ARITHMETIC CHOW RINGS AND DELIGNE-BEILINSON COHOMOLOGY

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Introduction.

The arithmetic Chow rings have been introduced by Gillet and Soulé in [G-S] in order to generalize to higher dimensions the Arakelov intersection theory. The Archimedean component of these arithmetic Chow rings is handled by defining Green currents associated with algebraic cycles and, when the cycles intersect properly, by defining a $*$-product between Green currents. This product corresponds to the intersection product of cycles and is commutative and associative. In the same paper Gillet and Soulé extend their definition of arithmetic Chow rings to the quasi-projective case, and, in the projective case, they relate Green currents with Deligne-Beilinson cohomology. Nevertheless they suggest that, in the quasi-projective case, the definition of arithmetic Chow rings can be improved by taking into account the Hodge theory of quasi-projective complex varieties. In this way, the relationship with Deligne-Beilinson cohomology should also be true in the quasi-projective case.

In a previous paper [Bu 2] the author has introduced some spaces of Green forms and a $*$-product between them. These spaces were defined only for projective varieties by using $\partial\bar{\partial}$-cohomology. With this definition, the space of Green forms associated with an algebraic cycle is naturally isomorphic to the space of Green currents for the same cycle.

The aim of this paper is to extend the definition of arithmetic Chow rings to the quasi-projective case. To this end, we give a new definition of the space of Green forms associated with an algebraic cycle using Deligne-Beilinson cohomology. This definition coincides with the old one for projective varieties. But it has the following advantages: it is easily extended to the quasi-projective case. The existence of a Green form for an algebraic cycle is a direct consequence of the existence of the cycle class in Deligne-Beilinson cohomology. The proof of the commutativity and associativity of the $*$-product is purely formal and is simpler than the previous ones. Finally this definition is very flexible. Changing the complexes used to define Green forms one can construct Green objects with different properties.

The paper is organized as follows:

In §1 we recall the definition and some properties of real Deligne-Beilinson cohomology. In §2 we recall the definition of a complex which computes real Deligne-Beilinson cohomology. This complex allows us to represent a class in real Deligne-Beilinson cohomology by a single differential form. In §3 we define a multiplicative structure on this complex which is

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associative up to homotopy and commutative. This multiplicative structure gives the usual product in Deligne-Beilinson cohomology. In §4 we define the truncated relative cohomology groups which are the basis for the construction of Green forms. In §5 we define Green forms and study their properties. In §6 we recall some relationships between K-theory and Deligne-Beilinson cohomology. Finally in §7 we give a definition of arithmetic Chow rings. In the case of arithmetic projective varieties these arithmetic Chow rings coincide with the arithmetic Chow rings defined by Gillet and Soulé, whereas in the quasi-projective case the Chow ring defined here satisfies a homotopy property (Theorem 7.5).

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§1. Real Deligne-Beilinson Cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$. Throughout this paper we shall always work with the analytic topology of X. In this section we shall recall the definition of real Deligne-Beilinson cohomology of X. See [Be], [E-V] and [J] for details. Let us choose a smooth compactification $j : X \longrightarrow \overline{X}$, with $D = \overline{X} - X$ a divisor with normal crossings. Let Λ be a subring of $\mathbb{R}$. We write $\Lambda(p) = (2\pi i)^p \Lambda \subset \mathbb{C}$. We will denote also by Λ and $\Lambda(p)$ the corresponding constant sheaves in the analytic topology. Let Ω^*_X be the sheaf of holomorphic forms on X and let $\Omega^*_{\overline{X}}(\log D)$ be the sheaf of holomorphic forms on $\overline{X}$ with logarithmic singularities along D [De]. Let F^p be the Hodge filtration of $\Omega^*_{\overline{X}}(\log D)$:

$$F^p \Omega^*_{\overline{X}}(\log D) = \bigoplus_{p' \geq p} \Omega^{p'}_{\overline{X}}(\log D).$$

Since j is affine, $Rj_* \Omega^*_X = j_* \Omega^*_X$. Moreover, in the derived category, there are natural maps

$$u_1 : Rj_* \Lambda(p) \longrightarrow j_* \Omega^*_X \qquad \text{and} \qquad u_2 : \Omega^*_{\overline{X}}(\log D) \longrightarrow j_* \Omega^*_X.$$

If (A^*, d) is a complex we shall write $A[k]^*$ for the complex $A[k]^n = A^{k+n}$, with differential $(-1)^k d$. If $f : A^* \longrightarrow B^*$ is a morphism of complexes, the simple of f is the complex

$$s(f)^* = A^* \oplus B[-1]^*,$$

with differential $d(a, b) = (da, f(a) - db)$.

The Deligne-Beilinson complex of the pair $(X, \overline{X})$ is

$$\Lambda(p)_{\mathcal{D}} = s(u : Rj_* \Lambda(p) \oplus F^p \Omega^*_{\overline{X}}(\log D) \longrightarrow j_* \Omega^*_X),$$

where $u(a, \omega) = u_2(\omega) - u_1(a)$.

The Λ-Deligne-Beilinson cohomology groups of X are defined by

$$H^*_{\mathcal{D}}(X, \Lambda(p)) = \mathbb{H}^*(\overline{X}, \Lambda(p)_{\mathcal{D}}).$$

These groups are independent of the compactification $\overline{X}$.

If $Y \subset X$ is a closed algebraic subset, then there are also defined Deligne-Beilinson cohomology groups of X with supports on Y, denoted by $H^*_{\mathcal{D},Y}(X, \Lambda(p))$. Moreover, using simplicial techniques, we can define Deligne-Beilinson cohomology groups for singular varieties. There is also the dual notion of Deligne-Beilinson homology groups denoted by $H^{\mathcal{D}}_*(X, \Lambda(p))$. Deligne-Beilinson cohomology and homology groups form a twisted Poincaré duality theory in the sense of Bloch and Ogus [B-O].

We can compare Λ-Deligne-Beilinson cohomology with cohomology with coefficients in Λ by means of the exact sequence

$$0 \to s(F^p\Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega^*_X) \to \Lambda(p)_{\mathcal{D}} \to Rj_*\Lambda(p) \to 0.$$

From this sequence and the degeneracy of the spectral sequence associated to the Hodge filtration, we obtain:

1.1. Proposition. *Let X be a smooth variety over $\mathbb{C}$ and let Λ a subring of $\mathbb{R}$. Then there is a cohomology long exact sequence*

$$\ldots \to H^{n-1}(X,\mathbb{C}) \big/ F^pH^{n-1}(X,\mathbb{C}) \to H^n_{\mathcal{D}}(X,\Lambda(p)) \to H^n(X,\Lambda(p)) \to \ldots.$$

Λ-Deligne-Beilinson cohomology studies the relationship between the Λ-structure and the Hodge filtration in cohomology. In general, we do not have a complex which gives us both the Λ-structure and the Hodge filtration. For this reason, we have to construct Deligne-Beilinson cohomology from a diagram of complexes. On the other hand, in the case of real Deligne-Beilinson cohomology, we shall see that we can construct a complex, $E^*_{\log}(X)$, which carries the real structure and the Hodge filtration.

Let X now denote a complex manifold and let D be a divisor with normal crossings on X. Let us write $V = X - D$ and let $j : V \longrightarrow X$ be the inclusion. Let $\mathcal{E}^*_X$ be the sheaf of complex C^∞ differential forms on X. The complex of sheaves $\mathcal{E}^*_X(\log D)$ (see [Bu 1]) is the sub-$\mathcal{E}^*_X$ algebra of $j_*\mathcal{E}^*_V$ generated locally by the sections

$$\log z_i\overline{z}_i,\ \frac{dz_i}{z_i},\ \frac{d\overline{z}_i}{\overline{z}_i},\quad \text{for } i = 1,\ldots,M,$$

where $z_1 \ldots z_M = 0$ is a local equation of D.

Let us write $E^*_X(\log D) = \Gamma(X, \mathcal{E}^*_X(\log D))$, and let $E^*_{X,\mathbb{R}}(\log D)$ be the subcomplex of real forms.

Let X be again a smooth algebraic variety over $\mathbb{C}$ and let I be the category of all smooth compactifications of X. That is, an element $(\widetilde{X}_\alpha, \imath_\alpha)$ of I is a smooth complex variety $\widetilde{X}_\alpha$ and an immersion $\imath_\alpha : X \hookrightarrow \widetilde{X}_\alpha$ such that $D_\alpha = \widetilde{X}_\alpha - \imath_\alpha(X)$ is a normal crossing divisor. The morphisms of I are the maps $f : \widetilde{X}_\alpha \longrightarrow \widetilde{X}_\beta$ such that $f \circ \imath_\alpha = \imath_\beta$. The opposed category, I^o is directed (see [De]).

1.2. Definition. *The complex of differential forms with logarithmic singularities along infinity* is

$$E^*_{\log}(X) = \varinjlim_{\alpha \in I^o} E^*_{\widetilde{X}_\alpha}(\log D_\alpha).$$

This complex is a subcomplex of $E^*_X = \Gamma(X, \mathcal{E}^*_X)$. We shall denote by $E^*_{\log,\mathbb{R}}(X)$ the corresponding real subcomplex.

The complex $E^*_{\log}(X)$ has a natural bigrading

$$E^*_{\log}(X) = \bigoplus E^{p,q}_{\log}(X).$$

The Hodge filtration of this complex is defined by

$$F^p E^n_{\log}(X) = \bigoplus_{\substack{p' \geq p \\ p'+q'=n}} E^{p',q'}_{\log}(X).$$

By the results of [Bu 1], if f is a morphism of I, the morphism

$$f^* : (E^*_{\widetilde{X}_\alpha}(\log D_\alpha), F) \longrightarrow (E^*_{\widetilde{X}_\beta}(\log D_\beta), F)$$

is a real filtered quasi-isomorphism. Moreover, for all $\alpha \in I^o$, the filtration F induces in $H^*(E^*_{X_\alpha}(\log D_\alpha)) = H^*(X - D, \mathbb{C})$ the Hodge filtration of the mixed Hodge structure of $H^*(X - D, \mathbb{Z})$ introduced by Deligne in [De].

Since I^o is directed, all the induced morphisms

$$(E^*_{\widetilde{X}_\alpha}(\log D_\alpha), F) \longrightarrow (E^*_{\log}(X), F)$$

are filtered quasi-isomorphisms.

Choose $\overline{X}$ to be a smooth compactification of X, with $D = \overline{X} - X$ a divisor with normal crossings. We write $E^*_{\log,\mathbb{R}}(X, p) = (2\pi i)^p E^*_{\log,\mathbb{R}}(X) \subset E^*_{\log}(X)$. Then, in the derived category of complexes of vector spaces, there are isomorphisms

$$\begin{aligned} &R\Gamma R j_* \mathbb{R}_X(p) \longrightarrow E^*_{\log,\mathbb{R}}(X, p), \\ &R\Gamma j_* \Omega^*_X \longrightarrow E^*_{\log}(X) \qquad \text{and} \\ &R\Gamma F^p \Omega^*_{\overline{X}}(\log D) \longrightarrow F^p E^*_{\log}(X). \end{aligned}$$

Let us write

$$E^*_{\log,\mathbb{R}}(X, p)_{\mathcal{D}} := s(u : E^*_{\log,\mathbb{R}}(X, p) \oplus F^p E^*_{\log}(X) \longrightarrow E^*_{\log}(X)),$$

where $u(a, b) = b - a$. Then we have:

1.3. Proposition. *The real Deligne-Beilinson cohomology groups of X can be computed as the cohomology of the complex $E^*_{\log,\mathbb{R}}(X, p)_{\mathcal{D}}$. That is*

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) = H^*(E^*_{\log,\mathbb{R}}(X, p)_{\mathcal{D}}).$$

1.4. Remark. In the above definition we can use real analytic forms instead of C^∞ forms obtaining the complexes $A^*_X(\log D)$ and $A^*_{\log}(X)$. The first one was introduced by Navarro Aznar in [N]. The cohomological properties of the differentiable complexes and of the real analytic complexes are the same. Therefore, throughout the construction of arithmetic Chow groups, C^∞ forms can be replace by real analytic forms. In particular, this implies the existence of real analytic Green forms.

§2. A Deligne-Beilinson Complex.

The goal of this section is to obtain a simpler version of the complex $E^*_{\log,\mathbb{R}}(X,p)_{\mathcal{D}}$. That is to construct a complex $\mathfrak{D}^*(E^*_{\log}(X),p)$ which allows us to represent Deligne-Beilinson cohomology classes by a single differential form. This construction has been done by X. Wang ([Wa]) in the projective case.

Let us recall the construction of the connection morphism of an exact sequence. Let

$$0 \to A^* \xrightarrow{\iota} B^* \xrightarrow{\pi} C^* \to 0$$

be an exact sequence of complexes of vector spaces. Let us choose a linear section σ of π. Then we can obtain a retraction τ of ι by

$$\tau(b) = \iota^{-1}(b - \sigma\pi b).$$

The connection morphism is induced by the morphism of complexes

$$\mathrm{Res}_\sigma : C^*[-1] \longrightarrow A^*,$$

defined by

$$\mathrm{Res}_\sigma(c) = \iota^{-1}(\sigma dc - d\sigma c).$$

If there is no danger of confusion we will write simply Res instead of Res_σ. It is straightforward to check that $\sigma dc - d\sigma c$ belongs to $\mathrm{Im}\,\iota$ and that Res is a morphism of complexes. Moreover, the induced morphism $\mathrm{Res} : H^*(C[-1]) \longrightarrow H^*(A)$ is the composition of the natural morphisms

$$H^*(C[-1]) \to H^*(s(B \longrightarrow C)) \xrightarrow{\cong} H^*(A).$$

If σ' is another section then the morphisms Res_σ and $\mathrm{Res}_{\sigma'}$ are homotopically equivalent.

We can also obtain Res from the retraction τ by the formula

$$\mathrm{Res}(\pi b) = d\tau b - \tau db.$$

Let now $u : A^* \longrightarrow B^*$ be a morphism of complexes of vector spaces. We can decompose u into two exact sequences

$$0 \to \mathrm{Im}(u)^* \xrightarrow{\iota} B^* \xrightarrow{\pi} \mathrm{Coker}(u)^* \to 0.$$

and

$$0 \to \mathrm{Ker}(u)^* \xrightarrow{j} A^* \xrightarrow{u'} \mathrm{Im}(u)^* \to 0$$

Let σ_1 and σ_2 be linear sections of π and u' respectively. Let τ_1 and τ_2 be the corresponding retractions of ι and j. Let us write $\mathrm{Res}_1 = \mathrm{Res}_{\sigma_1}$ and $\mathrm{Res}_2 = \mathrm{Res}_{\sigma_2}$.

We define a complex

$$\widehat{s}(u)^* = \mathrm{Ker}(u)^* \oplus \mathrm{Coker}(u)^*[-1]$$

with differential $d(a,b) = (da + \mathrm{Res}_2\,\mathrm{Res}_1\, b, -db)$.

Observe that the complex $\widehat{s}(u)^*$ depends on the choice of the sections σ_1 and σ_2, but a different choice of sections gives a homotopically equivalent complex. The main result concerning the complex $\widehat{s}(u)^*$ is the following.

