ARITHMETIC GEOMETRY OF TORIC VARIETIES. METRICS, MEASURES AND HEIGHTS

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Abstract. We show that the height of a toric variety with respect to a toric metrized line bundle can be expressed as the integral over a polytope of a certain adelic family of concave functions. To state and prove this result, we study the Arakelov geometry of toric varieties. In particular, we consider models over a discrete valuation ring, metrized line bundles, and their associated measures and heights. We show that these notions can be translated in terms of convex analysis, and are closely related to objects like polyhedral complexes, concave functions, real Monge-Ampère measures, and Legendre-Fenchel duality.

We also present a closed formula for the integral over a polytope of a function of one variable composed with a linear form. This allows us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes. We also compute the height of toric projective curves with respect to the Fubini-Study metric, and of some toric bundles.

Contents

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1. Introduction

Systems of polynomial equations appear in a wide variety of contexts in both pure and applied mathematics. Systems arising from applications are not random but come with a certain structure. When studying those systems, it is important to be able to exploit that structure.

A relevant result in this direction is the Bernštein-Kušnirenko-Khovanskii theorem [Kuš76, Ber75]. Let K be a field with algebraic closure $\overline{K}$. Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope and $f_1, \dots, f_n \in K[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ a family of Laurent polynomials whose Newton polytope is contained in Δ. The BKK theorem implies that the number (counting multiplicities) of isolated common zeros of $f_1, \dots, f_n$ in $(\overline{K}^{\times})^n$ is bounded above by $n!$ times the volume of Δ, with equality when $f_1, \dots, f_n$ is generic among the families of Laurent polynomials with Newton polytope contained in Δ. This shows how a geometric problem (the counting of the number of solutions of a system of equations) can be translated into a combinatorial, simpler one. It is commonly used to predict when a given system of polynomial equations has a small number of solutions. As such, it is a cornerstone of polynomial equation solving and has motivated a large amount of work and results over the past 25 years, see for instance [GKZ94, Stu02, PS08b] and the references therein.

A natural way to study polynomials with prescribed Newton polytope is to associate to the polytope Δ a toric variety X over K equipped with an ample line bundle L. The polytope conveys all the information about the pair (X, L). For instance, the degree of X with respect to L is given by the formula

$$\deg_L(X) = n! \operatorname{vol}(\Delta), \tag{1.1}$$

where vol denotes the Lebesgue measure of $\mathbb{R}^n$. The Laurent polynomials f_i can be identified with global sections of L, and the BKK theorem is equivalent to this formula. Indeed, there is a dictionary which allows to translate algebro-geometric properties of toric varieties in terms of combinatorial properties of polytopes and fans, and formula (1.1) is one entry in this "toric dictionary".

The central motivation for this text is an arithmetic analogue for heights of this formula, which is Theorem 1.2 below. The height is a basic arithmetic invariant of a proper variety over the field of rational numbers. Together with its degree, it measures the amount of information needed to represent this variety, for instance, via its Chow form. Hence, this invariant is also relevant in computational algebraic geometry, see for instance [GHH+97, AKS07, DKS10]. The notion of height of varieties generalizes the height of points already considered by Siegel, Northcott, Weil and others, and is a key tool in Diophantine geometry, see for instance [BG06] and the references therein.

Assume that the pair (X, L) is defined over $\mathbb{Q}$. Let $\mathfrak{M}_{\mathbb{Q}}$ denote the set of places of $\mathbb{Q}$, and $\{\vartheta_v\}_{v\in\mathfrak{M}_{\mathbb{Q}}}$ a family of concave functions on Δ such that $\vartheta_v \equiv 0$ for all but a finite number of v. We will show that, to this data, one can associate an adelic family of metrics $\{\|\cdot\|_v\}_v$ on L. Write $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ for the resulting metrized line bundle.

Theorem 1.2. *The height of X with respect to $\overline{L}$ is given by*

$$\mathrm{h}_{\overline{L}}(X) = (n+1)! \sum_{v\in\mathfrak{M}_{\mathbb{Q}}} \int_{\Delta} \vartheta_v \,\mathrm{d}\,\mathrm{vol}\,.$$

This theorem was announced in [BPS09] and we prove it in the present text. To establish it in a wide generality, we have been led to study the Arakelov geometry of toric varieties. In the course of our research, we have found that a large part of the arithmetic geometry of toric varieties can be translated in terms of convex analysis. In particular, we have added a number of new entries to the arithmetic geometry chapter of the toric dictionary, including models of toric varieties over a discrete valuation ring, metrized line bundles, and their associated measures and heights. These objects are closely related to objects of convex analysis like polyhedral complexes, concave functions, Monge-Ampère measures and Legendre-Fenchel duality.

These additions to the toric dictionary are very concrete and well-suited for computations. In particular, they provide a new wealth of examples in Arakelov geometry where constructions can be made explicit and properties tested. In this direction, we also present a closed formula for the integral over a polytope of a function of one variable composed with a linear form. This formula allows us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes. Some of these heights are related to the average entropy of a simple random process on the polytope. We also compute the height of toric projective curves with respect to the Fubini-Study metric and of some toric bundles.

There are many other arithmetic invariants of toric varieties that may be studied in terms of convex analysis. For instance, one can give criteria for positivity properties of toric metrized line bundles, like having or being generated by small sections, a formula for its arithmetic volume, and an arithmetic analogue of the BKK theorem bounding the height of the solutions of a system of sparse polynomial equations with rational coefficients. In fact, we expect that the results of this text are just the starting point of a program relating the arithmetic geometry of toric varieties and convex analysis.

In the rest of this introduction, we will present the context and the contents of our results. We will refer to the body of the text for the precise definitions and statements.

Arakelov geometry provides a framework to define and study heights. We leave for a moment the realm of toric varieties, and we consider a projective variety X over $\mathbb{Q}$ of dimension n. Let $\mathcal{X}$ be a proper integral model of X, and $X(\mathbb{C})$ the analytic space over the complex numbers associated to X. The main idea behind Arakelov geometry is that the pair $(\mathcal{X}, X(\mathbb{C}))$ should behave like a compact variety of dimension $n+1$ [Ara74]. Following this philosophy, Gillet and Soulé have developed an arithmetic intersection theory [GS90]. As an application of this theory, one can introduce a very general and precise definition, with a geometric flavor, of the height of a variety [BGS94]. To the variety X, one associates the arithmetic intersection ring $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$. This ring is equipped with a trace map $\int\colon \widehat{\mathrm{CH}}^{n+1}(X)_{\mathbb{Q}} \to \mathbb{R}$. Given a line bundle L on X, an arithmetic line bundle $\overline{L}$ is a pair $(\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ which is an integral model of L, and $\|\cdot\|$ is a smooth metric on the analytification of L. In this setting, the analogue of the first Chern class of L is the arithmetic first Chern class $\widehat{\mathrm{c}}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. The *height* of X with respect to $\overline{L}$ is then defined as

$$\mathrm{h}_{\overline{L}}(X) = \int \widehat{\mathrm{c}}_1(\overline{L})^{n+1} \in \mathbb{R}.$$

This is the arithmetic analogue of the degree of X with respect to L. This formalism has allowed to obtain arithmetic analogues of important results in algebraic geometry like the Bézout's theorem, the Riemman-Roch theorem, the Lefschetz fixed point formula, the Hilbert-Samuel formula, etc.

This approach has two technical issues. In the first place, it only works for smooth varieties and smooth metrics. In the second place, it depends on the existence of an integral model, which puts the Archimedean and non-Archimedean places in different footing. For the definition of heights, both issues were addressed by Zhang [Zha95b] by taking an adelic point of view and considering uniform limits of semipositive metrics.

Many natural metrics that arise when studying line bundles on toric varieties are not smooth, but are particular cases of the metrics considered by Zhang. This is the case for the canonical metric of a toric line bundle, see §5.2. The associated canonical height of subvarieties plays an important role in Diophantine approximation in tori, in particular in the generalized Bogomolov and Lehmer problems, see for instance [DP99, AV09] and the references therein. Maillot has extended the arithmetic intersection theory of Gillet and Soulé to this kind of metrics at the Archimedean place, while maintaining the use of an integral model to handle the non-Archimedean places [Mai00].

The adelic point of view of Zhang was developed by Gubler [Gub02, Gub03] and by Chambert-Loir [Cha06]. From this point of view, the height is defined as a sum of local contributions. We outline this procedure, that will be recalled with more detail in §2.

For the local case, let K be either $\mathbb{R}$, $\mathbb{C}$, or a field complete with respect to a nontrivial non-Archimedean absolute value. Let X be a proper variety over K and L a line bundle on X, and consider their analytifications, respectively denoted by X^{an} and L^{an}. In the Archimedean case, X^{an} is the complex space $X(\mathbb{C})$ (equipped with an anti-linear involution, if $K = \mathbb{R}$), whereas in the non-Archimedean case it is the Berkovich space associated to X. The basic metrics that can be put on L^{an} are the smooth metrics in the Archimedean case, and the algebraic metrics in the non-Archimedean case, that is, the metrics induced by an integral model of a pair $(X, L^{\otimes e})$ with $e \geq 1$. There is a notion of semipositivity for smooth and and for algebraic metrics, and the uniform limit of such semipositive metrics leads to the notion of *approachable* metric on L^{an}. More generally, a metric on L^{an} is *integrable* if it is the quotient of two approachable metrics.

Let $\overline{L}$ be an integrable metrized line bundle on X and Y a d-dimensional cycle of X. These data induce a (signed) measure on X^{an}, denoted $c_1(\overline{L})^{\wedge d} \wedge \delta_Y$ by analogy with the Archimedean smooth case, where it corresponds with the current of integration along Y^{an} of the d-th power of the first Chern form. This measure plays an important role in the distribution of points of small height in the direction of the Bogomolov conjecture and its generalizations, see for instance [Bil97, SUZ97, Yua08]. Furthermore, if we have sections s_i, $i = 0, \dots, d$, that meet Y properly, one can define a notion of *local height* $\mathrm{h}_{\overline{L}}(Y; s_0, \dots, s_d)$. The metrics and their associated measures and local heights are related by the Bézout-type formula:

$$\mathrm{h}_{\overline{L}}(Y \cdot \mathrm{div}(s_d); s_0, \dots, s_{d-1}) = \mathrm{h}_{\overline{L}}(Y; s_0, \dots, s_d) + \int_{X^{\mathrm{an}}} \log \|s_d\| \, c_1(\overline{L})^{\wedge d} \wedge \delta_Y .$$

For the global case, consider a proper variety X over $\mathbb{Q}$ and a line bundle L on X. For simplicity, assume that X is projective, although this hypothesis is not really necessary. An *integrable quasi-algebraic* metric on L is a family of integrable metrics $\|\cdot\|_v$ on the analytic line bundles L_v^{an}, $v \in \mathfrak{M}_{\mathbb{Q}}$, such that there is an integral model of $(X, L^{\otimes e})$, $e \geq 1$, which induces $\|\cdot\|_v$ for all but a finite number of v. Write $\overline{L} = (L, \{\|\cdot\|_v\}_v)$, and $\overline{L}_v = (L_v, \|\cdot\|_v)$ for each $v \in \mathfrak{M}_{\mathbb{Q}}$. Given a d-dimensional cycle Y of X, its *global height* is defined as

$$\mathrm{h}_{\overline{L}}(Y) = \sum_{v \in \mathfrak{M}_{\mathbb{Q}}} \mathrm{h}_{\overline{L}_v}(Y; s_0, \dots, s_d),$$

for any family of sections s_i, $i = 0, \dots, d$, meeting Y properly. The fact that the metric is quasi-algebraic implies that the right-hand side has only a finite number of nonzero terms, and the product formula implies that this definition does not depend on the choice of sections. This notion can be extended to number fields, function fields and, more generally, to M-fields [Zha95b, Gub03].

Now we review briefly the elements of the construction of toric varieties from combinatorial data, see §4 for more details. Let K be a field and $\mathbb{T} \simeq \mathbb{G}_m^n$ a split torus over K. Let $N = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \simeq \mathbb{Z}^n$ be the lattice of one-parameter subgroups of $\mathbb{T}$ and $M = N^{\vee}$ the dual lattice of characters of $\mathbb{T}$. Set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. To a fan Σ on $N_{\mathbb{R}}$ one can associate a toric variety X_Σ of dimension n. It is a normal variety that contains $\mathbb{T}$ as a dense open subset, denoted $X_{\Sigma,0}$, and there is an action of $\mathbb{T}$ on X_Σ which extends the natural action of the torus on itself. In particular, every toric variety has a distinguished point x_0 that corresponds to the identity element of $\mathbb{T}$. The variety X_Σ is proper whenever the underlying fan is complete. For sake of simplicity, in this introduction we will restrict to the proper case.

A Cartier divisor invariant under the torus action is called a $\mathbb{T}$-Cartier divisor. In combinatorial terms, a $\mathbb{T}$-Cartier divisor is determined by a virtual support function on Σ, that is, a continuous function $\Psi \colon N_{\mathbb{R}} \to \mathbb{R}$ whose restriction to each cone of Σ is an element of M. Let D_Ψ denote the $\mathbb{T}$-Cartier divisor of X_Σ determined by Ψ. A *toric line bundle* on X_Σ is a line bundle L on this toric variety, together with the choice of a nonzero element $z \in L_{x_0}$. The total space of a toric line bundle has a natural structure of toric variety whose distinguished point agrees with z. A rational section of a toric line bundle is called *toric* if it is regular and nowhere zero on the principal open subset $X_{\Sigma,0}$, and $s(x_0) = z$. Given a virtual support function Ψ, the line bundle $L_\Psi = \mathcal{O}(D_\Psi)$ has a natural structure of toric line bundle and a canonical toric section s_Ψ such that $\mathrm{div}(s_\Psi) = D_\Psi$. Indeed, any line bundle on X_Σ is isomorphic to a toric line bundle of the form L_Ψ for some Ψ. The line bundle L_Ψ

is generated by global sections (respectively, is ample) if and only if Ψ is concave (respectively, Ψ is strictly concave on Σ).

Consider the lattice polytope

$$\Delta_\Psi := \{x \in M_\mathbb{R} : \langle x, u\rangle \geq \Psi(u) \text{ for all } u \in N_\mathbb{R}\} \subset M_\mathbb{R}.$$

This polytope encodes a lot of information about the pair (X_Σ, L_Ψ). In case the virtual support function Ψ is concave, it is determined by this polytope, and the formula (1.1) can be written more precisely as

$$\deg_{L_\Psi}(X_\Sigma) = n! \operatorname{vol}_M(\Delta_\Psi),$$

where the volume is computed with respect to the Haar measure vol_M on $M_\mathbb{R}$ normalized so that M has covolume 1.

In this text we extend the toric dictionary to metrics, measures and heights as considered above. For the local case, let K be either $\mathbb{R}$, $\mathbb{C}$, or a field complete with respect to a nontrivial non-Archimedean absolute value associated to a discrete valuation. In this latter case, let K° be the valuation ring, $K^{\circ\circ}$ its maximal ideal and ϖ a generator of $K^{\circ\circ}$. Let $\mathbb{T}$ be an n-dimensional split torus over K, $\mathbb{T}^{\mathrm{an}}$ its analytification and $\mathbb{S}^{\mathrm{an}}$ the compact torus of $\mathbb{T}^{\mathrm{an}}$. Let X be a toric variety over K with torus $\mathbb{T}$ and L a toric line bundle on X. The compact torus $\mathbb{S}^{\mathrm{an}}$ is a closed subgroup of the analytic torus $\mathbb{T}^{\mathrm{an}}$ and it acts on X^{an}. A metric $\|\cdot\|$ on L^{an} is *toric* if, for every toric section s, the function $\|s\|$ is invariant under the action of $\mathbb{S}^{\mathrm{an}}$.

The correspondence that to a virtual support function assigns a toric line bundle with a toric section can be extended to approachable and integrable metrics. Assume that Ψ is concave, and let X_Σ, L_Ψ and s_Ψ be as before. For short, write $X = X_\Sigma$, $L = L_\Psi$ and $s = s_\Psi$. There is a fibration $\operatorname{val}_K\colon X_0^{\mathrm{an}} \to N_\mathbb{R}$ whose fibers are the orbits of the action of $\mathbb{S}^{\mathrm{an}}$ on X_0^{an}. Now let $\psi\colon N_\mathbb{R} \to \mathbb{R}$ be a continuous function. We define a metric on the restriction $L^{\mathrm{an}}|_{X_0^{\mathrm{an}}}$ by setting

$$\|s(p)\|_\psi = e^{\lambda_K \psi(\operatorname{val}_K(p))},$$

with $\lambda_K = 1$ if $K = \mathbb{R}$ or $\mathbb{C}$, and $\lambda_K = -\log|\varpi|$ otherwise.

Our first addition to the toric dictionary is the following classification result. Assume that the function ψ is concave and that $|\psi - \Psi|$ is bounded. Then $\|\cdot\|_\psi$ extends to an approachable toric metric on L^{an} and, moreover, every approachable toric metric on L^{an} arises in this way (Theorem 5.73(1)). There is a similar characterization of integrable toric metrics in terms of differences of concave functions (Corollary 5.83) and a characterization of toric metrics that involves the topology of the variety with corners associated to X_Σ (Proposition 5.16). As a consequence of these classification results, we obtain a new interpretation of the canonical metric of L^{an} as the metric associated to the concave function Ψ under this correspondence.

We can also classify approachable metrics in terms of concave functions on polytopes: there is a bijective correspondence between the space of continuous concave functions on Δ_Ψ and the space of approachable toric metrics on L^{an} (Theorem 5.73(2)). This correspondence is induced by the previous one and the Legendre-Fenchel duality of concave functions. Namely, let $\|\cdot\|$ be an approachable toric metric on L^{an}, write $\overline{L} = (L, \|\cdot\|)$ and let ψ be the corresponding concave function. The associate *roof function* $\vartheta_{\overline{L},s}\colon \Delta_\Psi \to \mathbb{R}$ is the concave function defined as λ_K times the Legendre-Fenchel dual $\psi^\vee$. One of the main outcomes of this text is that the pair $(\Delta_\Psi, \vartheta_{\overline{L},s})$ plays, in the arithmetic geometry of toric varieties, a role analogous to that of the polytope in its algebraic geometry.

Our second addition to the dictionary is the following characterization of the measure associated to an approachable toric metric. Let X, $\overline{L}$ and ψ be as before, and write $\mu_\psi = c_1^{\wedge n+1}(\overline{L}) \wedge \delta_{X_\Sigma}$ for the induced measure on X^{an}. Then

$$(\mathrm{val}_K)_*(\mu_\psi|_{X_0^{\mathrm{an}}}) = n!\,\mathcal{M}_M(\psi),$$

where $\mathcal{M}_M(\psi)$ is the (real) Monge-Ampère measure of ψ with respect to the lattice M (Definition 3.92). The measure μ_ψ is determined by this formula, and the conditions of being invariant under the action of $\mathbb{S}^{\mathrm{an}}$ and that the set $X^{\mathrm{an}} \setminus X_0^{\mathrm{an}}$ has measure zero. This gives a direct and fairly explicit expression for the measure associated to an approachable toric metric.

The fact that each toric line bundle has a canonical metric allows us to introduce a notion of local toric height that is independent of a choice of sections. Let X be an n-dimensional projective toric variety and $\overline{L}$ an approachable toric line bundle as before, and let $\overline{L}^{\mathrm{can}}$ be the same toric line bundle L equipped with the canonical metric. The *toric local height* of X with respect to $\overline{L}$ is defined as

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X) = \mathrm{h}_{\overline{L}}(X; s_0, \dots, s_n) - \mathrm{h}_{\overline{L}^{\mathrm{can}}}(X; s_0, \dots, s_n),$$

for any family of sections s_i, $i = 0, \dots, d$, that meet properly on X (Definition 6.1). Our third addition to the toric dictionary is the following formula for this toric local height in terms of the roof function introduced above (Theorem 6.6):

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X) = (n+1)! \int_{\Delta_\Psi} \vartheta_{\overline{L},s} \,\mathrm{d}\,\mathrm{vol}_M\,.$$

More generally, the toric local height can be defined for a family of $n+1$ integrable toric line bundles on X. The formula above can be extended by multilinearity to compute this local toric height in terms of the mixed integral of the associated roof functions (Remark 6.23).

For the global case, let Σ and Ψ be as before, and consider the associated toric variety X over $\mathbb{Q}$ equipped with a toric line bundle L and toric section s. Given a family of concave functions $\{\psi_v\}_{v\in\mathfrak{M}_\mathbb{Q}}$ such that $|\psi_v - \Psi|$ is bounded for all v and such that $\psi_v = \Psi$ for all but a finite number of v, the metrized toric line bundle $\overline{L} = (L, \{\|\cdot\|_{\psi_v}\}_v)$ is quasi-algebraic. Moreover, every approachable quasi-algebraic toric metric on L arises in this way (Theorem 5.85). Write $\overline{L}^v = (L, \|\cdot\|_{\psi_v})$ for the metrized toric line bundle corresponding to a place v. The associated roof functions $\vartheta_{\overline{L}^v,s}\colon \Delta_\Psi \to \mathbb{R}$ are identically zero except for a finite number of places. Then, the global height of X with respect to $\overline{L}$ can be computed as (Theorem 6.37)

$$\mathrm{h}_{\overline{L}}(X) = \sum_{v\in\mathfrak{M}_\mathbb{Q}} \mathrm{h}^{\mathrm{tor}}_{\overline{L}^v}(X_v) = (n+1)! \sum_{v\in\mathfrak{M}_\mathbb{Q}} \int_{\Delta_\Psi} \vartheta_{\overline{L}^v,s} \,\mathrm{d}\,\mathrm{vol}_M,$$

which precises Theorem 1.2 at the beginning of this introduction.

