chow aritméticos son covariantes con respecto a morfismos propios que son lisos en la fibra genérica.

Para poder definir imágenes inversas y productos de intersección, se demuestra que una corriente de Green g, para el ciclo Z, se puede representar mediante una forma diferencial, que también denotaremos con la letra g, que es C^∞ sobre $X_\infty - Z_\infty$ y que tiene singularidades de tipo logarítmico a lo largo de Z.

Veamos como se define la componente arquimediana del producto de intersección. Sean (Y, g_Y) y (Z, g_Z) ciclos aritméticos tales que Y_∞ y Z_∞ se cortan propiamente, es decir

$$\operatorname{codim} Z_\infty \cap Y_\infty = \operatorname{codim} Y_\infty + \operatorname{codim} Z_\infty.$$

Podemos suponer que g_Z es una forma diferencial con singularidades logarítmicas. Entonces la corriente $\delta_Y \wedge g_Z$ está bien definida. Nótese que, en general, el producto de dos corrientes no está definido, por tanto, es fundamental que g_Z se represente con una forma diferencial. Pondremos $\alpha_Z = dd^c g_Z + \delta_Z$. El producto $*$ de corrientes de Green se define por

$$g_Y * g_Z = g_Y \wedge \alpha_Z + \delta_Y \wedge g_Z.$$

Se puede demostrar que este producto es conmutativo y asociativo. Además, si X es una variedad aritmética quasi-proyectiva, el producto $*$ proporciona la componente arquimediana de un producto de intersección

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^{p+q}(X) \otimes \mathbb{Q}.$$

Con este producto de intersección, $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \bigoplus \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ es un anillo asociativo, conmutativo y unitario.

Estos grupos de Chow aritméticos se pueden incluir en varias sucesiones exactas. Por ejemplo:

$$\mathrm{CH}^{p,p-1}(X) \to \widetilde{A}^{p-1,p-1}(X) \to \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X) \to 0 \quad \text{y}$$
$$\mathrm{CH}^{p,p-1}(X) \to H^{p-1,p-1}(X) \to \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X) \oplus Z^{p,p}(X) \to H^{p,p}(X) \to 0,$$

donde $\widetilde{A}^{p-1,p-1}$ es un grupo de formas diferenciales módulo $\operatorname{Im} \partial + \operatorname{Im} \overline{\partial}$, $Z^{p,p}(X)$ es un grupo de formas diferenciales cerradas y $H^{p,p}(X)$ denota un cierto grupo de cohomología de X (para una definición precisa de los términos que aparecen en estas sucesiones exactas se puede consultar [G-S 2] o [S-A-B-K]). La segunda de estas sucesiones exactas sólo es válida si X_K es proyectiva y, debido a esto, Gillet y Soulé indican en [G-S 2] que la definición de grupos de Chow aritméticos para variedades quasi-proyectivas no es la óptima posible. Además sugieren que, utilizando formas diferenciales con crecimiento logarítmico, se podría obtener una definición mejor.

Gillet y Soulé también desarrollan, en [G-S 3], una teoría de clases características para fibrados hermíticos, esto es, fibrados vectoriales sobre una variedad aritmética X, provistos de una métrica hermítica sobre el fibrado vectorial inducido en la variedad X_∞. Estas clases características satisfacen las propiedades usuales de las clases de Chern de los fibrados vectoriales. En el mismo artículo se introduce el grupo de Grothendieck de fibrados vectoriales hermíticos, $\widehat{K}_0(X)$ y se demuestra que el carácter de Chern induce un isomorfismo

$$\widehat{ch} : \widehat{K}_0(X) \otimes \mathbb{Q} \longrightarrow \widehat{\mathrm{CH}}^*(X) \otimes \mathbb{Q}.$$

Finalmente, mencionaremos que, en [G-S 4], Gillet y Soulé demuestran un análogo aritmético del Teorema de Riemann-Roch-Grothendieck. Sea $f : X \longrightarrow Y$ un morfismo propio entre variedades aritméticas quasi-proyectivas sobre $\mathbb{Z}$. Supongamos que el morfismo inducido $f : X_\infty \longrightarrow Y_\infty$ es liso. Sea $\overline{E} = (E, h)$ un fibrado hermítico sobre X. El determinante de la cohomología $\lambda(E) = \det Rf_*(E)$ es un fibrado de línea sobre Y. Elijamos una métrica hermítica h_f, invariante por conjugación, en el fibrado tangente relativo Tf, tal que la restricción de h_f a cada fibra de f sobre $Y(\mathbb{C})$ es una métrica de Kähler. Entonces el fibrado de línea $\lambda(E)$ puede ser provisto con la métrica de Quillen h_Q (ver [Q 2], [Bi-G-S] o [S-A-B-K]).

El Teorema de Riemann-Roch Aritmético de Gillet y Soulé afirma que

$$\widehat{c}_1(\lambda(E), h_Q) = f_* \left(\widehat{ch}(E, h)\widehat{Td}(Tf, h_f) - a(ch(E_\mathbb{C})Td(Tf_\mathbb{C})R(Tf_\mathbb{C})) \right)^{(1)},$$

donde $\alpha^{(1)}$ designa la componente de grado uno de $\alpha \in \widehat{\mathrm{CH}}(Y)_\mathbb{Q}$, $\widehat{c}_1$, $\widehat{ch}$ y $\widehat{Td}$ denotan la primera clase de Chern aritmética, el carácter de Chern aritmético y la clase de Todd aritmética de un fibrado hermítico, los símbolos ch y Td designan la forma carácter de Chern y la forma de Todd y $R(Tf_\mathbb{C})$ es una clase característica de corrección, definida por una serie (véanse las definiciones precisas en [G-S 3] y [G-S 4], una introducción a esta teoría se puede encontrar en [S-A-B-K]).

§2. Descripción del Contenido de la Presente Memoria.

Vamos a dar un resumen por capítulos del contenido de esta tesis.

Capítulo I.

El punto de partida de este trabajo, es el hecho de que las corrientes de Green, asociadas a un ciclo algebraico, se pueden representar por formas diferenciales que tienen singularidades logarítmicas a lo largo del soporte del ciclo. Para estudiar este tipo de formas, en este capítulo se introduce un complejo de formas diferenciales con singularidades logarítmicas, el complejo de Dolbeault logarítmico C^∞, y se estudian sus propiedades cohomológicas.

Más explícitamente, sea X una variedad compleja lisa de dimensión d, sea $D \subset X$ un divisor con cruces normales, es decir, un divisor que en coordenadas locales $(z_1, \dots, z_d)$, admite la ecuación

$$z_1 \dots z_k = 0.$$

Sea $V = X - D$ y sea $j : V \longrightarrow X$ la inclusión. Designemos por $\Omega^*_X(\log D)$ el complejo de formas holomorfas sobre X con singularidades logarítmicas a lo largo de D. Recordemos que $\Omega^*_X(\log D)$ es el Ω^*_X-álgebra generada localmente por las secciones dz_i/z_i, para $i = 1, \dots, k$. Este complejo, junto con su filtración de Hodge, F, y su filtración por el peso, W, es una pieza clave en la construción de Deligne ([De 1]) de las estructuras de Hodge mixtas de la cohomología de las variedades algebraicas.

Denotaremos por $\mathcal{E}^*_X$ el complejo de haces de formas diferenciales C^∞ a valores en $\mathbb{C}$ y por $\mathcal{E}^*_{X,\mathbb{R}}$ el subcomplejo de formas diferenciales reales. El complejo de formas diferenciales sobre X con singularidades logarítmicas a lo largo de D, $\mathcal{E}^*_X(\log D)$, es la sub-$\mathcal{E}^*_X$-álgebra de $j_*\mathcal{E}^*_V$ generada localmente por las secciones

$$\begin{aligned} &\log z_i\bar{z}_i, \quad \frac{dz_i}{z_i}, \quad \frac{d\bar{z}_i}{\bar{z}_i}, \quad \text{si } i \in [1,k] \quad \text{y} \\ &dz_i, \quad d\bar{z}_i, \quad \text{si } i \notin [1,k], \end{aligned}$$

donde $z_1 \dots z_k = 0$ es una ecuación local de D.

Este complejo tiene una estructura real dada por el subcomplejo de formas reales, $\mathcal{E}^*_{X,\mathbb{R}}(\log D)$, una filtración de Hodge, F, y una filtración por el peso, W, definida sobre $\mathbb{R}$. El complejo $\mathcal{E}^*_X(\log D)$ es una variante con estructura real y C^∞ del complejo $\Omega^*_X(\log D)$. En este sentido, el principal resultado del primer capítulo es que el morfismo natural

$$(\Omega^*_X(\log D), F, W) \longrightarrow (\mathcal{E}^*_X(\log D), F, W)$$

es un quasi-isomorfismo bifiltrado. Más aún, teniendo en cuenta la estructura real del complejo $\mathcal{E}^*_X(\log D)$ obtenemos que el triple

$$((\mathcal{E}^*_{X,\mathbb{R}}(\log Y), W), (\mathcal{E}^*_X(\log Y), W, F), Id)$$

es un $\mathbb{R}$-complejo de Hodge mixto cohomológico que induce en $H^*(V, \mathbb{R})$ una estructura de Hodge mixta real.

Tanto la definición de este complejo, como la demostración de sus propiedades cohomológicas, están basadas en una construcción análoga introducida por Navarro Aznar en [N]. En dicho artículo se introduce un complejo de Dolbeault logarítmico analítico-real sobre X, $\mathcal{A}^*_X(\log D)$, y se demuestra el mismo resultado para dicho complejo. Obsérvese que las propiedades cohomológicas del complejo de Dolbeault logarítmico C^∞ y de este complejo de Dolbeault logarítmico analítico-real son las mismas. En particular, se pueden reemplazar formas C^∞ por formas analítico-reales en todas la construcciones posteriores, obteniéndose una teoría análoga, que puede tener consecuencias interesantes gracias a la rigidez de las formas analítico-reales.

En la última sección del primer capítulo, se estudia la relación entre funciones de Green sobre una curva y los complejos de Dolbeault logarítmicos, tanto el analítico-real como el C^∞. Por comodidad del lector, repetiremos la definición de función de Green. Sea C una curva compleja proyectiva y lisa, sea p un punto de C y sea ω una $(1,1)$-forma diferencial real C^∞ (o analítico-real) sobre C con la normalización

$$\int_C \omega = 1.$$

Entonces una función de Green g para p con respecto a ω es una función real y C^∞ (o analítico-real) sobre $C - \{p\}$ tal que:

(G1) $dd^c g = \omega$ en $C - \{p\}$.

(G2) Si z es un parámetro local de C alrededor de p, entonces

$$g(z) = -\log z\bar{z} + \varphi(z),$$

donde φ es una función real C^∞ (o analítica real) definida en un entorno de p.

(G3) Satisface la condición de normalización

$$\int_C g\omega = 0.$$

En lo sucesivo, para simplificar la notación, designaremos los haces mediante letras mayúsculas en cursiva y los grupos de secciones globales mediante la misma letra en tipografía romana, por ejemplo

$$E^*_{\widetilde{X}}(\log D) = \Gamma(\widetilde{X}, \mathcal{E}^*_{\widetilde{X}}(\log D)).$$

La primera relación entre funciones de Green y los complejos de Dolbeault logarítmicos es que la condición (G2) se puede substituir por la condición más débil:

(G2') La funcion g pertenece a $E^0_C(\log p)$ (o a $A^0_C(\log p)$),

obteniéndose una definición equivalente.

La otra relación que se discute es el hecho de que se puede demostrar la existencia de funciones de Green, a partir de las propiedades cohomológicas de los complejos de Dolbeault logarítmicos.

Los resultados de este capítulo apareceran publicados en [Bu 1].

Capítulo II.

Obsérvese que si g es una función de Green para p con respecto a ω, entonces g es una función localmente integrable. Sea $[g]$ la corriente asociada. La condición (G2) implica la ecuación de corrientes

$$dd^c[g] + \delta_p = \omega.$$

Por tanto $[g]$ es una corriente de Green para el ciclo p en el sentido de Gillet y Soulé recordado en el §1. Por otro lado, si queremos que la definición de función de Green sea independiente de la elección de la forma ω podemos usar la siguiente definición alternativa.

Una función de Green g para el punto p es una función definida sobre $C - \{p\}$ tal que:

(G1") $g \in E^0_{C,\mathbb{R}}(\log p)$ y $dd^c g \in E^2_{C,\mathbb{R}}$.

(G2") El par $(dd^c g, d^c g)$ representa la clase de cohomología del punto p en el grupo $H^2_p(C, \mathbb{R})$.

De hecho, toda función que cumpla (G1") y (G2") es localmente integrable y su corriente asociada es una corriente de Green para el punto p. Recíprocamente, toda corriente de Green para el punto p se puede obtener a partir de una función de este tipo. La generalización de este hecho a dimensión superior es el objetivo del este capítulo.

Sea X una variedad compleja, lisa y proyectiva, y sea $Y \subset X$ un subconjunto algebraico cerrado. Sea $\pi : \widetilde{X} \longrightarrow X$ un morfismo propio de variedades lisas, con $\pi^{-1}(Y) = D$ un divisor con cruces normales, y tal que $\pi|_{\widetilde{X}-D} : \widetilde{X} - D \longrightarrow X - Y$ es un isomorfismo. El par $(\widetilde{X}, D)$ se denomina una resolución de singularidades de (X, Y).

El grupo de formas de Green sobre X con soporte singular a lo largo de Y se define como

$$GE^n_{X,Y} = \left\{ g \in E^{n-2}_{\widetilde{X}}(\log D) \mid dd^c g \in E^n_{\widetilde{X}} \right\} \big/ (\operatorname{Im} \partial + \operatorname{Im} \overline{\partial})$$

Estos grupos no dependen de la elección de la resolución de singularidades $(\widetilde{X}, D)$ y están provistos de una estructura real natural y de una bigraduación. Además, existe un morfismo bien definido

$$\mathrm{cl} : GE^*_{X,Y} \longrightarrow H^*_Y(X, \mathbb{C}).$$

Mediante el uso de la estructura de Hodge mixta de $H^*_Y(X, \mathbb{C})$ se demuestra que este morfismo es exhaustivo.

Dados dos subconjuntos cerrados Y y Z se define el producto $*$:

$$GE^n_{X,Y} \otimes GE^m_{X,Z} \xrightarrow{*} GE^{n+m}_{X,Y\cap Z}.$$

Este producto es compatible con el cup-producto en cohomología con soportes, es asociativo y conmutativo en el sentido graduado.

También se estudian en este capítulo las propiedades functoriales de estos grupos: Son contravariantes para cualquier morfismo y covariantes para morfismos lisos.

Además el morfismo imagen-inversa respeta el producto y el morfismo imagen-directa satisface la fórmula de proyección usual.

Sea ahora y un ciclo algebraico de X de codimensión p, cuyo soporte es Y y cuya clase de cohomología es $\{y\} \in H_Y^{2p}(X, \mathbb{C})$. Se define el espacio de formas de Green asociadas a y como

$$GE_X(y) = \left\{ g \in GE_{X,Y}^{p,p} \mid \begin{array}{c} g \text{ es real y} \\ \mathrm{cl}(g) = \{y\} \end{array} \right\}.$$

En la última sección del capítulo II, se demuestra que el espacio de formas de Green para un ciclo algebraico es naturalmente isomorfo al espacio de corrientes de Green para dicho ciclo, en el sentido de [G-S 2]. En particular, obtenemos una nueva demostración de la existencia de corrientes de Green. Además se prueba la compatibilidad del producto $*$, y de los morfismos imagen-directa e imagen-inversa definidos para formas de Green y para corrientes de Green. En la construción del anterior isomorfismo, juegan un papel importante las propiedades de la filtración por el peso del complejo de Dolbeault logarítmico, ya que permiten encontrar representantes adecuados de las formas de Green.

Obsérvese que hemos reemplazado la condición (CG) de la página 13, en la definición de corrientes de Green, por una condición cohomológica. Es decir, una forma de Green es un elemento de un cierto complejo que representa una clase de cohomología determinada. De esta construcción se sigue que el complejo que se use para definir formas de Green es, en cierta forma, secundario. Otro complejo que calcule la misma cohomología puede dar lugar a una noción diferente de formas de Green con nuevas propiedades.

Los resultados de este capítulo apareceran publicados en [Bu 2].

Capítulo III.

La base del capítulo III es la observación, hecha en [G-S 2], de que en el caso de variedades proyectivas, la teoría de cohomología que subyace bajo el concepto de corriente de Green, y por tanto de forma de Green, para un ciclo, es la cohomología de Deligne real. Este punto de vista es también la base de la construcción de Beilinson ([Be 2]), de la componente arquimediana del apareamiento por las alturas.

El objetivo de dicho capítulo es dar una definición más transparente del espacio de formas de Green, que ponga de manifiesto el papel de la cohomología de Deligne real. Así mismo, se pretende que la extensión del concepto de formas de Green a variedades algebraicas complejas lisas, no necesariamente proyectivas, tenga en cuenta la cohomología de Deligne-Beilinson de la variedad.

Sea X una variedad algebraica compleja lisa. En este capítulo se introduce un complejo de Dolbeault de formas diferenciales con singularidades logarítmicas en el infinito, $E^*_{\log}(X)$. Este complejo se construye como límite directo de los complejos $E^*_{\overline{X}}(\log D)$, donde $\overline{X}$ es una compactificación lisa de X y $D = \overline{X} - X$ es un divisor con cruces normales.

A partir de $E^*_{\log}(X)$, se construye un nuevo complejo, $\mathfrak{D}(E^*_{\log}(X), p)$, cuya cohomología es la cohomología de Deligne-Beilinson real de la variedad X. De hecho, se construyen equivalencias homotópicas explícitas entre dicho complejo y un complejo de Deligne-Beilinson de X. El interes del complejo $\mathfrak{D}(E^*_{\log}(X), p)$ radica en que proporciona representantes más simples, de las clases de cohomología de Deligne real, que el complejo de Deligne-Beilinson.

El complejo $\mathfrak{D}(E^*_{\log}(X),p)$ es una generalización de un complejo usado por Wang en [Wa], para construir la cohomología de Deligne real de una variedad proyectiva lisa. Una variante de estos complejos ha sido usada por Demailly en [Dem], para estudiar las propiedades de la cohomología $\partial\overline{\partial}$ de las variedades complejas compactas, no necesariamente kählerianas.

Mediante las equivalencias homotópicas entre el complejo $\mathfrak{D}(E^*_{\log}(X),p)$ y el complejo de Deligne-Beilinson, se dota al primero de una estructura multiplicativa que induce en su cohomología el producto usual en cohomología de Deligne.

En la siguiente sección de este capítulo se introduce el concepto de grupos de cohomología truncada relativa. Estos grupos, que están asociados a un morfismo de complejos $f: A^* \longrightarrow B^*$, se denotan por $\widehat{H}^*(A^*,B^*)$ y estan definidos por:

$$\widehat{H}^n(A^*,B^*) = \left\{(a,\widetilde{b}) \in ZA^n \oplus \widetilde{B}^{n-1} \mid f(a) = db\right\},$$

donde ZA^n es el subgrupo de ciclos de A^n y $\widetilde{B}^{n-1} = B^{n-1}/\operatorname{Im} d$. Varias definiciones de clases características secundarias se pueden dar en términos de estos grupos. Un ejemplo clásico es el grupo de caracteres diferenciales de Cheeger-Simons ([C-S]).

Los grupos de cohomología truncada relativa vienen provistos de un morfismo exhaustivo

$$\mathrm{cl}: \widehat{H}^*(A^*,B^*) \longrightarrow H^*(A^*,B^*).$$

Además, si los complejos A^* y B^* poseen algún tipo de estructura multiplicativa, esta estructura induce una estructura multiplicativa en $\widehat{H}^*(A^*,B^*)$, compatible con la inducida en $H^*(A^*,B^*)$.

Sea ahora y un ciclo algebraico de X de codimensión p. Escribimos $Y = \operatorname{supp} y$ e $\{y\} \in H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$, su clase en el grupo de cohomología de Deligne real con soporte en Y (ver, por ejemplo, [Be 1], [E-V] o [J]). Entonces definimos el espacio de formas de Green asociadas al ciclo y como el conjunto de elementos

$$g \in \widehat{H}^{2p}(\mathfrak{D}(E^*_{\log}(X),p), \mathfrak{D}(E^*_{\log}(X-Y),p))$$

tales que

$$\mathrm{cl}(g) = \{y\} \in H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)).$$

Esta definición coincide con la definición dada en el capítulo II en el caso de variedades proyectivas. Esto implica que la existencia de formas de Green asociadas a un ciclo, es equivalente a la existencia de una clase característica del ciclo en cohomología de Deligne real, compatible con la clase característica en cohomología a valores reales.

Por último recuperamos el producto $*$ como el producto inducido por la estructura multiplicativa de los complejos $\mathfrak{D}(E^*_{\log}(X),\cdot)$. En particular, esto relaciona el producto $*$ con el producto en cohomología de Deligne.

Capítulo IV.

En este capítulo abordamos la construcción de los grupos de Chow aritméticos. En primer lugar recordamos algunas nociones sobre la homología de Deligne real y cómo se puede construir mediante el uso de corrientes.

En segundo lugar recordamos algunas de las relaciones entre la teoría K algebraica y la cohomología de Deligne. Consideremos una variedad algebraica lisa X

sobre $\mathbb{C}$ de dimensión d. Denotemos por $X^{(p)}$ el conjunto de subvariedades irreducibles de X de codimensión p y sea $Z^p = Z^p(X)$ el grupo de ciclos algebraicos de codimensión p.

Sean

$$R_p^i = R_p^i(X) = \bigoplus_{x \in X^{(i)}} K_{p-i}(k(x))$$

los grupos del término E_1 de la sucesión espectral de Brown-Gersten-Quillen ([Q 1]). En particular $R_p^p(X) = Z^p(X)$. Denotaremos por $d : R_p^i \longrightarrow R_p^{i+1}$ la diferencial de esta sucesión espectral.

Dado que $K_1(k(x)) = k(x)^*$ es el grupo de unidades de $k(x)$, todo elemento $f \in R_p^{p-1}$ es de la forma

$$f = \sum_{x \in X^{(i)}} f_x,$$

con $f_x \in k(x)^*$. Y como además se tiene $df = \operatorname{div} f = \sum \operatorname{div} f_x$, resulta que

$$R_p^p(X) \big/ dR_p^{p-1}(X) = \mathrm{CH}^p(X),$$

es el grupo de Chow de X de codimensión p.

Sea $\mathcal{Z}^p = \mathcal{Z}^p(X)$ el conjunto de todos los subconjuntos algebraicos de X de codimensión $\geq p$ ordenados por inclusión, y sea $\mathcal{Z}^p/\mathcal{Z}^{p+1}$ el conjunto de todos los pares $(Z, Z') \in \mathcal{Z}^p \times \mathcal{Z}^{p+1}$ tales que $Z' \subset Z$, también ordenado por inclusión.

Siguiendo a Bloch y Ogus ([B-O]) escribimos

$$H^n_{\mathcal{D},\mathcal{Z}^p/\mathcal{Z}^{p+1}}(X, \mathbb{R}(q)) = \varinjlim_{(Z,Z') \in \mathcal{Z}^p/\mathcal{Z}^{p-1}} H^n_{\mathcal{D},Z-Z'}(X - Z', \mathbb{R}(q)).$$

Entonces damos una demostración del siguiente resultado, que es la pieza clave en la construcción de los grupos de Chow aritméticos:

Existe un diagrama conmutativo

$$\begin{array}{ccccc}
R_p^{p-2} & \xrightarrow{d} & R_p^{p-1} & \xrightarrow{d} & R_p^p \\
\rho\downarrow & & \rho\downarrow & & \rho\downarrow \\
H^{2p-2}_{\mathcal{D},\mathcal{Z}^{p-2}/\mathcal{Z}^{p-1}}(X, \mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p-1}_{\mathcal{D},\mathcal{Z}^{p-1}/\mathcal{Z}^{p}}(X, \mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p}_{\mathcal{D},\mathcal{Z}^{p}/\mathcal{Z}^{p+1}}(X, \mathbb{R}(p)),
\end{array}$$

donde el morfismo vertical de la derecha es la aplicación ciclo y los morfismos horizontales inferiores son morfismos de conexión.

En la tercera sección de este capítulo damos una definición alternativa de los grupos de Chow aritméticos $\widehat{\mathrm{CH}}^*(X)$. Los grupos definidos de esta forma, en el caso de variedades proyectivas, son naturalmente isomorfos a los definidos por Gillet y Soulé en [G-S 2]. Por otro lado, en el caso de las variedades quasi-proyectivas, los grupos aquí definidos, tienen en cuenta la estructura de Hodge de la cohomología de estas variedades. En particular, tenemos la sucesión exacta

$$\begin{aligned}
\mathrm{CH}^{p,p-1}(X) \to & H^{2p-1}_{\mathcal{D}}(X, \mathbb{R}(p)) \to \widehat{\mathrm{CH}}^p(X) \to \\
& \mathrm{CH}^p(X) \oplus Z^{p,p}(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p)) \to 0.
\end{aligned}$$

Estos grupos tienen prácticamente las mismas propiedades que los definidos en [G-S 2]: Son contravariantes con respecto a morfismos de variedades aritméticas. Además, $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^*(X) \otimes \mathbb{Q}$ tiene una estructura de anillo conmutativo. Sin embargo, en el caso de variedades quasi-proyectivas, las hipótesis necesarias para definir el morfismo imagen-directa son más fuertes que para los grupos definidos en [G-S 2].

Para dar un ejemplo de la flexibilidad del proceso de construcción de los grupos de Chow aritméticos, y para poder definir imágenes directas para morfismos propios en general, en la siguiente sección introducimos los grupos de Chow aritméticos homológicos, $\widehat{\mathrm{CH}}_*(X)$. Estos grupos se obtienen reemplazando los complejos $E^*_{\log}(X)$ por unos complejos de corrientes. Las principales propiedades de estos grupos son:

1) Si X tiene dimensión d existe un morfismo natural

$$\widehat{\mathrm{CH}}^p(X) \longrightarrow \widehat{\mathrm{CH}}_{d-p}(X)$$

que es un isomorfismo si la dimensión relativa del morfismo estructural de X es cero.

2) Los grupos $\widehat{\mathrm{CH}}_*(X)$ son covariantes para morfismos propios de variedades aritméticas.

3) $\widehat{\mathrm{CH}}_*(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}_*(X) \otimes \mathbb{Q}$ es un $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$-módulo.

4) Existe una sucesión exacta

$$\mathrm{CH}_{p,p-1}(X) \to \widetilde{D}(X_{\mathbb{R}}) \to \widehat{\mathrm{CH}}_p(X) \to \mathrm{CH}_p(X) \to 0,$$

donde $\widetilde{D}(X_{\mathbb{R}})$ es un grupo definido a partir de las corrientes en X.

5) Dado un ciclo algebraico y de X de dimensión p, existe una clase canónica $\theta(y) \in \widehat{\mathrm{CH}}_p(X)$.

Sin embargo, señalemos que la construcción de anillos de Chow aritméticos homológicos presentada aquí no es óptima, en el sentido de que el complejo de corrientes utilizado no tiene en cuenta la estructura de Hodge de las variedades quasi-proyectivas. Sería interesante encontrar un análogo al complejo $E^*_{\log}(X)$ en términos de corrientes, y utilizarlo en una definición mejorada de los anillos de Chow aritméticos homológicos.

En la última sección de este trabajo se obtiene un análogo aritmético de la fórmula de Riemann-Hurwitz. Para ello introducimos una noción de métrica singular en un fibrado de línea sobre una curva. Sea C una curva proyectiva y sea $\mathcal{L}$ un fibrado de línea sobre C. Sea s una sección no nula de $\mathcal{L}$ definida en un abierto. El tipo de métricas singulares que admitimos son aquellas que, localmente alrededor de cualquier punto p, satisfacen

$$\|s\| = (z\overline{z})^q h(z),$$

donde $q \in \mathbb{Q}$, z es un parámetro local alrededor de p y h es una función estrictamente positiva, que es C^∞ en un entorno de p, salvo en el punto p, donde sólo exigimos que sea contínua. Este tipo de métricas aparecen de forma natural al considerar morfismos ramificados de curvas.

Sea $f : X \longrightarrow Y$ un morfismo de curvas complejas ramificado de grado d. Sea $\mathcal{L}$ un fibrado de línea sobre X provisto de una métrica hermítica lisa $\| \cdot \|$. Entonces

$f_*\mathcal{L}$ es un fibrado vectorial de rango d, y sobre $f_*\mathcal{L}$ podemos inducir una métrica singular poniendo

$$\langle s, s' \rangle_y = \sum_{f(x)=y} r_x \langle s, s' \rangle_x ,$$

siendo r_x el indice de ramificación de f en x. Esta métrica induce una métrica del tipo anterior sobre el fibrado de línea $\det f_*\mathcal{L}$.

Por otra parte, si TX y TY designan los fibrados tangentes de X e Y y suponemos que TY está provisto de una métrica lisa, el morfismo

$$df : TX \longrightarrow f^*TY$$

induce una métrica singular sobre TX que también es del tipo considerado anteriormente.

Sea ahora X una superficie aritmética y $\overline{\mathcal{L}}$ un fibrado de línea sobre X provisto de una métrica singular en $\mathcal{L}_{\mathbb{C}}$. Entonces, existe una clase característica $\widehat{c}_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{CH}}_1(X)$, que extiende la noción de primera clase de Chern para un fibrado de línea hermítico introducida en [G-S 3].

Por otra parte, si $f : X \longrightarrow Y$ es un morfismo finito de superficies aritméticas, se puede construir la primera clase de Chern del haz tangente relativo $\widehat{c}_1(Tf)$. Si R es el divisor de ramificación, entonces se tiene

$$\widehat{c}_1(Tf) = -\theta(R).$$

La fórmula de Riemann-Hurwitz aritmética, que se demuestra en la última sección de este trabajo, establece que, si $\overline{\mathcal{L}}$ es un fibrado de línea sobre X, provisto de una métrica hermítica lisa, entonces

$$\widehat{c}_1(\det f_*\overline{\mathcal{L}}) = f_* \left(\widehat{c}_1(\overline{\mathcal{L}}) + \frac{1}{2}\widehat{c}_1(Tf) \right)$$

en $\widehat{\mathrm{CH}}_1(Y)$.

CAPITULO I

A C^∞ Logarithmic Dolbeault Complex

§1. Preliminaries

Let X be a complex smooth manifold. Let $Y \subset X$ be a divisor with normal crossings (in the sequel DNC). We shall write $V = X - Y$ and denote the inclusion by $j : V \hookrightarrow X$. Let $x \in X$. We shall say that U is a coordinate neighbourhood of x adapted to Y if x has coordinates $(0, \dots, 0)$ and $Y \cap U$ is defined by the equation $z_{i_1} \cdots z_{i_M} = 0$. In particular if $x \notin Y$, then $Y \cap U = \emptyset$. When U and Y are fixed we shall write $I = \{i_1, \dots, i_M\}$.

Let $\mathcal{O}_X$ be the structural sheaf of holomorphic functions and let Ω^*_X be the $\mathcal{O}_X$-module of holomorphic forms. Let us recall the definition of the holomorphic logarithmic complex, denoted $\Omega^*_X(\log Y)$ (cf. [De 1]). The sheaf $\Omega^*_X(\log Y)$ is the sub-$\mathcal{O}_X$-algebra of $j_*\Omega^*_V$ generated in each coordinate neighbourhood adapted to Y by the sections dz_i/z_i, for $i \in I$, and dz_i, for $i \notin I$.

There are two filtrations defined on $\Omega^*_X(\log Y)$. The Hodge filtration is the decreasing filtration defined by:

$$F^p\Omega^*_X(\log Y) = \bigoplus_{p' \geq p} \Omega^{p'}_X(\log Y).$$

The weight filtration is the multiplicative increasing filtration obtained by giving weight 0 to the sections of Ω^*_X and weight 1 to the sections dz_i/z_i, for $i \in I$.

Given a complex K^*, let $\tau_{\leq}$ be the canonical filtration:

$$\tau_{\leq p}(K)^n = \begin{cases} K^n, & \text{if } n < p, \\ \operatorname{Ker} d, & \text{if } n = p, \\ 0, & \text{if } n > p. \end{cases}$$

Deligne has proven in [De 1] the following theorem:

Theorem 1.1. *Let X be a proper smooth algebraic variety over $\mathbb{C}$. There is a filtered quasi-isomorphism*

$$\alpha : (Rj_*\mathbb{R}, \tau_{\leq}) \otimes \mathbb{C} \longrightarrow (\Omega^*_X(\log Y), W).$$

Moreover, the triple

$$((Rj_*\mathbb{R}, \tau_{\leq}), (\Omega^*_X(\log Y), W, F), \alpha)$$

is an $\mathbb{R}$-cohomological mixed Hodge complex which induces in $H^*(V, \mathbb{R})$ an $\mathbb{R}$-mixed Hodge structure functorial on V. This mixed Hodge structure is independent on X.

We refer the reader to [De 1] for the definitions and properties of mixed Hodge structures (MHS), mixed Hodge complexes (MHC) and cohomological mixed Hodge complexes (CMHC).

Remark. Actually, in [De 1] a stronger theorem is proven involving the rational and integer structures of $H^*(V)$. Nevertheless in this work we shall deal only with the real structure.

We shall denote by $\mathcal{A}_{X,\mathbb{R}}$ (resp. $\mathcal{E}_{X,\mathbb{R}}$) the sheaf of real analytic functions (resp. real C^∞ functions) over X, by $\mathcal{A}^*_{X,\mathbb{R}}$ (resp. $\mathcal{E}^*_{X,\mathbb{R}}$) the $\mathcal{A}_{X,\mathbb{R}}$-algebra (resp. $\mathcal{E}_{X,\mathbb{R}}$-algebra) of differential forms. We shall write $\mathcal{A}_X = \mathcal{A}_{X,\mathbb{R}} \otimes \mathbb{C}$ and $\mathcal{A}^*_X = \mathcal{A}^*_{X,\mathbb{R}} \otimes \mathbb{C}$. (resp. $\mathcal{E}_X = \mathcal{E}_{X,\mathbb{R}} \otimes \mathbb{C}$ and $\mathcal{E}^*_X = \mathcal{E}^*_{X,\mathbb{R}} \otimes \mathbb{C}$.) The complex structure of X induces bigradings:

$$\mathcal{A}^n_X = \bigoplus_{p+q=n} \mathcal{A}^{p,q}_X$$

and

$$\mathcal{E}^n_X = \bigoplus_{p+q=n} \mathcal{E}^{p,q}_X.$$

An example of bifiltered acyclic resolution of $\Omega^*_X(\log Y)$ is the following ([De 1]): Let K^* be the simple complex associated to the double complex formed by $\Omega^*_X(\log Y) \otimes_{\mathcal{O}} \mathcal{E}^{0,*}$. This complex is filtered by the subcomplexes $F^p(\Omega^*_X(\log Y)) \otimes \mathcal{E}^{0,*}$ and $W_n(\Omega^*_X(\log Y)) \otimes \mathcal{E}^{0,*}$. The sheaves $Gr_F\, Gr^W\, (K^n)$ are $\mathcal{E}_X$-modules, hence fine and therefore acyclic for the functor $\Gamma(X,\cdot)$. Using the fact that $\mathcal{E}_X$ is a flat $\mathcal{O}_X$-module ([M]) one can prove that

$$(\Omega^*_X(\log Y), W, F) \longrightarrow (K^*, W, F)$$

is a bifiltered quasi-isomorphism. This construction is not symmetrical under conjugation. Hence this resolution does not have a real structure.

In [N] Navarro Aznar introduced the analytic logarithmic Dolbeault complex, denoted $\mathcal{A}^*_X(\log Y)$, of which we recall the definition. The sheaf $\mathcal{A}^*_{X,\mathbb{R}}(\log Y)$ is the sub-$\mathcal{A}_{X,\mathbb{R}}$-algebra of $j_*\mathcal{A}^*_{V,\mathbb{R}}$ generated in each coordinate neighbourhood U adapted to Y by the sections

$$\begin{aligned} \log z_i\bar{z}_i, \quad \operatorname{Re}\frac{dz_i}{z_i}, \quad \operatorname{Im}\frac{dz_i}{z_i}, \quad &\text{for } i \in I, \text{ and} \\ \operatorname{Re} dz_i, \quad \operatorname{Im} dz_i, \quad &\text{for } i \notin I. \end{aligned}$$

The weight filtration of this complex, also noted W, is the multiplicative increasing filtration obtained by assigning weight 0 to the sections of $\mathcal{A}^*_{X,\mathbb{R}}$ and weight 1 to the sections

$$\log z_i\bar{z}_i, \quad \operatorname{Re}\frac{dz_i}{z_i}, \quad \operatorname{Im}\frac{dz_i}{z_i}, \quad \text{for } i \in I.$$

Consider the sheaf $\mathcal{A}^*_X(\log Y) = \mathcal{A}^*_{X,\mathbb{R}}(\log Y) \otimes \mathbb{C}$. It has a weight filtration induced by the weight filtration of $\mathcal{A}^*_{X,\mathbb{R}}(\log Y)$ and a bigrading induced by the bigrading of $j_*\mathcal{A}^*_V$:

$$\mathcal{A}^*_X(\log Y) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}_X(\log Y)$$

where $\mathcal{A}^{p,q}_X(\log Y) = A^*_X(\log Y) \cap j_*\mathcal{A}^{p,q}_V$.

The Hodge filtration of $\mathcal{A}^*_X(\log Y)$ is defined by

$$F^p\mathcal{A}^*_X(\log Y) = \bigoplus_{p' \geq p} \mathcal{A}^{p',q}_X(\log Y).$$

It follows easily from the definitions that the inclusion

$$(\Omega^*_X(\log Y), W, F) \hookrightarrow (\mathcal{A}^*_X(\log Y), W, F)$$

is a bifiltered morphism.

In [N], the following result is proven:

Theorem 1.2. i) *There is a filtered quasi-isomorphism*

$$\beta : (Rj_*\mathbb{R}, \tau_{\leq}) \longrightarrow (\mathcal{A}^*_{X,\mathbb{R}}(\log Y), W).$$

ii) *The inclusion*

$$\iota : (\Omega^*_X(\log Y), W, F) \hookrightarrow (\mathcal{A}^*_X(\log Y), W, F)$$

is a bifiltered quasi-isomorphism.

iii) *The quasi-isomorphisms ι, α and β are compatible, i.e. $\iota \circ \alpha = \beta \otimes \mathbb{C}$*

As a consequence of Theorem 1.2 we have

Corollary 1.3. *Let X be a smooth proper algebraic variety over $\mathbb{C}$ and let Y be a DNC. Then the triple*

$$((\mathcal{A}^*_{X,\mathbb{R}}(\log Y), W), (\mathcal{A}^*_X(\log Y), W, F), Id)$$

is a $\mathbb{R}$-CMHC which induces in $H^*(V, \mathbb{R})$ the $\mathbb{R}$-MHS given by Theorem 1.1.

§2. The C^∞ Logarithmic Dolbeault Complex

Throughout this section we shall use the notations of §1.

Let us consider the sheaves

$$\mathcal{P}^*_{X,\mathbb{R}}(\log Y) := \mathcal{E}_{X,\mathbb{R}} \otimes_{\mathcal{A}_{X,\mathbb{R}}} \mathcal{A}^*_{X,\mathbb{R}}(\log Y)$$

and

$$\mathcal{P}^*_X(\log Y) := \mathcal{E}_X \otimes_{\mathcal{A}_X} \mathcal{A}^*_X(\log Y).$$

There is a natural morphism

$$\mu : \mathcal{P}^*_X(\log Y) \longrightarrow j_*\mathcal{E}^*_V$$

given by multiplication: $\mu(f \otimes \omega) = f \cdot \omega$, for $\omega \in \mathcal{A}^*_X(\log Y)$ and $f \in \mathcal{E}_X$.

Definition 2.1. The C^∞ logarithmic Dolbeault complex, noted $\mathcal{E}^*_X(\log Y)$, is the image of μ:

$$\mathcal{E}^*_X(\log Y) = \mu(\mathcal{P}^*_X(\log Y)) \subset j_*\mathcal{E}^*_V.$$

This complex has a real structure given by

$$\mathcal{E}^*_{X,\mathbb{R}}(\log Y) := \mu(\mathcal{P}^*_{X,\mathbb{R}}(\log Y)).$$

The weight filtration of $\mathcal{A}^*_{X,\mathbb{R}}(\log Y)$ tensored with the trivial filtration of $\mathcal{E}_{X,\mathbb{R}}$ defines a weight filtration of $\mathcal{P}^*_{X,\mathbb{R}}(\log Y)$. The weight filtration of $\mathcal{E}^*_{X,\mathbb{R}}(\log Y)$ is defined by

$$W_n\mathcal{E}^*_X(\log Y) := \mu(W_n\mathcal{P}^*_X(\log Y)).$$

The complexes $\mathcal{P}^*_X(\log Y)$ and $j_*\mathcal{E}^*_V$ have bigradings induced by the complex structure of X and the morphism μ is a bigraded morphism. Hence the complex $\mathcal{E}^*_X(\log Y)$ has a natural bigrading:

$$\mathcal{E}^*_X(\log Y) = \bigoplus_{p+q=n} \mathcal{E}^{p,q}_X(\log Y),$$

where

$$\mathcal{E}^{p,q}_X(\log Y) = \mathcal{E}^*_X(\log Y) \cap j_*\mathcal{E}^{p,q}_V = \mu(\mathcal{P}^{p,q}_X(\log Y)).$$

The Hodge filtration of $\mathcal{E}^*_X(\log Y)$ is defined by

$$F^p\mathcal{E}^*_X(\log Y) = \bigoplus_{p' \geq p} \mathcal{E}^{p',q}_X(\log Y).$$

The sheaves $Gr_F Gr^W \mathcal{E}_X^*(\log Y)$ are acyclic because they are $\mathcal{E}_X$-modules, hence fine.

We have the following diagram of bifiltered complexes and bifiltered morphisms

$$\begin{array}{ccc} (\Omega_X^*(\log Y), W, F) & \rightarrow & (\mathcal{A}_X^*(\log Y), W, F) \\ \downarrow & \swarrow & \\ (\mathcal{E}_X^*(\log Y), W, F) & & \end{array}$$

where the upper arrow is a bifiltered quasi-isomorphism.

The main result of this chapter is

Theorem 2.2. *The inclusion*

$$(\Omega_X^*(\log Y), W, F) \hookrightarrow (\mathcal{E}_X^*(\log Y), W, F)$$

is a bifiltered quasi-isomorphism.

Remark. By the notations it may seem that Theorem 2.2 contradicts [N, (8.11)]. However it should be noted that the sheaf called $\mathcal{E}_X^*(\log Y)$ in [N] is called here $\mathcal{P}_X^*(\log Y)$. And, with our notations, the morphism $\mu : \mathcal{P}_X^*(\log Y) \longrightarrow \mathcal{E}_X^*(\log Y)$ is not an isomorphism (see Corollary 4.2 below).

As a consequence of Theorem 2.2 the morphism

$$(\mathcal{A}_X^*(\log Y), W, F) \longrightarrow (\mathcal{E}_X^*(\log Y), W, F)$$

is a bifiltered quasi-isomorphism and, $\mathbb{C}$ being a faithfully flat $\mathbb{R}$-module,

$$(\mathcal{A}_{X,\mathbb{R}}^*(\log Y), W) \longrightarrow (\mathcal{E}_{X,\mathbb{R}}^*(\log Y), W)$$

is a filtered quasi-isomorphism. Thus, we have

Corollary 2.3. *Let X be a smooth proper algebraic variety over $\mathbb{C}$ and let Y be a DNC. Then the triple*

$$((\mathcal{E}_{X,\mathbb{R}}^*(\log Y), W), (\mathcal{E}_X^*(\log Y), W, F), Id)$$

is a $\mathbb{R}$-CMHC which induces in $H^*(V, \mathbb{R})$ the $\mathbb{R}$-MHS given by Theorem 1.1.

In the rest of this section, in order to prove Theorem 2.2, we shall follow the proof of part ii) of Theorem 1.2 given in [N] and point out where some modifications are needed. The result is that Theorem 2.2 is a consequence of two key lemmas whose proof will be delayed until §5.

By definition of bifiltered quasi-isomorphism, Theorem 2.1 is equivalent to

Proposition 2.4. *The sequence*

$$0 \rightarrow W_n\Omega_X^p(\log Y) \xrightarrow{i} W_n\mathcal{E}_X^{p,0}(\log Y) \xrightarrow{\bar\partial} W_n\mathcal{E}_X^{p,1}(\log Y) \xrightarrow{\bar\partial} \cdots$$

is an exact sequence of sheaves.

Proof. Let $x \in X$. Let U be a coordinate neighbourhood of x adapted to Y. Put $I = \{i_1, \ldots, i_M\}$ as in §1. We shall prove the exactness on stalks.

Let $n,p,q \geq 0$. For each $J \subset I$ we denote by $W^{p,q}_{n,J}$ the intersection of the subalgebra $W_n\mathcal{E}^{p,q}_X(\log Y)_x$ with the algebra generated by

$$\begin{aligned} &\frac{dz_i}{z_i},\ \frac{d\bar{z}_i}{\bar{z}_i},\ \log z_i\bar{z}_i, && \text{for } i \in J,\\ &\frac{dz_i}{z_i},\ d\bar{z}_i, && \text{for } i \notin J,\ i \in I \text{ and}\\ &dz_i,\ d\bar{z}_i, && \text{for } i \notin I. \end{aligned}$$

If there is no danger of confusion we shall omit the superindexes p,q. Let $W_{n,J,k}$ be the subset of $W_{n,J}$ composed by the elements of $W_{n,J}$ such that, for at least one $m \in J$, their weight on $d\bar{z}_m/\bar{z}_m$ and $\log z_m\bar{z}_m$ is less than or equal to k.

One has the following relations:

$$\begin{aligned} W_n &= W_{n,I},\\ W_{n,J} &= \bigcup_{k\geq 0} W_{n,J,k}, \\ W_{n,J,0} &= \bigcup_{K\subsetneq J} W_{n,K}. \end{aligned} \tag{2.5}$$

Let $\omega \in W_n\mathcal{E}^{p,q}(\log Y)_x$ be such that $\bar{\partial}\omega = 0$. We need to prove that $\omega = \bar{\partial}\eta$ with $\eta \in W_n\mathcal{E}^{p,q-1}(\log Y)_x$. There is $J \subset I$ and $k \in \mathbb{Z}$ such that $\omega \in W_{n,J,k}$. We shall make the proof by induction over k and over the cardinal of J. If $J = \emptyset$ the result follows from the next lemma.

Lemma 2.6. *The sequence*

$$0 \longrightarrow W_n\Omega^p_X(\log Y)_x \xrightarrow{i} W^{p,0}_{n,\emptyset} \xrightarrow{\bar{\partial}} W^{p,1}_{n,\emptyset} \xrightarrow{\bar{\partial}} \cdots$$

is exact.

Proof. By definition one has

$$W^{p,q}_{n,\emptyset} = W_n\Omega^p_X(\log Y)_x \otimes \mathcal{E}^{0,q}_{X,x}.$$

The exactness of this sequence has already been discussed after Theorem 1.1.

Let us continue the proof of Proposition 2.4. After Lemma 2.6 and the relations 2.5 it is enough to prove that, if $k > 0$, then there exists an element $\eta \in W_n\mathcal{E}^{p,q-1}_X(\log Y)_x$ such that $\omega - \bar{\partial}\eta \in W_{n,J,k-1}$.

Assume that $1 \in J$ and that the weight of ω on $d\bar{z}_1/\bar{z}_1$ and $\log z_1\bar{z}_1$ is less than or equal to k. For simplicity we shall write $\lambda_1 = \log z_1\bar{z}_1$. We have a decomposition

$$\omega = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \gamma \wedge \lambda_1^{k-1}\frac{d\bar{z}_1}{\bar{z}_1} + \rho,$$

where $\alpha, \beta, \gamma \in W_{n-k,J-\{1\}}$ do not contain $d\bar{z}_1$ and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$. We must show that, adding to ω elements of $\bar{\partial}W_n$, we can eliminate the first three terms.

For the first step we have that $\frac{1}{k}\gamma\lambda_1^k \in W_n\mathcal{E}^{p,q-1}_X(\log Y)_x$, and

$$\bar{\partial}(\frac{1}{k}\gamma\lambda_1^k) = \frac{1}{k}\bar{\partial}\gamma\lambda_1^k + (-1)^{p+q-1}\gamma \wedge \lambda_1^{k-1}\frac{d\bar{z}_1}{\bar{z}_1}.$$

Hence we can write

$$\omega = \alpha'\lambda_1^k + \beta' \wedge \lambda_1^k d\bar{z}_1 + \rho, \qquad \text{mod } \bar{\partial}W_n,$$

where α' and β' satisfy the same conditions that α and β.

For the next step we need the following lemma which will be proven in §5:

Lemma 2.7. *Let $\beta \in W_{n-k,J-\{1\}}$ be a form which does not contain $d\bar{z}_1$, then there exists a form $\varphi \in W_{n-k}$ such that*

$$\bar{\partial}(\bar{z}\varphi\lambda_1^k) = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \rho,$$

where $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$.

Using this lemma one has that

$$\omega = \alpha''\lambda_1^k + \rho', \qquad \mod \bar{\partial}W_n,$$

where α'' and ρ' satisfy the same conditions as α and ρ respectively.

For the last step we need another lemma which will also be proven in §5:

Lemma 2.8. *Let $\omega = \alpha\lambda_1^k + \rho$ be a form such that $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$ and $\bar{\partial}\omega = 0$. Then $\omega \in W_{n,J,k-1}$.*

Clearly this lemma concludes the proof of Theorem 2.2.

Remarks. a) In the case of analytic functions, Lemma 2.7 is proven in [N] solving the equation

$$\frac{\partial}{\partial \bar{z}_1}(\bar{z}_1\varphi) = \beta, \tag{2.9}$$

integrating the power series that defines the components of β in a neighbourhood of x. In general, the equation 2.9 cannot be solved in the case of C^∞ functions. For example ([N]), let $\tilde{f} \in \mathbb{C}[\![z]\!]$ be a non-convergent formal power series. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function which has $\tilde{f}$ as Taylor series at 0. This function exists by Borel extension Theorem (see Theorem 3.3 below). Then the equation

$$\frac{\partial}{\partial \bar{z}}(\bar{z}g) = \frac{\partial}{\partial \bar{z}} f$$

does not have any solution: If g were a solution, then $f - \bar{z}g$ would be a holomorphic function with non-convergent Taylor series.

b) On the other hand, in the real analytic case, Lemma 2.8 can be strengthened saying that α is actually 0. This is a consequence of the following: Let $\{f_i\}$ be a finite family of real analytic functions in a neighbourhood of x such that

$$\sum_i f_i\lambda_1^i = 0, \tag{2.10}$$

then, for all i, the functions f_i are zero. But this is not true in the case of C^∞ functions (cf. Corollary 4.2 below).

Roughly speaking, the idea of the proof of Lemma 2.7 and of Lemma 2.8 in the differentiable case is, first, to obtain a solution of 2.9 up to a flat function using Borel's relative extension Theorem; second, to prove that equation 2.10 implies, in the differentiable case, that the functions f_i are flat and finally, to show that the smoother property of flat functions gives us the proof.

§3. Whitney Functions

In this section we recall some results of the theory of Whitney functions that we shall use throughout this chapter. A complete treatment of this subject, including the proofs omitted here, can be found in [M] or in [T]. The notations that we shall use differ slightly from those of these texts. In fact all the constructions given here depend only on the differentiable structure, thus they can be formulated in terms of real differentiable manifolds. We state them in a complex setting due to the use we shall give them in the remainder of the chapter.

The results we shall need are Borel's relative extension Theorem, Theorem 3.3 below, and Theorem 3.4 which relates the ideal of flat functions on the intersection of two analytic sets with the ideals of flat functions on each set.

Consider the space $\mathbb{C}^d$ with complex coordinates $(z_1, \ldots, z_d)$. We shall use double multi-index notation. Let $\alpha = (\alpha_1, \ldots, \alpha_d, \alpha'_1, \ldots, \alpha'_d)$, $\alpha_i, \alpha'_i \in \mathbb{Z}_{\geq 0}$, then we shall note

$$|\alpha| = \sum_{i \in \sigma} \alpha_i + \alpha'_i\,, \quad z^\alpha = \prod_{i \in \sigma} z^{\alpha_i} \bar{z}^{\alpha'_i}\,, \quad \alpha! = \prod_{i \in \sigma} \alpha_i! \alpha'_i! \quad \text{and} \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial z^\alpha}\,.$$

Let $U \subset \mathbb{C}^d$ be an open set, and let A be a closed subset of U. The space of (complex) jets of order m over A, $J^m(A)$, is defined as the set of all sequences $F = (F^\alpha)_{|\alpha| \leq m}$, where the F^α are continuous complex functions over A.

The space of jets over A is defined as

$$J(A) = \varprojlim J^m(A).$$

Let $\mathcal{E}_{\mathbb{C}^d}(U) = \Gamma(U, \mathcal{E}_{\mathbb{C}^d})$ be the ring of complex C^∞ functions over U. For each $m \in \mathbb{Z}_{\geq 0}$, there is a morphism

$$J^m : \mathcal{E}_{\mathbb{C}^d}(U) \longrightarrow J^m(A)$$

defined by

$$J^m(f)^\alpha = \frac{1}{\alpha!} \partial^\alpha f \Big|_A\,.$$

Taking limits, they give a morphism

$$J : \mathcal{E}_{\mathbb{C}^d}(U) \longrightarrow J(A).$$

If it is necessary to precise the closed set over which the jets are defined we shall write J_A.

Let $x \in A$ and $F \in J^m(A)$. The Taylor polynomial of F, of degree m, centred at x is the polynomial

$$T_x^m F(z) = \sum_{|\alpha| \leq m} (z-x)^\alpha F^\alpha(x).$$

The remainder of F at x of degree m is the jet over A defined by

$$R_x^m(F) = F - J^m(T_x^m F) \in J^m(A).$$

If $F \in J(A)$ then there are obvious definitions of Taylor polynomial and remainder of F of all degrees.

The Taylor approximation Theorem implies that if $F = J^m(f)$ is the jet of a C^∞ function then it satisfies the condition:

W. For all compact $K \subset A$ then

$$(R_x^m F)^\alpha(z) = o(|z-x|^{m-|\alpha|}),$$

for $x, z \in K$ and $|\alpha| \leq m$, when $|z-x| \to 0$.

A jet $F \in J^m(A)$ is said to be a Whitney function of order m, denoted $F \in \mathcal{W}^m(A)$, if it satisfies the condition **W**.

The space of Whitney functions is defined as

$$\mathcal{W}(A) = \varprojlim \mathcal{W}^m(A).$$

Thus the jet of a C^∞ function is a Whitney function. The interest of the Whitney functions is given by the following theorem which says that they are exactly the image of J.

Theorem 3.1. (Whitney's extension Theorem, see [M] or [T]) *The morphism*

$$J_A : \mathcal{E}_{\mathbb{C}^d}(U) \longrightarrow \mathcal{W}(A)$$

is an epimorphism.

Given a jet $F \in J(A)$, after Theorem 3.1, to know whether it is the jet of a differentiable function we must check the conditions **W** for all m. If A is a hyperplane, using C^∞ functions over A instead of continuous functions, we can give a different definition of Whitney functions avoiding the condition **W**. In this case Whitney's extension Theorem specializes in a relative version of Borel's extension Theorem.