2.1. Proposition. *Let $u: A^* \longrightarrow B^*$ be a morphism of complexes of vector spaces. Then the maps $\varphi : \widehat{s}(u) \longrightarrow s(u)$ and $\psi : s(u) \longrightarrow \widehat{s}(u)$ given by*

$$\begin{aligned} \varphi(a,b) &= (j(a) - \sigma_2 \operatorname{Res}_1 b, \sigma_1 b) \quad \textit{and} \\ \psi(a,b) &= (\tau_2 a + \operatorname{Res}_2(\tau_1 b), \pi b), \end{aligned}$$

are morphisms of complexes. Moreover, they are homotopy equivalences, one the inverse of the other. More explicitly, we have $\psi\varphi = Id$ and

$$\varphi\psi - Id = dh + hd,$$

where $h: s(u)^n \longrightarrow s(u)^{n-1}$ is given by $h(a,b) = (-\sigma_2\tau_1 b, 0)$.

Proof. All the checks are straightforward. For instance let us check that $\varphi\psi - \mathrm{Id} = dh + hd$. We have

$$\begin{aligned} \varphi\psi(a,b) &= \varphi(\tau_2 a + \operatorname{Res}_2 \tau_1 b, \pi b) \\ &= (j\tau_2 a + j \operatorname{Res}_2 \tau_1 b - \sigma_2 \operatorname{Res}_1 \pi b, \sigma_1 \pi b). \end{aligned}$$

Therefore

$$\varphi\psi(a,b) - (a,b) = (j\tau_2 a - a + j \operatorname{Res}_2 \tau_1 b - \sigma_2 \operatorname{Res}_1 \pi b, \sigma_1 \pi b - b).$$

On the other hand

$$\begin{aligned} dh(a,b) + hd(a,b) &= d(-\sigma_2\tau_1 b, 0) + h(da, ua - db) \\ &= (-d\sigma_2\tau_1 b - \sigma_2\tau_1 ua + \sigma_2\tau_1 db, -u\sigma_2\tau_1 b). \end{aligned}$$

Hence the result follows from the equalities

$$\begin{aligned} u\sigma_2\tau_1 b &= \iota\tau_1 b = b - \sigma_1\pi b, \\ \sigma_2\tau_1 ua &= \sigma_2 u' a = a - j\tau_2 a \quad \text{and} \\ \operatorname{Res}_2 \tau_1 b\sigma_2 - \operatorname{Res}_1 \pi b &= \sigma_2\tau_1 db - \sigma_2 d\tau_1 b + \operatorname{Res}_2 \tau_1 b \\ &= \sigma_2\tau_1 db - d\sigma_2\tau_1 b. \end{aligned}$$

We want to apply 2.1 to the complex $E^*_{\log,\mathbb{R}}(X,p)_{\mathcal{D}}$. Since the process of simplification depends only on the relationship between the real structure, the differential and the bigrading, we shall work with an abstract Dolbeault complex.

2.2. Definition. A *Dolbeault (cochain) complex* is a complex of real vector spaces $(A^*_{\mathbb{R}}, d)$ provided with a bigrading on $A^*_{\mathbb{C}} = A^*_{\mathbb{R}} \otimes \mathbb{C}$:

$$A^d_{\mathbb{C}} = \bigoplus_{p+q=d} A^{p,q},$$

such that

DC1. The differential d can be decomposed as a sum of operators $d = \partial + \overline{\partial}$ of type $(1,0)$ and $(0,1)$.

DC2. It satisfies the symmetry property

$$\overline{A^{p,q}} = A^{q,p},$$

where $\overline{}$ denotes complex conjugation.

By **DC2** the operator $\overline{\partial}$ is the complex conjugate of ∂.

Let A be a Dolbeault complex. The *Hodge filtration* F of A^* is

$$F^p A^n = \bigoplus_{\substack{p' \geq p \\ q'+p'=n}} A^{p',q'}.$$

We denote by $\overline{F}$ the filtration complex conjugate of F. That is

$$\overline{F}^p A^n = \bigoplus_{\substack{p' \geq p \\ q'+p'=n}} A^{q',p'}.$$

Examples of Dolbeault complexes are the complex of C^∞ (or real analytic) differential forms on a complex manifold and the complex of C^∞ differential forms with logarithmic singularities at infinity.

Let A^* be a Dolbeault complex. We write $A^*_{\mathbb{R}}(p) = (2\pi i)^p A^*_{\mathbb{R}} \subset A^*_{\mathbb{C}}$ and

$$A^*_{\mathbb{R}}(p)_{\mathcal{D}} = s(A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \xrightarrow{u} A^*_{\mathbb{C}}),$$

where $u(a,b) = b - a$. For example, if X is a smooth variety over $\mathbb{C}$ and $A^* = E^*_{\log}(X)$, then we have seen that

$$H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}}) = H^*_{\mathcal{D}}(X, \mathbb{R}(p)).$$

On the other hand, if $A^* = E^*_X$ is the complex of C^∞ differential forms on X, then the groups $H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}})$ are called analytic Deligne cohomology groups.

Let us apply Proposition 2.1 to the morphism $u : A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \longrightarrow A^*_{\mathbb{C}}$. The first step is to compute $\operatorname{Ker} u$ and $\operatorname{Coker} u$.

2.3. Lemma. *Let A^* be a Dolbeault complex. The morphism*

$$u : A^n_{\mathbb{R}}(p) \oplus F^p A^n_{\mathbb{C}} \longrightarrow A^n_{\mathbb{C}}$$

is injective for $n \leq 2p-1$ and surjective for $n \geq 2p-1$. In particular for $n = 2p-1$ it is an isomorphism. Moreover we have

$$\begin{aligned} \operatorname{Coker}(u)^n &= A^n_{\mathbb{C}} \big/ (A^n_{\mathbb{R}}(p) + F^p A^n_{\mathbb{C}} + \overline{F}^p A^n_{\mathbb{C}}) \\ &\cong A^n_{\mathbb{R}}(p-1) \big/ (A^n_{\mathbb{R}}(p-1) \cap (F^p A^n_{\mathbb{C}} + \overline{F}^p A^n_{\mathbb{C}})) \\ &\cong A^n_{\mathbb{R}}(p-1) \cap \bigoplus_{\substack{p'+q'=n \\ p'<p,\ q'<p}} A^{p',q'}, \end{aligned}$$

and

$$\begin{aligned}\operatorname{Ker}(u)^n &\cong A^n_{\mathbb{R}}(p)\cap F^pA^n_{\mathbb{C}}\cap \overline{F}^pA^n_{\mathbb{C}}\\ &= A^n_{\mathbb{R}}(p)\cap \bigoplus_{\substack{p'+q'=n\\ p'\geq p,\ q'\geq p}} A^{p',q'}.\end{aligned}$$

Proof. Since for a Dolbeault complex we have

$$\begin{aligned}A^n_{\mathbb{C}} &= F^pA^n_{\mathbb{C}}+\overline{F}^qA^n_{\mathbb{C}} && \text{for } p+q\leq n+1, \text{ and}\\ \{0\} &= F^pA^n_{\mathbb{C}}\cap\overline{F}^qA^n_{\mathbb{C}} && \text{for } p+q\geq n+1,\end{aligned}$$

it is enough to prove the descriptions of $\operatorname{Coker}(u)$ and $\operatorname{Ker}(u)$.

Clearly

$$\operatorname{Im} u\subset A^n_{\mathbb{R}}(p)+F^pA^n_{\mathbb{C}}+\overline{F}^pA^n_{\mathbb{C}}.$$

Let $x\in\overline{F}^pA^n_{\mathbb{C}}$. Then $\overline{x}\in F^pA^n_{\mathbb{C}}$ and $x+(-1)^p\overline{x}\in A^n_{\mathbb{R}}(p)$. Therefore

$$A^n_{\mathbb{R}}(p)+F^pA^n_{\mathbb{C}}+\overline{F}^pA^n_{\mathbb{C}}=A^n_{\mathbb{R}}(p)+F^pA^n_{\mathbb{C}},$$

and

$$\operatorname{Coker}(u)^n=A^n_{\mathbb{C}}\big/\left(A^n_{\mathbb{R}}(p)+F^pA^n_{\mathbb{C}}+\overline{F}^pA^n_{\mathbb{C}}\right).$$

If $(a,b)\in\operatorname{Ker} u$, then $a=b$ and $a\in A^n_{\mathbb{R}}(p)\cap F^pA^n_{\mathbb{C}}$. Therefore $a=(-1)^p\overline{a}\in\overline{F}^pA^n_{\mathbb{C}}$. Hence

$$\operatorname{Ker}(u)^n\cong A^n_{\mathbb{R}}(p)\cap F^pA^n_{\mathbb{C}}\cap\overline{F}^pA^n_{\mathbb{C}}.$$

The next step is to choose linear sections of the maps

$$\pi: A^*_{\mathbb{C}}\longrightarrow\operatorname{Coker}(u)^*\quad\text{and}\quad u': A^*_{\mathbb{R}}(p)\oplus F^pA^*_{\mathbb{C}}\longrightarrow\operatorname{Im}(u)^*.$$

In order to give explicit expressions of these sections, let us introduce some maps. Let

$$\pi_p: A^*_{\mathbb{C}}\longrightarrow A^*_{\mathbb{R}}(p)$$

be the projection obtained from the direct sum decomposition $A^*_{\mathbb{C}}=A^*_{\mathbb{R}}(p)\oplus A^*_{\mathbb{R}}(p-1)$. Namely, we have

$$\pi_px=\frac{1}{2}(x+(-1)^p\overline{x}).$$

Let $x=\sum x^{p,q}\in A^*_{\mathbb{C}}$. We will denote by

$$F^px=\sum_{p'\geq p}x^{p',q},$$

the projection over $F^pA^*_{\mathbb{C}}$ and by

$$F^{p,p}x = \sum_{\substack{p'\geq p\\ q'\geq p}} x^{p',q'},$$

the projection over $F^pA^*_{\mathbb{C}} \cap \overline{F}^pA^*_{\mathbb{C}}$.

By Lemma 2.3, $\operatorname{Coker}(u)^n$ may be identified with the subgroup of $A^n_{\mathbb{C}}$

$$A^n_{\mathbb{R}}(p-1) \cap \bigoplus_{\substack{p'+q'=n\\ p'<p,\ q'<p}} A^{p',q'} = F^{n-p+1} \cap \overline{F}^{n-p+1} \cap A^n_{\mathbb{R}}(p-1).$$

Let us write $q = n-p+1$. Then, with the above identification, the morphism $\pi : A^n_{\mathbb{C}} \longrightarrow \operatorname{Coker}(u)^n$ is

$$\pi(x) = \pi_{p-1}(F^{q,q}x).$$

This gives us a natural way to choose a section σ_1 of π: the inclusion

$$F^qA^n_{\mathbb{C}} \cap \overline{F}^qA^n_{\mathbb{C}} \cap A^n_{\mathbb{R}}(p-1) \longrightarrow A^n_{\mathbb{C}}.$$

With this choice of σ_1 we have

$$\tau_1(x) = \begin{cases} x - \pi_{p-1}(F^{q,q}x), & \text{for } n \leq 2p-2,\\ x, & \text{for } n \geq 2p-1.\end{cases}$$

And

$$\operatorname{Res}_1(x) = -\pi_{p-1}(F^p dx), \qquad \text{for } x \in \operatorname{Coker}(u)^n \text{ and } n \leq 2p-2.$$

Let us look for a section σ_2 of u'. In this case it is simpler to look for a retraction τ_2 of the map

$$j : \operatorname{Ker}(u)^n \longrightarrow A^n_{\mathbb{R}}(p) \oplus F^pA^n_{\mathbb{C}}.$$

Since for $n < 2p$ the map u is injective, then $\tau_2(a,f) = 0$ in this case.

By lemma 2.3 we have

$$\operatorname{Ker}(u)^n \cong A^n_{\mathbb{R}}(p) \cap F^pA^n_{\mathbb{C}} \cap \overline{F}^pA^n_{\mathbb{C}}.$$

Let us write $q = n-p+1$. We have the following direct sum decompositions

$$\begin{aligned} A^n_{\mathbb{R}}(p) &= F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p) \,\oplus\, (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p) \text{ and}\\ F^pA^n_{\mathbb{C}} &= F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p) \,\oplus\, F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \,\oplus\, F^q. \end{aligned}$$

Thus we can impose the condition

(2.4) $\qquad \tau_2(a,f) = 0,$ for $a \in (F^q + \overline{F}^q) \cap A^n_{\mathbb{R}}(p)$ and $f \in F^p \cap \overline{F}^p \cap A^n_{\mathbb{R}}(p-1) \oplus F^q$.

There are several choices of retraction τ_2 satisfying condition 2.4; among them we choose

$$\tau_2(a,f) = F^{p,p}a.$$

With this choice we obtain

$$\sigma_2 x = (-2\pi_p(x - F^p x), 2F^p(\pi_{p+1}x) - (-1)^{p-1}F^{p,p}\overline{x}),$$

and

$$\mathrm{Res}_2(x) = \begin{cases} 0, & \text{for } n < 2p-1, \\ 2\pi_p(\partial x^{p-1,n-p+1}), & \text{for } n \geq 2p+1. \end{cases}$$

2.5. Definition. Let A^* be a Dolbeault complex. The Deligne-Beilinson complex associated with A is the complex

$$\mathfrak{D}^*(A,p) = \widehat{s}(A^*_{\mathbb{R}}(p) \oplus F^p A^*_{\mathbb{C}} \xrightarrow{u} A^*_{\mathbb{C}}).$$

The differential of this complex will be denoted by $d_{\mathfrak{D}}$.

Let us summarize the results of this section.

2.6. Theorem. *Let A^* be a Dolbeault complex. Then*

1) The complex $\mathfrak{D}^*(A,p)$ is given by

$$\mathfrak{D}^n(A,p) = \begin{cases} A^{n-1}_{\mathbb{R}}(p-1) \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p,\ q'<p}} A^{p',q'}, & \textit{for } n \leq 2p-1, \\ A^n_{\mathbb{R}}(p) \cap \bigoplus\limits_{\substack{p'+q'=n \\ p'\geq p,\ q'\geq p}} A^{p',q'}, & \textit{for } n \geq 2p. \end{cases}$$

For $x \in \mathfrak{D}^n(A,p)$ the differential $d_{\mathfrak{D}}$ is given by

$$d_{\mathfrak{D}} x = \begin{cases} -\pi(dx), & \textit{for } n < 2p-1 \textit{ and} \\ -2\partial\overline{\partial}x, & \textit{for } n = 2p-1, \\ dx, & \textit{for } n \geq 2p, \end{cases}$$

where $\pi : A^* \longrightarrow \mathrm{Coker}(u)^*$ is the projection.

