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Hermitian vector bundles and characteristic classes

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Abstract. This paper reviews the realization of Beilinson's regulator map using hermitian differential geometry. This construction is a generalization of Chern-Weil theory of characteristic classes of vector bundles, to higher K-theory.

1. Introduction

Chern classes of complex vector bundles can be constructed using different techniques. For instance, in algebraic topology, one can use the Euler class of an oriented sphere bundle to define the top Chern class and then define the other classes inductively (see [**25**]). Or, in differential geometry, we can use Chern-Weil theory, which produces Chern classes in de Rham cohomology by means of the curvature of a connection (see for example [**5**], [**34**] or [**16**]). Another aproach, due to Grothendieck [**17**], starts with the first Chern class of a line bundle, and then uses an explicit formula for the cohomology of a projective bundle to define higher Chern classes. This approach is very useful in algebraic geometry and can be used to produce Chern classes in the Chow ring, in étale cohomology or in Deligne-Beilinson cohomology.

The Grothendieck method has been generalized by Gillet [**11**] to produce characteristic classes from higher K-theory to any arbitrary cohomology satisfying certain axioms. In the particular case when the characteristic class is the Chern character and the cohomology theory is real Deligne-Beilinson cohomology, the map obtained is Beilinson's regulator map [**2**]. This map is a generalization of Borel's regulator and it is involved in very deep conjectures in Arithmetic Geometry.

Beilinson's regulator is still a very mysterious map, in part because higher K-theory is a rich and complex world. Thus it is useful to have as many approaches to Beilinson's regulator as possible. Gillet and Soulé [**13**] have given a description of Beilinson's regulator for K_1 using Bott-Chern forms. This description can be seen as a generalization of Chern-Weil theory to K_1. In the paper [**8**], S. Wang and the author have extended this description of Beilinson's regulator to higher K-theory.

The aim of this paper is to review the construction of characteristic classes for higher K-theory. This paper is meant to be introductory, and so the focus is placed

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on the basic ideas of the theory. But references are given where the reader can find detailed proofs and general statements.

The main object of study will be algebraic vector bundles over smooth complex algebraic varieties. These objects can be studied from the point of view of complex geometry or from algebraic geometry. Since our objective is to give a complex geometric construction of an object defined in the algebraic geometry setting we will shift from one point of view to the other in different sections.

The plan of the study is as follows. In section 2, Grothendieck's construction of Chern classes is presented in the particular case of sheaf cohomology with integer coefficients. In section 3, we recall the Chern-Weil theory of characteristic classes of hermitian vector bundles. Section 2 is devoted to a simple version of Gillet's construction of characteristic classes for higher K-theory. In section 5, we review real Deligne cohomology. Bott-Chern forms are the topic of section 6. In section 7, we introduce the complex of exact cubes. By a result of R. McCarthy [**24**] the rational homology of this complex is isomorphic to rational K-theory. Finally, section 8 is devoted to the definition of characteristic classes for higher K-theory using exact cubes of hermitian vector bundles. For simplicity, we only discuss the case of projective varieties. But note that an important ingredient in the comparison between this construction and Gillet's construction, is the extension of this theory to quasi-projective varieties.

2. Chern classes of vector bundles

There are many different constructions of Chern classes of vector bundles (see for example [**25**], [**34**] or [**18**]). In this section we will review a very general one due to Grothendieck [**17**]. In this construction, the properties of the cohomology theory that are needed to define Chern classes are given as axioms for a cohomology. These axioms are satisfied by many theories, for instance, Chow rings of algebraic varieties. Moreover Grothendieck's construction is the basis of Gillet's construction of Chern classes for higher algebraic K-theory [**11**].

In this section, we will specialize the construction of Chern classes to the case of sheaf cohomology of smooth complex varieties with integer coefficients. We will use the classical topology. To stress this point we will work with holomorphic vector bundles. In section 4 we will discuss the axiomatic approach in the algebraic geometry context.

Let us introduce the first Chern class of a holomorphic line bundle. This will act as a normalization for the Chern classes. Let X be a complex manifold, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let $\mathcal{O}_X^*$ be the sheaf of invertible holomorphic functions. Then there is an isomorphism

$$\left\{\begin{matrix}\text{Isomorphism classes}\\ \text{of holomorphic}\\ \text{line bundles}\end{matrix}\right\} \longleftrightarrow H^1(X,\mathcal{O}_X^*).$$

The exponential sequence

$$0 \longrightarrow (2\pi i)\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

gives us a long exact sequence in cohomology

$$H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}(1)) \longrightarrow H^2(X,\mathcal{O}_X),$$

where we have written $\mathbb{Z}(1)$ for $(2\pi i)\mathbb{Z} \subset \mathbb{C}$.

DEFINITION 2.1. The *first Chern class* of a line bundle $\mathcal{L}$, denoted $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}(1))$, is the image of its class in $H^1(X, \mathcal{O}_X^*)$ by the connection morphism δ.

Once we have defined the first Chern class, the main tool to define higher Chern classes will be the Dold-Thom isomorphism. Let E be a rank n holomorphic vector bundle over X. Let $\mathbb{P}(E)$ be the associated projective bundle and let us denote by $p : \mathbb{P}(E) \longrightarrow X$ the projection. Then the vector bundle p^*E has a tautological subbundle, whose fibre at each point is the line determined by that point. Let us denote by $\mathcal{O}_{\mathbb{P}(E)}(-1)$ this line bundle and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the dual line bundle. Let ξ be the first Chern class of $\mathcal{O}_{\mathbb{P}(E)}(1)$. Let us denote by $\mathbb{Z}(p)$ the constant sheaf $(2\pi i)^p\mathbb{Z} \subset \mathbb{C}$.

THEOREM 2.2 (Dold-Thom isomorphism). *For each pair of integers i, m, the map*

$$\sum_{k=0}^{n-1} p^*(\cdot) \cup \xi^k : \bigoplus_{k=0}^{n-1} H^{m-2k}(X, \mathbb{Z}(i-k)) \longrightarrow H^m(\mathbb{P}(E), \mathbb{Z}(i))$$

is an isomorphism.

PROOF. When X is one point, the result is the classical formula for the cohomology of the projective space. The general case follows from the fact that the existence of the global classes ξ^k, implies the triviality of the Leray spectral sequence of the morphism $p : \mathbb{P}(E) \longrightarrow X$. □

This theorem allows us to define Chern classes in the following way.

DEFINITION 2.3. The Chern classes of the vector bundle E are the classes $c_i(E) \in H^{2i}(X, \mathbb{Z}(i))$ determined by the equation

$$p^*(c_n(E)) + p^*(c_{n-1}(E)) \cup \xi + \cdots + p^*(c_1(E)) \cup \xi^{n-1} + \xi^n = 0. \tag{2.1}$$

The total Chern class is the sum

$$c(E) = 1 + c_1(E) + \cdots + c_n(E).$$

The Chern classes are characterized by the first Chern class and the behaviour under inverse images and exact sequences. This property is very useful, for instance, when comparing different definitions of Chern classes.

THEOREM 2.4. *There exists a unique way to assign, to each holomorphic vector bundle E, a total Chern class $c(E)$ satisfying the following properties:*

1. *Normalization: If $\mathcal{L}$ is a line bundle then $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$, where $c_1(\mathcal{L})$ is defined in 2.1.*
2. *Functoriality: For any morphism of complex varieties $f : X \longrightarrow Y$ we have $c(f^*E) = f^*c(E)$.*
3. *Whitney sum formula: For any exact sequence of vector bundles*

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0 \tag{2.2}$$

we have $c(E) = c(S) \cup c(Q)$.

SKETCH OF PROOF. Let us start by proving the uniqueness. Assume that there exists a theory of Chern classes satisfying conditions 1, 2 and 3. Let E be a vector bundle over a smooth complex manifold X. Let us write $Q = p^*E/\mathcal{O}_{\mathbb{P}(E)}(-1)$.

Then Q is a rank $n-1$ vector bundle over $\mathbb{P}(E)$. The basic idea is to use the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow p^*E \longrightarrow Q \longrightarrow 0.$$

First, using induction and properties 1 and 3 one easily proves that $c_i(E) = 0$ for $i > n$. Then, since the first Chern class of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is $-\xi$, properties 1, 2 and 3 imply that

$$p^*c(E) = (1-\xi)c(Q).$$

Since $c_n(Q) = 0$ and $(1-\xi)^{-1} = 1 + \xi + \xi^2 + \xi^3 + \dots$, the Chern classes should satisfy the formula (2.1). By the Dold-Thom isomorphism this equation determines the Chern classes.

To prove the existence we need to show that Chern classes, as defined in 2.3 satisfy the conditions 1, 2 and 3. The first condition follows from the definition. The second one is easy. For the third one we need to use the existence and some properties of the Gysin morphism (see [**17**] for more details). □

3. Chern classes of hermitian vector bundles

In this section we will explain the Chern-Weil construction of Chern classes using complex differential geometry. This theory is explained in many places (see for instance [**5**], [**34**] or [**16**]) and we refer the reader to them for details.

Let B be a subring of $\mathbb{R}$. We will write $B(p) = (2\pi i)^p B$. The symmetric group on n elements, $\mathfrak{S}_n$, acts on $B[T_1, \dots, T_n]$ by permuting the variables. Let $IP(n)$ be the subset of invariant polynomials. Then $IP(n) = B[\sigma_1, \dots, \sigma_n]$, where σ_i is the degree i symmetric elementary function in n variables. An analogous result is true if we replace polynomials by formal power series. We will denote by $\widehat{IP}(n)$ the set of invariant power series in n variables and by $IP(n)_k$ the space of invariant homogeneous polynomials of degree k.

Let be $\mathfrak{M}_n$ the vector space of $n \times n$ complex matrices. Let $\varphi : \mathfrak{M}_n \longrightarrow \mathbb{C}$ be a map such that $\varphi(A)$ is a homogeneous polynomial of degree k in the entries of A, with coefficients in B. We say that φ is invariant if, for all $A \in \mathfrak{M}_n$ and $g \in GL_n(\mathbb{C})$ we have

$$\varphi(A) = \varphi(gAg^{-1}).$$

Let us denote by $I_k(\mathfrak{M}_n)$ the space of invariant homogeneous polynomials of degree k. There is an isomorphism $I_k(\mathfrak{M}_n) \longrightarrow IP(n)_k$ which sends any φ to its value in the diagonal matrix with entries $T_1, \dots, T_n$. Due to this isomorphism we will identify both spaces.

Let X be a complex manifold. Let $\mathcal{E}^*$ denote the sheaf of complex smooth differential forms, $\mathcal{E}^{p,q}$ the sheaf of (p,q)-forms, $\mathcal{E}^*_{\mathbb{R}}$ the subcomplex of real forms and $\mathcal{E}^*_{\mathbb{R}}(p) = (2\pi i)^p \mathcal{E}^*_{\mathbb{R}}$. Let $\mathcal{E}^*(X)$, $\mathcal{E}^{p,q}(X)$, $\mathcal{E}^*_{\mathbb{R}}(X)$ and $\mathcal{E}^*_{\mathbb{R}}(p)(X)$ denote the corresponding groups of global sections. Let E be a holomorphic vector bundle on X. Then we will denote by $\mathcal{E}^*(E)$ the sheaf of E-valued smooth differential forms and by $\mathcal{E}^*(X,E)$ the corresponding space of global sections. $\mathcal{E}^{p,q}(X,E)$ will denote the space of forms of type (p,q) with values in E.

A connection on E is a $\mathbb{C}$-linear map

$$D : \mathcal{E}^0(X,E) \longrightarrow \mathcal{E}^1(X,E)$$

satisfying the Leibnitz rule

$$D(\phi\xi) = d\phi\xi + \phi D\xi, \tag{3.1}$$

for any $\phi \in \mathcal{E}^0(X)$ and $\xi \in \mathcal{E}^0(X, E)$.