Let Y_1 be the hyperplane of equation $z_1 = 0$. We define the morphism

$$\begin{aligned} J_1 : \mathcal{E}_{\mathbb{C}^d}(U) &\longrightarrow \mathcal{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!] \\ f &\longmapsto \sum_{i,j} \frac{1}{i!j!} \left.\frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\right|_{Y_1} z_1^i \bar{z}_1^j, \end{aligned}$$

and the morphism

$$\begin{aligned} \delta : \mathcal{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!] &\longrightarrow J(Y_1) \\ \sum_{i,j} F^{i,j} z_1^i \bar{z}_1^j &\longmapsto \left(\frac{1}{\hat{\alpha}_1!} \partial^{\hat{\alpha}_1} F^{\alpha_1, \alpha_1'} \right)_\alpha , \end{aligned}$$

where, if $\alpha = (\alpha_1, \ldots, \alpha_d, \alpha_1', \ldots, \alpha_d')$, then $\hat{\alpha}_1 = (0, \alpha_2, \ldots, \alpha_d, 0, \alpha_2', \ldots, \alpha_d')$.

It follows from the definitions that δ is injective and that

$$J_{Y_1} = \delta \circ J_1.$$

Hence $\mathcal{W}(Y_1 \cap U) \subset \operatorname{Im} \delta$.

Using Taylor's approximation Theorem it is easy to show that an element of $\operatorname{Im} \delta$ satisfies the condition **W** for all m. And so $\operatorname{Im} \delta \subset \mathcal{W}(Y_1 \cap U)$. Thus we obtain

Proposition 3.2. *The morphism δ is an isomorphism between $\mathcal{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!]$ and $\mathcal{W}(U \cap Y_1)$.*

In this situation Theorem 3.1 can be restated as follows:

Theorem 3.3. (Borel's relative extension Theorem) *The morphism J_1 is an epimorphism, i.e. if*

$$F = \sum_{i,j} F^{i,j} z_1^i \bar{z}_1^j$$

is a formal power series, where the coefficients $F^{i,j}$ are complex C^∞ functions over $Y_1 \cap U$, then there exists a complex C^∞ function f over U, such that

$$\left. \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j} \right|_{Y_1} = F^{i,j}.$$

Now that we have a characterization of $\operatorname{Im} J$ let us look at $\operatorname{Ker} J$. Recall that a function f on U is said to be flat on A if $J_A(f) = 0$. The flat functions form an ideal which we denote by $\mathfrak{m}_A^\infty(U)$.

A useful property of flat functions is the following theorem by Łojasiewicz.

Theorem 3.4. *Let A_1 and A_2 be two closed analytic subsets of U. Then*

$$\mathfrak{m}_{A_1 \cap A_2}^\infty(U) = \mathfrak{m}_{A_1}^\infty(U) + \mathfrak{m}_{A_2}^\infty(U).$$

Proof. Usually this result is formulated in other terms which we recall here.

Let $B_1 \subset B_2$ be two closed subsets of U. There is an obvious restriction morphism $\mathcal{W}(B_2) \longrightarrow \mathcal{W}(B_1)$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathbb{C}^d}(U) & \rightarrow & \mathcal{W}(B_2) \\ & \searrow & \downarrow \\ & & \mathcal{W}(B_1). \end{array}$$

Theorem 3.1 implies that all such restriction morphisms are epimorphisms.

Let now B_1 and B_2 be two different closed subsets. We can construct the following sequence

$$0 \longrightarrow \mathcal{W}(B_1 \cup B_2) \xrightarrow{\rho} \mathcal{W}(B_1) \oplus \mathcal{W}(B_2) \xrightarrow{\pi} \mathcal{W}(B_1 \cap B_2) \longrightarrow 0,$$

where $\rho(F) = (F|_{B_1}, F|_{B_2})$ and $\pi(F, G) = F|_{B_1 \cap B_2} - G|_{B_1 \cap B_2}$. It is clear that ρ is injective, π is surjective and that $\pi \circ \rho = 0$. But in general this sequence is not exact. It is said that B_1 and B_2 are regularly situated if this sequence is exact.

The usual formulation of the Łojasiewicz result is the following.

Theorem 3.5. (Łojasiewicz, see [M]) *If A_1 and A_2 are two real analytic closed sets of U, then they are regularly situated.*

Theorem 3.5 is equivalent to Theorem 3.4 because, by Theorem 3.1,

$$\mathcal{W}(A) \cong \mathcal{E}_{\mathbb{C}^d}(U)/\,\mathfrak{m}_A^\infty(U).$$

§4. Flat functions and logarithmic singularities.

Recall the notations of §1. Let X be a complex manifold of dimension d and let Y be a DNC. Let $x \in X$. From now on we shall fix a coordinate neighbourhood U of x adapted to Y, with coordinates $(z_1, \dots, z_d)$. If Y is defined by the equation $z_{i_1} \cdots z_{i_M} = 0$ set $I = \{i_1 \dots i_M\}$. For shorthand let us write $\lambda_i = \log z_i \bar{z}_i$. We denote by Y_i the hyperplane of equation $z_i = 0$.

In this section we shall relate the kernel of the morphism $\mu : \mathcal{P}_X^*(\log Y) \longrightarrow \mathcal{E}_X^*(\log Y)$ with the flat functions. The results we shall need in the sequel are Proposition 4.1 and Proposition 4.3.

Roughly speaking, the flat functions act as smoothers: let h be a differentiable function, singular along a closed set A, let f be a function flat on A. If the singularity of h is not "too bad", then $f \cdot h$ can be extended to a smooth function flat over A. (cf. for example [T, IV.4.2] for a precise statement.) In particular, we have the following easy result.

Proposition 4.1. *Let f be a complex C^∞ function on U, flat on Y_i, then for all $k \geq 0$ the function*

$$f \cdot \frac{\lambda_i^k}{P(z_i, \bar{z}_i)},$$

where $P(z_i, \bar{z}_i)$ is a monomial, can be extended to a C^∞ function flat on Y_i.

Proposition 4.1 is the reason for the morphism μ not being an isomorphism. For instance, let us consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(z) = e^{-\frac{1}{z\bar{z}}}.$$

It is a function flat on 0. By Proposition 4.1 the function $f(z) \cdot \log z\bar{z}$ is a C^∞ function over $\mathbb{C}$. Thus

$$s = f \otimes \log z\bar{z} - f \cdot \log z\bar{z} \otimes 1$$

is a nonzero section of $\mathcal{P}_X^0(\log 0)$ and $\mu(s) = 0$. Generalizing this example we obtain the following result.

Corollary 4.2. *The ideal* $\operatorname{Ker} \mu$ *contains the elements*

$$f \otimes \lambda_i - f \cdot \lambda_i \otimes 1,$$

where $i \in I$ and f is flat on Y_i.

Let us introduce some notations. A single multi-index of length d is an ordered set $a = (a_1, a_2, \ldots, a_d)$, with $a_i \in \mathbb{Z}_{\geq 0}$. The set of all single multi-indexes of length d is $\mathbb{Z}^d_{\geq 0}$. This is a partially ordered set: Put $b \geq a$ if $b_i \geq a_i$, $\forall i$. For $a \in \mathbb{Z}^d_{\geq 0}$ we shall write

$$\lambda^a = \prod_i \lambda_i^{a_i}, \text{ and } |a| = \sum_i a_i.$$

We define the support of $a \in \mathbb{Z}^d_{\geq 0}$ as

$$\operatorname{supp}(a) = \{i \mid a_i \neq 0\}$$

Let $\Lambda \subset \mathbb{Z}^d_{\geq 0}$ be a finite subset. We define the support of Λ as

$$\operatorname{supp}(\Lambda) = \bigcup_{a \in \Lambda} \operatorname{supp}(a)\,.$$

If $J = \{j_1, \ldots, j_N\}$ is a subset of I we put

$$Y_J = \bigcap_{j \in J} Y_j\,.$$

If $a \in \Lambda$ let us write

$$Y_{a,\Lambda} = \bigcap_{\substack{b \in \Lambda \\ b \geq a}} Y_{\operatorname{supp}(b)}.$$

Note that if $\Lambda' \subset \Lambda$ then $Y_{a,\Lambda} \subset Y_{a,\Lambda'}$, that $Y_{a,\Lambda} \subset Y_{\operatorname{supp}(a)}$ and that if a is maximal in Λ then $Y_{a,\Lambda} = Y_{\operatorname{supp}(a)}$.

Proposition 4.3. *Let $\Lambda \subset \mathbb{Z}^d_{\geq 0}$ be a finite set of multi-indexes. Let $\{f_a\}_{a \in \Lambda}$ be a family of C^∞ functions on U. Then the equation*

$$\sum_{a \in \Lambda} f_a \lambda^a = 0$$

implies that the functions f_a are flat on $Y_{a,\Lambda}$. In particular, if a is maximal in Λ then f_a is flat on $Y_{\operatorname{supp}(a)}$.

Proof. Let us prove first the case $\#\operatorname{supp}(\Lambda) = 1$. We can assume that $\operatorname{supp}(\Lambda) = \{1\}$. In this case we have to prove that, if $f_k \in \mathcal{E}_X(U)$ and

$$\sum_k f_k(z_1, \ldots, z_d) \cdot \lambda_1^k = 0, \tag{4.4}$$

then the functions f_k are flat on Y_1.

Let $y = (0, x_2, \ldots, x_d)$ be a point of $Y_1 \cap U$. Consider the functions

$$h_k(z) = f_k(z, x_2, \ldots, x_d).$$

We shall write $r^2 = z_1 \bar{z}_1$. If we see that, for all n, $h_k = O(r^n)$, i.e. that $h(z)/r^n$ is bounded when $r \longrightarrow 0$, then the functions f_k and all their derivatives with respect to z_1 and $\bar{z}_1$ will be zero in y. Varying the point y, we shall obtain that f_k is flat on Y_1.

Let $n, l \geq 0$. Suppose that, for $k > l$ one has $h_k = O(r^n)$ and, for $k \leq l$ one has $h_k = O(r^{n-1})$. This is true for $n = 1$ and l large enough. Making the quotient of 4.4 by $r^{n-1} \log^l r^2$ we obtain

$$0 = h_0 \frac{1}{r^{n-1} \log^l r^2} + \cdots + h_{l-1} \frac{1}{r^{n-1} \log r^2} + h_l \frac{1}{r^{n-1}} + h_{l+1} \frac{\log r^2}{r^{n-1}} + \cdots .$$

In this equation all terms tend to zero when z tends to zero except perhaps the l-th term. Therefore it also tends to zero, i.e.

$$\lim_{z \to 0} h_l \frac{1}{r^{n-1}} = 0.$$

Thus $h_l = O(r^n)$. By inverse induction over l and induction over n we have that $h_l = O(r^n)$ for all n and all l.

Suppose now that $\# \operatorname{supp}(\Lambda) > 1$. We shall prove first, by induction over $\# \operatorname{supp}(\Lambda)$, that the functions f_a with a maximal are flat on $Y_{\operatorname{supp}(a)}$. Let $a' \in \Lambda$ be a maximal element and assume that $1 \in \operatorname{supp}(a')$.

Let us write

$$\sum_a f_a \lambda^a = \sum_k (\sum_b f_{k,b} \lambda^b) \lambda_1^k = 0.$$

Put $V = U - \bigcup_{i \in I - \{1\}} Y_i$. For each k, the functions

$$\sum_b f_{k,b} \lambda^b$$

are C^∞ functions on V. By the case $\# \operatorname{supp}(\Lambda) = 1$, they are flat on $Y_1 \cap V$. Hence, for all $p, q \in \mathbb{Z}_{\geq 0}$

$$\left. \frac{\partial^{p+q}}{\partial z_1^p \partial \bar{z}_1^q} \sum_b f_{k,b} \lambda^b \right|_{Y_1} = 0\,.$$

By induction hypothesis, for b maximal, the functions

$$\left. \frac{\partial^{p+q}}{\partial z_1^p \partial \bar{z}_1^q} f_{k,b} \right|_{Y_1}$$

are flat on $Y_{\operatorname{supp}(b)}$. If $a' = (k', b')$ is maximal in Λ, then b' is maximal in the set $\{b \mid (k', b) \in \Lambda\}$. Therefore the function $f_{a'}$ is flat on $Y_{\operatorname{supp}(a')} = Y_{\operatorname{supp}(b')} \cap Y_1$.

Finally let us prove the general statement by induction over $\max\{|a| \mid a \in \Lambda\}$. Without loss of generality we can assume that if $a \in \Lambda$ then all the elements $b \leq a$ also belong to Λ.

Set $\Lambda' = \{a \in \Lambda \mid a \text{ is not maximal}\}$. Then $\max\{|a| \mid a \in \Lambda'\} < \max\{|a| \mid a \in \Lambda\}$. For each $a \in \Lambda$ maximal, f_a is flat on $Y_{\operatorname{supp}(a)}$. By Theorem 3.4 we can write

$$f_a = \sum_{i \in \operatorname{supp}(a)} f_{a,i}$$

where $f_{a,i}$ is flat on Y_i. Using Proposition 4.1 and reorganizing terms we obtain

$$\sum_{a \in \Lambda} f_a \lambda^a = \sum_{a \in \Lambda'} g_a \lambda^a = 0.$$

By construction $f_a - g_a$ is flat on $Y_{a,\Lambda}$ and by induction hypothesis g_a is flat on $Y_{a,\Lambda'} \supset Y_{a,\Lambda}$. Hence f_a is flat on $Y_{a,\Lambda}$.

Now we can give a precise characterization of $\operatorname{Ker} \mu$.

Proposition 4.5. *The ideal* $\operatorname{Ker}\mu$ *is generated by the elements*

$$f \otimes \lambda_i - f \cdot \lambda_i \otimes 1,$$

where $i \in I$ *and* f *is flat on* Y_i.

Proof. We shall denote by $\mathcal{J}$ the ideal generated by the elements

$$f \otimes \lambda_i - f \cdot \lambda_i \otimes 1,$$

with $i \in I$ and f flat on Y_i.

Let $\eta \in \operatorname{Ker}\mu$. We can assume that $\eta \in \mathcal{P}^0_X(\log Y)_x$. Let us write

$$\eta = \sum_{a \in \Lambda} g_a \otimes \lambda^a.$$

We shall do the proof by induction over the weight w of η: $w(\eta) = \max\{|a| \mid a \in \Lambda\}$. If $w = 0$ then $\eta = 0$ because $\mu(\eta) = \eta$.

If $w > 0$ it is enough to show that adding elements of $\mathcal{J}$ we can lower the weight of η. Let $a \in \Lambda$ with $|a| = w$. Then a is a maximal element of Λ. Hence, by Proposition 4.3, g_a is flat on $Y_{\mathrm{supp}(a)}$. Thus we can write

$$g_a = \sum_{i \in \mathrm{supp}(a)} g_{a,i},$$

where $g_{a,i}$ is flat on Y_i. Let $\hat{a}_i = (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_d)$. Then

$$\begin{aligned} g_a \otimes \lambda^a &= \sum_{i \in \mathrm{supp}(a)} g_{a,i} \otimes \lambda^a \\ &= \sum_{i \in \mathrm{supp}(a)} g_{a,i} \cdot \lambda^{a_i} \otimes \lambda^{\hat{a}_i}\,, \quad \mod \mathcal{J}. \end{aligned}$$

Repeating this process for each a with $|a| = w$ we have the inductive step.

§5. Proof of Lemmas 2.7 and 2.8

In this section we shall end the proof of Theorem 2.1. We follow the notations of §2 and of §4. We also use the following notation:

$$\xi_i = \frac{dz_i}{z_i}, \text{ for } i \in I$$
$$\xi_i = dz_i, \text{ for } i \notin I.$$

If $L = (\{l_1, \ldots, l_p\}, \{l'_1, \ldots, l'_q\})$ is a pair of ordered subsets of $[1, d]$ we shall note

$$\xi^L = \xi_{l_1} \wedge \cdots \wedge \xi_{l_p} \wedge \bar{\xi}_{l'_1} \wedge \cdots \wedge \bar{\xi}_{l'_q}.$$

Let us recall Lemma 2.7:

Lemma. *Let $\beta \in W^{p,q-1}_{n-k,J-\{1\}}$ be a form which does not contain $d\bar{z}_1$, then there exists a form $\varphi \in W_{n-k}$ such that*

$$\bar{\partial}(\bar{z}\varphi\lambda_1^k) = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \rho,$$

where $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$.

Proof. We shall see first that we can solve the equation

$$\frac{\partial}{\partial \bar{z}_1}(\bar{z}_1 g) = f$$

up to a flat function.

Lemma 5.1. *Let $f : U \longrightarrow \mathbb{C}$ be a C^∞ function, then there exists a C^∞ function $g : U \longrightarrow \mathbb{C}$ such that the function*

$$f - \frac{\partial}{\partial \bar{z}_1}(\bar{z}_1 g)$$

is flat on Y_1.

Proof. The jet of f on Y_1 is the formal power series

$$J_1(f) = \sum_{i,j} \frac{1}{i!j!} \left.\frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\right|_{Y_1} z_1^i \bar{z}_1^j.$$

Integrating this series with respect $\bar{z}_1$ and dividing by $\bar{z}_1$ we get the series

$$\hat{g} = \sum_{i,j} \frac{1}{i!(j+1)!} \frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\bigg|_{Y_1} z_1^i \bar{z}_1^j.$$

By Theorem 3.3 there exists a function g on U whose jet on Y_1 is $\hat{g}$. This is the desired function.

Let us continue the proof of Lemma 2.7. Set

$$\beta = \sum_{a,L} f_{a,L} \lambda^a \xi^L.$$

Applying Lemma 5.1 to the functions $f_{a,L}$ we obtain functions $g_{a,L}$. With them we can write

$$\varphi = \sum_{a,L} g_{a,L} \lambda^a \xi^L.$$

Note that $\varphi \in W^{p,q-1}_{n-k,J-\{1\}}$ because $\beta \in W^{p,q-1}_{n-k,J-\{1\}}$.

Let l be the degree of φ, i.e. $l = p + q - 1$. We have

$$\begin{aligned} \beta \wedge \lambda_1^k d\bar{z}_1 - (-1)^l \bar{\partial}(\bar{z}_1 \varphi \lambda_1^k) &= \left(\beta - \frac{\partial(\bar{z}_1\varphi)}{\partial \bar{z}_1}\right) \wedge \lambda_1^k d\bar{z}_1 \\ &\quad - \left((-1)^l \bar{\partial}(\bar{z}_1\varphi) - \frac{\partial(\bar{z}_1\varphi)}{\partial \bar{z}_1} \wedge d\bar{z}_1\right) \lambda_1^k \\ &\quad - k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1. \end{aligned}$$

By construction $\beta - \partial(\bar{z}_1\varphi)/\partial\bar{z}_1$ is flat on Y_1. Hence, by Proposition 4.1, the weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ of $(\beta - \partial(\bar{z}_1\varphi)/\partial\bar{z}_1) \wedge \lambda_1^k d\bar{z}_1$ is zero.

The weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ of $k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1$ is $k-1$. Thus we can write

$$\rho = \left(\beta - \frac{\partial \bar{z}_1 \varphi}{\partial \bar{z}_1}\right) \wedge \lambda_1^k d\bar{z}_1 - k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1.$$

On the other hand the form

$$\alpha = (-1)^l \bar{\partial}(\bar{z}_1\varphi) - \frac{\partial \bar{z}_1 \varphi}{\partial \bar{z}_1} \wedge d\bar{z}_1$$

does not contain $d\bar{z}_1$ and belongs to $W_{n-k,J-\{1\}}$. Therefore $(-1)^l\varphi$ is the form we are looking for.

Recall now Lemma 2.8:

Lemma. *Let $\omega = \alpha\lambda_1^k + \rho \in W^{p,q}_{n,J,k}$ be a form such that $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$ and $\bar{\partial}\omega = 0$. Then $\omega \in W_{n,J,k-1}$.*

Proof. Set

$$\alpha = \sum_{a \in \Lambda} \sum_{L} f_{a,L} \lambda^a \xi^L.$$

We shall do the proof by induction over $\max\{|a| \mid a \in \Lambda\}$, the weight of α on λ. Let $V = U - \bigcup_{i \in I-\{1\}} Y_i$. By hypothesis

$$\bar{\partial}\omega = \lambda_1^k \bar{\partial}\alpha + (-1)^l k\alpha \wedge \lambda_1^{k-1} \frac{d\bar{z}_1}{\bar{z}_1} + \bar{\partial}\rho = 0,$$

where $l = p + q$ is the degree of α.

For each L, the function

$$h_L = \frac{\partial}{\partial \bar{z}_1} (\sum_{a \in \Lambda} f_{a,L} \lambda^a)$$

is C^∞ in V and is the coefficient of $\lambda_1^k dz_1 \wedge \xi^L$ in $\bar{\partial}\omega$. So by Proposition 4.3 h_L is flat on $Y_1 \cap V$.

Look now at the terms with λ_1^{k-1}. Since ρ has weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ less than or equal to $k-1$, the coefficient of $\lambda_1^{k-1}\bar{\xi}_1$ in $\bar{\partial}\rho$ must be divisible by $\bar{z}_1$. Applying Proposition 4.3 to the coefficient of $\lambda_1^{k-1} dz_1 \wedge \xi^L$ we have that

$$\sum_{a \in \Lambda} f_{a,L} \lambda^a + \bar{z}_1 g$$

is flat on $Y_1 \cap V$. This fact and the corresponding fact for h_L implies that

$$\sum_{a \in \Lambda} f_{a,L} \lambda^a$$

is flat on $Y_1 \cap V$. Considering the partial derivatives of this function as in the case $\#\operatorname{supp}(\Lambda) > 1$ of the proof of Proposition 4.3, we obtain that the functions $f_{a,L}$, with a maximal, are flat on $Y_{\operatorname{supp}(a)} \cap Y_1$.

By Theorem 3.4 we can write, for a maximal in Λ,

$$f_{a,L} = \sum_{i \in \operatorname{supp}(a)} f_{i,a,L} + f_{1,a,L},$$

where $f_{i,a,L}$ is flat on Y_i. Hence, for $i \in \operatorname{supp}(a)$ we have

$$f_{i,a,L}\lambda^a = (f_{i,a,L}\lambda_i^{a_i})\lambda^{\hat{a}_i}$$

and $(f_{i,a,L}\lambda_i^{a_i})$ is a C^∞ function on U. On the other hand $f_{1,a,L}\lambda_1^k\lambda^a \in W_{n,J-\{1\}} \subset W_{n,J,k-1}$. Therefore we can write $\alpha = \alpha' + \alpha''$, where the weight of α' on λ is less than those of α and $\alpha'' \in W_{n,J,k-1}$. Thus we have $\omega = \alpha' + \rho'$, where $\rho' = \rho + \alpha''$ satisfies the same conditions as ρ and α' the same as α but with less weight on λ. This concludes the inductive step.

If $\max\{|a| \mid a \in \Lambda\} = 0$ we proceed in the same way but now we obtain $\alpha' = 0$, hence the result.

This finishes the proof of Theorem 2.2.

§6. Green Functions and Logarithmic Dolbeault Complexes

Since the fundamental work of Néron and Arakelov in Arithmetic Intersection Theory, Green functions have been widely used in the study at infinity of arithmetic divisors.

In this section we shall examine the relationships between Green functions and logarithmic Dolbeault complexes, suggesting that these complexes may be a useful tool in the study and generalization of Green functions.

Let X be a complex manifold and let D be an irreducible divisor. We shall denote by $|D|$ the support of D. Let ω be a real (1,1) form which represents the cohomology class of D. Then a Green function for D with respect to ω is a function

$$g_{D} \in \Gamma(X - |D|, \mathcal{E}^{0}_{X,\mathbb{R}})$$

with logarithmic singularities along $|D|$ and such that

$$dd^{c} g_{D} = \omega,$$

where d^c is the real differential operator defined by $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

The meaning of the words logarithmic singularities may vary from one work to another, ranging from logarithmic growth conditions to a more precise description of the singularity.

A well known method to construct Green functions is the following. Let L be the line bundle associated to D. Let $\|\cdot\|$ be a hermitian metric in L and s be a section of L such that $D = (s)$. Then a Green function for D is

$$g_{D} = -\log \|s\|^{2}. \tag{6.1}$$

It is also well known that $\omega = dd^{c} g_{D}$ is the first Chern form of $(L, \|\cdot\|)$ and that ω represents the cohomology class of D. To obtain Green functions with respect to another form in the same cohomology class, say ω', it is enough to apply the $\partial\bar{\partial}$-Lemma to the exact form $\omega - \omega'$.

From now on, we shall use the following convention. The global sections of a sheaf will be denoted by the same letter as the sheaf but in roman script instead of calligraphic, for instance

$$E^{p,q}_{X}(\log Y) := \Gamma(X, \mathcal{E}^{p,q}_{X}(\log Y)).$$

Let Y be a divisor with normal crossings, $Y = \bigcup Y_k$ with Y_k a smooth divisor for each k. Set $V = X - Y$. A first relationship between logarithmic Dolbeault complexes and Green forms is that $E^*_X(\log Y)$ can be characterized as being the minimum sub-E_X-algebra of E^*_V, closed under ∂ and $\bar{\partial}$, that contains a Green function of the type 6.1 for each smooth divisor Y_k.

Let us examine specifically the case of curves. The general case will be the topic of the next chapter. Let C be a compact Riemann surface. Choose a point x of C and assume that ω is a differentiable (resp. real analytic) volume form on C normalized in such a way that

$$\int_C \omega = 1.$$

By De Rham duality this is equivalent to saying that ω represents the cohomology class of x viewed as a divisor. In this case the usual definition of Green functions is the following (cf. for example [L 3] or [Gro]):

A Green function for x with respect to the form ω is a differentiable (resp. real analytic) function

$$g_x : C - \{x\} \longrightarrow \mathbb{R}$$

such that

G1. $dd^c g_x = \omega$.

G2. If z is a local parameter for x in a neighbourhood U of x then

$$g_x(z) = -\log z\bar{z} + \varphi(z),$$

where φ is a real differentiable (resp. real analytic) function defined in the whole U.

G3. It satisfies

$$\int_C g_x \omega = 0.$$

It is well known that the conditions **G1** and **G2** determine g_x up to an additive constant and that this constant is fixed by **G3**.

The condition **G2** obviously implies the condition

G2′. The function g_x belongs to $E^0_C(\log\{x\})$.

In fact, in presence of **G1**, the statements **G2** and **G2′** are equivalent, i.e. we have the following regularity lemma.

Proposition 6.2. *Let $g \in E^0_C(\log\{x\})$ be a solution of the differential equation* **G1**. *Then, up to an additive constant, g is a Green function for x with respect to ω.*

Proof. We only need to show that g satisfies **G2** in a neighbourhood U of x. Let z be a local parameter for x in U. Put $\lambda = \log z\bar{z}$. We have a (non-unique) decomposition

$$g = \sum_{k=0}^{n} f_k \lambda^k,$$

where the functions f_k are smooth on x. The fact that g satisfies **G1** and Proposition 4.3 implies that the functions f_k are flat on x for $k > 1$, and that there exists a constant a such that $f_1 - a$ is flat on x. Hence, by Proposition 4.1, we can write

$$g = a\lambda + \varphi,$$

where φ is a C^∞ function in the whole U.

It only remains to determine the value of the constant a. This constant is determined by the cohomology class of ω. Let us consider in U the standard metric of $\mathbb{C}$. Let S_ε be the sphere of centre x and radius ε. We have, using Stokes' Theorem, that

$$1 = \int_C \omega = \int_C dd^c g = -\lim_{\varepsilon \to 0} \int_{S_\varepsilon} d^c g = -a.$$

Therefore, $a = -1$, concluding the proof of the lemma.

In view of Proposition 6.2, to prove the existence of a Green function for x it is enough to solve the equation **G1** in the complex $E_C^*(\log\{x\})$. Let us give a proof of the existence of such solutions which does not depend on the existence of metrics on line bundles (see also [L 3] and [Gro]).

Proposition 6.3. *Let ω be a real (1,1) form on C. Then, for each $x \in C$, there exists a real function $g \in E_C^0(\log\{x\})$ such that $dd^c g = \omega$. This function is unique up to an additive constant.*

Proof. The uniqueness follows from the fact that g satisfies **G1** and **G2**.

The form ω is exact in the complex $E_C^*(\log\{x\})$ because $H^2(C - \{x\}, \mathbb{C}) = \{0\}$. Since the spectral sequence of $E_C^*(\log\{x\})$ with the filtration F degenerates at E_1, the differential d is strictly compatible with F. Hence there exists an element $\varphi \in F^1 E_C^1(\log\{x\})$ such that $d\varphi = \omega$. Then $\varphi \in E_C^{1,0}(\log x)$, $\partial\varphi = 0$ and $\bar\partial\varphi = \omega$. Thus we have that $\bar\partial\bar\varphi = 0$ and, ω being real, that $\partial\bar\varphi = \omega$.

Now the form $\bar\varphi - \varphi$ is closed and represents an element of $H^1(C - \{x\}, \mathbb{C})$. Since C is smooth the Hodge filtration of this complex satisfies ([De 1]):

$$H^1(C - \{x\}, \mathbb{C}) = F^1 + \bar{F}^1.$$

Therefore there exist forms $\psi_1 \in E_C^{1,0}(\log\{x\})$ and $\psi_2 \in E_C^{0,1}(\log\{x\})$, with $d\psi_1 = d\psi_2 = 0$ and a function $f \in E_C^{0,0}(\log\{x\})$ such that

$$df + \psi_1 + \psi_2 = \bar\varphi - \varphi.$$

Hence $\partial\bar\partial f = \omega$. Writing

$$g = \frac{\pi}{\sqrt{-1}}(f - \bar f)$$

we have the desired function.

Remarks. a) This proposition is a version of the $\partial\bar\partial$-Lemma. (Compare for example with [D-G-M-S]). The properties of elliptic differential operators usually used to prove the existence of Green functions are hidden here in the mixed Hodge structure of the cohomology groups of C and in the degeneracy of the spectral sequence associated with the filtration F.

b) All the results of this section remain true if we replace the C^∞ complexes for real analytical ones. In particular, if ω is a real analytic (1,1) form then there exists a real analytic Green function with respect to ω for any point $x \in C$. In this case, by uniqueness, any Green function with respect to ω is real analytic.

c) The definition of Green function has been generalized by Gillet and Soulé (cf. [G-S 2]) in the concept of Green forms and Green currents associated with arithmetic cycles. They also introduced the star product of Green currents which corresponds to the intersection product of cycles. The techniques of this section will be generalized in the next chapter to give alternative definitions of Green forms for cycles and of the star product between them. They will also be used to prove the existence of these Green forms.

CAPITULO II

Green Forms and Their Product

§1. A First Definition of Green Forms.

Let X be a complex projective manifold of dimension d, and let D be a divisor of X with normal crossings (in the sequel DNC). We shall write $V = X - D$ and denote by $j : V \hookrightarrow X$ the inclusion. Let $\mathcal{E}_X$ be the sheaf of complex C^∞ functions on X and let $\mathcal{E}_X^*$ be the $\mathcal{E}_X$-algebra of differential forms. The complex structure of X induces a bigrading: $\mathcal{E}_X^n = \bigoplus_{p+q=n} \mathcal{E}_X^{p,q}$. Under this bigrading the differential can be decomposed as $d = \partial + \bar\partial$ with ∂ of type $(1,0)$ and $\bar\partial$ of type $(0,1)$. We shall denote $d^c = \frac{i}{4\pi}(\bar\partial - \partial)$.

Let us recall the definition of the C^∞ logarithmic Dolbeault complex given in the last previous chapter. $\mathcal{E}_X^*(\log D)$ is the sub-$\mathcal{E}_X$-algebra of $j_*\mathcal{E}_V^*$ generated, in each coordinate neighbourhood U in which D has the equation $z_1 \cdots z_M = 0$, by the sections

$$\begin{aligned} \log z_i\bar z_i, \quad \frac{dz_i}{z_i}, \quad \frac{d\bar z_i}{\bar z_i}, &\quad \text{for } i \in [1,M] \quad \text{and} \\ dz_i, \quad d\bar z_i, &\quad \text{for } i \notin [1,M]. \end{aligned}$$

This sheaf is a real subsheaf of $j_*\mathcal{E}_V^*$, hence it has a real structure. We shall denote by $\mathcal{E}_{X,\mathbb{R}}^*(\log D)$ the corresponding subsheaf of real forms.

The weight filtration, W, of this sheaf is the multiplicative increasing filtration obtained by assigning weight 0 to the sections of $\mathcal{E}_X^*$ and weight 1 to the sections

$$\log z_i\bar z_i, \quad \frac{dz_i}{z_i} \quad \text{and} \quad \frac{d\bar z_i}{\bar z_i} \quad \text{for} \quad i \in [1,M].$$

This filtration is defined over $\mathbb{R}$.

The sheaf $\mathcal{E}_X^*(\log D)$ has a bigrading induced by the bigrading of $j_*\mathcal{E}_V^*$:

$$\mathcal{E}_X^*(\log D) = \bigoplus_{p+q=n} \mathcal{E}_X^{p,q}(\log D),$$

where $\mathcal{E}_X^{p,q}(\log D) = \mathcal{E}_X^*(\log D) \cap j_*\mathcal{E}_V^{p,q}$.

The Hodge filtration of $\mathcal{E}_X^*(\log D)$ is defined by

$$F^p\mathcal{E}_X^*(\log D) = \bigoplus_{p' \geq p} \mathcal{E}_X^{p',q}(\log D).$$

Let $(\Omega_X^*(\log D), F, W)$ be the logarithmic De Rham complex with the usual Hodge and weight filtrations ([De 1]). The natural inclusion

$$(\Omega_X^*(\log D), F, W) \longrightarrow (\mathcal{E}_X^*(\log D), F, W)$$

is a bifiltered quasi-isomorphism. Thus $((\mathcal{E}^*_{X,\mathbb{R}}(\log D), W), (\mathcal{E}^*_X(\log D), W, F), Id)$ is a $\mathbb{R}$-cohomological mixed Hodge complex which induces in $H^*(V, \mathbb{R})$ the $\mathbb{R}$-mixed Hodge structure defined by Deligne in [De 1].

We refer the reader to [De 1] for the definitions and properties of mixed Hodge structures, Hodge complexes, mixed Hodge complexes and cohomological mixed Hodge complexes. For an introduction to mixed Hodge structures see [Du]. Other references are [Gr-S] and [G-N-P-P].

From now on let us fix a closed algebraic subset, $Y \subsetneq X$ and let us denote by $V = X - Y$ and by j the inclusion $V \hookrightarrow X$. Following Hironaka ([Hi]), one can obtain a proper modification $\widetilde{X}$ of X:

$$\begin{array}{ccc} D & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \pi \\ Y & \longrightarrow & X, \end{array}$$

where $\widetilde{X}$ is smooth, $D = \pi^{-1}(Y)$ is a DNC and $\pi|_{\widetilde{X}-D}$ is an isomorphism. The pair $(\widetilde{X}, D)$ is called a resolution of singularities of (X, Y).

Let us also fix one such resolution. Then we shall write $\mathcal{E}^*_X(\log Y) = \pi_* \mathcal{E}^*_{\widetilde{X}}(\log D)$. Note that this complex actually depends on the resolution chosen.

In the sequel we shall use the following convention. To denote sheaves we shall use calligraphic letters whereas to denote the group of global sections we shall use the same roman letter. For instance $E^*_X(\log Y) := \Gamma(X, \mathcal{E}^*_X(\log Y))$.

The cohomology groups $H^*(X, \mathbb{C})$ and $H^*(V, \mathbb{C})$ with their real mixed Hodge structure can be computed by the complexes E^*_X and $E^*_X(\log Y)$ respectively, because the corresponding sheaves are fine. In both cases, if ω is a closed form, we shall denote by $\{\omega\}$ the cohomology class it represents.

In addition, the cohomology of X with supports in Y, $H^*_Y(X, \mathbb{C})$, is the cohomology of the pair (X, V), and so it can be computed as the cohomology of the simple of the morphism

$$j^* : E^*_X \longrightarrow E^*_X(\log Y).$$

We shall denote by $\mathrm{S}^*_{X;Y}$ the simple of j^* and by $\mathcal{S}^*_{X;Y}$ the corresponding sheaf.

Let us recall briefly the definition of the simple of j^*. The complex is

$$\mathrm{S}^n_{X;Y} = E^n_X \oplus E^{n-1}_X(\log Y)$$

and the differential is

$$d(\omega, \eta) = (d\omega, j^*\omega - d\eta).$$

The complex $\mathrm{S}^*_{X;Y}$ can be provided with a natural structure of real mixed Hodge complex ([De 1, 8.1.15]). It induces in $H^*_Y(X, \mathbb{C})$ a real mixed Hodge structure in such a way that the cohomology long exact sequence of the pair (X, V) is an exact sequence of real mixed Hodge structures.

As before, if $d(\omega, \eta) = 0$ we shall denote by $\{(\omega, \eta)\}$ the cohomology class that it represents.

Let us see that we can represent a class in $H^n_Y(X, \mathbb{C})$ by a single element of $E^{n-2}_X(\log Y)$.

Proposition 1.1. *Let $x \in H^n_Y(X,\mathbb{C})$ be a cohomology class. Then there exists a form $g \in E^{n-2}_X(\log Y)$ with $dd^c g \in E^n_X$ such that $x = \{(dd^c g, d^c g)\}$.*

Proof. Since X is smooth, the mixed Hodge structure of $H^n(X,\mathbb{C})$ satisfies

$$Gr^W_r H^n(X,\mathbb{C}) = 0$$

for $r < n$ ([De 1, 3.2.15]). Moreover $W_{n-1}H^{n-1}(V,\mathbb{C}) = \operatorname{Im} j^*$. (This follows from [De 1, 3.2.17].) And taking into account that

$$H^{n-1}(X,\mathbb{C}) \xrightarrow{j^*} H^{n-1}(V,\mathbb{C}) \longrightarrow H^n_Y(X,\mathbb{C}) \longrightarrow H^n(X,\mathbb{C})$$

is an exact sequence real mixed Hodge structures we obtain that $Gr^W_r H^n_Y(X,\mathbb{C}) = 0$ for $r < n$. Let us show that this implies that

$$H^n_Y(X,\mathbb{C}) = \sum_{p+q=n} (F^p \cap \bar{F}^q) H^n_Y(X,\mathbb{C}). \tag{1.2}$$

Let $x \in H^n_Y(X,\mathbb{C})$. Then $x \in W_{n+r}H^n_Y(X,\mathbb{C})$. We shall prove 1.2 by induction over r.

If $x \in W_n H_Y(X,\mathbb{C}) = Gr^W_n H_Y(X,\mathbb{C})$ then, since the filtrations F and $\bar{F}$ induced in $Gr^W_n H^n_Y(X,\mathbb{C})$ are n-opposite, we have that

$$x \in \sum_{p+q=n} (F^p \cap \bar{F}^q) H^n_Y(X,\mathbb{C}).$$

Assume now that $x \in W_{n+r}H^n_Y(X,\mathbb{C})$. Since the filtrations F and $\bar{F}$ induced in $Gr^W_{n+r}H^n_Y(X,\mathbb{C})$ are $n+r$-opposite we have

$$\begin{aligned} Gr^W_{n+r}H^n_Y(X,\mathbb{C}) &= \sum_{p+q=n+r} (F^p \cap \bar{F}^q) Gr^W_{n+r}H^n_Y(X,\mathbb{C}) \\ &\subset \sum_{p+q=n} (F^p \cap \bar{F}^q) Gr^W_{n+r}H^n_Y(X,\mathbb{C}). \end{aligned}$$

Let $\alpha \in (F^p \cap \bar{F}^q) Gr^W_{n+r}H^n_Y(X,\mathbb{C})$. Then we can represent α by $y \in F^p$ or by $z \in F^q$ with $y - z \in W_{n+r-1}$. By induction hypothesis we have

$$y - z = \sum_{p'+q'=n} w^{p',q'}$$

with $w^{p,q} \in F^p \cap F^q \cap W_{n+r-1}$. Then

$$y - \sum_{p' \geq p} w^{p',q'} = z + \sum_{q' > q} w^{p',q'} \in F^p \cap F^q \cap W_{n+r}$$

represents α. Therefore

$$x \in \sum_{p+q=n} (F^p \cap \bar{F}^q) H^n_Y(X,\mathbb{C}) + W_{n+r-1}H^n_Y(X,\mathbb{C}).$$

Using again the induction hypothesis we have 1.2.

By 1.2 we can assume that $x \in F^p \cap \bar{F}^q$. Then x can be represented by a pair (ω_1, η_1) with $\omega_1 \in F^p E^n_X$ and $\eta_1 \in F^p E^{n-1}_X(\log Y)$ or by a pair (ω_2, η_2) with $\omega_2 \in \bar{F}^q E^n_X$ and $\eta_2 \in \bar{F}^q E^{n-1}_X(\log Y)$. Since both pairs represent the same cohomology class, there exists a pair (a, c) such that

$$d(a, c) = (\omega_1, \eta_1) - (\omega_2, \eta_2). \tag{1.3}$$

Let $a = \sum a^{p,q}$ and $c = \sum c^{p,q}$ be the decomposition of a and c in pure forms. Let us write

$$F^p a = \sum_{p' \geq p} a^{p',q} \quad \text{and} \quad F^p c = \sum_{p' \geq p} c^{p',q},$$

and let $g = -2\pi i c^{p-1,q-1}$. Then, taking in 1.3 the part which belongs to F^p we obtain

$$(\omega_1, \eta_1) = d(F^p a, F^p c) + (\partial a^{p-1,q}, -\partial c^{p-1,q-1}) + (\omega_2^{p,q}, 0). \tag{1.4}$$

The pair (ω_2, η_2) is closed and both forms belong to $\bar{F}^q$. Hence $\omega_2^{p,q} = \partial \eta_2^{p-1,q}$. From 1.3 we get $\eta_2^{p-1,q} = -a^{p-1,q} + \partial c^{p-2,q} + \bar{\partial} c^{p-1,q-1}$. Thus $\omega_2^{p,q} = -\partial a^{p-1,q} + \partial\bar{\partial} c^{p-1,q-1}$. Substituting in 1.4 and reorganizing terms we obtain

$$(\omega_1, \eta_1) = (dd^c g, d^c g) + d(F^p a, F^p c) + d(0, \frac{c^{p-1,q-1}}{2}).$$

Therefore $(dd^c g, d^c g)$ represents x.

Reciprocally, a form $g \in E^{n-2}_X(\log Y)$ can be used to represent a cohomology class in $H^n_Y(X, \mathbb{C})$ provided that $dd^c g \in E^n_X$. This leads us to the following definition.

Definition 1.5. The space of *Green forms over X with singular support on Y* is the $\mathbb{C}$-vector space

$$GE^*_{X,Y} = \{g \in E^{*-2}_X(\log Y) \mid dd^c g \in E^*_X\} / (dE^{*-3}_X(\log Y) + d^c E^{*-3}_X(\log Y)).$$

Note that we are assuming $Y \subsetneq X$. If $Y = X$ we can define

$$GE^*_{X,X} = \{\omega \in E^*_X \mid d\omega = d^c\omega = 0\}.$$

We shall leave it to the reader to make explicit the case $Y = X$ in all the definitions and results below.

The total space of *Green forms over X* is

$$GE^*_X = \bigoplus_{\substack{Y \subseteq X \\ Y \text{ closed}}} GE^*_{X,Y}.$$

Given that $E^*_X(\log Y)$ depends on the resolution of singularities chosen, this definition of Green forms may depend on it. We shall see in Corollary 1.10 that, in fact, this is not the case.

If $g \in E^*_X(\log Y)$ satisfies $dd^c g \in E^*_X$ we shall note by $\widetilde{g}$ the element of $GE^*_{X,Y}$ that it represents.

Since dd^c is a real bigraded operator and $dE^*_X(\log Y) + d^cE^*_X(\log Y)$ is a real bihomogeneous subgroup, then $GE^*_{X,Y}$ has a bigrading and a real structure induced from those of $E^*_X(\log Y)$. The weight filtration of $E^*_X(\log Y)$ induces a filtration of $GE^*_{X,Y}$, also called the weight filtration and denoted W.

If a form $g \in E^*_X(\log Y)$ belongs to $\operatorname{Im} d + \operatorname{Im} d^c$ then $dd^cg = 0$ and $d^cg \in \operatorname{Im} d$. Thus $\{(dd^cg, d^cg)\} = 0$. Therefore we have a well defined morphism

$$\mathrm{cl} : GE^*_{X,Y} \longrightarrow H^*_Y(X, \mathbb{C}).$$

Since d and d^c are real operators the morphism cl is real and by 1.1 this morphism is surjective.

Example 1.6. Let L be the linear subvariety of $\mathbb{P}^n(\mathbb{C})$ of equation $X_0 = \cdots = X_{p-1} = 0$, where $(X_0, \ldots, X_n)$ are homogeneous coordinates. Let us write

$$\tau = \log(|X_0|^2 + \ldots |X_n|^2),\ \sigma = \log(|X_0|^2 + \ldots |X_{p-1}|^2),\ \alpha = dd^c\tau \text{ and } \beta = dd^c\sigma.$$

Then

$$\Lambda = (\tau - \sigma)(\sum_{\nu=0}^{p-1} \alpha^\nu \wedge \beta^{p-1-\nu})$$

is a well defined differential form on $\mathbb{P}^n(\mathbb{C}) - L$. It is easy to see that $\Lambda \in E^{p-1,p-1}_{\mathbb{P}^n(\mathbb{C})}(\log L)$ and that $dd^c\Lambda = \alpha^p \in E^{p,p}_{\mathbb{P}^n(\mathbb{C})}$. Thus Λ is a Green form. The form Λ was introduced by Levine ([Lv], see also [G-S 2]).

Since the morphism $H^{2p}_L(\mathbb{P}^n(\mathbb{C}), \mathbb{C}) \longrightarrow H^{2p}(\mathbb{P}^n(\mathbb{C}), \mathbb{C})$ is an isomorphism and the form α^p represents the cohomology class of L in $H^{2p}(\mathbb{P}^n(\mathbb{C}), \mathbb{C})$, we obtain that $\mathrm{cl}(\Lambda)$ is the cohomology class of L in $H^{2p}_L(\mathbb{P}^n(\mathbb{C}), \mathbb{C})$.

Let us write

$$\begin{aligned} B^*_{X,Y} &= \{\omega \in E^*_X \mid \omega = d\alpha + d^c\beta,\ \alpha, \beta \in E^*_X(\log Y)\} \\ Z^*_{X,Y} &= \{\omega \in B^*_{X,Y} \mid d\omega = d^c\omega = 0\} \\ GE^*_{X,Y,0} &= \{\widetilde{g} \in GE^*_{X,Y} \mid dd^cg = 0\}. \end{aligned}$$

The three groups are real and have a natural bigrading.

We can relate these groups and the space of Green forms with the following morphisms:

a) The morphism $dd^c : GE^n_{X,Y} \longrightarrow E^{n+2}_X$ given by $dd^c(\widetilde{g}) = dd^cg \in E^{n+2}_X$. By definition $GE_{X,Y,0} = \operatorname{Ker} dd^c$. It is clear that $\operatorname{Im} dd^c \subset Z^{n+2}_{X,Y}$.

b) The morphism $\widehat{\alpha} : E^{p,q}_X \longrightarrow GE^{p,q}_{X,Y}$ induced by the inclusion $E^{p,q}_X \hookrightarrow E^{p,q}_X(\log Y)$. Then $B^{p,q}_{X,Y} = \operatorname{Ker} \widehat{\alpha}$. Thus we have an induced morphism $\alpha : E^{p,q}_X / B^{p,q}_{X,Y} \longrightarrow GE^{p,q}_{X,Y}$.

c) The morphism $\beta : H^n(V, \mathbb{C}) \longrightarrow GE^n_{X,Y,0}$ given by $\beta(\{g\}) = \widetilde{g}$, where $g \in E^n_X(\log Y)$ and $dg = 0$.

d) The morphism cl restricted to $GE^*_{X,Y,0}$ factorizes through the morphism $\varphi : GE^n_{X,Y,0} \longrightarrow H^{n+1}(V, \mathbb{C})$ given by $\varphi(\widetilde{g}) = \{d^cg\}$.

It is easy to show that all these morphisms are well defined.

The following lemma, which includes a version of the dd^c-lemma, will be used in the study of the behavior of the morphisms given above.

Lemma 1.7.

1) *(dd^c-lemma) Let $\omega \in E_X^{p,q}(\log Y)$ be a d-exact form. Then there exists a form $g \in E_X^{p-1,q-1}(\log Y)$ such that $dd^c g = \omega$. Moreover if ω is real we can choose g real.*
2) *If $\omega \in E_X^*$ belongs to $dE_X^*(\log Y) + d^c E_X^*(\log Y)$, then there exist forms $a, b \in E_X^*$ such that $\omega + da + d^c b$ is d and d^c closed. In other words*

$$B_{X,Y}^* = Z_{X,Y}^* + dE_X^* + d^c E_X^*.$$

3) *If $\omega \in E_X^{p,q}$ belongs to $dE_X^*(\log Y) + d^c E_X^*(\log Y)$ and is closed, then ω is d-exact in $E_X^*(\log Y)$. Equivalently, $Z_{X,Y}^{p,q}$ is the space of closed (p,q)-forms α such that $j^*\{\alpha\} = 0$.*
4) *If $g \in E_X^{p,q}(\log Y)$ satisfies $d^c g = dg'$ for a form $g' \in E_X^{p+q}(\log Y)$, then*

$$g \in \operatorname{Ker} d \cap E_X^{p,q} + dE_X^{p+q-1}(\log Y) + d^c E_X^{p+q-1}(\log Y).$$

Proof. Let us prove 1). Since the spectral sequence associated with the Hodge filtration F of $E_X^*(\log Y)$ degenerates at E_1 the morphism d is strictly compatible with the filtration F. As this complex is symmetrical under complex conjugation, the differential d is also strictly compatible with the filtration $\bar{F}$. Thus, ω being d-exact and of type (p,q), we can write $\omega = da = db$, with $a \in F^p$ and $b \in \bar{F}^q$. Therefore $d(a-b) = 0$. Now, since V is smooth we have, by an argument similar to the proof of 1.2,

$$H^{p+q-1}(V,\mathbb{C}) = F^p H^{p+q-1}(V,\mathbb{C}) + \bar{F}^q H^{p+q-1}(V,\mathbb{C}).$$

Hence the class of $a-b$ is the sum of a class in F^p and a class in $\bar{F}^q$. Thus $a - b = a_1 + b_1 + dc$ with $a_1 \in F^p$, $b_1 \in \bar{F}^q$ and $da_1 = db_1 = 0$. If we take the terms of type $(p, q-1)$ we get $a^{p,q-1} = a_1^{p,q-1} + \partial c^{p-1,q-1} + \bar{\partial} c^{p,q-2}$. Therefore

$$\omega = \bar{\partial} a^{p,q-1} = -\partial\bar{\partial} c^{p-1,q-1} = dd^c(2\pi i c^{p-1,q-1}).$$

So it is enough to take $g = 2\pi i c^{p-1,q-1}$. If ω is real of type (p,p) then

$$g = \pi i (c^{p-1,p-1} - \bar{c}^{p-1,p-1})$$

is real of type $(p-1,p-1)$ and $\omega = dd^c g$.

Let us prove 2). Since $\omega \in \operatorname{Im} d + \operatorname{Im} d^c$ in the complex $E_X^*(\log Y)$ then $dd^c\omega = 0$ in E_X^*. But one of the formulations of the dd^c-Lemma for this last complex says ([D-G-M-S, §5])

$$\operatorname{Ker} dd^c = \operatorname{Ker} d \cap \operatorname{Ker} d^c + \operatorname{Im} d + \operatorname{Im} d^c.$$

To prove 3) we can write, by hypothesis, $\omega = \partial a + \bar{\partial} b = -\bar{\partial} a - \partial b + d(a+b)$ with a and b of type $(p-1,q)$ and $(p,q-1)$ respectively. Hence

$$\{\omega\} = -\{\bar{\partial} a\} - \{\partial b\}.$$

But

$$\{\omega\} \in (W_{p+q} \cap F^p \cap \bar{F}^q) H^{p+q}(V,\mathbb{C}) \text{ and } -\{\bar{\partial} a\} - \{\partial b\} \in (\bar{F}^{q+1} + F^{p+1}) H^{p+q}(V,\mathbb{C}).$$

Since in a mixed Hodge structure

$$F^p \cap \bar{F}^q \cap W_{p+q} \cap (F^{p+1} + \bar{F}^{q+1}) = 0,$$

the form ω is d-exact in $E^*_X(\log Y)$.

Let us now prove 4). We have to show that adding to g elements of the form $\partial a + \bar{\partial} b$ we can obtain $g \in \operatorname{Ker} d \cap E^{p,q}_X$.

Since $\bar{\partial} g$ is pure and

$$\bar{\partial} g = \frac{1}{2} dg - 2\pi i d^c g = d(\frac{1}{2} g - 2\pi i g'),$$

we can apply part 1) to $\bar{\partial} g$ to obtain a form $a \in E^{p-1,q}_X(\log Y)$ such that $\bar{\partial} g = \partial\bar{\partial} a$. Analogously we can obtain a form $b \in E^{p,q-1}_X(\log Y)$ such that $\partial g = \partial\bar{\partial} b$. Then

$$d(g + \partial a - \bar{\partial} b) = d^c(g + \partial a - \bar{\partial} b) = 0.$$

So we can suppose that $dg = 0$.

Since V is smooth, the mixed Hodge structure of $H(V, \mathbb{C})$ satisfies

$$(F^p \cap \bar{F}^q \cap W_{p+q} + F^{p+1} + \bar{F}^{q+1}) H^{p+q}(V, \mathbb{C}) = H^{p+q}(V, \mathbb{C}).$$

But $W_{p+q} H^{p+q}(V, \mathbb{C})$ is the image of $H^{p+q}(X, \mathbb{C})$ by j^*. Therefore there exist $\omega \in E^{p,q}_X$, $g_1 \in F^{p+1} E^{p+q}_X(\log Y)$ and $g_2 \in \bar{F}^{q+1} E^{p+q}_X(\log Y)$, with $d\omega = dg_1 = dg_2 = 0$ such that $g = \omega + g_1 + g_2 + d\eta$. Taking the part of bidegree (p, q) one obtains

$$g = \omega + \partial \eta^{p-1,q} + \bar{\partial} \eta^{p,q-1}.$$

This concludes the proof the Lemma 1.7.

In the next proposition we shall give some exact sequences which involve the space of Green forms.

Proposition 1.8. *Let $p, q \in \mathbb{N}$, put $n = p + q$. Then the following sequences are exact:*

1) $\quad 0 \to GE^n_{X,Y,0} \to GE^n_{X,Y} \xrightarrow{dd^c} Z^{n+2}_{X,Y} \to 0$

2) $\quad 0 \to Gr^p_F Gr^W_n H^n(V, \mathbb{C}) \xrightarrow{\beta} GE^{p,q}_{X,Y,0} \xrightarrow{\varphi} F^{p+1} \cap \bar{F}^{q+1} H^{n+1}(V, \mathbb{C}) \to 0$

3) $\quad 0 \to E^{p,q}_X / B^{p,q}_{X,Y} \xrightarrow{\alpha} GE^{p,q}_{X,Y} \xrightarrow{\mathrm{cl}} F^{p+1} \cap \bar{F}^{q+1} H^{n+2}_Y(X, \mathbb{C}) \to 0$

4) $\quad 0 \to E^n_X / B^n_{X,Y} \xrightarrow{\alpha} W_1 GE^n_{X,Y} \xrightarrow{\mathrm{cl}} W_{n+2} H^{n+2}_Y(X, \mathbb{C}) \to 0$

Proof. By the definition of $GE^n_{X,Y,0}$, to prove 1) it is enough to prove the surjectivity of the morphism dd^c. Let $\omega \in Z^{n+2}_{X,Y}$. Since $Z^{n+2}_{X,Y}$ is a bihomogeneous subgroup of E^{n+2}_X we can assume that ω is pure of type (p, q). By part 3) of Lemma 1.7 ω is d-exact in $E^*_X(\log Y)$ and by part 1) of the same lemma ω is dd^c-exact. Thus dd^c is surjective.