2) The complexes $A^*_{\mathbb{R}}(p)_{\mathcal{D}}$ and $\mathfrak{D}^*(A,p)$ are homotopically equivalent. The homotopy equivalences $\psi : A^n_{\mathbb{R}}(p)_{\mathcal{D}} \longrightarrow \mathfrak{D}^n(A,p)$ and $\varphi : \mathfrak{D}^n(A,p) \longrightarrow A^n_{\mathbb{R}}(p)_{\mathcal{D}}$ are given by

$$\psi(a,f,\omega) = \begin{cases} \pi(\omega), & \textit{for } n \leq 2p-1 \textit{ and} \\ F^{p,p}a + 2\pi_p(\partial\omega^{p-1,n-p+1}), & \textit{for } n \geq 2p, \end{cases}$$

and

$$\varphi(x) = \begin{cases} (\partial x^{p-1,n-p} - \overline{\partial}x^{n-p,p-1}, 2\partial x^{p-1,n-p}, x), & \textit{for } n \leq 2p-1 \textit{ and} \\ (x,x,0), & \textit{for } n \geq 2p. \end{cases}$$

Moreover $\psi\varphi = Id$ *and* $\varphi\psi - Id = dh + hd$*, where* $h : A^n_{\mathbb{R}}(p)_{\mathcal{D}} \longrightarrow A^{n-1}_{\mathbb{R}}(p)_{\mathcal{D}}$ *is given by*

$$h(a,f,\omega) = \begin{cases} (\pi_p(\overline{F}^p\omega + \overline{F}^{n-p}\omega), -2F^p(\pi_{p-1}\omega), 0), & \text{for } n \leq 2p-1 \text{ and} \\ (2\pi_p(\overline{F}^{n-p}\omega), -F^{p,p}\omega - 2F^{n-p}(\pi_{p-1}\omega), 0), & \text{for } n \geq 2p. \end{cases}$$

3) *The natural morphism* $H^*(A^*_{\mathbb{R}}(p)_{\mathcal{D}}) \longrightarrow H^*(A^*_{\mathbb{R}}(p))$ *is induced by the morphism of complexes*

$$r_p : \mathfrak{D}^*(A,p) \longrightarrow A^*_{\mathbb{R}}(p)$$

given by

$$r_p x = \begin{cases} 2\pi_p(F^p dx) = \partial x^{p-1,n-p} - \overline{\partial} x^{n-p,p-1}, & \text{for } n \leq 2p-1 \text{ and} \\ x, & \text{for } n \geq 2p. \end{cases}$$

2.7. Corollary. *Let* X *be a smooth variety over* $\mathbb{C}$*. then*

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) = H^*(\mathfrak{D}^*(E^*_{\log}(X), p)).$$

2.8. Remark. Let A be a Dolbeault complex. By construction, the cohomology groups $H^{2p}(\mathfrak{D}^*(A,p))$ are

$$H^{2p}(\mathfrak{D}^*(A,p)) = \left\{x \in A^{p,p} \cap A^{2p}_{\mathbb{R}}(p) \mid dx = 0\right\} / \mathrm{Im}(\partial\overline{\partial})$$

Therefore they are the $\mathbb{R}(p)$-part of the $\partial\overline{\partial}$-cohomology of A. In particular we have a relation between $\partial\overline{\partial}$ cohomology and real Deligne-Beilinson cohomology. On the other hand we have

$$H^{2p-1}(\mathfrak{D}^*(A,p)) = \left\{x \in A^{p-1,p-1} \cap A^{2p-2}_{\mathbb{R}}(p-1) \mid \partial\overline{\partial} x = 0\right\} / (\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}) \,.$$

A variant of this complex has been used in [Dem] to study the properties of $\partial\overline{\partial}$-cohomology.

2.9. Remark. The complex $\mathfrak{D}^*(A,p)$, the maps φ and r_p and the map ψ, for $n < 2p$, do not depend on the choice of the section σ_2. Only the map ψ for $n \geq 2p$ depends on the choice of σ_2. Moreover the maps φ, ψ and the homotopy h are natural. That is, given a morphism $A \longrightarrow B$ between Dolbeault complexes there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{D}^*(A,p) & \xrightarrow{\varphi} & A^*_{\mathbb{R}}(p)_{\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathfrak{D}^*(B,p) & \xrightarrow{\varphi} & B^*_{\mathbb{R}}(p)_{\mathcal{D}} \end{array}$$

and analogous diagrams for ψ and h.

§3. Multiplicative Structure of Deligne-Beilinson Cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$ and $\overline{X}$ a smooth compactification of X with $\overline{X} - X$ a divisor with normal crossings. For each real number $0 \leq \alpha \leq 1$, there is defined a product $\cup_\alpha$ on the Deligne-Beilinson complex $\Lambda(p)_{\mathcal{D}}$ (see [Be] or [E-V, §3]). All these products are homotopically equivalent. Moreover the product obtained for $\alpha = 1/2$ is graded-commutative and the products obtained for $\alpha = 0$ and $\alpha = 1$ are associative. Therefore they induce an associative and commutative product in Deligne-Beilinson cohomology denoted $\cup$. We want to transport this multiplicative structure to the complex $\mathfrak{D}^*(A, \cdot)$.

3.1. Definition. Let A be a Dolbeault complex. We say that A is a *Dolbeault algebra* if there is a product

$$A^*_{\mathbb{R}} \otimes A^*_{\mathbb{R}} \xrightarrow{\wedge} A^*_{\mathbb{R}}$$

such that $A^*_{\mathbb{R}}$ is a differential associative graded-commutative algebra, and the induced product on $A^*_{\mathbb{C}}$ is compatible with the bigrading. That is

$$A^{p,q} \wedge A^{p',q'} \subset A^{p+p',q+q'}.$$

Let $(A, d, \wedge)$ be a Dolbeault algebra and let $0 \leq \alpha \leq 1$ be a real number. The product $\cup_\alpha$ of the Deligne-Beilinson complex corresponds to the product

$$A^n_{\mathbb{R}}(p)_{\mathcal{D}} \otimes A^m_{\mathbb{R}}(q)_{\mathcal{D}} \xrightarrow{\cup_\alpha} A^{n+m}_{\mathbb{R}}(p+q)_{\mathcal{D}}$$

defined, for

$$(a_p, f_p, \omega_p) \in A^n_{\mathbb{R}}(p)_{\mathcal{D}} = (2\pi i)^p A^n_{\mathbb{R}} \oplus F^p A^n \oplus A^{n-1}$$

and

$$(a_q, f_q, \omega_q) \in A^m_{\mathbb{R}}(q)_{\mathcal{D}} = (2\pi i)^q A^m_{\mathbb{R}} \oplus F^q A^m \oplus A^{m-1},$$

by

$$\begin{aligned}(a_p, f_p, \omega_p) \cup_\alpha (a_q, f_q, \omega_q) &= \\ &(a_p \wedge a_q, f_p \wedge f_q, \alpha(\omega_p \wedge a_q + (-1)^n f_p \wedge \omega_q) + (1-\alpha)(\omega_p \wedge f_q + (-1)^n a_p \wedge \omega_q)).\end{aligned}$$

In order to define a product in $\mathfrak{D}^*(A, \cdot)$ we shall use the following result.

3.2. Proposition. *Let A^* and B^* be complexes of modules over a ring, such that there are homotopy equivalences $\varphi : A^* \longrightarrow B^*$ and $\psi : B^* \longrightarrow A^*$, one the inverse of the other. Assume furthermore that there is defined a product in B^*. That is, a morphism of complexes*

$$B^* \otimes B^* \xrightarrow{\cup_B} B^*.$$

Then
1) The map

$$A^* \otimes A^* \xrightarrow{\cup_A} A^*,$$

defined by $x \cup_A y = \psi(\varphi x \cup_B \varphi y)$ *is a morphism of complexes.*

2) *If the product* $\cup_B$ *is associative or associative up to homotopy then the product* $\cup_A$ *is associative up to homotopy.*
3) *If the product* $\cup_B$ *is graded commutative, the same is true for* $\cup_A$. *If it is graded commutative up to homotopy, then* $\cup_A$ *is graded commutative up to homotopy.*

Proof. To prove that $\cup_A$ is a morphism of complexes we use that φ, ψ and $\cup_B$ are morphisms of complexes. The statement about commutativity follows easily from the definition of $\cup_A$.

Assume now that $\cup_B$ is associative. Let h be the homotopy between $\varphi\psi$ and Id. That is

$$\varphi\psi - \mathrm{Id} = hd + dh.$$

Let us define a map

$$A^n \otimes A^m \otimes A^l \xrightarrow{h_{\mathrm{a}}} A^{n+m+l-1}$$

by

$$h_{\mathrm{a}}(a \otimes b \otimes c) = \psi(h(\varphi a \cup_B \varphi b) \cup_B \varphi c) + (-1)^{n+1}\psi(\varphi a \cup_B h(\varphi b \cup_B \varphi c)).$$

Then we can check easily that

$$(a \cup_A b) \cup_A c - a \cup_A (b \cup_A c) = h_{\mathrm{a}} d(a \otimes b \otimes c) + dh_{\mathrm{a}}(a \otimes b \otimes c).$$

The case when $\cup_B$ is only associative up to homotopy is analogous.

Applying Proposition 3.2. to $A^*_{\mathbb{R}}(p)_{\mathcal{D}}$ and $\mathfrak{D}(A^*, p)$ we obtain

3.3. Theorem. *Let* $(A, d, \wedge)$ *be a Dolbeault algebra, and let* $\alpha \in [0,1]$. *Let the map*

$$\mathfrak{D}^n(A,p) \otimes \mathfrak{D}^m(A,q) \xrightarrow{\cdot} \mathfrak{D}^{n+m}(A, p+q)$$

be defined by $x \cdot y = \psi(\varphi x \cup_\alpha \varphi y)$. *Then:*

1) *It is a morphism of complexes and does not depend on* α. *It is also independent of the section* σ_2 *used to define* $\mathfrak{D}^*(A,*)$, *provided this section satisfies the condition 2.4. Moreover it induces the product* $\cup$ *in real Deligne-Beilinson cohomology.*
2) *This product is graded commutative and it is associative up to a natural homotopy.*
3) *Let* $x \in \mathfrak{D}^n(A,p)$ *and* $y \in \mathfrak{D}^m(A,q)$. *Let us write* $l = n+m$ *and* $r = p+q$. *Then*

$$x \cdot y = \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & \text{for } n < 2p \text{ and } m < 2q, \\ \pi(x \wedge y), & \text{for } n < 2p,\ m \geq 2q,\ l < 2r, \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r \partial((x \wedge y)^{r-1,l-r}), & \text{for } n < 2p,\ m \geq 2q,\ l \geq 2r, \\ x \wedge y, & \text{for } n \geq 2p \text{ and } m \geq 2q, \end{cases}$$

where $r_p(x) = 2\pi_p(F^p dx)$ *(see 2.6.3) and* π *is the projection* $A^*_{\mathbb{C}} \longrightarrow \mathrm{Coker}\, u$ *(see §2).*

4) *If* $x \in \mathfrak{D}^{2p}(A,p)$ *is a cocycle, then for all* y, z *we have*

$$x \cdot y = y \cdot x \quad \text{and}$$
$$y \cdot (x \cdot z) = (y \cdot x) \cdot z = x \cdot (y \cdot z).$$

Proof. Let us first check the formulae of 3). If $n < 2p$ and $m < 2q$ we have

$$\begin{aligned}\psi(\varphi x \cup_\alpha \varphi y) &= \psi\left((r_p(x), 2F^p(dx), x) \cup_\alpha (r_q(y), 2F^q(dy), y)\right) \\ &= \psi\big(r_p(x) \wedge r_q(y), 2F^p(dx) \wedge 2F^q(dy), \\ &\qquad \alpha(x \wedge r_q(y) + (-1)^n 2F^p(dx) \wedge y) + (1-\alpha)(x \wedge 2F^q(dy) + (-1)^n r_p(x) \wedge y)\big) \\ &= \pi(\alpha(x \wedge r_q(y) + (-1)^n 2F^p(dx) \wedge y) + (1-\alpha)(x \wedge 2F^q(dy) + (-1)^n r_p(x) \wedge y)).\end{aligned}$$

But

$$\pi(x \wedge r_q(y)) = x \wedge r_q(y) \quad \text{and} \quad \pi(x \wedge 2F^q(dy)) = x \wedge r_q(y).$$

The same is true for the other two terms. Therefore

$$x \cdot y = (-1)^n r_p(x) \wedge y + x \wedge r_q(y).$$

Note that this result does not depend on α, nor on the choice of σ_2, because we have used ψ only for $l < 2r$.

If $n < 2p$, $m \geq 2q$ and $r \geq 2r$, we have

$$\begin{aligned}\psi(\varphi x \cup_\alpha \varphi y) &= \psi\left((r_p(x), 2F^p(dx), x) \cup_\alpha (y, y, 0)\right) \\ &= (r_p(x) \wedge y, 2F^p(dx) \wedge y, x \wedge y) \\ &= F^{r,r}(r_p(x) \wedge y) + 2\pi_r(\partial(x \wedge y)^{r-1,l-r}).\end{aligned}$$

This result does not depend on α either. Nor does this formula depend on the choice of σ_2 satisfying 2.4 because $x \wedge y \in A^{l-1}_{\mathbb{R}}(r-1)$ and $u(r_p(x) \wedge y, 2F^p(dx) \wedge y) \in A^l_{\mathbb{R}}(r-1)$. The other cases are analogous.

The remainder of the proposition is a consequence of these formulae and of Proposition 3.2, except for the fact that the homotopy for the associativity is natural, which follows from the naturality of φ, ψ and the homotopy h.

§4. Truncated Relative Cohomology Groups.

In this section, we introduce some groups of secondary cohomology classes associated with a morphism of complexes. These groups will be called truncated relative cohomology groups and are a generalization of the group of differential characters ([C-S]) and of the group of Green currents ([G-S], see also [Bu 2]).

4.1. Definition. Let R be a ring and let $f : A^* \longrightarrow B^*$ be a morphism of complexes of R-modules. Let us denote by ZA^* the submodule of cocycles of A^* and by $\widetilde{B}^* = B^* / \operatorname{Im} d$. If $b \in B^*$ we write $\widetilde{b}$ for its class in $\widetilde{B}^*$. The *truncated relative cohomology groups* associated with f are

$$\widehat{H}^n(A^*, B^*) = \left\{(a, \widetilde{b}) \in ZA^n \oplus \widetilde{B}^{n-1} \mid f(a) = db\right\}.$$

These groups are R-modules in a natural way. If the morphism f is injective we write $\widetilde{b}$ instead of $(a, \widetilde{b})$.

4.2. Examples.

1) If $B=0$ then $\widehat{H}^n(A^*,B^*)=ZA^n$. If $A=0$ then $\widehat{H}^n(A^*,B^*)=H^{n-1}(B^*)$.
2) ([C-S]) Let M be a differentiable manifold. Let A^* be the complex of real valued differential forms on M. Let $\Lambda\subset\mathbb{R}$ be a proper subring and let $C^*(M,\mathbb{R}/\Lambda)$ be the complex of $\mathbb{R}/\Lambda$-valued smooth cochains. There is an injective morphism

$$f: A^* \longrightarrow C^*(M,\mathbb{R}/\Lambda)$$

defined by integration. Then the group $\widehat{H}^n(A^*,C^*(M,\mathbb{R}/\Lambda))$ coincides with the group of differential characters of M, $\widehat{H}^{n-1}(M,\mathbb{R}/\Lambda)$.