A remarkable feature of these results is that they read exactly the same in the Archimedean and in the non-Archimedean cases. For general metrized line bundles, these two cases are analogous but not identical. By contrast, the classification of toric metrics and the formulae for the associated measures and for the local heights are the same in both cases. We also point out that these results holds in greater generality than explained in this introduction: in particular, they hold for proper toric varieties which are not necessarily projective and, in the global case, for general adelic fields (Definition 2.48). By contrast, we content ourselves with the case when the torus is split. For the computation of heights, one can always reduce to the split case by considering a suitable field extension. Still, it would be interesting to extend our results to the non-split case by considering the corresponding Galois actions as, for instance, in [ELST10].

The toric dictionary in arithmetic geometry is very concrete and well-suited for computations. For instance, let K be a local field, X a toric variety and $\varphi\colon X \to$

$\mathbb{P}^r$ an equivariant map. Let $\overline{L}$ be the toric approachable metrized line bundle on X induced by the canonical metric on the universal line bundle of $\mathbb{P}^r$, and s a toric section of L. The concave function $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ corresponding to this metric is piecewise affine (Example 5.26). Hence, it defines a polyhedral complex in $N_{\mathbb{R}}$, and it turns out that $(\mathrm{val}_K)_*(\mu_\psi|_{X_0^{\mathrm{an}}})$, the direct image under val_K of the measure induced by $\overline{L}$, is a discrete measure on $N_{\mathbb{R}}$ supported on the vertices of this polyhedral complex (Proposition 3.95). The roof function $\vartheta_{\overline{L},s}$ is the function parameterizing the upper envelope of a polytope in $M_{\mathbb{R}} \times \mathbb{R}$ associated to φ and the section s (Example 6.31). The toric local height of X with respect to $\overline{L}$ can be computed as the integral of this piecewise affine concave function.

Another nice example is given by toric bundles on a projective space. For a finite sequence of integers $a_r \geq \dots \geq a_0 \geq 1$, we consider the vector bundle on $\mathbb{P}^n_{\mathbb{Q}}$

$$E := \mathcal{O}(a_0) \oplus \dots \oplus \mathcal{O}(a_r).$$

The toric bundle $\mathbb{P}(E) \to \mathbb{P}^n_{\mathbb{Q}}$ is defined as the bundle of hyperplanes of the total space of E. This is an $(n+r)$-dimensional toric variety over $\mathbb{Q}$ which can be equipped with an ample universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, see §8.2 for details.

We equip $\mathcal{O}_{\mathbb{P}(E)}(1)$ with an approachable adelic toric metric as follows: the Fubini-Study metrics on each line bundle $\mathcal{O}(a_j)$ induces a semipositive smooth toric metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ for the Archimedean place of $\mathbb{Q}$, whereas for the finite places we consider the corresponding canonical metric. We show that both the corresponding concave functions ψ_v and roof functions ϑ_v can be described in explicit terms (Lemma 8.17 and Proposition 8.20). We can then compute the height of $\mathbb{P}(E)$ with respect to this metrized line bundle as (Proposition 8.26)

$$\mathrm{h}_{\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}}(\mathbb{P}(E)) = \mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n) \sum_{\substack{\boldsymbol{i}\in\mathbb{N}^{r+1}\\ |\boldsymbol{i}|=n+1}} \boldsymbol{a}^{\boldsymbol{i}} + \sum_{\substack{\boldsymbol{i}\in\mathbb{N}^{r+1}\\ |\boldsymbol{i}|=n}} A_{n,r}(\boldsymbol{i})\, \boldsymbol{a}^{\boldsymbol{i}},$$

where for $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$, we set $|\boldsymbol{i}| = i_0 + \dots + i_r$, $\boldsymbol{a}^{\boldsymbol{i}} = a_0^{i_0} \dots a_r^{i_r}$ and $A_{n,r}(\boldsymbol{i}) = \sum_{m=0}^{r}(i_m+1)\sum_{j=i_m+2}^{n+r+1}\frac{1}{2j}$, while $\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n_{\mathbb{Q}}) = \sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{2j}$ denotes the height of the projective space with respect to the Fubini-Study metric. In particular, the height of $\mathbb{P}(E)$ is a positive rational number.

The Fubini-Study height of the projective space was computed by Bost, Gillet and Soulé [BGS94, Lemma 3.3.1]. Other early computations for the Fubini-Study height of some toric hypersurfaces where obtained in [CM00, Dan97]. Mourougane has determined the height of Hirzebruch surfaces, as a consequence of his computations of Bott-Chern secondary classes [Mou06]. A Hirzebruch surface is a toric bundle over $\mathbb{P}^1_{\mathbb{Q}}$, and the result of Mourougane is a particular case of our computations for the height of toric bundles, see Remark 8.27.

The fact that the canonical height of a toric variety is zero is well-known. It results from its original construction by a limit process on the direct images of the variety under the so-called "powers maps". Maillot has studied the Arakelov geometry of toric varieties and line bundles with respect to the canonical metric, including the associated Chern currents and their product [Mai00].

In [PS08a], Philippon and Sombra gave a formula for the canonical height of a "translated" toric projective variety, a projective variety which is the closure of a translate of a subtorus, defined over a number field. In [PS08b], they also obtain a similar formula for the function field case. Both results are particular cases of our general formula, see Remark 6.40. Indeed, part of our motivation for the present text was to understand and generalize this formula in the framework of Arakelov geometry.

For the Archimedean smooth case, our constructions are related to the Guillemin-Abreu classification of Kähler structures on symplectic toric varieties [Abr03]. The roof function corresponding to a smooth metrized line bundle on a smooth toric variety coincides, up to a sign, with the so-called "symplectic potential" of a Kähler toric variety, see Remark 5.74. In the Archimedean continuous case, Boucksom and Chen have recently considered a similar construction in their study of arithmetic Okounkov bodies [BC09]. It would be interesting to further explore the connection with these results.

We now discuss the contents of each section, including some other results of interest.

Section 2 is devoted to the first half of the dictionary. Namely, we review integrable metrized line bundles both in the Archimedean and in the non-Archimedean cases. For the latter case, we recall the basic properties of Berkovich spaces of schemes. We then explain the associated measures and heights following [Zha95b, Cha06, Gub03]. For simplicity, the theory presented is not as general as the one in [Gub03]: in the non-Archimedean case we restrict ourselves to discrete valuation rings and in the global case to adelic fields, while in *loc. cit.* the theory is developed for arbitrary valuations and for M-fields, respectively.

Section 3 deals with the second half of the dictionary, that is, convex analysis with emphasis on polyhedral sets. Most of the material in this section is classical. We have gathered all the required results, adapting them to our needs and adding some new ones. We work with concave functions, which are the functions which naturally arise in the theory of toric varieties. For latter reference, we have translated many of the notions and results of convex analysis, usually stated for convex functions, in terms of concave functions.

We first recall the basic definitions about convex sets and convex decompositions, and then we study concave functions and the Legendre-Fenchel duality. We introduce a notion of Legendre-Fenchel correspondence for general closed concave functions, as a duality between convex decompositions (Definition 3.31 and Theorem 3.33). This is the right generalization of both the classical Legendre transform of strictly concave differentiable functions, and the duality between polyhedral complexes induced by a piecewise affine concave function. We also consider the interplay between Legendre-Fenchel duality and operations on concave functions like, for instance, the direct and inverse images by affine maps. This latter study will be important when considering the functoriality with respect to equivariant morphisms between toric varieties. We next particularize to two extreme cases: differentiable concave functions whose stability set is a polytope that will be related to semipositive smooth toric metrics in the Archimedean case, and to piecewise affine concave functions that will correspond to semipositive algebraic toric metrics in the non-Archimedean case. Next, we treat differences of concave functions, that will be related to integrable metrics. We end this section by studying the Monge-Ampère measure associated to a concave function. There is an interesting interplay between Monge-Ampère measures and Legendre-Fenchel duality. In this direction, we prove a combinatorial analogue of the arithmetic Bézout's theorem (Theorem 3.97), which is a key ingredient in the proof of our formulae for the height of a toric variety.

In §4 we study the algebraic geometry of toric varieties over a field and of toric schemes over a discrete valuation ring (DVR). We start by recalling the basic constructions and results on toric varieties, including Cartier and Weil divisors, toric line bundles and sections, orbits and equivariant morphisms, and positivity properties. Toric schemes over a DVR where first considered by Mumford in [KKMS73], who studied and classified them in terms of fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. In the proper case, these schemes can be alternatively classified in terms of complete polyhedral complexes in $N_{\mathbb{R}}$ [BS10]. Given a complete fan Σ in $N_{\mathbb{R}}$, the models over a DVR of the proper toric variety X_Σ are classified by complete polyhedral complexes on $N_{\mathbb{R}}$ whose recession fan (Definition 3.7) coincides with Σ (Theorem 4.60). Let Π be such a polyhedral complex, and denote by $\mathcal{X}_\Pi$ the corresponding model of X_Σ. Let (L,s) be a toric line bundle on X_Σ with a toric section defined by a virtual support function Ψ. We show that the models of (L,s) over $\mathcal{X}_\Pi$ are classified by functions that are rational piecewise affine on Π and whose recession function is Ψ (Theorem 4.81). We also prove a toric version of the Nakai-Moishezon criterion for toric schemes over a DVR, which implies that semipositive models translate into concave functions under the above correspondence (Theorem 4.95).

In §5 we study toric metrics and their associated measures. For the discussion, consider a local field K, a complete fan Σ on $N_{\mathbb{R}}$ and a virtual support function Ψ on Σ, and let (X,L) denote the corresponding proper toric variety over K and toric line bundle. We first introduce a variety with corners N_Σ which is a compactification of $N_{\mathbb{R}}$, together with a proper map $\operatorname{val}_K\colon X_\Sigma^{\mathrm{an}} \to N_\Sigma$ whose fibers are the orbits of the action of $\mathbb{S}^{\mathrm{an}}$ on X_Σ^{an}, and we prove the classification theorem for toric metrics on L^{an} (Proposition 5.16). We next treat smooth metrics in the Archimedean case. A toric smooth metric is semipositive if and only if the associated function ψ is concave (Proposition 5.29). We make explicit the associated measure in terms of the Hessian of this function, hence in terms of the Monge-Ampère measure of ψ (Theorem 5.33). We also observe that an arbitrary smooth metric on L can be turned into a toric smooth metric by averaging it by the action of $\mathbb{S}^{\mathrm{an}}$. If the given metric is semipositive, so is the obtained toric smooth metric.

Next, in the same section, we consider algebraic metrics in the non-Archimedean case. We first show how to describe the reduction map for toric schemes over a DVR in terms of the corresponding polyhedral complex and the map val_K (Lemma 5.39). We then study the triangle formed by toric metrics, rational piecewise affine functions and toric models (Proposition 5.41 and Theorem 5.49), the problem of obtaining a toric metric from a non-toric one (Proposition 5.51) and the effect of a field extension (Proposition 5.53). Next, we treat in detail the one-dimensional case, were one can write in explicit terms the metrics, associated functions and measures. Back to the general case, we use these results to complete the characterization of toric semipositive algebraic metrics in terms of piecewise affine concave functions (Proposition 5.67). We also describe the measure associated to a semipositive toric algebraic metric in terms of the Monge-Ampère measure of its associated concave function (Theorem 5.70).

Once we have studied smooth metrics in the Archimedean case and algebraic metrics in the non-Archimedean case, we can study approachable toric metrics. We show that the same classification theorem is valid in the Archimedean and non-Archimedean cases (Theorem 5.73). Moreover, the associated measure is described in exactly the same way in both cases (Theorem 5.81). We end this section by introducing and classifying adelic toric metrics (Definition 5.84 and Corollary 5.83).

In §6, we prove the formulae for the toric local height and for the global height of toric varieties (Theorem 6.6 and Theorem 6.37). By using the functorial properties of the height, we recover, from our general formula, the formulae for the canonical height of a translated toric projective variety in [PS08a, Théorème 0.3] for number fields and in [PS08b, Proposition 4.1] for function fields.

In §7, we consider the problem of integrating functions on polytopes. We first present a closed formula for the integral over a polytope of a function of one variable composed with a linear form, extending in this direction Brion's formula for the case of a simplex [Bri88] (Proposition 7.3 and Corollary 7.14). This allows us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes (Proposition 7.27). We can interpret some of these heights as the average entropy of a simple random process defined by the polytope (Proposition 7.34).

In §8 we study some further examples. We first consider translated toric curves in $\mathbb{P}^n_{\mathbb{Q}}$. For these curves, we consider the line bundle obtained from the restriction of $\mathcal{O}(1)$ to the curve, equipped with the metric induced by the Fubiny-Study metric at the place at infinity and by the canonical metric for the finite places. We compute the corresponding concave function ψ and toric local height in terms of the roots of a univariate polynomial (Theorem 8.7). We finally consider toric bundles as explained before, and compute the relevant concave functions, measure and height.

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2. Metrized line bundles and their associated heights

In this section we will recall the adelic theory of heights as introduced by Zhang [Zha95b] and developed by Gubler [Gub02, Gub03] and Chambert-Loir [Cha06]. These heights generalize the ones that can be obtained from the arithmetic intersection theory of Gillet and Soulé [GS90, BGS94].

To explain the difference between both points of view, consider a smooth variety X over $\mathbb{Q}$. In Gillet-Soulé's theory, we choose a regular proper model $\mathcal{X}$ over $\mathbb{Z}$ of X, and we also consider the real analytic space X^{an} given by the set of complex points $X(\mathbb{C})$ and the anti-linear involution induced by the complex conjugation. By contrast, in the adelic point of view we consider the whole family of analytic spaces X^{an}_v, $v \in \mathfrak{M}_{\mathbb{Q}}$. For the Archimedean place, X^{an}_v is the real analytic space considered before, while for the non-Archimedean places, this is the associated Berkovich space [Ber90]. Both points of view have advantages and disadvantages. In the former point of view, there exists a complete formalism of intersection theory and characteristic classes, with powerful theorems like the arithmetic Riemann-Roch theorem and the Lefschetz fixed point theorem, but one is restricted to smooth varieties and needs an explicit integral model of X. In the latter point of view, one can define heights, but does not dispose yet of a complete formalism of intersection theory. Its main advantages are that it can be easily extended to non-smooth varieties and that there is no need of an integral model of X. Moreover, all places, Archimedean and non-Archimedean, are set on a similar footing.

2.1. **Smooth metrics in the Archimedean case.**

Let X be an algebraic variety over $\mathbb{C}$ and X^{an} its associated complex analytic space. We recall the definition of differential forms on X^{an} introduced by Bloom and Herrera [BH69]. The space X^{an} can be covered by a family of open subsets $\{U_i\}_i$ such that each U_i can be identified with a closed analytic subset of an open ball in $\mathbb{C}^r$ for some r. On each U_i, the differential forms are defined as the restriction to this subset of smooth complex-valued differential forms defined on an open neighbourhood of U_i in $\mathbb{C}^r$. Two differential forms on U_i are identified if they coincide on the non-singular locus of U_i. We denote by $\mathscr{A}^*(U_i)$ the complex of differential forms of U_i, which is independent of the chosen embedding. In particular, if U_i is non-singular, we recover the usual complex of differential forms. These complexes glue together to define a sheaf $\mathscr{A}^*_{X^{\mathrm{an}}}$. This sheaf is equipped with differential operators d, $\mathrm{d^c}$, ∂, $\bar{\partial}$, an external product and inverse images with respect to analytic morphisms: these operations are defined locally on each $\mathscr{A}^*(U_i)$ by extending the differential forms to a neighbourhood of U_i in $\mathbb{C}^r$ as above and applying the corresponding operations for $\mathbb{C}^r$. We write $\mathcal{O}_{X^{\mathrm{an}}}$ and $C^\infty_{X^{\mathrm{an}}} = \mathscr{A}^0_{X^{\mathrm{an}}}$ for the sheaves of analytic functions and of smooth functions of X^{an}, respectively.

Let L be an algebraic line bundle on X and L^{an} its analytification.

Definition 2.1. A *metric* on L^{an} is an assignment that, to each local section s of L^{an} on an open subset $U \subset X^{\mathrm{an}}$, associates a continuous function

$$\|s(\cdot)\|\colon U \longrightarrow \mathbb{R}_{\geq 0}$$

such that, for all $p \in U$,

(1) $\|s(p)\| = 0$ if and only if $s(p) = 0$;
(2) for any $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(U)$, it holds $\|(\lambda s)(p)\| = |\lambda(p)|\,\|s(p)\|$.

The pair $\overline{L} := (L, \|\cdot\|)$ is called a *metrized line bundle*.The metric $\|\cdot\|$ is *smooth* if for every local section s of L^{an}, the function $\|s(\cdot)\|^2$ is smooth.

We remark that what we call “metric” in this text is called “continuous metric” in other contexts.

Let $\overline{L} = (L, \|\cdot\|)$ be a smooth metrized line bundle. Given a local section s of L^{an} on an open subset U, the *first Chern form* of $\overline{L}$ is the $(1,1)$-form defined on U as

$$\mathrm{c}_1(\overline{L}) = \partial\bar{\partial}\log\|s\|^2 \in \mathscr{A}^{1,1}(U).$$

It does not depend on the choice of local section and can be extended to a global closed $(1,1)$-form. Observe that we are using the algebro-geometric convention, and so $\mathrm{c}_1(\overline{L})$ determines a class in $H^2(X^{\mathrm{an}}, 2\pi i\,\mathbb{Z})$.

Example 2.2. Let $X = \mathbb{P}^n_{\mathbb{C}}$ and $L = \mathcal{O}(1)$, the universal line bundle of $\mathbb{P}^n_{\mathbb{C}}$. A rational section s of $\mathcal{O}(1)$ can be identified with a homogeneous rational function $\rho_s \in \mathbb{C}(x_0, \dots, x_n)$ of degree 1. The poles of this section coincide which those of ρ_s. For a point $p = (p_0 : \cdots : p_n) \in \mathbb{P}^n(\mathbb{C})$ outside this set of poles, the *Fubini-Study metric* of $\mathcal{O}(1)^{\mathrm{an}}$ is defined as

$$\|s(p)\|_{\mathrm{FS}} = \frac{|\rho_s(p_0, \dots, p_n)|}{(\sum_i |p_i|^2)^{1/2}}.$$

Clearly, this definition does not depend on the choice of a representative of p. The pair $(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$ is a metrized line bundle.

Many smooth metrics can be obtained as the inverse image of the Fubini-Study metric. Let X be a variety over $\mathbb{C}$ and L a line bundle on X, and assume that there is an integer $e \geq 1$ such that $L^{\otimes e}$ is generated by global sections. Choose a basis of the space of global sections $\Gamma(X, L^{\otimes e})$ and let $\varphi\colon X \to \mathbb{P}^M_{\mathbb{C}}$ be the induced morphism. Given a local section s of L, let s' be a local section of $\mathcal{O}(1)$ such that $s^{\otimes e} = \varphi^* s'$. Then, the smooth metric on L^{an} obtained from the Fubini-Study metric by inverse image is given by

$$\|s(p)\| = \|s'(\varphi(p))\|^{1/e}_{\mathrm{FS}}$$

for any $p \in X^{\mathrm{an}}$ which is not a pole of s.

Definition 2.3. Let $\overline{L}$ be a smooth metrized line bundle and $\mathbb{D} = \{z \in \mathbb{C} | \, |z| \leq 1\}$, the unit disk of $\mathbb{C}$. We say that $\overline{L}$ is *semipositive* if, for every holomorphic map $\varphi\colon \mathbb{D} \longrightarrow X^{\mathrm{an}}$,

$$\frac{1}{2\pi i}\int_{\mathbb{D}} \varphi^* \, \mathrm{c}_1(\overline{L}) \geq 0.$$

We say that $\overline{L}$ is *positive* if this integral is strictly positive for all non-constant holomorphic maps as before.

Example 2.4. The Fubini-Study metric (Example 2.2) is positive because its first Chern form defines a smooth metric on the holomorphic tangent bundle of $\mathbb{P}^n(\mathbb{C})$ [GH94, Chapter 0, §2]. All metrics obtained as inverse image of the Fubini-Study metric are semipositive.

A family of smooth metrized line bundles $\overline{L}_0, \dots, \overline{L}_{d-1}$ on X and a d-dimensional cycle Y of X define a signed measure on X^{an} as follows. First suppose that Y is a subvariety of X and let δ_Y denote the current of integration along the analytic subvariety Y^{an}, defined as $\delta_Y(\omega) = \frac{1}{(2\pi i)^d}\int_{Y^{\mathrm{an}}} \omega$ for $\omega \in \mathscr{A}^{2d}_{X^{\mathrm{an}}}$. Then the current

$$\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$$

is a signed measure on X^{an}. This notion extends by linearity to $Y \in Z_d(X)$. If $\overline{L}_i$, $i = 0, \dots, d-1$, are semipositive and Y is effective, this signed measure is a measure.