Given any connection D we can extend it to obtain operators

$$D : \mathcal{E}^k(X, E) \longrightarrow \mathcal{E}^{k+1}(X, E)$$

by imposing the Leibnitz rule. But in general $(\mathcal{E}^*(X, E), D)$ is not a complex, because the operator

$$K = D^2 : \mathcal{E}^0(X, E) \longrightarrow \mathcal{E}^2(X, E)$$

may be non zero. In fact the operator K is $\mathcal{E}^0(X)$-linear and thus it can be seen as a section $K \in \mathcal{E}^2(X, \operatorname{End}(E))$. This operator is called the curvature of the connection.

Due to the decomposition $\mathcal{E}^1(X, E) = \mathcal{E}^{0,1}(X, E) \oplus \mathcal{E}^{1,0}(X, E)$ we can decompose $D = D^{0,1} + D^{1,0}$, with

$$\begin{aligned} D^{0,1} &: \mathcal{E}^0(X, E) \longrightarrow \mathcal{E}^{0,1}(X, E) \\ D^{1,0} &: \mathcal{E}^0(X, E) \longrightarrow \mathcal{E}^{1,0}(X, E). \end{aligned}$$

Since E is holomorphic, there is a well defined $\overline{\partial}$-operator, which in a local frame is given by

$$\overline{\partial}(f_1, \dots, f_n) = (\overline{\partial} f_1, \dots, \overline{\partial} f_n).$$

Let us assume that E is provided with a hermitian metric h. Let us denote by $\langle \cdot, \cdot \rangle$ the corresponding inner product. Then there is a unique connection that is compatible with both the complex structure and the hermitian metric. That is, there is a unique connection $D = D(h)$ satisfying the following conditions:

1. "Compatibility with the hermitian metric": For any $\xi, \eta \in \mathcal{E}^0(X, E)$

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle .$$

2. "Compatibility with the complex structure": $D^{0,1} = \overline{\partial}$.

Let us write $K = K(h) = D^2$, the curvature of the metric h. Let f be a local frame, if we denote by $h(f)$ the matrix of the metric h in this frame, then (see [**34**] pag. 82).

$$K = \overline{\partial}(h^{-1}(f)\partial h(f)). \tag{3.2}$$

From this equation it is clear that $K \in \mathcal{E}^{1,1}(X, \operatorname{End}(E))$. Thus in a local frame it is given by a matrix of $(1,1)$ forms.

Let $\varphi \in \widehat{IP}(n)$ and let us denote by φ_k its component of degree k. The invariance of φ_k implies that we have a well defined element $\widetilde{\varphi}_k(E, h) = \varphi_k(-K) \in \mathcal{E}^{2k}(X)$. To see this, one first defines $\varphi_k(-K)$ in a local frame, where K is a matrix of $(1,1)$ forms and then one uses the invariance to glue together these local definitions (see for instance [**34**] III.3). Since $I = \bigoplus_{k \geq 1} \mathcal{E}^k(X)$ is a nilpotent ideal, we have also a well defined form $\widetilde{\varphi}(E, h) = \bigoplus \widetilde{\varphi}_k(E, h)$.

THEOREM 3.1. *Let $E \longrightarrow X$ be a holomorphic vector bundle and let h be a hermitian metric. Then*

1. *The form $\widetilde{\varphi}(E, h)$ is closed.*

2. *The form $\widetilde{\varphi}(E,h)$ satisfies*
$$\widetilde{\varphi}(E,h)\in\bigoplus_{p\geq 0}\mathcal{E}^{2p}_{\mathbb{R}}(p)(X).$$
3. *The class of $\widetilde{\varphi}(E,h)$ in de Rham cohomology is independent of the metric h.*

Since any holomorphic vector bundle admits a hermitian metric the above theorem allows us to use hermitian metrics to define characteristic classes.

Definition 3.2. Let X be a complex manifold. Let E be a rank n holomorphic vector bundle and let $\varphi\in\mathbb{R}[[T_1,\dots,T_n]]$ be an invariant power series. Let us choose any hermitian metric h on E. Then the cohomology class of $\widetilde{\varphi}(E,h)$, denoted
$$\varphi^{dR}(E)\in\bigoplus_p H^{2p}_{dR}(X,\mathbb{R}(p))$$
will be called the de Rham Chern class of E associated to the power series φ. The differential form $\widetilde{\varphi}(E,h)$ will be called the Chern form.

Remark 3.3. This distinction between Chern classes and de Rham Chern classes is provisional. We will distinguish between them until we see that they agree.

Examples 3.4.

1. If $\varphi=\sigma_i$, the i-th elementary symmetric function then $c_i^{dR}(E)=\varphi^{dR}(E)$ is the i-th de Rham Chern class of E (see the remark above).
2. If $\varphi=1+\sum\sigma_i$ then $c^{dR}(E)=\varphi^{dR}(E)$ is called the total de Rham Chern class. Observe that the total Chern form is given by
$$\widetilde{c}(E,h)=\det(1-K).$$
3. If B contains the field $\mathbb{Q}$, then the Chern character is defined by the power series
$$\operatorname{ch}(T_1,\dots,T_n)=\sum_{i=1}^{n}\exp(T_i).$$

The natural inclusion $\mathbb{Z}(p)\longrightarrow\mathcal{E}^0_{\mathbb{R}}(p)$ induces a morphism $\psi:H^*(X,\mathbb{Z}(p))\longrightarrow H^*_{dR}(X,\mathbb{R}(p))$. In order to see that the two definitions of Chern classes are compatible we have to compare $\psi(c(E))$ with $c^{dR}(E)$. The key to compare both definitions of Chern classes is the theorem 2.4. Thus we only need to show that the de Rham Chern classes also satisfy the properties given in theorem 2.4.

Theorem 3.5.

1. *Let $f:X\longrightarrow Y$ be a morphism of complex manifolds. Let (E,h) be a hermitian vector bundle on Y. Then*
$$f^*\widetilde{c}(E,h)=\widetilde{c}(fE,f^*h).$$
2. *Let $\mathcal{L}$ be a holomorphic line bundle. Then*
$$c_1^{dR}(\mathcal{L})=\psi\left(c_1(\mathcal{L})\right).$$
3. *Let*
$$0\longrightarrow S\longrightarrow E\longrightarrow Q\longrightarrow 0$$
be an exact sequence of holomorphic vector bundles on X. Then
$$c(E)=c(S)\wedge c(Q).$$

PROOF. The proof of properties 1 and 3 are standard and can be found in any reference for Chern-Weil theory. For instance, see [**34**]. Since property 2 is more specific of the comparison we are making, we will provide a proof. The idea is to represent $c_1(\mathcal{L})$ as a Čech cocycle and $c_1^{dR}(\mathcal{L})$ as a differential form. So we will use the comparison between Čech and de Rham cohomologies to compare the two classes.

Let us start representing $c_1(\mathcal{L})$ using Čech cohomology. Let us assume that $\mathfrak{U} = \{U_i\}$ is a good cover of X (see for instance [**6**]). This means that the open sets U_i and all their finite intersections are contractible. This implies that Čech cohomology for the constant sheaf $\mathbb{Z}(1)$ agrees with sheaf cohomology. Given a sheaf of abelian groups $\mathcal{F}$, we will denote by $C^*(\mathfrak{U}, \mathcal{F})$ the complex of Čech cochains with respect to the open cover $\mathfrak{U}$.

Let $\{(U_i, s_i)\}$ be a trivialization of the line bundle $\mathcal{L}$. Thus s_i is a non vanishing section of $\Gamma(U_i, \mathcal{L})$. For each pair i, j let us write $U_{i,j} = U_i \cap U_j$. Let $g_{ij} = s_j/s_i$ be the transition functions. So g_{ij} belongs to $\Gamma(U_{i,j}, \mathcal{O}^*)$ and $\{g_{ij}\} \in C^1(\mathfrak{U}, \mathcal{O}^*)$ is a 1-cocycle that represents the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}^*)$. We have to apply the connection morphism to this class. Since the open set $U_{i,j}$ is contractible, we can choose a determination of the logarithm $\log g_{ij}$ over $U_{i,j}$. Over $U_{ijk} = U_i \cap U_j \cap U_k$ let us write $t_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij}$. Then t_{ijk} is constant and it is an integer multiple of $2\pi i$. The set $\{t_{ijk}\}$ is a Čech 2-cocycle for the sheaf $\mathbb{Z}(1)$ that represents $c_1(\mathcal{L})$.

Next we want to represent $c_1^{dR}(\mathcal{L})$ as an explicit differential form. Since $\mathcal{L}$ has rank one, the curvature $K \in \mathcal{E}^{1,1}(X, \operatorname{End}(\mathcal{L})) = \mathcal{E}^{1,1}(X)$ is a differential form. By 3.2 this form is given, in each open U_i, by

$$K = \overline{\partial}(h(s_i)^{-1}\partial h(s_i)) = \overline{\partial}\partial \log(h(s_i)).$$

Since this form does not depend on the section s_i it is a global differential form. Thus the first de Rham Chern class is represented by the form $\widetilde{c}_1(\mathcal{L}, h) = -K = \partial\overline{\partial}\log(h(s_i))$.

To compare the two classes we will follow the comparison between Čech and de Rham cohomologies given in [**6**]. The main tool is the double complex $C^*(\mathfrak{U}, \mathcal{E}^*)$. It has natural morphisms from $\mathcal{E}^*(X)$ and from $C^*(\mathfrak{U}, \mathbb{C})$ and is quasi-isomorphic to both complexes. Let

$$\begin{aligned} d' &: C^p(\mathfrak{U}, \mathcal{E}^q) \longrightarrow C^{p+1}(\mathfrak{U}, \mathcal{E}^q) \\ d'' &: C^p(\mathfrak{U}, \mathcal{E}^q) \longrightarrow C^p(\mathfrak{U}, \mathcal{E}^{q+1}) \end{aligned}$$

denote the differentials, where d' is the differential of Čech cochains and d'' is $(-1)^p$ times the differential of forms.

Then $\{t_{ijk}\} = d'\{\log g_{ij}\}$, and

$$d''\{\log g_{ij}\} = \{-d(g_{ij}/g_{ij})\} = \{-s_i/s_j\, d(s_j/s_i)\}.$$

therefore the Čech cochain $\{s_i/s_j\, d(s_j/s_i)\}$ also represents $c_1(\mathcal{L})$. On the other hand, $-K = -d''\{\partial \log(h(s_i))\}$, and

$$\begin{aligned} d'\{-\partial\log(h(s_i))\} =& \{-\partial\log(h(s_j)/h(s_i))\} = \\ & \{-\partial\log(s_j\overline{s}_j/s_i\overline{s}_i)\} = \\ & \{-s_i/s_j\, d(s_j/s_i)\}. \end{aligned}$$

Therefore $-K$ represents also the class $c_1(\mathcal{L})$. □

As a consequence of this theorem we see that $\psi(c(E)) = c^{dR}(E)$. In particular this implies that the Chern class c_i corresponds to the elementary symmetric function σ_i. This justifies the following definition.

DEFINITION 3.6. Let X be a complex manifold. Let E be a rank n holomorphic vector bundle and let $\varphi \in B[[T_1, \dots, T_n]]$ be an invariant power series. Let

$$\varphi(T_1, \dots, T_n) = \phi(\sigma_i, \dots, \sigma_n)$$

be the expression of φ in terms of symmetric elementary functions. Then *the Chern class of E associated to the power series φ* is:

$$\varphi(E) = \phi(c_1(E), \dots, c_n(E)) \in \bigoplus_p H^{2p}(X, B(p)).$$

COROLLARY 3.7. *Let X be a complex manifold and let E be a holomorphic vector bundle of rank n. Let $\varphi \in B[[\sigma_1, \dots, \sigma_n]]$ be an invariant power series. Then*

$$\varphi^{dR}(E) = \psi\left(\varphi(E)\right).$$

In view of this result we will drop the superscript ^{dR} from the notation.

The Chern character (see example 3.4.3) is one of the most interesting power series of characteristic classes. The main advantage of the Chern character class is that it behaves very well under exact sequences and tensor products. For the proof of the next proposition we refer also to [**34**].