Let us give another description of the truncated relative cohomology groups which explains their name. Let σ denote the "bête" filtration. That is, given a complex A^*, then

$$\sigma^p A^n=\begin{cases} A^n, & \text{if } n\geq p,\\ 0, & \text{if } n<p.\end{cases}$$

Let $s(\cdot)$ denote the simple of a morphism of complexes. Then

$$H^n(s(\sigma^pA^*\longrightarrow B^*))=\begin{cases} H^{n-1}(B^*), & \text{if } n<p,\\ \widehat{H}^n(A^*,B^*), & \text{if } n=p \text{ and}\\ H^n(A^*,B^*), & \text{if } n>p.\end{cases}$$

From this description we can obtain exact sequences involving truncated relative cohomology groups. Let us first define some maps involving these groups:

$$\mathrm{cl}:\widehat{H}^n(A^*,B^*)\longrightarrow H^n(A^*,B^*),\qquad \mathrm{cl}(a,\widetilde{b})=\{(a,b)\},$$

where $\{\cdot\}$ denotes cohomology class.

$$\omega:\widehat{H}^n(A^*,B^*)\longrightarrow ZA^n,\qquad \omega(a,\widetilde{b})=a.$$

$$\mathrm{a}:\widetilde{A}^{n-1}\longrightarrow \widehat{H}^n(A^*,B^*),\qquad \mathrm{a}(\widetilde{a})=(da,f(a)\widetilde{\ }).$$

$$\mathrm{b}:H^{n-1}(B^*)\longrightarrow \widehat{H}^n(A^*,B^*),\qquad \mathrm{b}(\{b\})=(0,\widetilde{b}).$$

We shall also denote by a the induced morphism $\mathrm{a}:H^{n-1}(A^*)\longrightarrow \widehat{H}^n(A^*,B^*)$.

4.3. Proposition. *Let $f:A^*\longrightarrow B^*$ be a morphism of complexes. Then there are exact sequences*

1) $H^{n-1}(A^*,B^*)\to\widetilde{A}^{n-1}\xrightarrow{\mathrm{a}}\widehat{H}^n(A^*,B^*)\xrightarrow{\mathrm{cl}}H^n(A^*,B^*)\to 0$

2) $0\to H^{n-1}(B^*)\xrightarrow{\mathrm{b}}\widehat{H}^n(A^*,B^*)\xrightarrow{\omega}ZA^n\to H^n(B^*)$

3) $H^{n-1}(A^*,B^*)\to H^{n-1}(A^*)\xrightarrow{\mathrm{a}}\widehat{H}^n(A^*,B^*)\xrightarrow{\mathrm{cl}\oplus\omega}$
$H^n(A^*,B^*)\oplus ZA^n\to H^n(A^*)\longrightarrow 0$

Proof. These exact sequences follow respectively from the exact sequences of complexes

$$0\to s(\sigma^nA^*\longrightarrow B^*)\to s(A^*\longrightarrow B^*)\to A^*/\sigma^nA^*\to 0,$$
$$0\to B^*[-1]\to s(\sigma^nA^*\longrightarrow B^*)\to\sigma^nA^*\to 0\qquad\text{and}$$
$$0\to s(\sigma^nA^*\longrightarrow B^*)\to s(A^*\longrightarrow B^*)\oplus\sigma^nA^*\to A^*\to 0.$$

Let $\underline{2}$ denote the category associated to the ordered set $\{0,1\}$. A morphism of complexes will also be called a $\underline{2}$-complex because it can be considered as a functor from the category $\underline{2}$ to the category of complexes. The $\underline{2}$-complex $f : A^* \longrightarrow B^*$ will be noted by (A^*, B^*, f) or simply by f. A morphism of $\underline{2}$-complexes $g : f_1 \longrightarrow f_2$ is a commutative diagram

$$\begin{array}{ccc} A_1^* & \xrightarrow{f_1} & B_1^* \\ \Big\downarrow g_A & & \Big\downarrow g_B \\ A_2^* & \xrightarrow{f_2} & B_2^*. \end{array}$$

If g_A and g_B have degree e, we say that g has degree e. For each n, the n-th truncated relative cohomology group is a covariant functor from the category of $\underline{2}$-complexes of R-modules to the category of R-modules. If $g = (g_A, g_B)$ is a morphism of $\underline{2}$-complexes, then there is an induced morphism

$$\begin{array}{rccc} \widehat{g} = \widehat{H}^*(g) : & \widehat{H}^*(A_1^*, B_1^*) & \longrightarrow & \widehat{H}^*(A_2^*, B_2^*) \\ & \widetilde{(a, b)} & \longmapsto & (g_A(a), (g_B(b))\widetilde{\ \ }). \end{array}$$

If g has degree e, then the induced morphism $\widehat{g}$ is also of degree e.

4.4. Proposition. *Let $g = (g_A, g_B)$ be a morphism of $\underline{2}$-complexes. If g_A is an isomorphism and g_B is a quasi-isomorphism then $\widehat{g}$ is an isomorphism.*

Proof. It is a direct consequence of 4.3.2.

This proposition reflects the asymmetry between the complexes A^* and B^*. We can freely replace the complex B^* by a quasi-isomorphic complex without changing the truncated relative cohomology groups. On the other hand, if we change A^* by a quasi-isomorphic complex, then we can change the properties of these groups.

Let us recall now how to construct a product on relative cohomology groups from a product at the level of complexes. We shall extend this construction to truncated relative cohomology groups.

Let $f : A^* \longrightarrow B^*$ and $g : C^* \longrightarrow D^*$ be a morphism of complexes. We can construct the complex

$$s(f) \otimes s(g) = s(A^* \longrightarrow B^*) \otimes s(C^* \longrightarrow D^*)$$

or consider the simple of the diagram

$$A^* \otimes C^* \xrightarrow{(f \otimes \mathrm{Id}, \mathrm{Id} \otimes g)} B^* \otimes C^* \oplus A^* \otimes D^* \xrightarrow{-\mathrm{Id} \otimes g + f \otimes \mathrm{Id}} B^* \otimes D^*.$$

There is an isomorphism of complexes

$$s(f) \otimes s(g) \longrightarrow s(A^* \otimes C^* \to B^* \otimes C^* \oplus A^* \otimes D^* \to B^* \otimes D^*.)$$

If $(a, b) \in s(f)^n$ and $(c, d) \in s(g)^m$ then this isomorphism is given by

$$(a, b) \otimes (c, d) \longmapsto (a \otimes c, b \otimes c + (-1)^n a \otimes d, (-1)^n b \otimes d).$$

Suppose that there is a morphism of commutative diagrams

$$\begin{array}{ccc} A^*\otimes C^* & \longrightarrow & A^*\otimes D^* \\ \downarrow & & \downarrow \\ B^*\otimes C^* & \longrightarrow & B^*\otimes D^* \end{array} \quad \xrightarrow{\cdot} \quad \begin{array}{ccc} E_1^* & \longrightarrow & E_3^* \\ \downarrow & & \downarrow \\ E_2^* & \longrightarrow & E_4^*. \end{array}$$

Then there is an induced product

$$s(f)\otimes s(g) \xrightarrow{\cdot} s(E_1^* \to s(E_2^*\oplus E_3^* \to E_4^*)).$$

Hence a product

$$H^n(A^*,B^*)\otimes H^m(C^*,D^*) \to H^{n+m}(E_1^*, s(E_2^*\oplus E_3^* \to E_4^*)).$$

If $\{(a,b)\}\in H^n(A^*,B^*)$ and $\{(c,d)\}\in H^m(C^*,D^*)$, this product is given by

$$\{(a,b)\}\otimes\{(c,d)\} \longmapsto \{(a\cdot c, b\cdot c+(-1)^n a\cdot d, (-1)^n b\cdot d)\}.$$

Here $\{\cdot\}$ denotes cohomology class.

4.5. Definition. With the above hypothesis, the $*$-product of truncated relative cohomology groups:

$$\widehat{H}^n(A^*,B^*)\otimes \widehat{H}^m(C^*,D^*) \xrightarrow{*} \widehat{H}^{n+m}(E_1^*, s(E_2^*\oplus E_3^* \to E_4^*))$$

is defined by

$$(a,\widetilde{b})*(c,\widetilde{d}) = (a\cdot c, (b\cdot c+(-1)^n a\cdot d, (-1)^n b\cdot d)^{\sim}).$$

4.6. Proposition. *The $*$-product of truncated relative cohomology groups is well defined, i.e. it does not depend on the choice of representatives b and d of $\widetilde{b}$ and $\widetilde{d}$. Moreover there are commutative diagrams*

$$\begin{array}{ccc} \widehat{H}^n(A^*,B^*)\otimes \widehat{H}^m(C^*,D^*) & \xrightarrow{\ *\ } & \widehat{H}^{n+m}(E_1^*, s(E_2^*\oplus E_3^* \longrightarrow E_4^*)) \\ \downarrow{\scriptstyle \omega\otimes\omega} & & \downarrow{\scriptstyle \omega} \\ A^n\otimes C^m & \xrightarrow{\ \cdot\ } & E_1^{n+m}, \end{array}$$

and

$$\begin{array}{ccc} \widehat{H}^n(A^*,B^*)\otimes \widehat{H}^m(C^*,D^*) & \xrightarrow{\ *\ } & \widehat{H}^{n+m}(E_1^*, s(E_2^*\oplus E_3^* \longrightarrow E_4^*)) \\ \downarrow{\scriptstyle \mathrm{cl}\otimes\mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^n(A^*,B^*)\otimes H^m(C^*,D^*) & \xrightarrow{\ \cdot\ } & H^{n+m}(E_1^*, s(E_2^*\oplus E_3^* \longrightarrow E_4^*)). \end{array}$$

Proof. Follows from the definitions.

§5. A Definition of Green Forms Using Deligne-Beilinson Cohomology.

In this section we shall see that the space of Green forms can be obtained as a truncated relative cohomology group of the Deligne-Beilinson complex. Moreover, the $*$-product of Green forms is induced by the product of the Deligne-Beilinson complex.

Let X be a smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{Z}^p = \mathcal{Z}^p(X)$ be the set of algebraic subsets of codimension $\geq p$, ordered by inclusion. Let us write

$$E^*_{\log}(X \setminus \mathcal{Z}^p) = \varinjlim_{Z \in \mathcal{Z}^p} E^*_{\log}(X - Z).$$

This complex is a Dolbeault complex and there is a natural injective map

$$E^*_{\log}(X) \longrightarrow E^*_{\log}(X \setminus \mathcal{Z}^p).$$

We shall write

$$\begin{aligned} H^*_{\mathcal{D}}(X \setminus \mathcal{Z}^p, \mathbb{R}(p)) &= H^*(\mathfrak{D}^*(E^*_{\log}(X \setminus \mathcal{Z}^p), p)) \quad \text{and} \\ H^*_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) &= H^*(s(\mathfrak{D}^*(E^*_{\log}(X), p) \longrightarrow \mathfrak{D}^*(E^*_{\log}(X \setminus \mathcal{Z}^p), p))). \end{aligned}$$

Since $\mathcal{Z}^p$ is a directed set we have

$$\begin{aligned} H^*_{\mathcal{D}}(X \setminus \mathcal{Z}^p, \mathbb{R}(p)) &= \varinjlim_{Z \in \mathcal{Z}^p} H^*_{\mathcal{D}}(X - Z, \mathbb{R}(p)) \quad \text{and} \\ H^*_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) &= \varinjlim_{Z \in \mathcal{Z}^p} H^*_{\mathcal{D},Z}(X, \mathbb{R}(p)). \end{aligned}$$

5.1. Definition. The *space of Green forms on X with codimension p singular support* is

$$GE^p(X) = \widehat{H}^{2p}(\mathfrak{D}^*(E^*_{\log}(X), p), \mathfrak{D}^*(E^*_{\log}(X \setminus \mathcal{Z}^p), p)).$$

Let $(\omega, \widetilde{g}) \in GE^p(X)$. Since the map $\mathfrak{D}^*(E^*_{\log}(X), p) \longrightarrow \mathfrak{D}^*(E^*_{\log}(X \setminus \mathcal{Z}^p), p)$ is injective, ω is determined by $\widetilde{g}$. Thus we shall sometimes represent $(\omega, \widetilde{g})$ by $\widetilde{g}$.

By the definition of the Deligne-Beilinson complex we have

$$\mathfrak{D}^{2p-1}(E^*_{\log}(X), p) \big/ \operatorname{Im} d_{\mathfrak{D}} = E^{p-1,p-1}_{\log}(X) \cap E^{2p-2}_{\log,\mathbb{R}}(p-1) \Big/ (\operatorname{Im} \partial + \operatorname{Im} \bar{\partial}).$$

We shall denote this group by $\widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X)$. Analogously we write

$$\widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X \setminus \mathcal{Z}^p) = \mathfrak{D}^{2p-1}(E^*_{\log}(X \setminus \mathcal{Z}^p), p) \big/ \operatorname{Im} d_{\mathfrak{D}}.$$

We also have that the subgroup of cocycles of $\mathfrak{D}^{2p}(E^*_{\log}(X), p)$ is

$$\left\{ \omega \in E^{p,p}_{\log}(X) \cap E^{2p}_{\log,\mathbb{R}}(X, p) \mid d\omega = 0 \right\}.$$

This group will be denoted by $ZE^{p,p}_{\log,\mathbb{R}}(X)$.

Then

$$\begin{aligned} GE^p(X) &= \left\{(\omega,\widetilde{g}) \in ZE^{p,p}_{\log,\mathbb{R}}(X) \oplus \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X\backslash \mathcal{Z}^p) \mid -2\partial\overline{\partial} g = \omega\right\} \\ &= \left\{\widetilde{g} \in \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X\backslash \mathcal{Z}^p) \mid \partial\overline{\partial} g \text{ is smooth on } X\right\}. \end{aligned}$$

5.2. Definition. If $Z \subset X$ is a codimension p algebraic subset of X, then the *space of Green forms on X with singular support contained on Z* is

$$GE^p_Z(X) = \widehat{H}^{2p}(\mathfrak{D}^*(E^*_{\log}(X), p), \mathfrak{D}^*(E^*_{\log}(X - Z), p)).$$

Since $\mathcal{Z}^p$ is a directed set and the codimension p algebraic subsets of X is a cofinal subset of $\mathcal{Z}^p$, the group $GE^p(X)$ is the direct limit of the groups $GE^p_Z(X)$ for Z of codimension p.

5.3. Definition. Let $\widetilde{g} \in GE^p(X)$. Then the *singular support of $\widetilde{g}$* is the intersection of all Z such that $\widetilde{g}$ has a representative in $GE^p_Z(X)$. We shall denote the singular support of $\widetilde{g}$ by $\operatorname{supp}\widetilde{g}$.

Since $GE^p(X)$ are truncated relative cohomology groups we can define maps

$$\begin{aligned} \mathrm{cl} &: GE^p(X) \longrightarrow H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)), \\ \omega &: GE^p(X) \longrightarrow ZE^{p,p}_{\log,\mathbb{R}}(X), \\ \mathrm{a} &: \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \longrightarrow GE^p(X) \quad \text{and} \\ \mathrm{b} &: H^{2p-1}_{\mathcal{D}}(X\backslash\mathcal{Z}^p, \mathbb{R}(p)) \longrightarrow GE^p(X), \end{aligned}$$

as in §4. We shall also denote by a the induced morphism

$$\mathrm{a} : H^{2p-1}_{\mathcal{D}}(X, \mathbb{R}(p)) \longrightarrow GE^p(X).$$

5.4. Proposition. *Let X be a smooth variety over $\mathbb{C}$. Then there are exact sequences*

1) $0 \to \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \xrightarrow{\mathrm{a}} GE^p(X) \xrightarrow{\mathrm{cl}} H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) \to 0.$

2) $0 \to H^{2p-1}_{\mathcal{D}}(X\backslash\mathcal{Z}^p, \mathbb{R}(p)) \xrightarrow{\mathrm{b}} GE^p(X) \xrightarrow{\omega} ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X\backslash\mathcal{Z}^p, \mathbb{R}(p)).$

3) $0 \to H^{2p-1}_{\mathcal{D}}(X, \mathbb{R}(p)) \xrightarrow{\mathrm{a}} GE^p(X) \xrightarrow{\mathrm{cl}\oplus\omega}$
$H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) \oplus ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p)) \to 0.$

Proof. This is a translation of Proposition 4.3. taking into account that Deligne-Beilinson cohomology satisfies

$$H^{2p-1}_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) = 0.$$

This can be proved using the exact sequence of Proposition 1.1. and the fact that, if Z is a codimension p algebraic subset of X then

$$H^n_Z(X, R) = 0$$

for $n < 2p$ and $R = \mathbb{R}$ or $\mathbb{C}$.