Let us prove 2). A cohomology class in $H^{n+1}(V, \mathbb{C})$ belongs to $F^{p+1} \cap \bar{F}^{q+1}$ if and only if it can be represented by a form $\eta_1 \in F^{p+1} E^{n+1}_X(\log Y)$ and by a form $\eta_2 \in \bar{F}^{q+1} E^{n+1}_X(\log Y)$.

Let $\widetilde{g} \in GE^{p,q}_{X,Y,0}$, since

$$d^c g = \frac{i}{2\pi}\bar{\partial} g - \frac{i}{4\pi} dg = -\frac{i}{2\pi}\partial g + \frac{i}{4\pi} dg,$$

we obtain $\{d^c g\} = \{\frac{i}{2\pi}\bar{\partial} g\} = \{-\frac{i}{2\pi}\partial g\}$. But $\{\frac{i}{2\pi}\bar{\partial} g\} \in \bar{F}^{q+1}$ and $\{-\frac{i}{2\pi}\partial g\} \in F^{p+1}$. Therefore $\varphi(\widetilde{g}) \in F^{p+1} \cap \bar{F}^{q+1} H^{n+1}(V, \mathbb{C})$.

Conversely, if $x \in F^{p+1} \cap \bar{F}^{q+1} H^{n+1}(V, \mathbb{C})$ let $\eta_1 \in F^{p+1}$ and $\eta_2 \in \bar{F}^{q+1}$ be representatives of x. Then $\eta_1 - \eta_2 = dc$. Let us write $g = -2\pi i c^{p,q}$ and let $F^p c$ be as in the proof of Proposition 1.1. Then

$$\eta_1 = d^c g + dF^p c + \frac{dc^{p-1,q-1}}{2}.$$

Therefore $\varphi(\widetilde{g}) = \{d^c g\} = x$ and φ is surjective.

To prove the injectivity of β recall that

$$\begin{aligned} Gr^p_F Gr^W_n H^n(V, \mathbb{C}) &= \operatorname{Im}(Gr^p_F H^n(X, \mathbb{C}) \longrightarrow Gr^p_F H^n(V, \mathbb{C})) \\ &\cong \{\omega \in E^{p,q}_X \mid d\omega = 0\}/(dE^{n-1}_X(\log Y) \cap E^{p,q}_X). \end{aligned}$$

And that the morphism β is defined by $\beta(\{g\}) = \widetilde{g}$, for $g \in E^{p,q}_X$ with $dg = 0$. But if $\beta(\{g\}) = 0$ then $g \in dE^*_X(\log Y) + d^c E^*_X(\log Y)$ and, by part 3) of Lemma 1.7, this implies that $\{g\} = 0$.

The composition $\varphi\beta = 0$ because if g is pure of type (p, q) and $dg = 0$ then $d^c g = 0$.

Finally let $\widetilde{g} \in GE^{p,q}_{X,Y,0}$ be such that $\varphi(\widetilde{g}) = 0$. Then $d^c g = dg'$ for a $g' \in E^{p+q}_X(\log Y)$. By part 4) of Lemma 1.8 there exists $g_1 \in E^{p,q}_X$ with $dg_1 = 0$, such that $\widetilde{g}_1 = \widetilde{g}$. Hence $\widetilde{g} \in \operatorname{Im}\beta$. This concludes the proof of the exactness of 2).

The surjectivity of the morphism cl in 3) has been proved in Proposition 1.1. The remainder of the proof of the exactness of 3) follows in a way similar to the proof of 1) and 2) using Lemma 1.7.

In the proof of 4) the only difficulty is the surjectivity of cl, i.e. that any cohomology class of $H^{n+2}_Y(X, \mathbb{C})$ of weight $n + 2$ can be obtained from a Green form of weight one. For this we shall use the following Lemma.

Lemma 1.9. *Let K^* be the simple of the morphism $E^*_X \longrightarrow W_1 E^*(\log Y)$ with his natural structure of mixed Hodge complex. Then the induced morphism*

$$W_n H^n(K^*) \longrightarrow W_n H^n_Y(X, \mathbb{C})$$

is a surjective morphism of real Hodge structures.

Proof. It is a morphism of mixed Hodge structures because it is induced by a real bifiltered morphism between the corresponding complexes. These Hodge structures are pure because, by construction, in both cases the part of weight $n - 1$ is zero.

For brevity we shall write $W^*_1 = W_1 E^*_X(\log Y)$. Taking the graded part of weight n in the cohomology long exact sequence of the pair (X, V) and in the long exact sequence associated to K^* we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Gr^W_n H^{n-1}(W^*_1) & \longrightarrow & W_n H^n(K^*) & \longrightarrow & H^n(X) & \longrightarrow & W_n H^n(W^*_1) \\ & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & Gr^W_n H^{n-1}(V) & \longrightarrow & W_n H^n_Y(X) & \longrightarrow & H^n(X) & \longrightarrow & W_n H^n(V). \end{array}$$

The first vertical arrow is an epimorphism by the definition of the weight filtration in cohomology.

In [De 1, 1.3.2] it is proved that if (K^*, d) is a complex povided with a decreasing filtration F then the spectral sequence associated to this filtration degenerates at E_1 if and only if the differential is strictly compatible with the filtration. That is, if

$$\operatorname{Im} d \cap F^p = dF^p.$$

It can be shown (using [De 1, 1.3.4]) that the spectral sequence associated to F degenerates at E_2 if and only if

$$\operatorname{Im} d \cap F^p \subset dF^{p-1}.$$

Therefore, since the spectral sequence associated with the filtration W degenerates at E_2 the last arrow is an isomorphism. Thus the lemma follows from the Five Lemma.

Let now x be a cohomology class in $W_{n+2}H_Y^{n+2}(X, \mathbb{C})$, since this group is bigraded and in this case cl is a bihomogeneous morphism of bidegree (1,1) we can assume that x is pure of type $(p+1, q+1)$.

By Lemma 1.9 there exists a cohomology class $\widehat{x} \in W_{n+2}H^{n+2}(K^*)$ of type $(p+1, q+1)$ whose image in $W_{n+2}H_Y^{n+2}(X, \mathbb{C})$ is x. Now $\widehat{x} \in F^{p+1} \cap \bar{F}^{q+1}$, thus there exist closed pairs (ω_1, η_1) and (ω_2, η_2) with $\omega_1 \in F^{p+1}E_X^{n+2}$, $\omega_2 \in \bar{F}^{q+1}E_X^{n+2}$, $\eta_1 \in W_1F^{p+1}E_X^{n+1}(\log Y)$ and $\eta_2 \in W_1\bar{F}^{q+1}E_X^{n+1}(\log Y)$ such that both represent $\widehat{x}$. Hence there exists a pair $(a, c) \in K^{n+1}$ such that $d(a, c) = (\omega_1, \eta_1) - (\omega_2, \eta_2)$. Putting $g = 2\pi i c^{p,q} \in W_1E_X^{p,q}(\log Y)$ we have, as in 1.1. $\operatorname{cl}(\widetilde{g}) = x$. The remainder of the proof of the exactness of 4) follows analogously to the other cases.

Corollary 1.10. *The space of Green forms does not depend on the resolution of singularities. More precisely the spaces of Green forms obtained from two different resolutions are related by a unique natural isomorphism.*

Proof. Let $\pi' : (\widetilde{X}', D') \longrightarrow (X, Y)$ be another resolution of singularities. Since, for each two resolutions of singularities there is always a resolution of singularities that dominates both, we can assume that $(\widetilde{X}', D')$ dominates $(\widetilde{X}, D)$, i.e. there is a morphism $h : (\widetilde{X}', D') \longrightarrow (\widetilde{X}, D)$ which commutes with the projections over X. Then there is a morphism $h^* : E^*_{\widetilde{X}}(\log D) \longrightarrow E^*_{\widetilde{X}'}(\log D')$. Let us denote by $GE'^*_{X,Y}$ the space of Green forms obtained with the new resolution. By parts 2) and 3) of Lemma 1.7 the spaces $B^{p,q}_{X,Y}$ are independent from the resolution of singularities, and from Proposition 1.8 we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_X^{p,q}/B_{X,Y}^{p,q} \arrow[r, "\alpha"] \arrow[d, equal] & GE_{X,Y}^{p,q} \arrow[r, "\mathrm{cl}"] \arrow[d] & F^{p+1} \cap \bar{F}^{q+1} H_Y^{n+2}(X, \mathbb{C}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_X^{p,q}/B_{X,Y}^{p,q} \arrow[r, "\alpha"] & GE'^{p,q}_{X,Y} \arrow[r, "\mathrm{cl}"] & F^{p+1} \cap \bar{F}^{q+1} H_Y^{n+2}(X, \mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

from which the corollary follows.

Let us now study the functoriality of the space of Green forms. The key point is the functoriality of the logarithmic complex: Let $f : X' \longrightarrow X$ be a morphism of complex manifolds and let D' and D be divisors with normal crossings of X' and X respectively such that $f^{-1}(D) \subset D'$. Then the inverse image of forms induces a morphism

$$f^* : E^*_X(\log D) \longrightarrow E^*_{X'}(\log D').$$

Let us begin changing the closed set Y. Let $Y' \subset X$ be another closed algebraic subset such that $Y \subset Y'$. Let $(\widetilde{X}', D')$ be a resolution of singularities of (X, Y') which factorizes through $(\widetilde{X}, D)$. By the functoriality of the logarithmic complex we have a morphism

$$\rho_{Y',Y} : E^*_X(\log Y) \longrightarrow E^*_X(\log Y'),$$

which induces a morphism

$$\rho_{Y',Y} : GE^*_{X,Y} \longrightarrow GE^*_{X,Y'}.$$

We shall call it the change of support morphism.

Let us construct a pull-back morphism of Green forms. Let $f : X' \longrightarrow X$ be a morphism of complex projective manifolds. Recall that we have fixed an algebraic closed subset $Y \subset X$ and a resolution of singularities (X, D) of (X, Y). Put $Y' = f^{-1}(Y)$. Let $(\widetilde{X}', D')$ be a resolution of singularities of (X', Y') such that there is a commutative diagram

$$\begin{array}{ccc} (\widetilde{X}', D') & \xrightarrow{\pi} & (X', Y') \\ \Big\downarrow \widetilde{f} & & \Big\downarrow f \\ (\widetilde{X}, D) & \xrightarrow{\pi} & (X, Y). \end{array}$$

Again by the functoriality of the logarithmic complex, the pull-back morphism $f^* : E^*_X \longrightarrow E^*_{X'}$ can be extended to a morphism $f^* : E^*_X(\log Y) \longrightarrow E^*_{X'}(\log Y')$. This induces a morphism

$$f^* : GE^*_{X,Y} \longrightarrow GE^*_{X',Y'}.$$

Note that composing with the adequate change of support morphism we can construct a pull-back morphism $f^* : GE^*_{X,Y} \longrightarrow GE^*_{X',Y'}$ for any closed algebraic subset Y' such that $f^{-1}(Y) \subset Y'$.

Let us also construct a push-forward morphism for Green forms. Let $f : X' \longrightarrow X$ be a smooth morphism between complex projective manifolds of relative dimension e. As before, let $Y' = f^{-1}(Y)$. By [Hi], the resolution of singularities of (X, Y) is obtained by succesive blowing-ups over smooth centers. Since f is smooth, the inverse image of a smooth subvariety is also smooth. Thus we can resolve singularities of (X, Y) and of (X', Y') simultaneously obtaining a diagram

$$\begin{array}{ccc} (\widetilde{X}', D') & \xrightarrow{\pi} & (X', Y') \\ \Big\downarrow \widetilde{f} & & \Big\downarrow f \\ (\widetilde{X}, D) & \xrightarrow{\pi} & (X, Y), \end{array}$$

where $(\widetilde{X}, D)$ and $(\widetilde{X}', D')$ are resolutions of singularities of (X, Y) and (X', Y'). The morphism $\widetilde{f}$ is smooth and each component of D' is mapped smoothly over a component of D. Locally we have the following description of $\widetilde{f}$: Let Δ be the unit disk. Then every point $p \in \widetilde{X}'$ has a neighbourhood isomorphic to Δ^{d+e} such that D' has, in this neighbourhood, equation $z'_1 \dots z'_k = 0$, the point $\widetilde{f}(p)$ has a neighbourhood isomorphic to Δ^d such that D has equation $z_1 \dots z_k = 0$, and the morphism $\widetilde{f}$ is the projection over the first d variables.

In these conditions, integrating along the fibres, we obtain a commutative diagram

$$\begin{tikzcd} E^*_{X'} \arrow[r, "f_*"] \arrow[d] & E^{*-2e}_X \arrow[d] \\ E^*_{\widetilde{X}'}(\log D') \arrow[r, "\widetilde{f}_*"] & E^{*-2e}_{\widetilde{X}}(\log D). \end{tikzcd}$$

Hence an induced morphism

$$f_* : GE^*_{X',Y'} \longrightarrow GE^{*-2e}_{X,Y}.$$

If X' has several components and f has a different relative dimension in each component we extend this definition by linearity.

Again composing with the adequate change of support morphism we can construct push-forward morphisms $f_* : GE^*_{X',Y'} \longrightarrow GE^{*-2e}_{X,Y}$ whenever $f(Y') \subset Y$.

The first properties of these morphisms are summarized in the next proposition.

Proposition 1.11.

1) Let Y and Y' be closed algebraic subsets of X with $Y \subset Y'$. Then there is a change of support morphism $\rho_{Y',Y} : GE^*_{X,Y} \longrightarrow GE^*_{X,Y'}$ compatible with the change of support morphism in cohomology $H^*_Y(X, \mathbb{C}) \longrightarrow H^*_{Y'}(X, \mathbb{C})$.

2) Let $f : X' \longrightarrow X$ be a morphism of complex projective manifolds and let Y and Y' be closed algebraic subsets of X and X' respectively, such that $f^{-1}(Y) \subset Y'$. Then there is a pull-back morphism $f^* : GE^*_{X,Y} \longrightarrow GE^*_{X',Y'}$ compatible with the pull-back morphism in cohomology with supports and with the pull-back of differential forms.

3) Let $f : X' \longrightarrow X$ be a smooth morphism of complex projective manifolds and let Y and Y' be closed algebraic subsets of X and X' respectively, such that $f(Y') \subset Y$. Then there is a push-forward morphism $f_* : GE^*_{X',Y'} \longrightarrow GE^{*-e}_{X,Y}$. This morphism is compatible with the push-forward morphism in cohomology with supports and with the morphism integration along the fibre between differential forms.

4) If the morphisms are defined they satisfy $(fg)^* = g^* f^*$ and $(fg)_* = f_* g_*$.

5) Let

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, "h'"'] & Z \arrow[d, "h"'] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

be a Cartesian square with f smooth. Then f' is also smooth and $h^* f_* = f'_* h'^*$.

Proof. Follows easily from the definitions.

Let $\mathcal{A}^*_X$ be the sheaf of real analytic differential forms and let $\mathcal{A}^*_{\widetilde{X}}(\log D)$ be the real analytic logarithmic Dolbeault complex introduced in [N]. Let us write $A^*_X(\log Y) = \Gamma(\widetilde{X}, \mathcal{A}^*_{\widetilde{X}}(\log D))$.

Using these complexes we can define the space of real analytic Green forms:

$$GA^*_{X,Y} = \{g \in A^*_X(\log Y) | dd^c g \in A^*_X\}/(dA^*_X(\log Y) + d^c A^*_X(\log Y)).$$

All the results of this section remain true for this group, provided that we substitute every E^*_X-module by the corresponding A^*_X-module. In particular we have that, for every cohomology class in $H^*_Y(X, \mathbb{C})$, there exists a real analytic Green form.

Note that the spaces of differentiable Green forms and analytic Green forms are not isomorphic. In fact the inclusion $A^*_X(\log Y) \longrightarrow E^*_X(\log Y)$ induces an injective morphism $GA^*_{X,Y} \longrightarrow GE^*_{X,Y}$ and an isomorphism $GA^*_{X,Y,0} \longrightarrow GE^*_{X,Y,0}$. So, whereas the space of Green forms depends on the complex we are using to define it, the space $GE^*_{X,Y,0}$, which has a description purely in terms of cohomology, is a more intrinsic object.

Example. 1.12. Let y be a Weil divisor of X and denote by Y the support of y. A real Green form g_y such that $\mathrm{cl}(g_y)$ is the cohomology class of y in $H^2_Y(X, \mathbb{C})$ will be called a Green function for y. A standard way to obtain a Green function for y is the following: Let $\|\cdot\|$ be a hermitian norm on $\mathcal{O}(y)$. Let s be a rational section of $\mathcal{O}(y)$ whose associated divisor is y. Then, the Poincaré-Lelong formula implies that $g_y = -\log \|s\|^2$ is a Green function for y.

Suppose moreover that Y is smooth and irreducible and that $y = Y$. Let ω be a closed n-form on X. Let x be the cohomology class in $H^{n+2}_Y(X, \mathbb{C})$ obtained by the pull-back of $\{\omega\}$ to $H^n(Y, \mathbb{C})$ followed by the Gysin morphism $H^n(Y, \mathbb{C}) \longrightarrow H^{n+2}_Y(X, \mathbb{C})$. Then $g_y\omega$ is a Green form and $\mathrm{cl}(g_y\omega) = x$. See [Gr-S] for a realization of the Gysin morphism in terms of differential forms from which this example follows.

Until now we have used the logarithmic Dolbeault complex to obtain Green forms. Conversely we can use Green functions to give a global construction of the logarithmic Dolbeault complex. Note that if D is a smooth divisor of X with local equation $z = 0$ then a Green form λ for D provided by the example 1.12 is a function that can be written locally as

$$\lambda = -\log z\bar{z} + f$$

with f a smooth function.

Proposition 1.13. *Let X be a complex projective manifold, and let $D = \bigcup D_i$ be a DNC. For each i, let λ_i be a Green function for D_i as in the example 1.12. Then $E^*_X(\log D)$ is the sub-E^*_X-algebra of E^*_{X-D} generated by the sections λ_i, $\partial\lambda_i$ and $\bar{\partial}\lambda_i$ for each i. The weight filtration is the multiplicative increasing filtration obtained by giving weight zero to the sections of E^*_X and weight one to the sections λ_i, $\partial\lambda_i$ and $\bar{\partial}\lambda_i$ for each i.*

Proof. This is a consequence of the fact that the sections λ_i, $\partial\lambda_i$ and $\bar{\partial}\lambda_i$ generate the sheaf $\mathcal{E}^*_X(\log D)$ locally and that all the $\mathcal{E}_X$-modules are fine and thus acyclic.

Note that this result is analogous to the global characterization of $\Omega^*_X(\log D)\otimes\mathcal{E}^*_X$ given in [Gr-S]. This proposition will be used to obtain adequate representatives of Green forms associated to algebraic cycles.

§2. The $*$-Product.

Let X be a complex projective manifold and let Y and Z be two closed algebraic subsets of X. In this section we shall show how to compute the cup-product

$$H^n_Y(X,\mathbb{C}) \otimes H^m_Z(X,\mathbb{C}) \xrightarrow{\cup} H^{n+m}_{Y\cap Z}(X,\mathbb{C})$$

in terms of logarithmic forms. Then we shall give the definition of a product

$$GE^n_{X,Z} \otimes GE^m_{X,Y} \xrightarrow{*} GE^{n+m+2}_{X,Y\cap Z}$$

which is compatible with the cup-product:

$$\mathrm{cl}(\widetilde{g}_Y * \widetilde{g}_Z) = \mathrm{cl}(\widetilde{g}_Y) \cup \mathrm{cl}(\widetilde{g}_Z).$$

This product is called $*$-product because, as we shall see, it extends the $*$-product defined by Gillet and Soulé in [G-S 2].

In order to compute the cup-product with logarithmic forms we shall need resolutions of singularities of Y, Z, $Y\cap Z$ and $Y\cup Z$ in X.

In general, if Z is a closed algebraic subset of X, $\pi : \widetilde{X} \longrightarrow X$ is a resolution of singularities of Z and Y is another algebraic closed subset of X, then the strict transform of Y, denoted $\widehat{Y}$, is the adherence of $\pi^{-1}(Y-Z)$ in $\widetilde{X}$. Note that the actual meaning of $\widehat{Y}$ depends on the resolution we are considering.

Using Hironaka's resolution of singularities ([Hi]) we can construct the following diagram:

$$\begin{array}{ccccc}
\widetilde{X}_{Y\cup Z} & \xrightarrow{p_Y} & \widetilde{X}_Y & & \\
{\scriptstyle p_Z}\downarrow & & \downarrow{\scriptstyle \pi_Y} & & \\
\widetilde{X}_Z & \xrightarrow{\pi_Z} & \widetilde{X}_{Y\cap Z} & \xrightarrow{\pi} & X,
\end{array}$$

where $\pi : \widetilde{X}_{Y\cap Z} \longrightarrow X$ is a resolution of singularities of $Y\cap Z$ such that $\widehat{Y}\cap\widehat{Z}=\emptyset$, the map π_Y (resp. π_Z) is a resolution of singularities of $\widehat{Y}$ (resp. $\widehat{Z}$) such that $\pi\circ\pi_Y$ (resp. $\pi\circ\pi_Z$) is a resolution of singularities of Y (resp. Z) and $\widetilde{X}_{Y\cup Z}$ is the fibred product of $\widetilde{X}_Y$ and $\widetilde{X}_Z$ over $\widetilde{X}_{Y\cap Z}$.

Observe that, if we restrict the Cartesian square to $\widetilde{X}_{Y\cap Z}-\widehat{Y}$ then π_Y becomes an isomorphism, hence p_Z also becomes an isomorphism. In addition if we restrict it to $\widetilde{X}_{Y\cap Z}-\widehat{Z}$ then π_Z and p_Y become isomorphisms. Since $\widehat{Y}$ and $\widehat{Z}$ do not meet, then $\widetilde{X}_{Y\cup Z}$ is covered by two open sets which are mapped isomorphically to

$\widetilde{X}_Y - \pi_{_Y}^{-1}(\widehat{Z})$ and to $\widetilde{X}_Z - \pi_{_Z}^{-1}(\widehat{Y})$ respectively. This implies, in particular, that $\widetilde{X}_{Y\cup Z}$ is smooth and that the morphism $\widetilde{X}_{Y\cup Z} \longrightarrow X$ is a resolution of singularities of $Y\cup Z$.

Using the above resolutions, we can define the complexes of sheaves $\mathcal{E}_X^*(\log Y\cap Z)$, $\mathcal{E}_X^*(\log Y)$, $\mathcal{E}_X^*(\log Z)$ and $\mathcal{E}_X^*(\log Y\cup Z)$ and the complexes $E_X^*(\log Y\cap Z)$, $E_X^*(\log Y)$, $E_X^*(\log Z)$ and $E_X^*(\log Y\cup Z)$.

Lemma 2.1. *The sequence of sheaves on X*

$$0 \to \mathcal{E}_X^*(\log Y\cap Z) \xrightarrow{(\pi_Y^*,\pi_Z^*)} \mathcal{E}_X^*(\log Y)\oplus\mathcal{E}_X^*(\log Z) \xrightarrow{p_Y^*-p_Z^*} \mathcal{E}_X^*(\log Y\cup Z) \to 0$$

is exact.

Proof. All the sheaves involved are the direct image by π of a fine sheaf on $\widetilde{X}_{Y\cap Z}$. Since fine sheaves are acyclic for the functor π_* we can reduce the problem to a local computation in $\widetilde{X}_{Y\cap Z}$. Then the lemma follows easily from the definition of these sheaves.

The cohomology groups $H^*_{Y\cap Z}(X,\mathbb{C})$ can be computed as the cohomology of the complex $\mathrm{S}^*_{X;Y\cap Z}$ (see §1) which is the simple of the morphism $E_X^* \to E_X^*(\log Y\cap Z)$. By Lemma 2.1, these cohomology groups can also be computed as the cohomology of the simple of the double complex

$$E_X^* \to E_X^*(\log Y)\oplus E_X^*(\log Z) \to E_X^*(\log Y\cup Z),$$

which will be denoted by $\mathrm{S}^*_{X;Y,Z;Y\cup Z}$. The corresponding sheaf will be denoted $\mathcal{S}^*_{X;Y,Z;Y\cup Z}$. Let us recall the definition of this simple. The graded group is

$$\mathrm{S}^n_{X;Y,Z;Y\cup Z} = E_X^n \oplus E_X^{n-1}(\log Y)\oplus E_X^{n-1}(\log Z)\oplus E_X^{n-2}(\log Y\cup Z),$$

and the differential is given by

$$d(\omega,\eta_{_Y},\eta_{_Z},\tau) = (d\omega, \omega - d\eta_{_Y}, \omega - d\eta_{_Z}, \eta_{_Y}-\eta_{_Z}+d\tau).$$

We can define a morphism of complexes

$$\psi : \mathrm{S}^*_{X;Y\cap Z} \longrightarrow \mathrm{S}^*_{X;Y,Z;Y\cup Z}$$

by

$$\psi(\omega,\eta) = (\omega,\eta,\eta,0),$$

which, by Lemma 2.1 is a quasi-isomorphism.

In terms of the complex $\mathrm{S}^*_{X;Y,Z;Y\cup Z}$ we can describe the cup-product as follows.

Proposition 2.2. *The morphism*

$$\mathrm{S}^n_{X;Y}\otimes\mathrm{S}^m_{X;Z} \xrightarrow{\wedge} \mathrm{S}^{n+m}_{X;Y,Z;Y\cup Z}$$

given by

$$(\omega_{_Y},\eta_{_Y})\wedge(\omega_{_Z},\eta_{_Z}) = (\omega_{_Y}\wedge\omega_{_Z}, \eta_{_Y}\wedge\omega_{_Z}, (-1)^n\omega_{_Y}\wedge\eta_{_Z}, (-1)^n\eta_{_Y}\wedge\eta_{_Z}).$$

is a morphism of complexes and induces in cohomology with supports the cup-product

$$H^n_Y(X,\mathbb{C}) \otimes H^m_Z(X,\mathbb{C}) \xrightarrow{\cup} H^{n+m}_{Y\cap Z}(X,\mathbb{C}).$$

Proof. Let us check that $\wedge$ is a morphism of complexes.

$$\begin{aligned}
&\wedge d\left((\omega_Y,\eta_Y)\otimes(\omega_Z,\eta_Z)\right)\\
&\quad= \wedge\left(d(\omega_Y,\eta_Y)\otimes(\omega_Z,\eta_Z) + (-1)^n(\omega_Y,\eta_Y)\otimes d(\omega_Z,\eta_Z)\right)\\
&\quad= \left((d\omega_Y,\omega_Y - d\eta_Y)\wedge(\omega_Z,\eta_Z) + (-1)^n(\omega_Y,\eta_Y)\wedge(d\omega_Z,\omega_Z - d\eta_Z)\right)\\
&\quad= \big(d\omega_Y\wedge\omega_Z, \omega_Y\wedge\omega_Z - d\eta_Y\wedge\omega_Z,\\
&\qquad\qquad\qquad (-1)^{n+1}d\omega_Y\wedge\eta_Z, (-1)^{n+1}(\omega_Y\wedge\eta_Z - d\eta_Y\wedge\eta_Z)\big)\\
&\qquad + \big((-1)^n\omega_Y\wedge d\omega_Z, (-1)^n\eta_Y\wedge d\omega_Z,\\
&\qquad\qquad\qquad \omega_Y\wedge\omega_Z - \omega_Y\wedge d\eta_Z, \eta_Y\wedge\omega_Z - \eta_Y\wedge d\eta_Z\big)\\
&\quad= d\left(\omega_Y\wedge\omega_Z, \eta_Y\wedge\omega_Z, (-1)^n\omega_Y\wedge\eta_Z, (-1)^n\eta_Y\wedge\eta_Z\right)\\
&\quad= d\left((\omega_Y,\eta_Y)\wedge(\omega_Z,\eta_Z)\right).
\end{aligned}$$

Let $\mathbb{C}_X$ be the constant sheaf. We can consider $\mathbb{C}_X$ a complex concentrated in degree zero. There is a morphism of complex of sheaves $\mathbb{C}_X \longrightarrow \mathcal{S}^*_{X;Y}$ given by $\alpha \longmapsto (\alpha, 0)$. Then we have a commutative diagram of complexes of sheaves

$$\begin{array}{ccc}
\mathbb{C}_X\otimes\mathbb{C}_X & \xrightarrow{\ \cdot\ } & \mathbb{C}_X\\
\downarrow & & \downarrow\\
\mathcal{S}^*_{X;Y}\otimes\mathcal{S}^*_{X;Z} & \xrightarrow{\ \wedge\ } & \mathcal{S}^*_{X;Y,Z;Y\cup Z},
\end{array}$$

where the upper arrow is given by multiplication.

By the functoriality of the cup-product there is an induced commutative diagram

$$\begin{array}{ccccc}
H^*_Y(X,\mathbb{C}_X)\otimes H^*_Z(X,\mathbb{C}_X) & \longrightarrow & H^*_{Y\cap Z}(X,\mathbb{C}_X\otimes\mathbb{C}_X) & \xrightarrow{\ \cdot\ } & H^*_{Y\cap Z}(X,\mathbb{C}_X)\\
\downarrow & & \downarrow & & \downarrow\\
\mathbb{H}^*_Y(X,\mathcal{S}^*_{X;Y})\otimes\mathbb{H}^*_Z(X,\mathcal{S}^*_{X;Z}) & \longrightarrow & \mathbb{H}^*_{Y\cap Z}(X,\mathcal{S}^*_{X;Y}\otimes\mathcal{S}^*_{X;Z}) & \xrightarrow{\ \wedge\ } & \mathbb{H}^*_{Y\cap Z}(X,\mathcal{S}^*_{X;Y,Z;Y\cup Z}).
\end{array}$$

This in turn, by the cohomological properties of the sheaves $\mathcal{S}$, induces a commutative diagram

$$\begin{array}{ccc}
H^*_Y(X,\mathbb{C}_X)\otimes H^*_Z(X,\mathbb{C}_X) & \xrightarrow{\ \cup\ } & H^*_{Y\cap Z}(X,\mathbb{C}_X)\\
\downarrow & & \downarrow\\
H^*(\mathrm{S}^*_{X;Y})\otimes H^*(\mathrm{S}^*_{X;Z}) & \xrightarrow{\ \wedge\ } & H^*(\mathrm{S}^*_{X;Y,Z;Y\cup Z}),
\end{array}$$

where the vertical arrows are isomorphisms.

In order to have a description of the cup-product in terms of $\mathrm{S}^*_{X;Y\cap Z}$ we need to obtain an inverse of the homomorphism $\psi : \mathrm{S}^*_{X;Y\cap Z} \longrightarrow \mathrm{S}^*_{X;Y,Z;Y\cup Z}$ for closed forms.

Since the sheaves $\mathcal{E}^*_X(\log Y \cap Z)$ are acyclic, the map

$$E^*_X(\log Y) \oplus E^*_X(\log Z) \xrightarrow{p^*_Y - p^*_X} E^*_X(\log Y \cup Z)$$

is an epimorphism. Let ϕ be a section of this map, as a map of graded vector spaces with real structure. Let us write $\phi(\tau) = (\phi_{Y,Z}(\tau), -\phi_{Z,Y}(\tau))$. Then we have the relation $\phi_{Z,Y} = 1 - \phi_{Y,Z}$.

Assume now that

$$\alpha = (\omega, \eta_Y, \eta_Z, \tau) \in \mathrm{S}^n_{X;Y,Z;Y\cup Z}$$

is closed. In particular we have $d\tau = \eta_Z - \eta_Y$. Hence $\eta_Y + d\phi_{Y,Z}(\tau) = \eta_Z - d\phi_{Z,Y}(\tau)$. But the left hand side of this equation belongs to $E^{n-1}_X(\log Y)$ and the right hand side to $E^{n-1}_X(\log Z)$; hence both belong to $E^{n-1}_X(\log Y \cap Z)$. So $\beta = (\omega, \eta_Y + d\phi_{Y,Z}(\tau))$ is a well defined element of $\mathrm{S}^n_{X;Y\cap Z}$.

Lemma 2.3.. *If the element*

$$\alpha = (\omega, \eta_Y, \eta_Z, \tau) \in \mathrm{S}^n_{X;Y,Z;Y\cup Z}$$

is closed, then

$$\beta = (\omega, \eta_Y + d\phi_{Y,Z}(\tau)) \in \mathrm{S}^n_{X;Y\cap Z}$$

is also closed, and both represent the same cohomology class in $H^n_{Y\cap Z}(X, \mathbb{C})$.

Proof. We have that $d\beta = d(\omega, \eta_Y + d\phi_{Y,Z}(\tau)) = (d\omega, \omega - d\eta_Y) = 0$. Hence β is closed. Moreover

$$\begin{aligned}
\psi(\beta) &= (\omega, \eta_Y + d\phi_{Y,Z}(\tau), \eta_Y + d\phi_{Y,Z}(\tau), 0)\\
&= (\omega, \eta_Y + d\phi_{Y,Z}(\tau), \eta_Z - d\phi_{Z,Y}(\tau), 0)\\
&= (\omega, \eta_Y, \eta_Z, \tau) + d\,(0, -\phi_{Y,Z}(\tau), \phi_{Z,Y}(\tau), 0)\\
&\in \alpha + \operatorname{Im} d.
\end{aligned}$$

Therefore α and β represent the same cohomology class.

Summing up Proposition 2.2 and Lemma 2.3 we obtain

Proposition 2.4. *Suppose that $\alpha = (\omega_Y, \eta_Y) \in \mathrm{S}^n_{X;Y}$ and $\beta = (\omega_Z, \eta_Z) \in \mathrm{S}^m_{X;Z}$ are closed. Let $\{\alpha\}$ and $\{\beta\}$ be the cohomology class they represent. Then $\{\alpha\} \cup \{\beta\}$ is represented by $(\omega_Y \wedge \omega_Z, \eta_Y \wedge \omega_Z + (-1)^n d\phi_{Y,Z}(\eta_Y \wedge \eta_Z)) \in \mathrm{S}^{n+m}_{X;Y\cap Z}$.*

Let us now define the $*$-product.

Definition 2.5. Let X be a complex projective manifold and let Y and Z be closed algebraic subsets of X. Let the maps $\phi_{Y,Z}$ and $\phi_{Z,Y}$ be as above. If $\widetilde{g}_Y \in GE^n_{X,Y}$ and $\widetilde{g}_Z \in GE^m_{X,Z}$ are Green forms, then the *$*$-product*

$$GE^n_{X,Y} \otimes GE^m_{X,Z} \xrightarrow{*} GE^{n+m+2}_{X,Y\cap Z}$$

is defined by

$$\widetilde{g}_Y * \widetilde{g}_Z = \left(d\phi_{Y,Z}(d^c g_Y \wedge g_Z) + (-1)^{n+1} d^c \phi_{Z,Y}(g_Y \wedge dg_Z) + (-1)^n d^c g_Y \wedge dg_Z\right)^{\sim}.$$

This definition of the $*$-product can be applied in the case of real analytic Green forms because the sheaves of $\mathcal{A}_X$-modules are also acyclic. This shows, in particular, that the product of two real analytic Green forms is also real analytic. In 2.7 we shall give a simpler definition of the $*$-product in terms of partitions of unity valid in the C^∞ case.

If there are chosen representatives g_Y and g_Z of $\widetilde{g}_Y$ and $\widetilde{g}_Z$ we shall denote

$$(2.5)\quad g_Y * g_Z = d\phi_{Y,Z}(d^c g_Y \wedge g_Z) + (-1)^{n+1} d^c\phi_{Z,Y}(g_Y \wedge dg_Z) + (-1)^n d^c g_Y \wedge dg_Z.$$

It is a representative of $\widetilde{g}_Y * \widetilde{g}_Z$ and depends on the choice of ϕ. Note that, after 2.7, we shall use other representatives for the $*$-product.

Extending the $*$-product by linearity we obtain a product

$$GE_X^* \otimes GE_X^* \longrightarrow GE_X^*.$$

The rest of this section will be devoted to proving the following result.

Theorem 2.6. *Let X be a complex projective manifold and let Y and Z be closed algebraic subsets of X. Let $\widetilde{g}_Y \in GE^n_{X,Y}$ and $\widetilde{g}_Z \in GE^m_{X,Z}$ be Green forms with singular support on Y and Z respectively. Then:*

1) *The $*$-product $\widetilde{g}_Y * \widetilde{g}_Z$ is well defined, i.e. it does not depend on the section ϕ nor on the representatives g_Y and g_Z.*
2) *With this product GE_X^* is a graded-commutative, associative and unitary ring. The product is compatible with the real structure and the bigrading.*
3) *The morphism*
$$\mathrm{cl} : GE_X^* \longrightarrow \bigoplus_{\substack{Y\subseteq X\\ Y \text{ closed}}} H_Y^*(X,\mathbb{C})$$
is a ring homomorphism.
4) *Let Y' and Z' be other closed algebraic subsets of X such that $Y \subset Y'$ and $Z \subset Z'$. Then the $*$-product is compatible with the change of support morphism:*
$$\rho_{Y',Y}(g_Y) * \rho_{Z',Z}(g_Z) = \rho_{Y'\cap Z',Y\cap Z}(g_Y * g_Z).$$
5) *Let $f : X' \longrightarrow X$ be a morphism between complex projective manifolds. Then the pull-back morphism $f^* : GE_X^* \longrightarrow GE_{X'}^*$ is a ring homomorphism.*
6) *Let $f : X' \longrightarrow X$ be a smooth morphism between complex projective manifolds. Let Y' be a closed algebraic subset of X' such that $f(Y') \subset Y$ and Z' another closed algebraic subset of X' such that $f^{-1}(Z) \subset Z'$. If $\widetilde{g}_{Y'}$ is a Green form on X' with singularities along Y' then the following projection formula holds:*
$$f_*(\widetilde{g}_{Y'} * f^*(\widetilde{g}_Z)) = f_*(\widetilde{g}_{Y'}) * \widetilde{g}_Z.$$

Proof. Once one knows that it is well defined, the compatibility with the real structure and the bigrading, and the distributive property are immediate.

We begin showing the independence from the section ϕ. Let $\phi' = (\phi'_{Y,Z}, -\phi'_{Z,Y})$ be another choice for section ϕ. Let $\lambda = \phi_{Y,Z} - \phi'_{Y,Z} = \phi'_{Z,Y} - \phi_{Z,Y}$. If $g \in E_X^*(\log Y \cup Z)$ then

$$\lambda(g) \in E_X^*(\log Y) \cap E_X^*(\log Z) = E_X^*(\log Y \cap Z).$$

If we denote by $*'$ the product computed with this new section then

$$g_Y * g_Z - g_Y *' g_Z = d\lambda(d^c g_Y \wedge g_Z) + (-1)^{n+1} d^c \lambda(g_Y \wedge dg_Z)$$

which is zero in $GE^*_{X,Y\cap Z}$.

Once we know the independence from the section ϕ we shall use a particular type of section obtained by means of partitions of unity. Let $\{\sigma_{Y,Z}, \sigma_{Z,Y}\}$ be a partition of unity on $\widetilde{X}_{Y\cap Z}$ subordinate to the open cover $\{\widetilde{X}_{Y\cap Z} - \widehat{Z}, \widetilde{X}_{Y\cap Z} - \widehat{Y}\}$, i.e. $\sigma_{Y,Z}$ is a C^∞ function whose value is 1 in a neighbourhood of $\widehat{Y}$ and is 0 in a neighbourhood of $\widehat{Z}$, and $\sigma_{Z,Y} = 1 - \sigma_{Y,Z}$.

If $\tau \in E^*_X(\log Y \cup Z)$ then $\sigma_{Y,Z}\tau \in E^*_X(\log Y)$. Hence the desired section is $\phi(\tau) = (\sigma_{Y,Z}\tau, -\sigma_{Z,Y}\tau)$.

This kind of section is simpler to use because we have an explicit formula for the commutator:

$$[d, \sigma_{Y,Z}] = d\sigma_{Y,Z}.$$

Lemma 2.7. *With the above notations, we have the following formula for the $*$-product*

$$\widetilde{g}_Y * \widetilde{g}_Z = \big(dd^c(\sigma_{Y,Z} g_Y) \wedge g_Z + \sigma_{Z,Y} g_Y \wedge dd^c g_Z\big)^{\sim}.$$

Proof. Substituting $\phi(\tau) = (\sigma_{Y,Z}\tau, -\sigma_{Z,Y}\tau)$ in 2.9.1 we can obtain

$$\begin{aligned}
d\phi_{Y,Z}(d^c g_Y \wedge g_Z) &+ (-1)^{n+1} d^c \phi_{Z,Y}(g_Y \wedge dg_Z) + (-1)^n d^c g_Y \wedge dg_Z \\
&= dd^c(\sigma_{Y,Z} g_Y) \wedge g_Z + \sigma_{Z,Y} g_Y \wedge dd^c g_Z + d(d^c \sigma_{Z,Y} \wedge g_Y \wedge g_Z).
\end{aligned}$$

But the form $d^c\sigma_{Z,Y}$ is zero in a neighbourhood of $\widehat{Y} \cup \widehat{Z}$ on $\widetilde{X}_{Y\cap Z}$. Hence the form $d^c\sigma_{Z,Y} \wedge g_Y \wedge g_Z$ belongs to $E^{n+m+1}_X(\log Y \cap Z)$ and we have equality in $GE^{n+m+2}_{X,Y\cap Z}$.

In the sequel, each time we need a representative for $\widetilde{g}_Y * \widetilde{g}_Z$ we shall write

$$g_Y * g_Z = dd^c(\sigma_{Y,Z} g_Y) \wedge g_Z + \sigma_{Z,Y} g_Y \wedge dd^c g_Z. \tag{2.8}$$

Next we shall prove the commutativity in the graded sense, i.e.

$$\widetilde{g}_Y * \widetilde{g}_Z = (-1)^{nm} \widetilde{g}_Z * \widetilde{g}_Y.$$

Using the representatives of the $*$-product given by 2.8, we have

$$\begin{aligned}
&g_Y * g_Z - (-1)^{nm} g_Z * g_Y \\
&\quad = dd^c(\sigma_{Y,Z} g_Y) \wedge g_Z + \sigma_{Z,Y} g_Y \wedge dd^c g_Z - (-1)^{nm}\big(dd^c(\sigma_{Z,Y} g_Z) \wedge g_Y + \sigma_{Y,Z} g_Z \wedge dd^c g_Y\big) \\
&\quad = dd^c(\sigma_{Y,Z} g_Y) \wedge g_Z + \sigma_{Z,Y} g_Y \wedge dd^c g_Z - g_Y \wedge dd^c(\sigma_{Z,Y} g_Z) - \sigma_{Y,Z} dd^c g_Y \wedge g_Z \\
&\quad = d\sigma_{Y,Z} \wedge d^c g_Y \wedge g_Z + d(d^c \sigma_{Y,Z} \wedge g_Y) \wedge g_Z - g_Y \wedge d\sigma_{Z,Y} \wedge d^c g_Z - g_Y \wedge d(d^c \sigma_{Z,Y} \wedge g_Z) \\
&\quad = d(d^c \sigma_{Y,Z} \wedge g_Y \wedge g_Z) + d^c(d\sigma_{Z,Y} \wedge g_Y \wedge g_Z).
\end{aligned}$$

But, as in the proof of Lemma 2.7, $d^c\sigma_{Y,Z} \wedge g_Y \wedge g_Z$ and $d\sigma_{Z,Y} \wedge g_Y \wedge g_Z$ belong to $E^{n+m+1}_X(\log Y \cap Z)$ concluding the proof of the commutativity.

Now, by commutativity, to show the independence from the representatives g_Y and g_Z it suffices to show the independence from g_Z. Suppose that $g_Z = da$, with $a \in E_X^*(\log Z)$. Then

$$\begin{aligned} \widetilde{g}_Y * \widetilde{g}_Z &= \big(dd^c(\sigma_{Y,Z}g_Y) \wedge da\big)^{\sim} \\ &= (-1)^n d(dd^c(\sigma_{YZ}g_Y) \wedge a)^{\sim} \\ &= 0. \end{aligned}$$

If $g_Z = d^c a$ we proceed analogously.

Let us now see the compatibility with the cup-product. By definition

$$\mathrm{cl}(\widetilde{g}_Y * \widetilde{g}_Z) = \{(dd^c(g_Y * g_Z), d^c(g_Y * g_Z))\}.$$

Writing $\omega_Y = dd^c g_Y$, $\eta_Y = d^c g_Y$, $\omega_Z = dd^c g_Z$ and $\eta_Z = d^c g_Z$ we have

$$dd^c(g_Y * g_Z) = \omega_Y \wedge \omega_Z,$$

and, using again the representatives given by 2.8, we obtain

$$d^c(g_Y * g_Z) = \eta_Y \wedge \omega_Z + (-1)^n d(\sigma_{Y,Z}\eta_Y \wedge \eta_Z) + (-1)^n d(d^c(\sigma_{Y,Z}) \wedge g_Y \wedge \eta_Z)$$

But $d^c(\sigma_{Y,Z}) \wedge g_Y \wedge \eta_Z \in E_X^*(\log Y \cap Z)$. Thus, by Proposition 2.4

$$\begin{aligned} \mathrm{cl}(\widetilde{g}_Y * \widetilde{g}_Z) &= \{(\omega_Y \wedge \omega_Z, \eta_Y \wedge \omega_Z + (-1)^n d(\sigma_{Y,Z}\eta_Y \wedge \eta_Z)\} \\ &= \mathrm{cl}(\widetilde{g}_Y) \cup \mathrm{cl}(\widetilde{g}_Z). \end{aligned}$$

To prove the associativity we shall use an argument similar to that used in the proof of commutativity but longer. Let W be another closed algebraic subset of X and $\widetilde{g}_W \in GE^*_{X,W}$. We have to show that

$$g_Y * (g_Z * g_W) - (g_Y * g_Z) * g_W = da + d^c b,$$

where $a, b \in E_X^*(\log Y \cap Z \cap W)$.

In order to compute the triple product we need several resolutions of singularities that we shall organize in the following way: Let $\widetilde{X}_{Y\cap Z\cap W}$ be a resolution of singularities of $Y \cap Z \cap W$ such that $\widehat{Y} \cap \widehat{Z} \cap \widehat{W} = \emptyset$. Recall that $\widehat{Y}$ denotes the strict transform of Y in the resolution of singularities we are considering. Let $\widetilde{X}_{Y\cap Z}$ be a resolution of singularities of $\widehat{Y} \cap \widehat{Z}$ in $\widetilde{X}_{Y\cap Z\cap W}$ which is also a resolution of singularities of $Y \cap Z$ in X and such that $\widehat{Y} \cap \widehat{Z} = \emptyset$ in $\widetilde{X}_{Y\cap Z}$. Let $\widetilde{X}_{Z\cap W}$ and $\widetilde{X}_{W\cap Y}$ be varieties with properties analogous to those of $\widetilde{X}_{Y\cap Z}$. Let now $\widetilde{X}_{Z\cap(W\cup Y)}$ be defined by the Cartesian square

$$\begin{tikzcd} \widetilde{X}_{Z\cap(W\cup Y)} \arrow[r] \arrow[d] & \widetilde{X}_{Z\cap W} \arrow[d] \\ \widetilde{X}_{Y\cap Z} \arrow[r] & \widetilde{X}_{Y\cap Z\cap W}. \end{tikzcd}$$

Let $\widetilde{X}_Z$ be a resolution of singularities of $\widehat{Z}$ in $\widetilde{X}_{Z\cap(W\cup Y)}$ which is also a resolution of singularities of Z in X. We define $\widetilde{X}_Y$ and $\widetilde{X}_W$ in the same way. Let $\widetilde{X}_{Y\cup W}$ be the product of $\widetilde{X}_Y$ and $\widetilde{X}_W$ over $\widetilde{X}_{Y\cap W}$. Note that there is an induced morphism $\widetilde{X}_{Y\cup W} \longrightarrow \widetilde{X}_{Z\cap(Y\cup W)}$. Finally we define $\widetilde{X}_{Y\cup Z\cup W}$ by the Cartesian square

$$\begin{array}{ccc} \widetilde{X}_{Y\cup Z\cup W} & \longrightarrow & \widetilde{X}_{Y\cup W} \\ \big\downarrow & & \big\downarrow \\ \widetilde{X}_Z & \longrightarrow & \widetilde{X}_{Z\cap(W\cup Y)}. \end{array}$$

Note that all the above spaces exist by Hironaka's resolution of singularities and they are smooth.

Let now $\{\sigma_{Y,Z}, \sigma_{Z,Y}\}$ (resp. $\{\sigma_{W,Y}, \sigma_{Y,W}\}$) be a partition of unity of $\widetilde{X}_{Y\cap Z}$ (resp. $\widetilde{X}_{W\cap Y}$) subordinate to the open cover $\{\widetilde{X}_{Y\cap Z} - \widehat{Z}, \widetilde{X}_{Y\cap Z} - \widehat{Y}\}$ (resp. $\{\widetilde{X}_{W\cap Y} - \widehat{Y}, \widetilde{X}_{W\cap Y} - \widehat{W}\}$).

The functions $\sigma_{*,*}$ can be pulled-back to C^∞ functions on $\widetilde{X}_{Y\cup Z\cup W}$, denoted with the same letter. Then $\operatorname{supp}\sigma_{Z,Y} \cap \operatorname{supp}\sigma_{Y,W}$ is a closed subset of $\widetilde{X}_{Y\cup Z\cup W}$. Since the morphism

$$\widetilde{X}_{Y\cup Z\cup W} \longrightarrow \widetilde{X}_{Z\cap W}$$

is proper, the image of $\operatorname{supp}\sigma_{Z,Y} \cap \operatorname{supp}\sigma_{Y,W}$ is a closed set which will be denoted by K. In $\widetilde{X}_{Z\cap W}$ we have that $K \cap \widehat{W} = \emptyset$. Therefore $\{\widetilde{X}_{Z\cap W} - \widehat{W}, \widetilde{X}_{Z\cap W} - \widehat{Z} \cup K\}$ is an open cover of $\widetilde{X}_{Z\cap W}$. Let $\{\sigma_{Z,W}, \sigma_{W,Z}\}$ be a partition of unity on $\widetilde{X}_{Z\cap W}$ subordinate to this open cover.

With these choices, in $\widetilde{X}_{Y\cup Z\cup W}$, we have

$$\sigma_{Z,Y}\sigma_{Y,W}\sigma_{W,Z} = 0. \tag{2.9}$$

Let us write

$$\begin{aligned} \sigma_{Z\cap W,Y} &= \sigma_{Z,Y}\sigma_{W,Y} \\ \sigma_{Y,Z\cap W} &= 1 - \sigma_{Z\cap W,Y} \\ \sigma_{Y\cap Z,W} &= \sigma_{Z,W}\sigma_{Y,W} \quad \text{and} \\ \sigma_{W,Y\cap Z} &= 1 - \sigma_{Y\cap Z,W}. \end{aligned} \tag{2.10}$$

Lemma 2.11.

1) The functions $\sigma_{Y\cap Z,W}$ and $\sigma_{Z\cap W,Y}$ can be extended to C^∞ functions on $\widetilde{X}_{Y\cap Z\cap W}$.

2) The functions $\{\sigma_{Y\cap Z,W}, \sigma_{W,Y\cap Z}\}$ (resp. $\{\sigma_{Z\cap W,Y}, \sigma_{Y,Z\cap W}\}$) form a partition of unity subordinate to the open cover $\{\widetilde{X}_{Y\cap Z\cap W} - \widehat{W}, \widetilde{X}_{Y\cap Z\cap W} - (\widehat{Y} \cap \widehat{Z})\}$ (resp. $\{\widetilde{X}_{Y\cap Z\cap W} - \widehat{Y}, \widetilde{X}_{Y\cap Z\cap W} - (\widehat{Z} \cap \widehat{W})\}$).

3. They satisfy the identity:

$$\sigma_{Z\cap W,Y} = \sigma_{W,Y\cap Z}\sigma_{Z,Y}.$$

Proof. 1) and 2) are direct consequences of the definitions and 3) follows from 2.9.

By Lemma 2.11, we can use these partitions of unity to construct representatives of the $*$-products. Now, a long but straightforward computation, using part 3) of

Lemma 2.11 gives

$$
\begin{aligned}
&(g_Y * g_Z)*g_W - g_Y * (g_Z * g_W)\\
&\qquad = dd^c\Big(\big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge g_W\Big)\\
&\qquad\quad + (-1)^{n+m+1}d\Big(\big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge d^c g_W\Big)\\
&\qquad\quad + (-1)^{n+m}d^c\Big(\big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge dg_W\Big).
\end{aligned}
$$

Hence the associativity follows from the following lemma:

Lemma 2.12. *The forms*

$$
\begin{aligned}
a &= \big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge g_W,\\
b &= \big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge d^c g_W \qquad \textit{and}\\
c &= \big(\sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\big)\wedge g_Z \wedge dg_W
\end{aligned}
$$

belong to $E^*_X(\log Y\cap Z\cap W)$.

Proof. Clearly a, b and c belong to $E^*_X(\log Y\cup Z\cup W)$. Thus it suffices to show that the form

$$\zeta = \sigma_{W,Z}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)$$

is zero in a neighbourhood of $\widehat{Y}\cup\widehat{Z}\cup\widehat{W}$ in $\widetilde{X}_{Y\cap Z\cap W}$. This can be checked case by case. For instance, if $x\in\widehat{Z}-\widehat{Y}\cup\widehat{W}$ in $\widetilde{X}_{Y\cap Z\cap W}$, there is a neighbourhood of x where ζ is zero because this is true for the functions $\sigma_{W,Z}$ and $\sigma_{Y,Z}$. Or, in a neighbourhood of $x\in\widehat{W}-\widehat{Y}\cup\widehat{Z}$, we have, using 2.10,

$$
\begin{aligned}
\zeta &= \sigma_{WZ}dd^c(\sigma_{Y,Z\cap W}g_Y) - \sigma_{W,Y\cap Z}dd^c(\sigma_{Y,Z}g_Y)\\
&= dd^c(\sigma_{Y,Z}g_Y) - dd^c(\sigma_{Y,Z}g_Y)\\
&= 0.
\end{aligned}
$$

The other cases follow similarly.

Let us prove now the compatibility with the change of support morphism. We have to compare the $*$-product of g_Y and g_Z viewed as Green forms with singular support on Y and Z respectively with the product of the same forms viewed as Green forms with singular support on Y' and Z'. We leave it to the reader to make explicit the different resolutions of singularities we shall need. Let $\{\sigma_{Y',Z'},\sigma_{Z',Y'}\}$ be a partition of unity used to compute the $*$-product between $GE^*_{X,Y'}$ and $GE^*_{X,Z'}$. Then, using the representatives of the product given by 2.5, we have

$$
\begin{aligned}
&\rho_{Y',Y}(g_Y)*\rho_{Z',Z}(g_Z) - \rho_{Y'\cap Z',Y\cap Z}(g_Y * g_Z)\\
&\qquad = d((\sigma_{Y,Z}-\sigma_{Y',Z'})d^c g_Y\wedge g_Z) + (-1)^{n+1}d^c((\sigma_{Y,Z}-\sigma_{Y',Z'})g_Y\wedge dg_Z).
\end{aligned}
$$

It is clear that $(\sigma_{Y,Z}-\sigma_{Y',Z'})d^c g_Y\wedge g_Z$ and $(\sigma_{Y,Z}-\sigma_{Y',Z'})g_Y\wedge dg_Z$ belong to $E_X(\log Y'\cap Z')$.