PROPOSITION 3.8.

1. *Let $0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$ be an exact sequence of vector bundles. Then*

$$\mathrm{ch}(E) = \mathrm{ch}(S) + \mathrm{ch}(Q).$$

2. *Let E and F be vector buncles. Then*

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F).$$

Unlike the Chern character class, the Chern character form does not need to behave additively for exact sequences. Let

$$0 \longrightarrow (S, h') \longrightarrow (E, h) \longrightarrow (Q, h'') \longrightarrow 0$$

be an exact sequence of hermitian vector bundles. Let us write $(S, h') \oplus (Q, h'')$ for the orthogonal direct sum. Then

$$\widetilde{\mathrm{ch}}\left((S, h') \oplus (Q, h'')\right) = \widetilde{\mathrm{ch}}(S, h') + \widetilde{\mathrm{ch}}(Q, h'').$$

But in general

$$\widetilde{\mathrm{ch}}(E, h) \neq \widetilde{\mathrm{ch}}\left((S, h') \oplus (Q, h'')\right).$$

At first glance this may seem unfortunate. But, in section 8, we will see that, as Schechtmann pointed out ([**30**]), the lack of additivity at the level of forms can be used to construct characteristic classes for higher K-theory.

4. Chern classes for higher K theory

In this section we will review Gillet's construction of Chern classes for higher K-theory [**11**]. We will follow the simplified version given in [**31**]. We will assume that the reader has some familiarity with the language of simplicial objects (see for instance [**23**] or [**14**]).

In this section we will be in the algebraic geometry context. To stress this fact, instead of working with complex numbers, let us fix a ground field k. Let $\mathcal{V}$ be the category of all smooth quasi-projective schemes, equipped with the Zarisky topology. Let us denote by $D(\mathcal{V})$ (resp. $D^{+}(\mathcal{V})$) the derived category of complexes of abelian sheaves on $\mathcal{V}$ (resp. which are bounded below). Let us assume that we have a fixed graded complex of sheaves

$$\mathcal{F}^{\cdot}(*) = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^{\cdot}(i), \qquad \text{with } \mathcal{F}^{\cdot}(i) \in D^{+}(\mathcal{V}).$$

Thus for each smooth quasi-projective scheme $X \in \mathrm{Ob}(\mathcal{V})$ we have the cohomology groups $\mathbb{H}^{\cdot}(X, \mathcal{F}^{\cdot}(*))$. We need to assume certain properties of this cohomology theory in order to mimic the construction of Chern classes of section 2.

Let us denote by $\mathcal{O}^{*}$ the sheaf of invertible rational functions. It defines an element of $D^{+}(\mathcal{V})$ assuming that it is a complex concentrated in degree zero. Then $\mathbb{H}^{1}(X, \mathcal{O}^{*})$ parameterizes isomorphism classes of algebraic line bundles over X. The first property we need for our cohomology theory is:

P 1: There is a morphism in $D^{+}(\mathcal{V})$

$$\mathcal{O}^{*}[1] \longrightarrow \mathcal{F}^{\cdot}(1).$$

In particular, for each $X \in \mathrm{Ob}(\mathcal{V})$, we obtain a morphism $c_1 : \mathbb{H}^{1}(X, \mathcal{O}^{*}) \longrightarrow \mathbb{H}^{2}(X, \mathcal{F}^{\cdot}(1))$ which allows us to define the first Chern class of a line bundle.

The next property we need is a multiplicative structure for the cohomology.

P 2: For each $n,\ m \in \mathbb{Z}$, there are homomorphisms in $D^{+}(\mathcal{V})$

$$\cup : \mathcal{F}^{\cdot}(n) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{F}^{\cdot}(m) \longrightarrow \mathcal{F}^{\cdot}(n+m), \quad \text{and } e : \mathbb{Z} \longrightarrow \mathcal{F}^{\cdot}(0)$$

which make $\mathcal{F}^{\cdot}(*)$ an associative and graded commutative (with respect to the first degree) algebra with unit.

The third property we need is a formula for the cohomology of the projective space, in other words, we need the Dold-Thom isomorphism to be satisfied.

P 3: For $X \in \mathrm{Ob}(\mathcal{V})$, let $p : \mathbb{P}^{n}_{X} \longrightarrow X$ be the n-dimensional projective space over X. Let ξ be the first Chern class of the line bundle $\mathcal{O}(1)$. Then, for each pair of integers $i,\ m$, the morphism

$$\sum_{k=0}^{n-1} p^{*}() \cup \xi^{k} : \bigoplus_{k=0}^{n-1} \mathbb{H}^{m-2k}(X, \mathcal{F}^{\cdot}(p-k)) \longrightarrow \mathbb{H}^{m}(\mathbb{P}^{n}_{X}, \mathcal{F}^{\cdot}(p))$$

is an isomorphism.

By a Leray spectral sequence argument, from property **P 3** it follows that the Dold-Thom isomorphism is satisfied for any projective bundle.

With properties **P 1**, **P 2** and **P 3**, we can repeat the procedure of the section 2 and define, for each vector bundle E over $X \in \mathrm{Ob}(\mathcal{V})$, Chern classes $c_i(E) \in \mathbb{H}^{2i}(X, \mathcal{F}^{\cdot}(i))$. But in order to have the Whitney sum formula we need to assume a Gysin property (see [**31**] and [**17**]).

P 4: Let $i : Y \hookrightarrow X$ be a closed immersion in $\mathcal{V}$ of pure codimension 1 and let $[Y] \in H^1(X, \mathcal{O}^*)$ be the class of the divisor Y. Then for any $x \in \mathbb{H}^{2i}(X, \mathcal{F}^{\cdot}(i))$ such that $i^* x = 0$, we have
$$x \cup c_1([Y]) = 0.$$

The next step is to extend the definition of Chern classes to simplicial schemes. Let Δ be the category whose objects are the ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$ and whose morphisms are ordered maps. Let $\mathcal{SV}$ denote the category of simplicial objects in $\mathcal{V}$, that is, the category of contravariant functors between Δ and $\mathcal{V}$.

Definition 4.1. Let $X_{\cdot} \in \operatorname{Ob} \mathcal{SV}$ be a simplicial scheme. A *rank n vector bundle* over $X_{\cdot}$ is an object of $\mathcal{SV}$, $E_{\cdot}$, together with a morphism of simplicial schemes $\pi_{\cdot} : E_{\cdot} \longrightarrow X_{\cdot}$ satisfying the following conditions

1. For any $k \geq 0$ the morphism $\pi_k : E_k \longrightarrow X_k$ is a rank n vector bundle.
2. For any $j,\ k \geq 0$ and any $\rho \in \operatorname{Mor}_{\Delta}([j],[k])$ the commutative diagram
$$\begin{array}{ccc} E_k & \xrightarrow{E(\rho)} & E_j \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi_k} \\ X_k & \xrightarrow{X(\rho)} & X_j \end{array}$$
is a relative morphism of vector bundles.

Alternatively, following [**12**], we can define a vector bundle over $X_{\cdot}$ as a vector bundle E_0 over X_0, together with an isomorphism $\alpha : \delta_0^* E_0 \longrightarrow \delta_1^* E_0$ of vector bundles over E_1, such that $\delta_2^* \alpha \circ \delta_0^* \alpha = \delta_1^* \alpha$.

If $X_{\cdot}$ is a simplicial scheme then again $\mathbb{H}^1(X_{\cdot}, \mathcal{O}^*)$ parameterizes isomorphism classes of line bundles (see [**12**] ex 1.1). Moreover, the property **P 3** implies that the Dold-Thom isomorphism is satisfied for arbitrary projective bundles over simplicial schemes (see [**11**] Lemma 2.4). Therefore Grothendieck's construction of Chern classes can be applied to simplicial schemes. In particular it can be applied in the universal case.

Let $B_{\cdot}\mathbf{GL_n}/k$ be the classifying scheme of the group scheme $\mathbf{GL_n}$ over k. The simplicial scheme $B_{\cdot}\mathbf{GL_n}/k$ is provided with a universal rank n vector bundle, denoted by E_n. Let us denote by $c_i^{(n)} = c_i(E_n) \in \mathbb{H}^{2i}(B_{\cdot}\mathbf{GL_n}/k, \mathcal{F}^{\cdot}(i))$ the i-th Chern class of the universal bundle.

The next objective is to explain how classes in the cohomology of the classifying scheme give rise to maps between K-theory and cohomology of schemes.

Let $S_{\cdot}$ be a simplicial set and let X be a scheme. Then we can construct the simplicial scheme $S_{\cdot} \times X$ such that
$$(S_{\cdot} \times X)_n = \coprod_{p \in S_n} \{p\} \times X,$$
and the faces and degeneracies are induced by those of $S_{\cdot}$. Let A be a finitely generated k-algebra such that $U = \operatorname{Spec} A$ is a smooth scheme. Then $B_{\cdot}\mathbf{GL_n}(A)$ is a simplicial set. Thus we can construct the simplicial scheme $B_{\cdot}\mathbf{GL_n}(A) \times U$. Since an element of $B_j\mathbf{GL_n}(A)$ is a morphism between U and $B_j\mathbf{GL_n}/k$ we obtain a tautological morphism of simplicial schemes
$$\tau : B_{\cdot}\mathbf{GL_n}(A) \times U \longrightarrow B_{\cdot}\mathbf{GL_n}/k.$$
Thus, we obtain classes $\tau^*(c_i^{(n)}) \in \mathbb{H}^{2i}(B_{\cdot}\mathbf{GL_n}(A) \times U, \mathcal{F}^{\cdot}(i))$.

For a simplicial set $S.$, let us denote by $\mathbb{Z}S.$ the homological complex which, in degree j is the free abelian group generated by S_j, and whose differential is $d = \sum(-1)^i\delta_i$.

LEMMA 4.2. *Let $S.$ be a simplicial set and let X be a smooth quasi-projective scheme. Then there is a natural short exact sequence*

$$0 \longrightarrow \prod_{p+q=n-1} \operatorname{Ext}^1_{\mathbb{Z}}(H_p(\mathbb{Z}S.), \mathbb{H}^q(X, \mathcal{F}^{\cdot}(i))) \longrightarrow \mathbb{H}^n(S. \times X, \mathcal{F}^{\cdot}(i))$$
$$\xrightarrow{\alpha} \prod_{p+q=n} \operatorname{Hom}(H_p(\mathbb{Z}S.), \mathbb{H}^q(X, \mathcal{F}^{\cdot})) \longrightarrow 0.$$

PROOF. This result is the dual of the Künneth formula (see for instance [**29**] Thm. 11.32). □

Thus, from each class $c_i^{(n)} \in H^{2i}(B.\mathbf{GL_n}/k, \mathcal{F}^{\cdot}(i))$ we obtain a family of morphisms

$$\alpha \circ \tau(c_i^{(n)})_j : H_j(\mathbb{Z}B.\mathbf{GL_n}(A)) \longrightarrow \mathbb{H}^{2i-j}(U, \mathcal{F}^{\cdot}(i)).$$

Now we want to go to the limit when n goes to infinity. For $m > n$, let us consider the inclusion $i_{n,m}\mathbf{GL_n}/k \longrightarrow \mathbf{GL_{n+1}}/k$ given by

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity $m-n \times m-n$-matrix. Since

$$i^*_{n,m}E_m = E_n \oplus (\text{a trivial vector bundle}),$$

we have that $i^*_{n,m}c_i^{(m)} = c_i^{(n)}$. Therefore, the family of morphisms $\{\psi \circ \varphi(c_i^{(n)})_j\}_{n\geq 1}$ defines a morphism

$$c_{i,j} : H_j(GL(A), \mathbb{Z}) = \varinjlim_n H_j(\mathbb{Z}B.\mathbf{GL_n}(A)) \longrightarrow \mathbb{H}^{2i-j}(U, \mathcal{F}^{\cdot}(i)).$$

The K-theory groups of A are the homotopy groups of the $+$-construction of $B.GL(A)$. Since the $+$-construction is acyclic, we have that

$$H_*(B.GL(A)^+, \mathbb{Z}) = H_*(B.GL(A), \mathbb{Z}) = H_*(GL(A), \mathbb{Z}).$$

DEFINITION 4.3. Let A be a finitely generated k-algebra such that $U = \operatorname{Spec} A$ is a smooth scheme. For each pair of integers i, j, the i-th Chern class map in $K_j(A)$ is the composition

$$c_{i,j} : K_j(A) = \pi_j(B.GL(A)^+) \xrightarrow{\text{Hurewicz}} H_j(GL(A), \mathbb{Z}) \xrightarrow{c_{i,j}} \mathbb{H}^{2i-j}(U, \mathcal{F}^{\cdot}(i)).$$

The following result is an easy consequence of the definition.