Fixing the singular support we have an analogous result.

5.5. Proposition. *Let X be a smooth variety over $\mathbb{C}$ and $Z \subset X$ a codimension p algebraic subset. Then there are exact sequences*

1) $0 \to \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X) \xrightarrow{\mathrm{a}} GE^p_Z(X) \xrightarrow{\mathrm{cl}} H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \to 0.$

2) $0 \to H^{2p-1}_{\mathcal{D}}(X-Z,\mathbb{R}(p)) \xrightarrow{\mathrm{b}} GE^p_Z(X) \xrightarrow{\omega} ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X-Z,\mathbb{R}(p)).$

3) $0 \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{\mathrm{a}} GE^p_Z(X) \xrightarrow{\mathrm{cl}\oplus\omega}$

$$H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \oplus ZE^{p,p}_{\log,\mathbb{R}}(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)) \to 0.$$

5.6. Corollary. *The natural map*

$$GE^p_Z(X) \longrightarrow GE^p(X)$$

is injective. Moreover, if $\widetilde{g} \in GE^p(X)$ then $\operatorname{supp}\widetilde{g} = \operatorname{supp}\mathrm{cl}(\widetilde{g})$.

Proof. The injectivity follows from the injectivity of the morphism

$$H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p))$$

and the Five Lemma. Let us write $Y = \operatorname{supp}\mathrm{cl}(\widetilde{g})$ and $Y' = \operatorname{supp}\widetilde{g}$. Clearly $Y \subset Y'$. Then we have a morphism of change of support $\varphi : GE^p_Y(X) \longrightarrow GE^p_{Y'}(X)$ and a commutative diagram

$$\begin{array}{ccc} GE^p_Y(X) & \xrightarrow{\mathrm{cl}} & H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ GE^p_{Y'}(X) & \xrightarrow{\mathrm{cl}} & H^{2p}_{\mathcal{D},Y'}(X,\mathbb{R}(p)), \end{array}$$

where the horizontal arrows are surjective. Let $\widetilde{g}' \in GE^p_Y(X)$ with $\mathrm{cl}(\widetilde{g}') = \mathrm{cl}(\widetilde{g})$. By Proposition 5.5, there is an element $\alpha \in \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X)$ such that $a(\alpha) = \widetilde{g} - \varphi\widetilde{g}'$. But then $\widetilde{g}' + a(\alpha) \in GE^p_Y(X)$ and it represents $\widetilde{g}$. Thus $Y = Y'$.

5.7. Definition. Let y be a codimension p algebraic cycle. Then the *space of Green forms associated with y* is

$$GE^p_y(X) = \{\widetilde{g} \in GE^p(X) \mid \mathrm{cl}(\widetilde{g}) = \rho(y)\},$$

where $\rho(y)$ is the class of y in $H^{2p}_{\mathcal{D},\mathcal{Z}^p}(X,\mathbb{R}(p))$ (see [J] or §7).

A direct consequence of Corollary 5.6 is:

5.8. Corollary. *Let y be a codimension p algebraic cycle and let $Y = \operatorname{supp} y$. If $\widetilde{g}_y$ is a Green form associated to y, then the singular support of $\widetilde{g}_y$ is Y.*

5.9. Theorem. *Let X be a smooth projective variety over $\mathbb{C}$ and y a codimension p algebraic cycle. Let $GE_X(y)$ be the space of Green forms for y as defined in* [Bu 2]. *Then there is a natural isomorphism*

$$GE^p_y(X) \longrightarrow GE_X(y)$$

given by

$$\widetilde{g} \longmapsto \frac{2}{(2\pi i)^{p-1}}\widetilde{g}.$$

If X has dimension d and $GC_X(y)$ is the space of Green currents for y in the sense of Gillet and Soulé ([G-S]*, see also* [Bu 2]*) then there is a natural isomorphism*

$$GE^p_y(X) \longrightarrow GC_X(y).$$

Proof. Let us write $Y = \operatorname{supp} y$. By definition

$$GE_X(y) = \left\{ g \in E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y) \mid \begin{array}{c} dd^c g \in E^{p,p}_X, \\ \{dd^c g, d^c g\} = \{y\} \end{array} \right\} \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}),$$

where $\{dd^c g, d^c g\}$ is the cohomology class represented by $(dd^c g, d^c g)$ and $\{y\}$ is the cohomology class of y. Both classes are considered in $H^{2p}_Y(X,\mathbb{R})$.

On the other hand, by Corollary 5.8, if we write

$$E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y,p-1) = E^{p-1,p-1}_{\log}(X-Y) \cap E^{2p-2}_{\log,\mathbb{R}}(X-Y,p-1)$$

we have

$$GE^p_y(X) = \left\{ g \in E^{p-1,p-1}_{\log,\mathbb{R}}(X-Y,p-1) \mid \begin{array}{c} -2\partial\overline{\partial} g \in E^{2p}_X \\ \{-2\partial\overline{\partial} g, g\} = \rho(y) \end{array} \right\} \Big/ (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}),$$

where now $\{-2\partial\overline{\partial} g, g\}$ and $\rho(y)$ are cohomology classes in $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$. But the natural morphism $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_Y(X,\mathbb{R})$ is induced by a morphism of complexes (see 2.6.3):

$$\frac{1}{(2\pi i)^p} r_p : s(\mathfrak{D}^*(E^*_X,p),\mathfrak{D}^*(E^*_{\log}(X-Y),p)) \longrightarrow s(E^*_{X,\mathbb{R}} \longrightarrow E^*_{\log,\mathbb{R}}(X-Y)).$$

Which, in degree $2p$, satisfies

$$\frac{1}{(2\pi i)^p} r_p(\omega,g) = (\frac{1}{(2\pi i)^p}\omega, \frac{2}{(2\pi i)^{p-1}} d^c g).$$

Therefore, this morphism sends the class $\{-2\partial\overline{\partial} g, g\}$ to the class $\{dd^c g, d^c g\}$. Moreover, by the definition of $\rho(y)$, this class is mapped to $\{y\}$. Hence the map

$$\begin{array}{ccc} GE^p_y(X) & \longrightarrow & GE_X(y) \\ \widetilde{g} & \longmapsto & \dfrac{2}{(2\pi i)^{p-1}}\widetilde{g} \end{array}$$

is well defined. The inverse of this map is also well defined, because the morphism $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \longrightarrow H^{2p}_Y(X,\mathbb{R})$ is an isomorphism.

The second part of the Theorem follows from the first part and the comparison isomorphism between Green forms and Green currents proved in [Bu 2].

5.10. Remark. By the definition of the space of Green forms as a truncated relative homology group, the morphism

$$GE^p_Y(X) \longrightarrow H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p))$$

is an epimorphism. Therefore the existence of Green forms is a direct consequence of the existence of the cycle class in real Deligne-Beilinson cohomology.

Now we want to define a product of Green forms. Let X be a smooth variety over $\mathbb{C}$ and let Y and Z be closed algebraic subsets of codimension p and q respectively such that $Y \cap Z$ has codimension $p+q$. Let $r = p+q$ and let us write $\mathfrak{D}^*(X,r) = \mathfrak{D}^*(E^*_{\log}(X),r)$ and

$$\mathfrak{D}^*(X;Y,Z,r) = s(\mathfrak{D}^*(X-Y,r) \oplus \mathfrak{D}^*(X-Z,r) \xrightarrow{j} \mathfrak{D}^*(X-Y\cup Z,r)),$$

where $j(a,b) = b-a$. Then, using Definition 4.5, we obtain a morphism

$$GE^p_Y \otimes GE^q_Z(X) \xrightarrow{*} \widehat{H}^{2r}(\mathfrak{D}^*(X,r),\mathfrak{D}^*(X;Y,Z,r)).$$

given by

$$\begin{aligned}(\omega_1,\widetilde{g}_1) * (\omega_2,\widetilde{g}_2) &= (\omega_1\cdot\omega_2, (g_1\cdot\omega_2, \omega_1\cdot g_2, g_1\cdot g_2)\widetilde{\ }\,) \\ &= (\omega_1\wedge\omega_2, (g_1\wedge\omega_2, \omega_1\wedge g_2, -r_p(g_1)\wedge g_2 + g_1\wedge r_q g_2)\widetilde{\ }\,) \\ &= (\omega_1\wedge\omega_2, (g_1\wedge\omega_2, \omega_1\wedge g_2, -4\pi i d^c g_1\wedge g_2 + 4\pi i g_1\wedge d^c g_2)\widetilde{\ }\,)\end{aligned}$$

But since the map

$$\begin{array}{ccc}\mathfrak{D}^*(X-Y\cap Z,r) & \longrightarrow & \mathfrak{D}^*(X;Y,Z,r) \\ g & \longmapsto & (g,g,0)\end{array}$$

is a quasi-isomorphism, therefore there is a natural isomorphism

$$GE^r_{Y\cap Z}(X) \longrightarrow \widehat{H}^{2r}(\mathfrak{D}^*(X,r),\mathfrak{D}^*(X;Y,Z,r)).$$

Hence the following definition makes sense.

5.11. Definition. Let X be a smooth variety over $\mathbb{C}$ and let Y and Z be algebraic subsets of codimension p and q respectively such that $Y\cap Z$ has codimension $p+q$. Then the $*$-*product*

$$GE^p_Y(X) \otimes GE^q_Z(X) \xrightarrow{*} GE^p_{Y\cap Z}(X)$$

is the product in truncated relative cohomology groups induced by the product of the Deligne-Beilinson complex.

5.12. Theorem. *The $*$-product of Green forms is commutative and associative. It is compatible with the product in Deligne-Beilinson cohomology and with the cup product of differential forms. Moreover if X is projective then it is compatible with the $*$-product of Green forms defined in* [Bu 2] *and with the $*$-product of currents defined in* [G-S].

Proof. The compatibility with $\cup$ and $\wedge$ follows easily from the definitions.

Let Y and Z be closed algebraic subsets of X of codimension p and q, and let $\widetilde{g}_1 \in GE^p_Y(X)$ and $\widetilde{g}_2 \in GE^q_Z(X)$. Write $r = p + q$. Then

$$\widetilde{g}_1 * \widetilde{g}_2 \in \widehat{H}^{2r}(\mathfrak{D}^*(X, r), \mathfrak{D}^*(X; Y, Z, r)),$$

and

$$\widetilde{g}_2 * \widetilde{g}_1 \in \widehat{H}^{2r}(\mathfrak{D}^*(X, r), \mathfrak{D}^*(X; Z, Y, r)).$$

Both groups are naturally isomorphic. The isomorphism between them is induced by an isomorphism of complexes

$$\mathfrak{D}^n(X; Y, Z, r) \longrightarrow \mathfrak{D}^n(X; Z, Y, r),$$

given by

$$(a, b, c) \longmapsto (b, a, -c).$$

It is straightforward to check that this isomorphism sends $\widetilde{g}_1 * \widetilde{g}_2$ to $\widetilde{g}_2 * \widetilde{g}_1$.

Let W, Y and Z be algebraic subsets of X of codimension p, q and r respectively, such that the codimension of $W \cap Y \cap Z$ is $p + q + r$ and Y intersects properly with W and Z. Let $(\omega_1, \widetilde{g}_1) \in GE^p_W(X)$, $(\omega_2, \widetilde{g}_2) \in GE^q_Y(X)$ and $(\omega_3, \widetilde{g}_3) \in GE^r_Z(X)$. Let us write $s = p + q + r$ and

$$\begin{aligned}\mathfrak{D}^*(X; W, Y, Z, s) = s(&\mathfrak{D}^*(X - W, s) \oplus \mathfrak{D}^*(X - Y, s) \oplus \mathfrak{D}^*(X - Z, s) \xrightarrow{j} \\ &\mathfrak{D}^*(X - W \cup Y, s) \oplus \mathfrak{D}^*(X - W \cup Z, s) \oplus \mathfrak{D}^*(X - Y \cup Z, s) \xrightarrow{k} \\ &\mathfrak{D}^*(X - W \cup Y \cup Z, s)),\end{aligned}$$

where

$$j(a, b, c) = (b - a, c - a, b - c) \quad \text{and} \quad k(a, b, c) = a - b + c.$$

Both products, $\widetilde{g}_1 * (\widetilde{g}_2 * \widetilde{g}_3)$ and $(\widetilde{g}_1 * \widetilde{g}_2) * \widetilde{g}_3$ are defined in $\widehat{H}^{2s}(\mathfrak{D}^*(X, s), \mathfrak{D}^*(X; W, Y, Z, s))$.

We have

$$\begin{aligned}\widetilde{g}_1 * (\widetilde{g}_2 * \widetilde{g}_3) = (\omega_1 \cdot (\omega_2 \cdot \omega_3), (&g_1 \cdot (\omega_2 \cdot \omega_3), \omega_1 \cdot (g_2 \cdot \omega_3), \omega_1 \cdot (\omega_2 \cdot g_3), \\ &- g_1 \cdot (g_2 \cdot \omega_3), -g_1 \cdot (\omega_2 \cdot g_3), -\omega_1 \cdot (g_2 \cdot g_3), g_1 \cdot (g_2 \cdot g_3))^{\sim}).\end{aligned}$$

and

$$\begin{aligned}(\widetilde{g}_1 * \widetilde{g}_2) * \widetilde{g}_3 = ((\omega_1 \cdot \omega_2) \cdot \omega_3, (&(g_1 \cdot \omega_2) \cdot \omega_3, (\omega_1 \cdot g_2) \cdot \omega_3, (\omega_1 \cdot \omega_2) \cdot g_3, \\ &- (g_1 \cdot g_2) \cdot \omega_3, -(g_1 \cdot \omega_2) \cdot g_3, -(\omega_1 \cdot g_2) \cdot g_3, (g_1 \cdot g_2) \cdot g_3)^{\sim}).\end{aligned}$$

By Theorem 3.3, $\omega_1 \cdot (\omega_2 \cdot \omega_3) = (\omega_1 \cdot \omega_2) \cdot \omega_3$. Therefore

$$\widetilde{g}_1 * (\widetilde{g}_2 * \widetilde{g}_3) - (\widetilde{g}_1 * \widetilde{g}_2) * \widetilde{g}_3 = (0, \widetilde{x}),$$

with $x \in \mathfrak{D}^{2s-1}(X; W, Y, Z, s)$ Let h_{a} be the homotopy which makes the product on the Deligne-Beilinson complex associative. That is

$$a \cdot (b \cdot c) - (a \cdot b) \cdot c = d_{\mathfrak{D}} h_{\mathrm{a}}(a \otimes b \otimes c) + h_{\mathrm{a}} d_{\mathfrak{D}}(a \otimes b \otimes c).$$

Let us consider the element $y \in \mathfrak{D}(X; W, Y, Z, s)$ given by

$$\begin{aligned} y = (&h_{\mathrm{a}}(g_1 \otimes \omega_2 \otimes \omega_3), h_{\mathrm{a}}(\omega_1 \otimes g_2 \otimes \omega_3), h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes g_3), \\ &h_{\mathrm{a}}(g_1 \otimes g_2 \otimes \omega_3), h_{\mathrm{a}}(g_1 \otimes \omega_2 \otimes g_3), h_{\mathrm{a}}(\omega_1 \otimes g_2 \otimes g_3), h_{\mathrm{a}}(g_1 \otimes g_2 \otimes g_3)). \end{aligned}$$

By the naturality of h_{a} we have,

$$d_{\mathfrak{D}} y = x - (h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes \omega_3), h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes \omega_3), h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes \omega_3), 0, 0, 0, 0).$$

Therefore the associativity follows from the lemma:

5.13. Lemma. *Let $\omega_1 \in \mathfrak{D}^{2p}(X - W, p)$, $\omega_2 \in \mathfrak{D}^{2q}(X - Y, q)$, and $\omega_3 \in \mathfrak{D}^{2r}(X - Z, r)$. Then*

$$h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes \omega_3) = 0.$$

Proof. By definition (see §3)

$$h_{\mathrm{a}}(\omega_1 \otimes \omega_2 \otimes \omega_3) = \psi(h(\varphi\omega_1 \cup \varphi\omega_2) \cup \varphi\omega_3) + \psi(\varphi\omega_1 \cup h(\varphi\omega_2 \cup \varphi\omega_3)),$$

where ψ and φ are the homotopy equivalences between the Deligne-Beilinson complexes, h is the homotopy between $\varphi\psi$ and Id and $\cup$ is the product $\cup_0$ in the Deligne-Beilinson complex which is associative.