Remark 2.5. We can reduce the study of algebraic varieties and line bundles over the field of real numbers to the complex case by using the following standard technique. A variety X over $\mathbb{R}$ induces a variety $X_{\mathbb{C}}$ over $\mathbb{C}$ together with an anti-linear involution $\sigma\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ such that the diagram

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "\sigma"] \arrow[d] & X_{\mathbb{C}} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

commutes, where the arrow below denotes the map induced by complex conjugation. A line bundle L on X determines a line bundle $L_{\mathbb{C}}$ on $X_{\mathbb{C}}$ and an isomorphism $\alpha\colon \sigma^* L_{\mathbb{C}} \to L_{\mathbb{C}}$ such that a section s of $L_{\mathbb{C}}$ is real if and only if $\alpha(\sigma^* s) = s$. By a *metric* on L^{an} we will mean a metric $\|\cdot\|$ on $L^{\mathrm{an}}_{\mathbb{C}}$ such that the induced map $\sigma^*(L_{\mathbb{C}}, \|\cdot\|) \to (L_{\mathbb{C}}, \|\cdot\|)$ is an isometry.

In this way, the above definitions can be extended to metrized line bundles on varieties over $\mathbb{R}$. For instance, a real smooth metrized line bundle is semipositive if and only if its associated complex smooth metrized line bundle is semipositive. The corresponding signed measure is a measure over $X^{\mathrm{an}}_{\mathbb{C}}$ which is invariant under σ.

In the sequel, every time we have a real variety, we will work with the associated complex variety and quietly ignore the anti-linear involution σ, because it will play no role in our results.

2.2. **Berkovich spaces of schemes.** In this section we recall Berkovich's theory of analytic spaces. We will not present the most general theory developed in [Ber90] but we will content ourselves with the analytic spaces associated to algebraic varieties, that are simpler to define and enough for our purposes.

Let K be a field complete with respect to a nontrivial non-Archimedean absolute value $|\cdot|$. Such fields will be called *non-Archimedean fields*. Let $K^{\circ} = \{\alpha \in K \mid |\alpha| \leq 1\}$ be the valuation ring, $K^{\circ\circ} = \{\alpha \in K \mid |\alpha| < 1\}$ the maximal ideal and $k = K^{\circ}/K^{\circ\circ}$ the residue field.

Let X be a scheme of finite type over K. Following [Ber90, §1 and Remark 3.4.2], we can associate an *analytic space* X^{an} to the scheme X as follows. First assume that $X = \mathrm{Spec}(A)$, where A is a finitely generated K-algebra. Then, the points of X^{an} are the multiplicative seminorms of A that extend the absolute value of K, see [Ber90, §1.1]. Every element a of A defines a function $|a(\cdot)|\colon X^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$ given by evaluation of the seminorm. The topology of X^{an} is the coarsest topology that makes the functions $|a(\cdot)|$ continuous for all $a \in A$.

To each point $p \in X^{\mathrm{an}}$ we attach a prime ideal

$$\mathfrak{p}_p = \{a \in A \mid |a(p)| = 0\}.$$

This induces a map $\pi\colon X^{\mathrm{an}} \to X$ defined as $\pi(p) = \mathfrak{p}_p$. The point p is a multiplicative seminorm on A and so it induces a non-Archimedean absolute value on the field of fractions of $A/\mathfrak{p}_p$. We denote by $\mathscr{H}(p)$ the completion of this field with respect to that absolute value.

Let U be an open subset of X^{an}. An *analytic function* on U is a function

$$f\colon U \longrightarrow \coprod_{p\in U} \mathscr{H}(p)$$

such that, for each $p \in U$, $f(p) \in \mathscr{H}(p)$ and there is an open neigborhood $U' \subset U$ of p with the property that, for all $\varepsilon > 0$, there are elements $a, b \in A$ with $b \notin \mathfrak{p}_q$ and $|f(q) - a(q)/b(q)| < \varepsilon$ for all $q \in U'$. The analytic functions form a sheaf, denoted $\mathcal{O}_{X^{\mathrm{an}}}$, and $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is a locally ringed space [Ber90, §1.5 and Remark 3.4.2]. In particular, every element $a \in A$ determines an analytic function on X^{an}, also denoted a. The function $|a(\cdot)|$ can then be obtained by composing a with the absolute value map

$$|\cdot|\colon \coprod_{p\in X^{\mathrm{an}}} \mathscr{H}(p) \longrightarrow \mathbb{R}_{\geq 0},$$

which justifies its notation.

Now, if X is a scheme of finite type over K, the analytic space X^{an} is defined by gluing together the affine analytic spaces obtained from an affine open cover of X. If we want to stress the base field we will denote X^{an} by X^{an}_K.

Let K' be a complete extension of K and $X^{\mathrm{an}}_{K'}$ the analytic space associated to the scheme $X_{K'}$. There is a natural map $X^{\mathrm{an}}_{K'} \to X^{\mathrm{an}}_K$ defined locally by restricting seminorms.

Definition 2.6. A *rational point* of X^{an}_K is a point $p \in X^{\mathrm{an}}$ satisfying $\mathscr{H}(p) = K$. We denote by $X^{\mathrm{an}}(K)$ the set of rational points of X^{an}. More generally, for a complete extension K' of K, the set of *K'-rational points* of X^{an} is defined as $X^{\mathrm{an}}(K') = X^{\mathrm{an}}_{K'}(K')$. There is a map $X^{\mathrm{an}}(K') \to X^{\mathrm{an}}$, defined by the composing the inclusion $X^{\mathrm{an}}(K') \hookrightarrow X^{\mathrm{an}}_{K'}$ with the map $X^{\mathrm{an}}_{K'} \to X^{\mathrm{an}}_K$ as above. The set of *algebraic points* of X^{an} is the union of $X^{\mathrm{an}}(K')$ for all finite extensions K' of K. Its image in X^{an} is denoted $X^{\mathrm{an}}_{\mathrm{alg}}$. We have that $X^{\mathrm{an}}_{\mathrm{alg}} = \{p \in X \,|\, [\mathscr{H}(p) : K] < \infty\}$.

The basic properties of X^{an} are summarized in the following theorem.

Theorem 2.7. *Let X be a scheme of finite type over K and X^{an} the associated analytic space.*

(1) *X^{an} is a locally compact and locally arc-connected topological space.*
(2) *X^{an} is Hausdorff (respectively compact and Hausdorff, arc-connected) if and only if X is separated (respectively proper, connected).*
(3) *The map $\pi\colon X^{\mathrm{an}} \to X$ is continuous. A locally constructible subset $T \subset X$ is open (respectively closed, dense) if and only if $\pi^{-1}(T)$ is open (respectively closed, dense).*
(4) *Let $\psi\colon X \longrightarrow Y$ be a morphism of schemes of finite type over K and $\psi^{\mathrm{an}}\colon X^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$ its analytification. Then ψ is flat (respectively unramified, étale, smooth, separated, injective, surjective, open immersion, isomorphism) if and only if ψ^{an} has the same property.*

(5) *Let K' be a complete extension of K. Then the map $\pi_{K'}: X^{\mathrm{an}}_{K'} \to X_{K'}$ induces a bijection between $X^{\mathrm{an}}(K')$ and $X(K')$.*
(6) *Set $X_{\mathrm{alg}} = \{p \in X \,|\, [K(p) : K] < \infty\}$. Then π induces a bijection between $X^{\mathrm{an}}_{\mathrm{alg}}$ and X_{alg}. The subset $X^{\mathrm{an}}_{\mathrm{alg}} \subset X^{\mathrm{an}}$ is dense.*

Proof. The proofs can be found in [Ber90] and the next pointers are with respect to the numeration in this reference: (1) follows from Theorem 1.2.1, Corollary 2.2.8 and Theorem 3.2.1, (2) is Theorem 3.4.8, (3) is Corollary 3.4.5, (4) is Proposition 3.4.6, (5) is Theorem 3.4.1(i), while (6) follows from Theorem 3.4.1(i) and Proposition 2.1.15. □

Example 2.8. Let M be a finitely generated free $\mathbb{Z}$-module of rank n. Consider the associated group algebra $K[M]$ and the algebraic torus $\mathbb{T}_M = \mathrm{Spec}(K[M])$. The corresponding analytic space $\mathbb{T}_M^{\mathrm{an}}$ is the set of multiplicative seminorms of $K[M]$ that extend the absolute value of K. This is an analytic group. We warn the reader that the set of points of an analytic group is not an abstract group, hence some care has to be taken when speaking of actions and orbits. The precise definitions and basic properties can be found in [Ber90, §5.1].

Its analytification $\mathbb{T}_M^{\mathrm{an}}$ is an analytic torus as in [Ber90, §6.3]. The subset

$$\mathbb{S}^{\mathrm{an}} = \{p \in \mathbb{T}_M^{\mathrm{an}} \,|\, |\chi^m(p)| = 1 \text{ for all } m \in M\}.$$

is a compact subgroup, called the *compact torus* of $\mathbb{T}_M^{\mathrm{an}}$.

Remark 2.9. Not every analytic space in the sense of Berkovich can be obtained by the above procedure. The general theory is based on spectra of affinoid K-algebras, that provide compact analytic spaces that are the building blocks of the more general analytic spaces.

2.3. **Algebraic metrics in the non-Archimedean case.** Let K be a field complete with respect to a nontrivial non-Archimedean absolute value, as in the previous section. For simplicity, we will assume from now on that K° is a discrete valuation ring (DVR), and we will fix a generator ϖ of its maximal ideal $K^{\circ\circ}$. This is the only case we will need in the sequel and it allows us to use a more elementary definition of measures and local heights. Nevertheless, the reader can consult [Gub03, Gub07] for the general case.

Let X be an algebraic variety over K and L a line bundle on X. Let X^{an} and L^{an} be their respective analytifications.

Definition 2.10. A *metric* on L^{an} is an assignment that, to each local section s of L^{an} on an open subset $U \subset X^{\mathrm{an}}$, associates a continuous function

$$\|s(\cdot)\| \colon U \longrightarrow \mathbb{R}_{\geq 0},$$

such that, for all $p \in U$,

(1) $\|s(p)\| = 0$ if and only if $s(p) = 0$;
(2) for any $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(U)$, it holds $\|(\lambda s)(p)\| = |\lambda(p)|\,\|s(p)\|$.

The pair $\overline{L} := (L, \|\cdot\|)$ is called a *metrized line bundle*.

Models of varieties and line bundles give rise to an important class of metrics. To introduce and study these metrics, we first consider the notion of model of varieties. Write $S = \mathrm{Spec}(K^{\circ})$. The scheme S has two points: the special point o and the generic point η. Given a scheme $\mathcal{X}$ over S, we set $\mathcal{X}_o = \mathcal{X} \times \mathrm{Spec}(k)$ and $\mathcal{X}_\eta = \mathcal{X} \times \mathrm{Spec}(K)$ for its special fibre and its generic fibre, respectively.

Definition 2.11. A *model* over S of X is a flat scheme $\mathcal{X}$ of finite type over S together with a fixed isomorphism $X \simeq \mathcal{X}_\eta$. This isomorphism is part of the model, and so we can identify $\mathcal{X}_\eta$ with X. When X is proper, we say that the model is *proper* whenever the scheme $\mathcal{X}$ is proper over S.

Given a model $\mathcal{X}$ of X, there is a reduction map defined on a closed subset of X^{an} with values in $\mathcal{X}_o$ [Ber90, §2.4]. This map can be described as follows. Let $\{\mathcal{U}_i\}_{i\in I}$ be a finite open affine cover of $\mathcal{X}$ by schemes over S of finite type and, for each i, let $\mathcal{A}_i$ be a K°-algebra such that $\mathcal{U}_i = \operatorname{Spec}(\mathcal{A}_i)$. Set $U_i = \mathcal{U}_i \cap X$ and let C_i be the closed subset of U_i^{an} defined as

$$C_i = \{p \in U_i^{\mathrm{an}} \mid |a(p)| \leq 1, \forall a \in \mathcal{A}_i\} \tag{2.12}$$

For each $p \in C_i$, the prime ideal $\mathfrak{q}_p := \{a \in \mathcal{A}_i \mid |a(p)| < 1\} \subset \mathcal{A}_i$ contains $K^{\circ\circ}\mathcal{A}_i$ and so it determines a point $\operatorname{red}(p) := \mathfrak{q}_p/K^{\circ\circ}\mathcal{A}_i \in \mathcal{U}_{i,o} \subset \mathcal{X}_o$. Consider the closed subset $C = \bigcup_i C_i \subset X^{\mathrm{an}}$. The above maps glue together to define a map

$$\operatorname{red}\colon C \longrightarrow \mathcal{X}_o. \tag{2.13}$$

This map is surjective and anti-continuous, in the sense that the preimages of the open subsets are closed [Ber90, §2.4]. For each irreducible component V of $\mathcal{X}_o$, there is a unique point $\xi_V \in C$ such that

$$\operatorname{red}(\xi_V) = \eta_V, \tag{2.14}$$

where η_V denotes the generic point of V [Ber90, Proposition 2.4.4]. The finite subset $\{\xi_V\}_V \subset X^{\mathrm{an}}$ is called the *Shilov boundary* of X^{an}. Observe that it depends on the choice of $\mathcal{X}$.

If both X and $\mathcal{X}$ are proper, then $C = X^{\mathrm{an}}$ and the reduction map is defined on the whole of X^{an}. If both X and $\mathcal{X}$ are normal, we can compute the Shilov boundary. Let V be an irreducible component of $\mathcal{X}_o$ and choose a finite type affine open subset $\mathcal{U} = \operatorname{Spec}(\mathcal{A}) \subset \mathcal{X}$ containing η_V. Put $A = \mathcal{A} \otimes_{K^\circ} K$ and $U = \mathcal{U} \cap X$. Then the point $\xi_V \in U \subset X^{\mathrm{an}}$ is the multiplicative seminorm on A given by

$$|a(\xi_V)| = |\varpi|^{\operatorname{ord}_V(a)/\operatorname{ord}_V(\varpi)}, \tag{2.15}$$

for each $a \in A$, where $\operatorname{ord}_V(f)$ is the order of f at the generic point of V.

Next we recall the definition of models of line bundles. Let L be a line bundle on X.

Definition 2.16. A *model* over S of (X, L) is a triple $(\mathcal{X}, \mathcal{L}, e)$, where $\mathcal{X}$ is a model over S of X, $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $e \geq 1$ is an integer, together with a fixed isomorphism $\mathcal{L}|_X \simeq L^{\otimes e}$. When $e = 1$, the model $(\mathcal{X}, \mathcal{L}, 1)$ will be denoted $(\mathcal{X}, \mathcal{L})$ for short. A model of $(\mathcal{X}, \mathcal{L}, e)$ is called *proper* whenever $\mathcal{X}$ is proper.

We assume that the variety X is proper for the rest of this section. To a proper model of a line bundle we can associate a metric.

Definition 2.17. Let $(\mathcal{X}, \mathcal{L}, e)$ be a proper model of (X, L). Let s be a local section of L^{an} defined at a point $p \in X^{\mathrm{an}}$. Let $\mathcal{U} \subset \mathcal{X}$ be a trivializing open neighbourhood of $\operatorname{red}(p)$ and σ a generator of $\mathcal{L}|_{\mathcal{U}}$. Let $U = \mathcal{U} \cap X$ and $\lambda \in \mathcal{O}_{U^{\mathrm{an}}}$ such that $s^{\otimes e} = \lambda\sigma$ on U^{an}. Then, the *metric induced by the proper model* $(\mathcal{X}, \mathcal{L}, e)$ on L^{an},, denoted $\|\cdot\|_{\mathcal{X},\mathcal{L},e}$, is given by

$$\|s(p)\|_{\mathcal{X},\mathcal{L},e} = |\lambda(p)|^{1/e}.$$

This definition does neither depend on the choice of the open set $\mathcal{U}$ nor of the section σ, and it gives a metric on L^{an}. The metrics on L^{an} obtained in this way are called *algebraic*, and a pair $\overline{L} := (L, \|\cdot\|_{\mathcal{X},\mathcal{L},e})$ is called an *algebraic metrized line bundle*.

Different models may give rise to the same metric.

Proposition 2.18. *Let $(\mathcal{X},\mathcal{L},e)$ and $(\mathcal{X}',\mathcal{L}',e')$ be proper models of (X,L), and $f\colon \mathcal{X}'\to\mathcal{X}$ a morphism of models such that $(\mathcal{L}')^{\otimes e}\simeq f^*\mathcal{L}^{\otimes e'}$. Then the metrics on L^{an} induced by both models agree.*

Proof. Let s be a local section of L^{an} defined on a point $p\in X^{\mathrm{an}}$. Let $\mathcal{U}\subset\mathcal{X}$ be a trivializing open neighbourhood of $\mathrm{red}_{\mathcal{X}}(p)$, the reduction of p with respect to the model $\mathcal{X}$, and σ a generator of $\mathcal{L}|_{\mathcal{U}}$. Let λ be an analytic function on $(\mathcal{U}\cap X)^{\mathrm{an}}$ such that $s^{\otimes e}=\lambda\sigma$.

We have that $\mathrm{red}_{\mathcal{X}'}(p)=f^{-1}(\mathrm{red}(p))$ and $\mathcal{U}':=f^{-1}(\mathcal{U})$ is a trivializing open set of $\mathcal{L}'^{\otimes e}$ with generator $f^*\sigma^{\otimes e'}$. Then $s^{\otimes ee'}=\lambda^{e'}f^*\sigma^{\otimes e'}$ on $(\mathcal{U}'\cap X)^{\mathrm{an}}=(\mathcal{U}\cap X)^{\mathrm{an}}$. Now the proposition follows directly from Definition 2.17. □

The inverse image of an algebraic metric is algebraic.

Proposition 2.19. *Let $\varphi\colon X_1\to X_2$ be a morphism of proper algebraic varieties over K and $\overline{L}_2$ a line bundle on X_2 equipped with an algebraic metric. Assume that X_1 admits a proper model. Then $\varphi^*\overline{L}_2$, the inverse image under φ of $\overline{L}_2$, is a line bundle on X_1 equipped with an algebraic metric.*

Proof. Let $(\mathcal{X}_2,\mathcal{L}_2,e)$ be a proper model of (X_2,L_2) which induces the metric in $\overline{L}_2$, and $\mathcal{X}_1'$ be a proper model of X_1. Let $\mathcal{X}_1$ be the Zariski closure of the graph of φ in $\mathcal{X}_1'\times_S\mathcal{X}_2$. This is a proper model of X_1 equipped with a morphism $\varphi_S\colon\mathcal{X}_1\to\mathcal{X}_2$. Then $(\mathcal{X}_1,\varphi_S^*\mathcal{L}_2,e)$ is a proper model of (X_1,φ^*L_2) which induces the metric of $\varphi^*\overline{L}_2$. □

Next we give a second description of an algebraic metric. As before, let X be a proper variety over K and L a line bundle on X, and $\|\cdot\|_{\mathcal{X},\mathcal{L},e}$ an algebraic metric on L^{an}. Let $p\in X^{\mathrm{an}}$ and put $H=\mathscr{H}(p)$, which is a complete extension of K. Let H° be its valuation ring, and o and η the special and the generic point of $\mathrm{Spec}(H^\circ)$, respectively. The point p induces a morphism of schemes $\mathrm{Spec}(H)\to X$. By the valuative criterion of properness, there is a unique extension

$$\widetilde{p}\colon\ \mathrm{Spec}(H^\circ)\longrightarrow\mathcal{X}. \tag{2.20}$$

It satisfies $\widetilde{p}(\eta)=\pi(p)$, where $\pi\colon X^{\mathrm{an}}\to X$ is the natural map introduced at the beginning of §2.2, and $\widetilde{p}(o)=\mathrm{red}(p)$.