Let us prove the compatibility with the pull-back morphism. Let us write $Y' = f^{-1}(Y)$ and $Z' = f^{-1}(Z)$. Let $\widetilde{X}_{Y\cap Z}$ be as in 2.1 and let $\{\sigma_{Y,Z},\sigma_{Z,Y}\}$ be a partition of unity as in formula 2.8. Let us obtain $\widetilde{X}'_{Y'\cap Z'}$ resolving singularities of the cartesian product of X' and $\widetilde{X}_{Y\cap Z}$ over X. Then the strict transforms $\widehat{Y}'$ and $\widehat{Z}'$ of Y' and Z' in $\widetilde{X}'_{Y'\cap Z'}$ do not meet. Let us denote by $\widetilde{f}:\widetilde{X}'_{Y'\cap Z'}\longrightarrow\widetilde{X}_{Y\cap Z}$ the induced morphism. If we write $\sigma_{Y',Z'}=\widetilde{f}^*(\sigma_{Y,Z})$ and $\sigma_{Z',Y'}=\widetilde{f}^*(\sigma_{Z,Y})$ then $\{\sigma_{Y',Z'},\sigma_{Z',Y'}\}$ is a partition of unity subordinated to the open cover $\{\widetilde{X}'_{Y'\cap Z'}-\widehat{Z}',\widetilde{X}'_{Y'\cap Z'}-\widehat{Y}'\}$. Thus

$$\begin{aligned}
f^*(\widetilde{g}_Y*\widetilde{g}_Z)&=f^*(dd^c(\sigma_{Y,Z}g_Y)\wedge g_Z+\sigma_{Z,Y}g_Y\wedge dd^cg_Z)^{\sim}\\
&=(dd^c(\sigma_{Y',Z'}f^*(g_Y))\wedge f^*(g_Z)+\sigma_{Z',Y'}f^*(g_Y)\wedge dd^cf^*(g_Z))^{\sim}\\
&=f^*(\widetilde{g}_Y)*f^*(\widetilde{g}_Z).
\end{aligned}$$

Finally let us prove the projection formula. By the definition of the push-forward morphism and the compatibility with the change of support morphism, it suffices to prove the case when $Y'=f^{-1}(Y)$ and $Z'=f^{-1}(Z)$. Then using the same notations as in the proof of the compatibility with the pull-back morphism, we have

$$\begin{aligned}
f_*\big(\widetilde{g}_{Y'}*f^*(\widetilde{g}_Z)\big)-f_*(\widetilde{g}_{Y'})*\widetilde{g}_Z=&f_*\big(dd^c(\sigma_{Y',Z'}g_{Y'})\wedge f^*(g_Z)+\sigma_{Z',Y'}g_{Y'}\wedge dd^cf^*(g_Z)\big)^{\sim}\\
&-\big(dd^c(\sigma_{Y,Z}f_*(g_{Y'}))\wedge g_Z+\sigma_{Z,Y}f_*(g_{Y'})\wedge dd^cg_Z\big)^{\sim},
\end{aligned}$$

which is zero by the commutativity of f_* and f^* with the differential and the projection formula for differentiable forms.

§3. Logarithmic Forms and Currents.

Throughout this section X will denote a complex irreducible manifold not necessarily projective of dimension d.

Let $\mathcal{D}_X^n$ be the sheaf of $\mathbb{C}$-valued currents on X. Its space of sections, $\Gamma(U, \mathcal{D}_X^n)$, is the topological dual of $\Gamma_c(U, \mathcal{E}_X^{2d-n})$ with the C^∞-topology. Recall that the convergence in $\Gamma_c(U, \mathcal{E}_X^{2d-n})$ can be checked in coordinate charts and then component by component. So it is enough to study convergence in $\Gamma_c(U', \mathcal{E}_{\mathbb{C}^d})$ for U' an open set of $\mathbb{C}^d$.

The norm $\|\varphi\|_k$ of a function $\varphi \in \Gamma_c(U', \mathcal{E}_{\mathbb{C}^d})$ is

$$\|\varphi\|_k = \sup_{\substack{\zeta \in U' \\ |\alpha| \leq k}} (\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(\zeta)),$$

where $\alpha = (\alpha_1, \dots, \alpha_d, \alpha'_1, \dots, \alpha'_d) \in \mathbb{N}^{2d}$ is a multi-index, $|\alpha| = \sum \alpha_i + \alpha'_i$ and

$$\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha} = \frac{\partial^{|\alpha|}\varphi}{\partial z_1^{\alpha_1} \dots \partial z_d^{\alpha_d} \partial \bar{z}_1^{\alpha'_1} \dots \partial \bar{z}_d^{\alpha'_d}}.$$

A sequence $\{\varphi_m\}$ of functions belonging to $\Gamma_c(U', \mathcal{E}_{\mathbb{C}^d})$ tends to zero in the topology C^k if there is a compact $K \subset U'$ such that $\mathrm{supp}(\varphi_m) \subset K$ for all m, and the sequence $\{\|\varphi_m\|_k\}$ tends to zero. The same sequence tends to zero in the topology C^∞ if there is a compact $K \subset U'$ such that $\mathrm{supp}(\varphi_m) \subset K$ for all m, and, for all k, the sequence $\{\|\varphi_m\|_k\}$ tends to zero.

Let us recall some basic facts about currents. See for example [deR] and [G-H] for details. A differential is defined on the graded group $\mathcal{D}_X^*$ by $dT(\omega) = (-1)^{n+1}T(d\omega)$ for $T \in \mathcal{D}_X^n$ and $\omega \in \mathcal{E}_X^{2d-n}$. With it $\mathcal{D}_X^*$ is a complex. Moreover, it has a bigrading induced by the complex structure of X, and one can define a Hodge filtration F by

$$F^p \mathcal{D}_X^* = \bigoplus_{p' \geq p} \mathcal{D}_X^{p',q}.$$

Under this bigrading, the differential can be decomposed as usual in the sum of an operator of type (1,0) and one operator of type (0,1): $d = \partial + \bar{\partial}$.

For every locally integrable n-form ω there is a well defined current $[\omega] \in \mathcal{D}_X^n$ given by

$$[\omega](\varphi) = \int_X \omega \wedge \varphi.$$

This map induces a filtered quasi-isomorphism between $(\mathcal{E}_X^*, F)$ and $(\mathcal{D}_X^*, F)$.

If ω is a differentiable form defined in an open dense subset of X such that ω and $d\omega$ are locally integrable on X then the residue of ω is the current defined by (see [G-H, 3.1]):

$$\operatorname{Res}(\omega) = d[\omega] - [d\omega].$$

Let Y be a codimension p reduced subvariety of X. We shall denote by Y_{ns} the set of regular points of Y and by i_{ns} the inclusion $Y_{\mathrm{ns}} \longrightarrow X$. The current "integral along Y", denoted by $\delta_Y \in \Gamma(X, \mathcal{D}_X^{2p})$ is defined by

$$\delta_Y(\omega) = \int_{Y_{\mathrm{ns}}} i_{\mathrm{ns}}^* \omega.$$

This current was introduced by Lelong. In general if $y = \sum c_i Y_i$ is a codimension p algebraic cycle, we shall write $\delta_y = \sum c_i \delta_{Y_i}$. This current is closed and represents the cohomology class of y in $H^*(X, \mathbb{C})$.

Let Y be a codimension p reduced subvariety. Let $\Sigma_Y \mathcal{E}_X^*$ be the complex of forms which vanish when restricted to Y:

$$\Sigma_Y \mathcal{E}_X^* = \{\omega \in \mathcal{E}_X^* \mid i_{\mathrm{ns}}^* \omega = 0\}.$$

For any open set U let $\mathcal{D}_X^* \langle Y \rangle (U)$ be the subcomplex of $\Gamma(U, \mathcal{D}_X^*)$ composed by the currents which annihilates $\Gamma_c(U, \Sigma_Y \mathcal{E}_X^{2d-*})$. Let $\mathcal{D}_X^* \langle Y \rangle$ denote the corresponding sheaf. In particular, we have that $\mathcal{D}_X^n \langle Y \rangle = \{0\}$ for $n < 2p$. Note that this complex is nothing more than the complex of currents on Y, $\mathcal{D}_{*,Y}$, in the sense of Herrera and Lieberman ([H-L]). More precisely, we have that $\mathcal{D}_X^{2d-*} \langle Y \rangle = \mathcal{D}_{*,Y}$.

Let us denote by $\mathcal{D}_{X/Y}^*$ the quotient $\mathcal{D}_X^* / \mathcal{D}_X^* \langle Y \rangle$. As a consequence of the Hahn-Banach theorem, for any open set U, the space $\Gamma(U, \mathcal{D}_{X/Y}^n)$ is the topological dual of $\Gamma_c(U, \Sigma_Y \mathcal{E}_X^{2d-n})$ with the topology induced by the topology of $\Gamma_c(U, \mathcal{E}_X^{2d-n})$. The aim of this section is to relate the complex of sheaves $\mathcal{E}_X^*(\log Y)$ with $\mathcal{D}_{X/Y}^*$.

Proposition 3.1. *Let X be a complex manifold of dimension d and D a DNC. Then the map $[\cdot] : \mathcal{E}_X^*(\log D) \longrightarrow \mathcal{D}_{X/D}^*$ that, to each $\omega \in \Gamma(U, \mathcal{E}_X^n(\log D))$ and $\varphi \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^{2d-n})$ assigns*

$$[\omega](\varphi) = \int_U \omega \wedge \varphi$$

is a well defined morphism of double complexes.

Proof. We have to show the following:

1. The integral is convergent.
2. The functional $[\omega]$ is continuous.
3. It is a morphism of complexes.

Once it is known to be a morphism of complexes it is a morphism of double complexes because it is bigraded.

Since (1) and (2) can be checked locally using a partition of unity and (3) is local we may assume that $U = \Delta^d$, where $\Delta = \{z \in \mathbb{C} \mid z\bar{z} < 1\}$ and D has the equation $z_1 \cdots z_M = 0$. With these coordinates, a section φ belongs to $\Sigma_D \mathcal{E}_X^*$ if and only if it can be decomposed in a sum of monomials such that, for each i, $1 \leq i \leq M$, each monomial contains at least one of the terms z_i, $\bar{z}_i$, dz_i or $d\bar{z}_i$. This is a version in terms of differential forms of the fact that the ideal of differentiable functions which vanish along $D_i = \{z_i = 0\}$ is generated by z_i and $\bar{z}_i$.

Using this local characterization of $\Sigma_D \mathcal{E}_X^*$ and the definition of $\mathcal{E}_X^*(\log D)$ it is easy to see that

$$\Sigma_D \mathcal{E}_X^m \wedge \mathcal{E}_X^n(\log D) \subset \Sigma_D \mathcal{E}_X^0 \cdot \mathcal{E}_X^{n+m}(\log D).$$

For instance if D is given by the equation $z_1 = 0$ then $dz_1 \in \Sigma_D \mathcal{E}_X^1 \wedge \mathcal{E}_X^0(\log D)$ and we can write

$$dz_1 = z_1 \frac{dz_1}{z_1} \in \Sigma_D \mathcal{E}_X^0 \cdot \mathcal{E}_X^1(\log D).$$

Thus to prove (1) we suppose that $\varphi \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^0)$ and $\omega \in \Gamma(U, \mathcal{E}_X^{2d}(\log D))$.

Lema 3.2. *Let $X = \Delta^d$ and let D be the divisor defined by the equation $z_1 \cdots z_M = 0$. Let $\varphi \in \Gamma_c(X, \Sigma_D \mathcal{E}_X^0)$. Then, for all $z \in X$,*

$$|\varphi(z)| \leq 2^M \sup_{\substack{\zeta \in X \\ 1 \leq l \leq M \\ \{i_1, \dots, i_M\} = \{1, \dots, M\}}} \left| \frac{\partial^M \varphi}{\partial z_{i_1} \dots \partial z_{i_l} \partial \bar{z}_{i_{l+1}} \dots \partial \bar{z}_{i_M}}(\zeta) \right| |z_1 \cdots z_M|.$$

Proof. Restricting φ to the slice $z_{M+1} = c_{M+1}, \dots, z_d = c_d$, where $c_{M+1}, \dots, c_d$ are constants, we can assume that $M = d$. With this assumption we shall make the proof by induction over M.

If $M = 1$, then $\varphi : \Delta \longrightarrow \mathbb{C}$ is a function such that $\varphi(0) = 0$. Put $z = r\mathrm{e}^{i\theta}$. For each $z \neq 0$ there exists a ζ' with $|\zeta'| \leq |z|$ such that

$$|\varphi(z)| \leq \left| \frac{\partial \varphi}{\partial r}(\zeta') \right| |z| \, .$$

But

$$\frac{\partial \varphi}{\partial r}(z) = \mathrm{e}^{i\theta} \frac{\partial \varphi}{\partial z}(z) + \mathrm{e}^{-i\theta} \frac{\partial \varphi}{\partial \bar{z}}(z).$$

Thus

$$|\varphi(z)| \leq 2 \sup(\left| \frac{\partial \varphi}{\partial z}(\zeta') \right|, \left| \frac{\partial \varphi}{\partial \bar{z}}(\zeta') \right|)|z| \leq 2 \sup_{\zeta \in \Delta}(\left| \frac{\partial \varphi}{\partial z}(\zeta) \right|, \left| \frac{\partial \varphi}{\partial \bar{z}}(\zeta) \right|)|z|.$$

This concludes the proof in the case $M = 1$.

Let us assume that the result is true for $M - 1$. Let $z = (z_1, \dots, z_M) \in X$. Applying the case $M = 1$ in the disk $\{(z_1, \dots, z_{M-1})\} \times \Delta$ there exists a point $\zeta' = (z_1, \dots, z_{M-1}, \zeta_M)$ such that

$$|\varphi(z)| \leq 2 \sup(\left| \frac{\partial \varphi}{\partial z_M}(\zeta') \right|, \left| \frac{\partial \varphi}{\partial \bar{z}_M}(\zeta') \right|)|z_M|. \tag{3.3}$$

Applying the case $M-1$ to the function $\dfrac{\partial \varphi}{\partial z_M}$ restricted to the hyperplane $z_M = \zeta_M$, we can find a point $\zeta'' = (\zeta_1, \dots, \zeta_M)$ such that

$$\left| \frac{\partial \varphi}{\partial z_M}(\zeta') \right| \leq 2^{M-1} \sup \left| \frac{\partial^{M-1}}{\partial z_{i_1} \dots \partial z_{i_l} \partial \bar{z}_{i_{l+1}} \dots \partial \bar{z}_{i_{M-1}}} \frac{\partial \varphi}{\partial z_M}(\zeta'') \right| |z_1 \dots z_{M-1}|, \tag{3.4}$$

where the supreme runs over all partitions $\{i_1, \ldots, i_l\} \cup \{i_{l+1}, \ldots, i_{M-1}\}$ of the set $\{1, \ldots, M-1\}$. Substituting 3.4 and the analogous result for $\dfrac{\partial \varphi}{\partial \bar{z}_M}$ in 3.3 we have the lemma.

Returning to the proof of Proposition 3.1, let us prove (1). Let us write $r_i = |z_i|$. By Lemma 3.2 one has

$$|\varphi| \leq C_1 \prod_{i=1}^{M} r_i,$$

with $C_1 \in \mathbb{R}_{>0}$. Moreover, since $\omega \in \Gamma(U, \mathcal{E}_X^{2d}(\log D))$, the singularities of ω are, at worst, of the type

$$\prod_{i=1}^{M} \frac{(\log r_i)^N}{r_i^2}.$$

Hence we can write $\omega = \psi dV$ with dV a differential of volume and

$$|\psi| \leq C_2 \frac{\left| \prod_{i=1}^{M} \log r_i \right|^N}{\prod_{i=1}^{M} r_i^2},$$

where $C_2 \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ are constants. Therefore

$$\left| \int_U \varphi \omega \right| \leq \int_U C \frac{\left| \prod_{i=1}^{M} \log r_i \right|^N}{\prod_{i=1}^{M} r_i} dV,$$

where $C = C_1 C_2$. Since the integral on the right is convergent the same is true for the integral on the left.

Let us now prove (2). For this it suffices to show that if (φ_m) is a sequence with $\varphi_m \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^{2d-n})$, such that

$$\lim_{m \to 0} \varphi_m = 0$$

and $\omega \in \Gamma(U, \mathcal{E}_X^n(\log D))$, then

$$\lim_{m \to 0} [\omega](\varphi_m) = \lim_{m \to 0} \int_U \omega \wedge \varphi_m = 0.$$

By the inclusion of sheaves discussed before Lemma 3.2 we can write $\omega \wedge \varphi_m = \sum_k f_{m,k} \omega_k$, with $f_{m,k} \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^0)$ and $\omega_k \in \Gamma(U, \mathcal{E}_X^{2d}(\log D))$. The functions $f_{m,k}$ can be obtained multiplying the components of φ_m with factors of the type z_i and $\bar{z}_i$. Since convergence of forms is checked component by component, we can choose the functions $f_{m,k}$ such that

$$\lim_{m \to 0} f_{m,k} = 0.$$

So we are reduced to the case $\varphi_m \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^0)$.

Now, if (φ_m) tends to zero in the topology C^∞ it also tends to zero in the topology C^M. Thus we can apply Lemma 3.2 again to obtain

$$|\varphi_m| \leq C'_m \prod_{i=1}^{M} r_i,$$

where the sequence C'_m tends to zero. Then, as in the proof of (1) we have

$$\left|\int_U \varphi_m \omega\right| \leq \int_U C_m \frac{\left|\prod_{i=1}^M \log r_i\right|^N}{\prod_{i=1}^M r_i} dV$$

where the sequence (C_m) also tends to zero and therefore the integral on the right tends to zero. Hence the functional $[\omega]$ is continuous.

Note that we have shown that it is continuous with the C^M topology. One can construct examples with $M = 1$ where the functional $[\omega]$ is not continuous with the C^0 topology.

Let us prove (3). For $\omega \in \Gamma(U, \mathcal{E}_X^n(\log D))$ and $\varphi \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^{2d-n-1})$ we have

$$\begin{aligned} d[\omega](\varphi) - [d\omega](\varphi) &= (-1)^{n+1}[\omega](d\varphi) - [d\omega](\varphi) \\ &= \int_U (-1)^{n+1} \omega \wedge d\varphi - d\omega \wedge \varphi \\ &= \int_U -d(\omega \wedge \varphi). \end{aligned}$$

Therefore it is enough to prove the following Lemma.

Lemma 3.5. *If* $\omega \in \Gamma_c(U, \Sigma_D \mathcal{E}_X^0 \cdot \mathcal{E}_X^{2d-1}(\log D))$ *then*

$$\int_U d\omega = 0.$$

Proof. By (1) the forms ω and $d\omega$ are locally integrable. Now we have that

$$\int_U d\omega = -\lim_{\varepsilon \to 0} \int_{D_\varepsilon} \omega$$

where $D_\varepsilon = \{z \in U \mid \inf_{1 \leq i \leq M} |z_i| = \varepsilon\}$. But the domain D_ε can be decomposed as

$$D_\varepsilon = \bigcup_j D_{\varepsilon,j},$$

where $D_{\varepsilon,j} = \{z \in D_\varepsilon \mid |z_j| = \varepsilon\}$. Therefore

$$\lim_{\varepsilon \to 0} \int_{D_\varepsilon} \omega = \sum_j \lim_{\varepsilon \to 0} \int_{D_{\varepsilon,j}} \omega$$

Since for any form η we have

$$\int_{D_{\varepsilon,j}} \eta \wedge dz_j \wedge d\bar{z}_j = 0,$$

the only monomials that contribute to the integral along $D_{\varepsilon,j}$ of ω are those that do not contain $dz_j \wedge d\bar{z}_j$. Thus, using Lemma 3.2 once more, we have

$$\begin{aligned}\left|\int_{D_{\varepsilon,j}} \omega\right| &\leq \left|\int_{D_{\varepsilon,j}} C_1 |\log r_j|^N dz_j \wedge \prod_{\substack{i=1,M\\ i\neq j}} \frac{|\log r_i|^N}{r_i} \bigwedge_{\substack{i=1,d\\ i\neq j}} dz_i \wedge d\bar{z}_i\right| \\ &\leq \left|\int_{r_j=\varepsilon} C_2 |\log r_j|^N dz_j\right|\end{aligned}$$

For suitable constants $N \in \mathbb{N}$ and $C_1, C_2 \in \mathbb{R}_{\geq 0}$. This last integral tends to zero when ε tends to zero concluding the proof of Lemma 3.5 and of Proposition 3.1.

Let us derive some consequences of Proposition 3.1. Let $i : Y \longrightarrow X$ be a codimension p subvariety, and let $\pi : (\widetilde{X}, D) \longrightarrow (X, Y)$ be a resolution of singularities. Recall that, by definition, $\mathcal{E}^*_X(\log Y) = \pi_* \mathcal{E}^*_{\widetilde{X}}(\log D)$.

Corollary 3.6. *The map $[\cdot] : \mathcal{E}^*_X(\log Y) \longrightarrow \mathcal{D}^*_{X/Y}$ that, to each pair of differential forms $\omega \in \Gamma(U, \mathcal{E}^n_X(\log Y))$ and $\varphi \in \Gamma_c(U, \Sigma_Y E^{2d-n}_X)$ assigns*

$$[\omega](\varphi) = \int_U \omega \wedge \varphi$$

is a well defined morphism of double complexes.

Proof. We have that

$$\pi^*(\omega) \in \Gamma(\pi^{-1}(U), \mathcal{E}^n_{\widetilde{X}}(\log D)) \text{ and } \pi^*(\varphi) \in \Gamma_c(\pi^{-1}(U), \Sigma_D \mathcal{E}^{2d-n}_{\widetilde{X}}).$$

Moreover, since π is an isomorphism out of a set of measure zero

$$\int_X \omega \wedge \varphi = \int_{\tilde{X}} \pi^*(\omega) \wedge \pi^*(\varphi),$$

and the corollary follows from Proposition 3.1.

Remark 3.7. The morphism of Proposition 3.1 is not only a morphism of complexes, but a quasi-isomorphism and even a filtered quasi-isomorphism with respect to the Hodge filtration. This can be seen using the techniques of [Fuj]. In the next chapter we shall give a proof of this fact.

On the other hand there are examples with Y singular such that the morphism of Corollary 3.6 is not a quasi-isomorphism. This is related to the fact that, for Y singular, the cohomology of the complex of currents on Y is not necessarily isomorphic to $H^*(Y, \mathbb{C})$ ([H-L]). Nevertheless, if Y is smooth or has normal crossings the morphism of Corollary 3.6 is a quasi-isomorphism.

Corollary 3.8. *Let X be a complex manifold and let Y be a codimension p closed subvariety. Let $\omega \in \Gamma(X, \mathcal{E}^n_X(\log Y))$.*
1) If ω and $d\omega$ are locally integrable then $\mathrm{Res}(\omega) \in \Gamma(X, \mathcal{D}^{n+1}_X \langle Y \rangle)$.
2) The form ω is locally integrable for $n < 2p$.

3) If $n < 2p-1$ then $\mathrm{Res}(\omega) = 0$.

Proof. By 3.6, for each form $\varphi \in \Gamma_c(U, \Sigma_Y E_X^{2d-n+1})$ we have $\mathrm{Res}(\omega)(\varphi) = 0$. Thus $\mathrm{Res}(\omega) \in \Gamma(X, \mathcal{D}_X^{n+1}\langle Y\rangle)$. Statements 2) and 3) follow from Corollary 3.6 and the fact that for $n < 2p$ the morphism $\mathcal{D}_X^n \longrightarrow \mathcal{D}_{X/Y}^n$ is an isomorphism.

The forms of weight one are locally integrable. Thus the residue is always defined and we can obtain an explicit formula for it. See for example [N] for a real analytic analog.

Proposition 3.9. *Let $D = D_1 \cup \cdots \cup D_n$ be a DNC, and for each i, let λ_i be a Green function for D_i. For all i, let us denote by $a_i : D_i \longrightarrow X$ the inclusion. Let $\omega \in W_1 E_X^r(\log D)$. By Proposition 1.13 we can write*

$$\omega = \sum_{i=1}^{n} \alpha_i \lambda_i + \beta_i \wedge d\lambda_i + \gamma_i \wedge d^c \lambda_i + \eta,$$

where α_i, β_i, γ_i and η belong to E_X^*. Then

$$\mathrm{Res}(\omega) = (-1)^{r+1} \sum_{i=1}^{n} a_{i*}[a_i^*(\gamma_i)].$$

Proof. Let φ be a test form. Since

$$\mathrm{Res}(\omega)(\varphi) = (-1)^{r+1} \int_X \omega \wedge d\varphi - \int_X d\omega \wedge \varphi = -\int_X d(\omega \wedge \varphi),$$

we have to show that

$$\int_X d(\omega \wedge \varphi) = (-1)^r \sum_{i=0}^{n} \int_{D_i} a_i^*(\gamma_i \wedge \varphi).$$

By a partition of unity argument, we can assume that φ has compact support contained in an open coordinate subset U such that, for all i, the divisor D_i has equation $z_i = 0$. Then, in U, we have $\lambda_i = -\log z_i \bar{z}_i + b_i$ with b_i smooth.

Now, using the notations of the proof of Lemma 3.5, we have

$$\begin{aligned}
\int_X d(\omega \wedge \varphi) &= -\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \int_{D_{\varepsilon,j}} \sum_{i=1}^{n} (\alpha_i \lambda_i + \beta_i \wedge d\lambda_i + \gamma_i \wedge d^c \lambda + \eta) \wedge \varphi \\
&= \lim_{\varepsilon \to 0} \sum_{i=1}^{n} \int_{D_{\varepsilon,i}} \beta_i \wedge d \log z_i \bar{z}_i \wedge \varphi + \gamma_i \wedge d^c \log z_i \bar{z}_i \wedge \varphi \\
&= \sum_{i=1}^{n} \int_{D_i} \lim_{\varepsilon \to 0} \int_{|z_i| = \varepsilon} \beta_i \wedge \left(\frac{dz_i}{z_i} + \frac{d\bar{z}_i}{\bar{z}_i}\right) \wedge \varphi + \frac{\sqrt{-1}}{4\pi} \gamma_i \wedge \left(\frac{d\bar{z}_i}{\bar{z}_i} - \frac{dz_i}{z_i}\right) \wedge \varphi \\
&= \sum_{i=1}^{n} (-1)^r \int_{D_i} a_i^*(\gamma_i \wedge \varphi).
\end{aligned}$$

Hence the proposition.

§4. Comparison Between Green Forms and Green Currents.

In this section we shall compare the notions of Green current introduced by Gillet and Soulé in [G-S 2] and the notion of Green form. We refer the reader to [G-S 2] for details about Green currents. For simplicity we shall only deal with complex manifolds. The case of real manifolds follows the same pattern taking into account an antilinear involution F_∞.

In this section X will denote a complex projective manifold of dimension d. We shall write

$$\widetilde{E}^*_X = E^*_X/(\operatorname{Im}\partial + \operatorname{Im}\bar\partial) \qquad \text{and} \qquad \widetilde{D}^*_X = D^*_X/(\operatorname{Im}\partial + \operatorname{Im}\bar\partial).$$

Both graded spaces have a natural bigrading and a real structure and we shall denote by $\widetilde{E}^*_{X,\mathbb{R}}$ and $\widetilde{D}^*_{X,\mathbb{R}}$ the corresponding real spaces. Note that by the regularity Lemma the induced morphism $\widetilde{E}^*_X \longrightarrow \widetilde{D}^*_X$ is injective.

Let us begin giving the definitions of the spaces we shall compare.

Definition 4.1. ([G-S 2]) Let y be a codimension p algebraic cycle. Then the space of Green currents associated to y is

$$GC_X(y) = \{\widetilde{T} \in \widetilde{D}^{p-1,p-1}_{X,\mathbb{R}} \mid dd^cT + \delta_y \in [E^{p,p}_X]\},$$

where $\widetilde{D}^{p-1,p-1}_{X,\mathbb{R}} = \widetilde{D}^{p-1,p-1}_X \cap \widetilde{D}^*_{X,\mathbb{R}}$, δ_y is the current integration along y (see §3) and $[E^{p,p}_X]$ is the image of $E^{p,p}_X$ in D^*_X.

By a Green current for y we shall mean an element $\widetilde{T}_y \in GC_X(y)$ or a current T_y representing it, which is defined up to $\operatorname{Im}\partial + \operatorname{Im}\bar\partial$.

Let us write $Y = \operatorname{supp} y$.

Definition 4.2. The *space of Green forms associated to* y is defined by

$$GE_X(y) = \{\widetilde{g} \in GE^{p,p}_{X,Y} \mid \mathrm{cl}(\widetilde{g}) = \{y\} \text{ and } \widetilde{g} \text{ is real}\},$$

where $\{y\}$ is the cohomology class of y in $H^{2p}_Y(X,\mathbb{C})$.

As for Green currents, by a Green form for y we shall mean an element $\widetilde{g}_y \in GE_X(y)$ or a form representing it.

Let $g \in E^{p-1,p-1}_X(\log Y)$. By part 2) of Corollary 3.8 this form is locally integrable. Thus it defines a current $[g] \in D^{p-1,p-1}_X$. Now the aim of this section is to prove:

Theorem 4.3. *Let y be a codimension p algebraic cycle and let $Y = \operatorname{supp}(y)$. Then:*

1) *The map $\widetilde{g} \mapsto [g]^{\sim}$ induces a well defined morphism between $GE_X(y)$ and $GC_X(y)$ which is an isomorphism.*
2) *If z is a codimension q algebraic cycle which intersects with y properly, then the $*$-product of Green forms and the $*$-product of Green currents are compatible.*
3) *Let $f : X' \longrightarrow X$ be a morphism of complex projective manifolds and assume that $\operatorname{codim}(f^{-1}(Y)) = p$. Then the pull-back morphisms of Green forms and of Green currents are compatible.*
4) *If $f : X \longrightarrow X'$ is a smooth morphism of complex projective manifolds such that $\dim(f(Y)) = \dim(Y)$, then the push-forward morphisms of Green forms and of Green currents are compatible.*

Proof. By part 3) of Corollary 3.8, if $a, b \in E_X^{2p-3}(\log Y)$ then $[da + d^c b] = d[a] + d^c[b]$. Therefore the map $\widetilde{g} \mapsto [g]^{\sim}$ induces a well defined morphism $GE_X(y) \longrightarrow \widetilde{D}_X^{2p-2}$.

In fact we can obtain a morphism of this type defined on all the space of Green forms. Let us write

$$\widetilde{D}^*_{X/Y} = \frac{D^*_{X/Y}}{\operatorname{Im} d + \operatorname{Im} d^c}.$$

Then by Corollary 3.6 we have:

Lemma 4.4. *The map $[\cdot] : GE^*_{X,Y} \longrightarrow \widetilde{D}^*_{X/Y}$ which sends $\widetilde{g}$ to $[g]^{\sim}$ is a well defined morphism.*

Now we have to show that if g_y is a Green form associated to y then $[g_y]$ is a Green current associated to y. For simplicity we shall assume that Y is irreducible and that $y = Y$.

To proceed further we need a special kind of representatives of Green forms provided by the next lemma.

Lemma 4.5. *Let $(\widetilde{X}, D)$ be a resolution of singularities of (X, Y), with $D = D_0 \cup \cdots \cup D_k$ a DNC. Let λ_i be a Green function for D_i, $i = 0, \ldots, k$. If $\widetilde{g}_y$ is a Green form for y, then there are smooth $(p-1, p-1)$-forms on $\widetilde{X}$, α_i, for $i = 0, \ldots, k$, with $\alpha_i|_{D_i}$ ∂ and $\bar{\partial}$ closed, and β such that*

$$g_y = \sum_{i=0}^{k} \alpha_i \lambda_i + \beta$$

is a representative of $\widetilde{g}_y$.

Proof. Since the cohomology class of y belongs to $Gr_F^p W_{2p} H_Y^{2p}(X, \mathbb{C})$, by part 4) of Proposition 1.8, there is a representative g'_y of $\widetilde{g}_y$ of weight one and type $(p-1, p-1)$. By proposition 1.13 we can write g'_y as

$$g'_y = \sum_{i=0}^{k} \alpha'_i \lambda_i + \gamma_i \wedge \partial \lambda_i + \delta_i \wedge \bar{\partial} \lambda_i + \beta,$$

where β and α'_i are smooth $(p-1,p-1)$-forms, γ_i are smooth $(p-2,p-1)$-forms and δ_i are smooth $(p-1,p-2)$-forms. Writing $\alpha_i = \alpha'_i + \partial\gamma_i + \bar{\partial}\delta_i$ we have that

$$g_y = \sum_{i=0}^{k} \alpha_i \lambda_i + \beta$$

also represents $\widetilde{g}_y$ because $g'_y - g_y \in \operatorname{Im}\partial + \operatorname{Im}\bar{\partial}$.

To prove that the restriction of α_i to D_i is ∂ and $\bar{\partial}$ closed recall that, by the definition of Green forms,

$$\partial\bar{\partial}\widetilde{g}_y = \partial\bar{\partial}g_y = \sum_{i=0}^{k} \partial\bar{\partial}\alpha_i\lambda_i - \bar{\partial}\alpha_i \wedge \partial\lambda_i + \partial\alpha_i \wedge \bar{\partial}\lambda_i + \partial\bar{\partial}\beta$$

is a smooth form. But since $\partial\lambda_i$ (resp. $\bar{\partial}\lambda_i$) has a singularity of the type dz_i/z_i (resp. $d\bar{z}_i/\bar{z}_i$) along D_i the form $\bar{\partial}\alpha_i$ (resp. $\partial\alpha_i$) vanish when restricted to D_i.

We shall call *basic* Green forms for y to the Green forms provided by this lemma. Note that basic Green forms should not be confused with Green forms of logarithmic type defined in [G-S 2, 1.3.2]. See 4.12 below.

Notation 4.6. We can obtain a resolution of singularities of (X,Y), say $(\widetilde{X},D)$ with D a DNC ([Hi]), obtaining first a pair $(\widetilde{X}',\widetilde{Y}\cup D')$ where $\widetilde{Y}$ is a resolution of singularities of Y, D' is a DNC which is mapped over the singular locus of Y and $\widetilde{Y}\cup D'$ has normal crossings. And then we obtain $\widetilde{X}$ as the blow up of $\widetilde{X}'$ along $\widetilde{Y}$. Let us now write $D' = D_1 \cup \cdots \cup D_k$ and denote by D_0 the exceptional divisor of this last blow up.

Let us give names to the different morphisms we shall need. Let π be the morphism $\widetilde{X} \longrightarrow X$ and let π_i be the restriction $\pi|_{D_i}$. Let a_i be the inclusion $D_i \longrightarrow \widetilde{X}$. Let φ be the induced morphism $\widetilde{Y} \longrightarrow X$ and finally let us write π'_0 for the induced morphism $D_0 \longrightarrow \widetilde{Y}$.

Let $\widetilde{g}_y \in GE_X(y)$ be a Green form for y and let

$$g_y = \sum_{i=0}^{k} \alpha_i \lambda_i + \beta$$

be a basic Green form representing it. By part 3) of Corollary 3.8 $d^c[g_y] = [d^c g_y]$. Thus

$$[dd^c g_y] - dd^c[g_y] = [dd^c g_y] - d[d^c g_y] = \operatorname{Res}(d^c g_y).$$

By Proposition 3.9 we have

$$\operatorname{Res}(d^c g_y) = \sum_i \pi_{i*}[a_i^*\alpha_i].$$

Given that the forms α_i have type $(p-1,p-1)$ and, for $i=1,\dots,k$ the fibres of π_i have dimension greater or equal than p, we have $\pi_{i*}[a_i^*\alpha_i] = 0$. Therefore

$$\operatorname{Res}(d^c g_y) = \pi_{0*}[a_0^*\alpha_0] = \varphi_*\pi'_{0*}[a_0^*\alpha_0].$$

But $\pi'_{0*}[a_0^*\alpha_0]$ is a closed current of degree zero on $\widetilde{Y}$, thus it is of the form $c\delta_{\widetilde{Y}}$ for a constant c. Hence

$$\mathrm{Res}(d^c g_y) = c\delta_Y.$$

Let us see now that c is determined by the cohomology class of $(dd^c g_y, d^c g_y)$. There is a commutative diagram

$$\begin{array}{ccc} E_X^* & \longrightarrow & E_X^*(\log Y) \\ \big\downarrow & & \big\downarrow \\ D_X^* & \longrightarrow & D_{X-Y}^*, \end{array}$$

where the vertical arrows are quasi-isomorphisms. In the simple of the morphism $D_X^* \longrightarrow D_{X-Y}^*$ we have

$$([dd^c g_y], [d^c g_y]) = (\mathrm{Res}(d^c g_y), 0) + d([d^c g_y], 0).$$

Therefore the cohomology class $\mathrm{cl}(\widetilde{g}_y) = \{(dd^c g_y, d^c g_y)\} \in H_Y^*(X, \mathbb{C})$ is equal to $\{(c\delta_Y, 0)\}$. Given that $\mathrm{cl}(\widetilde{g}_y)$ is the cohomology class of $y = Y$, then $\mathrm{Res}(d^c g_y) = \delta_y$. Hence, writing $\omega_y = dd^c g_y \in E_X^{p,p}$, we have

$$dd^c[g_y] + \delta_y = \omega_y.$$

So the current $[g_y]$ is a Green current for y.

Note that we have also proved that, for $\eta \in E_X^{d-p+1,d-p+1}$,

$$\int_{D_0} a_0^*(\alpha_0 \wedge \pi^*(\eta)) = \int_{\widetilde{Y}} \varphi^*(\eta). \tag{4.7}$$

This formula also holds for a locally integrable form η, differentiable in an open dense subset U, with $U \cap Y$ dense and such that its pull-back to $\widetilde{Y}$ is locally integrable.

Once we know that $[\cdot] : GE_X(y) \longrightarrow GC_X(y)$ is a well defined morphism, let us show that it is an isomorphism.

It is proved in [G-S 2, 1.2] that the space of Green currents for y is an affine space over the vector space

$$\widetilde{E}_{X,\mathbb{R}}^{p-1,p-1} = \frac{E_{X,\mathbb{R}}^{p-1,p-1}}{\mathrm{Im}\, d + \mathrm{Im}\, d^c},$$

i.e. for every y there exists a Green current and if $\widetilde{T}_y$ is a fixed Green current then

$$GC_X(y) = \widetilde{T}_y + [\widetilde{E}_{X,\mathbb{R}}^{p-1,p-1}].$$

On the other hand, as a consequence of part 3) of Proposition 1.8 the space $GE_X(y)$ is an affine space over the vector space $E_{X,\mathbb{R}}^{p-1,p-1}/B$, where B is the subgroup

$$B = E_{X,\mathbb{R}}^{p-1,p-1} \cap (dE_X^*(\log Y) + d^c E_X^*(\log Y)).$$

Let $\omega \in B$. By part 2) of Lemma 1.7

$$\omega = \omega_1 + da + d^c b,$$

with $a, b \in E^*_X$, $d\omega_1 = 0$ and such that $j^*(\{\omega_1\}) = 0$ in $H^{2p-2}(X - Y, \mathbb{C})$. But since $j^* : H^{2p-2}(X, \mathbb{C}) \longrightarrow H^{2p-2}(X - Y, \mathbb{C})$ is an isomorphism ω_1 is d-exact in E^*_X. Therefore

$$B = E^{p-1,p-1}_{X,\mathbb{R}} \cap (dE^*_X + d^c E^*_X).$$

Hence the space $GE_X(y)$ is also an affine space over the vector space $\widetilde{E}^{p-1,p-1}_{X,\mathbb{R}}$. Since the morphism $[\cdot]$ preserves these structures of affine spaces it is an isomorphism.

In order to prove the compatibility of the push-forward morphisms we shall prove first a preliminary result. Let $f : X \longrightarrow X'$ be a smooth morphism of relative dimension e. Let Y a be a closed algebraic subset of X and let us write $Y' = f(Y)$. Since f is proper and $Y \subset f^{-1}(Y')$ there is and induced morphism of double complexes $f_* : D^{*,*}_{X/Y} \longrightarrow D^{*-e,*-e}_{X'/Y'}$ which induces a morphism $f_* : \widetilde{D}^{*,*}_{X/Y} \longrightarrow \widetilde{D}^{*-e,*-e}_{X'/Y'}$.

Lemma 4.8. *The diagram*

$$\begin{array}{ccc} GE^n_{X,Y} & \xrightarrow{[\cdot]} & \widetilde{D}^{n-2}_{X/Y} \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ GE^{n-2e}_{X',Y'} & \xrightarrow{[\cdot]} & \widetilde{D}^{n-2-2e}_{X'/Y'}. \end{array}$$

is commutative.

Proof. Let $\widetilde{g} \in GE^n_{X,Y}$. Let us compute $[f_*\widetilde{g}]$. Recall that to obtain $f_*\widetilde{g}$ we have to use first the restriction morphism $GE^n_{X,Y} \longrightarrow GE^n_{X,f^{-1}(Y')}$ and then compute the direct image.

Let $(\widetilde{X}, D)$ be the resolution of singularities used to define g; thus we have $g \in E^{n-2}_{\widetilde{X}}(\log D)$. Let us construct a cartesian square

$$\begin{array}{ccc} (\widetilde{X}_1, D_1) & \longrightarrow & (X, f^{-1}(Y)) \\ {\scriptstyle \widetilde{f}}\Big\downarrow & & {\scriptstyle f}\Big\downarrow \\ (\widetilde{X}', D') & \longrightarrow & (X', Y'), \end{array}$$

where the horizontal arrows are resolutions of singularities and the vertical arrows are smooth. Finally let $(\widetilde{X}_2, D_2)$ be a resolution of singularities of $(X, f^{-1}(Y))$ which dominates $(\widetilde{X}, D)$ and $(\widetilde{X}_1, D_1)$. Then there are forms $a, b \in E^{n-3}_{\widetilde{X}_2}(\log D_2)$ such that

$$g' = g + \partial a + \bar{\partial} b \in E^{n-2}_{\widetilde{X}_1}(\log D_1). \tag{4.9}$$

The push-forward of $\widetilde{g}$ is defined by

$$f_*\widetilde{g} = (f_* g')^{\sim}.$$

By 4.9, Corollary 3.6 and the compatibility of f_* with the differential we have, in $\widetilde{D}^{n-2e}_{X'/Y'}$,

$$(f_*[g])^{\sim} = (f_*[g'])^{\sim}.$$

Therefore we have

$$f_*[\widetilde{g}] = (f_*[g])^{\sim} = (f_*[g'])^{\sim} = [(f_*g')^{\sim}] = [f_*\widetilde{g}].$$

Let us now prove the compatibility of the push-forward morphism for Green currents and for Green forms. Let y be a codimension p algebraic cycle on X and let $Y = \operatorname{supp} y$. For simplicity we shall assume that Y is irreducible, $y = Y$ and that $\dim(f(Y)) = \dim(Y)$. Let $\widetilde{g}_y \in GE_X(y) \subset GE^{p,p}_{X,Y}$. By Lemma 4.8 we have in $\widetilde{D}^{p-e-1,p-e-1}_{X'/Y'}$

$$[f_*\widetilde{g}_y] = f_*[\widetilde{g}_y]. \tag{4.10}$$

Since $\operatorname{codim}(Y') = p - e$ we have an isomorphism

$$\widetilde{D}^{p-e-1,p-e-1}_{X'} \longrightarrow \widetilde{D}^{p-e-1,p-e-1}_{X'/Y'}.$$

Therefore the equality 4.10 holds in $GC_{X'}(f_*y)$.

Observe that if $\dim(f(Y)) < \dim(Y)$ then $f_*(y) = 0$ but the argument also works.

Next we shall prove the compatibility with the pull-back morphisms. Both morphisms have formally the same formula ([G-S 2, 2.1.3.1]):

$$[f^*\widetilde{g}_y] = [f^*g_y]^{\sim}. \tag{4.11}$$

Nevertheless we still have to solve a technical problem because Green forms of logarithmic type are used to define pull-back of Green currents. And, in general Green forms of logarithmic type are not Green forms with the definition given here. We shall now discuss the relationship between the two classes of forms.

For the convenience of the reader let us quote the definition of forms of logarithmic type.

Definition 4.12. ([G-S 2, 1.3.2]) Let $Y \subset X$ be a closed algebraic subset. A smooth form η on $X - Y$ is said to be a *form of logarithmic type* along Y if there exist a projective morphism

$$\pi : X' \longrightarrow X$$

and a smooth form φ on $X' - \pi^{-1}(Y)$ such that

(i) X' is smooth, $\pi^{-1}(Y)$ is a divisor with normal crossings and π is smooth over $X - Y$.

(ii) η is the direct image of the restriction of φ to $X' - \pi^{-1}(Y)$.

(iii) For any point $x \in X'$ there is an open neighbourhood U of x, and a system of holomorphic coordinates $(z_1, \dots, z_n)$ of U centred at x such that $\pi^{-1}(Y) \cap U$ has equation $z_1 \cdots z_k = 0$ for some $k \leq n$, and there exist smooth ∂ and $\bar{\partial}$-closed forms α_i on U, $i = 1, \dots, k$, and a smooth form β on U with

$$\varphi|_U = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta.$$

A Green form of logarithmic type for a cycle y is a form of logarithmic type, g, such that $[g]$ is a Green current for y.

Note that a Green form of logarithmic type on X is the direct image of a basic Green form on X', but since the morphism $X' \longrightarrow X$ cannot be factored in general by a smooth morphism followed by a birrational morphism it is not a Green form. On the other hand a basic Green form satisfies all the conditions above except for the functions α_i of (iii) being ∂ and $\bar{\partial}$-closed. In fact one can construct examples where, if one imposes X' being birrational with X, then the condition of being closed in (iii) cannot be satisfied.

Let $\widetilde{g}_y$ be a Green form for y and let $[\widetilde{g}_y]$ the corresponding Green current. Following the construction of Green forms of logarithmic type ([G-S, 1.3.5]) one can obtain a diagram

$$(\widetilde{X}', D') \xrightarrow{\pi_2} (X', Y') \xrightarrow{f} (\widetilde{X}, D) \xrightarrow{\pi_1} (X, Y),$$

where $(\widetilde{X}, D)$ is a resolution of singularities of (X, Y), with $D = D_1 \cup \cdots \cup D_k$. The manifold X' is the disjoint union of $\widetilde{X}$ and the varieties $D_i \times \widetilde{X}$ for each i. The closed subset Y' is the graph of the inclusion $\coprod D_i \longrightarrow X'$. And $(\widetilde{X}', D')$ is a resolution of singularities of (X', Y'). Note that f is smooth and $f(Y') = D$.

Then there exists a form φ on $\widetilde{X}'$ satisfying the conditions of Definition 4.12 and, if we write $\pi : \widetilde{X}' \longrightarrow X$ for the composition map, then $(\pi_*[\varphi])^{\widetilde{}} = [\widetilde{g}_y]$ in $\widetilde{D}_X^{p-1,p-1}$.

By construction $\varphi \in E^*_{X'}(\log Y')$ and $dd^c\varphi \in E^*_{X'}$. So it defines a Green form $\widetilde{\varphi} \in GE^*_{X',Y'}$. Since f is smooth, there is also defined a Green form $f_*\widetilde{\varphi} \in GE^*_{\widetilde{X},D}$.

Using Proposition 1.8, it is easy to see that the morphism $\pi_1^* : GE^*_{X,Y} \longrightarrow GE^*_{\widetilde{X},D}$ is injective and its image is the set of those Green forms $\widetilde{g}$ such that $dd^c g \in E^*_X$. Therefore there is a uniquely determined Green form $\widetilde{g}'_y \in GE^*_{X,Y}$ such that $\pi_1^*\widetilde{g}'_y = f_*\widetilde{\varphi}$. Moreover by Lemma 4.4 we have, in $\widetilde{D}^*_{\widetilde{X}/D}$, $[\pi_1^*\widetilde{g}'_y] = [f_*\widetilde{\varphi}]$.

Using Lemma 4.8 we obtain

$$\pi_{1*}[\pi_1^*\widetilde{g}'_y] = \pi_{1*}[f_*\widetilde{\varphi}] = \pi_{1*}f_*[\widetilde{\varphi}] = (\pi_*[\varphi])^{\widetilde{}} = [\widetilde{g}_y].$$

Now by the isomorphism between the space of Green currents for y and the space of Green forms for y and the fact that π_1 is birrational we obtain $\widetilde{g}'_y = \widetilde{g}_y$. Therefore

$$f_*\widetilde{\varphi} = \pi_1^*\widetilde{g}_y. \tag{4.13}$$

Let $h : Z \longrightarrow X$ be a morphism with Z a complex irreducible projective manifold such that $h(Z) \not\subset Y$. We can obtain a diagram

$$\begin{array}{ccccccc}
\widetilde{Z}' & \xrightarrow{\pi_2'} & Z' & \xrightarrow{f'} & \widetilde{Z} & \xrightarrow{\pi_1'} & Z \\
{\scriptstyle \widetilde{h}'}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle \widetilde{h}}\downarrow & & {\scriptstyle h}\downarrow \\
\widetilde{X}' & \xrightarrow{\pi_2} & X' & \xrightarrow{f} & \widetilde{X} & \xrightarrow{\pi_1} & X,
\end{array}$$

where π_1' and π_2' are resolutions of singularities of $(Z, h^{-1}(Y))$ and $(Z', h'^{-1}(Y'))$ respectively and the centre square is a Cartesian square. Hence f' is smooth. Let us write π' for the composition map $\widetilde{Z}' \longrightarrow Z$. Then we have

$$\begin{aligned} h^*[\pi_*\varphi]^{\sim} &= [h^*\pi_*\varphi]^{\sim} && \text{by [G-S 2, 2.1.3.1]} \\ &= [\pi'_*\widetilde{h}'^*\varphi]^{\sim} \\ &= \pi'_{1*}[f'_*h'^*\widetilde{\varphi}] && \text{by Lemma 4.8} \\ &= \pi'_{1*}[\widetilde{h}^*f_*\widetilde{\varphi}] && \text{by 1.11.5} \\ &= \pi'_{1*}[\widetilde{h}^*\pi_1^*\widetilde{g}_y] && \text{by 4.13} \\ &= \pi'_{1*}[\pi_1'^*h^*\widetilde{g}_y] \\ &= [h^*\widetilde{g}_y]. \end{aligned}$$

Hence we have compatibility of the pull-back morphisms.

Let us now prove the compatibility of the $*$-product of forms and the $*$-product of currents. For simplicity we shall suppose that $y = Y$ and $z = Z$ with Y and Z irreducible subvarieties.

Let $\widetilde{X}_{Y\cap Z}$, $\widetilde{X}_Y$, $\widetilde{X}_Z$ and $\widetilde{X}_{Y\cup Z}$ be as in §2. To simplify notations we shall write $\widetilde{X} = \widetilde{X}_{Y\cup Z}$ and denote by π the morphism $\pi : \widetilde{X} \longrightarrow X$. Then $\pi^{-1}(Y)$, $\pi^{-1}(Z)$ and $\pi^{-1}(Y\cap Z)$ are unions of irreducible components of the exceptional divisor. Thus we can write

$$\begin{aligned} D_Y &= \pi^{-1}(Y) = D_0 \cup \cdots \cup D_k, \\ D_Z &= \pi^{-1}(Z) = D_r \cup \cdots \cup D_s, \\ D_{Y\cap Z} &= \pi^{-1}(Y\cap Z) = D_r \cup \cdots \cup D_k \text{ and} \\ D_{Y\cup Z} &= \pi^{-1}(Y\cup Z) = D_0 \cup \cdots \cup D_s. \end{aligned}$$

Let us assume furthermore that D_0 is the only component of D which is mapped dominantly over Y. We shall also assume that $\widetilde{X}_Y$ is obtained as in 4.6 and let $\widetilde{Y}$ and $\varphi : \widetilde{Y} \longrightarrow X$ have the same meaning as in 4.6. Note that the morphism φ factorizes through a morphism $\widetilde{Y} \longrightarrow \widetilde{X}_Z$. This last morphism will also be called φ.

If g_z is a basic Green form we can define, analogously to [G-S 2, 2.1.3.2], a current $g_z \wedge \delta_Y$ by

$$g_z \wedge \delta_Y = \varphi_*[\varphi^* g_z].$$

Lemma 4.14. *Let Y and Z be irreducible subvarieties of codimensions p and q, which intersect properly. Let $y = Y$ and $z = Z$ be the corresponding algebraic cycles and let g_y and g_z be basic Green forms for y and z respectively. Let us write $\omega_z = dd^c g_z$. Then we have the equality of currents*

$$[g_y * g_z] = \delta_Y \wedge g_z + [g_y \wedge \omega_z] + d[d^c(\sigma_{Y,Z}\, g_y) \wedge g_z] + d^c[\sigma_{Y,Z}\, g_y \wedge dg_z].$$

Thus, in $\widetilde{D}_X^{p+q-1,p+q-1}$ we have

$$[\widetilde{g}_y * \widetilde{g}_z] = (\delta_Y \wedge g_z + g_y \wedge \omega_z)^{\sim}.$$

Proof. Using the representatives of the $*$-product given by 2.8 we have

$$\begin{aligned} g_y * g_z &= dd^c(\sigma_{Y,Z}\, g_y) \wedge g_z + \sigma_{Z,Y}\, g_y \wedge dd^c g_z \\ &= d(d^c(\sigma_{Y,Z}\, g_y) \wedge g_z) + d^c(\sigma_{Y,Z}\, g_y \wedge dg_z) + g_y \wedge dd^c g_z. \end{aligned}$$

Thus the lemma is a consequence of the equalities

$$[d(d^c(\sigma_{Y,Z}g_y)\wedge g_z)] = d[(d^c(\sigma_{Y,Z}g_y)\wedge g_z)] + \delta_Y \wedge g_z \tag{4.15}$$

and

$$[d^c(\sigma_{Y,Z}g_y \wedge dg_z)] = d^c[(\sigma_{Y,Z}g_y \wedge dg_z)]. \tag{4.16}$$

Let us prove these equalities. Since g_y and g_z are basic Green forms we can write

$$g_y = \sum_{i=1}^{k} \alpha_i\lambda_i + \beta \quad \text{and}$$
$$g_z = \sum_{i=r}^{s} \alpha'_i\lambda_i + \beta',$$

with λ_i a Green function for D_i. Let $\eta \in E_X^{d-p-q+1,d-p-q+1}$ be a test form. Then

$$[d(d^c(\sigma_{Y,Z}g_y)\wedge g_z)](\eta) - d[(d^c(\sigma_{Y,Z}g_y)\wedge g_z)](\eta) = \int_{\widetilde{X}} d(d^c(\sigma_{Y,Z}g_y)\wedge g_z \wedge \eta).$$

By a partition of unity argument we can assume that η has compact support contained in an open coordinate set and that, for all i, the divisor D_i has equation $z_i = 0$. Then, with the notations of 3.5, we have

$$\int_{\widetilde{X}} d(d^c(\sigma_{Y,Z}g_y)\wedge g_z \wedge \eta) = -\sum_{j=0}^{s} \lim_{\varepsilon\to 0}\int_{D_{\varepsilon,j}} d^c(\sigma_{Y,Z}g_y)\wedge g_z \wedge \eta.$$

By the argument used in the proof of Proposition 3.9 we obtain

$$\lim_{\varepsilon\to 0}\int_{D_{\varepsilon,0}} d^c(\sigma_{Y,Z}g_y)\wedge g_z \wedge \eta = \int_{D_0} (\alpha_0 \wedge g_z \wedge \eta)|_{D_0}.$$

Now observe that g_z is a differentiable form defined on $\widetilde{X}_Z - D_Z$. Thus $\varphi^* g_z$ is well defined and locally L^1. By 4.6 we obtain

$$\int_{D_0} (\alpha_0 \wedge g_z \wedge \eta)|_{D_0} = (\delta_Y \wedge g_z)(\eta).$$

To prove (4.15) it remains to be shown that, for $j = 1, \ldots, s$,

$$\lim_{\varepsilon\to 0}\int_{D_{\varepsilon,j}} d^c(\sigma_{Y,Z}g_y)\wedge g_z \wedge \eta = 0.$$

For $j = 1, \ldots, r-1$ is due to the fact that $g_y \wedge \pi^*(\eta)$ is a $(d-p, d-p)$-form defined on $\widetilde{X}_Z$ and the image of D_j in $\widetilde{X}_Z$ has dimension strictly less than $d-p$. Hence the restriction of $g_y \wedge \pi^*(\eta)$ to D_j is zero.