But

$$\begin{aligned} h(\varphi\omega_1 \cup \varphi\omega_2) &= h((\omega_1, \omega_1, 0) \cup (\omega_2, \omega_2, 0)) \\ &= h(\omega_1 \wedge \omega_2, \omega_1 \wedge \omega_2, 0) \\ &= 0. \end{aligned}$$

Therefore we obtain the Lemma.

Let us show now that the $*$-product defined here is compatible with the $*$-product defined in [Bu 2]. Let Y and Z be closed algebraic subsets of X of codimension p and q respectively which intersect properly. Write $r = p+q$. Let $\widetilde{X}$ be a resolution of singularities of $Y \cap Z$ such that the strict transforms of Y and Z do not meet. Write $\widehat{Y}$ for the strict transform of Y and $\widehat{Z}$ for that of Z. Let $\sigma_{Y,Z}$ be a smooth function on $\widetilde{X}$ such that it takes the value 1 in a neighbourhood of $\widehat{Y}$ and the value 0 in a neighbourhood of $\widehat{Z}$. Let $\sigma_{Z,Y} = 1 - \sigma_{Y,Z}$. Let us denote by $*'$ the $*$-product of Green forms defined in [Bu 2]. Then

$$\begin{aligned} \widetilde{g}_2 *' \widetilde{g}_2 &= 4\pi i (dd^c(\sigma_{Y,Z} g_1) \wedge g_2 + \sigma_{Z,Y} g_1 \wedge dd^c g_2)^{\sim} \\ &= (d_{\mathfrak{D}}(\sigma_{Y,Z} g_1) \cdot g_2 + \sigma_{Z,Y} g_1 \cdot d_{\mathfrak{D}} g_2)^{\sim}. \end{aligned}$$

The factor $4\pi i$ comes from the normalization for Green forms used here which differs from that used in [Bu 2].

The isomorphism

$$\varphi : \widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X - Y \cap Z, r)) \longrightarrow \widehat{H}^{2r}(\mathfrak{D}^*(X,r), \mathfrak{D}^*(X,Y,Z,r))$$

sends $(d_{\mathfrak{D}}(\sigma_{Y,Z} g_1) \cdot g_2 + \sigma_{Z,Y} g_1 \cdot d_{\mathfrak{D}} g_2)^{\sim}$ to

$$(d_{\mathfrak{D}}(\sigma_{Y,Z} g_1) \cdot g_2 + \sigma_{Z,Y} g_1 \cdot d_{\mathfrak{D}} g_2, d_{\mathfrak{D}}(\sigma_{Y,Z} g_1) \cdot g_2 + \sigma_{Z,Y} g_1 \cdot d_{\mathfrak{D}} g_2, 0)^{\sim}.$$

Then

$$\begin{aligned}\varphi(\widetilde{g}_1 *' \widetilde{g}_2) - \widetilde{g}_1 * \widetilde{g}_2 &= (d_{\mathfrak{D}}(\sigma_{Y,Z} g_1 \cdot g_2), -d_{\mathfrak{D}}(\sigma_{Z,Y} g_1 \cdot g_2), -g_1 \cdot g_2)^{\sim} \\ &= \big(d_{\mathfrak{D}}(\sigma_{Y,Z} g_1 \cdot g_2, -\sigma_{Z,Y} g_1 \cdot g_2, 0)\big)^{\sim} \\ &= 0.\end{aligned}$$

Therefore $\widetilde{g}_1 * \widetilde{g}_2$ and $\widetilde{g}_1 *' \widetilde{g}_2$ represents the same Green form.

In [Bu 2, §4], the compatibility of the $*$-product of Green currents with the product $*'$ of Green forms is proved. Therefore the product of Green currents is also compatible with the product defined here.

5.15 Remark. The key point in the proof of the associativity is Lemma 5.13. We can even use a weaker version of this lemma, assuming that ω_i, $i = 1, 2, 3$, are closed. It may be convenient to replace the complexes $\mathfrak{D}$ by other complexes in order to obtain Green forms with different properties. Then to prove the associativity of the product of these new Green forms we only need to check Lemma 5.13. in that case.

5.16 Remark. In the proof of Theorem 5.12. we have assumed that the intersections are proper, because we have defined $GE^p_Z(X)$ only for closed subsets Z of codimension $\geq p$. With the obvious definition of $GE^p_Z(X)$ for Z of arbitrary codimension, the Theorem also holds, except for the comparison between Green forms and Green currents.

§6. K-Chains and Real Deligne-Beilinson Cohomology.

Our construction of arithmetic Chow groups is based in the relationship between algebraic K-theory and Deligne-Beilinson cohomology.

Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension d. By [Be] (see also [Gi] and [Sch]) we have a Chern character

$$\mathrm{ch} : K_j(X) \longrightarrow \bigoplus_i H^{2i-j}_{\mathcal{D}}(X, \mathbb{R}(i)).$$

Using the techniques of [Gi] one can also define a Chern character map from the Brown-Gersten-Quillen spectral sequence for the K-theory of X and the Bloch-Ogus spectral sequence for the Deligne-Beilinson cohomology. The aim of this section is to write down explicitly the last stages of this map.

Let us denote by $X^{(p)}$ the set of irreducible subvarieties of codimension p. The groups of the E_1 term of the Brown-Gersten-Quillen spectral sequence (see [Q 1], [Gi] [Gr 1] and [Gr 2]) are given by:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K^x_{-p-q}(X),$$

where

$$K^x_n(X) = \lim_{x \in U} K_n^{\overline{\{x\}} \cap U}(U).$$

Whereas the groups of the Bloch-Ogus spectral sequence (see [B-O] and [Gi]) are

$$E_1^{p,q} = \bigoplus_i \bigoplus_{x \in X^{(p)}} H^{2i+p+q}_{\mathcal{D},x}(X, \mathbb{R}(i)),$$

where

$$H^n_{\mathcal{D},x}(X, \mathbb{R}(i)) = \lim_{x \in U} H^n_{\mathcal{D},\overline{\{x\}} \cap U}(U, \mathbb{R}(i)).$$

Note that this spectral sequence is a direct sum of spectral sequences. One for each i, and each of them is shifted in order to make

$$\mathrm{ch} : \bigoplus_{x \in X^{(p)}} K^x_{-p-q}(X) \longrightarrow \bigoplus_i \bigoplus_{x \in X^{(p)}} H^{2i+p+q}_{\mathcal{D},x}(X, \mathbb{R}(i))$$

bihomogeneous.

6.1. Proposition. *There is a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{x \in X^{(p)}} K^x_{-p-q}(X) & \xrightarrow{\mathrm{ch}} & \bigoplus_i \bigoplus_{x \in X^{(p)}} H^{2i+p+q}_{\mathcal{D},x}(X, \mathbb{R}(i)) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in X^{(p)}} K_{-p-q}(k(x)) & \xrightarrow{\mathrm{ch}} & \bigoplus_i \bigoplus_{x \in X^{(p)}} H^{2i-p+q}_{\mathcal{D}}(x, \mathbb{R}(i-p)), \end{array}$$

where the vertical arrows are isomorphisms and the horizontal arrows are morphisms of spectral sequences.

Proof. The proof is analogous to the proof of [Gi, 3.9]. The fact that the vertical arrows are isomorphism come from the localization and purity theorems for K-theory and Deligne-Beilinson cohomology. For the commutativity one uses that each $x \in X^{(p)}$ has a neigborhood U such that

$$\overline{\{x\}} \cap U \longrightarrow U$$

is a smooth inmersion with trivial normal bundle. In this case, by Riemann-Roch ([Gi]), the Chern character commutes with direct images. The result is obtained taking the limit for such neighborhoods.

Let us write

$$R^i_p = R^i_p(X) = \bigoplus_{x \in X^{(i)}} K_{p-i}(k(x)).$$

Then $R_p^p(X) = Z^p(X)$, the group of codimension p algebraic cycles. The elements of R_p^{p-1} will be called K_1-chains and the elements of R_p^{p-2}, K_2-chains. Let us denote by $d : R_p^i \longrightarrow R_p^{i+1}$ the differential of this spectral sequence.

Recall that $K_1(k(x)) = k(x)^*$ is the group of units of $k(x)$. If f is a K_1-chain then

$$f = \sum_{x \in X^{(i)}} f_x,$$

with $f_x \in k(x)^*$. And $df = -\operatorname{div} f = \sum -\operatorname{div} f_x$. Therefore

$$R_p^p(X) \big/ dR_p^{p-1}(X) = CH^p(X),$$

is the codimension p Chow group of X. Note that the sign of df differs from [Gi] but d is a connection homomorphism and its sign depends on the conventions used to define the simple of a morphism of complexes.

On the other hand, the group $K_2(k(X))$ can be described as

$$K_2(k(X)) = k(X)^* \underset{\mathbb{Z}}{\otimes} k(X)^* \Big/ R,$$

where R is the subgroup generated by the elements of the form $f \otimes (1-f)$. The element of $K_2(k(X))$ represented by $f \otimes g$ will be denoted by $\{f, g\}$.

The differential is given by the tame symbol. Let Y be a divisor of X, ν_Y the corresponding valuation. Then the Y-th component of $d\{f, g\}$ is

$$\overline{\left\{ (-1)^{\nu_Y(f)\nu_Y(g)} \frac{f^{\nu_Y(g)}}{g^{\nu_Y(f)}} \right\}},$$

where $\overline{\{\cdot\}}$ denotes the class in $k(Y)^*$.

Let $\mathcal{Z}^p = \mathcal{Z}^p(X)$ denote the set of all closed algebraic subsets of X of codimension $\geq p$ ordered by inclusion. Let $\mathcal{Z}^p \backslash \mathcal{Z}^{p+1}$ denote the set of all pairs $(Z, Z') \in \mathcal{Z}^p \times \mathcal{Z}^{p+1}$ such that $Z' \subset Z$. We consider this set ordered by inclusion.

Following [B-O], let us write

$$H^n_{\mathcal{D},\mathcal{Z}^p \backslash \mathcal{Z}^{p+1}}(X, \mathbb{R}(q)) = \varinjlim_{(Z,Z') \in \mathcal{Z}^p \backslash \mathcal{Z}^{p+1}} H^n_{\mathcal{D}, Z - Z'}(X - Z', \mathbb{R}(q)).$$

Since $\mathcal{Z}^p \backslash \mathcal{Z}^{p+1}$ is a directed set, we can obtain these groups as the cohomology groups of the complex

$$\varinjlim_{(Z,Z') \in \mathcal{Z}^p \backslash \mathcal{Z}^{p+1}} s(\mathfrak{D}^*(E_{\log}(X - Z'), q) \longrightarrow \mathfrak{D}^*(E_{\log}(X - Z), q)).$$

Observe that we have

$$H^n_{\mathcal{D},\mathcal{Z}^p \backslash \mathcal{Z}^{p+1}}(X, \mathbb{R}(q)) = \bigoplus_{x \in X^{(p)}} H^n_{\mathcal{D},x}(X, \mathbb{R}(q)).$$

6.2. Theorem. *There is a commutative diagram*

$$\begin{array}{ccccc}
R_p^{p-2} & \xrightarrow{d} & R_p^{p-1} & \xrightarrow{d} & R_p^p \\
\rho\downarrow & & \rho\downarrow & & \rho\downarrow \\
H^{2p-2}_{\mathcal{D},\mathcal{Z}^{p-2}\setminus\mathcal{Z}^{p-1}}(X,\mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p-1}_{\mathcal{D},\mathcal{Z}^{p-1}\setminus\mathcal{Z}^{p}}(X,\mathbb{R}(p)) & \xrightarrow{\partial} & H^{2p}_{\mathcal{D},\mathcal{Z}^{p}\setminus\mathcal{Z}^{p+1}}(X,\mathbb{R}(p)),
\end{array}$$

where the map ρ is the only non zero component of the map ch *composed with the Gysin morphism. This diagram is covariant for equidimensional projective morphisms.*

Proof. Let $x \in X^{(p)}$ and let $K_i^{(j)}(k(x))$ be the j-graded part of $K_i(k(x))$ with respect to the γ-filtration. One knows that $\mathrm{ch}(K_i^{(j)}(k(x))) \subset H^{2j-i}_{\mathcal{D}}(x,\mathbb{R}(j))$ (see for example [Sch]). Moreover, being $k(x)$ a field, we have $K_i(k(x)) = K_i^{(i)}(k(x))$ for $i \leq 2$ ([Sou]). Therefore the Chern character induces maps

$$K_i(k(x)) \longrightarrow H^i_{\mathcal{D}}(x,\mathbb{R}(i)), \text{ for } i \leq 2.$$

Composing these maps with the Gysin morphism we obtain the map ρ.

The commutativity of the diagram is then a direct consequence of Proposition 6.1.

For the covariance for projective morphisms let $f : X \longrightarrow Y$ be a projetive morphism. Let $x \in X^{(p)}$. If $\dim \overline{\{x\}} \neq \dim \overline{\{f(x)\}}$ then $f_*(\alpha) = 0$ for $\alpha \in K_i(k(x))$ or $\alpha \in H^i_{\mathcal{D}}(x,\mathbb{R}(i))$. On the other hand, if $\dim \overline{\{x\}} = \dim \overline{\{f(x)\}}$ then there are neighbourhoods U of x and V of $f(x)$ such that f induces a morphism $f' : U \longrightarrow V$ which is projective, étale and finite. Therefore the relative tangent bundle is trivial and the Riemann-Roch Theorem implies that the Chern character commutes with direct images.