Proposition 2.21. *With notation as above, let s be a local section of L in a neighbourhood of $\pi(p)$. Then*

$$\|s(p)\|_{\mathcal{X},\mathcal{L},e}=\inf\left\{|a|^{1/e}\,\middle|\,a\in H^\times,\,a^{-1}\widetilde{p}^*s^{\otimes e}\in\widetilde{p}^*\mathcal{L}\right\}. \tag{2.22}$$

Proof. Write $\|\cdot\|=\|\cdot\|_{\mathcal{X},\mathcal{L},e}$ for short. Let $\mathcal{U}=\mathrm{Spec}(\mathcal{A})\ni\mathrm{red}(p)$ be an open affine trivializing set of $\mathcal{L}$ and σ be a generator of $\mathcal{L}|_{\mathcal{U}}$. Then $s^{\otimes e}=\lambda\sigma$ with λ in the fraction field of $\mathcal{A}$. We have that $\lambda(p)\in H$ and, by definition, $\|s(p)\|=|\lambda(p)|^{1/e}$. If $\lambda(p)=0$, the equation is clearly satisfied. Denote temporarily by C the right-hand side of (2.22). If $\lambda(p)\neq0$,

$$\lambda(p)^{-1}\widetilde{p}^*s^{\otimes e}=\widetilde{p}^*\sigma\in\widetilde{p}^*\mathcal{L}.$$

Hence $\|s(p)\|\geq C$. Moreover, if $a\in H^\times$ is such that $a^{-1}\widetilde{p}^*s^{\otimes e}\in\widetilde{p}^*\mathcal{L}$, then there is an element $\alpha\in H^\circ\setminus\{0\}$ with $a^{-1}\widetilde{p}^*s^{\otimes e}=\alpha\widetilde{p}^*\sigma$. Therefore, $a=\lambda(p)/\alpha$ and $|a|^{1/e}=|\lambda(p)|^{1/e}/|\alpha|^{1/e}\geq|\lambda(p)|^{1/e}$. Thus, $\|s(p)\|\leq C$. □

We give a third description of an algebraic metric in terms of intersection theory that makes evident the relationship with higher dimensional Arakelov theory. Let $(\mathcal{X},\mathcal{L},e)$ be a proper model of (X,L) and $\iota\colon\mathcal{Y}\to\mathcal{X}$ a closed algebraic curve. Let $\widetilde{\mathcal{Y}}$ be the normalization of $\mathcal{Y}$ and $\widetilde{\iota}\colon\widetilde{\mathcal{Y}}\to\mathcal{X}$ and $\rho\colon\widetilde{\mathcal{Y}}\to\mathrm{Spec}(K^\circ)$ the induced morphisms. Let s be a rational section of $\mathcal{L}$ such that $\operatorname{div}(s)$ intersects properly $\mathcal{Y}$. Then the intersection number $(\iota\cdot\operatorname{div}(s))$ is defined as

$$(\iota\cdot\operatorname{div}(s))=\deg(\rho_*(\operatorname{div}(\widetilde{\iota}^*s))).$$

Proposition 2.23. *With the above notation, let $p\in X^{\mathrm{an}}_{\mathrm{alg}}$. Let $\widetilde{p}$ as in* (2.20)*. This is a closed algebraic curve. Let s be a local section of L defined at p and such that $s(p)\neq 0$. Then*

$$\frac{\log\|s(p)\|}{\log|\varpi|}=\frac{(\widetilde{p}\cdot\operatorname{div}(s^{\otimes e}))}{e[\mathscr{H}(p):K]}.$$

Proof. We keep the notation in the proof of Proposition 2.21. In particular, $s^{\otimes e}=\lambda\sigma$ with λ in the fraction field of $\mathcal{A}$, and $\mathscr{H}(p)=H$. We verify that

$$\frac{\log\|s(p)\|}{\log|\varpi|}=\frac{\log|\lambda(p)|}{e\log|\varpi|}=\frac{\log|\operatorname{N}_{H/K}(\lambda(p))|}{e[H:K]\log|\varpi|}=\frac{\operatorname{ord}_{\varpi}(\operatorname{N}_{H/K}(\lambda(p)))}{e[H:K]}$$

and

$$\begin{aligned}(\widetilde{p}\cdot\operatorname{div}(s^{\otimes e}))=\deg(\rho_*(\operatorname{div}(\widetilde{p}^*s^{\otimes e})))&=\deg(\rho_*(\operatorname{div}(\lambda(p))))\\&=\deg(\operatorname{div}(\operatorname{N}_{H/K}(\lambda(p))))=\operatorname{ord}_{\varpi}(\operatorname{N}_{H/K}(\lambda(p))),\end{aligned}$$

which proves the statement. □

Example 2.24. Let $X=\mathbb{P}^0_K=\operatorname{Spec}(K)$. A line bundle L on X is necessarily trivial, that is, $L\simeq K$. Consider the model $(\mathcal{X},\mathcal{L},e)$ of (X,L) given by $\mathcal{X}=\operatorname{Spec}(K^\circ)$, $e\geq 1$, and $\mathcal{L}$ a free K°-submodule of $L^{\otimes e}$ of rank one. Let $v\in L^{\otimes e}$ be a basis of $\mathcal{L}$. For a section s of L we can write $s^{\otimes e}=\alpha v$ with $\alpha\in K$. Hence,

$$\|s\|=|\alpha|^{1/e}.$$

All algebraic metrics on L^{an} can be obtained in this way.

Example 2.25. Let $X=\mathbb{P}^n_K$ and $L=\mathcal{O}(1)$, the universal line bundle of $\mathbb{P}^n_K$. As a model for (X,L) we consider $\mathcal{X}=\mathbb{P}^n_{K^\circ}$, the projective space over $\operatorname{Spec}(K^\circ)$, $\mathcal{L}=\mathcal{O}_{\mathbb{P}^n_{K^\circ}}(1)$, and $e=1$. A rational section s of L can be identified with a homogeneous rational function $\rho_s\in K(x_0,\dots,x_n)$ of degree 1.

Let $p=(p_0:\dots:p_n)\in(\mathbb{P}^n_K)^{\mathrm{an}}\setminus\operatorname{div}(s)$ and set $H=\mathscr{H}(p)$. Let i_0 be such that $|p_{i_0}|=\max_i\{|p_i|\}$. Take $U\simeq\mathbb{A}^n_K$ (respectively $\mathcal{U}\simeq\mathbb{A}^n_{K^\circ}$) as the affine set $x_{i_0}\neq 0$ over H (respectively H°). The point p corresponds to the algebraic morphism

$$p^*\colon K[X_0,\dots,X_{i_0-1},X_{i_0+1},\dots,X_n]\longrightarrow H$$

that sends X_i to p_i/p_{i_0}. The extension $\widetilde{p}$ factors through the algebraic morphism

$$\widetilde{p}^*\colon K^\circ[X_1,\dots,X_{i_0-1},X_{i_0+1},\dots,X_n]\longrightarrow H^\circ,$$

with the same definition. Then

$$\begin{aligned}||s(p)||&=\inf\left\{|z|\ \middle|\ z\in H^\times, z^{-1}\widetilde{p}^*s\in\widetilde{p}^*\mathcal{L}\right\}\\&=\inf\left\{|z|\ \middle|\ z\in H^\times, z^{-1}\rho_s(p_0/p_{i_0},\dots,1,\dots,p_n/p_{i_0})\in H^\circ\right\}\\&=\left|\frac{\rho_r(p_0,\dots,p_n)}{p_{i_0}}\right|\\&=\frac{|\rho_r(p_0,\dots,p_n)|}{\max_i\{|p_i|\}}.\end{aligned}$$

We call this the *canonical metric* of $\mathcal{O}(1)^{\mathrm{an}}$ and we denote it by $\|\cdot\|_{\mathrm{can}}$.

Many other algebraic metrics can be obtained from Example 2.25, by considering maps of varieties to projective spaces. Let X be a proper variety over K equipped with a line bundle L such that $L^{\otimes e}$ is generated by global sections for an integer $e \geq 1$. A set of global sections in $\Gamma(X, L^{\otimes e})$ that generates $L^{\otimes e}$ induces a morphism $\varphi\colon X \to \mathbb{P}^n_K$ and, by inverse image, a metric $\varphi^* \|\cdot\|_{\mathrm{can}}$ on L. If X admits a a proper model, Proposition 2.19 shows that this metric is algebraic.

Now we recall the notion of semipositivity for algebraic metrics. A curve C in $\mathcal{X}$ is *vertical* if it is contained in $\mathcal{X}_o$.

Definition 2.26. Let $\|\cdot\|$ be an algebraic metric on L and set $\overline{L} = (L, \|\cdot\|)$. We say that $\overline{L}$ is *semipositive* if there is a model $(\mathcal{X}, \mathcal{L}, e)$ of (X, L) that induces the metric such that, for every vertical curve C in $\mathcal{X}$,

$$\deg_{\mathcal{L}}(C) \geq 0.$$

With the hypothesis in Proposition 2.19, the inverse image of a semipositive algebraic metric is also a semipositive algebraic metric.

Example 2.27. The canonical metric in Example 2.25 is semipositive: for a vertical curve C, its degree with respect to $\mathcal{O}_{\mathbb{P}^n_{K^\circ}}(1)$ equals its degree with respect to the restriction of this model to the special fibre. This restriction identifies with $\mathcal{O}_{\mathbb{P}^n_k}(1)$, the universal line bundle of $\mathbb{P}^n_k$, which is ample. Hence all the metrics obtained by inverse image of the canonical metric of $\mathcal{O}(1)^{\mathrm{an}}$ are also semipositive.

Finally, we recall the definition of the signed measures associated with algebraic metrics.

Definition 2.28. Let $\overline{L}_i$, $i = 0, \dots, d-1$, be line bundles on X equipped with algebraic metrics. For each i, choose a model $(\mathcal{X}_i, \mathcal{L}_i, e_i)$ that realizes the metric of $\overline{L}_i$. We can assume without loss of generality that the models $\mathcal{X}_i$ agree with a common model $\mathcal{X}$. Let Y be a d-dimensional subvariety of X and Y^{an} its analytification. Let $\mathcal{Y} \subset \mathcal{X}$ be the closure of Y, $\widetilde{\mathcal{Y}}$ be its normalization, $\widetilde{\mathcal{Y}}_o$ its special fibre, and $\widetilde{\mathcal{Y}}_o^{(0)}$ the set of irreducible components of this special fibre. For each $V \in \widetilde{\mathcal{Y}}_o^{(0)}$, consider the point $\xi_V \in Y^{\mathrm{an}}$ defined by (2.15). Let δ_{ξ_V} be the Dirac delta measure on X^{an} supported on ξ_V. We define a discrete signed measure on X^{an} by

$$\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y = \sum_{V \in \widetilde{\mathcal{Y}}_o^{(0)}} \mathrm{ord}_V(\varpi) \frac{\deg_{\mathcal{L}_0, \dots, \mathcal{L}_{d-1}}(V)}{e_0 \dots e_{d-1}} \delta_{\xi_V}. \tag{2.29}$$

This notion extends by linearity to the group of d-dimensional cycles of X.

This signed measure only depends on the metrics and not on the particular choice of models [Cha06, Proposition 2.7]. Observe that $\mathrm{ord}_V(\varpi)$ is the multiplicity of the component V in $\widetilde{\mathcal{Y}}_o$ and that the total mass of this measure equals $\deg_{L_0,\dots,L_{d-1}}(Y)$. If $\overline{L}_i$ is semipositive for all i and Y is effective, this signed measure is a measure.

Remark 2.30. The above measure was introduced by Chambert-Loir [Cha06]. For the subvarieties of a projective space equipped with the canonical metric, it is also possible to define similar measures through the theory of Chow forms, see [Phi94].

2.4. **Approachable and integrable metrics, measures and local heights.** Let K be either $\mathbb{R}$ or $\mathbb{C}$ (the *Archimedean case*) as in §2.1, or a complete field with respect to a nontrivial non-Archimedean absolute value (the *non-Archimedean case*) as in §2.3. Let X be a proper variety over K. Its analytification X^{an} will be a complex analytic space in the Archimedean case (equipped with an anti-linear involution when $K = \mathbb{R}$), or an analytic space in the sense of Berkovich, in the non-Archimedean case. A *metrized line bundle* on X is a pair $\overline{L} = (L, \|\cdot\|)$, where

L is a line bundle on X and $\|\cdot\|$ is a metric on L^{an}. Recall that the operations on line bundles of tensor product, dual and inverse image under a morphism extend to metrized line bundles.

Given two metrics $\|\cdot\|$ and $\|\cdot\|'$ on L^{an}, their quotient defines a continuous function $X^{\mathrm{an}} \to \mathbb{R}_{>0}$ given by $\|s(p)\|/\|s(p)\|'$ for any local section s of L not vanishing at p. The *distance* between $\|\cdot\|$ and $\|\cdot\|'$ is defined as the supremum of the absolute value of the logarithm of this function. In other words,

$$\mathrm{dist}(\|\cdot\|, \|\cdot\|') = \sup_{p \in X^{\mathrm{an}} \setminus \mathrm{div}(s)} |\log(\|s(p)\|/\|s(p)\|')|,$$

for any non-zero rational section s of L.

Definition 2.31. Let $\overline{L} = (L, \|\cdot\|)$ be a metrized line bundle on X. The metric $\|\cdot\|$ is *approachable* if there exists a sequence of semipositive smooth (in the Archimedean case) or semipositive algebraic (in the non-Archimedean case) metrics $(\|\cdot\|_l)_{l\geq 0}$ on L^{an} such that

$$\lim_{l\to\infty} \mathrm{dist}(\|\cdot\|, \|\cdot\|_l) = 0.$$

If this is the case, we say that $\overline{L}$ is *approachable*. This metrized line bundle is *integrable* if there are approachable line bundles $\overline{M}$, $\overline{N}$ such that $\overline{L} = \overline{M} \otimes \overline{N}^{-1}$.

The tensor product and the inverse image of approachable line bundles are also approachable. The tensor product, the dual and the inverse image of integrable line bundles are also integrable.

Example 2.32. Let $X = \mathbb{P}^n$ be the projective space over $\mathbb{C}$ and $L = \mathcal{O}(1)$. The *canonical metric* of $\mathcal{O}(1)^{\mathrm{an}}$ is the metric given, for $p = (p_0 : \dots : p_n) \in \mathbb{P}^n(\mathbb{C})$, by

$$\|s(p)\|_{\mathrm{can}} = \frac{|\rho_s(p_0, \dots, p_n)|}{\max_i\{|p_i|\}},$$

for any rational section s of L defined at p and the homogeneous rational function $\rho_s \in \mathbb{C}(x_0, \dots, x_n)$ associated to s.

This is an approachable metric. Indeed, consider the m-power map $[m] : \mathbb{P}^n \to \mathbb{P}^n$ defined as $[m](p_0 : \dots : p_n) = (p_0^m : \dots : p_n^m)$. The m-th root of the inverse image by $[m]$ of the Fubini-Study metric of $\mathcal{O}(1)^{\mathrm{an}}$ is the semipositive smooth metric on L^{an} given by

$$\|s(p)\|_m = \frac{|s(p_0, \dots, p_n)|}{(\sum_i |p_i|^{2m})^{1/2m}}.$$

The family of metrics obtained varying m converges uniformly to the canonical metric.

Proposition 2.33. *Let Y be a d-dimensional subvariety of X and $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, d-1$, a collection of approachable metrized line bundles on X. For each i, let $(\|\cdot\|_{i,l})_{l\geq 0}$ be a sequence of semipositive smooth (in the Archimedean case) or algebraic (in the non-Archimedean case) metrics on L_i^{an} that converge to $\|\cdot\|_i$. Then the measures $\mathrm{c}_1(L_0, \|\cdot\|_{0,l}) \wedge \dots \wedge \mathrm{c}_1(L_{d-1}, \|\cdot\|_{d-1,l}) \wedge \delta_Y$ converge weakly to a measure on X^{an}.*

Proof. The non-Archimedean case is proven in [Cha06, Proposition 2.7(b)] and in [Gub07, Proposition 3.12]. The Archimedean case can be proved similarly. □

Definition 2.34. Let $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, d-1$, be a collection of approachable metrized line bundles on X. For a d-dimensional subvariety $Y \subset X$, we denote by $\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$ the limit measure in Proposition 2.33. For integrable bundles $\overline{L}_i$ and a d-dimensional cycle Y of X, we can associate a signed measure $\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$ on X^{an} by multilinearity.

This signed measure behaves well under field extensions.

Proposition 2.35. *With the previous notation, let K' be a finite extension of K. Set $(X',Y') = (X,Y) \times \operatorname{Spec}(K')$ and let $\varphi\colon X'^{\mathrm{an}} \to X^{\mathrm{an}}$ be the induced map. Let $\varphi^*\overline{L}_i$, $i = 0,\dots,d-1$, be the line bundles with algebraic metrics on X' obtained by base change. Then*

$$\varphi_*\left(\mathrm{c}_1(\varphi^*\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\varphi^*\overline{L}_{d-1}) \wedge \delta_{Y'}\right) = \mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y.$$

Proof. This follows from [Gub07, Remark 3.10]. □

We also have the following functorial property.

Proposition 2.36. *Let $\varphi\colon X' \to X$ be a morphism of proper varieties over K, Y' a d-dimensional cycle of X', and $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0,\dots,d-1$, a collection of integrable metrized line bundles on X. Then*

$$\varphi_*\left(\mathrm{c}_1(\varphi^*\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\varphi^*\overline{L}_{d-1}) \wedge \delta_{Y'}\right) = \mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_{\varphi_*Y}.$$

Proof. In the non-Archimedean, this follows from [Gub07, Corollary 3.9(2)]. In the Archimedean case, this follows from the functoriality of Chern classes, the projection formula, and the continuity of direct image of measures. □

Definition 2.37. Let Y be a d-dimensional cycle of X and, for $i = 0,\dots,d$, L_i a line bundle on X and s_i a rational section of L_i. We say that $s_0,\dots,s_d$ *meet properly* Y if, for all $I \subset \{0,\dots,d\}$,

$$\dim\left(Y \cap \bigcap_{i\in I} |\operatorname{div} s_i|\right) = d - \#I.$$

The above signed measures allow to integrate continuous functions on X^{an}. Indeed, it is also possible to integrate certain functions with logarithmic singularities that play an important role in the definition of local heights. Moreover, this integration is continuous with respect to uniform convergence of metrics.

Theorem 2.38. *Let Y be a d-dimensional cycle of X, $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0,\dots,d$, a collection of approachable metrized line bundles, and s_i, $i = 0,\dots,d$, a collection of rational sections meeting Y properly.*

1. *The function $\log\|s_d\|_d$ is integrable with respect to the measure $\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$.*
2. *Let $(\|\cdot\|_{i,n})_{n\geq 1}$ be a sequence of approachable metrics that converge to $\|\cdot\|_i$ for each i. Then*

$$\int_{X^{\mathrm{an}}} \log\|s_d\|_d\, \mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$$
$$= \lim_{n_0,\dots,n_d\to\infty} \int_{X^{\mathrm{an}}} \log\|s_d\|_{d,n_d}\, \mathrm{c}_1(\overline{L}_{0,n_0}) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1,n_{d-1}}) \wedge \delta_Y.$$

Proof. In the Archimedean case, when X is smooth, this is proved in [Mai00, Théorèmes 5.5.2(2) and 5.5.6(6)]. For completions of number fields this is proved in [CT09, Theorem 4.1], both in the Archimedean and non-Archimedean cases. Their argument can be easily extended to cover the general case. □

Definition 2.39. The *local height* on X is the function that, to each d-dimensional cycle Y and each family of integrable metrized line bundles with sections $(\overline{L}_i, s_i)$, $i = 0,\dots,d$, such that the sections meet Y properly, associates a real number $\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0,\dots,s_d)$ determined inductively by the properties:

(1) $\mathrm{h}(\emptyset) = 0$;

(2) if Y is a cycle of dimension $d \geq 0$, then

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0,\dots,s_d) = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_{d-1}}(Y \cdot \operatorname{div} s_d; s_0,\dots,s_{d-1}) - \int_{X^{\mathrm{an}}} \log \|s_d\|_d \,\mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y.$$

In particular, for $p \in X(K) \setminus |\operatorname{div}(s_0)|$,

$$\mathrm{h}_{\overline{L}_0}(p; s_0) = -\log \|s_0(p)\|_0. \tag{2.40}$$

Remark 2.41. Definition 2.39 makes sense thanks to Theorem 2.38. We have choosen to introduce first the measures and then heights for simplicity of the exposition. Nevertheless, the approach followed in the literature is the inverse, because the proof of Theorem 2.38 relies on the properties of local heights. The interested reader can consult also [Cha10] for more details.

Remark 2.42. Definition 2.39 works better when the variety X is projective. In this case, for every cycle Y there exist sections that meet Y properly, thanks to the moving lemma. This does not necessarily occur for arbitrary proper varieties. Nevertheless, we will be able to define the global height (Definition 2.57) of any cycle of a proper variety by using Chow's lemma. Similarly we will be able to define the toric local height (Definition 6.1) of any cycle of a proper toric variety.

Remark 2.43. When X is regular and the metrics are smooth (in the Archimedean case) or algebraic (in the non-Archimedean case), the local heights of Definition 2.39 agree with the local heights that can be derived using the Gillet-Soulé arithmetic intersection product. In particular, in the Archimedean case, this local height agrees with the Archimedean contribution of the Arakelov global height introduced by Bost, Gillet and Soulé in [BGS94]. In the non-Archimedean case, the local height can be interpreted in terms of an intersection product. Assume that Y is prime and choose models $(\mathcal{X}_i, \mathcal{L}_i, e_i)$ of (X, L_i) that realize the algebraic metrics of $\overline{L}_i$. Without loss of generality, we may assume that all the models $\mathcal{X}_i$ agree with a common model $\mathcal{X}$. The sections $s_i^{\otimes e_i}$ can be seen as rational sections of $\mathcal{L}_i$ over $\mathcal{X}$. With the notations in Definition 2.28, the equation (2.15) implies that

$$\log \|s_d(\xi_V)\| = \frac{\log|\varpi| \operatorname{ord}_V(s_d^{\otimes e_d})}{e_d \operatorname{ord}_v(\varpi)}.$$

Therefore, in this case the equation in Definition 2.39(2) can be written as

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0,\dots,s_d) = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_{d-1}}(Y \cdot \operatorname{div}(s_d); s_0,\dots,s_{d-1}) - \frac{\log|\varpi|}{e_0\dots e_d} \sum_{V \in \widetilde{\mathcal{Y}}_0^{(0)}} \operatorname{ord}_V(s_d^{\otimes e_d}) \deg_{\mathcal{L}_0,\dots,\mathcal{L}_{d-1}}(V). \tag{2.44}$$

Remark 2.45. It is a fundamental observation by Zhang [Zha95b] that the non-Archimedean contribution of the Arakelov global height of a variety can be expressed in terms of a family of metrics. In particular, this global height only depends on the metrics and not on a particular choice of models, exhibiting the analogy between the Archimedean and non-Archimedean settings. The local heights were extended by Gubler [Gub02, Gub03] to non-necessarily discrete valuations and he also weakened the hypothesis of proper intersection.