For $j = r, \dots, k-1$ is true because the restriction of $\pi^*(\eta)$ to D_j is zero and the singularities of $d^c g_y \wedge g_z$ are of the type

$$\frac{dz_j}{z_j} \log z_j \bar{z}_j \qquad \text{or} \qquad \frac{d\bar{z}_j}{\bar{z}_j} \log z_j \bar{z}_j.$$

Finally for $j = k, \dots, s$ is true because $\sigma_{Y,Z}$ is zero in a neighbourhood of $D_k \cup \dots \cup D_s$.

The proof of (4.16) is similar.

Now the product of Green forms and of Green currents are given by formula formally by the same formula

$$\widetilde{g}_y * \widetilde{g}_z = (\delta_Y \wedge g_z + g_y \wedge \omega_z)^{\sim}.$$

In one case with basic Green forms and in the other case with Green forms of logarithmic type. Observe that $\delta_Y \wedge g_z$ in this formula is defined using pull-back of Green forms. Hence the compatibility of the $*$-product follows from the compatibility of pull-backs.

CAPITULO III

Green Forms and Deligne Cohomology

§1. Real Deligne Cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$. Throughout this chapter we shall work with smooth algebraic varieties over $\mathbb{C}$, but viewed as complex manifolds i.e. with the analytic topology. In this section we shall show how to construct real Deligne-Beilinson cohomology of X in terms of a single complex of differential forms: $\mathfrak{D}^*(E^*_{\log}(X),\cdot)$. In the case of X being a smooth projective variety this complex has been studied by Wang in [Wa].

A variant of this complex has also been used by Demailly in [Dem] to study the properties of $\partial\bar{\partial}$-cohomology.

Let us recall briefly the definition of Deligne-Beilinson cohomology. See [Be 1], [E-V] and [J] for details. Let us choose a smooth compactification $j : X \longrightarrow \overline{X}$, with $D = \overline{X} - X$ a divisor with normal crossings. Let Λ be a subring of $\mathbb{R}$. We write $\Lambda(p) = (2\pi i)^p\Lambda \subset \mathbb{C}$. We will denote also by Λ and $\Lambda(p)$ the corresponding constant sheaves in the analytic topology. Let Ω^*_X be the sheaf of holomorphic forms on X and let $\Omega^*_{\overline{X}}(\log D)$ be the sheaf of holomorphic forms on $\overline{X}$ with logarithmic singularities along D [De 1]. Let F^p be the Hodge filtration of $\Omega^*_{\overline{X}}(\log D)$:

$$F^p\Omega^*_{\overline{X}}(\log D) = \bigoplus_{p'\geq p} \Omega^{p'}_{\overline{X}}(\log D).$$

Since j is affine, $Rj_*\Omega^*_X = j_*\Omega^*_X$. Moreover, in the derived category, there are natural maps

$$u_1 : Rj_*\Lambda(p) \longrightarrow j_*\Omega^*_X \qquad \text{and} \qquad u_2 : \Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega^*_X.$$

If (A^*, d) is a complex we shall write $A[k]^*$ for the complex $A[k]^n = A^{k+n}$, with differential $(-1)^k d$. If $f : A^* \longrightarrow B^*$ is a morphism of complexes, the simple of f is the complex

$$s(f)^* = A^* \oplus B[-1]^*,$$

with differential $d(a, b) = (da, f(a) - db)$.

The Deligne-Beilinson complex of the pair $(X, \overline{X})$ is

$$\Lambda(p)_{\mathcal{D}} = s(u : Rj_*\Lambda(p) \oplus F^p\Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega^*_X),$$

where $u(a, \omega) = u_2(\omega) - u_1(a)$.

The Λ-Deligne cohomology groups of X are defined by

$$H^*_{\mathcal{D}}(X, \Lambda(p)) = \mathbb{H}^*(\overline{X}, \Lambda(p)_{\mathcal{D}}).$$

These groups are independent from the compactification $\overline{X}$.

If $Y \subset X$ is a closed algebraic subset, then there are also defined Deligne cohomology groups of X with supports on Y, denoted $H^*_{\mathcal{D},Y}(X,\Lambda(p))$. Moreover, using simplicial techniques, we can define Deligne cohomology groups for singular varieties. There is also the dual notion of Deligne homology groups denoted by $H^{\mathcal{D}}_*(X,\Lambda(p))$. Deligne cohomology and homology groups form a twisted Poincaré duality theory in the sense of Bloch and Ogus [B-O].

We can compare Λ-Deligne cohomology with cohomology with coefficients in Λ by means of the exact sequence

$$0 \longrightarrow s(F^p\Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega^*_X) \rightarrow \Lambda(p)_{\mathcal{D}} \rightarrow Rj_*\Lambda(p) \rightarrow 0.$$

From this sequence we obtain:

Proposition 1.1. *Let X be a smooth variety over $\mathbb{C}$ and let Λ a subring of $\mathbb{R}$. Then there is a cohomology long exact sequence*

$$H^{n-1}(X,\mathbb{C})\big/F^pH^{n-1}(X,\mathbb{C}) \rightarrow H^n_{\mathcal{D}}(X,\Lambda(p)) \rightarrow H^n(X,\Lambda(p)) \rightarrow .$$

Λ-Deligne cohomology studies the relationship between the Λ-structure and the Hodge filtration in cohomology. In general, we do not have a complex which gives us both the Λ-structure and the Hodge filtration. For this reason, we have to construct Deligne cohomology from a diagram of complexes. On the other hand, in the case of real Deligne cohomology we can construct a complex, $E^*_{\log}(X)$, which carries the real structure and the Hodge filtration. Using this, we can give simpler representatives of real Deligne cohomology classes.

Let X be a complex manifold and let D be a divisor with normal crossings on X. Let us write $V = X - D$ and let $j : V \longrightarrow X$ be the inclusion. Let $\mathcal{E}^*_X$ be the sheaf of complex C^∞ differential forms on X. The complex of sheaves $\mathcal{E}^*_X(\log D)$ (see chapter I) is the sub-$\mathcal{E}^*_X$ algebra of $j_*\mathcal{E}^*_V$ generated locally by the sections

$$\log z_i\bar{z}_i,\ \frac{dz_i}{z_i},\ \frac{d\bar{z}_i}{\bar{z}_i},\quad \text{for } i = 1,\dots,M,$$

where $z_1 \dots z_M = 0$ is a local equation of D.

Let us write $E^*_X(\log D) = \Gamma(X, \mathcal{E}^*_X(\log D))$, and let $E^*_{X,\mathbb{R}}(\log D)$ be the subcomplex of real forms.

Let X now be a smooth algebraic variety over $\mathbb{C}$ and let $X \hookrightarrow \overline{X}$ be a smooth compactification. Let us write $Y = \overline{X} - X$. Let I be the category of all diagrams

$$\begin{tikzcd} D_\alpha \arrow[r] \arrow[d] & \widetilde{X}_\alpha \arrow[d, "\pi_\alpha"] \\ Y \arrow[r] & \overline{X}, \end{tikzcd}$$

where $\widetilde{X}_\alpha$ is smooth, D_α is a divisor with normal crossings, π_α is proper and $\pi_\alpha|_{\widetilde{X}_\alpha - D_\alpha}$ is an isomorphism over $\overline{X} - Y$. Any such diagram is called a resolution of singularities of $(\overline{X}, Y)$. The morphisms of I are the maps $f : (\widetilde{X}_\alpha, D_\alpha) \longrightarrow (\widetilde{X}_\beta, D_\beta)$ such that $\pi_\alpha = \pi_\beta \circ f$.

Definition 1.2. The complex of differential forms with logarithmic singularities along infinity is

$$E^*_{\log}(X) = \varinjlim_{\alpha \in I} E^*_{\widetilde{X}_\alpha}(\log D_\alpha).$$

This complex is a subcomplex of $E^*_X = \Gamma(X, \mathcal{E}^*_X)$ and it is independent of the choice of the compactification $\overline{X}$. We shall denote by $E^*_{\log,\mathbb{R}}(X)$ the corresponding real subcomplex.

The complex $E^*_{\log}(X)$ has a natural bigrading

$$E^*_{\log}(X) = \bigoplus E^{p,q}_{\log}(X).$$

The Hodge filtration of this complex is defined by

$$F^p E^*_{\log}(X) = \bigoplus_{p' \geq p} E^{p',*}_{\log}(X).$$

Moreover, the weight filtration, W, of the complexes $E^*_{\widetilde{X}_\alpha}(\log D)$, induces a weight filtration on $E^*_{\log}(X)$ also denoted by W. By the results of chapter I, if f is a morphism of I, the morphism

$$f^* : E^*_{\widetilde{X}_\alpha}(\log D_\alpha) \longrightarrow E^*_{\widetilde{X}_\beta}(\log D_\beta)$$

is a real bifiltered quasi-isomorphism. Moreover, for all $\alpha \in I$, the pair

$$((E^*_{X_\alpha,\mathbb{R}}(\log D_\alpha), W), (E^*_{X_\alpha}(\log D_\alpha), W, F))$$

is a real mixed Hodge complex which induces in $H^*(X, \mathbb{R})$ the mixed Hodge structure introduced by Deligne in [De 1].

Since I is directed, all the induced morphisms

$$E^*_{\widetilde{X}_\alpha}(\log D_\alpha) \longrightarrow E^*_{\log}(X)$$

are bifiltered quasi-isomorphisms. So, the pair

$$((E^*_{\log,\mathbb{R}}(X), W), (E^*_{\log}(X), W, F))$$

is a real mixed Hodge complex which induces in $H^*(X, \mathbb{R})$ the mixed Hodge structure introduced in [De 1].

Remark 1.3. In the above definition we can use real analytic forms instead of C^∞ forms obtaining the complexes $A^*_X(\log D)$ and $A^*_{\log}(X)$. The first one was introduced by Navarro Aznar in [N]. The cohomological properties of the differentiable complexes and of the real analytic complexes are the same. Therefore, throughout the construction of arithmetic Chow groups, C^∞ forms can be replace by real analytic forms. In particular, this implies the existence of real analytic Green forms.

Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\overline{X}$ be a smooth compactification with $D = \overline{X} - X$ a divisor with normal crossings. Write $E^*_{\log,\mathbb{R}}(X, p) = (2\pi i)^p E^*_{\log,\mathbb{R}}(X) \subset E^*_{\log}(X)$. Then there are isomorphisms in the derived category

$$\begin{aligned} &R\Gamma R j_* \mathbb{R}_X(p) \longrightarrow E^*_{\log,\mathbb{R}}(X, p), \\ &R\Gamma j_* \Omega^*_X \longrightarrow E^*_{\log}(X) \qquad \text{and} \\ &R\Gamma F^p \Omega^*_{\overline{X}}(\log D) \longrightarrow F^p E^*_{\log}(X). \end{aligned}$$

Let us write

$$E^*_{\log,\mathbb{R}}(X, p)_{\mathcal{D}} := s(u : E^*_{\log,\mathbb{R}}(X, p) \oplus F^p E^*_{\log}(X) \longrightarrow E^*_{\log}(X)),$$

where $u(a, b) = b - a$. Then we have

Proposition 1.4. *The real Deligne cohomology groups of X can be computed as the cohomology of the complex $E^*_{\log,\mathbb{R}}(X,p)_{\mathcal{D}}$. That is*

$$H^*_{\mathcal{D}}(X,\mathbb{R}(p)) = H^*(E^*_{\log,\mathbb{R}}(X,p)_{\mathcal{D}}).$$

Our goal now is to give a simpler version of this complex. To this end we shall relate the simple of a morphism of complexes with the kernel and the cokernel of the morphism.

Let us recall the construction of the connection morphism of an exact sequence. Let

$$0 \to A^* \xrightarrow{\iota} B^* \xrightarrow{\pi} C^* \to 0$$

be an exact sequence of complexes of vector spaces. Let us choose a linear section σ of π. Then we can obtain a retraction τ of ι by

$$\tau(b) = \iota^{-1}(b - \sigma\pi b).$$

The connection morphism is induced by the morphism of complexes

$$\mathrm{Res}_\sigma : C^*[-1] \longrightarrow A^*,$$

defined by

$$\mathrm{Res}_\sigma(c) = \iota^{-1}(\sigma dc - d\sigma c).$$

If there is no danger of confusion we will write simply Res instead of Res_σ. It is straightforward to check that $\sigma dc - d\sigma c$ belongs to $\mathrm{Im}\,\iota$ and that Res is a morphism of complexes. Moreover, the induced morphism $\mathrm{Res} : H^*(C[1]) \longrightarrow H^*(A)$ is the composition of the natural morphisms

$$H^*(C[1]) \to H^*(s(B \longrightarrow C)) \xrightarrow{\cong} H^*(A).$$

We can also obtain Res from the retraction τ by the formula

$$\mathrm{Res}(\pi b) = d\tau b - \tau db.$$

Let now $u : A^* \longrightarrow B^*$ be a morphism of complexes of vector spaces. We can decompose u into two exact sequences

$$0 \to \mathrm{Ker}(u)^* \xrightarrow{j} A^* \xrightarrow{u'} \mathrm{Im}(u)^* \to 0$$

and

$$0 \to \mathrm{Im}(u)^* \xrightarrow{\iota} B^* \xrightarrow{\pi} \mathrm{Coker}(u)^* \to 0.$$

Let σ_1 and σ_2 be linear sections of π and u' respectively. Let τ_1 and τ_2 be the corresponding retractions of ι and j. Let us write $\mathrm{Res}_1 = \mathrm{Res}_{\sigma_1}$ and $\mathrm{Res}_2 = \mathrm{Res}_{\sigma_2}$.

We define a complex

$$\widehat{s}(u)^* = \mathrm{Ker}(u)^* \oplus \mathrm{Coker}(u)^*[-1]$$

with differential $d(a,b) = (da + \mathrm{Res}_2 \mathrm{Res}_1 b, -db)$. Then we have maps $\varphi : \widehat{s}(u) \longrightarrow s(u)$ and $\psi : s(u) \longrightarrow \widehat{s}(u)$ given by

$$\begin{aligned}\varphi(a,b) &= (j(a) - \sigma_2 \mathrm{Res}_1 b, \sigma_1 b) \quad \text{and}\\ \psi(a,b) &= (\tau_2 a + \mathrm{Res}_2(\tau_1 b), \pi b)\end{aligned}$$

Proposition 1.5. *The maps φ and ψ are morphisms of complexes. Moreover they are homotopy equivalences, one the inverse of the other. More explicitly, we have $\psi\varphi = Id$ and*

$$\varphi\psi - Id = dh + hd,$$

where $h : s(u)^n \longrightarrow s(u)^{n-1}$ is given by $h(a,b) = (-\sigma_2\tau_1 b, 0)$.

Proof. All the checks are straightforward. For instance let us check that $\varphi\psi - \mathrm{Id} = dh + hd$. We have

$$\begin{aligned}\varphi\psi(a,b) &= \varphi(\tau_2 a + \operatorname{Res}_2 \tau_1 b, \pi b)\\ &= (j\tau_2 a + j \operatorname{Res}_2 \tau_1 b - \sigma_2 \operatorname{Res}_1 \pi b, \sigma_1 \pi b).\end{aligned}$$

Therefore

$$\varphi\psi(a,b) - (a,b) = (j\tau_2 a - a + j \operatorname{Res}_2 \tau_1 b - \sigma_2 \operatorname{Res}_1 \pi b, \sigma_1 \pi b - b).$$

On the other hand

$$\begin{aligned}dh(a,b) + hd(a,b) &= d(-\sigma_2\tau_1 b, 0) + h(da, ua - db)\\ &= (-d\sigma_2\tau_1 b - \sigma_2\tau_1 ua + \sigma_2\tau_1 db, -u\sigma_2\tau_1 b).\end{aligned}$$

Hence the result follows from the equalities

$$\begin{aligned}u\sigma_2\tau_1 b &= \iota\tau_1 b = b - \sigma_1\pi b,\\ \sigma_2\tau_1 ua &= \sigma_2 u' a = a - j\tau_2 a \quad \text{and}\\ \operatorname{Res}_2 \tau_1 b\sigma_2 - \operatorname{Res}_1 \pi b &= \sigma_2\tau_1 db - \sigma_2 d\tau_1 b + \operatorname{Res}_2 \tau_1 b\\ &= \sigma_2\tau_1 db - d\sigma_2\tau_1 b.\end{aligned}$$

Let us come back to the complex $E^*_{\log,\mathbb{R}}(X,p)_{\mathcal{D}}$. Since the process of simplification depends only on the relationship between the real structure, the differential and the bigrading, we shall work with an abstract Dolbeault complex.

Definition 1.6. A *Dolbeault (cochain) complex* is a complex of real vector spaces $(A^*_{\mathbb{R}}, d)$ provided with a bigrading on $A^*_{\mathbb{C}} = A^*_{\mathbb{R}} \otimes \mathbb{C}$:

$$A^d_{\mathbb{C}} = \bigoplus_{p+q=d} A^{p,q},$$

such that

DC1. The differential d can be decomposed as a sum of operators $d = \partial + \bar{\partial}$ of type $(1,0)$ and $(0,1)$.

DC2. It satisfies the symmetry property

$$\overline{A^{p,q}} = A^{q,p},$$

where $\overline{}$ denotes complex conjugation.

By **DC2** the operator $\bar{\partial}$ is the complex conjugate of ∂.

Let A be a Dolbeault complex. The *Hodge filtration* F of A^* is

$$F^pA^* = \bigoplus_{p'\geq p} A^{p',*}.$$

We denote by $\overline{F}$ the filtration complex conjugate of F. That is

$$\overline{F}^pA^* = \bigoplus_{p'\geq p} A^{*,p'}.$$

Examples of Dolbeault complexes are the complex of C^∞ (or real analytic) differential forms on a complex manifold and the complex of C^∞ differential forms with logarithmic singularities at infinity.

Let A^* be a Dolbeault complex. We write $A^*_{\mathbb{R}}(p) = (2\pi i)^p A^*_{\mathbb{R}} \subset A^*_{\mathbb{C}}$ and

$$A^*_{\mathbb{R}}(p)_{\mathcal{D}} = s(A^*_{\mathbb{R}}(p) \oplus F^pA^*_{\mathbb{C}} \xrightarrow{u} A^*_{\mathbb{C}}),$$

where $u(a,b) = b-a$. For example, if X is a smooth variety over $\mathbb{C}$ and $A^* = E^*_{\log}(X)$, then we have seen that

$$H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}}) = H^*_{\mathcal{D}}(X, \mathbb{R}(p)).$$

On the other hand, if $A^* = E^*_X$ is the complex of C^∞ differential forms on X, then the groups $H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}})$ are called analytic Deligne cohomology groups.

Let us apply Proposition 1.5 to the morphism $u : A^*_{\mathbb{R}}(p) \oplus F^pA^*_{\mathbb{C}} \longrightarrow A^*_{\mathbb{C}}$. The first step is to compute $\operatorname{Ker} u$ and $\operatorname{Coker} u$.

Lemma 1.7. *Let A^* be a Dolbeault complex. The morphism*

$$u : A^n_{\mathbb{R}}(p) \oplus F^pA^n_{\mathbb{C}} \longrightarrow A^n_{\mathbb{C}}$$

is injective for $n \leq 2p-1$ and surjective for $n \geq 2p-1$. In particular for $n = 2p-1$ it is an isomorphism. Moreover we have

$$\begin{aligned}
\operatorname{Coker}(u)^n &= A^n_{\mathbb{C}} \big/ (A^n_{\mathbb{R}}(p) + F^pA^n_{\mathbb{C}} + \overline{F}^pA^n_{\mathbb{C}}) \\
&\cong A^n_{\mathbb{R}}(p-1) \big/ (A^n_{\mathbb{R}}(p-1) \cap (F^pA^n_{\mathbb{C}} + \overline{F}^pA^n_{\mathbb{C}})) \\
&\cong A^n_{\mathbb{R}}(p-1) \cap \bigoplus_{\substack{p'+q'=n \\ p'<p,\ q'<p}} A^{p',q'},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Ker}(u)^n &\cong A^n_{\mathbb{R}}(p) \cap F^pA^n_{\mathbb{C}} \cap \overline{F}^pA^n_{\mathbb{C}} \\
&= A^n_{\mathbb{R}}(p) \cap \bigoplus_{\substack{p'+q'=n \\ p'\geq p,\ q'\geq p}} A^{p',q'}.
\end{aligned}$$

Proof. Since in a Dolbeault complex we have

$$\begin{aligned}
A^n_{\mathbb{C}} &= F^pA^n_{\mathbb{C}} + \overline{F}^qA^n_{\mathbb{C}} \qquad &&\text{for } p+q \leq n+1, \text{ and} \\
\{0\} &= F^pA^n_{\mathbb{C}} \cap \overline{F}^qA^n_{\mathbb{C}} \qquad &&\text{for } p+q \geq n+1,
\end{aligned}$$

it is enough to prove the descriptions of $\operatorname{Coker}(u)$ and $\operatorname{Ker}(u)$.

Clearly
$$\operatorname{Im} u \subset A^n_{\mathbb{R}}(p) + F^p A^n_{\mathbb{C}} + \overline{F}^p A^n_{\mathbb{C}}.$$
Let $x \in \overline{F}^p A^n_{\mathbb{C}}$. Then $\overline{x} \in F^p A^n_{\mathbb{C}}$ and $x + (-1)^p \overline{x} \in A^n_{\mathbb{R}}(p)$. Therefore
$$A^n_{\mathbb{R}}(p) + F^p A^n_{\mathbb{C}} + \overline{F}^p A^n_{\mathbb{C}} = A^n_{\mathbb{R}}(p) + F^p A^n_{\mathbb{C}},$$
and
$$\operatorname{Coker}(u)^n = A^n_{\mathbb{C}} \big/ (A^n_{\mathbb{R}}(p) + F^p A^n_{\mathbb{C}} + \overline{F}^p A^n_{\mathbb{C}}) \ .$$
If $(a,b) \in \operatorname{Ker} u$, then $a = b$ and $a \in A^n_{\mathbb{R}}(p) \cap F^p A^n_{\mathbb{C}}$. Therefore $a = (-1)^p \overline{a} \in \overline{F}^p A^n_{\mathbb{C}}$. Hence
$$\operatorname{Ker}(u)^n \cong A^n_{\mathbb{R}}(p) \cap F^p A^n_{\mathbb{C}} \cap \overline{F}^p A^n_{\mathbb{C}}.$$

The next step is to choose linear sections of the maps
$$\pi : A^*_{\mathbb{C}} \longrightarrow \operatorname{Coker}(u)^* \quad \text{and} \quad u' : A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \longrightarrow \operatorname{Im}(u)^*.$$
In order to give explicit expressions of these sections, let us introduce some maps. Let
$$\pi_p : A^*_{\mathbb{C}} \longrightarrow A^*_{\mathbb{R}}(p)$$
be the projection obtained from the direct sum decomposition $A^*_{\mathbb{C}} = A^*_{\mathbb{R}}(p) \oplus A^*_{\mathbb{R}}(p-1)$. Namely, we have
$$\pi_p x = \frac{1}{2}(x + (-1)^p \overline{x}).$$
Let $x = \sum x^{p,q} \in A^*_{\mathbb{C}}$. We will denote by
$$F^p x = \sum_{p' \geq p} x^{p',q},$$
the projection over $F^p A^*_{\mathbb{C}}$ and by
$$F^{p,p} x = \sum_{\substack{p' \geq p \\ q' \geq p}} x^{p',q'},$$
the projection over $F^p A^*_{\mathbb{C}} \cap \overline{F}^p A^*_{\mathbb{C}}$.

By Lemma 1.7, $\operatorname{Coker}(u)^n$ may be identified with the subgroup of $A^n_{\mathbb{C}}$
$$A^n_{\mathbb{R}}(p-1) \cap \bigoplus_{\substack{p'+q'=n \\ p'<p,\ q'<p}} A^{p',q'} = F^{n-p+1} \cap \overline{F}^{n-p+1} \cap A^n_{\mathbb{R}}(p-1).$$
Let us write $q = n - p + 1$. Then, with the above identification, the morphism $\pi : A^n_{\mathbb{C}} \longrightarrow \operatorname{Coker}(u)^n$ is
$$\pi(x) = \pi_{p-1}(F^{q,q} x).$$
This gives us a natural way to choose a section σ_1 of π: The inclusion
$$F^q A^n_{\mathbb{C}} \cap \overline{F}^q A^n_{\mathbb{C}} \cap A^n_{\mathbb{R}}(p-1) \longrightarrow A^n_{\mathbb{C}}.$$

With this choice of σ_1 we have

$$\tau_1(x) = \begin{cases} x - \pi_{p-1}(F^{q,q}x), & \text{for } n \leq 2p-2, \\ x, & \text{for } n \geq 2p-1. \end{cases}$$

And

$$\mathrm{Res}_1(x) = -\pi_{p-1}(F^p dx), \quad \text{for } x \in \mathrm{Coker}(u)^n \text{ and } n \leq 2p-2.$$

Let us look for a section σ_2 of u'. For $n < 2p$ the map u is injective. Therefore the section σ_2 is unique. Let $x \in \mathrm{Im}(u)$. Then $x = -a_p + f_p$, where $a_p \in A^n_{\mathbb{R}}(p)$ and $f_p \in F^p A^*_{\mathbb{C}}$. We have

$$f_p = 2F^p(x - \pi_p x) \qquad \text{and} \qquad a_p = f_p - x.$$

Hence the section σ_2 is given by

$$\sigma_2 x = (2F^p(x - \pi_p x) - x, 2F^p(x - \pi_p x)).$$

Let now $n \geq 2p$. Let us write $q = n - p + 1 > p$. Then there are many possible choices for σ_2. Nevertheless, we have the following direct sum decompositions

$$\begin{aligned} A^n_{\mathbb{R}}(p) &= F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p) \,\oplus\, (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p), \\ F^p A^n_{\mathbb{C}} &= F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p) \,\oplus\, F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \,\oplus\, F^q \text{ and} \\ A^n_{\mathbb{C}} &= F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p) \,\oplus\, F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \,\oplus\, F^q \,\oplus\, (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p). \end{aligned}$$

Thus we can impose the condition

$$\sigma_2(x) = \begin{cases} (-x, 0), & \text{for } x \in (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p), \\ (0, x), & \text{for } x \in F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \oplus F^q. \end{cases} \tag{1.8}$$

Note that with this condition we have fixed the image of

$$A^n_{\mathbb{R}}(p-1) \,\oplus\, F^q A^n_{\mathbb{C}} = (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p) \,\oplus\, F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \,\oplus\, F^q A^n_{\mathbb{C}}.$$

Now the only ambiguity is how to distribute $F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p)$ between $A^n_{\mathbb{R}}(p)$ and $F^p A^n_{\mathbb{C}}$. Actually there are two extreme options depending on whether the equivalence $\psi : s(u) \longrightarrow \widehat{s}(u)$ is to factorize through $s(A^*_{\mathbb{R}}(p) \longrightarrow A^*_{\mathbb{C}}/F^p)$ or through $s(F^p A^*_{\mathbb{C}} \longrightarrow A^*_{\mathbb{R}}(p-1))$. In the first case we have to write $\sigma_2(x) = (-x, 0)$ for $x \in F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p)$ and in the second case $\sigma_2(x) = (0, x)$.

We fix the section

$$\sigma_2 x = (-2\pi_p(x - F^p x), 2\pi_p(F^p x) + (-1)^{p-1}\overline{x}),$$

which corresponds to the first option. The retraction

$$\tau_2 : A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \longrightarrow \mathrm{Ker}(u) \cong F^p A^*_{\mathbb{C}} \cap \overline{F}^p A^*_{\mathbb{C}} \cap A^*_{\mathbb{R}}(p),$$

associated to σ_2, is given by

$$\tau_2(a_p, f_p) = \begin{cases} 0, & \text{for } n < 2p, \\ F^{p,p} a_p, & \text{for } n \geq 2p. \end{cases}$$

And the morphism Res_2 is given by

$$\mathrm{Res}_2(x) = \begin{cases} 0, & \text{for } n < 2p-1, \\ 2\pi_p(\partial x^{p-1,n-p+1}), & \text{for } n \geq 2p+1. \end{cases}$$

Definition 1.9. Let A^* be a Dolbeault complex. We will call Deligne complex associated to A the complex

$$\mathfrak{D}^*(A,p) = \widehat{s}(A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \xrightarrow{u} A^*_{\mathbb{C}}).$$

The differential of this complex will be denoted by $d_{\mathfrak{D}}$.

Let us summarize the results of this section.

Theorem 1.10. *Let A^* be a Dolbeault complex. Then*

1) The complex $\mathfrak{D}^*(A,p)$ is given by

$$\mathfrak{D}^n(A,p) = \begin{cases} \operatorname{Coker}(u)^{n-1} = A^{n-1}_{\mathbb{R}}(p-1) \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p,\ q'<p}} A^{p',q'}, & \text{for } n \leq 2p-1, \\ \operatorname{Ker}(u)^n = A^n_{\mathbb{R}}(p) \cap \bigoplus\limits_{\substack{p'+q'=n \\ p'\geq p,\ q'\geq p}} A^{p',q'}, & \text{for } n \geq 2p. \end{cases}$$

For $x \in \mathfrak{D}^n(A,p)$ the differential $d_{\mathfrak{D}}$ is given by

$$d_{\mathfrak{D}} x = \begin{cases} dx, & \text{for } n \geq 2p, \\ -\pi(dx), & \text{for } n < 2p-1 \text{ and} \\ \operatorname{Res}_2 \operatorname{Res}_1 x = -2\partial\bar{\partial} x, & \text{for } n = 2p-1, \end{cases}$$

where $\pi : A^* \longrightarrow \operatorname{Coker}(u)^*$ is the projection.

2) The complexes $A^*_{\mathbb{R}}(p)_{\mathcal{D}}$ and $\mathfrak{D}^*(A,p)$ are homotopically equivalent. The homotopy equivalences $\psi : A^n_{\mathbb{R}}(p)_{\mathcal{D}} \longrightarrow \mathfrak{D}^n(A,p)$ and $\varphi : \mathfrak{D}^n(A,p) \longrightarrow A^n_{\mathbb{R}}(p)_{\mathcal{D}}$ are given by

$$\psi(a,f,\omega) = \begin{cases} \pi(\omega), & \text{for } n \leq 2p-1 \text{ and} \\ F^{p,p}a + 2\pi_p(\partial\omega^{p-1,n-p+1}), & \text{for } n \geq 2p, \end{cases}$$

and

$$\varphi(x) = \begin{cases} (\partial x^{p-1,n-p} - \bar{\partial} x^{n-p,p-1}, 2\partial x^{p-1,n-p}, x), & \text{for } n \leq 2p-1 \text{ and} \\ (x,x,0), & \text{for } n \geq 2p. \end{cases}$$

Moreover $\psi\varphi = Id$ and $\varphi\psi - Id = dh + hd$, where $h : A^n_{\mathbb{R}}(p)_{\mathcal{D}} \longrightarrow A^{n-1}_{\mathbb{R}}(p)_{\mathcal{D}}$ is given by

$$h(a,f,\omega) = \begin{cases} (\pi_p(\overline{F}^p\omega + \overline{F}^{n-p}\omega), -2F^p(\pi_{p-1}\omega), 0), & \text{for } n \leq 2p-1 \text{ and} \\ (2\pi_p(\overline{F}^{n-p}\omega), -F^{p,\bar{p}}\omega - 2F^{n-p}(\pi_{p-1}\omega), 0), & \text{for } n \geq 2p. \end{cases}$$

3) The natural morphism $H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}}) \longrightarrow H^*(A^*_{\mathbb{R}}(p))$ is induced by the morphism of complexes

$$r_p : \mathfrak{D}^*(A,p) \longrightarrow A^*_{\mathbb{R}}(p)$$

given by

$$r_p x = \begin{cases} 2\pi_p(F^p dx) = \partial x^{p-1,n-p} - \bar{\partial} x^{n-p,p-1}, & \text{for } n \leq 2p-1 \text{ and} \\ x, & \text{for } n \geq 2p. \end{cases}$$

Corollary 1.11. *Let X be a smooth variety over $\mathbb{C}$. then*

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) = H^*(\mathfrak{D}^*(E^*_{\log}(X), p)).$$

The fact that the cohomology of $\mathfrak{D}^*(E^*_X, p)$ is the Deligne cohomology of X, for X a projective complex manifold, has been proved in [Wa].

Remark 1.12. Let A be a Dolbeault complex. By construction, the cohomology groups $H^{2p}(\mathfrak{D}^*(A, p))$ are

$$H^{2p}(\mathfrak{D}^*(A, p)) = \left\{x \in A^{p,p} \cap A^{2p}_{\mathbb{R}}(p) \mid dx = 0\right\} / \mathrm{Im}(\partial\bar{\partial})$$

Therefore they are the $\mathbb{R}(p)$-part of the $\partial\bar{\partial}$-cohomology of A. In particular we have a relation between $\partial\bar{\partial}$ cohomology and real Deligne cohomology. On the other hand we have

$$H^{2p-1}(\mathfrak{D}^*(A, p)) = \left\{x \in A^{p-1,p-1} \cap A^{2p-2}_{\mathbb{R}}(p) \mid \partial\bar{\partial}x = 0\right\} / (\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}) \ .$$

A variant of this complex has been used in [Dem] to study the properties of $\partial\bar{\partial}$-cohomology.

Remark 1.13. The complex $\mathfrak{D}^*(A, p)$, the maps φ and r_p and the map ψ, for $n < 2p$, do not depend on the choice of the section σ_2. Only the map ψ for $n \geq 2p$ depends on the choice of σ_2. Moreover the maps φ, ψ and the homotopy h are natural. That is, given a morphism $A \longrightarrow B$ between Dolbeault complexes there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{D}^*(A, p) & \xrightarrow{\varphi} & A^*_{\mathbb{R}}(p)_{\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathfrak{D}^*(B, p) & \xrightarrow{\varphi} & B^*_{\mathbb{R}}(p)_{\mathcal{D}} \end{array}$$

and analogous diagrams for ψ and h.

§2. Multiplicative Structure of Deligne Cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$ and $\overline{X}$ a smooth compactification of X with $\overline{X} - X$ a divisor with normal crossings. For each real number $0 \leq \alpha \leq 1$, there is defined a product $\cup_\alpha$ on the Deligne-Beilinson complex $\Lambda(p)_{\mathcal{D}}$ (see [Be 1] or [E-V, §3]). All these products are homotopically equivalent. Moreover the product obtained for $\alpha = 1/2$ is commutative and the products obtained for $\alpha = 0$ and $\alpha = 1$ are associative. Therefore they induce an associative and commutative product in Deligne cohomology denoted $\cup$. We want to transport this multiplicative structure to the complex $\mathfrak{D}^*(A, \cdot)$.

Definition 2.1. Let A be a Dolbeault complex. We say that A is a *Dolbeault algebra* if there is a product

$$A^*_{\mathbb{R}} \otimes A^*_{\mathbb{R}} \xrightarrow{\wedge} A^*_{\mathbb{R}}$$

such that $A^*_{\mathbb{R}}$ is a differential associative graded-commutative algebra, and the induced product on $A^*_{\mathbb{C}}$ is compatible with the bigrading. That is

$$A^{p,q} \wedge A^{p',q'} \subset A^{p+p',q+q'}.$$

Let $(A, d, \wedge)$ be a Dolbeault algebra and let $0 \leq \alpha \leq 1$ be a real number. The product $\cup_\alpha$ of the Deligne-Beilinson complex corresponds to the product

$$A^n_{\mathbb{R}}(p)_{\mathcal{D}} \otimes A^m_{\mathbb{R}}(q)_{\mathcal{D}} \xrightarrow{\cup_\alpha} A^{n+m}_{\mathbb{R}}(p+q)_{\mathcal{D}}$$

defined by

$$\begin{aligned}&(a_p, f_p, \omega_p) \cup_\alpha (a_q, f_q, \omega_q) = \\ &\quad (a_p \wedge a_q, f_p \wedge f_q, \alpha(\omega_p \wedge a_q + (-1)^n f_p \wedge \omega_q) + (1-\alpha)(\omega_p \wedge f_q + (-1)^n a_p \wedge \omega_q))\,.\end{aligned}$$

In order to define a product in $\mathfrak{D}^*(A, \cdot)$ we shall use the following result.

Proposition 2.2. *Let A^* and B^* be complexes of modules over a ring, such that there are homotopy equivalences $\varphi : A^* \longrightarrow B^*$ and $\psi : B^* \longrightarrow A^*$, one the inverse of the other. Assume furthermore that there is defined a product in B^*. That is, a morphism of complexes*

$$B^* \otimes B^* \xrightarrow{\cup_B} B^*.$$

Then
1) The map

$$A^* \otimes A^* \xrightarrow{\cup_A} A^*,$$

defined by $x \cup_{A} y = \psi(\varphi x \cup_{B} \varphi y)$ *is a morphism of complexes.*

2) *If the product* $\cup_{B}$ *is associative or associative up to homotopy then the product* $\cup_{A}$ *is associative up to homotopy.*
3) *If the product* $\cup_{B}$ *is graded commutative, the same is true for* $\cup_{A}$. *If it is graded commutative up to homotopy, then* $\cup_{A}$ *is graded commutative up to homotopy.*

Proof. To prove that $\cup_{A}$ is a morphism of complexes we use that φ, ψ and $\cup_{B}$ are morphisms of complexes. The statement about commutativity follows easily from the definition of $\cup_{A}$.

Assume now that $\cup_{B}$ is associative. Let h be the homotopy between $\varphi\psi$ and Id. That is

$$\varphi\psi - \mathrm{Id} = hd + dh.$$

Let us define a map

$$A^{n} \otimes A^{m} \otimes A^{l} \xrightarrow{h_{\mathrm{a}}} A^{n+m+l-1}$$

by

$$h_{\mathrm{a}}(a \otimes b \otimes c) = \psi(h(\varphi a \cup_{B} \varphi b) \cup_{B} \varphi c) + (-1)^{n+1}\psi(\varphi a \cup_{B} h(\varphi b \cup_{B} \varphi c)).$$

Then we can check easily that

$$(a \cup_{A} b) \cup_{A} c - a \cup_{A} (b \cup_{A} c) = h_{\mathrm{a}} d(a \otimes b \otimes c) + dh_{\mathrm{a}}(a \otimes b \otimes c).$$

The case when $\cup_{B}$ is only associative up to homotopy is analogous.

Applying Proposition 2.2. to $A^{*}_{\mathbb{R}}(p)_{\mathcal{D}}$ and $\mathfrak{D}(A^{*}, p)$ we obtain

Theorem 2.3. *Let* $(A, d, \wedge)$ *be a Dolbeault algebra, and let* $\alpha \in [0, 1]$. *Let the map*

$$\mathfrak{D}^{n}(A, p) \otimes \mathfrak{D}^{m}(A, q) \overset{\cdot}{\longrightarrow} \mathfrak{D}^{n+m}(A, p + q)$$

be defined by $x \cdot y = \psi(\varphi x \cup_{\alpha} \varphi y)$. *Then:*

1) *It is a morphism of complexes and does not depend on* α. *It is also independent of the section* σ_2, *provided this section satisfies the condition 1.8. Moreover it induces the product* $\cup$ *in real Deligne cohomology.*
2) *This product is graded commutative and it is associative up to a natural homotopy.*
3) *Let* $x \in \mathfrak{D}^{n}(A, p)$ *and* $y \in \mathfrak{D}^{m}(A, q)$. *Let us write* $l = n + m$ *and* $r = p + q$. *Then*

$$x \cdot y = \begin{cases} (-1)^{n} r_p(x) \wedge y + x \wedge r_q(y), & \text{for } n < 2p \text{ and } m < 2q, \\ \pi(x \wedge y), & \text{for } n < 2p,\ m \geq 2q,\ l < 2r, \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r \partial((x \wedge y)^{r-1, l-r}), & \text{for } n < 2p,\ m \geq 2q,\ l \geq 2r, \\ x \wedge y, & \text{for } n \geq 2p \text{ and } m \geq 2q, \end{cases}$$

where $r_p(x) = 2\pi_p(F^{p} dx)$ *(see 1.10.3) and* π *is the projection* $A^{*}_{\mathbb{C}} \longrightarrow \operatorname{Coker} u$ *(see §1).*

4) *If* $x \in \mathfrak{D}^{2p}(A, p)$ *is a cycle, then for all* y, z *we have*

$$x \cdot y = y \cdot x \quad \text{and}$$
$$y \cdot (x \cdot z) = (y \cdot x) \cdot z = x \cdot (y \cdot z).$$

Proof. Let us first check the formulae of 3). If $n < 2p$ and $m < 2q$ we have

$$\begin{aligned}
\psi(\varphi x \cup_\alpha \varphi y) &= \psi\left((r_p(x), 2F^p(dx), x) \cup_\alpha (r_q(y), 2F^q(dy), y)\right) \\
&= \psi\big(r_p(x) \wedge r_q(y), 2F^p(dx) \wedge 2F^q(dy), \\
&\qquad \alpha(x \wedge r_q(y) + (-1)^n 2F^p(dx) \wedge y) + (1-\alpha)(x \wedge 2F^q(dy) + (-1)^n r_p(x) \wedge y)\big) \\
&= \pi(\alpha(x \wedge r_q(y) + (-1)^n 2F^p(dx) \wedge y) + (1-\alpha)(x \wedge 2F^q(dy) + (-1)^n r_p(x) \wedge y)).
\end{aligned}$$

But

$$\pi(x \wedge r_q(y)) = x \wedge r_q(y) \quad \text{and} \quad \pi(x \wedge 2F^q(dy)) = x \wedge r_q(y).$$

The same is true for the other two terms. Therefore

$$x \cdot y = (-1)^n r_p(x) \wedge y + x \wedge r_q(y).$$

Note that this result does not depend on α nor on the choice of σ_2 because we have used ψ only for $l < 2r$.

If $n < 2p$, $m \geq 2q$ and $r \geq 2r$, we have

$$\begin{aligned}
\psi(\varphi x \cup_\alpha \varphi y) &= \psi\left((r_p(x), 2F^p(dx), x) \cup_\alpha (y, y, 0)\right) \\
&= (r_p(x) \wedge y, 2F^p(dx) \wedge y, x \wedge y) \\
&= F^{r,\overline{r}}(r_p(x) \wedge y) + 2\pi_r(\partial(x \wedge y)^{r-1,l-r}).
\end{aligned}$$

This result does not depend on α either. Nor does this formula depend on the choice of σ_2 satisfying 1.8 because $x \wedge y \in A^{l-1}_{\mathbb{R}}(r-1)$ and $u(r_p(x) \wedge y, 2F^p(dx) \wedge y) \in A^l_{\mathbb{R}}(r-1)$. The other cases are analogous.

The remainder of the proposition is a consequence of these formulae and of Proposition 2.2, except for the fact that the homotopy for the associativity is natural, which follows from the naturality of φ, ψ and the homotopy h.

In [Wa], X. Wang constructed higher order arithmetic characteristic classes for K-theory. A key ingredient in his constructions is a set of differential forms, denoted B_n. The remainder of this section relates these differential forms with the product on the Deligne complex and will not be used in the other sections.

Let X be a complex manifold and let E^*_X be the Dolbeault algebra of complex differential forms. For $i = 1, \ldots, n$, let $u_i \in \mathfrak{D}^1(E^*_X, 1) = E^0_{X,\mathbb{R}}$.

Let us write

$$S^n_i = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma u_{\sigma(1)} \partial u_{\sigma(2)} \wedge \cdots \wedge \partial u_{\sigma(i)} \wedge \overline{\partial} u_{\sigma(i+1)} \wedge \cdots \wedge \overline{\partial} u_{\sigma(n)},$$

where $\mathfrak{S}_n$ is the symmetric group of n-elements and $(-1)^\sigma$ is the sign of the permutation σ.

Then B_n is defined by

$$B_n(u_1, \ldots, u_n) = \sum_{i=1}^n (-1)^{i-1} S^n_i.$$

An important property of these forms is the inductive formula (proved in [Wa, 2.2.1])

$$d_{\mathfrak{D}} B_n(u_1, \ldots, u_n) = n \sum_{l=1}^n (-1)^{l-1} \partial\overline{\partial} u_l \wedge B_{n-1}(u_1, \ldots, \widehat{u_l}, \ldots, u_n). \tag{2.4}$$

The difference of signs between this formula and [Wa, 2.2.1] comes from the fact that the differential $d_{\mathfrak{D}}$ and the differential of the complex used in [Wa] differ in the sign for $n < 2p-1$.

The definition of B_n and the proof of 2.4 is purely algebraic and can be applied to any Dolbeault algebra.

The product defined in Theorem 2.3 allows us to give an interpretation of the forms B_n and of the formula 2.4. Namely, B_n is obtained as a symmetrized product of the u_i and 2.4 is a consequence of the Leibnitz rule.

Proposition 2.5. *Let A be a Dolbeault algebra and let $u_i \in \mathfrak{D}^1(A,1)$, $i = 1,\dots,n$. Let us write*

$$C_n(u_1,\dots,u_n) = (-1/2)^{n-1} \sum_{\sigma\in\mathfrak{S}_n} (-1)^\sigma u_{\sigma(1)} \cdot (u_{\sigma(2)} \dots (u_{\sigma(n-1)} \cdot u_{\sigma(n)}) \dots).$$

Then $B_n(u_1,\dots,u_n) = C_n(u_1,\dots,u_n)$.

Proof. Let us see that C_n verifies also 2.4.

$$\begin{aligned}
d_{\mathfrak{D}} C_n &= (-1/2)^{n-1} \sum_{\sigma\in\mathfrak{S}_n} (-1)^\sigma d_{\mathfrak{D}} \left(u_{\sigma(1)} \cdot (u_{\sigma(2)} \dots (u_{\sigma(n-1)} \cdot u_{\sigma(n)}) \dots)\right) \\
&= (-1/2)^{n-1} \sum_{\sigma\in\mathfrak{S}_n} (-1)^\sigma \sum_{i=1}^{n} (-1)^{i-1} u_{\sigma(1)} \cdot (u_{\sigma(2)} \dots (d_{\mathfrak{D}} u_{\sigma(i)} \dots u_{\sigma(n)}) \dots) \\
&= (-1/2)^{n-1} \sum_{i=1}^{n} (-1)^{i-1} \sum_{j=1}^{n} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(i)=j}} (-1)^\sigma u_{\sigma(1)} \cdot (u_{\sigma(2)} \dots (d_{\mathfrak{D}} u_{\sigma(i)} \dots u_{\sigma(n)}) \dots).
\end{aligned}$$

Since $d_{\mathfrak{D}} u_j \in \mathfrak{D}^2(A,1)$ and is closed, we can commute it with the other elements. Therefore

$$d_{\mathfrak{D}} C_n = (-1/2)^{n-1} \sum_{i=1}^{n} (-1)^{i-1} \sum_{j=1}^{n} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(i)=j}} (-1)^\sigma d_{\mathfrak{D}} u_j \cdot u_{\sigma(1)} \cdot (u_{\sigma(2)} \dots (\widehat{u_{\sigma(i)}} \dots u_{\sigma(n)}) \dots).$$

For each i,j let $\varepsilon = \varepsilon_{i,j}$ be the unique permutation such that $\varepsilon(j) = i$ and

$$\varepsilon|_{\{1,n\}-\{j\}} : \{1,n\} - \{j\} \longrightarrow \{1,n\} - \{i\}$$

is an increasing function. If $\sigma \in \mathfrak{S}_n$ with $\sigma(i) = j$, let τ be the permutation $\tau = \sigma\varepsilon_{i,j}$. Then $(-1)^\tau = (-1)^\sigma(-1)^{i-j}$ and $\tau(j) = j$. We can consider τ as an element of $\mathfrak{S}_{n-1}$. The correspondence $\sigma \longmapsto \tau$ is a bijection between the set of $\sigma \in \mathfrak{S}_n$ with $\sigma(i) = j$ and $\mathfrak{S}_{n-1}$. Therefore we have

$$\begin{aligned}
d_{\mathfrak{D}} C_n &= (-1/2)^{n-1} \sum_{i=1}^{n} (-1)^{i-1} \sum_{j=1}^{n} \sum_{\tau\in\mathfrak{S}_{n-1}} (-1)^\tau (-1)^{i-j} d_{\mathfrak{D}} u_j \cdot u_{\tau(1)} \cdot (u_{\tau(2)} \dots u_{\tau(n)}) \dots) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{j-1} \partial\bar{\partial} u_j (-1/2)^{n-2} \cdot \sum_{\tau\in\mathfrak{S}_{n-1}} (-1)^\tau u_{\tau(1)} \cdot (u_{\tau(2)} \dots u_{\tau(n)}) \dots) \\
&= n \sum_{j=1}^{n} (-1)^{j-1} \partial\bar{\partial} u_j \wedge C_{n-1}(u_i,\dots,\widehat{u_j},\dots,u_n).
\end{aligned}$$

Let us now prove that $B_n = C_n$. By the definition of the product in the complex $\mathfrak{D}$ it can be checked by induction that C_n is a linear combination of elements of the form

$$u_{k_1}\partial u_{k_2} \wedge \cdots \wedge \partial u_{k_i} \wedge \overline{\partial} u_{k_{i+1}} \wedge \cdots \wedge \overline{\partial} u_{k_n}.$$

Moreover C_n is symmetric under the action of the symmetric group. Therefore it is a linear combination of the elements S_i^n, $i = 1, \ldots, n$. Let us write

$$C_n = \sum_{i=1}^{n} c_{i,n} S_i^n.$$

We want to see that $c_{i,n} = (-1)^{i-1}$ for all i. By the formula 2.4 for C_n we obtain that $d_{\mathfrak{D}} C_n$ does not contain any term of the form

$$\partial u_1 \wedge \cdots \wedge \partial u_i \wedge \overline{\partial} u_{i+1} \wedge \cdots \wedge \overline{\partial} u_n.$$

This implies that

$$\begin{aligned} c_{1,n} &= -c_{2,n}, \\ c_{2,n} &= -c_{3,n}, \\ &\cdots \\ c_{n-1,n} &= -c_{n,n}. \end{aligned}$$

So it is enough to show that $c_{1,n} = 1$. Since $\overline{F}^{n-1} C_n = c_{1,n} S_1^n$ we only need to compare $\overline{\partial} S_1^n$ with $\overline{\partial}\, \overline{F}^{n-1} C_n$.

On the one hand we have

$$\begin{aligned} \overline{\partial} S_1^n &= \overline{\partial} \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \overline{\partial} u_{\sigma(2)} \wedge \cdots \wedge \overline{\partial} u_{\sigma(n)} \\ &= n! \overline{\partial} u_1 \wedge \cdots \wedge \overline{\partial} u_n. \end{aligned}$$

On the other hand, If $a \in \mathfrak{D}^p(A, p)$ and $b \in \mathfrak{D}^q(A, q)$ then

$$\begin{aligned} \overline{\partial}\, \overline{F}^{p+q-1}(a \cdot b) &= \overline{\partial}\, \overline{F}^{p+q-1} \left(a \wedge (\partial b^{q-1,0} - \overline{\partial} b^{0,q-1}) + (-1)^n (\partial a^{p-1,0} + \overline{\partial} a^{0,p-1}) \wedge b \right) \\ &= \overline{\partial}(-a^{0,p-1} \wedge \overline{\partial} b^{0,q-1} - (-1)^n \overline{\partial} a^{0,p-1} \wedge b^{0,q-1}) \\ &= -2 \overline{\partial}\, \overline{F}^{p-1} a \wedge \overline{\partial}\, \overline{F}^{q-1} b. \end{aligned}$$

Therefore

$$\begin{aligned} \overline{\partial}\, \overline{F}^{n-1} C_n &= \overline{\partial}\, \overline{F}^{n-1} (-1/2)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \cdot (u_{\sigma(2)} \ldots u_{\sigma(n)}) \\ &= n! \overline{\partial} u_1 \wedge \cdots \wedge \overline{\partial} u_n. \end{aligned}$$

Hence $c_{1,n} = 1$ and $B_n = C_n$.

§3. Truncated Relative Cohomology Groups.

Generalizing the definitions of differential characters ([C-S]) and of Green currents ([G-S 2], see also chapter II), in this section we introduce some groups of secondary cohomology classes associated with a morphism of complexes. These groups will be called truncated relative cohomology groups.

Definition 3.1. Let R be a ring and let $f : A^* \longrightarrow B^*$ be a morphism of complexes of R-modules. Let us denote by ZA^* the submodule of cycles of A^* and by $\widetilde{B}^* = B^* / \operatorname{Im} d$. If $b \in B^*$ we write $\widetilde{b}$ for its class in $\widetilde{B}^*$. The *truncated relative cohomology groups* associated to f are

$$\widehat{H}^n(A^*, B^*) = \left\{(a, \widetilde{b}) \in ZA^n \oplus \widetilde{B}^{n-1} \mid f(a) = db\right\}.$$

These groups are R-modules in a natural way. If the morphism f is injective we write $\widetilde{b}$ instead of $(a, \widetilde{b})$.

Examples 3.2.
1) If $B = 0$ then $\widehat{H}^n(A^*, B^*) = ZA^n$. If $A = 0$ then $\widehat{H}^n(A^*, B^*) = H^{n-1}(B^*)$.
2) ([C-S]) Let M be a differential manifold. Let A^* be the complex of real valued differential forms on M. Let $\Lambda \subset \mathbb{R}$ be a proper subring and let $C^*(M, \mathbb{R}/\Lambda)$ be the complex of $\mathbb{R}/\Lambda$-valued smooth cochains. There is an injective morphism

$$f : A^* \longrightarrow C^*(M, \mathbb{R}/\Lambda)$$

defined by integration. Then the group $\widehat{H}^n(A^*, C^*(M, \mathbb{R}/\Lambda))$ coincides with the group of differential characters of M, $\widehat{H}^{n-1}(M, \mathbb{R}/\Lambda)$.

Let us give another description of the truncated relative cohomology groups which explains their name. Let σ denote the "bête" filtration. That is, given a complex A^*, then

$$\sigma^p A^n = \begin{cases} A^n, & \text{if } n \geq p \text{ and} \\ 0, & \text{if } n < p. \end{cases}$$

Let $s(\cdot)$ denote the simple of a morphism of complexes. Then

$$H^n(s(\sigma^p A^* \longrightarrow B^*)) = \begin{cases} H^{n-1}(B^*), & \text{if } n < p, \\ \widehat{H}^n(A^*, B^*), & \text{if } n = p \text{ and} \\ H^n(A^*, B^*), & \text{if } n > p. \end{cases}$$

From this description we can obtain exact sequences involving truncated relative cohomology groups. Let us first define some maps involving these groups:

$$\begin{array}{ll}
\mathrm{cl} : \widehat{H}^n(A^*,B^*) \longrightarrow H^n(A^*,B^*), & \mathrm{cl}(a,\widetilde{b}) = \{(a,b)\}, \\
\quad \text{where } \{\cdot\} \text{ denotes cohomology class.} & \\
\omega : \widehat{H}^n(A^*,B^*) \longrightarrow ZA^n, & \omega(a,\widetilde{b}) = a. \\
\mathrm{a} : \widetilde{A}^{n-1} \longrightarrow \widehat{H}^n(A^*,B^*), & \mathrm{a}(\widetilde{a}) = (da, f(a)^{\sim}). \\
\mathrm{b} : H^{n-1}(B^*) \longrightarrow \widehat{H}^n(A^*,B^*), & \mathrm{b}(\{b\}) = (0,\widetilde{b}).
\end{array}$$

We shall also denote by a the induced morphism $\mathrm{a} : H^{n-1}(A^*) \longrightarrow \widehat{H}^n(A^*,B^*)$.