Now we want to determine the map ρ. At the level of cycles, ρ is the cycle class map. At the level of K_1-chains, if $x \in X^{(p-1)}$, the morphism

$$\rho : K_1(k(x)) \longrightarrow H^{2p-1}_x(X,\mathbb{R}(p))$$

is the composition

$$K_1(k(x)) \xrightarrow{c_{1,1}} H^1(x,\mathbb{R}(1)) \xrightarrow{j_!} H^{2p-1}_x(X,\mathbb{R}(p)),$$

where $j_!$ is the Gysin morphism and $c_{1,1}$ is the first Chern class. On the other hand, we know ([Be]) that $c_{1,1}$ is given by the natural map

$$c : \mathcal{O}^*[-1] \longrightarrow \mathbb{R}_{\mathcal{D}}(*).$$

In terms of the complexes $\mathfrak{D}^*(E_{\log}(\cdot),*)$, we have that, for $f \in \mathcal{O}^*(X)$, $c(f)$ is represented by

$$\frac{1}{2}\log f\overline{f} \in \mathfrak{D}^1(E_{\log}(X),1).$$

At the level of K_2-chains, we can use the multiplicativity of the Chern character and we have that, if $f, g \in \mathcal{O}^*(X)$ then $\mathrm{ch}\{f, g\}$ is represented by

$$\frac{1}{2}\log f\overline{f} \cdot \frac{1}{2}\log g\overline{g} \in \mathfrak{D}^2(E_{\log}(X), 2).$$

With this description of the map ρ, Theorem 6.2 can be checked directly (see [Bu 3]).

Let us write

$$\mathrm{CH}^{p,p-1}(X) = \frac{\operatorname{Ker} d : R_p^{p-1}(X) \longrightarrow R_p^p(X)}{\operatorname{Im} d : R_p^{p-2}(X) \longrightarrow R_p^{p-1}(X)}.$$

As a consequence of Theorem 6.2 we have

6.3. Corollary. *There are well defined maps*

$$\begin{aligned} \rho : CH^p(X) &\longrightarrow H_{\mathcal{D}}^{2p}(X, \mathbb{R}(p)), \quad \textit{and} \\ \rho : CH^{p,p-1}(X) &\longrightarrow H_{\mathcal{D}}^{2p-1}(X, \mathbb{R}(p)). \end{aligned}$$

The first is the cycle class map and the second is the Beilinson regulator map (see [G-S, 3.5.]*). Moreover these maps are covariant for proper morphisms and contravariant for flat morphisms.*

Proof. The case of Chow groups is well known.

Let $\alpha \in R_p^{p-1}$ with $d\alpha = 0$. Then there is an open set $U \subset X$ and a subvariety $Z \subset U$ of codimension $p-1$ such that

$$\rho(\alpha) \in H_{\mathcal{D},Z}^{2p-1}(U, \mathbb{R}(p)).$$

Moreover, we can assume that $Y = X - U$ has codimension p. Then $\rho(\alpha)$ defines a class in $H_{\mathcal{D}}^{2p-1}(U, \mathbb{R}(p))$. By the purity of Deligne-Beilinson cohomology, we have an exact sequence

$$0 \to H_{\mathcal{D}}^{2p-1}(X, \mathbb{R}(p)) \to H_{\mathcal{D}}^{2p-1}(U, \mathbb{R}(p)) \xrightarrow{\partial} H_{\mathcal{D},Y}^{2p}(X, \mathbb{R}(p)).$$

Since $d\alpha = 0$, by Theorem 6.2 $\partial\rho(\alpha) = 0$. Then we have a well defined class, also denoted by $\rho(\alpha) \in H_{\mathcal{D}}^{2p-1}(X, \mathbb{R}(p))$.

Assume that $\alpha = d\beta$. Let us write $V = U - Z$. Then $\rho(\beta) \in H_{\mathcal{D}}^{2p-2}(V, \mathbb{R}(p))$ and $\rho(\alpha) = \partial\rho(\beta)$, where ∂ is the connection morphism

$$H_{\mathcal{D}}^{2p-2}(V, \mathbb{R}(p)) \longrightarrow H_{\mathcal{D},Z}^{2p-1}(U, \mathbb{R}(p)).$$

Therefore the class $\rho(\alpha) \in H_{\mathcal{D}}^{2p-1}(U, \mathbb{R}(p))$ is zero. Thus ρ is well defined.

6.4. Remark. The morphism ρ is also compatible with invere images and intersection products. See [Fu §19] for the case of cycles and [G-S 4.2] for a precise statement at the level of K_1-chains.

6.5. Remark. Let $X_{\mathbb{R}}$ be a smooth real algebraic variety, equivalently $X_{\mathbb{R}}$ is a pair (X, F_∞), where X is a smooth complex variety and F_∞ is an antilinear involution. Then all the results of this section remain valid, provided that we substitute K-chains by real defined K-chains and every complex $A(X)$ by the subcomplex

$$A(X_{\mathbb{R}}) = \{x \in A(X) \mid F_\infty^*(x) = \overline{x}\}.$$

See for example [E-V, 2.1].

In particular, if $A(X)$ is a Dolbeault complex, we shall write

$$\mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) = \begin{cases} \left\{x \in \mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) \mid F_\infty^* x = (-1)^{p-1} x\right\}, & \text{if } n \leq 2p-1 \text{ and} \\ \{x \in \mathfrak{D}^n(A^*(X_{\mathbb{R}}), p) \mid F_\infty^* x = (-1)^p x\}, & \text{if } n \geq 2p. \end{cases}$$

We shall also write

$$H^n_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) = H^n(\mathfrak{D}(E^*_{\log}(X_{\mathbb{R}}), p)).$$

§7. Arithmetic Chow Rings.

Let (A, Σ, F_∞) be an arithmetic ring (See [G-S, §3]). That is, A is an excellent Noetherian domain, Σ is a nonempty set of monomorphisms $\sigma : A \longrightarrow \mathbb{C}$ and F_∞ is a conjugate-linear involution of $\mathbb{C}$-algebras $F_\infty : \mathbb{C}^\Sigma \longrightarrow \mathbb{C}^\Sigma$, such that the image of A in $\mathbb{C}^\Sigma$ is invariant under F_∞. Let us denote by K the quotient field of A. The first examples of such arithmetic rings A are

1) $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, with Σ containing only the inclusion.
2) $A = \mathbb{C}$, with $\Sigma = \{\mathrm{Id}, \sigma\}$, where Id is the identity and σ is the conjugation.
3) $A = \mathcal{O}_K$, the ring of integers of a number field K, with Σ the set of complex immersions of K.

Let X be a regular separated flat A-scheme of finite type, with generic fibre X_K regular over K. X is called an arithmetic variety over A, or arithmetic variety if A is fixed. If $\sigma \in \Sigma$ we write $X_\sigma = X \underset{\sigma}{\otimes} \mathbb{C}$ and $X_\Sigma = \coprod X_\sigma$. Let X_∞ be the complex manifold determined by X_Σ. We denote by F_∞ the anti-linear involution of X_Σ induced by F_∞. Finally we denote by $X_{\mathbb{R}}$ the real manifold (X_∞, F_∞).

In this section we shall use Green forms to define cohomological arithmetic Chow groups of X. In the case when X_K is proper over K then these arithmetic Chow groups are naturally isomorphic to the arithmetic Chow groups defined in [G-S], whereas in the quasi-projective case the groups defined here have better Hodge theoretic properties.

To take into account the structure of real variety of X_∞ (see Remark 6.5) we write

$$GE^p(X_{\mathbb{R}}) = \{\widetilde{g} \in GE^p(X_\infty) \mid F_\infty^* g = \overline{g}\}.$$

Note that, since $g \in \mathfrak{D}^{2p-1}(E^*_{\log}(X_\infty/\mathcal{Z}^p), p)$, we have $\overline{g} = (-1)^{p-1} g$.

We also write

$$\begin{aligned} H^*_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) &= H^*(\mathfrak{D}(E^*_{\log}(X_{\mathbb{R}}), p)), \\ \widetilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X_{\mathbb{R}}) &= \left\{ g \in \mathfrak{D}^{2p-1}(E^*_{\log}(X_\infty), p) \mid F^*_\infty g = \overline{g} \right\} / (\operatorname{Im} d_{\mathfrak{D}}) \\ &= \left\{ g \in E^{p-1,p-1}_{\log}(X_\infty) \cap E^{2p-2}_{\log,\mathbb{R}}(X_\infty, p) \mid F^*_\infty g = (-1)^{p-1} g \right\} \Big/ (\operatorname{Im} \partial + \operatorname{Im} \overline{\partial}) \end{aligned}$$

and

$$\begin{aligned} ZE^{p,p}_{\log,\mathbb{R}}(X_{\mathbb{R}}) &= \left\{ \omega \in \mathfrak{D}^{2p}(E^*_{\log}(X_\infty), p) \mid d_{\mathfrak{D}} \omega = 0,\ F^*_\infty \omega = \overline{\omega} \right\} \\ &= \left\{ \omega \in E^{p,p}_{\log}(X_\infty) \cap E^{2p}_{\log,\mathbb{R}}(X_\infty, p) \mid d\omega = 0,\ F^*_\infty \omega = (-1)^p \omega \right\}. \end{aligned}$$

Observe that Proposition 5.4 remains valid provided we use the corresponding groups for $X_{\mathbb{R}}$.

Let $Z^p(X)$ denote the set of codimension p algebraic cycles on X. For each $y \in Z^p(X)$ there is a well defined cycle $y_K \in Z^p(X_K)$. Hence a cycle $y_{\mathbb{R}} \in Z^p(X_{\mathbb{R}})$. We shall write $\rho(y) = \rho(y_{\mathbb{R}}) \in H^{2p}_{\mathcal{D}, \operatorname{supp} y}(X_{\mathbb{R}}, \mathbb{R}(p))$. Then the space of Green forms for y is defined by:

$$GE^p_y(X_{\mathbb{R}}) = \{ \widetilde{g} \in GE^p(X_{\mathbb{R}}) \mid \operatorname{cl}(\widetilde{g}) = \rho(y) \}.$$

And the group of codimension p arithmetic cycles is defined by

$$\begin{aligned} \widehat{\mathrm{Z}}^p(X) &= \left\{ (y, \widetilde{g}) \in Z^p(X) \oplus GE^p(X_{\mathbb{R}}) \mid \widetilde{g} \in GE^p_y(X_{\mathbb{R}}) \right\} \\ &= \{ (y, \widetilde{g}) \in Z^p(X) \oplus GE^p(X_{\mathbb{R}}) \mid \operatorname{cl}(\widetilde{g}) = \rho(y)) \}. \end{aligned}$$

That is, a codimension p arithmetic cycle is a pair $(y, \widetilde{g})$, where y is a codimension p algebraic cycle, and $\widetilde{g}$ is the class in $\mathfrak{D}^{2p-1}(E^*_{\log}((X \backslash \mathcal{Z}^p)_{\mathbb{R}}), p) / \operatorname{Im} d_{\mathfrak{D}}$ of a form $g \in \mathfrak{D}^{2p-1}(E^*_{\log}((X \backslash \mathcal{Z}^p)_{\mathbb{R}}), p)$, such that

$$\omega = d_{\mathfrak{D}} g = -2\partial\overline{\partial} g \in \mathfrak{D}^{2p}(E^*_{\log}(X_{\mathbb{R}}), p),$$

and the pair (ω, g) represents the class $\rho(y) \in H^{2p}_{\mathcal{D}, \mathcal{Z}^p}(X_{\mathbb{R}}, \mathbb{R}(p))$.

Let us now define rational equivalence in this setting. Let W be a codimension $p-1$ irreducible subvariety of X and let $f \in k(W)^*$. Let us write $Y = \operatorname{supp} \operatorname{div} f$. We have a well defined subvariety W_∞ of X_∞ (which may be empty) and a function $f_\infty \in k(W_\infty)^*$. Since f is defined over K, the function f_∞ satisfies $F^*_\infty f = \overline{f}$. Hence the map ρ (see §2) gives us a class

$$\rho(f) = \rho(f_\infty) \in H^{2p-1}_{\mathcal{D}}((X - Y)_{\mathbb{R}}, \mathbb{R}(p)).$$

Therefore we have an element

$$\mathrm{b}(\rho(f)) \in GE^p_{\operatorname{div} f}(X_{\mathbb{R}}),$$

where $\mathrm{b} : H^{2p-1}_{\mathcal{D}}((X - Y)_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow GE^p_{\operatorname{div} f}(X_{\mathbb{R}})$ is the map introduced after Definition 5.3.

Then we write

$$\widehat{\operatorname{div}} f = (\operatorname{div} f, -\operatorname{b}(\rho(f))).$$

By Theorem 6.2, $\widehat{\operatorname{div}} f \in \widehat{\mathrm{Z}}^p(X)$. We denote by $\widehat{\mathrm{Rat}}^p$ the subgroup of $\widehat{\mathrm{Z}}^p$ generated by the elements of the form $\widehat{\operatorname{div}} f$.

7.1. Definition. *The arithmetic Chow groups of X are*

$$\widehat{\mathrm{CH}}^p(X) = \widehat{\mathrm{CH}}^p(X, \mathfrak{D}(E_{\log})) = \widehat{\mathrm{Z}}^p(X) \Big/ \widehat{\mathrm{Rat}}^p.$$

We shall write $\widehat{\mathrm{CH}}^p(X, \mathfrak{D}(E_{\log}))$ when we want to stress the complex used to define the Green objects, or when we want to differentiate them from the arithmetic Chow groups defined by Gillet and Soulé.

We shall write

$$\widehat{\mathrm{CH}}^*(X) = \bigoplus_p \widehat{\mathrm{CH}}^p(X).$$

Before stating the comparison theorem between the Chow groups defined here and in [G-S], let us recall some results about currents.

For X, a complex variety of dimension d, let $\mathcal{D}_X^n$ denote the sheaf of complex valued currents on X. That is, for an open subset $U \subset X$, $\Gamma(U, \mathcal{D}_X^n)$ is the topological dual of $\Gamma_c(U, \mathcal{E}_X^{2d-n})$.

The sheaf $\mathcal{D}_X^n$ has a real structure $\mathcal{D}_{X,\mathbb{R}}^n$ and a natural bigrading

$$\mathcal{D}_X^n = \bigoplus_{p+q=n} \mathcal{D}_X^{p,q}.$$

From this we can define the Hodge filtration as usual. We shall write $D_X^n = \Gamma(X, \mathcal{D}_X^n)$.

There is a map

$$\begin{array}{rccc} [\cdot]: & E_X^n & \longrightarrow & D_X^{2d-n} \\ & \omega & \longmapsto & [\omega], \end{array}$$

defined by

$$[\omega](\omega') = \frac{1}{(2\pi i)^d} \int_X \omega' \wedge \omega.$$

More generally, if ω is a locally L^1 form, then we define $[\omega]$ by the same formula.

We can turn D_X^* into a chain complex by writing, for $T \in D_X^n$,

$$dT(\omega) = (-1)^n T(d\omega).$$

By Stokes' Theorem, the map $[\cdot] : E_X^* \longrightarrow D_X^*$ is a morphism of complexes. Moreover, it is a filtered quasi-isomorphism with respect to the Hodge filtration.

If $Y \subset X$ is a subvariety, we shall denote by $\Sigma_Y E_X^*$ the subcomplex of E_X^* composed by the forms which vanish when restricted to each irreducible component of Y. Then the complex of currents on Y ([H-L]) is defined by:

$$D_Y^n = \{T \in D_X^n \mid T(\omega) = 0,\ \forall \omega \in \Sigma_Y E_X^n\}.$$

Up to a shift in the graduation, this complex only depends on Y and, when Y is smooth, coincides with the usual complex of currents.