Remark 2.46. The local heights of Definition 2.39 agree with the local heights introduced by Gubler, see [Gub03, Proposition 3.5] for the Archimedean case and [Gub03, Remark 9.4] for the non-Archimedean case. In the Archimedean case, the local height in [Gub03] is defined in terms of a refined star product of Green currents based on [Bur94]. The hypothesis needed in Gubler's definition of local heights are weaker than the ones we use. We have chosen the current definition because it is more elementary and suffices for our purposes.

Theorem 2.47. *The local height function satisfies the following properties.*

(1) *It is symmetric and multilinear with respect to $\otimes$ in the pairs $(\overline{L}_i, s_i)$, $i = 0, \dots, d$, provided that all terms are defined.*
(2) *Let $\varphi\colon X' \to X$ be a morphism of proper varieties over K, Y a d-dimensional cycle of X', and $(\overline{L}_i, s_i)$ an integrable metrized line bundle on X and a section, $i = 0, \dots, d$. Then*
$$\mathrm{h}_{\varphi^*\overline{L}_0,\dots,\varphi^*\overline{L}_d}(Y; \varphi^* s_0, \dots, \varphi^* s_d) = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(\varphi_* Y; s_0, \dots, s_d),$$
provided that both terms are defined.
(3) *Let Z be the zero-cycle $Y \cdot \operatorname{div}(s_0) \cdots \operatorname{div}(s_{d-1})$ and f a rational function such that the section fs_d meets Z properly. Then*
$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d) - \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, fs_d) = \log|f(Z)|,$$
where, if $Z = \sum_l m_l p_l$, then $f(Z) = \prod_l f(p_l)^{m_l}$.
(4) *Let $\overline{L'}_d = (L_d, \|\cdot\|')$ be another choice of metric. Then*
$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d) - \mathrm{h}_{\overline{L}_0,\dots,\overline{L}'_d}(Y; s_0, \dots, s_d) = \\ -\int_{X^{\mathrm{an}}} \log(\|s_d(p)\|/\|s_d(p)\|')\, \mathrm{c}_1(\overline{L}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{L}_{d-1}) \wedge \delta_Y$$
is independent of the choice of sections.

Proof. In the Archimedean case, statement (1) is [Gub03, Proposition 3.4], statement (2) is [Gub03, Proposition 3.6]. In the non-Archimedean case, statement (1) and (2) are [Gub03, Remark 9.3]. The other two statements follow easily from the definition. □

2.5. **Adelic metrics and global heights.** ??? `modificar la definicion de cuerpo global como en el segundo papel y aadir la invariancia de la altura por cambio de cuerpo, ditto caso torico.`

To define global heights, we first introduce the notion of adelic field, which is a generalization of the notion of global field. In [Gub03] one can find a more general theory of global heights based on the concept of M-fields.

Definition 2.48. Let $\mathbb{K}$ be a field and $\mathfrak{M}_{\mathbb{K}}$ a family of absolute values on $\mathbb{K}$ with positive real weights. For each $v \in \mathfrak{M}_{\mathbb{K}}$ we denote by $|\cdot|_v$ the corresponding absolute value, by $n_v \in \mathbb{R}_{>0}$ the weight, and by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to $|\cdot|_v$. We say that $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}})$ is an *adelic field* if

(1) for each $v \in \mathfrak{M}_{\mathbb{K}}$, the absolute value $|\cdot|_v$ is Archimedean or associated to a nontrivial discrete valuation;
(2) for each $\alpha \in \mathbb{K}^\times$, $|\alpha|_v = 1$ except a for a finite number of v.

Observe that the complete fields $\mathbb{K}_v$ are either $\mathbb{R}$, $\mathbb{C}$ or of the kind of fields considered in §2.3.

Definition 2.49. Let $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}})$ be an adelic field. For $\alpha \in \mathbb{K}^\times$, the *defect* of α is
$$\operatorname{def}(\alpha) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \log|\alpha|_v.$$
Since $\operatorname{def}\colon \mathbb{K}^\times \to \mathbb{R}$ is a group homomorphism, we have that $\operatorname{def}(\mathbb{K}^\times)$ is a subgroup of $\mathbb{R}$. If $\operatorname{def}(\mathbb{K}^\times) = 0$, then $\mathbb{K}$ is said to satisfy the *product formula*. The *group of global heights* of $\mathbb{K}$ is $\mathbb{R}/\operatorname{def}(\mathbb{K}^\times)$.

Let $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ be an adelic field and $\mathbb{F}$ a finite extension of $\mathbb{K}$. For each $v\in\mathfrak{M}_{\mathbb{K}}$, put $\mathfrak{M}_v$ for the set of absolute values $|\cdot|_w$ of $\mathbb{F}$ that extend $|\cdot|_v$, with weight

$$n_w=\frac{[\mathbb{F}_w:\mathbb{K}_v]}{[\mathbb{F}:\mathbb{K}]}n_v.$$

Set $\mathfrak{M}_{\mathbb{F}}=\coprod_v\mathfrak{M}_v$. Then $(\mathbb{F},\mathfrak{M}_{\mathbb{F}})$ is an adelic field and $\operatorname{def}(\mathbb{F}^\times)\subset\frac{1}{[\mathbb{F}:\mathbb{K}]}\operatorname{def}(\mathbb{K}^\times)$. In particular, if $\mathbb{K}$ satisfies the product formula so does $\mathbb{F}$.

Example 2.50. Let $\mathfrak{M}_{\mathbb{Q}}$ be the set of places of $\mathbb{Q}$, where the corresponding absolute values are normalized in the standard way. Then $(\mathbb{Q},\mathfrak{M}_{\mathbb{Q}})$ is an adelic field that satisfies the product formula. If $\mathbb{K}$ is a number field, by the construction above, we obtain an adelic field $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ which satisfies the product formula too.

Example 2.51. Let B be an irreducible projective variety over a field k, which is regular in codimension 1, and L an ample line bundle on B. Set $\mathbb{K}=k(B)$. For a prime divisor v on B and $\alpha\in\mathbb{K}^\times$, we denote by $\operatorname{ord}_v(\alpha)$ the order of α at v. Fix a constant $c>1$ and denote by $\mathfrak{M}_{\mathbb{K}}$ the set of prime divisors on B. For each $v\in\mathfrak{M}_{\mathbb{K}}$, the corresponding absolute value and weight are defined as

$$|\alpha|_v=c^{-\operatorname{ord}_v(\alpha)},\quad n_v=\deg_L(v).$$

Then $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ is an adelic field. Moreover, $\mathbb{K}$ satisfies the product formula, since the degree of a principal divisor is zero.

Definition 2.52. The adelic fields in examples 2.50 and 2.51 will be called *global fields*. For a finite subset $S\subset\mathfrak{M}_{\mathbb{K}}$ containing the Archimedean places, we consider the Noetherian ring $\mathbb{K}_S^\circ=\{\alpha\in\mathbb{K}\,|\,|\alpha|_v\leq 1,\forall v\notin S\}$.

Definition 2.53. Let $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ be an adelic field. Let X be a proper variety over $\mathbb{K}$ and L a line bundle on X. For each $v\in\mathfrak{M}_{\mathbb{K}}$ set $X_v=X\times\operatorname{Spec}(\mathbb{K}_v)$ and $L_v=L\times\operatorname{Spec}(\mathbb{K}_v)$.

(1) A *metric* on L is a family of metrics $\|\cdot\|_v$, $v\in\mathfrak{M}_{\mathbb{K}}$, where $\|\cdot\|_v$ is a metric on L_v^{an}. We will denote by $\overline{L}=(L,(\|\cdot\|_v)_v)$ the corresponding metrized line bundle. The metric is said to be *approachable* (respectively *integrable*) if the metrics $\|\cdot\|_v$ are approachable (respectively integrable) for all $v\in\mathfrak{M}_{\mathbb{K}}$.
(2) Suppose that $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ is a global field. A metric on L is called *quasi-algebraic* if there exists a finite subset $S\subset\mathfrak{M}_{\mathbb{K}}$ containing the Archimedean places, an integer $e\geq 1$ and a proper model $(\mathcal{X},\mathcal{L},e)$ over $\mathbb{K}_S^\circ$ of (X,L) such that, for each $v\notin S$, the metric $\|\cdot\|_v$ is induced by the localization of this model at v.

Definition 2.54. Let $(\mathbb{K},\mathfrak{M}_{\mathbb{K}})$ be an adelic field, X a proper variety over $\mathbb{K}$ and $\overline{L}_i$, $i=0,\dots,d$, a family of integrable metrized line bundles on X. Let Y be a d-dimensional cycle of X. We say that Y is *integrable* with respect to $\overline{L}_0,\dots,\overline{L}_d$ if there is a proper map $\varphi\colon X'\to X$, a cycle Y' of X' such that $\varphi_*Y'=Y$, and rational sections s_i of φ^*L_i, $i=0,\dots,d$, that intersect Y' properly and such that for all but a finite number of $v\in\mathfrak{M}_{\mathbb{K}}$,

$$\mathrm{h}_{v,\overline{\varphi}^*L_0,\dots,\overline{\varphi}^*L_d}(Y';s_0,\dots,s_d)=0,\tag{2.55}$$

where h_v denotes the local height function on X_v.

The notion of integrability of cycles is stable under tensor product and inverse image of integrable metrized line bundles, thanks to Theorem 2.47(1,2). For an integrable cycle Y, the condition (2.55) is satisfied for any choice of morphism φ, cycle Y' and sections that intersect Y' properly, thanks to the definition of adelic field and Theorem 2.47(3).

We are mainly interested in global fields and quasi-algebraic metrics. In this case, all cycles are integrable.

Proposition 2.56. *Let $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}})$ be a global field and X a proper variety over $\mathbb{K}$ of dimension n. Let $d \le n$ and let $\overline{L}_i$, $i = 0, \dots, d$, be a family of line bundles with quasi-algebraic integrable metrics. Then every d-dimensional cycle of X is integrable with respect to $\overline{L}_0, \dots, \overline{L}_d$.*

Proof. It is enough to prove that every prime cycle is integrable. Applying the Chow Lemma to the support of the cycle and using that the inverse image of a quasi-algebraic metric is quasi-algebraic, we are reduced to the case when X is projective.

We proceed by induction on d. For $d = -1$, the statement is clear, and so we consider the case when $d \ge 0$. Let Y be a d-dimensional cycle of X and s_i, $i = 0, \dots, d$, rational sections of L_i that intersect Y properly. Let $(\mathcal{X}, \mathcal{L}_d)$ be a proper model over $\mathbb{K}_S^\circ$ of $(X, L_d^{\otimes e_d})$. Then $s_d^{\otimes e_d}$ is a non-zero rational section of $\mathcal{L}_d$ and so it defines a finite number of vertical components. Hence, for all places $v \notin S$ which are not below any of these vertical components,

$$\mathrm{h}_{v,\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d) = \mathrm{h}_{v,\overline{L}_0,\dots,\overline{L}_{d-1}}(Y \cdot \mathrm{div}(s_d); s_0, \dots, s_{d-1}),$$

thanks to the equation (2.44). The statement follows then from the inductive hypothesis. □

Definition 2.57. Let X be a proper variety over $\mathbb{K}$, $\overline{L}_0, \dots, \overline{L}_d$ integrable metrized line bundles on X, and Y an integrable d-dimensional cycle of X. Let X', Y' and $s_0, \dots, s_d$ be as in Definition 2.54. The global height of Y with respect to $s_0, \dots, s_d$ is defined as

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \, \mathrm{h}_{v,\varphi^*\overline{L}_0,\dots,\varphi^*\overline{L}_d}(Y'; s_0, \dots, s_d) \in \mathbb{R}.$$

The *global height* of Y, denoted $\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y)$, is the class of $\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d)$ in the quotient group $\mathbb{R}/\mathrm{def}(\mathbb{K}^\times)$.

The global height is well-defined as an element of $\mathbb{R}/\mathrm{def}(\mathbb{K}^\times)$ because of Theorem 2.47(3). In particular, if $\mathbb{K}$ satisfies the product formula, the global height is a well-defined real number.

Theorem 2.58. *The global height of integrable cycles satisfies the following properties.*

(1) *It is symmetric and multilinear with respect to tensor products of integrable metrized line bundles.*
(2) *Let $\varphi\colon X' \to X$ be a morphism of proper varieties over K, $\overline{L}_i$, $i = 0, \dots, d$, integrable metrized line bundles on X, and Y an integrable d-dimensional cycle of X'. Then*

$$\mathrm{h}_{\varphi^*\overline{L}_0,\dots,\varphi^*\overline{L}_d}(Y) = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(\varphi_* Y).$$

Proof. This follows readily from Theorem 2.47(1,2). □

3. The Legendre-Fenchel duality

In this section we explain the notions of convex analysis that we will use in our study of the arithmetic of toric varieties. The central theme is the Legendre-Fenchel duality of concave functions. A basic reference in this subject is the classical book by Rockafellar [Roc70] and we will refer to it for many of the proofs.

Although the usual references in the literature deal with convex functions, we will work instead with *concave* functions. These are the functions which arise in the theory of toric varieties. In this respect, we remark that the functions which are called "convex" in the classical books on toric varieties [KKMS73, Ful93] are concave in the sense of convex analysis.

3.1. **Convex sets and convex decompositions.** Let $N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a real vector space of dimension n and $M_{\mathbb{R}} = N_{\mathbb{R}}^{\vee}$ its dual space. The pairing between $x \in M_{\mathbb{R}}$ and $u \in N_{\mathbb{R}}$ will be alternatively denoted by $\langle x, u\rangle$, $x(u)$ or $u(x)$.

A non-empty subset C of $N_{\mathbb{R}}$ is *convex* if, for each pair of points $u_1, u_2 \in C$, the line segment

$$\overline{u_1u_2} = \{tu_1 + (1-t)u_2 \mid 0 \leq t \leq 1\}$$

is contained in C. Throughout this text, convex sets are assumed to be non-empty. A non-empty subset $\sigma \subset N_{\mathbb{R}}$ is a *cone* if $\lambda\sigma = \sigma$ for all $\lambda \in \mathbb{R}_{>0}$.

The *affine hull* of a convex set C, denoted $\operatorname{aff}(C)$, is the minimal affine space which contains it. The *dimension* of C is defined as the dimension of its affine hull. The *relative interior* of C, denoted $\operatorname{ri}(C)$, is defined as the interior of C relative to its affine hull. The *recession cone* of C, denoted by $\operatorname{rec}(C)$, is the set

$$\operatorname{rec}(C) = \{u \in N_{\mathbb{R}} \mid C + u \subset C\}.$$

It is a cone of $N_{\mathbb{R}}$. The *cone* of C is defined as

$$\operatorname{c}(C) = \overline{\mathbb{R}_{>0}(C \times \{1\})} \subset N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}.$$

It is a closed cone. If C is closed, then $\operatorname{rec}(C) = \operatorname{c}(C) \cap (N_{\mathbb{R}} \times \{0\})$.

Definition 3.1. Let C be a convex set. A convex subset $F \subset C$ is called a *face* of C if, for every closed line segment $\overline{u_1u_2} \subset C$ such that $\operatorname{ri}(\overline{u_1u_2}) \cap F \neq \emptyset$, the inclusion $\overline{u_1u_2} \subset F$ holds. A face of C of codimension 1 is called a *facet*. A non-empty subset $F \subset C$ is called an *exposed face* of C if there exists $x \in M_{\mathbb{R}}$ such that

$$F = \{u \in C \mid \langle x, u\rangle \leq \langle x, v\rangle, \forall v \in C\}.$$

Any exposed face of a convex set is a face, and the facets of a convex set are always exposed. However, a convex set may have faces which are not exposed. For instance, think about the four points of junction of the straight lines and bends of the boundary of the inner area of a racing track in a stadium.

Definition 3.2. Let Π be a non-empty collection of convex subsets of $N_{\mathbb{R}}$. The collection Π is called a *convex subdivision* if it satisfies the conditions:

(1) every face of an element of Π is also in Π;
(2) every two elements of Π are either disjoint or they intersect in a common face.

If Π satisfies only (2), then it is called a *convex decomposition*. The *support* of Π is defined as the set $|\Pi| = \bigcup_{C\in\Pi} C$. We say that Π is *complete* if its support is the whole of $N_{\mathbb{R}}$. For a given set $E \subset N_{\mathbb{R}}$, we say that Π is a convex subdivision (or decomposition) *in* E whenever $|\Pi| \subset E$. A convex subdivision in E is called *complete* if $|\Pi| = E$.

For instance, the collection of all faces of a convex set defines a convex subdivision of this set. The collection of all exposed faces of a convex set is a convex decomposition, but it is not necessarily a convex subdivision.

In this text, we will be mainly concerned with the polyhedral case.

Definition 3.3. A *convex polyhedron* of $N_{\mathbb{R}}$ is a convex set defined as the intersection of a finite number of closed halfspaces. It is called *strongly convex* if it does not contain any line. A *convex polyhedral cone* is a convex polyhedron σ such that $\lambda\sigma = \sigma$ for all $\lambda > 0$. A *polytope* is a bounded convex polyhedron.

For a convex polyhedron, there is no difference between faces and exposed faces.

By the Minkowski-Weyl theorem, polyhedra can be explicitly described in two dual ways, either by the *H-representation*, as an intersection of half-spaces, or by the *V-representation*, as the Minkowski sum of a cone and a polytope [Roc70, Theorem 19.1]. An H-representation of a polyhedron Λ in $N_{\mathbb{R}}$ is a finite set of affine equations $\{(a_j, \alpha_j)\}_{1\leq j\leq k} \subset M_{\mathbb{R}} \times \mathbb{R}$ so that

$$\Lambda = \bigcap_{1\leq j\leq k} \{u \in N_{\mathbb{R}} \mid \langle a_j, u\rangle + \alpha_j \geq 0\}. \tag{3.4}$$

With this representation, the recession cone can be written as

$$\operatorname{rec}(\Lambda) = \bigcap_{1\leq j\leq k} \{u \in N_{\mathbb{R}} \mid \langle a_j, u\rangle \geq 0\}.$$

A V-representation of a polyhedron Λ' in $N_{\mathbb{R}}$ consists in a set of vectors $\{b_j\}_{1\leq j\leq k}$ in the tangent space $T_0N_{\mathbb{R}}(\simeq N_{\mathbb{R}})$ and a non-empty set of points $\{b_j\}_{k+1\leq j\leq l} \subset N_{\mathbb{R}}$ such that

$$\Lambda' = \operatorname{cone}(b_1, \dots, b_k) + \operatorname{conv}(b_{k+1}, \dots, b_l) \tag{3.5}$$

where

$$\operatorname{cone}(b_1, \dots, b_k) := \left\{ \sum_{j=1}^{k} \lambda_j b_j \,\middle|\, \lambda_j \geq 0 \right\}$$

is the cone generated by the given vectors (with the convention that $\operatorname{cone}(\emptyset) = \{0\}$) and

$$\operatorname{conv}(b_{k+1}, \dots, b_l) := \left\{ \sum_{j=k+1}^{l} \lambda_j b_j \,\middle|\, \lambda_j \geq 0, \ \sum_{j=k+1}^{l} \lambda_j = 1 \right\}$$

is the convex hull of the given set of points. With this second representation, the recession cone can be obtained as

$$\operatorname{rec}(\Lambda') = \operatorname{cone}(b_1, \dots, b_k).$$

Definition 3.6. A *polyhedral complex* in $N_{\mathbb{R}}$ is a finite convex subdivision whose elements are convex polyhedra. A polyhedral complex is called *strongly convex* if all of its polyhedra are strongly convex. It is called *conic* if all of its elements are cones. A strongly convex conic polyhedral complex is called a *fan*. If Π is a polyhedral complex, we will denote by Π^i the subset of i-dimensional polyhedra of Ψ. In particular, if Σ is a fan, Σ^i is its subset of i-dimensional cones.

There are two natural processes for linearizing a polyhedral complex.

Definition 3.7. The *recession* of Π is defined as the collection of polyhedral cones of $N_{\mathbb{R}}$ given by

$$\operatorname{rec}(\Pi) = \{\operatorname{rec}(\Lambda) \mid \Lambda \in \Pi\}.$$

The *cone* of Π is defined as the collection of cones in $N_{\mathbb{R}} \times \mathbb{R}$ given by

$$\mathrm{c}(\Pi) = \big\{ \mathrm{c}(\Lambda) \mid \Lambda \in \Pi \big\} \cup \big\{ \sigma \times \{0\} \mid \sigma \in \operatorname{rec}(\Pi) \big\}.$$

It is natural to ask whether the recession or the cone of a given polyhedral complex is a complex too. The following example shows that this is not always the case.

Example 3.8. Let Π be the polyhedral complex in $\mathbb{R}^3$ containing the faces of the polyhedra

$$\Lambda_1 = \{(x_1, x_2, 0) |\, x_1, x_2 \geq 0\}, \quad \Lambda_2 = \{(x_1, x_2, 1) |\, x_1 + x_2, x_1 - x_2 \geq 0\}.$$

Then $\operatorname{rec}(\Lambda_1)$ and $\operatorname{rec}(\Lambda_2)$ are two cones in $\mathbb{R}^2 \times \{0\}$ whose intersection is the cone $\{(x_1, x_2, 0) | x_2, x_1 - x_2 \geq 0\}$. This cone is neither a face of $\operatorname{rec}(\Lambda_1)$ nor of $\operatorname{rec}(\Lambda_2)$. Hence $\operatorname{rec}(\Pi)$ is not a complex and, consequently, neither is $\mathrm{c}(\Pi)$. In Figure 1 we see the polyhedron Λ_1 in light grey, the polyhedron Λ_2 in darker grey and $\operatorname{rec}(\Lambda_2)$ as dashed lines.