PROPOSITION 4.4. *Let $f^\sharp : A \longrightarrow B$ be a morphism of finitely generated k algebras such that $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are smooth schemes. Let $f : V \longrightarrow U$ be the corresponding morphism of k-schemes. Then the following diagram*

$$\begin{array}{ccc} K_j(A) & \xrightarrow{c_{i,j}} & \mathbb{H}^{2i-j}(U, \mathcal{F}.(i)) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ K_j(B) & \xrightarrow{c_{i,j}} & \mathbb{H}^{2i-j}(V, \mathcal{F}.(i)) \end{array}$$

is commutative.

In order to extend the construction of Chern classes to arbitrary smooth quasi-projective schemes we will require a property of invariance under homotopy. This will allow us to use Jouanolou's trick. But strictly speaking this property is not necessary (see [**11**]).

P 5: For any scheme $X \in \mathrm{Ob}(\mathcal{V})$, the natural map $\mathbb{A}^1_X \longrightarrow X$ induces an isomorphism

$$\mathbb{H}^*(X, \mathcal{F}^{\cdot}(*)) \longrightarrow \mathbb{H}^*(\mathbb{A}^1_X, \mathcal{F}^{\cdot}(*)).$$

This property implies that, for any fibre bundle $p: Y \longrightarrow X$, with fibre $\mathbb{A}^n$, (not necessarily a vector bundle), the map $p^*\mathbb{H}^*(X, \mathcal{F}^{\cdot}(*)) \longrightarrow \mathbb{H}^*(Y, \mathcal{F}^{\cdot}(*))$ is an isomorphism.

For any scheme $X \in \mathcal{V}$, by Jouanolou's lemma ([**20**] lemma 1.5) there is a fibre bundle $p: Y \longrightarrow X$, with Y affine, which is a torsor over a vector bundle.

Definition 4.5. Let X be a smooth scheme over k. Let us choose an affine torsor $p: Y \longrightarrow X$. For each pair of integers i, j, the *i-th Chern class map for $K_j(X)$* is the composition

$$c_{i,j}: K_j(X) \overset{p^*}{\cong} K_j(Y) \xrightarrow{c_{i,j}} \mathbb{H}^{2i-j}(Y, \mathcal{F}^{\cdot}(i)) \overset{p^*}{\cong} \mathbb{H}^{2i-j}(X, \mathcal{F}^{\cdot}(i)).$$

Thanks to the naturality of the map $c_{i,j}$ in the affine case (proposition 4.4) and [**20**] proposition 1.6, this definition does not depend on the choice of the affine torsor Y.

So far we have constructed Chern classes for higher K-theory. Now we want to construct characteristic classes for arbitrary power series. Let again B be a subring of $\mathbb{R}$ and assume that $\mathcal{F}$ is a sheaf of B-modules. Observe that, replacing T_{n+1} by 0, we have a morphism between the ring of invariant power series

$$B[[T_1, \dots, T_{n+1}]]^{\mathfrak{S}_{n+1}} \longrightarrow B[[T_1, \dots, T_n]]^{\mathfrak{S}_n}.$$

In terms of elementary symmetric functions the above morphism is the morphism

$$B[[\sigma_1, \dots, \sigma_{n+1}]] \longrightarrow B[[\sigma_1, \dots, \sigma_n]]$$

that sends σ_{n+1} to 0. We will denote by

$$i^*_{n,m}: \widehat{IP}(m) \longrightarrow \widehat{IP}(n)$$

the induced morphisms. These morphisms make $\{\widehat{IP}(n)\}_n$ an inverse system. Observe that we have a commutative diagram

$$\begin{array}{ccc} \{\widehat{IP}(m)\}_n & \longrightarrow & \prod_i \mathbb{H}^{2i}(B.\mathbf{GL_m}/k, \mathcal{F}^{\cdot}(i)) \\ {\scriptstyle i^*_{n,m}}\downarrow & & {\scriptstyle i^*_{n,m}}\downarrow \\ \{\widehat{IP}(n)\}_n & \longrightarrow & \prod_i \mathbb{H}^{2i}(B.\mathbf{GL_n}/k, \mathcal{F}^{\cdot}(i)) \end{array}$$

where the horizontal arrows send the elementary symmetric function σ_i to the i-th Chern class.

Definition 4.6. A *stable invariant power series* is an element

$$\{\varphi^{(n)}\} \in \varprojlim \widehat{IP}(n).$$

Observe that any stable invariant power series is given by an element

$$\varphi \in B[[\sigma_1, \dots, \sigma_n, \dots]].$$

Proceeding as with the Chern classes, for any invariant stable power series φ, we obtain characteristic classes

$$\varphi : K_j(X) \longrightarrow \bigoplus \mathbb{H}^{2i-j}(X, \mathcal{F}(i)).$$

EXAMPLE 4.7. Assume that $B \supset \mathbb{Q}$. Let us denote by $\overline{\mathrm{ch}}^{(n)}$ the reduced Chern character of rank n. That is

$$\overline{\mathrm{ch}}^{(n)}(T_1, \dots, T_n) = \mathrm{ch}(T_1, \dots T_n) - n.$$

Since $\overline{\mathrm{ch}}^{(n+1)}(T_1, \dots, T_n, 0) = \overline{\mathrm{ch}}^{(n)}(T_1, \dots, T_n)$, then $\{\overline{\mathrm{ch}}^{(n)}\}$ is a stable invariant power series. Thus it defines a reduced Chern character class, denoted $\overline{\mathrm{ch}}$.

DEFINITION 4.8. The Chern character is the map

$$\mathrm{ch} : K_i(\cdot) \longrightarrow \bigoplus_j \mathbb{H}^{2j-i}(\cdot, \mathcal{F}(j)),$$

given by

$$\mathrm{ch} = \begin{cases} \overline{\mathrm{ch}}, & \text{if } i > 0, \\ \mathrm{rank} + \overline{\mathrm{ch}}, & \text{if } i = 0. \end{cases}$$

Recall that there is a product structure ([**21**])

$$K_i \otimes K_j \longrightarrow K_{i+j}$$

that extends the multiplicative structure induced in K_0 by the tensor product. The Chern character is also compatible with this product. For a proof of the following result see [**11**] or [**31**].

THEOREM 4.9. *For $x \in K_i$ and $y \in K_j$ then*

$$\mathrm{ch}(x \cdot y) = \mathrm{ch}(x) \cup \mathrm{ch}(y).$$

5. Real Deligne cohomology

Let us recall the definition of real Deligne cohomology and some complexes that can be used to compute it. Again let X be a smooth proper complex variety. As in section 3 let B be a subring of $\mathbb{R}$. Let us write $B(p) = (2\pi i)^p B \subset \mathbb{C}$. We will denote also by $B(p)$ the constant sheaf. Let Ω_X^* be the sheaf of holomorphic differential forms on X.

DEFINITION 5.1. The *(B-)Deligne cohomology* of X, denoted $H_{\mathcal{D}}(X, B(p))$, is the hypercohomology of the complex of sheaves

$$B(p) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^{p-1}.$$

This definition has been extended by Beilinson to the case of smooth complex algebraic varieties, not necessarily proper (see [**2**], [**1**], [**10**] and [**19**]). This extension is known as Deligne-Beilinson cohomology. Beilinson also showed that Deligne-Beilinson cohomology can be written as sheaf cohomology for a sheaf in the Zariski topology, satisfying the properties 1 to 5 of last section. Thus we can apply the construction of the last section and obtain characteristic classes from higher K-theory to Deligne-Beilinson cohomology. In particular, if we take $B = \mathbb{R}$ then the

Chern character in real Deligne-Beilinson cohomology is called Beilinson's regulator and it is a generalization of Borel's regulator (see [**4**], [**2**], [**27**]).

Let F denote the Hodge filtration:

$$F^p\Omega_X^* = \bigoplus_{p'\geq p} \Omega_X^{p'}.$$

Then real Deligne cohomology can be defined also as the hypercohomology of the simple complex associated to the morphism of complexes

$$\underline{\mathbb{R}}(p) \oplus F^p\Omega_X^* \longrightarrow \Omega_X^*.$$

We want to have real Deligne cohomology as the cohomology of an explicit complex, as de Rham cohomology is the cohomology of the complex of differential forms. To this end we will resolve the constant sheaf and the sheaves of holomorphic forms using smooth differential forms. We will use the same notation as in section 3 for the sheaves and complexes of differential forms.

The Hodge filtration of the complex $\mathcal{E}^*$ is given by

$$F^p\mathcal{E}^n = \bigoplus_{p'\geq p} \mathcal{E}^{p',n-p'}.$$

The sheaves of differential forms are fine, hence acyclic. Since $\mathcal{E}_{\mathbb{R}}^*(p)$ is a resolution of the constant sheaf $\mathbb{R}(p)$, and $\mathcal{E}^*$ is a resolution of Ω_X^*, compatible with the Hodge filtration, we obtain that real Deligne cohomology of X is the cohomology of the simple complex associated to the morphism of complexes

$$u_p : \mathcal{E}_{\mathbb{R}}^*(p)(X) \oplus F^p\mathcal{E}^*(X) \longrightarrow \mathcal{E}^*(X),$$

given by $u_p(r, f) = f - r$. Let us denote the simple complex associated to u_p as

$$s(u_p) = s(\mathcal{E}_{\mathbb{R}}^*(p)(X) \oplus F^p\mathcal{E}^*(X) \longrightarrow \mathcal{E}^*(X)).$$

An element of $s^n(u_p)$ is given by a triple

$$(r, f, \omega) \in \mathcal{E}_{\mathbb{R}}^n(p)(X) \oplus F^p\mathcal{E}^n(X) \oplus \mathcal{E}^{n-1}(X),$$

and the differential is given by $d(r, f, \omega) = (dr, df, f - r - d\omega)$.

Following Deligne [**9**] we can use a simpler complex to compute real Deligne cohomology.

Definition 5.2. Let $\mathcal{D}^*(X, p)$ denote the complex given by

$$\mathcal{D}^n(X,p) = \begin{cases} \mathcal{E}_{\mathbb{R}}^{n-1}(p-1)(X) \cap \bigoplus\limits_{\substack{p'+q'=n-1\\ p'<p,\ q'<p}} \mathcal{E}^{p',q'}(X), & \text{for } n \leq 2p-1 \\ \mathcal{E}_{\mathbb{R}}^{n}(p)(X) \cap \bigoplus\limits_{\substack{p'+q'=n\\ p'\geq p,\ q'\geq p}} \mathcal{E}^{p',q'}(X), & \text{for } n \geq 2p. \end{cases}$$

The differential of this complex, denoted by $d_{\mathcal{D}}$, is induced by d in degree greater or equal than $2p$, by $-d$ in degree less or equal than $2p-2$ and is equal to $-2\partial\overline{\partial}$ in degree $2p-1$.