Let us write $D^*_{X/Y} = D^*_X / D^*_Y$. If Y is a divisor with normal crossings, there is a morphism

$$[\cdot] : E^*_X(\log Y) \longrightarrow D^*_{X/Y}$$

which is a filtered quasi-isomorphism with respect to the Hodge filtration (see [Bu 2] and [Bu 3]).

7.2. Theorem. *Let X be an arithmetic variety, with X_K proper over K and* $\dim X_K = d$. *Then there is a natural isomorphism*

$$\widehat{CH}^p(X, \mathfrak{D}(E^*_{\log})) \longrightarrow \widehat{CH}^p(X),$$

where the group on the right hand side is the arithmetic Chow group defined in [G-S]. *This isomorphism is given by*

$$(y, \widetilde{g}) \longmapsto (y, 2(2\pi i)^{d-p+1}[g]^{\sim}),$$

where g is a representative of $\widetilde{g}$.

Proof. By [Bu 2, 3.8.2], any representative g of $\widetilde{g}$ is locally integrable in the whole X. Therefore the current $[g]$ is well defined. By Theorem 5.9, the map

$$(y, \widetilde{g}) \longmapsto (y, 2(2\pi i)^{d-p+1}[g]^{\sim}),$$

gives us an isomorphism between the group of arithmetic cycles defined here and the group of arithmetic cycles in the sense of Gillet and Soulé. Thus we only need to check that the two concepts of rational equivalence coincide.

Let us recall the definition of rational equivalence in [G-S]. Let W be a codimension $p-1$ irreducible subvariety of X and let $f \in k(W)^*$. Let $\widetilde{W}_\infty$ be a resolution of singularities of W_∞ and let $j : \widetilde{W}_\infty \longrightarrow X_\infty$ be the induced map. The function f induces a well defined function, also denoted by $f \in k(\widetilde{W}_\infty)^*$. Then $\widehat{\operatorname{div}} f$ in the sense of Gillet and Soulé is defined by

$$\widehat{\operatorname{div}} f = (\operatorname{div} f, -(2\pi i)^{d-p+1} j_*[\log f\overline{f}]).$$

The factor $(2\pi i)^{d-p}$ comes from the different definition of $[\cdot]$ here and in [G-S].

Therefore we are reduced to proving that there is a representative g of $-\operatorname{b}(\rho f)$ such that, if $[g]$ is the associated current on X, then

$$2[g] + j_*[\log f\overline{f}] \in \operatorname{Im} \partial + \operatorname{Im} \overline{\partial}$$

in the complex $D^*_{X_\infty}$. Since this statement only depends on the complex variety X_∞ we will assume that X is a complex variety of dimension d.

Let $Y = \operatorname{supp} \operatorname{div} f$. Let $\pi : (\widetilde{X}, D) \longrightarrow (X, Y)$ be a resolution of singularities, with $D = \pi^{-1}(Y)$ a divisor with normal crossings. Then the class $\rho(f) \in H^{2p-1}_{\mathcal{D}}(X - Y, \mathbb{R}(d-p))$ is represented by the current $j_*[-(1/2) \log f\overline{f}]$. Therefore, in the complex $D^*_{\widetilde{X}/D}$, we have the equation

$$2[g] + j_*[\log f\overline{f}] = \partial a + \overline{\partial} b.$$

By [Bu 2, 1.9] we may assume that g is of weight one. Therefore it is locally integrable in the whole $\widetilde{X}$. Let us also denote by $[g]$ the associated current in the complex $D^*_{\widetilde{X}}$. Let a' and b' be elements of $D^*_{\widetilde{X}}$ which are mapped to a and b. Then in the complex $D^*_{\widetilde{X}}$ we have

$$2[g] + j_*[\log f\overline{f}] = \partial a' + \overline{\partial} b' + c,$$

where $c \in D_D^{p-1,p-1}$. So $\pi_* c \in D_Y^{p-1,p-1} = \{0\}$ because $\operatorname{codim} Y = p$. Hence, in the complex D^*_X we have

$$2[g] + j_*[\log f\overline{f}] = \partial \pi_* a' + \overline{\partial} \pi_* b'.$$

This concludes the proof of the theorem.

Our next objective is to fit the groups $\widehat{\mathrm{CH}}^*(X)$ in some exact sequences. Recall that we have defined a morphism (see 6.3)

$$\rho : \mathrm{CH}^{p,p-1}(X) \longrightarrow H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)).$$

We will also denote by ρ the composition

$$\rho : \mathrm{CH}^{p,p-1}(X) \longrightarrow H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow \widetilde{E}_{\log}^{p-1,p-1}(X_{\mathbb{R}}).$$

We have maps

$$\begin{aligned}
\zeta &: \widehat{\mathrm{CH}}^p(X) \longrightarrow \mathrm{CH}^p(X), && \zeta(y, \widetilde{g}) = y, \\
\rho &: \mathrm{CH}^p(X) \longrightarrow H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)), && \text{see 6.3}, \\
a &: \widetilde{E}_{\log}^{p-1,p-1}(X_{\mathbb{R}}) \longrightarrow \widehat{\mathrm{CH}}^p(X), && a(\widetilde{g}) = (0, \widetilde{g}), \\
\omega &: \widehat{\mathrm{CH}}^p(X) \longrightarrow ZE_{\log}^{p,p}(X_{\mathbb{R}}), && \omega(y, \widetilde{g}) = -2\partial\overline{\partial} g \qquad \text{and} \\
h &: ZE_{\log}^{p,p}(X_{\mathbb{R}}) \longrightarrow H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)), && h(\alpha) = \{\alpha\},
\end{aligned}$$

where $\{\alpha\}$ is the cohomology class of α.

Let us write

$$\begin{aligned}
&\widehat{\mathrm{CH}}^p(X)_0 = \mathrm{Ker}(\omega) \qquad \text{and} \\
&\mathrm{CH}^p(X)_0 = \{y \in \mathrm{CH}^p(X) \mid y_\infty \underset{\text{hom}}{\sim} 0\}.
\end{aligned}$$

Then the analogue of [G-S, Theorem 3.3.5] is:

Theorem 7.3. *Let X be an arithmetic variety. Then we have exact sequences:*

$$\begin{aligned}
&(i) \quad CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{E}_{\log}^{p-1,p-1}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0, \\
&(ii) \quad CH^{p,p-1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X) \\
&\qquad\qquad \xrightarrow{(\zeta,-\omega)} CH^p(X) \oplus ZE_{\log}^{p,p}(X_{\mathbb{R}}) \xrightarrow{\rho+h} H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \to 0, \\
&(iii) \quad CH^{p,p-1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X)_0 \xrightarrow{\zeta} CH^p(X)_0 \to 0.
\end{aligned}$$

Proof. The proof of the exactness of the three sequences is similar. So we shall write only the first.

The fact that the composition of two consecutive morphisms is zero follows easily from the definitions.

The surjectivity of ζ is equivalent to the existence of Green forms for a cycle and is a consequence of the surjectivity of the map cl (Proposition 5.4.1.)

Assume now that $\zeta(y,\widetilde{g})=0$. Then $y=\sum \operatorname{div} f_i$ and $(y,\widetilde{g})-\sum \widehat{\operatorname{div}} f_i=(0,\widetilde{g}')$. Then $\operatorname{cl}\widetilde{g}'=0$. By Proposition 5.4.1, $\widetilde{g}'\in \operatorname{Im} a$.

If $\widetilde{g}\in \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}})$ with $a(\widetilde{g})=0$, then $(0,\widetilde{g})=\sum \widehat{\operatorname{div}} f_i$. Therefore $\sum \operatorname{div} f_i=0$ and $f=\sum f_i$ determines an element of $\mathrm{CH}^{p,p-1}(X)$ and $\widetilde{g}=\rho(f)$.

Example 7.4. In [G-S, 3.4] there are some examples of explicit arithmetic Chow groups. Since these examples are given for arithmetic varieties with projective X_∞, they are also examples for the arithmetic Chow groups introduced here.

Let us give a simple example where the groups obtained here and the groups obtained in [G-S] differ. Let $X=\mathbb{A}^1_{\mathbb{Z}}=\operatorname{Spec}(\mathbb{Z}[t])$. Then X is an arithmetic variety over $\mathbb{Z}$. We have that $\mathrm{CH}^1(X)=0$ and $\mathrm{CH}^{1,0}(X)=\{-1,1\}$, but $\rho(\mathrm{CH}^{1,0}(X))=0$. Therefore

$$\widehat{\mathrm{CH}}^1(X,\mathfrak{D}(E_{\log}))\cong E^0_{\log}(\mathbb{A}^1_{\mathbb{R}}).$$

That is, the space of F_∞-invariant, real valued C^∞ functions on $\mathbb{A}^1_{\mathbb{C}}$ which have logarithmic singularities at infinity. Moreover we have

$$\widehat{\mathrm{CH}}^1(X,\mathfrak{D}(E_{\log}))_0=H^1_{\mathcal{D}}(\mathbb{A}^1_{\mathbb{R}},\mathbb{R}(1))=\mathbb{R}.$$

In particular, the morphism

$$\pi^*:\widehat{\mathrm{CH}}^*(\operatorname{Spec}\mathbb{Z},\mathfrak{D}(E_{\log}))_0\longrightarrow \widehat{\mathrm{CH}}^*(X,\mathfrak{D}(E_{\log}))_0$$

is an isomorphism.

On the other hand, the groups $\widehat{\mathrm{CH}}^1(X)_0$ as defined in [G-S] are isomorphic to the analytic Deligne cohomology of $\mathbb{A}^1_{\mathbb{R}}$, $H^1_{\mathcal{D}^{\mathrm{an}}}(\mathbb{A}^1_{\mathbb{R}},\mathbb{R}(1))$, which is an infinite dimensional real vector space.

Let us give a generalization of the above example.

Theorem 7.5. *Let X be an arithmetic variety and let $\pi:M\longrightarrow X$ be a geometric vector bundle. Then the induced morphism*

$$\pi^*:\widehat{CH}^*(X,\mathfrak{D}(E_{\log}))_0\longrightarrow \widehat{CH}^*(M,\mathfrak{D}(E_{\log}))_0$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{CH}^{p,p-1}(X) & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) & \longrightarrow & \widehat{\mathrm{CH}}^p(X)_0 & \longrightarrow & \mathrm{CH}^p(X)_0 & \longrightarrow & 0\\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\
\mathrm{CH}^{p,p-1}(M) & \longrightarrow & H^{2p-1}_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(p)) & \longrightarrow & \widehat{\mathrm{CH}}^p(M)_0 & \longrightarrow & \mathrm{CH}^p(M)_0 & \longrightarrow & 0.
\end{array}$$

At the level of CH^p and $\mathrm{CH}^{p,p-1}$, the morphism π^* is an isomorphism by [Gi, Th 8.3]. At the level of Deligne-Beilinson cohomology, the morphism π^* is an isomorphism because

$$\pi^* : E^*_{\log}(X_{\mathbb{C}}) \longrightarrow E^*_{\log}(M_{\mathbb{C}})$$

is a real filtered quasi-isomorphism with respect to the Hodge filtration. Therefore π^* is also an isomorphism at the level of $\widehat{\mathrm{CH}}^p_0$.

Let us summarize the properties of cohomological Chow groups. These properties can be proved as in [G-S] substituting Green currents by Green forms.

Let $(y, \widetilde{g}_y)$ and $(z, \widetilde{g}_z)$ be two arithmetic cycles such that y and z intersect properly. Then the singular support of $\widetilde{g}_y$ and the singular support of $\widetilde{g}_z$ intersect properly. Therefore the product $\widetilde{g}_y * \widetilde{g}_z$ is defined and is a Green form for $y \cdot z$. We can define an intersection product by

$$(y, \widetilde{g}_y) \cdot (z, \widetilde{g}_z) = (y \cdot z, \widetilde{g}_y * \widetilde{g}_z). \tag{7.6}$$

Let us write

$$\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^*(X) \otimes \mathbb{Q}.$$

Then we have (see [G-S, Theorem 4.2.3] for a more precise statement):

Theorem 7.7. *Let A be an arithmetic ring with fraction field K and let X be an arithmetic variety with X_K quasi-projective. Then, for each pair of non-negative integers p, q, there is an intersection pairing*

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \longrightarrow \widehat{CH}^{p+q}(X)_{\mathbb{Q}},$$

which is given by formula 7.6. for cycles intersecting properly.

This product induces in $\widehat{CH}^*(X)_{\mathbb{Q}}$ a structure of commutative and associative ring. Moreover, the induced maps

$$\zeta : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow CH^*(X) \otimes \mathbb{Q}$$

and

$$\omega : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow \bigoplus_p E^{p,p}_{\log}(X_{\mathbb{R}}, p)$$

are morphisms of rings. Therefore the subgroup $\widehat{CH}^*(X)_{0,\mathbb{Q}} = \mathrm{Ker}(\omega)$ is an ideal of $\widehat{CH}^*(X)_0$.

The functorial properties of the cohomological Chow groups are summarized in the following theorem. For proofs see [G-S, Theorem 3.6.1] and [G-S, Theorem 4.4.3]. Note that, in the case of arithmetic varieties which are not proper over A, we have to impose stronger conditions for the existence of a push-forward map. This is done to ensure that the direct image of a logarithmic form is again a logarithmic form (see the construction of a push forward of Green forms in [Bu 2, 1.14]).

Theorem 7.8. *Let A be an arithmetic ring.*

1. *Let $f : X' \longrightarrow X$ be a morphism of regular quasi-projective arithmetic varieties. Then there is a pull-back morphism*

$$f^* : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^p(X'),$$

such that, if $(y, \widetilde{g}_y) \in \widehat{Z}^p(X)$ and $f^{-1}(y)$ is equidimensional of codimension p then

$$f^*(y, \widetilde{g}_y) = (f^*y, f^*\widetilde{g}_y),$$

with f^*y defined as in [Se]*. If $g : X'' \longrightarrow X'$ is another such morphism then $(fg)^* = g^*f^*$. Moreover f^* induces a ring homomorphism*

$$f^* : \widehat{CH}^*(X)_{\mathbb{Q}} \longrightarrow \widehat{CH}^*(X')_{\mathbb{Q}}.$$

2. *Let $f : X' \longrightarrow X$ be a proper morphism of equidimensional regular arithmetic varieties. Assume that there are smooth compactifications $\overline{X}'_\infty$ of X'_∞ and $\overline{X}_\infty$ of X_∞, such that $f_\infty : X'_\infty \longrightarrow X_\infty$ can be extended to a smooth map $\overline{f}_\infty : \overline{X}'_\infty \longrightarrow \overline{X}_\infty$. Let $e = \dim(X') - \dim(X)$. Then there is a push-forward morphism*

$$f_* : \widehat{CH}^p(X') \longrightarrow \widehat{CH}^{p-e}(X),$$

such that $f_*(y, \widetilde{g}_y) = (f_*y, f_*g_y)$. If $g : X'' \longrightarrow X'$ is another such morphism then $(fg)_* = f_*g_*$. Moreover, if $\alpha \in \widehat{CH}^p(X')$ and $\beta \in \widehat{CH}^q(X)$, then

$$f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta \in \widehat{CH}^{p+q-e}(X)_{\mathbb{Q}}.$$

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