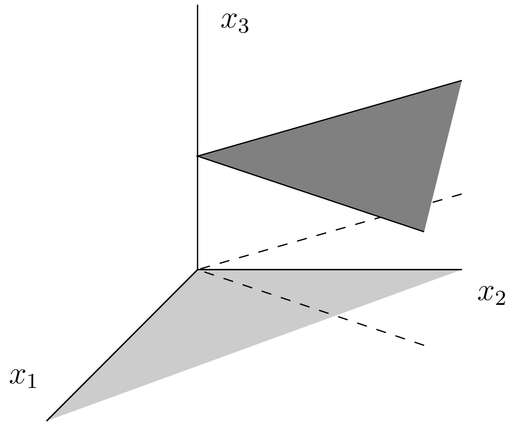

FIGURE 1.

Therefore, to assure that $\mathrm{rec}(\Pi)$ or $\mathrm{c}(\Pi)$ are complexes, we need to impose some condition on Π. This question has been addressed in [BS10]. Because our applications, we are mostly interested in the case when Π is complete. It turns out that this assumption is enough to avoid the problem raised in Example 3.8.

Proposition 3.9. *Let Π be a complete polyhedral complex in $N_{\mathbb{R}}$. Then $\mathrm{rec}(\Pi)$ and $\mathrm{c}(\Pi)$ are complete conic polyhedral complexes in $N_{\mathbb{R}}$ and $N_{\mathbb{R}}\times\mathbb{R}_{\geq 0}$, respectively. If, in addition, Π is strongly convex, then both $\mathrm{rec}(\Pi)$ and $\mathrm{c}(\Pi)$ are fans.*

Proof. This is a particular case of [BS10, Theorem 3.4]. □

Definition 3.10. Let Π_1 and Π_2 be two polyhedral complexes in $N_{\mathbb{R}}$. The *complex of intersections* of Π_1 and Π_2 is defined as the collection of polyhedra

$$\Pi_1 \cdot \Pi_2 = \{\Lambda_1 \cap \Lambda_2 | \Lambda_1 \in \Pi_1, \Lambda_2 \in \Pi_2\}.$$

Lemma 3.11. *The collection $\Pi_1 \cdot \Pi_2$ is a polyhedral complex. If Π_1 and Π_2 are complete, then*

$$\mathrm{rec}(\Pi_1 \cdot \Pi_2) = \mathrm{rec}(\Pi_1) \cdot \mathrm{rec}(\Pi_2).$$

Proof. Using the H-representation of polyhedra, one verifies that, if Λ_1 and Λ_2 are polyhedra with non-empty intersection, then any face of $\Lambda_1 \cap \Lambda_2$ is the intersection of a face of Λ_1 with a face of Λ_2. This implies that $\Pi_1 \cdot \Pi_2$ is a polyhedral complex.

Now suppose that Π_1 and Π_2 are complete. Let $\sigma \in \mathrm{rec}(\Pi_1 \cdot \Pi_2)$. This means that $\sigma = \mathrm{rec}(\Lambda)$ and $\Lambda = \Lambda_1 \cap \Lambda_2$ with $\Lambda_i \in \Pi_i$. It is easy to verify that $\Lambda \neq \emptyset$ implies $\mathrm{rec}(\Lambda) = \mathrm{rec}(\Lambda_1) \cap \mathrm{rec}(\Lambda_2)$. Therefore $\sigma \in \mathrm{rec}(\Pi_1) \cdot \mathrm{rec}(\Pi_2)$. This shows

$$\mathrm{rec}(\Pi_1 \cdot \Pi_2) \subset \mathrm{rec}(\Pi_1) \cdot \mathrm{rec}(\Pi_2).$$

Since both complexes are complete, they agree. □

We consider now an integral structure in $N_{\mathbb{R}}$. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n such that $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Set $M = N^{\vee} = \mathrm{Hom}(N, \mathbb{Z})$ for its dual lattice so $M_{\mathbb{R}} = M \otimes \mathbb{R}$. We also set $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$.

Definition 3.12. Let Λ be a polyhedron in $N_{\mathbb{R}}$. We say that Λ is a *lattice polyhedron* if it admits a V-representation with integral vectors and points. We say that it is *rational* if it admits a V-representation with rational coefficients.

Observe that any rational polyhedron admits an H-representation with integral coefficients.

Definition 3.13. Let Π be a strongly convex polyhedral complex in $N_{\mathbb{R}}$. We say that Π is *lattice* (respectively *rational*) if all of its elements are lattice (respectively rational) polyhedra. For short, a strongly convex rational polyhedral complex is called an *SCR polyhedral complex.* A conic SCR polyhedral complex is called a *rational fan.*

Remark 3.14. The statement of Proposition 3.9 is compatible with rational structures. Namely, if Π is rational, the same is true for $\mathrm{rec}(\Pi)$ and $\mathrm{c}(\Pi)$.

Corollary 3.15. *The correspondence $\Pi \mapsto \mathrm{c}(\Pi)$ is a bijection between the set of complete polyhedral complexes in $N_{\mathbb{R}}$ and the set of complete conical polyhedral complexes in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. Its inverse is the correspondence that, to each conic polyhedral complex Σ in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ corresponds the complex in $N_{\mathbb{R}}$ obtained by intersecting Σ with the hyperplane $N_{\mathbb{R}} \times \{1\}$. These bijections preserve rationality and strong convexity.*

Proof. This is [BS10, Corollary 3.12]. □

3.2. **The Legendre-Fenchel dual of a concave function.** Let $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ be as in the previous section.

Set $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ with the natural order and arithmetic operations. Unless otherwise stated, we will use the conventions $(-\infty) - (-\infty) = 0$ and $0 \cdot (-\infty) = 0$. A function $f\colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$ is *concave* if

$$f(tu_1 + (1-t)u_2) \geq tf(u_1) + (1-t)f(u_2)$$

for all $u_1, u_2 \in N_{\mathbb{R}}$, $0 < t < 1$ and f is not identically $-\infty$. Observe that a function f is concave in our sense if and only if $-f$ is a proper convex function in the sense of [Roc70]. The *effective domain* $\mathrm{dom}(f)$ of such a function is the subset of points of $N_{\mathbb{R}}$ where f takes finite values. It is a convex set. A concave function $f\colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$ defines a concave function with finite values $f\colon \mathrm{dom}(f) \to \mathbb{R}$. Conversely, if $f\colon C \to \mathbb{R}$ is a concave function defined on some convex set C, we can extend it to the whole of $N_{\mathbb{R}}$ by declaring that its value at any point of $N_{\mathbb{R}} \setminus C$ is $-\infty$. We will move freely from the point of view of concave functions on the whole of $N_{\mathbb{R}}$ with possibly infinite values to the point of view of real-valued concave functions on arbitrary convex sets.

A concave function is *closed* if it is upper semicontinuous. This includes the case of continuous concave functions defined on closed convex sets. Given an arbitrary concave function, there exists a unique minimal closed concave function above f. This function is called the *closure* of f and is denoted by $\mathrm{cl}(f)$.

Let f be a concave function on $N_{\mathbb{R}}$. The *Legendre-Fenchel dual* of f is the function

$$f^{\vee}\colon M_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}}, \quad x \longmapsto \inf_{u \in N_{\mathbb{R}}} (\langle x, u\rangle - f(u)).$$

It is a closed concave function. The Legendre-Fenchel duality is an involution between such functions: if f is closed, then $f^{\vee\vee} = f$ [Roc70, Cor. 12.2.1]. In fact, for any concave function f we have $f^{\vee\vee} = \mathrm{cl}(f)$.

The effective domain of $f^{\vee}$ is called the *stability set* of f. It can be described as

$$\mathrm{stab}(f) = \mathrm{dom}(f^{\vee}) = \{x \in M_{\mathbb{R}} \mid \langle x, u\rangle - f(u) \text{ is bounded below}\}.$$

Example 3.16. The *indicator function* of a convex set $C \subset N_{\mathbb{R}}$ is the concave function ι_C defined as $\iota_C(u) = 0$ for $u \in C$ and $\iota_C(u) = -\infty$ for $u \notin C$. Observe that ι_C is the logarithm of the characteristic function of C. This function is closed if and only if C is a closed set.

The *support function* of a convex set C is the function

$$\Psi_C\colon M_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad x \longmapsto \inf_{u\in C}\langle x,u\rangle.$$

It is a closed concave function. A function $f\colon M_{\mathbb{R}} \to \mathbb{R}$ is called *conical* if $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$. The support function Ψ_C is conical. The converse is also true: all conical closed concave functions are of the form Ψ_C for a closed convex set C.

We have $\iota_C^\vee = \Psi_C$ and $\Psi_C^\vee = \mathrm{cl}(\iota_C) = \iota_{\overline{C}}$. Thus, the Legendre-Fenchel duality defines a bijective correspondence between indicator functions of closed convex subsets of $N_{\mathbb{R}}$ and closed concave conical functions on $M_{\mathbb{R}}$.

Next result shows that the Legendre-Fenchel duality is monotonous.

Proposition 3.17. *Let f and g be concave functions such that $g(u) \leq f(u)$ for all $u \in N_{\mathbb{R}}$. Then $\mathrm{dom}(g) \subset \mathrm{dom}(f)$, $\mathrm{stab}(g) \supset \mathrm{stab}(f)$ and $g^\vee(x) \geq f^\vee(x)$ for all $x \in M_{\mathbb{R}}$.*

Proof. It follows directly from the definitions. □

The Legendre-Fenchel duality is continuous with respect to uniform convergence.

Proposition 3.18. *Let $(f_i)_{i\geq 1}$ be a sequence of concave functions which converges uniformly to a function f. Then f is a concave function and the sequence $(f_i^\vee)_{i\geq 1}$ converges uniformly to $f^\vee$. In particular, there is some $i_0 \geq 1$ such that $\mathrm{dom}(f_i) = \mathrm{dom}(f)$ and $\mathrm{stab}(f_i) = \mathrm{stab}(f)$ for all $i \geq i_0$.*

Proof. It is a direct consequence of Proposition 3.17. □

The classical Legendre duality of strictly concave differentiable functions can be described in terms of the gradient map ∇f, called in this setting the "Legendre transform". We will next show that the Legendre transform can be extended to the general concave case as a correspondence between convex decompositions.

Let f be a concave function on $N_{\mathbb{R}}$. The *sup-differential* of f at a point $u \in N_{\mathbb{R}}$ is defined as the set

$$\partial f(u) = \{x \in M_{\mathbb{R}} \mid \langle x, v-u\rangle \geq f(v) - f(u) \text{ for all } v \in N_{\mathbb{R}}\}.$$

For an arbitrary concave function, the sup-differential is a generalization of the gradient. In general, $\partial f(u)$ may contain more than one point, so the sup-differential has to be regarded as a multi-valued function.

We say that f is *sup-differentiable* at a point $u \in N_{\mathbb{R}}$ if $\partial f(u) \neq \emptyset$. The *effective domain* of ∂f, denoted $\mathrm{dom}(\partial f)$, is the set of points where f is sup-differentiable. For a subset $E \subset N_{\mathbb{R}}$ we define

$$\partial f(E) = \bigcup_{u\in E} \partial f(u).$$

In particular, the *image* of ∂f is defined as $\mathrm{im}(\partial f) = \partial f(N_{\mathbb{R}})$.

The sup-differential $\partial f(u)$ is a closed convex set for all $u \in \mathrm{dom}(\partial f)$. It is bounded if and only if $u \in \mathrm{ri}(\mathrm{dom}(f))$. Hence, in the particular case when $\mathrm{dom}(f) = N_{\mathbb{R}}$, we have that $\partial f(u)$ is a bounded closed convex subset of $M_{\mathbb{R}}$ for all $u \in N_{\mathbb{R}}$. The effective domain of the sup-differential is not necessarily convex but it differs very little from being convex, in the sense that it satisfies

$$\mathrm{ri}(\mathrm{dom}(f)) \subset \mathrm{dom}(\partial f) \subset \mathrm{dom}(f). \tag{3.19}$$

Let f be a closed concave function and consider the pairing

$$P_f\colon M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}}, \quad (u,x) \longmapsto f(u) + f^\vee(x) - \langle x,u\rangle. \tag{3.20}$$

This pairing satisfies $P_f(u,x) \leq 0$ for all u, x.

Proposition 3.21. *Let f be a closed concave function on $N_{\mathbb{R}}$. For $u \in N_{\mathbb{R}}$ and $x \in M_{\mathbb{R}}$, the following conditions are equivalent:*

1. $x \in \partial f(u)$;
2. $u \in \partial f^{\vee}(x)$;
3. $P_f(u, x) = 0$.

Proof. This is proved in [Roc70, Theorem 23.5]. □

If f is closed, then $\operatorname{im}(\partial f) = \operatorname{dom}(\partial f^{\vee})$ and so the image of the sup-differential is close to be a convex set, in the sense that

$$\operatorname{ri}(\operatorname{stab}(f)) \subset \operatorname{im}(\partial f) \subset \operatorname{stab}(f). \tag{3.22}$$

Definition 3.23. We denote by $\Pi(f)$ the collection of all sets of the form

$$C_x := \partial f^{\vee}(x)$$

for some $x \in \operatorname{stab}(f)$.

Lemma 3.24. *Let $x \in \operatorname{stab}(f)$. Then $C_x = \{u \in N_{\mathbb{R}} \mid P_f(u, x) = 0\}$. In other words, the set C_x is characterized by the condition*

$$f(u) = \langle x, u\rangle - f^{\vee}(x) \text{ for } u \in C_x \quad \text{and} \quad f(u) < \langle x, u\rangle - f^{\vee}(x) \text{ for } u \notin C_x. \tag{3.25}$$

Thus the restriction of f to C_x is an affine function with linear part given by x, and C_x is the maximal subset where this property holds.

Proof. The first statement follows from the equivalence of (2) and (3) in Proposition 3.21. The second statement follows from the definition of P_f and its non-positivity. □

The *hypograph* of a concave function f is defined as the set

$$\operatorname{hypo}(f) = \{(u, \lambda) \mid u \in N_{\mathbb{R}}, \lambda \le f(u)\} \subset N_{\mathbb{R}} \times \mathbb{R}.$$

A face of the hypograph is called *non-vertical* if it projects injectively in $N_{\mathbb{R}}$.

Proposition 3.26. *Let f be a closed concave function on $N_{\mathbb{R}}$. For a subset $C \subset N_{\mathbb{R}}$, the following conditions are equivalent:*

1. $C \in \Pi(f)$;
2. $C = \{u \in N_{\mathbb{R}} \mid x \in \partial f(u)\}$ *for a* $x \in M_{\mathbb{R}}$;
3. *there exist $x_C \in M_{\mathbb{R}}$ and $\lambda_C \in \mathbb{R}$ such that the set $\{(u, \langle x_C, u\rangle - \lambda_C) \mid u \in C\}$ is an exposed face of the hypograph of f.*

In particular, the correspondence

$$C_x \mapsto \{(u, \langle x, u\rangle - f^{\vee}(x)) \mid u \in C_x\}$$

is a bijection between $\Pi(f)$ and the set of non-vertical exposed faces of $\operatorname{hypo}(f)$.

Proof. The equivalence between the conditions (1) and (2) comes directly from Proposition 3.21. The equivalence with the condition (3) follows from (3.25). □

Proposition 3.27. *Let f be a closed concave function. Then $\Pi(f)$ is a convex decomposition of* $\operatorname{dom}(\partial f)$.

Proof. The collection of non-vertical exposed faces of $\operatorname{hypo}(f)$ forms a convex decomposition in $N_{\mathbb{R}} \times \mathbb{R}$. Using Proposition 3.26 we obtain that $\Pi(f)$ is a convex decomposition of $|\Pi(f)| = \operatorname{dom}(\partial f)$. □

We need the following result in order to properly define the Legendre-Fenchel correspondence for an arbitrary concave function as a bijective correspondence between convex decompositions.

Lemma 3.28. *Let f be a closed concave function and $C \in \Pi(f)$. Then for any $u_0 \in \mathrm{ri}(C)$,*

$$\bigcap_{u \in C} \partial f(u) = \partial f(u_0).$$

Proof. Fix $x_0 \in \mathrm{dom}(\partial f^\vee)$ such that $C = C_{x_0}$ and $u_0 \in \mathrm{ri}(C)$. Let $x \in \partial f(u_0)$. Then

$$\langle x, v - u_0 \rangle \geq f(v) - f(u_0) \quad \text{for all } v \in N_\mathbb{R}. \tag{3.29}$$

Let $u \in C$. By (3.25), we have $f(u) - f(u_0) = \langle x_0, u - u_0 \rangle$ and so the above inequality implies $\langle x, u - u_0 \rangle \geq \langle x_0, u - u_0 \rangle$. The fact $u_0 \in \mathrm{ri}(C)$ implies $u_0 + \lambda(u_0 - u) \in C$ for some small $\lambda > 0$. Applying the same argument to this element we obtain the reverse inequality $\langle x, u - u_0 \rangle \leq \langle x_0, u - u_0 \rangle$ and so

$$\langle x - x_0, u - u_0 \rangle = 0. \tag{3.30}$$

In particular, $f(u) - f(u_0) = \langle x_0, u - u_0 \rangle = \langle x, u - u_0 \rangle$ and from (3.29) we obtain

$$\langle x, v - u \rangle = \langle x, v - u_0 \rangle + f(u_0) - f(u) \geq f(v) - f(u) \quad \text{for all } v \in N_\mathbb{R}.$$

Hence $x \in \bigcap_{u \in C} \partial f(u)$ and so $\partial f(u_0) \subset \bigcap_{u \in C} \partial f(u)$, which implies the stated equality. □

Definition 3.31. Let f be a closed concave function. The *Legendre-Fenchel correspondence* of f is defined as

$$\mathcal{L}f \colon \Pi(f) \longrightarrow \Pi(f^\vee), \quad C \longmapsto \bigcap_{u \in C} \partial f(u).$$

By Lemma 3.28, $\mathcal{L}f(C) = \partial f(u_0)$ for any $u_0 \in \mathrm{ri}(C)$. Hence,

$$\mathcal{L}f(C) \in \Pi(f^\vee).$$

Definition 3.32. Let E, E' be subsets of $N_\mathbb{R}$ and $M_\mathbb{R}$ respectively, and Π, Π' convex decompositions of E and E', respectively. We say that Π and Π' are *dual convex decompositions* if there exists a bijective map $\Pi \to \Pi', C \mapsto C^*$ such that

(1) for all $C, D \in \Pi$ we have $C \subset D$ if and only if $C^* \supset D^*$;
(2) for all $C \in \Pi$ the sets C and C^* are contained in orthogonal affine spaces of $N_\mathbb{R}$ and $M_\mathbb{R}$, respectively.

Theorem 3.33. *Let f be a closed concave function, then $\mathcal{L}f$ is a duality between $\Pi(f)$ and $\Pi(f^\vee)$ with inverse $(\mathcal{L}f)^{-1} = \mathcal{L}f^\vee$.*

Proof. We will prove first that $\mathcal{L}f^\vee = (\mathcal{L}f)^{-1}$. Fix $C \in \Pi(f)$ and set $C' = \mathcal{L}f(C)$. Let $y_0 \in M_\mathbb{R}$ such that $C = C_{y_0}$ and let $u_0 \in \mathrm{ri}(C)$. Hence $u_0 \in C_{y_0} = \partial f^\vee(y_0)$ and so $y_0 \in \partial f(u_0) = C'$ by Proposition 3.21 and Lemma 3.28. Hence

$$\mathcal{L}f^\vee(\mathcal{L}f(C)) = \mathcal{L}f^\vee(C') = \bigcap_{x \in C'} \partial f^\vee(x) \subset \partial f^\vee(y_0) = C.$$

On the other hand, let $x_0 \in \mathrm{ri}(C')$. In particular, $x_0 \in \partial f(u_0)$ and so $u_0 \in \partial f^\vee(x_0) = \mathcal{L}f^\vee(C')$ for all $u_0 \in C$. It implies

$$C \subset \mathcal{L}f^\vee(C') = \mathcal{L}f^\vee(\mathcal{L}f(C)).$$

Thus $\mathcal{L}f^\vee(\mathcal{L}f(C)) = C$ and applying the same argument to $f^\vee$ we conclude that $\mathcal{L}f^\vee = (\mathcal{L}f)^{-1}$ and that $\mathcal{L}f$ is bijective.

Now we have to prove that $\mathcal{L}$ is a duality between $\Pi(f)$ and $\Pi(f^\vee)$. Let $C, D \in \Pi(f)$ such that $C \subset D$. Clearly, $\mathcal{L}f(C) \supset \mathcal{L}f(D)$. The reciprocal follows by applying the same argument to $f^\vee$. The fact that C and $\mathcal{L}f(C)$ lie in orthogonal affine spaces has already been shown during the proof of Lemma 3.28 above, see (3.30). □

Definition 3.34. Let f be a closed concave function. The pair of convex decompositions $(\Pi(f),\Pi(f^\vee))$ will be called the *dual pair of convex decompositions* induced by f.