A proof that the cohomology of this complex is Deligne cohomology can be found on [**7**]. This proof is based in the following facts:

1. The morphism $u : \mathcal{E}^n_{\mathbb{R}}(p)(X) \oplus F^p\mathcal{E}^n(X) \longrightarrow \mathcal{E}^n(X)$ is injective for $n \leq 2p-1$ and the cokernel is
$$\mathcal{E}^{n-1}_{\mathbb{R}}(p-1)(X) \cap \bigoplus_{\substack{p'+q'=n-1\\ p'<p,\ q'<q}} \mathcal{E}^{p',q'}(X).$$
2. The above morphism u is surjective for $n \geq 2p-1$ and the kernel is
$$\mathcal{E}^n_{\mathbb{R}}(p)(X) \cap \bigoplus_{\substack{p'+q'=n\\ p'\geq p,\ q'\geq q}} \mathcal{E}^{p',q'}(X).$$
3. In particular, for $n = 2p-1$ the morphism u is an isomorphism. Moreover, if $\omega \in \mathcal{E}^{2p-2}(X)$, then $du^{-1}d\omega = -2\partial\bar{\partial}\omega$.

EXAMPLE 5.3. Observe that $\mathcal{D}^{2p}(X,p) = \mathcal{E}^{2p}_{\mathbb{R}}(p)(X) \cap \mathcal{E}^{p,p}(X)$. Thus if (E,h) is a hermitian holomorphic vector bundle of rank n and $\varphi \in \mathbb{R}[[T_1,\dots,T_n]]$ is a real invariant power series, then
$$\widetilde{\varphi}(E,h) \in \bigoplus_p \mathcal{D}^{2p}(X,p).$$

In order to write down explicit morphisms between $\mathcal{D}^*(X,p)$ and $s(u_p)$ we need to introduce some notations. Given a differential form a, we will write $a = \sum a^{p,q}$ its decomposition in forms of pure bidegree. Let $F^{p,p}$ be the morphism defined by
$$F^{p,p}a = \sum_{\substack{p'\geq p\\ q'\geq p}} a^{p',q'}.$$
Let π_p be the morphism given by
$$\pi_p a = (a + (-1)^p\overline{a})/2.$$
Observe that π_p is the projection of $\mathcal{E}^*(X)$ onto $\mathcal{E}^*_{\mathbb{R}}(p)(X)$, and that, for $n < 2p$, $\rho_p = \pi_{p-1} \circ F^{n-p+1,n-p+1}$ is the projection of $\mathcal{E}^n(X)$ onto the cokernel of u_p.

Let $\psi : s^n(u_p) \longrightarrow \mathcal{D}^n(X,p)$ and $\varphi : \mathcal{D}^n(X,p) \longrightarrow s^n(u_p)$ be the morphisms given by
$$\psi(r,f,\omega) = \begin{cases} \rho_p(\omega), & \text{for } n \leq 2p-1,\\ F^{p,p}r + 2\pi_p(\partial\omega^{p-1,q+1}), & \text{for } n \geq 2p,\end{cases}$$
$$\varphi(x) = \begin{cases} (\partial x^{p-1,q} - \bar{\partial}x^{q,p-1}, 2\partial x^{p-1,q}, x), & \text{for } n \leq 2p-1,\\ (x,x,0), & \text{for } n \geq 2p,\end{cases}$$
where $q = n-p$. Then ψ and φ are homotopy equivalences inverse to each other (see [**7**]).

Real Deligne cohomology has a product [**2**], [**1**] that can be described, in terms of the complexes $s(u_p)$ in the following way. Let $0 \leq \alpha \leq 1$ be a real number. For $(r,f,\omega) \in s^n(u_p)$ and $(s,g,\eta) \in s^m(u_q)$, let us write
$$\begin{aligned}(r,f,\omega) \cup_\alpha (s,g,\eta) = (&r\wedge s, f\wedge g,\\ &\alpha(\omega\wedge s + (-1)^n f\wedge\eta) + (1-\alpha)(\omega\wedge g + (-1)^n r\wedge\eta)).\end{aligned}$$
This is a family of products, all of them homotopically equivalent. Moreover, for $\alpha = 1/2$ this product is graded commutative, whereas for $\alpha = 0,1$ this product is associative. Therefore they induce a graded commutative, associative product in

Deligne cohomology. This product induces a product in the complex $\mathcal{D}^*(X,p)$ (see [**7**]). This complex is only associative or graded commutative up to homotopy.

In order to have a complex with a graded commutative and associative product we will use the Thom-Whitney simple, introduced in [**26**]. The Thom-Whitney simple associates, to a strict cosimplicial differential complex, a new differential complex. This new differential complex is homotopically equivalent to the total complex of the original cosimplicial complex. The main interest of the Thom-Whitney simple is that, if we start with a strict cosimplicial graded commutative associative algebra, the complex that we obtain is again a differential graded commutative associative algebra.

In our case, we can consider the morphism u_p a strict cosimplicial complex $u_p^{\cdot}$ writing

$$\begin{aligned} u_p^0 &= \mathcal{E}^*_{\mathbb{R}}(p)(X) \oplus F^p\mathcal{E}^*(X), \\ u_p^1 &= \mathcal{E}^*(X), \\ u_p^i &= 0, \qquad\qquad \text{for } i \geq 2, \end{aligned}$$

with morphisms given by

$$\delta^0(r,f) = f, \qquad \delta^1(r,f) = r,$$

and the other morphisms equal to zero.

Let us describe the Thom-Whitney simple in this case. Let L_1^* be the complex of algebraic forms in the affine line $\mathbb{A}^1_{\mathbb{R}}$. That is, $L_1^0 = \mathbb{R}[t]$, and $L_1^1 = \mathbb{R}[t]dt$. Let $\delta_0, \delta_1 : L_1^* \longrightarrow \mathbb{R}$ be the morphisms given by evaluation at 0 and 1 respectively. That is $\delta_0(f(t) + g(t)dt) = f(0)$, and $\delta_1(f(t) + g(t)dt) = f(1)$.

Definition 5.4. The *Thom-Whitney simple* of $u_p^{\cdot}$, denoted $s_{TW}(u_p^{\cdot})$ is the subcomplex of

$$\mathcal{E}^*_{\mathbb{R}}(p)(X) \oplus F^p\mathcal{E}^*(X) \oplus L_1^* \otimes \mathcal{E}^*(X)$$

formed by the elements (r,f,ω) such that

$$\begin{aligned} f &= (\delta_0 \otimes \mathrm{Id})(\omega), \\ r &= (\delta_1 \otimes \mathrm{Id})(\omega). \end{aligned}$$

The differential and the product of the Thom-Whitney simple are given componentwise:

$$d(r,f,\omega) = (dr, df, d\omega)$$

and

$$(r,f,\omega) \wedge (s,g,\eta) = (r \wedge s, f \wedge g, \omega \wedge \eta).$$

With these definitions of differential and product, the direct sum $\bigoplus_p s_{TW}(u_p)$ is a differential graded commutative associative algebra.

We can construct explicit equivalences (see [**26**])

$$s_{TW}(u_p) \underset{E}{\overset{I}{\rightleftarrows}} s(u_p)$$

given by

$$E(r,f,\omega) = (r, f, t \otimes f + (1-t) \otimes r + dt \otimes \omega),$$

and

$$I(r,f,(h(t)+g(t)dt)\otimes\omega)=\left(r,f,\int_0^1 g(t)dt\omega\right).$$

We will write $I'=\psi\circ I: s_{TW}(u_p)\longrightarrow \mathcal{D}^*(X,p)$ and $E'=E\circ\varphi:\mathcal{D}^*(X,p)\longrightarrow s_{TW}(u_p)$. We will later use that I' and E' is a pair of homotopy equivalences, inverse to each other, between $\mathcal{D}^*(X,*)$ and a complex with an associative and graded commutative product.

6. Bott-Chern forms

For any stable invariant power series φ and any exact sequence

$$\overline{\xi}:0\longrightarrow (S,h')\xrightarrow{f}(E,h)\longrightarrow(Q,h'')\longrightarrow 0$$

of hermitian vector bundles, the Whitney sum formula implies that the associated Chern classes satisfy

$$\varphi(E)=\varphi(S\oplus Q). \tag{6.1}$$

But in general this equation is no longer true for the Chern forms. That is, the form

$$\widetilde{\varphi}(E,h)-\widetilde{\varphi}((S,h')\oplus(Q,h''))\in\bigoplus\mathcal{D}^{2p}(X,p)$$

may be non zero. Nevertheless, equation 6.1 and the $\partial\overline{\partial}$-lemma ([**16**]) imply that there exists a differential form $\widetilde{\varphi}_1$ such that

$$\widetilde{\varphi}(E,h)-\widetilde{\varphi}((S,h')\oplus(Q,h''))=-2\partial\overline{\partial}\widetilde{\varphi}_1. \tag{6.2}$$

Since $\mathcal{D}^{2p}(X,p)=\mathcal{E}^{2p}_{\mathbb{R}}(p)(X)\cap\mathcal{E}^{p,p}(X)$ and $-2\partial\overline{\partial}$ is a purely imaginary operator, bihomogeneous of bidegree $(1,1)$, we can choose

$$\widetilde{\varphi}_1\in\bigoplus_p\mathcal{E}^{2p-2}_{\mathbb{R}}(p-1)(X)\cap\mathcal{E}^{p-1,p-1}(X)=\bigoplus_p\mathcal{D}^{2p-1}(X,p).$$

In other words, the form $\widetilde{\varphi}(E,h)-\widetilde{\varphi}((S,h')\oplus(Q,h''))$ is exact in the complex $\bigoplus\mathcal{D}^*(X,p)$ The aim of this section is to solve the equation 6.2 in a functorial way.

We will say that the exact sequence $\overline{\xi}$ is split if (E,h) is the orthogonal direct sum (S,h') and (Q,h'').

THEOREM 6.1 (Gillet and Soulé [**13**]). *Let φ be a stable invariant power series. To each exact sequence of hermitian vector bundle*

$$\overline{\xi}:0\longrightarrow (S,h')\xrightarrow{f}(E,h)\longrightarrow(Q,h'')\longrightarrow 0$$

we can assign a differential form $\widetilde{\varphi}_1(\overline{\xi})\in\bigoplus_p\mathcal{D}^{2p-1}(X,p)$, called the Bott-Chern form. satisfying the following properties.

1. $-2\partial\overline{\partial}\widetilde{\varphi}_1=\widetilde{\varphi}(E,h)-\widetilde{\varphi}((S,h')\oplus(Q,h''))$.
2. *If $f:X\longrightarrow Y$ is a morphism of complex manifolds then*

$$\widetilde{\varphi}_1(f^*\overline{\xi})=f^*\widetilde{\varphi}_1(\overline{\xi}).$$

3. *If $\overline{\xi}$ is a split exact sequence of hermitian bundles, then $\widetilde{\varphi}_1(\overline{\xi})=0$.*

Moreover, these properties characterize $\widetilde{\varphi}_1$ up to $\operatorname{Im}\partial+\operatorname{Im}\overline{\partial}$. That is, up to a boundary in the complex $\bigoplus_p\mathcal{D}^*(X,p)$.

The proof that these properties characterize Bott-Chern forms can be found in [**3**]. Let us give a construction of Bott-Chern forms. The method we will follow is a modification of the method of Gillet and Soulé that avoids the need to choose a partition of unity. In this way we obtain well defined Bott-Chern forms satisfying the conditions of theorem 6.1 and not only classes up to $\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$.

A standard procedure to prove that two differential forms α and β, defined on a differential variety Y, are cohomologous is the following. First, one constructs a geometric homotopy between α and β. That is, a differential form η defined on $Y \times \mathbb{R}$ such that $\eta|_{Y\times\{1\}} = \alpha$ and $\eta|_{Y\times 0} = \beta$. From this homotopy one obtains a primitive for $\alpha - \beta$ by integration:

$$d\int_0^1 \eta = \alpha - \beta. \tag{6.3}$$

Let us denote by Δ^1 the current

$$\Delta^1(\eta) = \int_0^1 \eta.$$

Then the equation 6.3 can be expressed as the equation in currents

$$d\Delta^1 = \delta_1 - \delta_0, \tag{6.4}$$

where δ_0 and δ_1 are the Dirac delta currents centered at 0 and 1 respectively. We will adapt the above procedure to Deligne cohomology.