Proposition 3.3. *Let $f : A^* \longrightarrow B^*$ be a morphism of complexes. Then there are exact sequences*

1) $H^{n-1}(A^*,B^*) \to \widetilde{A}^{n-1} \xrightarrow{\mathrm{a}} \widehat{H}^n(A^*,B^*) \xrightarrow{\mathrm{cl}} H^n(A^*,B^*) \to 0$

2) $0 \to H^{n-1}(B^*) \xrightarrow{\mathrm{b}} \widehat{H}^n(A^*,B^*) \xrightarrow{\omega} ZA^n \to H^n(B^*)$

3) $H^{n-1}(A^*,B^*) \to H^{n-1}(A^*) \xrightarrow{\mathrm{a}} \widehat{H}^n(A^*,B^*) \xrightarrow{\mathrm{cl}\oplus\omega}$
$$H^n(A^*,B^*) \oplus ZA^n \to H^n(A^*) \longrightarrow 0$$

Proof. These exact sequences follow respectively from the exact sequences of complexes

$$\begin{array}{l}
0 \to s(\sigma^n A^* \longrightarrow B^*) \to s(A^* \longrightarrow B^*) \to A^*/\,\sigma^n A^* \to 0, \\
0 \to B^*[-1] \to s(\sigma^n A^* \longrightarrow B^*) \to \sigma^n A^* \to 0 \qquad \text{and} \\
0 \to s(\sigma^n A^* \longrightarrow B^*) \to s(A^* \longrightarrow B^*) \oplus \sigma^n A^* \to A^* \to 0.
\end{array}$$

A morphism of complexes will also be called a $\underline{2}$-complex because it can be considered as a functor from the category $\underline{2}$ to the category of complexes. The $\underline{2}$-complex $f : A^* \longrightarrow B^*$ will be noted by (A^*,B^*,f) or simply by f. A morphism of $\underline{2}$-complexes $g : f_1 \longrightarrow f_2$ is a commutative diagram

$$\begin{array}{ccc}
A_1^* & \xrightarrow{f_1} & B_1^* \\
\Big\downarrow{\scriptstyle g_A} & & \Big\downarrow{\scriptstyle g_B} \\
A_2^* & \xrightarrow{f_2} & B_2^*.
\end{array}$$

If g_A and g_B have degree e, we say that g has degree e. For each n, the n-th truncated relative cohomology group is a covariant functor from the category of $\underline{2}$-complexes of R-modules to the category of R-modules. If $g = (g_A, g_B)$ is a morphism of $\underline{2}$-complexes, then there is an induced morphism

$$\begin{array}{rccc}
\widehat{g} = \widehat{H}^*(g) : & \widehat{H}^*(A_1^*,B_1^*) & \longrightarrow & \widehat{H}^*(A_2^*,B_2^*) \\
& (a,\widetilde{b}) & \longmapsto & (g_A(a), (g_B(b))^{\sim}).
\end{array}$$

If g has degree e, then the induced morphism $\widehat{g}$ is also of degree e.

Proposition 3.4. *Let $g = (g_A, g_B)$ be a morphism of $\underline{2}$-complexes. If g_A is an isomorphism and g_B is a quasi-isomorphism then $\widehat{g}$ is an isomorphism.*

Proof. A direct consequence of 3.3.2.

This proposition reflects the asymmetry between the complexes A^* and B^*. We can freely replace the complex B^* by a quasi-isomorphic complex without changing the truncated relative cohomology groups. On the other hand, if we change A^* by a quasi-isomorphic complex, then we can change the properties of these groups.

Let us recall now how to construct a product on relative cohomology groups from a product at the level of complexes. We shall extend this construction to truncated relative cohomology groups.

Let $f : A^* \longrightarrow B^*$ and $g : C^* \longrightarrow D^*$ be a morphism of complexes. We can construct the complex

$$s(f) \otimes s(g) = s(A^* \longrightarrow B^*) \otimes s(C^* \longrightarrow D^*)$$

or consider the simple of the diagram

$$A^* \otimes C^* \xrightarrow{(f\otimes \mathrm{Id}, \mathrm{Id}\otimes g)} B^* \otimes C^* \oplus A^* \otimes D^* \xrightarrow{-\mathrm{Id}\otimes g + f \otimes \mathrm{Id}} B^* \otimes D^*.$$

There is an isomorphism of complexes

$$s(f) \otimes s(g) \longrightarrow s(A^* \otimes C^* \to B^* \otimes C^* \oplus A^* \otimes D^* \to B^* \otimes D^*.)$$

If $(a, b) \in s(f)^n$ and $(c, d) \in s(g)^m$ then this isomorphism is given by

$$(a, b) \otimes (c, d) \longmapsto (a \otimes c, b \otimes c + (-1)^n a \otimes d, (-1)^n b \otimes d).$$

Suppose that there is a morphism of commutative diagrams

$$\begin{array}{ccccccc} A^* \otimes C^* & \longrightarrow & A^* \otimes D^* & & E_1^* & \longrightarrow & E_3^* \\ \downarrow & & \downarrow & \stackrel{\cdot}{\longrightarrow} & \downarrow & & \downarrow \\ B^* \otimes C^* & \longrightarrow & B^* \otimes D^* & & E_2^* & \longrightarrow & E_4^*. \end{array}$$

Then there is an induced product

$$s(f) \otimes s(g) \stackrel{\cdot}{\to} s(E_1^* \to s(E_2^* \oplus E_3^* \to E_4^*)).$$

Hence a product

$$H^n(A^*, B^*) \otimes H^m(C^*, D^*) \to H^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \to E_4^*)).$$

If $\{(a, b)\} \in H^n(A^*, B^*)$ and $\{(c, d)\} \in H^m(C^*, D^*)$, this product is given by

$$\{(a, b)\} \otimes \{(c, d)\} \longmapsto \{(a \cdot c, b \cdot c + (-1)^n a \cdot d, (-1)^n b \cdot d)\}.$$

Here $\{\cdot\}$ denotes cohomology class.

Definition 3.5. With the above hypothesis, the $*$-product of truncated relative cohomology groups:

$$\widehat{H}^n(A^*, B^*) \otimes \widehat{H}^m(C^*, D^*) \xrightarrow{*} \widehat{H}^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \to E_4^*))$$

is defined by

$$(a, \widetilde{b}) * (c, \widetilde{d}) = (a \cdot c, (b \cdot c + (-1)^n a \cdot d, (-1)^n b \cdot d)^{\sim}).$$

Proposition 3.5. *The $*$-product of truncated relative cohomology groups is well defined, i.e. it does not depend on the choice of representatives b and d of $\widetilde{b}$ and $\widetilde{d}$. Moreover there are commutative diagrams*

$$\begin{array}{ccc} \widehat{H}^n(A^*, B^*) \otimes \widehat{H}^m(C^*, D^*) & \xrightarrow{\quad * \quad} & \widehat{H}^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \longrightarrow E_4^*)) \\ \Big\downarrow{\scriptstyle \omega \otimes \omega} & & \Big\downarrow{\scriptstyle \omega} \\ A^n \otimes C^m & \xrightarrow{\quad \cdot \quad} & E_1^{n+m}, \end{array}$$

and

$$\begin{array}{ccc} \widehat{H}^n(A^*, B^*) \otimes \widehat{H}^m(C^*, D^*) & \xrightarrow{\quad * \quad} & \widehat{H}^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \longrightarrow E_4^*)) \\ \Big\downarrow{\scriptstyle \mathrm{cl} \otimes \mathrm{cl}} & & \Big\downarrow{\scriptstyle \mathrm{cl}} \\ H^n(A^*, B^*) \otimes H^m(C^*, D^*) & \xrightarrow{\quad \cdot \quad} & H^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \longrightarrow E_4^*)). \end{array}$$

Proof. Follows from the definitions.

§4. A Definition of Green Forms Using Deligne Cohomology.

In this section we shall see that the space of Green forms can be obtained as a truncated relative cohomology group of the Deligne complex. Moreover, the $*$-product of Green forms is induced by the product of the Deligne complex.

Let X be a smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{Z}^p = \mathcal{Z}^p(X)$ be the set of algebraic subsets of codimension $\geq p$, ordered by inclusion. Let us write

$$E^*_{\log}(X\backslash\mathcal{Z}^p) = \varinjlim_{Z\in\mathcal{Z}^p} E^*_{\log}(X-Z).$$

This complex is a Dolbeault complex and there is a natural injective map

$$E^*_{\log}(X) \longrightarrow E^*_{\log}(X\backslash\mathcal{Z}^p).$$

We shall write

$$\begin{aligned} H^*_{\mathcal{D}}(X\backslash\mathcal{Z}^p,\mathbb{R}(p)) &= H^*(\mathfrak{D}^*(E^*_{\log}(X\backslash\mathcal{Z}^p),p)) \quad \text{and} \\ H^*_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)) &= H^*(s(\mathfrak{D}^*(E^*_{\log}(X),p) \longrightarrow \mathfrak{D}^*(E^*_{\log}(X\backslash\mathcal{Z}^p),p))). \end{aligned}$$

Since $\mathcal{Z}^p$ is a directed set we have

$$\begin{aligned} H^*_{\mathcal{D}}(X\backslash\mathcal{Z}^p,\mathbb{R}(p)) &= \varinjlim_{Z\in\mathcal{Z}^p} H^*_{\mathcal{D}}(X-Z,\mathbb{R}(p)) \quad \text{and} \\ H^*_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)) &= \varinjlim_{Z\in\mathcal{Z}^p} H^*_{\mathcal{D},Z}(X,\mathbb{R}(p)). \end{aligned}$$

Definition 4.1. The *space of Green forms on X with codimension p singular support* is

$$GE^p(X) = \widehat{H}^{2p}(\mathfrak{D}^*(E^*_{\log}(X),p),\mathfrak{D}^*(E^*_{\log}(X\backslash\mathcal{Z}^p),p)).$$

Let $(\omega,\widetilde{g}) \in GE^p(X)$. Since the map $\mathfrak{D}^*(E^*_{\log}(X),p) \longrightarrow \mathfrak{D}^*(E^*_{\log}(X\backslash\mathcal{Z}^p),p)$ is injective, ω is determined by $\widetilde{g}$. Thus we shall sometimes represent $(\omega,\widetilde{g})$ by $\widetilde{g}$.

By the definition of the Deligne complex we have

$$\mathfrak{D}^{2p-1}(E^*_{\log}(X),p)\big/\operatorname{Im} d_{\mathfrak{D}} = E^{p-1,p-1}_{\log}(X)\cap E^{2p-2}_{\log,\mathbb{R}}(p-1)\Big/(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}).$$

We shall denote this group by $\widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X)$. Analogously we write

$$\widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X\backslash\mathcal{Z}^p) = \mathfrak{D}^{2p-1}(E^*_{\log}(X\backslash\mathcal{Z}^p),p)\big/\operatorname{Im} d_{\mathfrak{D}}.$$

We also have that the subgroup of cycles of $\mathfrak{D}^{2p}(E^*_{\log}(X),p)$ is

$$\left\{\omega \in E^{p,p}_{\log}(X) \cap E^{2p}_{\log,\mathbb{R}}(X,p) \mid d\omega = 0\right\}.$$

This group will be denoted by $ZE^{p,p}_{\log,\mathbb{R}}(X)$.

Then

$$\begin{aligned} GE^p(X) &= \left\{(\omega,\widetilde{g}) \in ZE^{p,p}_{\log,\mathbb{R}}(X) \oplus \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X\backslash\mathcal{Z}^p) \mid -2\partial\overline{\partial}g = \omega\right\} \\ &= \left\{\widetilde{g} \in \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X\backslash\mathcal{Z}^p) \mid \partial\overline{\partial}g \text{ is smooth on } X\right\}. \end{aligned}$$

If $Z \subset X$ is a codimension p algebraic subset of X, then the *space of Green forms on X with singular support contained on Z* is

$$GE^p_Z(X) = \widehat{H}^{2p}(\mathfrak{D}^*(E^*_{\log}(X),p), \mathfrak{D}^*(E^*_{\log}(X-Z),p)).$$

Since $\mathcal{Z}^p$ is a directed set and the codimension p algebraic subsets of X is a cofinal subset of $\mathcal{Z}^p$, the group $GE^p(X)$ is the direct limit of the groups $GE^p_Z(X)$ for Z of codimension p.

Let $\widetilde{g} \in GE^p(X)$. Then the *singular support of* $\widetilde{g}$ is the intersection of all Z such that $\widetilde{g}$ has a representative in $GE^p_Z(X)$. We shall denote the singular support of $\widetilde{g}$ by $\operatorname{supp}\widetilde{g}$.

Since $GE^p(X)$ are truncated relative cohomology groups we can define maps

$$\begin{aligned} \mathrm{cl} &: GE^p(X) \longrightarrow H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)), \\ \omega &: GE^p(X) \longrightarrow ZE^{p,p}_{\log,\mathbb{R}}(X), \\ \mathrm{a} &: \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \longrightarrow GE^p(X) \quad \text{and} \\ \mathrm{b} &: H^{2p-1}_{\mathcal{D}}(X\backslash\mathcal{Z}^p,\mathbb{R}(p)) \longrightarrow GE^p(X), \end{aligned}$$

as in §3. We shall also denote by a the induced morphism

$$\mathrm{a} : H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \longrightarrow GE^p(X).$$

Proposition 4.2. *Let X be a smooth variety over $\mathbb{C}$. Then there are exact sequences*

1) $0 \to \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \xrightarrow{\mathrm{a}} GE^p(X) \xrightarrow{\mathrm{cl}} H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)) \to 0.$

2) $0 \to H^{2p-1}_{\mathcal{D}}(X\backslash\mathcal{Z}^p,\mathbb{R}(p)) \xrightarrow{\mathrm{b}} GE^p(X) \xrightarrow{\omega} ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X\backslash\mathcal{Z}^p,\mathbb{R}(p)).$

3) $0 \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{\mathrm{a}} GE^p(X) \xrightarrow{\mathrm{cl}\oplus\omega}$

$$H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)) \oplus ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)) \to 0.$$

Proof. This is a translation of Proposition 3.3. taking into account that Deligne cohomology satisfies

$$H^{2p-1}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)) = 0.$$

This can be proved using the exact sequence of Proposition 1.1. and the fact that, if Z is a codimension p algebraic subset of X then

$$H^n_Z(X,R) = 0$$

for $n < 2p$ and $R = \mathbb{R}$ or $\mathbb{C}$.

Fixing the singular support we have an analogous result.

Proposition 4.3. *Let X be a smooth variety over $\mathbb{C}$ and $Z \subset X$ a codimension p algebraic subset. Then there are exact sequences*

1) $0 \to \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \xrightarrow{\mathrm{a}} GE^p_Z(X) \xrightarrow{\mathrm{cl}} H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \to 0.$

2) $0 \to H^{2p-1}_{\mathcal{D}}(X - Z,\mathbb{R}(p)) \xrightarrow{\mathrm{b}} GE^p_Z(X) \xrightarrow{\omega} ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X - Z,\mathbb{R}(p)).$

3) $0 \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{\mathrm{a}} GE^p_Z(X) \xrightarrow{\mathrm{cl}\oplus\omega}$
$$H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \oplus ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)) \to 0.$$

Corollary 4.4. *The natural map*

$$GE^p_Z(X) \longrightarrow GE^p(X)$$

is injective. Moreover, if $\widetilde{g} \in GE^p(X)$ then $\operatorname{supp}\widetilde{g} = \operatorname{supp}\mathrm{cl}(\widetilde{g})$.

Proof. The injectivity follows from the injectivity of the morphism

$$H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p))$$

and the Five Lemma. Let us write $Y = \operatorname{supp}\mathrm{cl}(\widetilde{g})$ and $Y' = \operatorname{supp}\widetilde{g}$. Clearly $Y \subset Y'$. Then we have a morphism of change of support $\varphi : GE^p_Y(X) \longrightarrow GE^p_{Y'}(X)$ and a commutative diagram

$$\begin{array}{ccc} GE^p_Y(X) & \xrightarrow{\mathrm{cl}} & H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \\ \varphi\big\downarrow & & \big\downarrow \\ GE^p_{Y'}(X) & \xrightarrow{\mathrm{cl}} & H^{2p}_{\mathcal{D},Y'}(X,\mathbb{R}(p)), \end{array}$$

where the horizontal arrows are surjective. Let $\widetilde{g}' \in GE^p_Y(X)$ with $\mathrm{cl}(\widetilde{g}') = \mathrm{cl}(\widetilde{g})$. By Proposition 4.3, there is an element $\alpha \in \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X)$ such that $a(\alpha) = \widetilde{g} - \varphi\widetilde{g}'$. But then $\widetilde{g}' + a(\alpha) \in GE^p_Y(X)$ and it represents $\widetilde{g}$. Thus $Y = Y'$.

Definition 4.5. Let y be a codimension p algebraic cycle and let $Y = \operatorname{supp} y$. Then the *space of Green forms associated to y* is

$$GE^p_y(X) = \{\widetilde{g} \in GE^p(X) \mid \mathrm{cl}(\widetilde{g}) = \rho(y)\},$$

where $\rho(y)$ is the class of y in $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$ (see [J] or chapter IV).

A direct consequence of Corollary 4.4 is:

Corollary 4.6. *Let y be a codimension p algebraic cycle and let $Y = \operatorname{supp} y$. If $\widetilde{g}_y$ is a Green form associated to y, then the singular support of $\widetilde{g}_y$ is Y.*

Theorem 4.7. *Let X be a smooth projective variety over $\mathbb{C}$ and y a codimension p algebraic cycle. Let $GE_X(y)$ be the space of Green forms for y as defined in chap. II, §4. Then there is a natural isomorphism*

$$GE^p_y(X) \longrightarrow GE_X(y)$$

given by

$$\widetilde{g} \longmapsto \frac{2}{(2\pi i)^{p-1}}\widetilde{g}.$$

If X has dimension d and $GC_X(y)$ is the space of Green currents for y in the sense of Gillet and Soulé ([G-S 2]*, see also chap II, §4) then there is a natural isomorphism*

$$GE^p_y(X) \longrightarrow GC_X(y).$$

Proof. Let us write $Y = \operatorname{supp} y$. By definition

$$GE_X(y) = \left\{ g \in E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y) \mid \begin{array}{c} dd^c g \in E^{p,p}_X, \\ \{dd^c g, d^c g\} = \{y\} \end{array} \right\} \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}),$$

where $\{dd^c g, d^c g\}$ is the cohomology class represented by $(dd^c g, d^c g)$ and $\{y\}$ is the cohomology class of y. Both classes are considered in $H^{2p}_Y(X,\mathbb{R})$.

On the other hand, by Corollary 4.6, if we write

$$E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y,p-1) = E^{p-1,p-1}_{\log}(X-Y) \cap E^{2p-2}_{\log,\mathbb{R}}(X-Y,p-1)$$

we have

$$GE^p_y(X) = \left\{ g \in E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y,p-1) \mid \begin{array}{c} -2\partial\overline{\partial} g \in E^{2p}_X \\ \{-2\partial\overline{\partial} g, g\} = \rho(y) \end{array} \right\} \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}),$$

where now $\{-2\partial\overline{\partial} g, g\}$ and $\rho(y)$ are cohomology classes in $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$. But the natural morphism $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_Y(X,\mathbb{R})$ is induced by a morphism of complexes (see 1.10.3)

$$\frac{1}{(2\pi i)^p} r_p : s(\mathfrak{D}^*(E^*_X,p), \mathfrak{D}^*(E^*_{\log}(X-Y),p)) \longrightarrow s(E^*_{X,\mathbb{R}} \longrightarrow E^*_{\log,\mathbb{R}}(X-Y)).$$

Which, in degree $2p$, satisfies

$$\frac{1}{(2\pi i)^p} r_p(\omega, g) = (\frac{1}{(2\pi i)^p}\omega, \frac{2}{(2\pi i)^{p-1}} d^c g).$$

Therefore, this morphism sends the class $\{-2\partial\overline{\partial} g, g\}$ to the class $\{dd^c g, d^c g\}$. Moreover, by the definition of $\rho(y)$, this class is mapped to $\{y\}$. Hence the map

$$\begin{array}{ccc} GE^p_y(X) & \longrightarrow & GE_X(y) \\ \widetilde{g} & \longmapsto & \dfrac{2}{(2\pi i)^{p-1}} \widetilde{g} \end{array}$$

is well defined. The inverse of this map is also well defined, because the morphism $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_Y(X,\mathbb{R})$ is an isomorphism.

The second part of the Theorem follows from the first part and the comparison isomorphism between Green forms and Green currents proved in chapter II, §4.

Remark 4.8. By the definition of the space of Green forms as a truncated relative homology group the morphism

$$GE^p_Y(X) \longrightarrow H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$$

is an epimorphism. Therefore the existence of Green forms is a direct consequence of the existence of the cycle class in real Deligne cohomology. Reciprocally, the existence of Green forms implies the existence of the cycle class.

Definition 4.9. Let X be a smooth variety over $\mathbb{C}$ and let Y and Z be algebraic subsets of codimension p and q respectively such that $Y \cap Z$ has codimension $p+q$. Then the $*$*-product*

$$GE^p_Y(X) \otimes GE^q_Z(X) \xrightarrow{*} GE^p_{Y\cap Z}(X)$$

is the product in truncated relative cohomology groups induced by the product of the Deligne complex. That is, let $r = p+q$. Write $\mathfrak{D}^*(X,r) = \mathfrak{D}^*(E^*_{\log}(X), r)$ and

$$\mathfrak{D}^*(X; Y, Z, r) = s(\mathfrak{D}^*(X-Y,r) \oplus \mathfrak{D}^*(X-Z,r) \xrightarrow{j} \mathfrak{D}^*(X - Y \cup Z, r)),$$

where $j(a,b) = b-a$. Then the map

$$\begin{array}{ccc} \mathfrak{D}^*(X - Y \cap Z, r) & \longrightarrow & \mathfrak{D}^*(X;Y,Z,r) \\ g & \longmapsto & (g,g,0) \end{array}$$

is a quasi-isomorphism. Therefore there is a natural isomorphism

$$\widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X - Y\cap Z, r)) \longrightarrow \widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X;Y,Z,r)).$$

In terms of this last group we have

$$\begin{aligned}(\omega_1, \widetilde{g}_1) * (\omega_2, \widetilde{g}_2) &= (\omega_1 \cdot \omega_2, (g_1 \cdot \omega_2, \omega_1 \cdot g_2, g_1 \cdot g_2)^{\sim}) \\ &= (\omega_1 \wedge \omega_2, (g_1 \wedge \omega_2, \omega_1 \wedge g_2, -r_p(g_1) \wedge g_2 + g_1 \wedge r_q g_2)^{\sim}) \\ &= (\omega_1 \wedge \omega_2, (g_1 \wedge \omega_2, \omega_1 \wedge g_2, -4\pi i d^c g_1 \wedge g_2 + 4\pi i g_1 \wedge d^c g_2)^{\sim})\end{aligned}$$

Theorem 4.10. *The $*$-product of Green forms is commutative and associative. It is compatible with the product in Deligne cohomology and with the cup product of differential forms. Moreover if X is projective then it is compatible with the $*$-product of Green forms defined in chapter II and with the $*$-product of currents defined in* [G-S 2].

Proof. The compatibility with $\cup$ and $\wedge$ follows easily from the definitions.

Let Y and Z be closed algebraic subsets of X of codimension p and q, and let $\widetilde{g}_1 \in GE^p_Y(X)$ and $\widetilde{g}_2 \in GE^q_Z(X)$. Write $r = p+q$. Then

$$\widetilde{g}_1 * \widetilde{g}_2 \in \widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X;Y,Z,r)),$$

and

$$\widetilde{g}_2 * \widetilde{g}_1 \in \widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X;Z,Y,r)).$$

Both groups are naturally isomorphic. The isomorphism between them is induced by an isomorphism of complexes

$$\mathfrak{D}^n(X;Y,Z,r) \longrightarrow \mathfrak{D}^n(X;Z,Y,r),$$

given by

$$(a,b,c) \longmapsto (b,a,-c).$$

It is straightforward to check that this isomorphism sends $\widetilde{g}_1 * \widetilde{g}_2$ to $\widetilde{g}_2 * \widetilde{g}_1$.

Let W, Y and Z be algebraic subsets of X of codimension p, q and r respectively, such that the codimension of $W\cap Y\cap Z$ is $p+q+r$ and Y intersects properly with W and Z. Let $(\omega_1,\widetilde{g}_1)\in GE^p_W(X)$, $(\omega_2,\widetilde{g}_2)\in GE^q_Y(X)$ and $(\omega_3,\widetilde{g}_3)\in GE^r_Z(X)$. Let us write $s=p+q+r$ and

$$\begin{aligned}\mathfrak{D}^*(X;W,Y,Z,s) = s(&\mathfrak{D}^*(X-W,s)\oplus\mathfrak{D}^*(X-Y,s)\oplus\mathfrak{D}^*(X-Z,s)\xrightarrow{j}\\ &\mathfrak{D}^*(X-W\cup Y,s)\oplus\mathfrak{D}^*(X-W\cup Z,s)\oplus\mathfrak{D}^*(X-Y\cup Z,s)\xrightarrow{k}\\ &\mathfrak{D}^*(X-W\cup Y\cup Z,s)),\end{aligned}$$

where

$$j(a,b,c)=(b-a,c-a,b-c)\quad\text{and}\quad k(a,b,c)=a-b+c.$$

Both products, $\widetilde{g}_1*(\widetilde{g}_2*\widetilde{g}_3)$ and $(\widetilde{g}_1*\widetilde{g}_2)*\widetilde{g}_3$ are defined in $\widehat{H}^{2s}(\mathfrak{D}^*(X,s),\mathfrak{D}^*(X;W,Y,Z,s))$.

We have

$$\begin{aligned}\widetilde{g}_1*(\widetilde{g}_2*\widetilde{g}_3) = (&\omega_1\cdot(\omega_2\cdot\omega_3),(g_1\cdot(\omega_2\cdot\omega_3),\omega_1\cdot(g_2\cdot\omega_3),\omega_1\cdot(\omega_2\cdot g_3),\\ &-g_1\cdot(g_2\cdot\omega_3),-g_1\cdot(\omega_2\cdot g_3),-\omega_1\cdot(g_2\cdot g_3),g_1\cdot(g_2\cdot g_3))^{\sim}).\end{aligned}$$

and

$$\begin{aligned}(\widetilde{g}_1*\widetilde{g}_2)*\widetilde{g}_3 = (&(\omega_1\cdot\omega_2)\cdot\omega_3,((g_1\cdot\omega_2)\cdot\omega_3,(\omega_1\cdot g_2)\cdot\omega_3,(\omega_1\cdot\omega_2)\cdot g_3,\\ &-(g_1\cdot g_2)\cdot\omega_3,-(g_1\cdot\omega_2)\cdot g_3,-(\omega_1\cdot g_2)\cdot g_3,(g_1\cdot g_2)\cdot g_3)^{\sim}).\end{aligned}$$

By Theorem 2.3, $\omega_1\cdot(\omega_2\cdot\omega_3)=(\omega_1\cdot\omega_2)\cdot\omega_3$. Therefore

$$\widetilde{g}_1*(\widetilde{g}_2*\widetilde{g}_3)-(\widetilde{g}_1*\widetilde{g}_2)*\widetilde{g}_3=(0,\widetilde{x}),$$

with $x\in\mathfrak{D}^{2s-1}(X;W,Y,Z,s)$ Let h_{a} be the homotopy which makes the product on the Deligne complex associative (see 2.3). That is

$$a\cdot(b\cdot c)-(a\cdot b)\cdot c=d_{\mathfrak{D}}h_{\mathrm{a}}(a\otimes b\otimes c)+h_{\mathrm{a}}d_{\mathfrak{D}}(a\otimes b\otimes c).$$

Let us consider the element $y\in\mathfrak{D}(X;W,Y,Z,s)$ given by

$$\begin{aligned}y=(&h_{\mathrm{a}}(g_1\otimes\omega_2\otimes\omega_3),h_{\mathrm{a}}(\omega_1\otimes g_2\otimes\omega_3),h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes g_3),\\ &h_{\mathrm{a}}(g_1\otimes g_2\otimes\omega_3),h_{\mathrm{a}}(g_1\otimes\omega_2\otimes g_3),h_{\mathrm{a}}(\omega_1\otimes g_2\otimes g_3),h_{\mathrm{a}}(g_1\otimes g_2\otimes g_3)).\end{aligned}$$

By the naturality of h_{a} we have,

$$d_{\mathfrak{D}}y=x-(h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes\omega_3),h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes\omega_3),h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes\omega_3),0,0,0,0).$$

Therefore the associativity follows from the lemma:

Lemma 4.11. *Let $\omega_1\in\mathfrak{D}^{2p}(X-W,p)$, $\omega_2\in\mathfrak{D}^{2q}(X-Y,q)$, and $\omega_3\in\mathfrak{D}^{2r}(X-Z,r)$. Then*

$$h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes\omega_3)=0.$$

Proof. By definition (see §2)

$$h_{\mathrm{a}}(\omega_1\otimes\omega_2\otimes\omega_3)=\psi(h(\varphi\omega_1\cup\varphi\omega_2)\cup\varphi\omega_3)+\psi(\varphi\omega_1\cup h(\varphi\omega_2\cup\varphi\omega_3)),$$

where ψ and φ are the homotopy equivalences between the Deligne complexes, h is the homotopy between $\varphi\psi$ and Id and $\cup$ is the product $\cup_0$ in the Deligne-Beilinson complex which is associative.

But

$$\begin{aligned} h(\varphi\omega_1 \cup \varphi\omega_2) &= h((\omega_1,\omega_1,0)\cup(\omega_2,\omega_2,0)) \\ &= h(\omega_1\wedge\omega_2,\omega_1\wedge\omega_2,0) \\ &= 0. \end{aligned}$$

Therefore we obtain the Lemma.

Let us show now that the $*$-product defined here is compatible with the $*$-product defined in chapter II. Let Y and Z be closed algebraic subsets of X of codimension p and q respectively which intersect properly. Write $r=p+q$. Let $\widetilde{X}$ be a resolution of singularities of $Y\cap Z$ such that the strict transforms of Y and Z do not meet. Write $\widehat{Y}$ for the strict transform of Y and $\widehat{Z}$ for that of Z. Let $\sigma_{Y,Z}$ be a smooth function on $\widetilde{X}$ such that takes the value 1 in a neighbourhood of $\widehat{Y}$ and the value 0 in a neighbourhood of $\widehat{Z}$. Let $\sigma_{Z,Y}=1-\sigma_{Y,Z}$. Let us denote by $*'$ the $*$-product of Green forms defined in chapter II. Then

$$\begin{aligned} \widetilde{g}_2 *' \widetilde{g}_2 &= 4\pi i(dd^c(\sigma_{Y,Z}g_1)\wedge g_2+\sigma_{Z,Y}g_1\wedge dd^c g_2)^{\sim} \\ &= (d_{\mathfrak{D}}(\sigma_{Y,Z}g_1)\cdot g_2+\sigma_{Z,Y}g_1\cdot d_{\mathfrak{D}}g_2)^{\sim}. \end{aligned}$$

The factor $4\pi i$ comes from the normalization for Green forms used here which differs from that used in chap II (see 4.7).

The isomorphism

$$\varphi : \widehat{H}^{2r}(\mathfrak{D}^*(X,r),\mathfrak{D}^*(X-Y\cap Z,r)) \longrightarrow \widehat{H}^{2r}(\mathfrak{D}^*(X,r),\mathfrak{D}^*(X,Y,Z,r))$$

sends $(d_{\mathfrak{D}}(\sigma_{Y,Z}g_1)\cdot g_2+\sigma_{Z,Y}g_1\cdot d_{\mathfrak{D}}g_2)^{\sim}$ to

$$(d_{\mathfrak{D}}(\sigma_{Y,Z}g_1)\cdot g_2+\sigma_{Z,Y}g_1\cdot d_{\mathfrak{D}}g_2, d_{\mathfrak{D}}(\sigma_{Y,Z}g_1)\cdot g_2+\sigma_{Z,Y}g_1\cdot d_{\mathfrak{D}}g_2,0)^{\sim}.$$

Then

$$\begin{aligned} \varphi(\widetilde{g}_1 *' \widetilde{g}_2)-\widetilde{g}_1 * \widetilde{g}_2 &= (d_{\mathfrak{D}}(\sigma_{Y,Z}g_1\cdot g_2), -d_{\mathfrak{D}}(\sigma_{Z,Y}g_1\cdot g_2), -g_1\cdot g_2)^{\sim} \\ &= \big(d_{\mathfrak{D}}(\sigma_{Y,Z}g_1\cdot g_2, -\sigma_{Z,Y}g_1\cdot g_2, 0)\big)^{\sim} \\ &= 0. \end{aligned}$$

Therefore $\widetilde{g}_1 * \widetilde{g}_2$ and $\widetilde{g}_1 *' \widetilde{g}_2$ represents the same Green form.

In chapter II, §4, the compatibility of the $*$-product of Green currents with the product $*'$ of Green forms is proved. Therefore the product of Green currents is also compatible with the product defined here.

Remark 4.12. The key point in the proof of the associativity is Lemma 4.11. We can even weak its hipothesis assuming that ω_i, $i=1,2,3$, are closed. It may be convenient to replace the complexes $\mathfrak{D}$ by other complexes in order to obtain Green forms with different properties. Then to prove the associativity of the product of these new Green forms we only need to check Lemma 4.11 in that case.

Remark 4.13. In the proof of Theorem 4.10 we have assumed that the intersections are proper, because we have defined $GE^p_Z(X)$ only for closed subsets Z of codimension $\geq p$. With the obvious definition of $GE^p_Z(X)$ for Z of arbitrary codimension, the Theorem also holds, except for the comparison between Green forms and Green currents.

CAPITULO IV

Arithmetic Chow Groups

§1. Real Deligne Homology.

We are interested in relating algebraic cycles and algebraic K-theoretic chains with Deligne cohomology. This can be done using Deligne homology and the Poincaré Duality homomorphism. In this section we shall review how to use currents to obtain explicit descriptions of real Deligne homology groups. We shall follow the conventions of [J], except that we shall use homological notation.

Let us begin with the case of X, a proper smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{D}_n^X$ denote the sheaf of complex valued currents on X. That is, for an open subset $U \subset X$, $\Gamma(U, \mathcal{D}_n^X)$ is the topological dual of $\Gamma_c(U, \mathcal{E}_X^n)$. This sheaf is denoted in [J] by $'\Omega_{X_\infty}^{-n}$.

The sheaf $\mathcal{D}_n^X$ has a natural bigrading

$$\mathcal{D}_n^X = \bigoplus_{p+q=n} \mathcal{D}_{p,q}^X,$$

and a real structure $\mathcal{D}_n^{X,\mathbb{R}}$. We shall write $D_n^X = \Gamma(X, \mathcal{D}_n^X)$.

If X is equidimensional of dimension d, then there is a map

$$\begin{array}{cccc} [\cdot]: & E_X^n & \longrightarrow & D_{2d-n}^X \\ & \omega & \longmapsto & [\omega], \end{array}$$

defined by

$$[\omega](\omega') = \frac{1}{(2\pi i)^d} \int_X \omega' \wedge \omega.$$

More generally, if ω is a locally L^1 form, then we define $[\omega]$ by the same formula. Observe that this notation differs from the notation used in chapter II by the inclusion of the normalization factor.

We can turn D_*^X into a chain complex by writing, for $T \in D_n^X$,

$$dT(\omega) = (-1)^n T(d\omega).$$

If X is equidimensional of dimension d, we shall also write $D_X^n = D_{2d-n}^X$. In this case, by Stokes' Theorem, the map $[\cdot] : E_X^* \longrightarrow D_X^*$ is a morphism of complexes and a quasi-isomorphism with respect to the Hodge filtration. Moreover the product

$$E_X^n \otimes D_m^X \longrightarrow D_{m-n}^X,$$

defined by

$$\omega \wedge T(\omega') = T(\omega' \wedge \omega),$$

turns D_*^X into a left E_X^*-module.

Definition 1.1. A *Dolbeault chain complex* is a complex of real vector spaces $(A_*^{\mathbb{R}}, d)$ provided with a bigrading on $A_*^{\mathbb{C}} = A_*^{\mathbb{R}} \otimes \mathbb{C}$:

$$A_d^{\mathbb{C}} = \bigoplus_{p+q=d} A_{p,q},$$

such that

DC1. The differential d can be decomposed as a sum of operators $d = \partial + \overline{\partial}$ of type $(-1,0)$ and $(0,-1)$.

DC2. It satisfies the symmetry property

$$\overline{A_{p,q}} = A_{q,p}.$$

The chain complex D_*^X is a Dolbeault chain complex. By analogy with the case of Dolbeault cochain complexes introduced in chapter III, we define:

Definition 1.2. Let A be a chain Dolbeault complex. The *Hodge filtration* F of A_* is the increasing filtration

$$F_p A_* = \bigoplus_{p' \leq p} A_{p',*}.$$

We denote by $\overline{F}$ the filtration complex conjugate of F.

We shall write

$$A_*^{\mathbb{R}}(p) = (2\pi i)^{-p} A_*^{\mathbb{R}}.$$

The *Deligne complexes* associated to A are

$$A_*^{\mathbb{R}}(p)_{\mathcal{D}} = s(A_*^{\mathbb{R}}(p) \oplus F_p A_*^{\mathbb{C}} \xrightarrow{u} A_*^{\mathbb{C}}),$$

where $u(a,b) = b - a$. And

$$\mathfrak{D}_n(A,p) = \widehat{s}_n(u) = \begin{cases} A_n^{\mathbb{R}}(p+1) \cap \displaystyle\bigoplus_{\substack{p'+q'=n+1 \\ p'>p,\ q'>p}} A_{p',q'}, & \text{for } n \geq 2p+1, \\ A_n^{\mathbb{R}}(p) \cap \displaystyle\bigoplus_{\substack{p'+q'=n \\ p'\leq p,\ q'\leq p}} A_{p',q'}, & \text{for } n \leq 2p. \end{cases}$$

The differential of this complex will also be denoted by $d_{\mathfrak{D}}$ and we have,

$$d_{\mathfrak{D}} x = \begin{cases} dx, & \text{for } n \leq 2p, \\ -\pi(dx), & \text{for } n > 2p+1 \text{ and} \\ -2\partial\overline{\partial} x, & \text{for } n = 2p+1, \end{cases}$$

where π is the projection $A_*^{\mathbb{C}} \longrightarrow \operatorname{Coker} u$.

As in the case of Dolbeault cochain complexes, the complexes $A_*^{\mathbb{R}}(p)_{\mathcal{D}}$ and $\mathfrak{D}_*(A,p)$ are homotopically equivalent.

Let A^* be a Dolbeault cochain algebra an let B_* a Dolbeault chain complex which is a left A^*-module. Then, the formules of chapter III, §2 define a product

$$\mathfrak{D}^n(A,p) \otimes \mathfrak{D}_m(B,q) \longrightarrow \mathfrak{D}_{m-n}(B, q-p).$$

Which induces in $H^*(\mathfrak{D}_*(B,\cdot))$ a structure of left $H^*(\mathfrak{D}^*(A,\cdot))$-module.

Let us now see how to construct real Deligne homology in terms of currents in the case of a smooth and proper complex algebraic variety.

Theorem 1.3. *Let X be a proper smooth variety over C. Then there is a natural isomorphism*

$$H_n^{\mathcal{D}}(X, \mathbb{R}(p)) \longrightarrow H_n(\mathfrak{D}_*(D_*^X, p)).$$

Proof. Note that the groups $H_n^{\mathcal{D}}(X, \mathbb{R}(p))$ are denoted in [J] by ${}'H_{\mathcal{D}}^{-n}(X, \mathbb{R}(-p))$. Therefore the proposition follows from [J, 1.3] and the fact that, if $C_*(X, \mathbb{R})$ is the complex of smooth singular chains on X then the natural map

$$C_*(X, \mathbb{R}) \longrightarrow D_*^{X,\mathbb{R}}$$

is a quasi-isomorphism.

Let us now study the case of an open smooth variety and of a divisor with normal crossings. Let X be a proper smooth variety over $\mathbb{C}$, Y a divisor with normal crossings and $V = X - Y$. We shall always assume that a divisor with normal crossings is the union of its smooth irreducible components.

Let us denote by $\Sigma_Y E_X^*$ the subcomplex of E_X^* composed by the forms which vanish when restricted to each irreducible component of Y. Then the complex of currents on Y ([H-L]) is defined by:

$$D_n^Y = \{T \in D_n^X \mid T(\omega) = 0,\ \forall \omega \in \Sigma_Y E_X^n\}.$$

This complex only depends on Y and, when Y is smooth, coincides with the usual complex of currents.

Let us write $D_*^{X/Y} = D_*^X / D_*^Y$

Both complexes, D_*^Y and $D_*^{X/Y}$, have a structure of Dolbeault chain complexes induced by that of D_*^X.

Theorem 1.4. *Let X be an irreducible proper smooth variety over $\mathbb{C}$ of dimension d, Y a divisor with normal crossings on Y and $V = X - Y$. Then there are natural isomorphisms*

$$H_*^{\mathcal{D}}(Y, \mathbb{R}(p)) \longrightarrow H_*(\mathfrak{D}_*(D_*^Y, p)) \tag{1}$$

and

$$H_*^{\mathcal{D}}(V, \mathbb{R}(p)) \longrightarrow H_*(\mathfrak{D}_*(D_*^{X/Y}, p)). \tag{2}$$

Proof. Let $Y = Y_1 \cup \cdots \cup Y_r$ be the decomposition of Y in smooth irreducible components. For each $I = (i_1, \ldots, i_q)$, an ordered q-tuple with $1 \leq i_1 < \cdots < i_q \leq r$ we write $|I| = q$, $I_{\widehat{j}} = (i_1, \ldots, \widehat{i_j}, \ldots, i_q)$, where $\widehat{i_j}$ means the absence of this element, and

$$Y_I = \bigcap_{i \in I} Y_i.$$

We shall denote by

$$\delta_j^I : Y_I \longrightarrow Y_{I_{\widehat{j}}} \quad \text{and} \quad b_I : Y_I \longrightarrow Y,$$

the natural inclusions. Finally let

$$Y_{(q)} = \coprod_{|I|=q} Y_I, \quad \delta_j^q = \coprod_{|I|=q} \delta_j^I : Y_{(q)} \longrightarrow Y_{q-1} \quad \text{and} \quad b_{(q)} = \coprod_{|I|=q} b_I : Y_{(q)} \longrightarrow Y.$$

We have that $Y_{(\cdot)}$ is a strict simplicial scheme and that the natural morphism $b : Y_{(\cdot)} \longrightarrow Y$ has cohomological descent. Therefore the real Deligne homology of X can be constructed as follows. Let $K_*^X \langle Y \rangle$ be the simple of the complex of complexes

$$D_*^{Y_{(1)}} \xleftarrow{\delta^2} D_*^{Y_{(2)}} \xleftarrow{\delta^3} \dots \xleftarrow{\delta^r} D_*^{Y_{(r)}},$$

where $\delta^q = \sum (-1)^j \delta_j^q$. Then

$$K_n^X \langle Y \rangle = D_n^{Y_{(1)}} \oplus D_{n-1}^{Y_{(2)}} \oplus \cdots \oplus D_{n-r}^{Y_{(r)}}.$$

Since all the morphisms δ^q are real and compatible with the Hodge filtration, then $K_*^X \langle Y \rangle$ has a real structure, $K_*^{X,\mathbb{R}} \langle Y \rangle$, and a Hodge filtration, F. This filtration is given by

$$F_p K_n^X \langle Y \rangle = F_p D_n^{Y_{(1)}} \oplus F_p D_{n-1}^{Y_{(2)}} \oplus \cdots \oplus F_p D_{n-r}^{Y_{(r)}}.$$

By the definition of Deligne homology of a singular variety (see [J]) we have

$$H_n^{\mathcal{D}}(Y, \mathbb{R}(p)) = H_n(s(K_*^{X,\mathbb{R}} \langle Y \rangle (p) \oplus F_p K_*^X \langle Y \rangle \longrightarrow K_*^X \langle Y \rangle)).$$

Hence (1) is consequence of the following results.

Lemma 1.5. *([Fuj]) The sequence*

$$0 \longleftarrow D_*^Y \xleftarrow{b_{(1)}} D_*^{Y_{(1)}} \xleftarrow{\delta^2} D_*^{Y_{(2)}} \xleftarrow{\delta^3} \dots \xleftarrow{\delta^r} D_*^{Y_r} \longleftarrow 0$$

is exact.

Corollary 1.6. *The natural morphism*

$$(K_*^X \langle Y \rangle, F) \longrightarrow (D_*^Y, F)$$

is a filtered quasi-isomorphism. Moreover it induces a quasi-isomorphism between the corresponding real subsheaves.

Proof. From Lemma 1.5 and the fact that the morphisms δ^i and $b_{(1)}$ are bihomogeneous we have that, for each p, q, the sequence

$$0 \longleftarrow D_{p,q}^Y \xleftarrow{b_{(1)}} D_{p,q}^{Y_{(1)}} \xleftarrow{\delta^2} D_{p,q}^{Y_{(2)}} \xleftarrow{\delta^3} \dots \xleftarrow{\delta^r} D_{p,q}^{Y_r} \longleftarrow 0$$

is exact. This implies that the morphism $K_*^X \langle Y \rangle \longrightarrow D_*^Y$ is a filtered quasi-isomorphism.

The fact that it induces a real quasi-isomorphism is proved in the same way using that the morphisms δ^i and $b_{(1)}$ are real.

Now (2) is consequence of (1) and of the definition of $D_*^{X/Y}$.

Remark 1.7. Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension d. Let $Y \subset X$ be a divisor with normal crossings. Let us write $D_{X/Y}^n = D_{2d-n}^{X/Y}$. A direct consequence of Corollary 1.6 is that the morphism of complexes

$$E_X^*(\log Y) \longrightarrow D_{X/Y}^*$$

introduced in chapter II, §3 is a filtered quasi-isomorphism with respect to the Hodge filtration.

§2. K-Chains and Real Deligne Cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension d. Let us denote by $X^{(p)}$ the set of irreducible subvarieties of codimension p and let $Z^p = Z^p(X)$ be the group of algebraic cycles of codimension p.

Let

$$R_p^i = R_p^i(X) = \bigoplus_{x \in X^{(i)}} K_{p-i}(k(x))$$

be the groups of the E_1 term of the Brown-Gersten-Quillen spectral sequence (see [Q 1], [Gi] [Gr 1] and [Gr 2]). Then $R_p^p(X) = Z^p(X)$. The elements of R_p^{p-1} will be called K_1-chains and the elements of R_p^{p-2}, K_2-chains. Let us denote by $d : R_p^i \longrightarrow R_p^{i+1}$ the differential of this spectral sequence.

Recall that $K_1(k(x)) = k(x)^*$ is the group of units of $k(x)$. If f is a K_1-chain then

$$f = \sum_{x \in X^{(i)}} f_x,$$

with $f_x \in k(x)^*$. And $df = \operatorname{div} f = \sum \operatorname{div} f_x$. Therefore

$$R_p^p(X) \big/ dR_p^{p-1}(X) = CH^p(X),$$

is the codimension p Chow group of X.

If $f \in R_p^i$ we will denote its support by

$$\operatorname{supp}(f) = \bigcup_{\substack{x \in X^{(i)} \\ f_x \neq 1}} \overline{\{x\}}.$$

Note that for $i = p$ we write

$$\operatorname{supp}(f) = \bigcup_{\substack{x \in X^{(i)} \\ f_x \neq 0}} \overline{\{x\}}.$$

because $K_0(k(x)) \cong \mathbb{Z}$ with additive notation.

The first aim of this section is to prove the following theorem.

Theorem 2.1. *Let X be a smooth algebraic variety over $\mathbb{C}$ and let $f \in R_p^i(X)$ with $i = p, p-1, p-2$. Then there is defined a class*

$$\rho(f) \in H^{p+i}_{\mathcal{D}, \operatorname{supp} f - \operatorname{supp} df}(X - \operatorname{supp} df, \mathbb{R}(p)),$$

such that:

1) If $i = p$, f is a codimension p algebraic cycle, then

$$\rho(f) = \mathrm{cl}(f) \in H^{2p}_{\mathcal{D},\mathrm{supp}\, f}(X, \mathbb{R}(p))$$

is the cycle class (see for instance [J]*).*

2) We have $\rho(df) = \partial\rho(f)$, where

$$\partial : H^{p+i}_{\mathcal{D},\mathrm{supp}\, f - \mathrm{supp}\, df}(X - \mathrm{supp}\, df, \mathbb{R}(p)) \longrightarrow H^{p+i+1}_{\mathcal{D},\mathrm{supp}\, df}(X, \mathbb{R}(p))$$

is the connection homomorphism.

3) If $h : X \longrightarrow X'$ is a proper morphism we have $h_*(\rho f) = \rho(h_* f)$

Proof. The case of a cycle y is well known. Let us recall how we can characterize $\rho(y)$ in terms of currents. If Y is a dimension n subvariety of X, $\widetilde{Y}$ is a resolution of singularities of Y, and $\pi : \widetilde{Y} \longrightarrow X$ is the induced map, then the current δ_Y is defined by

$$\delta_Y(\omega) = \frac{1}{(2\pi i)^n} \int_{\widetilde{Y}} \pi^*(\omega).$$

Note that this definition differs from the definition used in chapter II by the inclusion of the normalization factor. If y is an algebraic cycle we define δ_y by linearity.

Let X be smooth and proper of dimension d and let $[X] \in Z^0(X)$ be the fundamental cycle. Then we have a class

$$\{X\} \in H^{\mathcal{D}}_{2d}(X, \mathbb{R}(d))$$

which is represented by the current

$$\delta_X \in D^{X,\mathbb{R}}_{d,d}(d) = \mathfrak{D}_{2d}(D^X_*, d).$$

Since X is smooth, the morphism

$$H^0_{\mathcal{D}}(X, \mathbb{R}(0)) \longrightarrow H^{\mathcal{D}}_{2d}(X, \mathbb{R}(d))$$

is an isomorphism and $\rho([X])$ is the preimage of $\{X\}$ by this isomorphism. Alternatively we can represent $\rho([X])$ directly by the function $1 \in \mathfrak{D}^0(E^*_X, 0)$.

Let $Y \subset X$ be a codimension p irreducible subvariety, let $[Y]$ be its fundamental cycle and let $\pi : \widetilde{Y} \longrightarrow Y$ be a resolution of singularities. Then we have a class $\{\widetilde{Y}\} \in H^{\mathcal{D}}_{2d-2p}(\widetilde{Y}, \mathbb{R}(d-p))$ and we obtain $\rho([Y])$ by the composition of morphisms

$$H^{\mathcal{D}}_{2d-2p}(\widetilde{Y}, \mathbb{R}(d-p)) \xrightarrow{\pi_*} H^{\mathcal{D}}_{2d-2p}(Y, \mathbb{R}(d-p)) \xrightarrow{\cong} H^{2p}_{\mathcal{D},Y}(X, \mathbb{R}(p)).$$

Or, in other words, $\rho([Y])$ is the image of $\rho([\widetilde{Y}])$ by the Gysin morphism

$$H^0_{\mathcal{D}}(\widetilde{Y}, \mathbb{R}(0)) \longrightarrow H^{2p}_{\mathcal{D},Y}(X, \mathbb{R}(p)).$$

If y is a codimension p algebraic cycle and $Y = \mathrm{supp}\, y$ then $\rho(y)$ is defined by linearity. Note that, if Y is a divisor with normal crossings, as a consequence of Theorem 1.4, we can represent $\{y\} \in H^{\mathcal{D}}_{2d-2}(Y, \mathbb{R}(d-1))$ by the current $\delta_y \in \mathfrak{D}_{2d-2}(D^Y_*, d-1)$.

Finally, if X is not proper and y is a codimension p algebraic cycle, let $\overline{X}$ be a smooth compactification of X and $\overline{y} \in Z^p(\overline{X})$ any cycle whose restriction to X is y. Then $\rho(y)$ is defined as the image of $\rho(\overline{y})$ by the restriction morphism

$$H^{2p}_{\mathcal{D},\operatorname{supp}\overline{y}}(\overline{X}, \mathbb{R}(p)) \to H^{2p}_{\mathcal{D},\operatorname{supp} y}(X, \mathbb{R}(p)).$$

This class is independent of the choice of the compactification $\overline{X}$.

Let us now study the case of K_1-chains. Let X be a proper smooth variety of dimension d and let $f \in R^0_1(X) = K_1(k(X)) = k(X)^*$, such that $Y = \operatorname{supp}(\operatorname{div} f)$ is a divisor with normal crossings. Let us write

$$\delta_f = \frac{-1}{2}[\log f\overline{f}] \in D^{X,\mathbb{R}}_{d,d}(d) = \mathfrak{D}_{2d-1}(D^X_*, d-1),$$

where this current is defined by (see §1)

$$\frac{-1}{2}[\log f\overline{f}](\omega) = \frac{1}{(2\pi i)^d}\int_X \frac{-1}{2}\log f\overline{f}\,\omega.$$

By the Poincaré-Lelong equation

$$\begin{aligned} d_{\mathfrak{D}}\delta_f &= -2\partial\overline{\partial}\frac{-1}{2}[\log f\overline{f}] \\ &= -\delta_{\operatorname{div} f} \end{aligned}$$

But this current belongs to $\mathfrak{D}_{2d-2}(D^Y_*, d-1)$. Therefore δ_f is a cycle in the group $\mathfrak{D}_{2d-1}(D^{X/Y}_*, d-1)$ and we obtain a class

$$\{\delta_f\} \in H^{\mathcal{D}}_{2d-1}(X-Y, \mathbb{R}(d-1)).$$

We define $\rho(f)$ as the preimage of $\{\delta_f\}$ by the isomorphism

$$H^1_{\mathcal{D}}(X-Y, \mathbb{R}(1)) \longrightarrow H^{\mathcal{D}}_{2d-1}(X-Y, \mathbb{R}(d-1)).$$

Alternatively we can represent $\rho(f)$ by the function

$$\frac{-1}{2}\log f\overline{f} \in E^{0,0}_{\log,\mathbb{R}}(X-Y, 0) = \mathfrak{D}^1(E^*_{\log}(X-Y), 1).$$

In order to prove 2), let us recall that we have a commutative diagram

$$\begin{array}{ccc} H^{\mathcal{D}}_{2d-1}(X-Y, \mathbb{R}(d-1)) & \xrightarrow{\partial} & H^{\mathcal{D}}_{2d-2}(Y, \mathbb{R}(d-1)) \\ \downarrow & & \downarrow \\ H^1_{\mathcal{D}}(X-Y, \mathbb{R}(1)) & \xrightarrow{\partial} & H^2_{\mathcal{D},Y}(X, \mathbb{R}(1)). \end{array}$$

Moreover $\partial\{\delta_f\}$ is represented by $\operatorname{Res}\delta_f = -d_{\mathfrak{D}}\delta_f = \delta_{\operatorname{div} f}$. Hence the result.

Let $W \subset X$ be an integral codimension $p-1$ subvariety and let $f \in K_1(k(W))$. Write $Y = \operatorname{supp}(\operatorname{div} f)$. Let $\pi : \widetilde{W} \longrightarrow W$ be a resolution of singularities such that $\widetilde{Y} = \pi^{-1}Y$ is a divisor with normal crossings. Then we define $\rho(f)$ as the image of $\rho(\pi^* f)$ by the Gysin morphism

$$H^1_{\mathcal{D}}(\widetilde{W} - \widetilde{Y}, \mathbb{R}(1)) \to H^{2p-1}_{\mathcal{D},W-Y}(X - Y, \mathbb{R}(p)).$$

The fact that $\partial\rho(f) = \rho(\operatorname{div} f)$ follows from the covariance of Deligne homology.