In particular, for $C\in\Pi(f)$ put $C^*:=\mathcal{L}f(C)$. For any $u_0\in\operatorname{ri}(C)$ and $x_0\in\operatorname{ri}(C^*)$, we have

$$C=\{u\in N_{\mathbb{R}}\mid P_f(u,x_0)=0\}\quad\text{and}\quad C^*=\{x\in M_{\mathbb{R}}\mid P_f(u_0,x)=0\}.$$

Following (3.25), the restrictions $f|_C$ and $f^\vee|_{C^*}$ are affine functions. Observe that we can recover the Legendre-Fenchel dual from the Legendre-Fenchel correspondence by writing, for $x\in C^*$ and any $u\in C$,

$$f^\vee(x)=\langle x,u\rangle-f(u). \tag{3.35}$$

Example 3.36. Let $\|\cdot\|_2$ denote the Euclidean norm on $\mathbb{R}^2$ and B_1 the unit ball. Consider the concave function $f\colon B_1\to\mathbb{R}$ defined as $f(u)=-\|u\|_2$. Then $\operatorname{stab}(f)=\mathbb{R}^2$ and the Legendre-Fenchel dual is the function defined by $f^\vee(x)=0$ if $\|x\|_2\le 1$ and $f^\vee(x)=1-\|x\|_2$ otherwise. The decompositions $\Pi(f)$ and $\Pi(f^\vee)$ consist of a collection of pieces of three different types and the Legendre-Fenchel correspondence $\mathcal{L}f\colon\Pi(f)\to\Pi(f^\vee)$ is given, for $z\in S^1$, by

$$\mathcal{L}f(\{0\})=B_1,\quad \mathcal{L}f([0,1]\cdot z)=\{z\},\quad \mathcal{L}f(\{z\})=\mathbb{R}_{\ge 1}\cdot z.$$

In the above example both decompositions are in fact subdivisions. But this is not always the case, as shown by the next example.

Example 3.37. Let $f\colon[0,1]\to\mathbb{R}$ the function defined by

$$f(u)=\begin{cases}-u\log(u), & \text{if } 0\le u\le \mathrm{e}^{-1},\\ \mathrm{e}^{-1}, & \text{if } \mathrm{e}^{-1}\le u\le 1-\mathrm{e}^{-1},\\ -(1-u)\log(1-u), & \text{if } 1-\mathrm{e}^{-1}\le u\le 1.\end{cases}$$

Then $\operatorname{stab}(f)=\mathbb{R}$ and the Legendre-Fenchel dual is the function $f^\vee(x)=x-\mathrm{e}^{x-1}$ for $x\le 0$ and $f^\vee(x)=-\mathrm{e}^{-x-1}$ for $x\ge 0$. Then $\operatorname{dom}(\partial f)=(0,1)$ and $\operatorname{dom}(\partial f^\vee)=\mathbb{R}$. Moreover,

$$\Pi(f)=(0,\mathrm{e}^{-1})\cup\{[\mathrm{e}^{-1},1-\mathrm{e}^{-1}]\}\cup(1-\mathrm{e}^{-1},1),\quad \Pi(f^\vee)=\mathbb{R}.$$

The Legendre-Fenchel correspondence sends bijectively $(0,\mathrm{e}^{-1})$ to $\mathbb{R}_{>0}$ and $(1-\mathrm{e}^{-1},1)$ to $\mathbb{R}_{<0}$, and sends the element $[\mathrm{e}^{-1},1-\mathrm{e}^{-1}]$ to the point $\{0\}$. In this example, $\Pi(f)$ is not a subdivision while $\Pi(f^\vee)$ is.

3.3. **Operations on concave functions and duality.** In this section we consider the basic operations on concave functions and their interplay with the Legendre-Fenchel duality.

Let f_1 and f_2 be two concave functions such that their stability sets are not disjoint. Their *sup-convolution* is the function

$$f_1\boxplus f_2\colon M_{\mathbb{R}}\longrightarrow\underline{\mathbb{R}},\quad v\longmapsto \sup_{u_1+u_2=v}(f_1(u_1)+f_2(u_2)).$$

This is a concave function whose effective domain is the Minkowski sum $\operatorname{dom}(f_1)+\operatorname{dom}(f_2)$. This operation is associative and commutative whenever the terms are defined.

The operations of pointwise addition and sup-convolution are dual to each other. When working with general concave functions, there are some technical issues in this duality that will disappear when considering uniform limits of piecewise affine concave functions.

Proposition 3.38. *Let $f_1,\dots,f_l$ be concave functions.*

(1) *If* $\mathrm{stab}(f_1)\cap\dots\cap\mathrm{stab}(f_l)\neq\emptyset$, *then*
$$(f_1\boxplus\dots\boxplus f_l)^\vee=f_1^\vee+\dots+f_l^\vee.$$
(2) *If* $\mathrm{dom}(f_1)\cap\dots\cap\mathrm{dom}(f_l)\neq\emptyset$, *then*
$$(\mathrm{cl}(f_1)+\dots+\mathrm{cl}(f_l))^\vee=\mathrm{cl}(f_1^\vee\boxplus\dots\boxplus f_l^\vee).$$
(3) *If* $\mathrm{ri}(\mathrm{dom}(f_1))\cap\dots\cap\mathrm{ri}(\mathrm{dom}(f_l))\neq\emptyset$, *then*
$$(f_1+\dots+f_l)^\vee=f_1^\vee\boxplus\dots\boxplus f_l^\vee.$$

Proof. This is proved in [Roc70, Theorem 16.4]. □

Remark 3.39. When some of the f_i, say $f_1,\dots,f_k$, are piecewise affine, the statement (3) of the previous proposition holds under the weaker hypothesis [Roc70, Theorem 20.1]
$$\mathrm{dom}(f_1)\cap\dots\cap\mathrm{dom}(f_k)\cap\mathrm{ri}(\mathrm{dom}(f_{k+1}))\cap\dots\cap\mathrm{ri}(\mathrm{dom}(f_l))\neq\emptyset.$$

Let f be a concave function. For $\lambda>0$, the *left* and *right scalar multiplication* of f by λ are the functions defined, for $u\in N_\mathbb{R}$, by $(\lambda f)(u)=\lambda f(u)$ and $(f\lambda)(u)=\lambda f(u/\lambda)$ respectively. For a point $u_0\in N_\mathbb{R}$, the *translate* of f by u_0 is the concave function defined as $(\tau_{u_0}f)(u)=f(u-u_0)$ for $u\in N_\mathbb{R}$.

Proposition 3.40. *Let f be a concave function on $N_\mathbb{R}$, $\lambda>0$, $u_0\in N_\mathbb{R}$ and $x_0\in M_\mathbb{R}$. Then*
(1) $\mathrm{dom}(\lambda f)=\mathrm{dom}(f)$, $\mathrm{stab}(\lambda f)=\lambda\,\mathrm{stab}(f)$ *and* $(\lambda f)^\vee=f^\vee\lambda$;
(2) $\mathrm{dom}(f\lambda)=\lambda\mathrm{dom}(f)$, $\mathrm{stab}(f\lambda)=\mathrm{stab}(f)$ *and* $(f\lambda)^\vee=\lambda f^\vee$;
(3) $\mathrm{dom}(\tau_{u_0}f)=\mathrm{dom}(f)+u_0$, $\mathrm{stab}(\tau_{u_0}f)=\mathrm{stab}(f)$ *and* $(\tau_{u_0}f)^\vee=f^\vee+u_0$;
(4) $\mathrm{dom}(f+x_0)=\mathrm{dom}(f)$, $\mathrm{stab}(f+x_0)=\mathrm{stab}(f)+x_0$ *and* $(f+x_0)^\vee=\tau_{x_0}f^\vee$.

Proof. This follows easily from the definitions. □

We next consider direct and inverse images of concave functions by affine maps. Let $Q_\mathbb{R}$ be a another finite dimensional real vector space and set $P_\mathbb{R}=Q_\mathbb{R}^\vee$ for its dual space. For a linear map $H\colon Q_\mathbb{R}\to N_\mathbb{R}$ we denote by $H^\vee\colon M_\mathbb{R}\to P_\mathbb{R}$ the dual map. We need the following lemma in order to properly define direct images.

Lemma 3.41. *Let $H\colon Q_\mathbb{R}\to N_\mathbb{R}$ be a linear map and g a concave function on $Q_\mathbb{R}$. If $\mathrm{stab}(g)\cap\mathrm{im}(H^\vee)\neq\emptyset$ then, for all $u\in N_\mathbb{R}$,*
$$\sup_{v\in H^{-1}(u)}g(v)<\infty.$$

Proof. Let $x\in M_\mathbb{R}$ such that $H^\vee(x)\in\mathrm{stab}(g)$. By the definition of the stability set, $\sup_{v\in Q_\mathbb{R}}(g(v)-\langle H^\vee(x),v\rangle)<\infty$. Thus, for any $u\in N_\mathbb{R}$,
$$\begin{aligned}\sup_{v\in Q_\mathbb{R}}(g(v)-\langle H^\vee(x),v\rangle)&=\sup_{v\in Q_\mathbb{R}}(g(v)-\langle x,H(v)\rangle)\\&\geq\sup_{v\in H^{-1}(u)}(g(v)-\langle x,H(v)\rangle)=\sup_{v\in H^{-1}(u)}g(v)-\langle x,u\rangle\end{aligned}$$
and so $\sup_{v\in H^{-1}(u)}g(v)$ is bounded above, as stated. □

Definition 3.42. Let $A\colon Q_\mathbb{R}\to N_\mathbb{R}$ be an affine map defined as $A=H+u_0$ for a linear map H and a point $u_0\in N_\mathbb{R}$. Let f be a concave function on $N_\mathbb{R}$ such that $\mathrm{dom}(f)\cap\mathrm{im}(A)\neq\emptyset$ and g a concave function on $Q_\mathbb{R}$ such that $\mathrm{stab}(g)\cap\mathrm{im}(H^\vee)\neq\emptyset$. Then the *inverse image* of f by A is defined as
$$A^*f\colon Q_\mathbb{R}\longrightarrow\mathbb{R},\quad v\longmapsto f\circ A(v),$$
and the *direct image* of g by A is defined as
$$A_*g\colon N_\mathbb{R}\longrightarrow\mathbb{R},\quad u\longmapsto\sup_{v\in A^{-1}(u)}g(v).$$

It is easy to see that the inverse image A^*f is concave with effective domain $\mathrm{dom}(A^*f) = A^{-1}(\mathrm{dom}(f))$. Similarly, the direct image A_*g is concave with effective domain $\mathrm{dom}(A_*g) = A(\mathrm{dom}(g))$, thanks to Lemma 3.41.

The inverse image of a closed function is also closed. In contrast, the direct image of a closed function is not necessarily closed: consider for instance the indicator function ι_C of the set $C = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 1, x > 0\}$, which is a closed concave function. Let $A\colon \mathbb{R}^2 \to \mathbb{R}$ be the first projection. Then $A_*\iota_C$ is the indicator function of the subset $\mathbb{R}_{>0}$, which is not a closed concave function.

We now turn to the behaviour of the sup-differential with respect to the basic operations. A first important property is the additivity.

Proposition 3.43. *For each $i = 1, \dots, l$, let f_i be a concave function and $\lambda_i > 0$ a real number. Then*

(1) $\partial\left(\sum_i \lambda_i f_i\right) \supset \sum_i \lambda_i \partial(f_i)$;
(2) *if* $\mathrm{ri}(\mathrm{dom}(f_1)) \cap \dots \cap \mathrm{ri}(\mathrm{dom}(f_l)) \neq \emptyset$, *then*

$$\partial\Big(\sum_i \lambda_i f_i\Big) = \sum_i \lambda_i \partial(f_i). \tag{3.44}$$

Proof. This is [Roc70, Theorem 23.8]. □

As in Remark 3.39, if $f_1, \dots, f_k$ are piecewise affine, then (3.44) holds under the weaker hypothesis

$$\mathrm{dom}(f_1) \cap \dots \cap \mathrm{dom}(f_k) \cap \mathrm{ri}(\mathrm{dom}(f_{k+1})) \cap \dots \cap \mathrm{ri}(\mathrm{dom}(f_l)) \neq \emptyset.$$

The following result gives the behaviour of the sup-differential with respect to linear maps

Proposition 3.45. *Let $H\colon Q_{\mathbb{R}} \to N_{\mathbb{R}}$ be a linear map, $u_0 \in N_{\mathbb{R}}$ and $A = H + u_0$ the associated affine map. Let f be a concave function on $N_{\mathbb{R}}$, then*

(1) $\partial(A^*f)(v) \supset H^\vee \partial f(Av)$ *for all* $v \in Q_{\mathbb{R}}$;
(2) *if either* $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{im}(A) \neq \emptyset$ *or f is piecewise affine and* $\mathrm{dom}(f) \cap \mathrm{im}(A) \neq \emptyset$, *then for all $v \in Q_{\mathbb{R}}$ we have*

$$\partial(A^*f)(v) = H^\vee \partial f(Av).$$

Proof. The linear case $u_0 = 0$ is [Roc70, Theorem 23.9]. The general case follows from the linear case and the commutativity of the sup-differential and the translation. □

We summarize the behaviour of direct and inverse images of affine maps with respect to the Legendre-Fenchel duality.

Proposition 3.46. *Let $A\colon Q_{\mathbb{R}} \to N_{\mathbb{R}}$ be an affine map defined as $A = H + u_0$ for a linear map H and a point $u_0 \in N_{\mathbb{R}}$. Let f be a concave function on $N_{\mathbb{R}}$ such that* $\mathrm{dom}(f) \cap \mathrm{im}(A) \neq \emptyset$ *and g a concave function on $Q_{\mathbb{R}}$ such that* $\mathrm{stab}(g) \cap \mathrm{im}(H^\vee) \neq \emptyset$. *Then*

(1) $\mathrm{stab}(A_*g) = (H^\vee)^{-1}(\mathrm{stab}(g))$ *and*
$$(A_*g)^\vee = (H^\vee)^*(g^\vee) + u_0;$$
(2) $H^\vee(\mathrm{stab}(f)) \subset \mathrm{stab}(A^*f) \subset \overline{H^\vee(\mathrm{stab}(f))}$ *and*
$$(A^*\mathrm{cl}(f))^\vee = \mathrm{cl}((H^\vee)_*(f^\vee - u_0));$$
(3) *if* $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{im}(A) \neq \emptyset$ *then* $\mathrm{stab}(A^*f) = H^\vee(\mathrm{stab}(f))$ *and, for all y in this set,*
$$(A^*f)^\vee(y) = (H^\vee)_*(f^\vee - u_0)(y) = \max_{x \in (H^\vee)^{-1}(y)} (f^\vee(x) - \langle x, u_0\rangle).$$

Moreover, for $y \in \mathrm{ri}(\mathrm{stab}(A^*f))$*, a point* $x \in (H^\vee)^{-1}(y)$ *realizes this maximum if and only if* $x \in \partial f(Av)$ *for a* $v \in Q_\mathbb{R}$ *such that* $y \in \partial(A^*f)(v)$*.*

Observe that the last assertion in the above proposition can be also expressed as

$$(A^*f)^\vee(\partial(A^*f)(v)) = f^\vee(\partial f(Av)) - \langle \partial f(Av), u_0\rangle. \tag{3.47}$$

Proof. By Proposition 3.40(3,4),

$$A^*(f) = (H+u_0)^*(f) = H^*(\tau_{-u_0}f), \quad A_*g = (H+u_0)_*g = \tau_{u_0}(H_*g).$$

Then, except for the last assertion, the result follows by combining this with the case when A is a linear map, treated in [Roc70, Theorem 16.3].

To prove the last assertion of the proposition, we first note that the concave function

$$(f^\vee - u_0)|_{(H^\vee)^{-1}(y)}$$

attains its maximum at a point x if and only if its sup-differential at x contains 0. We fix a point x_0 in $(H^\vee)^{-1}(y)$ and we consider the affine inclusion

$$\iota\colon \mathrm{Ker}(H^\vee) \hookrightarrow M_\mathbb{R}, \quad z \mapsto z + x_0.$$

We denote by $\iota^\vee\colon N_\mathbb{R} \to N_\mathbb{R}/\mathrm{im}(H)$ the dual of the linear part of ι. Set $F = \iota^*(f^\vee - u_0)$, then for $z \in \mathrm{Ker}(H^\vee)$, by Proposition 3.45, we have

$$\partial F(z) = \iota^\vee(\partial f^\vee(z+x_0) - u_0)$$

and so $0 \in \partial F(z)$ if and only if $\partial f^\vee(z+x_0) \cap \mathrm{im}(A) \neq \emptyset$. Hence $x = z + x_0$ realizes the maximum if and only if $x \in \partial f(Av)$ for some $v \in Q_\mathbb{R}$ such that $y \in \partial(A^*f)(v)$, as stated. □

In particular, the operations of direct and inverse image of *linear* maps are dual to each other. In the notation of Proposition 3.46 and assuming for simplicity $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{im}(H) \neq \emptyset$, we have

$$(H_*g)^\vee = (H^\vee)^*(g^\vee), \quad (H^*f)^\vee = (H^\vee)_*(f^\vee),$$

while the stability sets relate by $\mathrm{stab}(H_*g) = (H^\vee)^{-1}(\mathrm{stab}(g))$ and $\mathrm{stab}(H^*f) = H^\vee(\mathrm{stab}(f))$.

The last concept we recall in this section is the notion of recession of a concave function.

Definition 3.48. The *recession function* of a concave function $f\colon N_\mathbb{R} \to \underline{\mathbb{R}}$, denoted $\mathrm{rec}(f)$, is the function

$$\mathrm{rec}(f)\colon N_\mathbb{R} \longrightarrow \underline{\mathbb{R}}, \quad u \longmapsto \inf_{v\in\mathrm{dom}(f)}(f(u+v) - f(v)).$$

This is a concave conical function. If f is closed, its recession function can be defined as the limit

$$\mathrm{rec}(f)(u) = \lim_{\lambda\to\infty} \lambda^{-1} f(v_0 + \lambda u) \tag{3.49}$$

for any $v_0 \in \mathrm{dom}(f)$ [Roc70, Theorem 8.5].

It is clear from the definition that $\mathrm{dom}(\mathrm{rec}(f)) \subset \mathrm{rec}(\mathrm{dom}(f))$. The equality does not hold in general, as can be seen by considering the concave function $\mathbb{R} \to \mathbb{R}$, $u \mapsto -\exp(u)$.

If f is closed then the function $\mathrm{rec}(f)$ is closed [Roc70, Theorem 8.5]. Hence it is natural to regard recession functions as support functions.

Proposition 3.50. *Let* f *be a concave function. Then* $\mathrm{rec}(f^\vee)$ *is the support function of* $\mathrm{dom}(f)$*. If* f *is closed, then* $\mathrm{rec}(f)$ *is the support function of* $\mathrm{stab}(f)$*.*

Proof. This is [Roc70, Theorem 13.3]. □

3.4. **The differentiable case.** In this section we make explicit the Legendre-Fenchel duality for smooth concave functions, following [Roc70, Chapter 26].

In the differentiable and strictly concave case, the decompositions $\Pi(f)$ and $\Pi(f^\vee)$ consist of the collection of all points of $\operatorname{dom}(\partial f)$ and of $\operatorname{dom}(\partial f^\vee)$ respectively. The Legendre-Fenchel correspondence agrees with the gradient map, and it is called the Legendre transform in this context.

Recall that a function $f\colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$ is differentiable at a point $u \in N_{\mathbb{R}}$ with $f(u) > -\infty$, if there exists some linear form $\nabla f(u) \in M_{\mathbb{R}}$ such that

$$f(v) = f(u) + \langle \nabla f(u), v-u\rangle + o(||v-u||),$$

where $||\cdot||$ denotes any fixed norm on $N_{\mathbb{R}}$. This linear form $\nabla(f)(u)$ is the gradient of f in the classical sense. It can be shown that a concave function f is differentiable at a point $u \in \operatorname{dom}(f)$ if and only if $\partial f(u)$ consists of a single element. If this is the case, then $\partial f(u) = \{\nabla f(u)\}$ [Roc70, Theorem 25.1]. Hence, the gradient and the sup-differential agree in the differentiable case.

Let $C \subset N_{\mathbb{R}}$ be a convex set. A function $f\colon C \to \mathbb{R}$ is *strictly concave* if $f(tu_1+(1-t)u_2) > tf(u_1)+(1-t)f(u_2)$ for all different $u_1,u_2 \in C$ and $0<t<1$.

Definition 3.51. Let $C \subset N_{\mathbb{R}}$ be an open convex set and $||\cdot||$ any fixed norm on $M_{\mathbb{R}}$. A differentiable concave function $f\colon C\to\mathbb{R}$ is of *Legendre type* if it is strictly concave and $\lim_{i\to\infty}\|\nabla f(u_i)\| \to \infty$ for every sequence $(u_i)_{i\geq 1}$ converging to a point in the boundary of C. In particular, any differentiable and strictly concave function on $N_{\mathbb{R}}$ is of Legendre type.

The stability set of a function of Legendre type has maximal dimension. Therefore its relative interior agrees with its interior and, in this case, we will use the classical notation $\operatorname{stab}(f)^\circ$ for the interior of $\operatorname{stab}(f)$.

The following result summarizes the basics properties of the Legendre-Fenchel duality acting on functions of Legendre type.

Theorem 3.52. *Let $f\colon C\to\mathbb{R}$ be a concave function of Legendre type defined on an open set $C\subset N_{\mathbb{R}}$ and let $D=\nabla f(C)\subset M_{\mathbb{R}}$ be the image of the gradient map. Then*

(1) $D = \operatorname{stab}(f)^\circ$;
(2) $f^\vee|_D$ *is a concave function of Legendre type;*
(3) $\nabla f\colon C\to D$ *is a homeomorphism and* $(\nabla f)^{-1} = \nabla f^\vee$;
(4) *for all* $x\in D$ *we have* $f^\vee(x) = \langle x, (\nabla f)^{-1}(x)\rangle - f((\nabla f)^{-1}(x))$.