Let us assume for a while that the hermitian metric h'' on Q is the hermitian metric induced by the metric h of E. The first step is to construct a geometric homotopy between the hermitian vector bundles (E, h) and $(S, h') \oplus (Q, h'')$. This homotopy will be parametrized by the complex projective line instead of by the unit interval. Let $(x : y)$ be homogeneous coordinates of $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$. Then x and y are sections of the bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. The standard metric of $\mathbb{C}^2$ induces the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(1)$. Let us denote by g this metric. Then

$$g(x) = \frac{x\overline{x}}{x\overline{x} + y\overline{y}} \quad \text{and} \quad g(y) = \frac{y\overline{y}}{x\overline{x} + y\overline{y}}.$$

Let $p_1 : X \times \mathbb{P}^1 \longrightarrow X$ and $p_2 : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ denote the projections. Let us write $E(1) = p_1^* E \otimes p_2^* \mathcal{O}(1)$ and $S(1) = p_1^* S \otimes p_2^* \mathcal{O}(1)$. Let us consider the morphism

$$\begin{array}{rccc} \psi : & S & \longrightarrow & S(1) \oplus E(1) \\ & s & \longmapsto & s \otimes y + f(s) \otimes x. \end{array}$$

Observe that the vector bundle $S(1) \oplus E(1)$ has a metric induced by the metrics of S, E and $\mathcal{O}(1)$.

Definition 6.2. The *transgression bundle* associated to the exact sequence $\overline{\xi}$ is the hermitian vector bundle

$$\operatorname{tr}_1(\overline{\xi}) = \operatorname{coker}(\psi)$$

with the hermitian metric induced by the metric of $S(1) \oplus E(1)$.

The restrictions of the transgression bundle $\operatorname{tr}_1(\overline{\xi})$ are

$$\operatorname{tr}_1(\overline{\xi})|_{X\times(0:1)} = (E, h), \tag{6.5}$$

$$\operatorname{tr}_1(\overline{\xi})|_{X\times(1:0)} = (S, h') \oplus (Q, h''). \tag{6.6}$$

Thus $\operatorname{tr}_1(\overline{\xi})$ is a geometric homotopy between $(S,h')\oplus(Q,h'')$ and (E,h). Note that to obtain 6.5 we had to assume that h'' is the induced metric.

The second step is to obtain a geometric homotopy of differential forms. This homotopy is given by the form $\widetilde{\varphi}(\operatorname{tr}_1(\overline{\xi}))$. Then the functoriality of the Chern forms and the equations 6.5 and 6.6 imply that

$$\widetilde{\varphi}(\operatorname{tr}_1(\overline{\xi}))|_{X\times(0:1)} = \widetilde{\varphi}\left((E,h)\right), \tag{6.7}$$

$$\widetilde{\varphi}(\operatorname{tr}_1(\overline{\xi}))|_{X\times(1:0)} = \widetilde{\varphi}\left((S,h')\oplus(Q,h'')\right). \tag{6.8}$$

The third step is to integrate the geometric homotopy to obtain a primitive. Let $t=x/y$ be an absolute coordinate in $\mathbb{P}^1$. Let $[1/2\,\log t\overline{t}]$ denote the current defined by

$$[1/2\,\log t\overline{t}](\eta)=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{1}{2}\log t\overline{t}\eta.$$

The current $[1/2\,\log t\overline{t}]$ will play the role, in Deligne cohomology, that the current Δ^1 plays in de Rham cohomology. The analogue of equation 6.4 is the Poincaré-Lelong equation (see for instance [**16**])

$$-2\partial\overline{\partial}[1/2\,\log t\overline{t}]=\delta_{(0:1)}-\delta_{(1:0)}. \tag{6.9}$$

Definition 6.3. The *Bott-Chern form* associated to the exact sequence $\overline{\xi}$ and the power series φ is the differential form

$$\widetilde{\varphi}_1(\overline{\xi})=[1/2\,\log t\overline{t}](\operatorname{tr}_1(\xi)). \tag{6.10}$$

Equations 6.7, 6.8 and 6.9 imply the condition 1 of theorem 6.1. Moreover the functoriality (condition 2) is clear from the construction of Bott-Chern forms.

Lemma 6.4. *If $\overline{\xi}$ is a split exact sequence then $\widetilde{\varphi}_1(\overline{\xi})=0$.*

Proof. Let us consider the morphism $\iota:\mathbb{P}^1\longrightarrow\mathbb{P}^1$ given by $\iota(x:y)=(y:x)$. The line bundle $\mathcal{O}(1)$ with the Fubini-Study metric is invariant under ι^*. Since $\overline{\xi}$ is split, the map ψ in the definition of the transgression bundle is

$$\begin{array}{rccc}\psi: & S & \longrightarrow & S(1)\oplus S(1)\oplus Q(1)\\ & s & \longmapsto & s\otimes y+s\otimes x+0.\end{array}$$

Therefore, by the invariance of $\mathcal{O}(1)$, $\iota^*\operatorname{tr}_1(\overline{\xi})$ is the cokernel of the morphism

$$\begin{array}{rccc}\psi': & S & \longrightarrow & S(1)\oplus S(1)\oplus Q(1)\\ & s & \longmapsto & s\otimes x+s\otimes y+0,\end{array}$$

which is isometric to $\operatorname{tr}_1(\overline{\xi})$. Therefore $\iota^*\widetilde{\varphi}(\operatorname{tr}_1(\overline{\xi}))=\widetilde{\varphi}(\operatorname{tr}_1(\overline{\xi}))$. Thus it is an even form. On the other hand, the current $[1/2\,\log t\overline{t}]$ is odd: $\iota_*[1/2\,\log t\overline{t}]=-[1/2\,\log t\overline{t}]$. Hence $\widetilde{\varphi}_1(\overline{\xi})=[1/2\,\log t\overline{t}](\operatorname{tr}_1(\xi))=0$. □

Let us assume now that the metric h'' of Q is arbitrary. Let h''' be the hermitian metric on Q induced by the metric h. Then from the exact sequence $\overline{\xi}$ we can define two new exact sequences.

$$\lambda^1\overline{\xi}:0\longrightarrow(S,h')\longrightarrow(E,h)\longrightarrow(Q,h''')\longrightarrow0$$

$$\lambda^2\overline{\xi}:0\longrightarrow(Q,h'')\longrightarrow(Q,h''')\longrightarrow0\longrightarrow0.$$

In both exact sequences the third metric is induced by the second one.

Definition 6.5. Let

$$\overline{\xi}: 0 \longrightarrow (S, h') \longrightarrow (E, h) \longrightarrow (Q, h'') \longrightarrow 0$$

be an exact sequence of hermitian vector bundles. Let φ be a stable invariant power series. Then the *Bott-Chern form* associated to φ anf $\overline{\xi}$ is

$$\widetilde{\varphi}_1(\overline{\xi}) = \widetilde{\varphi}_1(\lambda^1\overline{\xi}) + \widetilde{\varphi}_1(\lambda^2\overline{\xi})$$

Remark 6.6. Since, by Lemma 6.4 the Bott-Chern of the exact sequence

$$0 \longrightarrow (Q, h) \longrightarrow (Q, h) \longrightarrow 0 \longrightarrow 0$$

is zero, if $\overline{\xi}$ is an exact sequence with the third metric induced by the second one then definition 6.3 gives the same result as definition 6.5.

7. Exact cubes

Let X be a smooth quasi-projective variety over $\mathbb{C}$. Let us fix a small full subcategory $\mathfrak{E} = \mathfrak{E}(X)$ of the category of algebraic vector bundles over X, which is equivalent to it.

Let $\langle -1, 0, 1\rangle$ be the category associated to the ordered set $\{-1, 0, 1\}$. Let $\langle -1, 0, 1\rangle^n$ be the n-th cartesian power, and let $\langle -1, 0, 1\rangle^0$ be the category with one element and one morphism. Following Loday [**22**], we define exact cubes as follows:

Definition 7.1. An exact n-cube of $\mathfrak{E}$ is a functor $\mathcal{F}$ from $\langle -1, 0, 1\rangle^n$ to $\mathfrak{E}$ such that, for all integers $1 \leq i \leq n$, and all $n-1$-tuples $(\alpha_1, \dots, \alpha_{n-1}) \in \langle -1, 0, 1\rangle^{n-1}$ the sequence

$$\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},-1,\alpha_i,\dots,\alpha_{n-1}} \longrightarrow \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_i,\dots,\alpha_{n-1}} \longrightarrow \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},1,\alpha_i,\dots,\alpha_{n-1}}$$

is a short exact sequence. We have written $\mathcal{F}_{\alpha_1,\dots,\alpha_n}$ for $\mathcal{F}(\alpha_1, \dots, \alpha_n)$.

We will denote by $\underline{C}_n\mathfrak{E}$ the category of exact n-cubes. It is a small exact category. We will write $C_n\mathfrak{E} = \mathrm{Ob}(\underline{C}_n\mathfrak{E})$.

Definition 7.2. Given an exact n-cube $\mathcal{F}$ and integers $i \in \{1, \dots, n\}$, $j \in \{-1, 0, 1\}$, *the face* $\partial_i^j\mathcal{F}$ is the exact $n-1$-cube defined by

$$\left(\partial_i^j\mathcal{F}\right)_{\alpha_1,\dots,\alpha_{n-1}} = \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},j,\alpha_i,\dots,\alpha_{n-1}}.$$

Examples 7.3.

1. An exact 0-cube is an element of $\mathrm{Ob}(\mathfrak{E})$.
2. An exact 1-cube is an exact sequence of objects of $\mathfrak{E}$.
3. For each $i \in \{1, \dots, n\}$, we can see an exact n-cube $\mathcal{F}$ as the exact sequence of exact $n-1$-cubes

$$0 \longrightarrow \partial_i^{-1}\mathcal{F} \longrightarrow \partial_i^0\mathcal{F} \longrightarrow \partial_i^1\mathcal{F} \longrightarrow 0$$

This exact sequence will be denoted $\partial_i^{\cdot}\mathcal{F}$. Note that $\mathcal{F}$ is characterized by any of the exact sequences $\partial_i^{\cdot}\mathcal{F}$.

Let $\mathbb{Z}C_n(\mathfrak{E})$ be the free abelian group generated by $C_n(\mathfrak{E})$. Let us define a differential $d : \mathbb{Z}C_n(\mathfrak{E}) \longrightarrow \mathbb{Z}C_{n-1}(\mathfrak{E})$ by the formula

$$d = \sum_{i=1}^{n}\sum_{j=-1}^{1}(-1)^{i+j+1}\partial_i^j.$$

It is easy to see that $d^2 = 0$; thus we have obtained a homology complex denoted $\mathbb{Z}C_*(\mathfrak{C})$. Since we are using a cubic theory, in order to obtain the right homology, we need to factor out by the degenerate elements.

For each exact $n-1$-cube $\mathcal{F}$, and each integer $i \in \{1, \dots, n\}$ we will denote by $s^i_{-1}\mathcal{F}$ the exact n-cube defined by the exact sequence (see example 7.3.3)

$$\partial_i^{\cdot}(s^i_{-1}\mathcal{F}) : 0 \longrightarrow 0 \longrightarrow \mathcal{F} \xrightarrow{\mathrm{Id}} \mathcal{F} \longrightarrow 0.$$

Analogously we define s^i_1 by the exact sequence

$$\partial_i^{\cdot}(s^i_1\mathcal{F}) : 0 \longrightarrow \mathcal{F} \xrightarrow{\mathrm{Id}} \mathcal{F} \longrightarrow 0 \longrightarrow 0.$$

The exact cubes in the image of s_{-1} and s_1 are called degenerate n-cubes. Clearly the differential d sends a degenerate cube to a linear combination of degenerate cubes. Therefore the subgroup generated by all degenerated cubes form a subcomplex of $\mathbb{Z}C_*(\mathfrak{C})$ denoted by D_*.