Let now $f \in R^{p-1}_p$ be an arbitrary K_1-chain, $f = \sum f_i$. Let us write

$$W = \bigcup_i \operatorname{supp} f_i, \quad Y = \operatorname{supp}(\operatorname{div} f) \quad \text{and} \quad Z = \bigcup_i \operatorname{supp}(\operatorname{div} f_i).$$

We have $Y \subset Z$. By linearity we obtain a class

$$\rho(f) \in H^{2p-1}_{\mathcal{D},W-Z}(X - Z, \mathbb{R}(p)).$$

But we have an exact sequence

$$\begin{aligned} &H^{2p-1}_{\mathcal{D},Z-Y}(X - Y, \mathbb{R}(p)) \to H^{2p-1}_{\mathcal{D},W-Y}(X - Y, \mathbb{R}(p)) \to H^{2p-1}_{\mathcal{D},W-Z}(X - Z, \mathbb{R}(p)) \to \\ &H^{2p}_{\mathcal{D},Z-Y}(X - Y, \mathbb{R}(p)) \to . \end{aligned}$$

And, since $Z - Y$ has codimension p, then $H^{2p-1}_{\mathcal{D},Z-Y}(X - Y, \mathbb{R}(p)) = 0$. Moreover, since ∂ is linear, $\partial\rho(f) = \rho(\operatorname{div} f)$ and this class has support on Y. Therefore we can lift $\rho(f)$ to a unique class also denoted $\rho(f) \in H^{2p-1}_{\mathcal{D},W-Y}(X - Y, \mathbb{R}(p))$.

As before, the case when X is not proper is done by restriction.

Let us prove the covariance of ρ at the level of K_1-chains. By the covariance of Deligne homology and K_1-chains, it is enough to check the case when $\pi : X \longrightarrow X'$ is a proper surjective morphism and $f \in k(X)^*$.

If $\dim X > \dim X'$ then $\pi_* f = 0$ and $\pi_* \delta_f = 0$.

If $\dim X = \dim X'$ then $\pi_* f = N(f)$, where N is the norm of the field extension $k(X') \longrightarrow k(X)$. In terms of functions

$$\pi_* f(x) = \prod_{\pi(y)=x} f(y)^{r(y)},$$

where $r(y)$ is the ramification index. On the other hand, if φ is a L^1 function on X then $\pi_*[\varphi] = [\pi_*\varphi]$, where

$$\pi_*\varphi(x) = \sum_{\pi(y)=x} r(y)\varphi(y).$$

Therefore $\delta_{\pi_* f} = \pi_*\delta_f$ and $\rho(\pi_* f) = \pi_*(\rho f)$.

The proof for K_2-chains will follow the same pattern as the proof for K_1-chains. Let X be a proper smooth variety over $\mathbb{C}$. Recall that the group $K_2(k(X))$ can be described as

$$K_2(k(X)) = k(X)^* \underset{\mathbb{Z}}{\otimes} k(X)^* \Big/ R,$$

where R is the subgroup generated by the elements of the form $f \otimes (1-f)$. The element of $K_2(k(X))$ represented by $f \otimes g$ will be denoted by $\{f, g\}$.

The differential of the Brown-Gersten-Quillen spectral sequence is given by the tame symbol. Let Y be a divisor of X, ν_Y the corresponding valuation. Then the Y-th component of $d\{f, g\}$ is given by

$$(-1)^{\nu_Y(f)\nu_Y(g)} \overline{\left\{ \frac{f^{\nu_Y(g)}}{g^{\nu_Y(f)}} \right\}},$$

where $\overline{\{\cdot\}}$ denotes the class in $k(Y)^*$.

Assume now that $f \otimes g \in k(X)^* \otimes k(X)^*$, such that $Z = \operatorname{div} f \cup \operatorname{div} g$ and $Y = \operatorname{supp} d\{f, g\}$ are divisors with normal crossings. We have $Y \subset Z$. Then we write

$$\begin{aligned} \lambda(f \otimes g) &= \frac{-1}{2} \log f\overline{f} \cdot \frac{-1}{2} \log g\overline{g} \\ &= \frac{1}{4}\left(-\left(\frac{df}{f} - \frac{d\overline{f}}{\overline{f}} \right) \log g\overline{g} + \left(\frac{dg}{g} - \frac{d\overline{g}}{\overline{g}} \right) \log f\overline{f} \right) \\ &\in \mathfrak{D}^2(E^*_{\log}(X - Z), 2), \end{aligned}$$

where $a \cdot b$ denotes the product in the Deligne cochain complex $\mathfrak{D}^*(E^*_{\log}(X-Z), \cdot)$.

We denote by $\delta_{f\otimes g}$ the current $[\lambda(f \otimes g)]$. That is

$$\delta_{f\otimes g}(\omega) = \frac{1}{(2\pi i)^d} \int_X \omega \wedge \lambda(f \otimes g).$$

Lemma 2.2.
1) The form $\lambda(f \otimes g)$ is closed in $\mathfrak{D}^2(E^*_{\log}(X - Z), 2)$.
2) If $g = 1 - f$, then the form $\lambda(f \otimes (1-f))$ is exact in $\mathfrak{D}^2(E^*_{\log}(X - Z), 2)$.
3) The current $\delta_{f\otimes g} \in \mathfrak{D}_{2d-2}(D^X_*, d-2)$ satisfies

$$d_{\mathfrak{D}} \delta_{f\otimes g} = -\delta_{d\{f,g\}}.$$

Proof. Let us prove 1). By the Leibnitz rule for the Deligne complex we have

$$d_{\mathfrak{D}} \lambda(f \otimes g) = d_{\mathfrak{D}} \frac{-1}{2} \log f\overline{f} \cdot \frac{-1}{2} \log g\overline{g} - \frac{-1}{2} \log f\overline{f} \cdot d_{\mathfrak{D}} \frac{-1}{2} \log g\overline{g} = 0.$$

In order to prove 2), we can consider f as a map $f : X \longrightarrow \mathbb{P}^1$. Then we have

$$\lambda(f \otimes (1-f)) = f^*\eta.$$

where η is the form

$$\begin{aligned} \eta &= \frac{1}{4}\left(\left(\frac{d\overline{x}}{\overline{x}} - \frac{dx}{x} \right) \log(1-x)(1-\overline{x}) + \left(\frac{d\overline{x}}{1-\overline{x}} - \frac{dx}{1-x} \right) \log x\overline{x} \right) \\ &\in \mathfrak{D}^2(E^*_{\log}(\mathbb{P}^1 - \{0, 1, \infty\}), 2). \end{aligned}$$

We can consider $\eta \in E^1_{\log,\mathbb{R}}(\mathbb{P}^1 - \{0,1,\infty\},1)$. Moreover a direct check shows that $d\eta = 0$. Let us prove that η is exact in the complex $E^*_{\log,\mathbb{R}}(\mathbb{P}^1 - \{0,1,\infty\},1)$. It is enough to show that the periods of η around 0, 1 and ∞ are zero. Since the three cases are analogous we only discuss the period of η around 0.

Let us write $x = re^{i\theta}$. Since η is closed

$$\int_{\|x\|=r} \eta = \lim_{r\to 0} \int_{\|x\|=r} \eta.$$

But it is easy to show that

$$\lim_{r\to 0} \| \int_{\|x\|=r} \eta \| \leq \lim_{r\to 0} C \int_{\|x\|=r} r \log r^2 d\theta = 0.$$

Therefore η is d-exact. Now, since

$$\mathfrak{D}^1(E^*_{\log}(\mathbb{P}^1 - \{0,1,\infty\}),2) = E^0_{\log,\mathbb{R}}(\mathbb{P}^1 - \{0,1,\infty\},1),$$

and for this degree $d_{\mathfrak{D}} = -d$ we have that η is $d_{\mathfrak{D}}$-exact.

We can obtain a primitive of the form η by means of the Bloch-Wigner dilogarithm. Let us explain the construction of this function (see [Bl 1] and [Z]). Let Li_2 be the the holomorphic function given, for $\|x\| < 1$, by the power series

$$\mathrm{Li}_2 = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

The function Li_2 can be extended analytically to $\mathbb{C} - (1,\infty)$ and we obtain

$$\mathrm{Li}_2 = -\int_0^x \log(1-u)\frac{du}{u}.$$

The Bloch-Wigner dilogarithm is the real function

$$D(x) = \Im(\mathrm{Li}_2(x)) + \arg(1-x)\log\|x\|,$$

where $\Im$ is the imaginary part and arg is the branch of the argument lying between $-\pi$ and π. This function is real analytic on $\mathbb{C}$ except at points 0 and 1, where it has singularities of the type $r \log r$. Then, we can check that,

$$idD(x) = \eta.$$

Observe that $\eta \in A^1_{\mathbb{P}^1_{\mathbb{C}}}(\log\{0,1,\infty\})$, the complex of real analytic forms on $\mathbb{P}^1_{\mathbb{C}}$ with logarithmic singularities on 0, 1 and ∞. Therefore, any primitive of η, which is determined up to a constant, lies in $A^0_{\mathbb{P}^1_{\mathbb{C}}}(\log\{0,1,\infty\})$. Thus $D(z)$ is real analytic on $\mathbb{C} - \{0,1\}$ and has logarithmic singularities.

Let us now prove 3). Let us write $Z = \operatorname{div} f \cup \operatorname{div} g = Z_1 \cup \cdots \cup Z_r$, with Z_i smooth irreducible divisors. Let ν_i be the valuation associated to Z_i and let $a_i : Z_i \longrightarrow X$ be the inclusion. We have that

$$d_{\mathfrak{D}}\delta_{f\otimes g} = \text{component of type } (d-1,d-1) \text{ of } -d\delta_{f\otimes g}.$$

Then

$$d_{\mathfrak{D}}\delta_{f\otimes g}(\omega) = \begin{cases} \frac{1}{(2\pi i)^d}\int_X d\omega\wedge\lambda(f\otimes g), & \text{if } \omega \text{ is of type } (d-1,d-1). \\ 0, & \text{if } \omega \text{ is of type } (d,d-2) \text{ or } (d-2,d), \end{cases}$$

Let $N(\varepsilon)$ be a tubular neighbourhood of Z of radius ε and let $V(\varepsilon)$ be the boundary of $N(\varepsilon)$. Then, by Stokes' theorem, if ω is a $(d-1,d-1)$ test form

$$\begin{aligned} \frac{1}{(2\pi i)^d}\int_X d\omega\wedge\lambda(f\otimes g) &= \frac{1}{(2\pi i)^d}\int_X d(\omega\wedge\lambda(f\otimes g)) \\ &= \lim_{\varepsilon\longrightarrow 0}\frac{1}{(2\pi i)^d}\int_{X-N(\varepsilon)} d(\omega\wedge\lambda(f\otimes g)) \\ &= \lim_{\varepsilon\longrightarrow 0}\frac{-1}{(2\pi i)^d}\int_{V(\varepsilon)} \omega\wedge\lambda(f\otimes g). \end{aligned}$$

On the other hand, if ω is of type $(d-2,d)$ or $(d,d-2)$, then $\delta_{d\{f,g\}}(\omega)=0$. And if ω is a test form of type $(d-1,d-1)$, then

$$\delta_{d\{f,g\}}(\omega) = \sum_i \frac{1}{(2\pi i)^{d-1}}\int_{Z_i} a_i^*\frac{-1}{2}\log\left(\frac{(f\overline{f})^{\nu_i(g)}}{(g\overline{g})^{\nu_i(f)}}\right)\omega.$$

Now the equality can be checked locally. So we can assume that ω has compact support on a neighbourhood of $0\in\mathbb{C}^d$. Since both terms are additive on f and g, we are reduced to the cases

1) $f=z_1$, $g=z_1$,
2) $f=z_1$, $g=z_2$,
3) $f=z_1$, g an inversible function in a neighbourhood of 0.

In the first case both terms of the equality are zero. The second and third cases are analogous so we shall write only the second case.

$$\begin{aligned} &\lim_{\varepsilon\longrightarrow 0}\frac{-1}{(2\pi i)^d}\int_{V(\varepsilon)}\omega\wedge\lambda(f\otimes g) \\ &\quad= \lim_{\varepsilon\longrightarrow 0}\frac{-1}{(2\pi i)^d}\int_{V(\varepsilon)}\omega\wedge\frac{1}{4}\left(\left(\frac{d\overline{z}_1}{\overline{z}_1}-\frac{dz_1}{z_1}\right)\log z_2\overline{z}_2 - \left(\frac{d\overline{z}_2}{\overline{z}_2}-\frac{dz_2}{z_2}\right)\log z_1\overline{z}_1\right) \\ &\quad= \frac{1}{(2\pi i)^{d-1}}\int_{z_1=0}\frac{-1}{2}\log z_2\overline{z}_2\,\omega + \frac{-1}{(2\pi i)^{d-1}}\int_{z_2=0}\frac{-1}{2}\log z_1\overline{z}_1\,\omega \\ &\quad= -\delta_{d\{z_1,z_2\}}(\omega). \end{aligned}$$

This concludes the proof of the lemma.

Let us return to the proof of Theorem 2.1. By part 1) of Lemma 2.2 the form $\lambda(f\otimes g)$ defines a class

$$\rho(f\otimes g)\in H^2_{\mathcal{D}}(X-Z,\mathbb{R}(2)).$$

By part 2) of the same lemma $\rho(f\otimes g)$ only depends on the class $\{f,g\}$.

Now we want to lift this class to a class in $H^2_{\mathcal{D}}(X-Y,\mathbb{R}(2))$. To this end, let us consider the exact sequence

$$H^2_{\mathcal{D},Z-Y}(X-Y,\mathbb{R}(2)) \longrightarrow H^2_{\mathcal{D}}(X-Y,\mathbb{R}(2)) \longrightarrow H^2_{\mathcal{D}}(X-Z,\mathbb{R}(2)) \xrightarrow{\partial}$$
$$H^3_{\mathcal{D},Z-Y}(X-Y,\mathbb{R}(2)).$$

The first group of this exact sequence can be included in another exact sequence:

$$H^1_{Z-Y}(X-Y,\mathbb{C})/F^2 \rightarrow H^2_{\mathcal{D},Z-Y}(X-Y,\mathbb{R}(2)) \rightarrow H^2_{Z-Y}(X-Y,\mathbb{R}(2)) \rightarrow$$
$$H^2_{Z-Y}(X-Y,\mathbb{C})/F^2.$$

Since Z has codimension one, the first group of the last sequence is zero. Moreover the last map of the same sequence is injective. Therefore $H^2_{\mathcal{D},Z-Y}(X-Y,\mathbb{R}(2))=0$.

By part 3) of the lemma $\partial\rho(f\otimes g)$ has support on Y. Hence we can lift $\rho(f\otimes g)$ to a unique class in $H^2_{\mathcal{D}}(X-Y,\mathbb{R}(2))$.

The remainder of the proof in the case of K_2-chains follows as in the case of K_1-chains.

Let us write

$$CH^{p,p-1}(X)=\frac{\operatorname{Ker} d: R_p^{p-1}(X)\longrightarrow R_p^p(X)}{\operatorname{Im} d: R_p^{p-2}(X)\longrightarrow R_p^{p-1}(X)}.$$

As a consequence of Theorem 2.1 we have

Corollary 2.3. *There are well defined maps*

$$\rho: CH^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)), \quad \textit{and}$$
$$\rho: CH^{p,p-1}(X) \longrightarrow H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

The first is the class cycle map and the second is, up to a normalization factor, the Beilinson regulator map (see [G-S 2, 3.5.]*). Moreover these maps are covariant for proper morphisms.*

We can write Theorem 2.1 in terms of a partial compatibility between the Brown-Gersten-Quillen spectral sequence and the Bloch-Ogus spectral sequence for Deligne cohomology.

Let $\mathcal{Z}^p=\mathcal{Z}^p(X)$ denote the set of all closed algebraic subsets of X of codimension $\geq p$ ordered by inclusion. Let $\mathcal{Z}^p\backslash\mathcal{Z}^{p+1}$ denote the set of all pairs $(Z,Z')\in\mathcal{Z}^p\times\mathcal{Z}^{p+1}$ such that $Z'\subset Z$. We consider this set ordered by inclusion.

Following [B-O], let us write

$$H^n_{\mathcal{D},\mathcal{Z}^p\backslash\mathcal{Z}^{p+1}}(X,\mathbb{R}(q)) = \varinjlim_{(Z,Z')\in\mathcal{Z}^p\backslash\mathcal{Z}^{p-1}} H^n_{\mathcal{D},Z-Z'}(X-Z',\mathbb{R}(q)).$$

Since $\mathcal{Z}^p\backslash\mathcal{Z}^{p-1}$ is a directed set, we can obtain these groups as the cohomology groups of the complex

$$\varinjlim_{(Z,Z')\in\mathcal{Z}^p\backslash\mathcal{Z}^{p-1}} s(\mathfrak{D}^*(E_{\log}(X-Z'),q)\longrightarrow\mathfrak{D}^*(E_{\log}(X-Z),q)).$$

We shall also write

$$H^n_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(q)) = \varinjlim_{Z\in\mathcal{Z}^p} H^n_{\mathcal{D},Z}(X,\mathbb{R}(q)) \qquad \text{and}$$
$$H^n_{\mathcal{D}}(X-\mathcal{Z}^p,\mathbb{R}(q)) = \varinjlim_{Z\in\mathcal{Z}^p} H^n_{\mathcal{D}}(X-Z,\mathbb{R}(q)).$$

Then Theorem 2.1 implies:

Theorem 2.4. *There is a commutative diagram*

$$\begin{array}{ccccc}
R_p^{p-2} & \xrightarrow{d} & R_p^{p-1} & \xrightarrow{d} & R_p^p \\
\rho\downarrow & & \rho\downarrow & & \rho\downarrow \\
H^{2p-2}_{\mathcal{D},\mathcal{Z}^{p-2}\setminus\mathcal{Z}^{p-1}}(X,\mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p-1}_{\mathcal{D},\mathcal{Z}^{p-1}\setminus\mathcal{Z}^{p}}(X,\mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p}_{\mathcal{D},\mathcal{Z}^{p}\setminus\mathcal{Z}^{p+1}}(X,\mathbb{R}(p)),
\end{array}$$

where the last map is the cycle class map. This diagram is covariant for proper maps.

Note that Theorem 2.1. is more precise than Theorem 2.4., in the sense that it specifies where each cohomology class is defined.

The last aim of this section is to study the compatibility of ρ with inverse images and intersection products. The case of cycles is well known and we have:

Theorem 2.5.

1) *Let $f : X' \longrightarrow X$ be a morphism of smooth algebraic varieties over $\mathbb{C}$ and let $Z \in Z^p(X)$ be an irreducible algebraic cycle such that $f^{-1}(Z)$ has codimension p. Then there is defined a cycle f^*Z. Moreover we have $\rho(f^*Z) = f^*\rho(Z)$.*
2) *Let Y and Z be two algebraic cycles of X which intersect properly. Then there is defined an intersection cycle $Y \cdot Z$ and we have $\rho(Y \cdot Z) = \rho(Y) \cup \rho(Z)$.*
3) *The morphism*

$$\rho : \bigoplus_p CH^p(X) \longrightarrow \bigoplus_p H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p))$$

is a natural transformation between covariant functors from the category of smooth complex varieties to the category of rings.

Proof. Using chapter III, Proposition 1.1 one can see that, for X smooth, the map

$$\varphi : H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)) \longrightarrow H^{2p}(X,\mathbb{C})$$

is injective, and the map $\varphi \circ \rho$ is, up to a normalization factor, the cycle class map. Then this theorem follows from the compatibility between Chow rings and cohomology (see for example [Ful, §19]).

The compatibility of ρ with inverse images and intersection products at the level of K_1-chains has been stated in [G-S 2, 4.2]. Let us recall their result.

Let us write $\widetilde{R}_p^{p-1} = R_p^{p-1}/\operatorname{Im} d$. Then the commutative diagram of Theorem 2.4 induces a commutative diagram

$$\begin{array}{ccc}
\widetilde{R}_p^{p-1}(X) & \xrightarrow{\mathrm{div}} & Z^p(X) \\
\rho\downarrow & & \rho\downarrow \\
H^{2p-1}_{\mathcal{D}}(X\setminus\mathcal{Z}^p,\mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p)).
\end{array}$$

Let $f = \sum f_W$ be a K_1-chain and let $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ be a collection of closed algebraic subsets. The K_1-chain h is said to meet $\mathcal{Z}$ properly if

1) Each W such that $f_W \neq 1$ meet Z properly for all $Z \in \mathcal{Z}$
2) $\operatorname{div} f_W$ meet Z properly for all W and all $Z \in \mathcal{Z}$.

If only condition 2) is satisfied we say that f and $\mathcal{Z}$ meet almost properly. Let $h : Y \longrightarrow X$ be a morphism of smooth complex varieties. Then $h(Y)$ is a finite union of locally closed subsets Z_i, $i = 1, \dots, N$ such that the fibres of h have the same dimension at the points of Z_i. The closure of each Z_i will be called a stratum of h.

2.6. Theorem. ([G-S 2, 4.2])

1) *Let $h : X' \longrightarrow X$ be a morphism of smooth complex varieties and let $f \in R_p^{p-1}(X)$. If f meets the set of strata of h properly, then there is defined a K_1-chain $h^*(f) \in R_p^{p-1}(X')$ such that* $\operatorname{div} h^*(f) = h^*(\operatorname{div} f)$ *and* $\rho h^*(f) = h^*\rho(f)$. *If h meets the set of strata of h almost properly, then the pull-back K_1-chain $h^*(f)$ is defined in $\widetilde{R}_p^{p-1}$ with the same properties.*
2) *Let X be a smooth complex variety, $f \in R_p^{p-1}$ a K_1-chain and Z a codimension q algebraic cycle. If f and Z meet properly, then there is defined a product K_1-chain $f \cdot Z \in R_{p+q}^{p+q-1}$ such that* $\operatorname{div}(f\cdot Z) = \operatorname{div}(f)\cdot Z$ *and* $\rho(f \cdot Z) = \rho(f) \cup \rho(Z)$. *If f and Z meet almost properly then the product $f \cdot Z$ is defined in $\widetilde{R}_{p+q}^{p+q-1}$ with the same properties.*
3) *Let $h : X' \longrightarrow X$ be a morphism of smooth quasi-projective complex varieties. Let $\widetilde{R}_p^{p-1}(X)_h$ be the subgroup of $\widetilde{R}_p^{p-1}$ generated by the K_1-chains f such that $h^{-1}(\operatorname{div} f)$ has codimension at least p. Then there is a well defined morphism*

$$h^* : \widetilde{R}_p^{p-1}(X)_h \longrightarrow \widetilde{R}_p^{p-1}(X'),$$

compatible with ρ and div.

4) *Let X be a smooth quasi-projective complex variety and Z a codimension q algebraic cycle. Let $\widetilde{R}_p^{p-1}(X)_Z$ be the subgroup of $\widetilde{R}_p^{p-1}$ generated by the K_1-chains f such that* $\operatorname{div} f$ *meets Z properly. Then there is a well defined morphism*

$$\cdot Z : \widetilde{R}_p^{p-1}(X)_Z \longrightarrow \widetilde{R}_{p+q}^{p+q-1}(X),$$

compatible with ρ and div.

Sketch of proof. Parts 1) and 2) are a reformulation of [G-S 2, Lemma 4.2.5] and part 3) and 4) are the consequence of the former and the Moving Lemma for K_1-chains ([G-S 2, Lemma 4.2.6], see the discussion after this Lemma).

Remark 2.7. Let $X_{\mathbb{R}}$ be a smooth real algebraic variety, equivalently $X_{\mathbb{R}}$ is a pair (X, F_∞), where X is a smooth complex variety and F_∞ is an antilinear involution. Then all the results of this secction remains valid, provided that we substitute K-chains by real defined K-chains and every complex $A(X)$ by the subcomplex

$$A(X_{\mathbb{R}}) = \{x \in A(X) \mid F_\infty^*(x) = \overline{x}\}.$$

See for example [E-V, 2.1].

In particular, If $A(X)$ is a Dolbeault complex, we shall write

$$\mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) = \begin{cases} \left\{x \in \mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) \mid F_\infty^* x = (-1)^{p-1}x\right\}, & \text{if } n \leq 2p-1 \text{ and} \\ \{x \in \mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) \mid F_\infty^* x = (-1)^{p}x\}, & \text{if } n \geq 2p. \end{cases}$$

We shall also write

$$H_{\mathcal{D}}^n(X_{\mathbb{R}}, \mathbb{R}(p)) = H^n(\mathfrak{D}(E_{\log}^*(X_{\mathbb{R}}), p)).$$

§3. Cohomological Arithmetic Chow Groups.

Let (A, Σ, F_∞) be an arithmetic ring (See [G-S 2, §3]). That is, A is an excellent Noetherian domain, Σ is a nonempty set of monomorphisms $\sigma : A \longrightarrow \mathbb{C}$ and F_∞ is a conjugate-linear involution of $\mathbb{C}$-algebras $F_\infty : \mathbb{C}^\Sigma \longrightarrow \mathbb{C}^\Sigma$, such that the image of A in $\mathbb{C}^\Sigma$ is invariant under F_∞. Let us denote by K the quotient field of A. The first examples of such arithmetic rings A are

1) $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, with Σ containing only the inclusion.
2) $A = \mathbb{C}$, with $\Sigma = \{\mathrm{Id}, \sigma\}$, where Id is the identity and σ is the conjugation.
3) $A = \mathcal{O}_K$, the ring of integers of a number field K, with Σ the set of complex immersions of K.

Let X be a regular separated flat A-scheme of finite type, with generic fibre X_K regular over K. X is called an arithmetic variety over A, or arithmetic variety if A is fixed. If $\sigma \in \Sigma$ we write $X_\sigma = X \underset{\sigma}{\otimes} \mathbb{C}$ and $X_\Sigma = \coprod X_\sigma$. Let X_∞ be the complex manifold determined by X_Σ. We denote by F_∞ the anti-linear involution of X_Σ induced by F_∞. Finally we denote by $X_\mathbb{R}$ the real manifold (X_∞, F_∞).

In this section we shall use Green forms to define cohomological arithmetic Chow groups of X. In the case when X_K is proper over K then this arithmetic Chow groups are naturally isomorphic to the arithmetic Chow groups defined in [G-S 2]. If X_K is not proper the groups defined here have better Hodge theoretic properties than the groups defined in [G-S 2]. In fact the existence of the groups introduced here was already predicted in [G-S 2].

To take into account the structure of real variety of X_∞ (see Remark 2.7) we write

$$GE^p(X_\mathbb{R}) = \{\widetilde{g} \in GE^p(X_\infty) \mid F_\infty^* g = \overline{g}\}\,.$$

Note that, since $g \in \mathfrak{D}^{2p-1}(E^*_{\log}(X_\infty/\mathcal{Z}^p), p)$, we have $\overline{g} = (-1)^{p-1} g$.

We also write

$$\begin{aligned}
H^*_\mathcal{D}(X_\mathbb{R}, \mathbb{R}(p)) &= H^*(\mathfrak{D}(E^*_{\log}(X_\mathbb{R}), p)),\\
\widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X_\mathbb{R}) &= \left\{g \in \mathfrak{D}^{2p-1}(E^*_{\log}(X_\infty), p) \mid F^*_\infty g = \overline{g}\right\} \big/ (\operatorname{Im} d_\mathfrak{D})\\
&= \left\{g \in E^{p-1,p-1}_{\log}(X_\infty) \cap E^{2p-2}_{\log,\mathbb{R}}(X_\infty, p) \mid F^*_\infty g = (-1)^{p-1} g\right\} \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial})
\end{aligned}$$

and

$$\begin{aligned}
ZE^{p,p}_{\log,\mathbb{R}}(X_\mathbb{R}) &= \left\{\omega \in \mathfrak{D}^{2p}(E^*_{\log}(X_\infty), p) \mid d_\mathfrak{D}\omega = 0,\ F^*_\infty \omega = \overline{\omega}\right\}\\
&= \left\{\omega \in E^{p,p}_{\log}(X_\infty) \cap E^{2p}_{\log,\mathbb{R}}(X_\infty, p) \mid d\omega = 0,\ F^*_\infty\omega = (-1)^p\omega\right\}.
\end{aligned}$$

Observe that Proposition 4.2 of chapter III remains valid provided we use the corresponding groups for $X_{\mathbb{R}}$.

Let $Z^p(X)$ denote the set of codimension p algebraic cycles on X. For each $y \in Z^p(X)$ there is a well defined cycle $y_K \in Z^p(X_K)$. Hence a cycle $y_\infty \in Z^p(X_\infty)$. We shall write $\rho(y) = \rho(y_\infty) \in H^{2p}_{\mathcal{D},\operatorname{supp} y}(X_{\mathbb{R}}, \mathbb{R}(p))$. Then the space of Green forms for y is defined by:

$$GE^p_y(X_{\mathbb{R}}) = \{\widetilde{g} \in GE^p(X_{\mathbb{R}}) \mid \operatorname{cl}(\widetilde{g}) = \rho(y)\}.$$

And the group of codimension p arithmetic cycles is defined by

$$\begin{aligned}\widehat{\mathrm{Z}}^p(X) &= \left\{(y, \widetilde{g}) \in Z^p(X) \oplus GE^p(X_{\mathbb{R}}) \mid \widetilde{g} \in GE^p_y(X_{\mathbb{R}})\right\} \\ &= \left\{(y, \widetilde{g}) \in Z^p(X) \oplus GE^p(X_{\mathbb{R}}) \mid \operatorname{cl}(\widetilde{g}) = \rho(y))\right\}.\end{aligned}$$

That is, a codimension p arithmetic cycle is a pair $(y, \widetilde{g})$, where y is a codimension p algebraic cycle, and $\widetilde{g}$ is the class in $\mathfrak{D}^{2p-1}(E^*_{\log}((X \backslash \mathcal{Z}^p)_{\mathbb{R}}), p) / \operatorname{Im} d_{\mathfrak{D}}$ of a form $g \in \mathfrak{D}^{2p-1}(E^*_{\log}((X \backslash \mathcal{Z}^p)_{\mathbb{R}}), p)$, such that

$$\omega = d_{\mathfrak{D}} g = -2\partial\overline{\partial} g \in \mathfrak{D}^{2p}(E^*_{\log}(X_{\mathbb{R}}), p),$$

and the pair (ω, g) represents the class $\rho(y) \in H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X_{\mathbb{R}}, \mathbb{R}(p))$.

Let us now define rational equivalence in this setting. Let W be a codimension $p-1$ irreducible subvariety of X and let $f \in k(W)^*$. Let us write $Y = \operatorname{supp} \operatorname{div} f$. We have a well defined subvariety W_∞ of X_∞ (which may be empty) and a function $f_\infty \in k(W_\infty)^*$. Since f is defined over K, the function f_∞ satisfies $F^*_\infty f = \overline{f}$. Hence the map ρ (see §2) gives us a class

$$\rho(f) = \rho(f_\infty) \in H^{2p-1}_{\mathcal{D}}((X - Y)_{\mathbb{R}}, \mathbb{R}(p)).$$

Therefore we have an element

$$\mathrm{b}(\rho(f)) \in GE^p_{\operatorname{div} f}(X_{\mathbb{R}}),$$

where $\mathrm{b} : H^{2p-1}_{\mathcal{D}}((X - Y)_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow GE^p_{\operatorname{div} f}(X_{\mathbb{R}})$ is the map introduced in chapter III, after Definition 4.1.

Then we write

$$\widehat{\operatorname{div}}\, f = (\operatorname{div} f, \mathrm{b}(\rho(f))) \in \widehat{\mathrm{Z}}^p(X).$$

We denote by $\widehat{\mathrm{Rat}}^p$ the subgroup of $\widehat{\mathrm{Z}}^p$ generated by the elements of the form $\widehat{\operatorname{div}}\, f$.

Definition. 3.1 The cohomological arithmetic Chow groups of X are

$$\widehat{\mathrm{CH}}^p(X) = \widehat{\mathrm{CH}}^p(X, \mathfrak{D}(E_{\log})) = \widehat{\mathrm{Z}}^p(X) \Big/ \widehat{\mathrm{Rat}}^p.$$

We shall write $\widehat{\mathrm{CH}}^p(X, \mathfrak{D}(E_{\log}))$ when we want to stress the complex used to define the Green objects, or when we want to differentiate them from the arithmetic Chow groups defined by Gillet and Soulé.

We shall write

$$\widehat{\mathrm{CH}}^*(X) = \bigoplus_p \widehat{\mathrm{CH}}^p(X).$$

Theorem 3.2. *Let X be an arithmetic variety, with X_K proper over K and $\dim X_K = d$. Then there is a natural isomorphism*

$$\widehat{CH}^p(X, \mathfrak{D}(E^*_{\log})) \longrightarrow \widehat{CH}^p(X),$$

where the group on the right hand side is the arithmetic Chow group defined in [G-S 2]*. This isomorphism is given by*

$$(y, \widetilde{g}) \longmapsto (y, 2(2\pi i)^{d-p+1}[g]^{\sim}),$$

where g is a representative of $\widetilde{g}$.

Proof. Any representative g of $\widetilde{g}$ is locally integrable in the whole X by chapter II, Corollary 3.8.2. Recall that $[g]$ means the current on X defined by

$$[g](\omega) = \frac{1}{(2\pi i)^d} \int_X \omega \wedge g.$$

By chapter III, Theorem 4.7, the map

$$(y, \widetilde{g}) \longmapsto (y, 2(2\pi i)^{d-p+1}[g]^{\sim}),$$

gives us an isomorphism between the group of arithmetic cycles defined here and the group of arithmetic cycles in the sense of Gillet and Soulé. Thus we only need to check that the two concepts of rational equivalence coincide.

Let us recall the definition of rational equivalence in [G-S 2]. Let W be a codimension $p-1$ irreducible subvariety of X and let $f \in k(W)^*$. Let $\widetilde{W}_\infty$ be a resolution of singularities of W_∞ and let $j : \widetilde{W}_\infty \longrightarrow X_\infty$ be the induced map. The function f induces a well defined function, also denoted by $f \in k(\widetilde{W}_\infty)^*$. Then $\widehat{\operatorname{div}} f$ in the sense of Gillet and Soulé is defined by

$$\widehat{\operatorname{div}} f = (\operatorname{div} f, -(2\pi i)^{d-p+1} j_*[\log f\overline{f}]).$$

The factor $(2\pi i)^{d-p}$ comes from the different definition of $[\cdot]$ here and in [G-S 2].

Therefore we are reduced to proving that: there is a representative g of $\mathrm{b}(\rho f)$ such that, if $[g]$ is the associated current on X, then

$$2[g] + j_*[\log f\overline{f}] \in \operatorname{Im} \partial + \operatorname{Im} \overline{\partial}$$

in the complex $D^*_{X_\infty}$. Since this statement only depends on the complex variety X_∞ we will assume that X is a complex variety of dimension d.

Let $Y = \operatorname{supp} \operatorname{div} f$. Let $\pi : (\widetilde{X}, D) \longrightarrow (X, Y)$ be a resolution of singularities, with $D = \pi^{-1}(Y)$ a divisor with normal crossings. Then the class $\rho(f) \in H^{2p-1}_{\mathcal{D}}(X - Y, \mathbb{R}(d-p))$ is represented by the current $j_*[-(1/2)\log f\overline{f}]$. Therefore, in the complex $D^*_{\widetilde{X}/D}$, we have the equation

$$2[g] + j_*[\log f\overline{f}] = \partial a + \overline{\partial} b.$$

By chapter II, Proposition 1.8 we may assume that g is of weight one. Therefore it is locally integrable in the whole $\widetilde{X}$. Let us also denote by $[g]$ the associated current in the complex $D^*_{\widetilde{X}}$. Let a' and b' be elements of $D^*_{\widetilde{X}}$ which are mapped to a and b. Then in the complex $D^*_{\widetilde{X}}$ we have

$$2[g] + j_*[\log f\overline{f}] = \partial a' + \overline{\partial} b' + c,$$

where $c \in D^{p-1,p-1}_D$. So $\pi_* c \in D^{p-1,p-1}_Y = \{0\}$ because $\operatorname{codim} Y = p$. Hence, in the complex D^*_X we have

$$2[g] + j_*[\log f\overline{f}] = \partial \pi_* a' + \overline{\partial} \pi_* b'.$$

This concludes the proof of the theorem.

Our next objective is to fit the groups $\widehat{\mathrm{CH}}^*(X)$ in some exact sequences. We shall denote by ρ the induced morphisms (see 2.3)

$$\begin{aligned} &\rho : CH^{p,p-1}(X) \longrightarrow H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \quad \text{and} \\ &\rho : CH^{p,p-1}(X) \longrightarrow \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}}). \end{aligned}$$

We have maps

$$\begin{aligned} \zeta : \widehat{\mathrm{CH}}^p(X) &\longrightarrow \mathrm{CH}^p(X), & \zeta(y, \widetilde{g}) &= y, \\ \rho : \mathrm{CH}^p(X) &\longrightarrow H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)), & &\text{see 2.3}, \\ a : \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}}) &\longrightarrow \widehat{\mathrm{CH}}^p(X), & a(\widetilde{g}) &= (0, \widetilde{g}), \\ \omega : \widehat{\mathrm{CH}}^p(X) &\longrightarrow ZE^{p,p}_{\log}(X_{\mathbb{R}}), & \omega(y, \widetilde{g}) &= -2\partial\overline{\partial} g \qquad \text{and} \\ h : ZE^{p,p}_{\log}(X_{\mathbb{R}}) &\longrightarrow H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)), & h(\alpha) &= \{\alpha\}, \end{aligned}$$

where $\{\alpha\}$ is the cohomology class of α.

Let us write

$$\begin{aligned} &\widehat{\mathrm{CH}}^p(X)_0 = \mathrm{Ker}(\omega) \qquad \text{and} \\ &\mathrm{CH}^p(X)_0 = \{y \in \mathrm{CH}^p(X) \mid y_\infty \underset{\text{hom}}{\sim} 0\}. \end{aligned}$$

Then the analogue of [G-S 2, Theorem 3.3.5] is:

Theorem 3.3. *Let X be an arithmetic variety. Then we have exact sequences:*

(i) $CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0,$

(ii) $CH^{p,p-1}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X)$
$\xrightarrow{(\zeta,-\omega)} CH^p(X) \oplus ZE^{p,p}_{\log}(X_{\mathbb{R}}) \xrightarrow{\rho+h} H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \to 0,$

(iii) $CH^{p,p-1}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X)_0 \xrightarrow{\zeta} CH^p(X)_0 \to 0.$

Proof. The proof of the exactness of the three sequences is similar. So we shall write only the first.

The fact that the composition of two consecutive morphisms iz zero, follows easily from the definitions.

The surjectivity of ζ is equivalent to the existence of Green forms for a cycle and is a consequence of the surjectivity of the map cl proved in chapter III, Proposition 4.2.

Assume now that $\zeta(y, \widetilde{g}) = 0$. Then $y = \sum \operatorname{div} f_i$ and $(y, \widetilde{g}) - \sum \widehat{\operatorname{div}} f_i = (0, \widetilde{g}')$. Then $\operatorname{cl} \widetilde{g}' = 0$. By III, Proposition 4.2.1, $\widetilde{g}' \in \operatorname{Im} a$.

If $\widetilde{g} \in \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}})$ with $a(\widetilde{g}) = 0$, then $(0, \widetilde{g}) = \sum \widehat{\operatorname{div}} f_i$. Therefore $\sum \operatorname{div} f_i = 0$ and $f = \sum f_i$ determines an element of $\mathrm{CH}^{p,p-1}(X)$ and $\widetilde{g} = \rho(f)$.

Let us prove that ρ is well defined. We have to show that, if $x \in R^{p-2}_p(X)$, then $\widehat{\operatorname{div}}(dx) = 0$. Let $x = \{f, g\}$ be an irreducible K_2-chain. Then $\operatorname{div}(dx) = d^2(x) = 0$. Hence it remains to show that $\mathrm{b}(\rho(dx)) = 0$. By Theorem 2.1, there exists an element

$$\rho\{f, g\} \in H^{2p-2}_{\mathcal{D}, \operatorname{supp} x - \operatorname{supp} dx}((X - \operatorname{supp} dx)_{\mathbb{R}}, \mathbb{R}(p))$$

such that $\rho(dx) = \partial \rho(x)$, where ∂ is the connection homomorphism

$$\partial : H^{2p-2}_{\mathcal{D}, \operatorname{supp} x - \operatorname{supp} dx}((X - \operatorname{supp} dx)_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow H^{2p-1}_{\mathcal{D}, \operatorname{supp} dx}(X_{\mathbb{R}}, \mathbb{R}(p)).$$

Therefore, the image of $\rho(dx)$ in the group $H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p))$ is zero. Hence

$$\mathrm{b}(\rho(dx)) = 0.$$

Note that $\rho\{f, g\}$ is constructed using the product in Deligne cohomology. By the relationship between the product in Deligne cohomology and the $*$-product, this proof is essentially the same as the proof given in [G-S 2]. This concludes the proof of the theorem.

Example 3.4. In [G-S 2, 3.4] there are some examples of explicit arithmetic Chow groups. Since these examples are given for arithmetic varieties with projective X_∞, they are also examples for the arithmetic Chow groups introduced here.

Let us give a simple example where the groups obtained here and the groups obtained in [G-S 2] differ. Let $X = \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[t])$. Then X is an arithmetic variety over $\mathbb{Z}$. We have that $\mathrm{CH}^1(X) = 0$ and $\mathrm{CH}^{1,0}(X) = \{-1, 1\}$, but $\rho(\mathrm{CH}^{1,0}(X)) = 0$. Therefore

$$\widehat{\mathrm{CH}}^1(X) \cong E^0_{\log}(\mathbb{A}^1_{\mathbb{R}}).$$

That is, the space of F_∞-invariant, real valued C^∞ functions on $A^1_{\mathbb{C}}$ which have logarithmic singularities at infinity. Moreover we have

$$\widehat{\mathrm{CH}}^1(X)_0 = H^1_{\mathcal{D}}(\mathbb{A}^1_{\mathbb{R}}, \mathbb{R}(1)) = \mathbb{R}.$$

In particular, the morphism

$$\pi^* : \widehat{\mathrm{CH}}^*(\operatorname{Spec} \mathbb{Z}) \longrightarrow \widehat{\mathrm{CH}}^*(X)_0$$

is an isomorphism.

On the other hand, the groups $\widehat{\mathrm{CH}}^1(X)_0$ as defined in [G-S 2] are isomorphic to the analytic Deligne cohomology of $\mathbb{A}^1_{\mathbb{R}}$, $H^1_{\mathcal{D}^{\mathrm{an}}}(\mathbb{A}^1_{\mathbb{R}}, \mathbb{R}(1))$, which is an infinite dimensional real vector space.

Let us give a generalization of the above example.

Theorem 3.5. *Let X be an arithmetic variety and let $\pi : M \longrightarrow X$ be a geometric vector bundle. Then the induced morphism*

$$\pi^* : \widehat{CH}^*(X)_0 \longrightarrow \widehat{CH}^*(M)_0$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{CH}^{p,p-1}(X) & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) & \longrightarrow & \widehat{\mathrm{CH}}^p(X)_0 & \longrightarrow & \mathrm{CH}^p(X)_0 & \longrightarrow & 0 \\
\downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \\
\mathrm{CH}^{p,p-1}(M) & \longrightarrow & H^{2p-1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(p)) & \longrightarrow & \widehat{\mathrm{CH}}^p(M)_0 & \longrightarrow & \mathrm{CH}^p(M)_0 & \longrightarrow & 0.
\end{array}$$

At the level of CH^p and $\mathrm{CH}^{p,p-1}$, the morphism π^* is an isomorphism by [Gi, Th 8.3]. At the level of Deligne cohomology, the morphism π^* is an isomorphism because

$$\pi^* : E^*_{\log}(X_{\mathbb{C}}) \longrightarrow E^*_{\log}(M_{\mathbb{C}})$$

is a real filtered quasi-isomorphism with respect to the Hodge filtration. Therefore π^* is also an isomorphism at the level of $\widehat{\mathrm{CH}}^p_0$.

Let us summarize the properties of cohomological Chow groups. These properties can be proved as in [G-S 2] substituting Green currents by Green forms.

Let $(y, \widetilde{g}_y)$ and $(z, \widetilde{g}_z)$ be two arithmetic cycles such that y and z intersect properly. Then the singular support of $\widetilde{g}_y$ and the singular support of $\widetilde{g}_z$ intersect properly. Therefore the product $\widetilde{g}_y * \widetilde{g}_z$ is defined and is a Green form for $y \cdot z$. We can define an intersection product by

$$(y, \widetilde{g}_y) \cdot (z, \widetilde{g}_z) = (y \cdot z, \widetilde{g}_y * \widetilde{g}_z). \tag{3.6}$$

Let us write

$$\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^*(X) \otimes \mathbb{Q}.$$

Then we have (see [G-S 2, Theorem 4.2.3] for a more precise statement):

Theorem 3.7. *Let A be an arithmetic ring with fraction field K and let X be an arithmetic variety with X_K quasi-projective. Then, for each pair of non-negative integers p, q, there is an intersection pairing*

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \longrightarrow \widehat{CH}^{p+q}(X)_{\mathbb{Q}},$$

which is given by formula 3.6. for cycles intersecting properly.

This product induces in $\widehat{CH}^*(X)_{\mathbb{Q}}$ a structure of commutative and associative ring. Moreover, the induced maps

$$\zeta : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow CH^*(X) \otimes \mathbb{Q}$$

and

$$\omega : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow \bigoplus_p E^{p,p}_{\log}(X_{\mathbb{R}}, p)$$

are morphisms of rings. Therefore the subgroup $\widehat{CH}^*(X)_{0,\mathbb{Q}} = \operatorname{Ker}(\omega)$ *is an ideal of* $\widehat{CH}^*(X)_0$.

The functorial properties of the cohomological Chow groups are summarized in the following theorem. For proofs see [G-S 2, Theorem 3.6.1] and [G-S 2, Theorem 4.4.3]. Note that, in the case of arithmetic varieties which are not proper over A, we have to impose stronger conditions for the existence of a push-forward map. This is done to ensure that the direct image of a logarithmic form is again a logarithmic form (see the construction of a push forward of Green forms in chapter II, 1.11).

Theorem 3.8. *Let A be an arithmetic ring with fraction field K.*

1. *Let $f : X' \longrightarrow X$ be a morphism of regular quasi-projective arithmetic varieties. Then there is a pull-back morphism*
$$f^* : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^p(X'),$$
such that, if $(y, \widetilde{g}_y) \in \widehat{Z}^p(X)$ and $f^{-1}(y)$ is equidimensional of codimension p then
$$f^*(y, \widetilde{g}_y) = (f^*y, f^*\widetilde{g}_y),$$
with f^*y defined as in [Se]. *If $g : X'' \longrightarrow X'$ is another such morphism then $(fg)^* = g^*f^*$. Moreover f^* induces a ring homomorphism*
$$f^* : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow \widehat{CH}^*(X')_{\mathbb{Q}}.$$
2. *Let $f : X' \longrightarrow X$ be a proper morphism of equidimensional regular arithmetic varieties. Assume that there are smooth compactifications $\overline{X}'_\infty$ of X'_∞ and $\overline{X}_\infty$ of X_∞, such that $f_\infty : X'_\infty \longrightarrow X_\infty$ can be extended to a smooth map $\overline{f}_\infty : \overline{X}'_\infty \longrightarrow \overline{X}_\infty$. Let $e = \dim(X') - \dim(X)$. Then there is a push-forward morphism*
$$f_* : \widehat{CH}^p(X') \longrightarrow \widehat{CH}^{p-e}(X),$$
such that $f_*(y, \widetilde{g}_y) = (f_*y, f_*g_y)$. If $g : X'' \longrightarrow X'$ is another such morphism then $(fg)_* = f_*g_*$. Moreover, if $\alpha \in \widehat{CH}^p(X')$ and $\beta \in \widehat{CH}^q(X)$, then
$$f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta \in \widehat{CH}^{p+q-e}(X)_{\mathbb{Q}}.$$

§4. Covariant Green Currents.

As we have seen, the push-forward morphisms of arithmetic Chow groups are only defined for maps which are smooth in the generic fibre. This suggests that the arithmetic Chow groups introduced in the last section are cohomological arithmetic Chow groups and they should be complemented with a notion of homological Chow groups. In this section and the next, we shall introduce a notion of homological Chow groups, which are covariant for arbitrary proper morphisms and are provided with a cap product with the cohomological Chow groups already defined.

The homological Chow groups are constructed in the same way than cohomological Chow groups replacing logarithmic forms by a suitable complex of currents. This section will be devoted to the discussion of the analogue of Green forms in this setting. These objects will be called Green currents but they should not be confused with the Green currents in the sense of [G-S 2].

The proofs in this section will be omitted because they are analogous to the corresponding proofs for Green forms.

Let X be a proper smooth variety over $\mathbb{C}$ and let D_*^X be the complex of currents on X as in §1. Let $Y \subset X$ be a closed algebraic subset. Let us write

$$D_n^{Y_\infty} = \left\{ T \in D_n^X \mid \operatorname{supp}(T) \subset Y \right\}.$$

Since $\operatorname{supp} dT \subset \operatorname{supp} T$ these groups form a complex. Note that the complex of currents on Y, D_*^Y (see §1 after Theorem 1.3) is a subcomplex of $D_*^{Y_\infty}$, but in general they do not agree. We write

$$D_*^{X/Y_\infty} = D_*^X \,/ D_*^{Y_\infty} .$$

If X is equidimensional of dimension d we shall write

$$D^n_{Y_\infty} = D^{Y_\infty}_{d-n} \qquad \text{and} \qquad D^n_{X/Y_\infty} = D^{X/Y_\infty}_{d-n}.$$

In this case the morphism of complexes $E^*_{\log}(X-Y) \longrightarrow D^*_{X/Y}$ (see chapter II, §3) induces a morphism

$$E^*_{\log}(X-Y) \longrightarrow D^*_{X/Y_\infty}.$$

By a result of Poly ([P]) this morphism is a quasi-isomorphism. On the other hand this morphism is not a filtered quasi-isomorphism with respect to the Hodge filtration. This will be a source of technical problems and indicates that the definition given here of homological Chow groups may be not optimal. The Hodge filtration of the complex D^*_{X/Y_∞} is related with the formal Hodge filtration studied by Ogus in [O].

Poly also proves in [P] that the complex D^{X/Y_∞}_* only depends on $X-Y$. Thus, if V is a smooth variety over $\mathbb{C}$ and X is a smooth compactification of V with $Y=X-V$, we shall write

$$D^{\log}_*(V)=D^{X/Y_\infty}_*.$$

This definition does not depend, up to a canonical isomorphism, on the compactification chosen.

Let now X be a smooth algebraic variety over $\mathbb{C}$. Since $D^{\log}_*(X)$ is a Dolbeault chain complex we can construct the complex $\mathfrak{D}_*(D^{\log}_*(X),p)$ as in §1. We shall write

$$H^{\mathcal{D}^{\text{for}}}_*(X,\mathbb{R}(p))=H_*(\mathfrak{D}(D^{\log}_*(X),p)).$$

The superindex ${}^{\text{for}}$ is included to remind us that this homology groups are not the Deligne homology groups of X. They will be called formal Deligne homology groups. Nevertheless, if $\overline{X}$ is a compactification of X with $Y=\overline{X}-X$ a divisor with normal crossings, the morphism

$$D^{\overline{X}/Y}_*\longrightarrow D^{\log}_*(X)$$

induces a morphism

$$H^{\mathcal{D}}_*(X,\mathbb{R}(p))\longrightarrow H^{\mathcal{D}^{\text{for}}}_*(X,\mathbb{R}(p)).$$

Let $\mathcal{Z}_p=\mathcal{Z}_p(X)$ be the set of closed algebraic subsets of X of dimension $\leq p$ ordered by inclusion, and let $\mathcal{Z}_p\backslash\mathcal{Z}_{p-1}$ denote the set of all pairs $(Z,Z')\in\mathcal{Z}_p\times\mathcal{Z}_{p-1}$ such that $Z'\subset Z$.

Analogously to §2. we define the groups

$$\begin{aligned}
H^{\mathcal{D}^{\text{for}},\mathcal{Z}_p\backslash\mathcal{Z}_{p+1}}_n(X,\mathbb{R}(q)) &= \varinjlim_{(Z,Z')\in\mathcal{Z}_p\backslash\mathcal{Z}_{p-1}} H^{\mathcal{D}^{\text{for}},Z-Z'}_n(X-Z',\mathbb{R}(q)),\\
H^{\mathcal{D}^{\text{for}},\mathcal{Z}_p}_n(X,\mathbb{R}(q)) &= \varinjlim_{Z\in\mathcal{Z}_p} H^{\mathcal{D}^{\text{for}},Z}_n(X,\mathbb{R}(q)) \qquad \text{and}\\
H^{\mathcal{D}^{\text{for}}}_n(X\backslash\mathcal{Z}_p,\mathbb{R}(q)) &= \varinjlim_{Z\in\mathcal{Z}_p} H^{\mathcal{D}^{\text{for}}}_n(X-Z,\mathbb{R}(q)).
\end{aligned}$$

If we write

$$\mathfrak{D}_*(D^{\log}_*(X\backslash\mathcal{Z}_p),q)=\varinjlim_{Z\in\mathcal{Z}_p}\mathfrak{D}_*(D^{\log}_*(X-Z_p),q),$$

then

$$H^{\mathcal{D}^{\text{for}}}_n(X\backslash\mathcal{Z}_p,\mathbb{R}(q))=H_n(\mathfrak{D}_*(D^{\log}_*(X\backslash\mathcal{Z}_p),q)).$$

If in the commutative diagram of Theorem 2.4 we replace dimension by codimension and use formal Deligne homology groups instead of Deligne cohomology groups, we also obtain a commutative diagram. In particular, if y is an algebraic cycle of dimension p and $Y=\operatorname{supp}y$, then the class

$$\rho(y)\in H^{\mathcal{D}^{\text{for}},Y}_{2p}(X,\mathbb{R}(p))=H_{2p}(\mathfrak{D}_*(D^{\log}_*(X),p),\mathfrak{D}_*(D^{\log}_*(X-Y),p))$$

is represented by the pair $(\delta_y, 0)$. We write $H_{2p}^{\mathcal{D}^{\mathrm{for}},Y}(X,\mathbb{R}(p))$ instead of $H_{2p}^{\mathcal{D}^{\mathrm{for}}}(Y,\mathbb{R}(p))$ because these groups depend on X.

Let $W \subset X$ be an irreducible subvariety of dimension $p+1$ and $f \in k(W)^*$. Let $\widetilde{W}$ be a resolution of singularities of W and $j : \widetilde{W} \longrightarrow X$ be the induced map. Let us write $Y = \mathrm{supp}(\mathrm{div}\, f)$. Then the class

$$\rho(f) \in H_{2p+1}^{\mathcal{D}^{\mathrm{for}},W-Y}(X-Y,\mathbb{R}(p))$$

is represented by the pair $(j_*[-\frac{1}{2}\log f\overline{f}], 0)$.

If $\overline{X}$ is a smooth compactification of X, then it can be shown that $D_n^{\log}(X)$ is the topological dual of the space of differential n-forms on $\overline{X}$ which are flat along $\overline{X}-X$ (see [M] for a proof when $n=0$). Let Y and Z be closed algebraic subsets of X. Since the product of a form with logarithmic singularities along Y by a form flat along Y is again flat along Y (see [Tu, IV.4.2]), we can define a product

$$E^n_{\log}(X-Y) \otimes D^{\log}_m(X-Z) \xrightarrow{\wedge} D^{\log}_{m-n}(X-(Y\cup Z))$$

by $\varphi \wedge T(\omega) = T(\omega \wedge \varphi)$, where $\varphi \in E^n_{\log}(X-Y)$, $T \in D_m^{X/Z_\infty}$ and ω is a $m-n$ form flat along $Y \cup Z$.

Note that this product does not exist if we replace D_*^{X/Y_∞} by the complex $D_*^{X/Y} = D_*^X/D_*^Y$ (see §1 for definitions). This is one of the reasons why we choose D_*^{X/Y_∞} to define homological Chow groups.