Definition 7.4. The *reduced cubical complex* of the category $\mathfrak{C}$ is

$$\widetilde{\mathbb{Z}}C_*\mathfrak{C} = \mathbb{Z}C_*\mathfrak{C}/\, D_*.$$

The homology of the complex $\mathbb{Z}C_*^{red}\mathfrak{C}$ is closely related to the K-theory of X. For instance, let $S_*\mathfrak{C}$ be the Waldhausen space associated with the category $\mathfrak{C}$ [**32**]. Then

$$K_i(X) = \pi_{i+1}\left|S_*\mathfrak{C}\right|.$$

Now, as in [**33**], [**8**] or [**24**], one can construct a morphism of complexes

$$\gamma : \mathbb{Z}S.\mathfrak{C}[1] \longrightarrow \widetilde{\mathbb{Z}}C_*\mathfrak{C}.$$

Composing with the Hurewicz morphism one obtains a natural map

$$K_i\mathfrak{C} \longrightarrow H_i(\widetilde{\mathbb{Z}}C_*\mathfrak{C}).$$

For the purpose of constructing characteristic classes this map is enough (see [**8**]). But R. McCarthy [**24**] has given a precise description of the homology of $\widetilde{\mathbb{Z}}C_*\mathfrak{C}$ that makes this complex much more interesting.

Theorem 7.5. *The homology of $\widetilde{\mathbb{Z}}C_*\mathfrak{C}$ is the homology of the algebraic K-theory spectrum of the category $\mathfrak{C}$. In particular*

$$K_i(\mathfrak{C}) \otimes \mathbb{Q} \cong H_i(\widetilde{\mathbb{Z}}C_*\mathfrak{C}) \otimes \mathbb{Q}.$$

Moreover, the use of cubes makes $\widetilde{\mathbb{Z}}C_*\mathfrak{C}$ very well behaved to study products.

Definition 7.6. Let $\mathcal{F}$ be an exact n-cube and let $\mathcal{G}$ be an exact m-cube. Then $\mathcal{F} \otimes \mathcal{G}$ is the exact $n+m$-cube given by

$$(\mathcal{F} \otimes \mathcal{G})_{\alpha_1,\dots,\alpha_{n+m}} = \mathcal{F}_{\alpha_1,\dots,\alpha_n} \otimes \mathcal{G}_{\alpha_{n+1},\dots,\alpha_{n+m}}.$$

This product makes $\widetilde{\mathbb{Z}}C_*\mathfrak{C}$ an associative differential algebra which is homotopically commutative. Therefore its homology has the structure of an associative and commutative algebra.

Theorem 7.7 (R. McCarthy [**24**].). *The morphism*

$$K_*(\mathfrak{C}) \longrightarrow H_*(\widetilde{\mathbb{Z}}C_*\mathfrak{C})$$

is multiplicative.

We want to introduce hermitian metrics in the vector bundles.

DEFINITION 7.8. Let $\overline{\mathfrak{E}} = \overline{\mathfrak{E}}(X)$ be the category with

$$\operatorname{Ob}\overline{\mathfrak{E}} = \left\{ (E,h) \middle| \begin{array}{l} E \in \operatorname{Ob}\mathfrak{E} \\ h \text{ hermitian metric on } E \end{array} \right\}$$

and

$$\operatorname{Hom}_{\overline{\mathfrak{E}}}((E,h),(F,g)) = \operatorname{Hom}_{\mathfrak{E}}(E,F).$$

For each vector bundle $E \in \mathfrak{E}$ let us choose a hermitian metric h_E. This gives us a functor $\mathfrak{F} : \mathfrak{E} \longrightarrow \overline{\mathfrak{E}}$. Let $\mathfrak{G} : \overline{\mathfrak{E}} \longrightarrow \mathfrak{E}$ be the functor *forget the metric*. These functors are equivalences inverse to each other. In particular, this implies the following result.

LEMMA 7.9. *The functors $\mathfrak{F}$ and $\mathfrak{G}$ induce morphisms of complexes*

$$\mathfrak{F} : \widetilde{\mathbb{Z}}C_*\mathfrak{E} \longrightarrow \widetilde{\mathbb{Z}}C_*\overline{\mathfrak{E}},$$
$$\mathfrak{G} : \widetilde{\mathbb{Z}}C_*\overline{\mathfrak{E}} \longrightarrow \widetilde{\mathbb{Z}}C_*\mathfrak{E},$$

which are homotopy equivalences, inverse one of the other.

PROOF. It is clear that $\mathfrak{G}\circ\mathfrak{F} = \mathrm{Id}$. In addition the homotopy $\mathfrak{h}$ between $\mathfrak{F}\circ\mathfrak{G}$ and the identity is given in the following way. Let $\overline{\mathcal{F}}$ be an element of $\widetilde{\mathbb{Z}}C_n\overline{\mathfrak{E}}$. Then $\mathfrak{h}\overline{\mathcal{F}}$ is the exact $n+1$ cube defined by the exact sequence

$$\partial_n^{\cdot}(\mathfrak{h}\overline{\mathcal{F}}) : 0 \longrightarrow \mathfrak{F}\circ\mathfrak{G}(\overline{\mathcal{F}}) \xrightarrow{\mathrm{Id}} \overline{\mathcal{F}} \longrightarrow 0 \longrightarrow 0.$$

□

To define higher Bott-Chern forms for exact cubes of hermitian vector bundles, extending the technique of section 6, we need the third metric in any short exact sequence to be induced by the middle metric. To this end we introduce the following notation.

DEFINITION 7.10. Let $\overline{\mathcal{F}} = \{(E_\alpha, h_\alpha)\}$ be an exact n-cube of hermitian vector bundles. We say that $\overline{\mathcal{F}}$ is an emi-n-cube, if, for each n-tuple $\alpha = (\alpha_1,\dots,\alpha_n)$, and each i with $\alpha_i = 1$, the metric h_α is induced by the metric $h_{(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)}$.

Let $\mathbb{Z}C^{emi}\overline{\mathfrak{E}}$ be the subcomplex of $\mathbb{Z}C\overline{\mathfrak{E}}$ generated by the the emi-n-cubes, and let D^{emi} be the subcomplex generated by the degenerate emi-n-cubes. We will write

$$\widetilde{\mathbb{Z}}C^{emi}\overline{\mathfrak{E}} = \mathbb{Z}C^{emi}\overline{\mathfrak{E}}\big/ D^{emi} \subset \widetilde{\mathbb{Z}}C\overline{\mathfrak{E}}.$$

Let us see that the complexes $\widetilde{\mathbb{Z}}C\overline{\mathfrak{E}}$ and $\widetilde{\mathbb{Z}}C^{emi}\overline{\mathfrak{E}}$ are homotopically equivalent. Let $\overline{\mathcal{F}} = \{(F_\alpha, h_\alpha)\} \in C_n\overline{\mathfrak{E}}$. For $i = 1,\dots,n$ let $\lambda_i^1\overline{\mathcal{F}}$ be defined by

$$\lambda_i^1\overline{\mathcal{F}}_\alpha = \begin{cases} (\mathcal{F}_\alpha, h_\alpha), & \text{if } \alpha_i = -1, 0, \\ (\mathcal{F}_\alpha, h'_\alpha), & \text{if } \alpha_i = 1, \end{cases}$$

where h'_α is the metric induced by $h_{(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)}$. Thus the operator λ_i^1 changes the metrics of the face $\partial_i^1\overline{\mathcal{F}}$ by those induced by the metrics of the face $\partial_i^0\overline{\mathcal{F}}$.

Let $\lambda_i^2\overline{\mathcal{F}}$ be the exact n-cube determined by the exact sequence

$$\partial_i^{\cdot}(\lambda_i^2\overline{\mathcal{F}}) : 0 \longrightarrow \partial_i^1\overline{\mathcal{F}} \longrightarrow \partial_i^1\lambda_i^1\overline{\mathcal{F}} \longrightarrow 0 \longrightarrow 0.$$

This n-cube measures the difference between $\overline{\mathcal{F}}$ and $\lambda_i^1\overline{\mathcal{F}}$.

Let us write $\lambda_i\overline{\mathcal{F}} = \lambda_i^1\overline{\mathcal{F}} + \lambda_i^2\overline{\mathcal{F}}$, and let us denote by λ the map

$$\begin{aligned} \lambda : \quad \mathbb{Z}C_n\overline{\mathfrak{E}} & \longrightarrow \quad \mathbb{Z}C_n\overline{\mathfrak{E}} \\ \overline{\mathcal{F}} \quad & \longmapsto \begin{cases} \lambda_n \dots \lambda_1\overline{\mathcal{F}}, & \text{if } n \geq 1, \\ \overline{\mathcal{F}}, & \text{if } n = 0. \end{cases} \end{aligned}$$

The map λ is a morphism of complexes. Moreover, the image of λ lies in the set of emi-n-cubes and λ sends degenerate cubes to degenerate cubes. Therefore the morphism λ induces a morphism of complexes

$$\widetilde{\lambda} : \widetilde{\mathbb{Z}}C\overline{\mathfrak{E}} \longrightarrow \widetilde{\mathbb{Z}}C^{emi}\overline{\mathfrak{E}}.$$

Proposition 7.11. *The morphism $\widetilde{\lambda}$ is a homotopy equivalence.*

Proof. Let us denote by

$$\iota : \widetilde{\mathbb{Z}}C^{emi}\overline{\mathfrak{E}} \longrightarrow \widetilde{\mathbb{Z}}C\overline{\mathfrak{E}}$$

the inclusion. Observe that, if $\overline{\mathcal{F}}$ is an emi-n-cube then

$$\lambda\overline{\mathcal{F}} = \overline{\mathcal{F}} + \text{ degenerate elements.}$$

Therefore $\widetilde{\lambda} \circ \iota = \mathrm{Id}$. Let $\mathfrak{F}$ and $\mathfrak{G}$ be the morphisms of complexes of lemma 7.9. Since $\mathfrak{F} \circ \mathfrak{G}$ is homotopically equivalent to the identity, we obtain the equivalence

$$\mathfrak{F} \circ \mathfrak{G} \circ \iota \circ \widetilde{\lambda} \sim \iota \circ \widetilde{\lambda}.$$

But, since the functor $\mathfrak{G}$ forgets the metric, $\mathfrak{G}(\iota \circ \lambda\overline{\mathcal{F}}) - \mathfrak{G}(\overline{\mathcal{F}})$ consists only in degenerate cubes. Therefore $\mathfrak{F} \circ \mathfrak{G} \circ \iota \circ \widetilde{\lambda} = \mathfrak{F} \circ \mathfrak{G}$. In consequence $\iota \circ \widetilde{\lambda} \sim \mathrm{Id}$. $\square$

8. Higher Bott-Chern forms

Let X be a smooth complex projective variety. The aim of this section is to give a morphism between the complex of emi-cubes and the complex $\mathcal{D}^*(X,*)$. This morphism will realize the characteristic classes from higher K-theory to real Deligne cohomology.

Observe that, since we want to realize the characteristic classes as a morphism of complexes of abelian groups, we will obtain a morphism of groups. In particular the induced map

$$K_0(X) \longrightarrow \bigoplus_p H_{\mathcal{D}}^{2p}(X,p)$$

will be additive. This forces us to choose the Chern character as our characteristic class. Nevertheless, since the Chern classes can be recovered from the components of the Chern character form, the formulae we obtain can be applied to any characteristic class.

The reason we restrict ourselves to projective varieties is to avoid the technical difficulties of the logarithmic singularities at infinity. But note that a main ingredient in the proof that higher Bott-Chern forms give Beilinson's regulator is the extension to quasi-projective varieties (see [**8**])

Let us see that Bott-Chern forms are the degree one step of the morphism of complexes we are looking for. If

$$\overline{\xi} : 0 \longrightarrow (S,h') \xrightarrow{f} (E,h) \longrightarrow (Q,h'') \longrightarrow 0$$

is an exact sequence of hermitian vector bundles, then

$$d\overline{\xi} = (E,h) - (Q,h'') - (S,h').$$

Therefore

$$\begin{aligned} d_{\mathcal{D}}\widetilde{\mathrm{ch}}_1(\overline{\xi}) &= -2\partial\overline{\partial}\widetilde{\mathrm{ch}}_1(\overline{\xi}) \\ &= \widetilde{\mathrm{ch}}(E,h) - \widetilde{\mathrm{ch}}(Q,h'') - \widetilde{\mathrm{ch}}(S,h') \\ &= \widetilde{\mathrm{ch}}(d\overline{\xi}). \end{aligned}$$

To extend this morphism to higher degrees, we will iterate the definition of Bott-Chern forms.

Let $\overline{\mathcal{F}} = \{\overline{\mathcal{F}}_\alpha\}$ be an emi-n-cube. The first step is to construct a geometric n-th order homotopy between the vertexes of $\overline{\mathcal{F}}$. The reason for calling it a homotopy will be clear in proposition 8.3. This homotopy will be a hermitian vector bundle defined over $X \times (\mathbb{P}^1)^n$ and will be called the n-th transgression bundle. Moreover, we want the n-th transgression bundle to be an exact functor. Let us define it inductively. If $n = 1$, an emi-1-cube is a short exact sequence $\overline{\xi}$ of hermitian vector bundles with the third metric induced by the second one. Then, the definition of the first transgression bundle is given in 6.2. It follows from the definition that tr_1 is an exact functor. Assume now that we have defined the functor tr_{n-1} and that it is an exact functor. As in 7.3.3, The emi-n-cube $\overline{\mathcal{F}}$ can be seen as an exact sequence of emi-$n-1$-cubes:

$$\partial_n^{\cdot}\overline{\mathcal{F}} : \partial_n^{\cdot}(\overline{\mathcal{F}})0 \longrightarrow \partial_n^{-1}\overline{\mathcal{F}} \longrightarrow \partial_n^{0}\overline{\mathcal{F}} \longrightarrow \partial_n^{1}\overline{\mathcal{F}} \longrightarrow 0.$$

Applying the functor tr_{n-1} to this exact sequence of emi-$n-1$-cubes, we obtain an exact sequence of hermitian vector bundles on $X \times (\mathbb{P}^1)^{n-1}$, denoted $\mathrm{tr}_{n-1}(\partial_n^{\cdot}\overline{\mathcal{F}})$.

Definition 8.1. Let $\overline{\mathcal{F}}$ be an emi-n-cube. Then the *n-th transgression bundle* is

$$\mathrm{tr}_n(\overline{\mathcal{F}}) = \mathrm{tr}_1(\mathrm{tr}_{n-1}(\partial_n^{\cdot}\overline{\mathcal{F}})).$$

Since tr_1 is an exact functor, and by induction hypothesis we may assume that tr_{n-1} is also an exact functor, we obtain that tr_n is also an exact functor.

Remark 8.2. An emi-n-cube can be seen as an exact sequence of emi-$n-1$-cubes in n different ways depending on which faces we take. Thus the above construction may depend, in principle, on the choice of an ordering of the subindexes. Nevertheless, the result is independent of this order. See for instance [**8**] Definition 3.8 for a more symmetric definition or [**28**] Proposition 2.1 for a proof of the invariance under permutations.

The basic property of the transgression bundle is the following.

Proposition 8.3 ([**8**] Proposition 3.9). *Let $\overline{\mathcal{F}}$ be an emi-n-cube. Let $(x_i : y_i)$ be homogeneous coordinates in the i-th factor of $(\mathbb{P}^1)^n$. Then*

$$\begin{aligned} \mathrm{tr}_n(\overline{\mathcal{F}})|_{\{x_i=0\}} &\cong \mathrm{tr}_{n-1}(\partial_i^0\overline{\mathcal{F}}), \\ \mathrm{tr}_n(\overline{\mathcal{F}})|_{\{y_i=0\}} &\cong \mathrm{tr}_{n-1}(\partial_i^{-1}\overline{\mathcal{F}}) \overset{\perp}{\oplus} \mathrm{tr}_{n-1}(\partial_i^1\overline{\mathcal{F}}). \end{aligned}$$

In view of this proposition, the n-th transgression bundle of an emi-n-cube is a homotopy between the $n-1$-transgression bundles of its faces.

The second step is to go from a homotopy of vector bundles to a homotopy of differential forms. This step is simple; the required homotopy is $\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))$ because by the functoriality of the Chern character form and Proposition 8.3 we obtain that

$$\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))|_{\{x_i=0\}} \cong \widetilde{\mathrm{ch}}(\mathrm{tr}_{n-1}(\partial_i^0\overline{\mathcal{F}})), \tag{8.1}$$

$$\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))|_{\{y_i=0\}} \cong \widetilde{\mathrm{ch}}(\mathrm{tr}_{n-1}(\partial_i^{-1}\overline{\mathcal{F}})) + \widetilde{\mathrm{ch}}(\mathrm{tr}_{n-1}(\partial_i^1\overline{\mathcal{F}})). \tag{8.2}$$

The third step is to integrate the differential form $\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))$ defined on $X \times (\mathbb{P}^1)^n$, to obtain a differential form, $\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})$, defined on X.

To this end we will introduce some currents on $(\mathbb{P}^1)^n$. Let us introduce the homological analogue of the complex $\mathcal{D}^*$, where these currents will live. For any smooth complex projective variety Y, let $D_*(Y)$ be the complex of currents on Y. That is, $D_n(Y)$ is the topological dual of $\mathcal{E}^n(Y)$. We will denote by $D_*^{\mathbb{R}}(Y)$ the subcomplex of real currents and by $D_{p,q}(Y)$ the currents of type p,q (i.e. the topological dual of $\mathcal{E}^{p,q}(Y)$). We will write $D_*^{\mathbb{R}}(Y)(p) = (2\pi i)^{-p}D_*^{\mathbb{R}}(Y)$.

Definition 8.4. Let $D_*(Y,*)$ be the complex defined by

$$\mathcal{D}_n(Y,p) = \begin{cases} D_n^{\mathbb{R}}(p)(Y) \cap \bigoplus\limits_{\substack{p'+q'=n \\ p'\le p,\ q'\le p}} D_{p',q'}(Y), & \text{for } n \le 2p. \\ D_{n+1}^{\mathbb{R}}(p+1)(X) \cap \bigoplus\limits_{\substack{p'+q'=n+1 \\ p'>p,\ q'>q}} D_{p',q'}(X), & \text{for } n \ge 2p+1. \end{cases}$$

The homology of the above complex is the Deligne homology of Y. If Y is equidimensional of dimension n then, for any form $\omega \in D^j(Y,p)$, we will denote by $[\omega] \in D_{2n-j}(Y, n-p)$ the current defined by

$$[\omega](\eta) = \frac{1}{(2\pi i)^n}\int_Y \omega \wedge \eta. \tag{8.3}$$

This morphism realizes the Poincaré duality. If ω is a locally integrable form, we will use also the notation $[\omega]$ to denote its associated current.

Let us denote by $d_j^i : (\mathbb{P}^1)^{n-1} \longrightarrow (\mathbb{P}^1)^n$, for $i = 1,\dots,n$ and $j = 0,\infty$ the inclusions given by

$$d_0^i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},(0:1),x_i,\dots,x_n)$$
$$d_\infty^i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},(1:0),x_i,\dots,x_n).$$

The currents we need in order to integrate the form $\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))$ are provided by the following result.

Theorem 8.5 ([**33**], see also [**8**] and [**15**]). *There exists a family of currents* $\{[W_n]\}_{n\ge 0}$ *with* $[W_n] \in \mathcal{D}_n((\mathbb{P}^1)^n, 0)$ *such that*

1. $[W_0] = 1$.
2. $d_{\mathcal{D}}[W_n] = \sum_{i=1}^n (-1)^i \left((d_0^i)_*[W_{n-1}] - (d_\infty^i)_*[W_{n-1}]\right)$.

Proof. By equation 6.9, we can write $[W_1] = [1/2 \log t\bar{t}]$. Let $p_i : (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1$ denote the projection over the i-th factor. Let us write $\lambda_i = p_i^*(1/2\log t\bar{t})$.

Then λ_i is a locally integrable function over $(\mathbb{P}^1)^n$. Let I' and E' be the homotopy equivalences introduced at the end of section 5. Then we will write

$$W_n = I'(E'(\lambda_1) \cup \cdots \cup E'(\lambda_n)) \tag{8.4}$$

which is a locally integrable form. The current $[W_n]$ is the associated current. Condition 2 is, formally, consequence of the equation 6.9 and Leibnitz rule (see [**8**] Proposition 6.7 and [**33**]). Explicitly, the forms W_n are given by (compare [**15**] 2.2)

$$W_n = \frac{(-1)^n}{2\,n!} \sum_{i=1}^{n} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{i-1} (-1)^{\sigma} \log(t_{\sigma(1)} \bar{t}_{\sigma(1)}) \\ \frac{dt_{\sigma(2)}}{t_{\sigma(2)}} \wedge \cdots \wedge \frac{dt_{\sigma(i)}}{t_{\sigma(i)}} \wedge \frac{d\bar{t}_{\sigma(i+1)}}{\bar{t}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{t}_{\sigma(n)}}{\bar{t}_{\sigma(n)}}.$$

□

Definition 8.6. Let $\overline{\mathcal{F}}$ be an emi-n-cube. Then the n-th Bott-Chern form of $\overline{\mathcal{F}}$ is

$$\begin{aligned} \widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}}) &= [W_n](\widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))) \\ &= \frac{1}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} W_n \wedge \widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}})). \end{aligned}$$

Let $\overline{\mathfrak{E}}$ be a small category of hermitian vector bundles over X (see section 7). Let us write $\widetilde{\mathbb{Z}}C^n_{emi}\mathfrak{E} = \widetilde{\mathbb{Z}}C^{emi}_{-n}\mathfrak{E}$. Then $\widetilde{\mathbb{Z}}C^*_{emi}\mathfrak{E}$ is a cohomological complex. The definition of higher Bott-Chern forms induces maps

$$\mathrm{ch} : \widetilde{\mathbb{Z}}C^n_{emi}\mathfrak{E} \longrightarrow \bigoplus_p \mathcal{D}^n(X,p)[2p].$$

Proposition 8.7. *The induced map*

$$\mathrm{ch} : \widetilde{\mathbb{Z}}C^*_{emi}\mathfrak{E} \longrightarrow \bigoplus_p \mathcal{D}^*(X,p)[2p].$$

is a morphism of complexes.

Proof. This proposition is a direct consequence of 8.1 and 8.5. □

The main result concerning higher Bott-Chern forms is

Theorem 8.8 ([**8**]). *The composition map*

$$K_i(X) \overset{Hurewicz}{\longrightarrow} H^{-i}(\widetilde{\mathbb{Z}}C^*_{emi}\mathfrak{E}) \longrightarrow \bigoplus_p H^{2p-i}_{\mathcal{D}}(X,\mathbb{R}(p))$$

agrees with Beilinson's regulator map.

Remark 8.9. The construction of higher Bott-Chern can be made working always with the Thom-Whitney simple. We define the Chern character form in the Thom-Whitney complex as

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{F}})_{TW} = E'(\widetilde{\mathrm{ch}}(\overline{\mathcal{F}})).$$

The analogues of the forms W_n are the forms

$$(W_n)_{TW} = E'(\lambda_1) \cup \cdots \cup E'(\lambda_n).$$

Then, for any emi-n-cube,$\overline{\mathcal{F}}$, the higher Bott-Chern form in the Thom-Whitney complex is

$$\widetilde{\mathrm{ch}}_n(\overline{\mathcal{F}})_{TW} = \frac{1}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} (W_n)_{TW} \cup \widetilde{\mathrm{ch}}(\mathrm{tr}_n(\overline{\mathcal{F}}))_{TW},$$

where the integral is computed componentwise. In this way we obtain a morphism which is multiplicative at the level of complexes (see [**8**]).

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