This product is compatible with the structures of Dolbeault complexes. That is, it is real and bigraded. Therefore it induces a product

$$\mathfrak{D}^n(E^*_{\log}(X-Y),p) \otimes \mathfrak{D}_m(D^{\log}_*(X-Z),q) \longrightarrow \mathfrak{D}_{m-n}(D^{\log}_*(X-(Y\cup Z)),q-p).$$

Which is given by the same formulas as those of chap. III, Theorem 2.3. The product of φ and T will be denoted by $\varphi \cdot T$.

Definition 4.1. The *space of Green currents on X with singular support on dimension p* is

$$GD_p(X) = \widehat{H}_{2p}(\mathfrak{D}_*(D^{\log}_*(X),p), \mathfrak{D}_*(D^{\log}_*(X/\mathcal{Z}_p),p)).$$

That is, an element of $GD_p(X)$ is a pair $(T,\widetilde{g})$, where

$$\begin{aligned} T \in & D^{\log}_{p,p}(X) \cap D^{\log,\mathbb{R}}_{2p}(X,p) \quad \text{and} \\ \widetilde{g} \in & D^{\log}_{p-1,p-1}(X\backslash\mathcal{Z}_p) \cap D^{\log,\mathbb{R}}_{2p-2}(X\backslash\mathcal{Z}_p,p-1)\Big/(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}). \end{aligned}$$

Such that $dT = 0$ and $-2\partial\overline{\partial}g = T|_{X-Z}$ for some $Z \in \mathcal{Z}_p$.

Note that in this case, if $(T,\widetilde{g}) \in GD_p(X)$, then T is not determined by $\widetilde{g}$. For instance if Y is a dimension p subvaritey, then the pair $(\delta_Y, 0)$ is a Green current. Moreover we shall see that it is a Green current for the cycle Y.

If $Z \subset X$ is a dimension p algebraic subset of X, then the *space of Green currents on X with singular support contained on Z* is

$$GD_p^Z(X) = \widehat{H}_{2p}(\mathfrak{D}_*(D^{\log}_*(X),p), \mathfrak{D}_*(D^{\log}_*(X-Z),p)).$$

As in the case of Green forms, $GD_p(X)$ is the direct limit of the groups $GD_p^Z(X)$ for Z of dimension p.

The *singular support* of $(T,\widetilde{g}) \in GD_p(X))$ is the intersection of all Z such that $(T,\widetilde{g})$ has a representative in $GD_p^Z(X)$. We shall denote the singular support of $(T,\widetilde{g})$ by $\operatorname{supp}(T,\widetilde{g})$.

Let us write

$$\widetilde{D}^{\log}_{p+1,p+1}(X) = D^{\log}_{p+1,p+1}(X) \cap D^{\log\mathbb{R}}_{2p+2}(X,p+1) \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial} + \operatorname{Im} H^{\mathcal{D}^{\mathrm{for}},\mathcal{Z}_p}_{2p+1}(X,\mathbb{R}(p))).$$

The term $\operatorname{Im} H^{\mathcal{D}^{\mathrm{for}},\mathcal{Z}_p}_{2p+1}(X,\mathbb{R}(p))$ is included in this definition because I do not know if there is a purity theorem for the formal Deligne cohomology groups. For the same reason we write

$$\begin{aligned}\widetilde{H}^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X,\mathbb{R}(p)) &= H^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X,\mathbb{R}(p)) \Big/ \operatorname{Im} H^{\mathcal{D}^{\mathrm{for}},\mathcal{Z}_p}_{2p+1}(X,\mathbb{R}(p)) \\ &= \operatorname{Im}(H^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X,\mathbb{R}(p)) \longrightarrow H^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X\backslash\mathcal{Z}_p,\mathbb{R}(p))).\end{aligned}$$

We also write

$$ZD^{\log,\mathbb{R}}_{p,p}(X) = \left\{T \in D^{\log}_{p,p}(X) \cap D^{\log,\mathbb{R}}_{2p}(X,p) \mid dT = 0\right\}.$$

Then the analogue of chap. III, Proposition 4.2 is:

Proposition 4.2. *Let X be a smooth variety over $\mathbb{C}$. Then there are exact sequences*

1. $0 \to \widetilde{D}^{\log,\mathbb{R}}_{p+1,p+1}(X) \xrightarrow{\mathrm{a}} GD_p(X) \xrightarrow{\mathrm{cl}} H^{\mathcal{D}^{\mathrm{for}},\mathcal{Z}_p}_{2p}(X,\mathbb{R}(p)) \to 0.$

2. $0 \to H^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X\backslash\mathcal{Z}_p,\mathbb{R}(p)) \xrightarrow{\mathrm{b}} GD_p(X) \xrightarrow{\omega} ZD^{\log,\mathbb{R}}_{p,p}(X) \to H^{\mathcal{D}^{\mathrm{for}}}_{2p}(X\backslash\mathcal{Z}_p,\mathbb{R}(p)).$

3. $0 \to \widetilde{H}^{\mathcal{D}^{\mathrm{for}}}_{2p+1}(X,\mathbb{R}(p)) \xrightarrow{\mathrm{a}} GD_p(X) \xrightarrow{\mathrm{cl}\oplus\omega}$
$H^{\mathcal{D}^{\mathrm{for}},\mathcal{Z}_p}_{2p}(X,\mathbb{R}(p)) \oplus ZD^{\log,\mathbb{R}}_{p,p}(X) \to H^{\mathcal{D}^{\mathrm{for}}}_{2p}(X,\mathbb{R}(p)) \to$
$0.$

Proposition 4.3. *Let X be a smooth variety over $\mathbb{C}$ of dimension d. Then there is a natural morphism*

$$GE^p(X) \longrightarrow GD_{d-p}(X)$$

given by $(\omega,\widetilde{g}) \longmapsto ([\omega],[g]^{\sim})$. This morphism is compatible with the exact sequences of Proposition 4.2 and chap III, Proposition 4.2. Moreover, if $d=0$ this morphism is an isomorphism.

Proposition 4.4. *Let $f : X' \longrightarrow X$ be a proper morphism of smooth varieties over $\mathbb{C}$. Then there is a push-forward morphism*

$$f_* : GD_p(X') \longrightarrow GD_p(X)$$

given by $f_*(T,\widetilde{g}) = (f_*T, f_*(g)^{\sim})$. This morphism is compatible with the push-forward of currents and the push-forward in homology. Moreover, assume that X and X' are equidimensional and that f can be extended to a smooth morphism between proper smooth algebraic varieties. Then the push-forward of Green currents is compatible with the push-forward of Green forms.

Our next objective is to define a product between Green forms and Green currents. We shall give three equivalent definitions for this product. The first is analogous to chap III, 4.9, the second to [G-S 2, 2.1.3] and the third to chap II, 2.7.

Let Y and Z be two closed algebraic subsets of X, with $\operatorname{codim} Y = p$ and $\dim Z = q$. Assume that Y and Z intersect properly. That is $\dim(Y \cap Z) = q - p$. Let $(\omega, \widetilde{g}_1) \in GE^p_Y(X)$ and $(T, \widetilde{g}_2) \in GD^Z_q(X)$.

Definition 4.5. Let us write $r = q - p$, $\mathfrak{D}_*(X, r) = \mathfrak{D}_*(D^{\log}_*(X), r)$ and

$$\mathfrak{D}_*(X; Y, Z, r) = s(\mathfrak{D}_*(X - Y, r) \oplus \mathfrak{D}_*(X - Z, r) \to \mathfrak{D}_*(X - Y \cup Z, r)).$$

Then there is a natural isomorphism

$$\widehat{H}_{2r}(\mathfrak{D}_*(X, r), \mathfrak{D}_*(X - Y \cap Z, r)) \longrightarrow \widehat{H}_{2r}(\mathfrak{D}_*(X, r), \mathfrak{D}_*(X; Y, Z, r)).$$

The $*$-*product* of $(\omega, \widetilde{g}_1)$ and $(T, \widetilde{g}_2)$ is given by

$$\begin{aligned}(\omega, \widetilde{g}_1) * (T, \widetilde{g}_2) &= (\omega_1 \cdot T, (g_1 \cdot T, \omega \cdot g_2, g_1 \cdot g_2)^{\widetilde{\ }}) \\ &= (\omega_1 \wedge T, (g_1 \wedge T, \omega \wedge g_2, -4\pi i d^c g_1 \wedge g_2 + 4\pi i g_1 \wedge d^c g_2)^{\widetilde{\ }}).\end{aligned}$$

Definition 4.6. Let us choose a current $g'_2 \in \mathfrak{D}_{2q+1}(D^{\log}_*(X), q)$ such that its image in $\mathfrak{D}_{2q+1}(D^{\log}_*(X - Z), q)$ is g_2. Let us write

$$\delta_2 = T - d_{\mathfrak{D}} g'_2 = T + 2\partial\bar{\partial} g'_2.$$

Then δ_2 is a current with support contained on Z. Therefore, there is a well defined current

$$g_1 \wedge \delta_2 \in \mathfrak{D}_{2q-2p+1}(D^{\log}_*(X - Y \cap Z), q - p).$$

The $*$-product between $(\omega, \widetilde{g}_1) \in GE^p_Y(X)$ and $(T, \widetilde{g}_2) \in GD^Z_q(X)$ is given by

$$(\omega, \widetilde{g}_1) * (T, \widetilde{g}_2) = \Big(\omega \wedge T, (\omega \wedge g'_2 + g_1 \wedge \delta_2)^{\widetilde{\ }}\Big)$$

Definition 4.7. Let now $(\widetilde{X}, D)$ be a resolution of singularities of $(X, Y \cap Z)$ such that the strict transforms of Y and Z do not meet. Write $\widehat{Y}$ for the strict transform of Y and $\widehat{Z}$ for that of Z. Let $\sigma_{Y,Z}$ be a smooth function on $\widetilde{X}$ such that it takes the value 1 in a neighbourhood of $\widehat{Y}$ and the value 0 in a neighbourhood of $\widehat{Z}$. Let $\sigma_{Z,Y} = 1 - \sigma_{Y,Z}$.

The $*$-*product* of $(\omega, \widetilde{g}_1)$ and $(T, \widetilde{g}_2)$ is given by

$$(\omega, \widetilde{g}_1) * (T, \widetilde{g}_2) = (\omega \wedge T, (d_{\mathfrak{D}}(\sigma_{Y,Z} g_1) \wedge g_2 + \sigma_{Z,Y} g_1 \wedge d_{\mathfrak{D}} g_2)^{\widetilde{\ }}) \in GD^{Y \cap Z}_{q-p}(X).$$

Proposition 4.8. *The three definitions of $*$-product are equivalent. Consequently, the second and third definitions are independent of the choices.*

Proposition 4.9. *The $*$-product between Green forms and Green currents is compatible with the $*$-product of Green forms, with the cap-product between $H^*_{\mathcal{D},Y}(X,\mathbb{R}(p))$ and $H_*^{\mathcal{D}^{\mathrm{for}},Z}(X,\mathbb{R}(p))$ and with the $\wedge$-product between forms and currents. Moreover, if $(\omega_1,\widetilde{g}_1)$, $(\omega_2,\widetilde{g}_2)$ are Green forms and $(T,\widetilde{g}_3)$ is a Green current, then*

$$((\omega_1,\widetilde{g}_1)*(\omega_2,\widetilde{g}_2))*(T,\widetilde{g}_3)=(\omega_1,\widetilde{g}_1)*((\omega_2,\widetilde{g}_2)*(T,\widetilde{g}_3))\,.$$

Proposition 4.10. *Let $f: X'\longrightarrow X$ be a proper morphism between smooth varieties over $\mathbb{C}$, $Y\subset X$ a codimension p algebraic subset of X with $f^{-1}(Y)$ of codimension p, $Z\subset X'$ a dimension q algebraic subset which intersects properly with $f^{-1}(Y)$. If $(\omega,\widetilde{g}_1)\in GE^p_Y(X)$ and $(T,\widetilde{g}_2)\in GD^Z_q(X)$, then*

$$f_*(f^*(\omega,\widetilde{g}_1)*(T,\widetilde{g}_2))=(\omega,\widetilde{g}_1)*f_*(T,\widetilde{g}_2).$$

§5. Homological Arithmetic Chow Groups.

Let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring and let X be an arithmetic variety. We shall assume that A is "good" in the sense of [G-S 4]. That is A is equicodimensional and Jacobson. Let $e = \dim \operatorname{Spec} A$. Following the case of cohomological Chow groups, we write

$$\begin{aligned} GD_p(X_{\mathbb{R}}) &= \left\{(T, \widetilde{g}) \in GD_p(X_\infty) \mid F_\infty^*(T, \widetilde{g}) = ((-1)^p T, (-1)^{p+1}\widetilde{g})\right\}, \\ H_*^{\mathcal{D}^{\text{for}}}(X_{\mathbb{R}}, \mathbb{R}(p)) &= H_*(\mathfrak{D}_*(D_*^{\log}(X_{\mathbb{R}}), p)), \\ \widetilde{D}^{\log,\mathbb{R}}_{p+1,p+1}(X_{\mathbb{R}}) &= \left\{\widetilde{g} \in \widetilde{D}^{\log,\mathbb{R}}_{p+1,p+1}(X_\infty) \mid F_\infty^*\widetilde{g} = (-1)^{p+1}\widetilde{g}\right\}, \\ ZD^{\log}_{p,p}(X_{\mathbb{R}}) &= \left\{T \in ZD^{\log,\mathbb{R}}_{p,p}(X_\infty) \mid F_\infty^* T = (-1)^p T\right\}. \end{aligned}$$

We also write

$$\widetilde{H}^{\mathcal{D}^{\text{for}}}_{2p+1}(X_{\mathbb{R}}, \mathbb{R}(p)) = \operatorname{Im}(H^{\mathcal{D}^{\text{for}}}_{2p+1}(X_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow H^{\mathcal{D}^{\text{for}}}_{2p+1}(X_{\mathbb{R}} \backslash \mathcal{Z}_p, \mathbb{R}(p))).$$

Let $Z_p(X)$ be the group of algebraic cycles of X of dimension p. If $y \in Z_p(X)$ is an irreducible divisor, then $y \cap X_{\mathbb{K}} = \emptyset$ or y is flat over $\operatorname{Spec} A$. In the first case we write $y_\infty = 0$. In the second, y determines a dimension $p - e$ cycle, y_∞ in X_∞. Then the space of Green currents for y is defined by

$$GD^y_{p-e}(X) = \{(T, \widetilde{g}) \in GD_{p-e}(X_{\mathbb{R}}) \mid \operatorname{cl}(T, \widetilde{g}) = \rho(y) = \rho(y_\infty)\}.$$

For instance the pair $(\delta_y, 0)$ is a Green current for the cycle y because it represents $\rho(y)$. As in the case of Green forms, if $(T, \widetilde{g})$ is a Green current for y then $\operatorname{supp}(T, \widetilde{g}) = \operatorname{supp} y_\infty$.

The group of dimension p arithmetic cycles is

$$\widehat{\mathrm{Z}}_p(X) = \left\{(y, (T, \widetilde{g})) \in Z_p(X) \oplus GD_{p-e}(X_{\mathbb{R}}) \mid (T, \widetilde{g}) \in GD^y_{p-e}(X_{\mathbb{R}})\right\}$$

Let W be a dimension $p+1$ irreducible subvariety of X and let $f \in k(W)^*$. Let $\widetilde{W}_\infty$ be a resolution of singularities of W_∞, $j : \widetilde{W}_\infty \longrightarrow X_\infty$ the induced map and let us denote by f_∞ the induced function in $\widetilde{W}_\infty$. Then we write

$$\begin{aligned} \widehat{\operatorname{div}} f &= (\operatorname{div} f, \mathrm{b}(\rho f)) \\ &= (\operatorname{div} f, (0, \frac{-1}{2} j_*[\log f_\infty \overline{f}_\infty])) \in \widehat{\mathrm{Z}}_p(X). \end{aligned}$$

This definition is compatible with the definition given for cohomological Chow groups.

We denote by $\widehat{\operatorname{Rat}}_p$ the subgroup of $\widehat{\mathrm{Z}}_p$ generated by the elements of the form $\widehat{\operatorname{div}} f$.

Definition 5.1 The homological arithmetic Chow groups of X are

$$\widehat{\mathrm{CH}}_p(X)=\widehat{\mathrm{CH}}_p(X,\mathfrak{D}(D^{\log}))=\widehat{\mathrm{Z}}_p(X)\Big/\widehat{\mathrm{Rat}}_p.$$

We shall write

$$\widehat{\mathrm{CH}}_*(X)=\bigoplus_p\widehat{\mathrm{CH}}_p(X).$$

Using the obvious definitions, we have the analogue of Theorem 3.3

Theorem 5.2. *Let* X *be an arithmetic variety. Let us write* $p'=p-e$*. Then we have exact sequences:*

(i) $CH_{p,p+1}(X)\xrightarrow{\rho}\widetilde{D}^{\log}_{p'+1,p'+1}(X_{\mathbb{R}})\xrightarrow{a}\widehat{CH}_p(X)\xrightarrow{\zeta}CH_p(X)\longrightarrow 0,$

(ii) $CH_{p,p+1}(X)\xrightarrow{\rho}\widetilde{H}^{\mathcal{D}^{\mathrm{for}}}_{2p'+1}(X_{\mathbb{R}},\mathbb{R}(p'))\xrightarrow{a}\widehat{CH}_p(X)$
$\xrightarrow{(\zeta,-\omega)}CH_p(X)\oplus ZD^{\log}_{p',p'}(X_{\mathbb{R}})\xrightarrow{c+h}H^{\mathcal{D}^{\mathrm{for}}}_{2p'}(X_{\mathbb{R}},\mathbb{R}(p'))\longrightarrow 0,$

(iii) $CH_{p,p-1}(X)\xrightarrow{\rho}\widetilde{H}^{\mathcal{D}^{\mathrm{for}}}_{2p'+1}(X_{\mathbb{R}},\mathbb{R}(p'))\xrightarrow{a}\widehat{CH}_p(X)_0\xrightarrow{\zeta}CH_p(X)_0\longrightarrow 0.$

Let us summarize the first properties of the homological Chow groups.

Theorem 5.3.

1) *Let* X *be an equidimensional arithmetic variety of dimension* d*. Then there is a cap-product morphism*

$$\widehat{CH}^p(X)\xrightarrow{\cap[X]}\widehat{CH}_{d-p}(X)$$

given by $(y,\widetilde{g}_y)\cap[X]=(y,([d_{\mathcal{D}}g_y],[g_y]^{\sim}))$*. If the dimension of* X_Σ *is zero then this morphism is an isomorphism.*

2) *Let* $f:X'\longrightarrow X$ *be a proper morphism of arithmetic varieties. Then there is defined a push-forward morphism*

$$f_*:\widehat{CH}_p(X')\longrightarrow\widehat{CH}_p(X)$$

given by $f_*(y,(T_y,\widetilde{g}_y))=(f_*y,(f_*T_y,f_*(g_y)^{\sim}))$*. Assume that* X *and* X' *are equidimensional, and that the induced morphism* $f_\infty:X'_\infty\longrightarrow X_\infty$ *can be extended to a smooth morphism between proper varieties. Then the push-forward morphism of homological Chow groups is compatible with the push-forward morphism of cohomological Chow groups.*

3) *There is a cap-product*

$$\widehat{CH}^p(X)\otimes\widehat{CH}_q(X)\longrightarrow\widehat{CH}_{q-p}(X)_{\mathbb{Q}},$$

which turns $\widehat{CH}_*(X)_{\mathbb{Q}}$ *into an* $\widehat{CH}^*(X)_{\mathbb{Q}}$ *module. When* X *is equidimensional of dimension* d*, there is a commutative diagram*

$$\begin{array}{ccc}\widehat{CH}^p\otimes\widehat{CH}^q & \longrightarrow & \widehat{CH}^{p+q}\\ \downarrow & & \downarrow\\ \widehat{CH}^p\otimes\widehat{CH}_{d-q} & \longrightarrow & \widehat{CH}_{d-q-p}.\end{array}$$

4) *Let* $f:X'\longrightarrow X$ *be a proper morphism between arithmetic varieties,* $x\in\widehat{CH}^p(X)$ *and* $y\in\widehat{CH}_q(X')$*. Then we have the projection formula*

$$f_*(f^*x\cdot y)=x\cdot f_*y.$$

Remark 5.4. Observe that, due to the lack of a complex of currents with all the properties we need, the definition given here of homological Chow groups is not optimal and should be considered as provisional. Nevertheless, by part 1) of the above theorem, if $\mathcal{O}_K$ is the ring of integers of a number field K, we have

$$\widehat{\mathrm{CH}}^*(\operatorname{Spec}\mathcal{O}_K) \cong \widehat{\mathrm{CH}}_{1-*}(\operatorname{Spec}\mathcal{O}_K).$$

Therefore, the intersection numbers computed with the homological arithmetic Chow groups are the correct ones.

To end this section we shall give an interpretation of the height of a cycle with respect to a metrized line bundle in terms of the homological arithmetic Chow groups. The reader is referred to [Bo-G-S] for a discussion of the different definitions of the height of a cycle.

Let X be an arithmetic variety of dimension d. Let y be an algebraic cycle of X of dimension p. Then there exists a Green form for y. Thus there exists an element $\widehat{y} \in \widehat{\mathrm{CH}}^{d-p}(X)$ which is mapped to the class of y in $\mathrm{CH}^{d-p}(X)$. But there is not a canonical way to choose this Green form. On the other hand, the simplest Green current for y is the pair $(\delta_y, 0)$. Therefore we have a natural way to assign to each algebraic cycle, an element of $\widehat{\mathrm{CH}}_p(X)$. Thus we have a map

$$\begin{array}{rcl} \theta : \mathrm{Z}_p(X) & \longrightarrow & \widehat{\mathrm{CH}}_p(X) \\ y & \longmapsto & (y, (\delta_y, 0)). \end{array}$$

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Then $\mathcal{O}_K$ is an arithmetic ring in a canonical way (see [G-S 2]). Let $S = \operatorname{Spec}\mathcal{O}_K$ and let X be a regular projective arithmetic variety over $\mathcal{O}_K$ of dimension d. Let us denote by $\pi : X \longrightarrow S$ the structural map.

The key point in the construction of heights in [Bo-G-S] is a biadditive pairing

$$\widehat{\mathrm{CH}}^q(X) \otimes Z_p(X) \longrightarrow \widehat{\mathrm{CH}}^{q-p+1}(S)_{\mathbb{Q}}.$$

This pairing is defined in the following way. Let $\widehat{x} \in \widehat{\mathrm{CH}}^q(X)$ and $y \in Z_p(X)$. Let us choose a Green form g_y for y and let us write $\widehat{y}$ for the class of $(y, \widetilde{g}_y)$ in $\widehat{\mathrm{CH}}^{d-p}(X)$. Then

$$(\widehat{x} \mid y) = \pi_*(\widehat{x} \cdot \widehat{y}) - a(\pi_*(\omega(\widehat{x}) \wedge g_y)) \in \widehat{\mathrm{CH}}^*(S)_{\mathbb{Q}}.$$

This pairing is independent of the choice of a Green form g_y. See [Bo-G-S, 2.3] for more details.

We shall give another description of this pairing in terms of the map θ.

Proposition 5.4. *Let $\widehat{x} \in \widehat{CH}^q(X)$ and $y \in Z_p(X)$. Then*

$$(\widehat{x} \mid y) = \pi_*(\widehat{x} \cdot \theta(y)) \in \widehat{CH}_{p-q}(S)_{\mathbb{Q}} \cong \widehat{CH}^{q-p+1}(S)_{\mathbb{Q}}.$$

Proof. If $p - q \neq 0, 1$ then both sides of the equation are 0. If $p - q = 1$ then both sides are equal to $\pi_*(\zeta(\widehat{x}) \cdot y) \in \widehat{\mathrm{CH}}_1(S)_{\mathbb{Q}} = \mathbb{Q}$, where $\zeta(\widehat{x})$ is the cycle of $\widehat{x}$ in

$\mathrm{CH}^*(X)$. Finally, if $p - q = 0$, let us choose a Green form $(\omega_y, \widetilde{g}_y)$ for y, and let $\widehat{y}$ be the class of $(y, ([\omega_y], [\widetilde{g}_y]))$ in $\widehat{\mathrm{CH}}_p(X)$. Then

$$\widehat{y} - \theta(y) = (0, ([\omega_y] - \delta_y, [\widetilde{g}_y])).$$

Assume that g_y is locally integrable on X_∞ and let γ be the current $[g_y]$ on X_∞. Then

$$a(\gamma) = (0, (d_{\mathfrak{D}}\gamma, [\widetilde{g}_y])) = \widehat{y} - \theta(y).$$

Therefore

$$\begin{aligned}
(\widehat{x} \mid y) &= \pi_*(\widehat{x} \cdot \widehat{y} - a(\omega(x) \wedge [g_y])) \\
&= \pi_*(\widehat{x} \cdot (\widehat{y} - a(\gamma))) \\
&= \pi_*(\widehat{x} \cdot \theta(y)).
\end{aligned}$$

§6. An Arithmetic Riemann-Hurwitz Formula.

Let $f : X \longrightarrow Y$ be a proper morphism between regular quasi-projective arithmetic varieties over $\mathbb{Z}$. Assume that the induced morphism $f : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$ is smooth. Let $\overline{E} = (E, h)$ be a metrized vector bundle over X (see [G-S 2]). The determinant of cohomology $\lambda(E) = \det Rf_*(E)$ is an algebraic line bundle over Y. Let us choose a hermitian metric h_f, invariant by conjugation, on the relative tangent space Tf such that the restriction of h_f to each fibre of f over $Y(\mathbb{C})$ is Kähler. Then the line bundle $\lambda(E)$ can be equipped with the Quillen metric h_Q (see [Q 2], [Bi-G-S] or [S-A-B-K]).

The arithmetic Riemann-Roch theorem of Gillet and Soulé ([G-S 3], see also [S-A-B-K] and [Fa 3]) states that

$$\widehat{c}_1(\lambda(E), h_Q) = f_* \left(\widehat{ch}(E, h)\widehat{Td}(Tf, h_f) - a(ch(E_{\mathbb{C}})Td(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}}))\right)^{(1)},$$

where $\alpha^{(1)}$ denotes the component of degree one of $\alpha \in \widehat{\mathrm{CH}}^*(Y)_{\mathbb{Q}}$, $\widehat{c}_1$, $\widehat{ch}$ and $\widehat{Td}$ denotes the arithmetic first Chern class, the arithmetic Chern character and the arithmetic Todd class of a metrized vector bundle, ch and Td denotes the Chern character form and the Todd form and $R(Tf_{\mathbb{C}})$ is a characteristic class (see [G-S 2], [G-S 3] for definitions.) Note that in [G-S 3] a more general theorem is proved.

We would like to remove the hypothesis of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$ being smooth. In a first step we can restrict ourselves to proper dominant morphisms. The main obstacles are the following:

1) The push-forward morphism $f_* : \widehat{\mathrm{CH}}^*(X) \longrightarrow \widehat{\mathrm{CH}}^*(Y)$ is only defined for a morphism f with $f_{\mathbb{Q}}$ smooth.
2) If $f_{\mathbb{Q}}$ is not smooth, then the metric h_Q is no longer a smooth metric but has singularities over the discriminant locus.
3) The relative tangent space $Tf_{\mathbb{C}}$ is no longer a vector bundle over $X(\mathbb{C})$.

The first difficulty can be overcome by replacing arithmetic Chow groups by homological arithmetic Chow groups which have defined push-forward morphisms for arbitrary proper morphisms.

In this section we shall show how to solve the other two difficulties for the simplest case. Namely, let $f : X \longrightarrow Y$ be a proper morphism between regular arithmetic surfaces such that $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$ is a branched covering between nonsingular curves.

In this case all the technical complexities of the Quillen metric disappear because we are in the relative dimension zero case. Nevertheless it illustrates some phenomena that might occur in the general case, for instance, that the characteristic classes of a vector bundle with a singular metric live in the homological Chow groups, as well as the characteristic classes of the relative tangent bundle of a morphism which is not smooth at generic fibre.

For simplicity, instead of working with general singular metrics as in [De 2], we shall consider only the following type of singular metric.

Definition 6.1. Let C be a compact complex curve, $\mathcal{L}$ a line bundle over C. A *singular hermitian metric* h on $\mathcal{L}$ is a smooth hermitian metric on $\mathcal{L}|_U$, where $U \subset C$ is a dense open Zariski subset, satisfying the following condition. For each $p \in C - U$, let s be a non-vanishing regular section of $\mathcal{L}$ on a neighbourhood of p and let z be a local coordinate around p. Then, in an open neighbourhood V of p we have

$$h(z) = \|s\|_z^2 = (z\overline{z})^\alpha f(z),$$

where $\alpha \in \mathbb{Q}$ and f is a real function, smooth on $V - p$, continuous on V, with $f(z) > 0$, and there exist constants $C, \varepsilon > 0$ such that

$$\left\| \frac{\partial f(z)}{\partial z} \right\| < \frac{C}{\|z\|^{1-\varepsilon}} \quad \text{and} \quad \left\| \frac{\partial^2 f(z)}{\partial z \partial \overline{z}} \right\| < \frac{C}{(z\overline{z})^{1-\varepsilon}}.$$

A metric which is smooth and strictly positive in every point will be called a smooth metric.

The point p is called a *singular point* of h and the number α is called the *index of singularity* of h at p. Note that this number is independent of the section and the local coordinate chosen. If h is a singular metric with singular points $p_1, \dots, p_k$ and indexes of singularity $\alpha_1, \dots, \alpha_k$ then the *singularity divisor* of h is

$$\operatorname{sing} h = \sum_{i=1}^{k} \alpha_i p_i \in \mathrm{Z}^1(C) \otimes \mathbb{Q}.$$

The associated current $\delta_{\operatorname{sing} h}$ will be called the *singularity current* of h.

Proposition 6.2. *Let C be a compact complex curve, $\mathcal{L}$ a line bundle on C with a singular metric $h = \| \cdot \|^2$ and s a rational section of $\mathcal{L}$ which is regular and non vanishing at the singular points of $\mathcal{L}$. Then the pair*

$$\begin{aligned} g(\mathcal{L}, h, s) &= ([-d_{\mathfrak{D}} \log \|s\|] - \delta_{\operatorname{sing} h}, -[\log \|s\|]) \\ &= ([2\partial\overline{\partial} \log \|s\|] - \delta_{\operatorname{sing} h}, -[\log \|s\|]) \\ &\in GD_1(C). \end{aligned}$$

is a Green current for the cycle $\operatorname{div} s$. *Therefore the current* $c_1(\mathcal{L}, h) = [-d_{\mathfrak{D}} \log \|s\|] - \delta_{\operatorname{sing} h}$ *represents the first Chern class of* $\mathcal{L}$ *and* $\deg \mathcal{L} = c_1(\mathcal{L}, h)(1)$.

Proof. By the definition of singular metric, the norm of the form $\partial\overline{\partial} \log \|s\|^2$ is bounded by the norm of

$$C \frac{dz \wedge d\overline{z}}{(z\overline{z})^{1-\varepsilon}}$$

for some constants $C, \varepsilon > 0$. Therefore the form $\partial\overline{\partial} \log \|s\|^2$ is locally integrable. The function $\log \|s\|^2$ is also locally integrable. Hence $g(\mathcal{L}, h, s)$ is well defined.

Let γ denote the current $-[\log\|s\|]$ considered as a current on C. Let us assume that $\operatorname{div} s = \sum k_p p$ and $\operatorname{sing} h = \sum \alpha_q q$. Let U be a neighbourhood of a singular point q of h, such that it does not contain any other singular point of h, nor a zero or a pole of s. Let φ be a function with compact support contained on U. Let $C(q,a)$ denote the circumference of centre p and radius a and let z be a local parameter for C around q. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_C d(\varphi\partial\log\|s\|^2) &= \frac{1}{2\pi i}\lim_{a\to 0}\int_{C(q,a)} -\varphi\left(\alpha_q\frac{dz}{z}+\frac{\partial f(z)}{\partial z}\frac{dz}{f}\right)\\ &= -\alpha_q\varphi(q).\end{aligned}$$

In the latter equality we have used the bound for $\partial f/\partial z$.

Let γ denote the current $-[\log\|s\|]$ considered as a current on C. Then, for any test function φ one has

$$\begin{aligned}d_{\mathfrak{D}}\gamma(\varphi) &= \frac{1}{2\pi i}\int_C 2\log\|s\|\partial\overline{\partial}\varphi\\ &= \frac{1}{2\pi i}\int_C d(\log\|s\|^2\overline{\partial}\varphi) - \frac{1}{2\pi i}\int_C \partial\log\|s\|^2\wedge\overline{\partial}\varphi\\ &= 0 + \frac{1}{2\pi i}\int_C d(\partial\log\|s\|^2\varphi) + \frac{1}{2\pi i}\int_C \partial\overline{\partial}\log\|s\|^2\varphi\\ &= \frac{1}{2\pi i}\int_C -d_{\mathfrak{D}}\log\|s\|\varphi - \sum k_p\varphi(p) - \sum\alpha_q\varphi(q)\\ &= [-d_{\mathfrak{D}}\log\|s\|](\varphi) - \delta_{divs}(\varphi) - \delta_{\operatorname{sing} h}(\varphi).\end{aligned}$$

Therefore $g(\mathcal{L},h,s)$ represents the same cohomology class as $(\delta_{\operatorname{div} s},0)$. Thus it is a Green current for $\operatorname{div} s$.

Let $f : C \longrightarrow C'$ be a non constant morphism of compact complex curves, with $\deg f = d$. Let $\overline{\mathcal{L}} = (\mathcal{L},h)$ be a metrized line bundle on C with h a smooth metric. Then $f_*\mathcal{L}$ is a vector bundle on C' of rank d. Recall that a local section of $f_*\mathcal{L}$, $s\in\Gamma(U,f_*\mathcal{L})$ is a section $s\in\Gamma(f^{-1}U,\mathcal{L})$. We can introduce a metric f_*h over $f_*\mathcal{L}$ by writing

$$\langle s,t\rangle_y = \sum_{f(x)=y} r_x\langle s,t\rangle_x\,,$$

where r_x is the ramification index of f at x. Note that f_*h is a smooth metric outside the discriminant locus of f.

Let us write $\det f_*h$ for the metric induced by f_*h in $\det f_*\mathcal{L}$. Let us also write $\lambda(\overline{\mathcal{L}}) = (\det f_*\mathcal{L}, \det f_*h)$ and let R be the ramification divisor of f.

Proposition 6.3. *The metric* $\det f_*h$ *on* $\det f_*\mathcal{L}$ *is a singular metric in the sense of Definition 6.1. Moreover*

$$\operatorname{sing}(\det f_*h) = \frac{1}{2}f_*R.$$

Proof. Outside the support of f_*R the map f is smooth. Therefore $\det f_*h$ is a smooth metric on an open Zariski subset. Let us now look at the ramification points. Let us consider the map

$$\begin{array}{rccc} f: & \mathbb{C} & \longrightarrow & \mathbb{C}\\ & x & \longmapsto & z = x^r.\end{array}$$

Let $\mathcal{O}_{\mathbb{C}}$ be the trivial bundle on $\mathbb{C}$. A metric $\|\cdot\|$ on $\mathcal{O}_{\mathbb{C}}$ is determined by the function

$$h(x) = \|1\|_x^2.$$

A basis of the vector bundle $f_*\mathcal{O}_{\mathbb{C}}$ is given by the sections $1, x, \dots, x^{r-1}$. Then for a point $p \in \mathbb{C}$, $p \neq 0$, we have

$$\left\langle x^i, x^j \right\rangle_p = \sum_{q^r=p} \left\langle x^i, x^j \right\rangle_q = \sum_{q^r=p} \overline{q}^i q^j h(q).$$

Let us write $a(p)_{i,j} = \left\langle x^{i-1}, x^{j-1} \right\rangle_p$. Then

$$\|1 \wedge x \wedge \cdots \wedge x^{r-1}\|_p^2 = \det(a(p)_{i,j}).$$

Let us write $I = \{1, \dots, r\}^r$. If $\tau \in I$ we shall write $\tau = (\tau(1), \dots, \tau(r))$. Given a point $p \neq 0$, let $\{q_1, \dots, q_r\}$ be the set of r-th roots of p. By multilinearity

$$\det(a(p)_{i,j}) = \sum_{\tau \in I} \det\left(h(q_{\tau(i)})\overline{q}_{\tau(i)}^{i-1} q_{\tau(i)}^{j-1}\right).$$

If $\tau(i) = \tau(i')$ then the i-th file of the matix $\left(h(q_{\tau(i)})\overline{q}_{\tau(i)}^{i-1} q_{\tau(i)}^{j-1}\right)$ is equal to the i'-th file times $\overline{q}_{\tau(i)}^{i-i'}$. Therefore the corresponding determinant is zero. Hence, if $\mathfrak{S}_r$ is the symmetric group of r elements, we have

$$\begin{aligned}
\det(a(p)_{i,j}) &= \sum_{\sigma \in \mathfrak{S}_r} \det(h(q_{\sigma(i)})\overline{q}_{\sigma(i)}^{i-1} q_{\sigma(i)}^{j-1}) \\
&= h(q_1)\dots h(q_r) \sum_{\sigma \in \mathfrak{S}_r} \det(\overline{q}_{\sigma(i)}^{i-1} q_{\sigma(i)}^{j-1}) \\
&= h(q_1)\dots h(q_r) \sum_{\sigma \in \mathfrak{S}_r} \overline{q}_{\sigma(2)}^{1} \dots \overline{q}_{\sigma(r)}^{r-1} \det(q_{\sigma(i)}^{j-1}) \\
&= h(q_1)\dots h(q_r) \det(q_i^{j-1}) \sum_{\sigma \in \mathfrak{S}_r} (-1)^{\sigma} \overline{q}_{\sigma(2)}^{1} \dots \overline{q}_{\sigma(r)}^{r-1} \\
&= h(q_1)\dots h(q_r) \det(q_i^{j-1}) \det(\overline{q}_i^{j-1}) \\
&= h(q_1)\dots h(q_r) \|\det(q_i^{j-1})\|^2.
\end{aligned}$$

Finally observe that $\det(q_i^{j-1})^2$ is the discriminant $D(1, \dots, x^{r-1})$ of the extension given by the polynomial $X^r - p$. Therefore (see [Sa, 2.7])

$$\|\det(q_i^{j-1})\|^2 = r^r (p\overline{p})^{\frac{r-1}{2}}$$

and

$$\|1 \wedge \cdots \wedge x^{r-1}\|_p = \det(a(p)_{i,j}) = r^r \|p\|^{r-1} h(q_1)\dots h(q_r). \tag{6.4}$$

Other way of making the same calculation is to observe that, if $\|\cdot\|'$ is another metric given by h' then

$$\|1 \wedge \cdots \wedge x^{r-1}\|_p' = \|1 \wedge \cdots \wedge x^{r-1}\|_p \frac{h(q_1)\dots h(q_r)}{h'(q_1)\dots h'(q_r)}$$

and then to do the computations with $h=1$.

Now it is clear that $k(p)=r^r h(q_1)\dots h(q_r)$ is smooth for $p\neq 0$ and continuous for $p=0$. Moreover in a compact neighbourhood of 0,

$$\begin{aligned}\left\|\frac{\partial k}{\partial z}(p)\right\| &= r^r\sum_{i=1}^{r}\prod_{j\neq i}\|h(q_j)\|\left\|\frac{\partial h}{\partial x}(q_i)\right\|\left\|\frac{\partial x}{\partial z}(p)\right\|\\ &\leq C'\left\|\frac{1}{p^{1-\frac{1}{r}}}\right\|.\end{aligned}$$

The bound for $\dfrac{\partial^2 k}{\partial\bar z\partial z}$ is proved in the same way.

It is clear from formula 6.4 that in this case

$$\operatorname{sing}(\det f_*h)=\frac{r-1}{2}[0]=\frac{1}{2}f_*R.$$

If $f: C\longrightarrow C'$ is a morphism of curves and $p\in C'$, then there is a neighbourhood U of p in C', biholomorphic to a disk, such that $f^{-1}(U)=V_1\coprod\dots\coprod V_l$ is a disjoint union of disks and in each V_i, $f|_{V_i}$ has ramification index r_i. Choosing a basis $\{s_{i,j}, i=1\dots l, j=1,\dots,r_i\}$ of $\Gamma(U,f_*\mathcal{L})$ such that $s_{i,j}$ vanishes on V_m for $m\neq i$, we have

$$\|s_{11}\wedge\dots\wedge s_{lr_l}\|=\prod_{i=1}^{l}\|s_{i1}\wedge\dots\wedge s_{ir_i}\|.$$

Therefore the proposition follows from the above computations.

Let TC denote the tangent bundle of C. There is a morphism

$$df: TC\longrightarrow f^*TC'.$$

Let h be a metric on TC' and let us denote by df^*h the metric on TC given by

$$\langle s,t\rangle=\langle df s, df t\rangle.$$

Proposition 6.5. *The metric df^*h on TC is a singular metric in the sense of Definition 6.1 and*

$$\operatorname{sing} df^*h=R,$$

where R is the ramification divisor of f.

Proof. Let ω be the differential form associated to the metric h. Then the metric df^*h has the associated form $f^*\omega$. Taking local coordinates we may assume that f is the morphism

$$\begin{array}{rccc} f: & \mathbb{C} & \longrightarrow & \mathbb{C}\\ & x & \longmapsto & z=x^r.\end{array}$$

In these local coordinates we have $\omega=h(z)dz\wedge d\bar z$ and

$$f^*\omega=h(x^r)dx^r\wedge d\bar x^r=h(x^r)(x\bar x)^{r-1}r^2dx\wedge d\bar x.$$

Hence we have the result.

Let C be a smooth complex projective curve and $\mathcal{L}$ a line bundle on C. Let h_1 and h_2 be two singular metrics on $\mathcal{L}$. Let us write $\overline{\mathcal{E}}$ for the sequence

$$\overline{\mathcal{E}}:\quad 0\to 0\to(\mathcal{L},h_1)\to(\mathcal{L},h_2)\to 0.$$

Definition 6.6. Let s be a rational section of $\mathcal{L}$ which is regular and non vanishing over a neighbourhood of $\operatorname{sing} h_1 \cup \operatorname{sing} h_2$. Then we write

$$\widetilde{c}_1(\overline{\mathcal{E}}) = g(\mathcal{L}, h_1, s) - g(\mathcal{L}, h_2, s).$$

Note that $\widetilde{c}_1(\overline{\mathcal{E}})$ does not depend on the choice of s. Moreover it is a Green current for the cycle 0. Therefore it is of the form $a(\gamma)$ for γ a current on C. If h_1 and h_2 are smooth metrics, then $\widetilde{c}_1(\overline{\mathcal{E}})$ is compatible with the definition given in [G-S 2].

Let X be a regular projective arithmetic surface over $\mathbb{Z}$.

Definition 6.7. A *singular metrized line bundle over* X is a pair $(\mathcal{L}, h)$, where $\mathcal{L}$ is a line bundle on X and h is a singular metric on $\mathcal{L}_\infty$ invariant under the action of F_∞. If $\overline{\mathcal{L}} = (\mathcal{L}, h)$ is a singular metrized line bundle then *the arithmetic first Chern class* of $\overline{\mathcal{L}}$ is

$$\widehat{c}_1(\overline{\mathcal{L}}) = (\operatorname{div} s, g(\mathcal{L}, h, s)) \in \widehat{\mathrm{CH}}_1(X),$$

for s any rational section of $\mathcal{L}$. This definition is independent of the choice of s. Note that if h is a smooth metric, this class is the image of the arithmetic first Chern class defined in [G-S 3].

Let $f : X \longrightarrow Y$ be a finite morphism between arithmetic surfaces. Let us choose a factorization of f

$$X \underset{i}{\longrightarrow} \mathbb{P}(E) \underset{p}{\longrightarrow} Y,$$

where i is a closed embedding and p is a projective bundle over Y. Let NX be the normal bundle of X in $\mathbb{P}(E)$ and i^*Tp the restriction to X of the relative tangent bundle to the projection p. Let us choose smooth hermitian metrics on $i^*Tp_{\mathbb{C}}$, $NX_{\mathbb{C}}$, $i^*T\mathbb{P}(E)_{\mathbb{C}}$, $TX_{\mathbb{C}}$ and $TY_{\mathbb{C}}$ invariant under the action of F_∞. These metrics will be denoted respectively by h_1, h_2, h_3, h_4 and h_5. Let us denote by $\overline{\mathcal{E}}_1$, $\overline{\mathcal{E}}_2$ and $\overline{\mathcal{E}}_3$ the metrized exact sequences

$$\begin{aligned}
\overline{\mathcal{E}}_1 : &\quad 0 \to (TX_{\mathbb{C}}, h_4) \to (i^*T\mathbb{P}(E)_{\mathbb{C}}, h_3) \to (NX_{\mathbb{C}}, h_2) \to 0,\\
\overline{\mathcal{E}}_2 : &\quad 0 \to (i^*Tp_{\mathbb{C}}, h_1) \to (i^*T\mathbb{P}(E)_{\mathbb{C}}, h_3) \to (f^*TY_{\mathbb{C}}, f^*h_5) \to 0 \qquad \text{and}\\
\overline{\mathcal{E}}_3 : &\quad 0 \to 0 \to (TX_{\mathbb{C}}, h_4) \to (TX_{\mathbb{C}}, df^*h_5) \to 0.
\end{aligned}$$

Let us denote by $\widetilde{c}_1(\overline{\mathcal{E}}_1)$ and $\widetilde{c}_1(\overline{\mathcal{E}}_2)$ the image in $\widehat{\mathrm{CH}}_1(X)$ of the classes with the same name defined in [G-S 3]. In other words, let

$$\overline{\mathcal{E}} : \qquad 0 \to A \to B \to C \to 0$$

be an exact sequence of vector bundles with smooth metrics. There is an isomorphism $\varphi_{\mathcal{E}} : \det A \otimes \det C \longrightarrow \det B$. Let s_1 and s_3 be sections of $\det A$ and $\det B$ and let $s_2 = \varphi_{\mathcal{E}}(s_1 \otimes s_3)$. Let us denote by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$ the metrics induced on $\det A$, $\det B$ and $\det C$. Then we have

$$\widetilde{c}_1(\overline{\mathcal{E}}) = (0, [d_{\mathfrak{D}} - \log \frac{\|s_2\|_2}{\|s_1\|_1 \|s_3\|_3}], [-\log \frac{\|s_2\|_2}{\|s_1\|_1 \|s_3\|_3}]) \in \widehat{\mathrm{CH}}_1(X).$$

Let us denote by $\widetilde{c}_1(\overline{\mathcal{E}}_3)$ the image in $\widehat{\mathrm{CH}}_1(X)$ of the class defined in 6.6.

Definition 6.8. The *arithmetic first Chern class of the relative tangent sheaf of f* is

$$\widehat{c}_1(Tf) = \widehat{c}_1(i^*Tp) - \widehat{c}_1(NX) - \widetilde{c}_1(\overline{\mathcal{E}}_1) + \widetilde{c}_1(\overline{\mathcal{E}}_2) - \widetilde{c}_1(\overline{\mathcal{E}}_3).$$

The natural morphism $i^*Tp \longrightarrow NX$ induces a morphism $\det i^*Tp \longrightarrow \det NX$ which is an isomorphism in an open Zariski set. It can be viewed as a rational section s of the line bundle $\det NX \otimes (\det i^*Tp)^\vee$, where $(\det i^*Tp)^\vee$ is the dual of $\det i^*Tp$. Let us denote by $R = \operatorname{div} s$. This is the ramification divisor of f and is independent of the factorization chosen for f.

Theorem 6.9. *(Riemann-Hurwitz formula) We have the equality*

$$\widehat{c}_1(Tf) = \theta(-R) \in \widehat{CH}_1(X).$$

In particular $\widehat{c}_1(Tf)$ does not depend on the choice of metrics.

Proof. Let us choose a rational section s_1 of $\det i^*Tp$ and let s_2 be the image of s_1 under the natural morphism $\det i^*Tp \longrightarrow \det NX$. We shall denote also by s_1 and s_2 the corresponding sections of $\det i^*Tp_{\mathbb{C}}$ and $\det NX_{\mathbb{C}}$. Let us choose a section s_3 of $\det T\mathbb{P}(E)_{\mathbb{C}}$ and let s_4 be the section of $TX_{\mathbb{C}}$ determined by $s_3 \otimes (s_2)^{-1}$ and the isomorphism

$$TX_{\mathbb{C}} \longrightarrow \det i^*T\mathbb{P}(E)_{\mathbb{C}} \otimes (\det NX_{\mathbb{C}})^\vee.$$

Finally let s_5 be the section of $f^*TY_{\mathbb{C}}$ determined by $s_3 \otimes (s_1)^{-1}$ and the isomorphism

$$f^*TY_{\mathbb{C}} \longrightarrow \det i^*T\mathbb{P}(E)_{\mathbb{C}} \otimes (\det i^*Tp_{\mathbb{C}})^\vee.$$

Then we have $df s_4 = s_5$.

Using these sections to compute $\widehat{c}_1(Tf)$ one obtains

$$\begin{aligned}
\widehat{c}_1(Tf) &= (\operatorname{div} s_1, [d_{\mathfrak{D}} - \log \|s_1\|_1], [-\log \|s_1\|_1]) \\
&\quad - (\operatorname{div} s_2, [d_{\mathfrak{D}} - \log \|s_2\|_2], [-\log \|s_2\|_2]) \\
&\quad - (0, [d_{\mathfrak{D}} - \log \frac{\|s_3\|_3}{\|s_2\|^2\|s_4\|_4}], [-\log \frac{\|s_3\|_3}{\|s_2\|_2\|s_4\|_4}]) \\
&\quad + (0, [d_{\mathfrak{D}} - \log \frac{\|s_3\|_3}{\|s_1\|_1\|s_5\|_5}], [-\log \frac{\|s_3\|_3}{\|s_1\|_1\|s_5\|_5}]) \\
&\quad - (0, [d_{\mathfrak{D}} - \log \frac{\|s_4\|_4}{\|df s_4\|_5}] + \delta_R, [-\log \frac{\|s_4\|_4}{\|df s_4\|_5}]) \\
&= (-R, -\delta_R, 0) \\
&= \theta(-R).
\end{aligned}$$

Let $f : X \longrightarrow Y$ be a finite morphism between arithmetic surfaces. Let $\overline{\mathcal{L}}$ be a line bundle on X provided with a smooth metric h. Let us denote by $\lambda(\overline{\mathcal{L}})$ the line bundle $\det f_*\mathcal{L}$ provided with the singular metric $\det f_*h$. Then we have the following Riemann-Roch theorem:

Theorem 6.10. *In $\widehat{CH}_1(Y)_{\mathbb{Q}}$ it holds the equality*

$$\widehat{c}_1(\lambda(\overline{\mathcal{L}})) = f_*\left(\widehat{c}_1(\overline{\mathcal{L}}) + \frac{1}{2}\widehat{c}_1(Tf)\right).$$

Proof. Let us choose a rational section s_1 of $\mathcal{L}$ and a rational section s_2 of $\det f_*\mathcal{L}$. Assume that both sections are regular and non vanishing on the ramification points. Let us denote by $\|\cdot\|_1$ the metric on $\mathcal{L}$ and by $\|\cdot\|_2$ the metric on $\det f_*\mathcal{L}$.

Let $Z_{\mathbb{Q}}$ be the discriminant locus of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$, let $Z = \overline{Z}_{\mathbb{Q}}$ be the adherence of $Z_{\mathbb{Q}}$. Then the morphism

$$f : X - f^{-1}(Z) \longrightarrow Y - Z$$

satisfies the hypothesis of the arithmetic Riemann-Roch theorem of [G-S 4]. Therefore there exists a rational function $\varphi \in K(Y)$ and a rational number α such that

$$\left(\operatorname{div} s_2 - f_* \operatorname{div} s_1 + \frac{1}{2} f_* R\right)|_{Y-Z} = \alpha \operatorname{div} \varphi|_{Y-Z} \qquad \text{and} \tag{6.11}$$

$$(\log \|s_2\|_2^2 - f_* \log \|s_1\|_1^2)|_{Y_{\mathbb{C}}-Z_{\mathbb{C}}} = \alpha \log \varphi\overline{\varphi}|_{Y_{\mathbb{C}}-Z_{\mathbb{C}}}. \tag{6.12}$$

We have the direct sum decomposition

$$Z_1(Y) = Z_1(Y)_{\mathrm{fin}} \oplus Z_1(Y_{\mathbb{Q}}),$$

where

$$Z_1(Y)_{\mathrm{fin}} = \{z \in Z_1(Y) \mid z \cap Y_{\mathbb{Q}} = \emptyset\}.$$

Moreover

$$Z_1(Y)_{\mathrm{fin}} = Z_1(Y - Z)_{\mathrm{fin}}.$$

Therefore to prove the equation 6.11 in the whole Y it is enough to prove it in $Y_{\mathbb{Q}}$. But $\operatorname{div}_{\mathbb{Q}} s_1 = \operatorname{div} s_1|_{Y_{\mathbb{Q}}}$ is determined by the singularities of $\log \|s_1\|_1^2$ and the same is true for $\operatorname{div}_{\mathbb{Q}} s_2$ and $\operatorname{div}_{\mathbb{Q}} \varphi$. Namely we have

(6.13)

$$\begin{aligned}
\delta_{\operatorname{div}_{\mathbb{Q}} s_1} &= [\partial\overline{\partial} \log \|s_1\|_1] - \partial\overline{\partial}[\log \|s_1\|_1],\\
\delta_{\operatorname{div}_{\mathbb{Q}} s_2} &= [\partial\overline{\partial} \log \|s_2\|_2] - \partial\overline{\partial}[\log \|s_2\|_2] - \frac{1}{2}\delta_{f_* R_{\mathbb{Q}}}, \qquad \text{by Proposition 6.3 and}\\
\delta_{\operatorname{div}_{\mathbb{Q}} \varphi} &= [\partial\overline{\partial} \log \|\varphi\|] - \partial\overline{\partial}[\log \|\varphi\|].
\end{aligned}$$

Hence, by 6.12,

$$\delta_{\operatorname{div}_{\mathbb{Q}} s_2} - f_*\delta_{\operatorname{div}_{\mathbb{Q}} s_1} + \frac{1}{2} f_*\delta_{R_{\mathbb{Q}}} = \alpha\delta_{\operatorname{div}_{\mathbb{Q}} \varphi}.$$

Which implies that equation 6.11 holds in the whole Y.

By Proposition 6.3 we have

$$\widehat{c}_1(\lambda(\overline{\mathcal{L}})) = (\operatorname{div} s_2, [\partial\overline{\partial} \log \|s_2\|_2^2] - \frac{1}{2} f_*\delta_R, [\frac{-1}{2} \log \|s_2\|_2^2]).$$

By Theorem 6.9

$$\widehat{c}_1(\overline{\mathcal{L}}) + \frac{1}{2}\widehat{c}_1(Tf) = (\operatorname{div} s_1 - \frac{1}{2}R, [\partial\overline{\partial} \log \|s_1\|_1^2] - \frac{1}{2}\delta_R, [\frac{-1}{2} \log \|s_1\|_1^2]).$$

Moreover

$$\widehat{\operatorname{div}}\,\varphi = (\operatorname{div} \varphi, 0, [\frac{-1}{2} \log \|\varphi\|^2]).$$

Therefore, by 6.11, 6.12 and 6.13 we have

$$\widehat{c}_1(\lambda(\overline{\mathcal{L}})) = f_*\left(\widehat{c}_1(\overline{\mathcal{L}}) + \frac{1}{2}\widehat{c}_1(Tf)\right) + \alpha\,\widehat{\operatorname{div}}\,\varphi.$$

This concludes the proof of the Theorem.

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