Proof. This follows from [Roc70, Theorem 26.5]. □

Example 3.53. Consider the function

$$f_{\mathrm{FS}}\colon \mathbb{R}^n \longrightarrow \mathbb{R},\quad u\longmapsto -\frac{1}{2}\log\Big(1+\sum_{i=1}^n \mathrm{e}^{-2u_i}\Big).$$

Let $\Delta^n = \{(x_1,\dots,x_n)\subset\mathbb{R}^n \mid x_i\geq 0, \sum x_i\leq 1\}$ be the *standard simplex* of $\mathbb{R}^n$. For $(x_1,\dots,x_n)\in\Delta^n$, write $x_0 = 1-\sum_{i=1}^n x_i$ and set

$$\varepsilon_n\colon \Delta^n \longrightarrow \mathbb{R},\quad x\longmapsto -\sum_{i=0}^m x_i\log(x_i). \tag{3.54}$$

We have $\nabla f_{\mathrm{FS}}(u) = \dfrac{1}{1+\sum_{i=1}^n \mathrm{e}^{-2u_i}}\left(\mathrm{e}^{-2u_1},\dots,\mathrm{e}^{-2u_n}\right)$ and so

$$\frac{1}{2}\varepsilon_n(\nabla f_{\mathrm{FS}}(u)) = \frac{\sum_{i=1}^n \mathrm{e}^{-2u_i}\,u_i}{1+\sum_{i=1}^n\mathrm{e}^{-2u_i}} + \frac{1}{2}\log\Big(1+\sum_{i=1}^n\mathrm{e}^{-2u_i}\Big) = \langle\nabla f_{\mathrm{FS}}(u),u\rangle - f_{\mathrm{FS}}(u),$$

which shows that $\operatorname{stab}(f_{\mathrm{FS}}) = \Delta^n$ and that $f_{\mathrm{FS}}^\vee = \frac{1}{2}\varepsilon_n$.

The fact that the sup-differential agrees with the gradient and is single-valued can simplify some statements. It is interesting to make explicit the computation of the Legendre-Fenchel dual of the inverse image by an affine map of a concave function of Legendre type.

Proposition 3.55. *Let $A\colon Q_{\mathbb{R}} \to N_{\mathbb{R}}$ be an affine map defined as $A = H + u_0$ for an injective linear map H and a point $u_0 \in N_{\mathbb{R}}$. Let $f\colon C \to \mathbb{R}$ be a concave function of Legendre type defined on an open convex set $C \subset N_{\mathbb{R}}$ such that $C \cap \operatorname{im}(A) \neq \emptyset$. Then A^*f is a concave function of Legendre type on $A^{-1}(C)$,*

$$\operatorname{stab}(A^*f)^\circ = \operatorname{im}(\nabla(A^*f)) = H^\vee(\operatorname{im}(\nabla f)) = H^\vee(\operatorname{stab}(f)^\circ),$$

and, for all $v \in A^{-1}C$,

$$(A^*f)^\vee(\nabla(A^*f)(v)) = f^\vee(\nabla f(Av)) - \langle \nabla f(Av), u_0\rangle.$$

Moreover, there is a section $\imath_{A,f}$ of $H^\vee|_{\operatorname{stab}(f)^\circ}$ such that the diagram

$$\begin{tikzcd}
 & A^{-1}C \arrow[dl, "A^*f"'] \arrow[rr, "\nabla(A^*f)"] \arrow[dd, "A"] & & \operatorname{stab}(A^*f)^\circ \arrow[dd, "\imath_{A,f}"'] \arrow[dr, "(A^*f)^\vee"] & \\
\mathbb{R} & & & & \mathbb{R} \\
 & C \arrow[ul, "f"] \arrow[rr, "\nabla f"'] & & \operatorname{stab}(f)^\circ \arrow[ur, "f^\vee - u_0"'] &
\end{tikzcd} \tag{3.56}$$

commutes.

Proof. This follows readily from Proposition 3.46. □

The section $\imath_{A,f}$ embeds $\operatorname{stab}(A^*f)^\circ$ as a real submanifold of $\operatorname{stab}(f)^\circ$. Varying u_0 in a suitable space of parameters, we obtain a foliation of $\operatorname{stab}(f)^\circ$ by "parallel" submanifolds. We illustrate this phenomenon with an example in dimension 2.

Example 3.57. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(u_1, u_2) = -\frac{1}{2}\log\left(1 + \mathrm{e}^{-2u_1} + \mathrm{e}^{-4u_1 - 2u_2} + \mathrm{e}^{-2u_1-4u_2}\right).$$

It is a concave function of Legendre type whose stability set is the polytope $\Delta = \operatorname{conv}((0,0),(1,0),(2,1),(1,2))$. The restriction of its Legendre-Fenchel dual to Δ° is also a concave function of Legendre type.

For $c \in \mathbb{R}$, consider the affine map

$$A_c\colon \mathbb{R} \to \mathbb{R}^2, \quad u \longmapsto (-u, u + c).$$

We write $A_c = H + (0, c)$ for a linear function H. The dual of H is the function $H^\vee\colon \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto x_2 - x_1$. Then $\operatorname{stab}(A_c^*f)^\circ = H^\vee(\Delta^\circ)$ is the open interval $(-1, 1)$. By Proposition 3.55, there is a map $\imath_{A_c,f}$ embedding $(-1,1)$ into Δ° in such a way that $\imath_{A_c,f} \circ \nabla(A_c^*f) = (\nabla f) \circ A_c$. For $u \in \mathbb{R}$,

$$\nabla(A_c^*f)(u) = \frac{\mathrm{e}^{-2u-4c} - \mathrm{e}^{2u} - \mathrm{e}^{2u-2c}}{1 + \mathrm{e}^{2u} + \mathrm{e}^{2u-2c} + \mathrm{e}^{-2u-4c}} \in (-1, 1),$$

$$(\nabla f) \circ A_c(u) = \frac{\left(\mathrm{e}^{2u} + 2\,\mathrm{e}^{2u-2c} + \mathrm{e}^{-2u-4c}, \mathrm{e}^{2u-2c} + 2\,\mathrm{e}^{-2u-4c}\right)}{1 + \mathrm{e}^{2u} + \mathrm{e}^{2u-2c} + \mathrm{e}^{-2u-4c}} \in \Delta^\circ.$$

From this, we compute $\imath_{A_c,f}(x) = (x_1, x_2)$ with

$$\begin{cases} x_1 = \dfrac{-\mathrm{e}^{-2c}}{2(1+\mathrm{e}^{-2c})}x + \dfrac{2+3\,\mathrm{e}^{-2c}}{2(1+\mathrm{e}^{-2c})}\left(\dfrac{x^2}{\sqrt{\rho_c^2+(1-\rho_c^2)x^2}+\rho_c} + \dfrac{\rho_c}{1+\rho_c}\right), \\ x_2 = \dfrac{2+\mathrm{e}^{-2c}}{2(1+\mathrm{e}^{-2c})}x + \dfrac{2+3\,\mathrm{e}^{-2c}}{2(1+\mathrm{e}^{-2c})}\left(\dfrac{x^2}{\sqrt{\rho_c^2+(1-\rho_c^2)x^2}+\rho_c} + \dfrac{\rho_c}{1+\rho_c}\right), \end{cases}$$

where we have set $\rho_c = 2\,\mathrm{e}^{-2c}\sqrt{1+\mathrm{e}^{-2c}}$ for short. In particular, the image of the map $\imath_{A_c,f}$ is an arc of conic: namely the intersection of Δ° with the conic of equation

$$(x_2 - x_1)^2 = (1-\rho_c^2)L_c(x_1,x_2)^2 + 2\rho_c L_c(x_1,x_2),$$

with $L_c(x_1,x_2) = \frac{2+\mathrm{e}^{-2c}}{2+3\,\mathrm{e}^{-2c}}x_1 + \frac{\mathrm{e}^{-2c}}{2+3\,\mathrm{e}^{-2c}}x_2 - \frac{\rho_c}{1+\rho_c}$. Varying $c \in \mathbb{R}$, these arcs of conics form a foliation of Δ°, they all pass through the vertex $(1,2)$ as $x \to 1$, and their other end as $x \to -1$ parameterizes the relative interior of the edge $\operatorname{conv}((1,0),(2,1))$, see Figure 2.

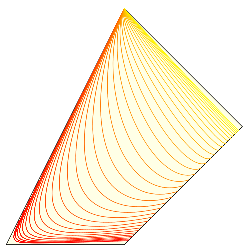

FIGURE 2. A foliation of Δ° by curves

3.5. **The piecewise affine case.** The Legendre-Fenchel duality for piecewise affine concave functions can be described in combinatorial terms. Moreover, some technical issues of the general theory disappear when dealing with piecewise affine concave functions on convex polyhedra and uniform limits of such functions.

Definition 3.58. Let $C \subset N_\mathbb{R}$ be a convex polyhedron. A function $f\colon C \to \mathbb{R}$ is *piecewise affine* if there a finite cover of C by closed subsets such that the restriction of f to each of these subsets is an affine function. A concave function $f\colon N_\mathbb{R} \to \underline{\mathbb{R}}$ is said to be *piecewise affine* if $\operatorname{dom}(f)$ is a convex polyhedron and the restriction $f|_{\operatorname{dom}(f)}$ piecewise affine.

Lemma 3.59. *Let f be a piecewise affine function defined on a convex polyhedron $C \subset N_\mathbb{R}$. Then there exists a polyhedral complex Π in C such that the restriction of f to each polyhedron of Π is an affine function.*

Proof. This is an easy consequence of the max-min representation of piecewise affine functions in [Ovc02]. □

Definition 3.60. Let C be a convex polyhedron, Π a polyhedral complex in C and $f\colon C \to \mathbb{R}$ a piecewise affine function. We say that Π and f are *compatible* if f is affine on each polyhedron of Π. Alternatively, we say that f is a piecewise affine function *on* Π. If the function f is concave, it is said to be *strictly concave on* Π if $\Pi = \Pi(f)$. The polyhedral complex Π is said to be *regular* if there exists a concave piecewise affine function f such that $\Pi = \Pi(f)$.

As was the case for convex polyhedra, piecewise affine concave functions can be described in two dual ways, which we refer as the H-representation and the V-representation. For the *H-representation*, we consider a convex polyhedron

$$\Lambda = \bigcap_{1\leq j\leq k} \{u \in N_{\mathbb{R}} \mid \langle a_j, u\rangle + \alpha_j \geq 0\}$$

as in (3.4) and a set of affine equations $\{(a_j, \alpha_j)\}_{k+1\leq j\leq l} \subset M_{\mathbb{R}} \times \mathbb{R}$. We then define a concave function on $N_{\mathbb{R}}$ as

$$f(u) = \min_{k+1\leq j\leq l} (\langle a_j, u\rangle + \alpha_j) \quad \text{ for } u \in \Lambda \tag{3.61}$$

and $f(u) = -\infty$ for $u \notin \Lambda$. With this representation, the recession function of f is given by

$$\operatorname{rec}(f)(u) = \min_{k+1\leq j\leq l} \langle a_j, u\rangle, \quad \text{ for } u \in \operatorname{rec}(\Lambda)$$

and $\operatorname{rec}(f)(u) = -\infty$ for $u \notin \operatorname{rec}(\Lambda)$. In particular,

$$\operatorname{dom}(\operatorname{rec}(f)) = \operatorname{rec}(\operatorname{dom}(f)), \quad \operatorname{stab}(\operatorname{rec}(f)) = \operatorname{stab}(f). \tag{3.62}$$

For the *V-representation*, we consider a polyhedron

$$\Lambda' = \operatorname{cone}(b_1, \dots, b_k) + \operatorname{conv}(b_{k+1}, \dots, b_l)$$

as in (3.5), a set of slopes $\{\beta_j\}_{1\leq j\leq k} \subset \mathbb{R}$ and a set of values $\{\beta_j\}_{k+1\leq j\leq l} \subset \mathbb{R}$. We then define a concave function on $M_{\mathbb{R}}$ as

$$g(u) = \sup\left\{ \sum_{j=1}^{l} \lambda_j \beta_j \,\middle|\, \lambda_j \geq 0,\ \sum_{j=k+1}^{l} \lambda_j = 1,\ \sum_{j=1}^{l} \lambda_j b_j = u \right\}. \tag{3.63}$$

With this second representation, we obtain the recession function as

$$\operatorname{rec}(g)(u) = \sup\left\{ \sum_{j=1}^{k} \lambda_j \beta_j \,\middle|\, \lambda_j \geq 0,\ \sum_{j=1}^{k} \lambda_j b_j = u \right\}.$$

As we have already mentioned, the Legendre-Fenchel duality of piecewise affine concave functions can be described in combinatorial terms.

Proposition 3.64. *Let Λ be a polyhedron in $N_{\mathbb{R}}$ and f a piecewise affine concave function with* $\operatorname{dom}(f) = \Lambda$ *given as*

$$\Lambda = \bigcap_{1\leq j\leq k} \{u \in N_{\mathbb{R}} \mid \langle a_j, u\rangle + \alpha_j \geq 0\},$$

$$f(u) = \min_{k+1\leq j\leq l} (\langle a_j, u\rangle + \alpha_j) \quad \textit{for } u \in \Lambda$$

with $a_j \in M_{\mathbb{R}}$ and $\alpha_j \in \mathbb{R}$. Then

$$\operatorname{stab}(f) = \operatorname{cone}(a_1, \dots, a_k) + \operatorname{conv}(a_{k+1}, \dots, a_l),$$

$$f^{\vee}(x) = \sup\left\{ \sum_{j=1}^{l} -\lambda_j \alpha_j \,\middle|\, \lambda_j \geq 0,\ \sum_{j=k+1}^{l} \lambda_j = 1,\ \sum_{j=1}^{l} \lambda_j a_j = x \right\} \quad \textit{for } x \in \operatorname{stab}(f).$$

Proof. This is proved in [Roc70, pp. 172-174]. □

Example 3.65. Let Λ be a convex polyhedron in $N_{\mathbb{R}}$. Then both the indicator function ι_Λ and the support function Ψ_Λ are concave and piecewise affine. We have $\Psi_\Lambda^{\vee} = \iota_\Lambda$. In particular, if we fix an isomorphism $N_{\mathbb{R}} \simeq \mathbb{R}^n$, the function

$$\Psi_{\Delta^n} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad (u_1, \dots, u_n) \longmapsto \min\{0, u_1, \dots, u_n\}$$

is the support function of the standard simplex $\Delta^n = \operatorname{conv}(\mathbf{0}, e_1^{\vee}, \dots, e_n^{\vee}) \subset M_{\mathbb{R}}$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ and $\{e_1^{\vee}, \dots, e_n^{\vee}\}$ is the dual basis. Hence, $\operatorname{stab}(\Psi_{\Delta^n}) = \Delta^n$ and $\Psi_{\Delta^n}^{\vee} = \iota_{\Delta^n}$.

Let Λ be a polyhedron in $N_{\mathbb{R}}$ and f a piecewise affine concave function with $\operatorname{dom}(f)=\Lambda$. Then $\operatorname{dom}(\partial f)=\Lambda$ and $\Pi(f)$ and $\Pi(f^{\vee})$ are convex decompositions of Λ and of $\Lambda' := \operatorname{stab}(f)$ respectively. By Theorem 3.33, the Legendre-Fenchel correspondence

$$\mathcal{L}f\colon \Pi(f) \longrightarrow \Pi(f^{\vee})$$

is a duality in the sense of Definition 3.32. However in the polyhedral case, these decompositions are dual in a stronger sense. We need to introduce some more definitions before we can properly state this duality.

Definition 3.66. Let Λ be a polyhedron and K a face of Λ. The *angle* of Λ at K is defined as

$$\angle(K,\Lambda)=\{t(u-v)\mid u\in\Lambda, v\in K, t\geq 0\}.$$

It is a polyhedral cone.

Definition 3.67. The *dual* of a convex cone $\sigma\subset N_{\mathbb{R}}$ is defined as

$$\sigma^{\vee}=\{x\in M_{\mathbb{R}}\mid \langle x,u\rangle\geq 0 \text{ for all } u\in\sigma\}.$$

This is a convex closed cone.

If σ is a convex closed cone, then $\sigma^{\vee\vee}=\sigma$. For a piecewise affine concave function f on $N_{\mathbb{R}}$, by Proposition 3.64 we have

$$\operatorname{rec}(\operatorname{dom}(f))^{\vee}=\operatorname{rec}(\operatorname{stab}(f)).$$

Definition 3.68. Let C, C' be convex polyhedra in $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively, and Π, Π' polyhedral complexes in C and C', respectively. We say that Π and Π' are *dual* polyhedral complexes if there is a bijective map $\Pi\to\Pi', \Lambda\mapsto\Lambda^*$ such that

(1) for all $\Lambda, K\in\Pi$, the inclusion $K\subset\Lambda$ hols if and only if $K^*\supset\Lambda^*$;
(2) for all $\Lambda, K\in\Pi$, if $K\subset\Lambda$, then $\angle(\Lambda^*,K^*)=\angle(K,\Lambda)^{\vee}$.

For $\Lambda\in\Pi$, the angle $\angle(\Lambda,\Lambda)$ is the linear subspace generated by differences of points in Λ. Condition (2) above implies that $\angle(\Lambda,\Lambda)$ and $\angle(\Lambda^*,\Lambda^*)$ are orthogonal. In particular, $\dim(\Lambda)+\dim(\Lambda^*)=n$.

Proposition 3.69. *Let f be a piecewise affine concave function with $\Lambda=\operatorname{dom}(f)$ and $\Lambda'=\operatorname{stab}(f)$. Then $\Pi(f)$ and $\Pi(f^{\vee})$ are polyhedral complexes in Λ and Λ' respectively. Moreover, they are dual of each other. In particular, the vertices of $\Pi(f)$ are in bijection with the polyhedra of $\Pi(f^{\vee})$ of maximal dimension.*

Proof. This is proved in [PR04, Proposition 1]. □

Example 3.70. Consider the standard simplex Δ^n of Example 3.65. Its indicator function induces the standard polyhedral complex in Δ^n consisting of the collection of its faces. The dual of ι_{Δ^n}, the support function Ψ_{Δ^n}, induces a fan $\Sigma_{\Delta^n}:=\Pi(\Psi_{\Delta^n})$ of $N_{\mathbb{R}}$. The duality between these polyhedral complexes can be made explicit as

$$\Pi(\iota_{\Delta^n})\longrightarrow\Sigma_{\Delta^n},\quad F\longmapsto\angle(F,\Delta^n)^{\vee}.$$

Example 3.71. The previous example can be generalized to an arbitrary polytope $\Delta\subset M_{\mathbb{R}}$. The indicator function ι_{Δ} induces the standard decomposition of Δ into its faces and dually, the support function Ψ_{Δ} induces a polyhedral complex $\Sigma_{\Delta}:=\Pi(\Psi_{\Delta})$ made of cones. If Δ is of maximal dimension, then Σ_{Δ} is a fan.

The faces of Δ are in one-to-one correspondence with the cones of Σ_{Δ} through the Legendre-Fenchel correspondence. For a face F of Δ, its corresponding cone is

$$\sigma_F:=F^*=\{u\in N_{\mathbb{R}}\mid \langle u,x-y\rangle\geq 0 \text{ for all } x\in\Delta, y\in F\}.$$

Reciprocally, to each cone σ corresponds a face of Δ of complementary dimension

$$F_\sigma := \sigma^* = \{x \in \Delta \mid \langle x, u\rangle = \Psi_\Delta(u) \text{ for all } u \in \sigma\}.$$

On a cone $\sigma \in \Sigma$, the function Ψ_Δ is defined by any vector m_σ in the affine space $\mathrm{aff}(F_\sigma)$. The cone σ is normal to F_σ.

For piecewise affine concave functions, the operations of taking the recession function and the associated polyhedral convex commute with each other.

Proposition 3.72. *Let f be a piecewise affine concave function on $N_\mathbb{R}$. Then*

$$\Pi(\mathrm{rec}(f)) = \mathrm{rec}(\Pi(f)).$$

Proof. Let $P_f(u,x) = f(u) + f^\vee(x) - \langle u, x\rangle$ be the function introduced in (3.20). For each $x \in \mathrm{stab}(f)$ write $P_{f,x}(u) = P(u,x)$. Let C_x be as in Definition 3.23. By Lemma 3.24,

$$C_x = \{u \in \mathrm{dom}(f) \mid P_{f,x}(u) = 0\}.$$

Write $P'(v) = \mathrm{rec}(f)(v) - \langle u, x\rangle$. Then $P' = \mathrm{rec}(P_{f,x})$.

We claim that, for each $x \in \mathrm{stab}(f)$,

$$\mathrm{rec}(C_x) = \{v \in \mathrm{dom}(\mathrm{rec}(f)) \mid P'(v) = 0\}.$$

Let $v \in \mathrm{rec}(C_x)$. Clearly $v \in \mathrm{dom}(\mathrm{rec}(f))$ and, since $x \in \mathrm{stab}(f)$, the set C_x is non-empty. Let $u_0 \in C_x$. Then, for each $\lambda > 0$, $u_0 + \lambda v \in C_x$. Therefore,

$$P'(v) = \lim_{\lambda\to\infty} \frac{P_{f,x}(u_0+\lambda v) - P_{f,x}(u_0)}{\lambda} = 0.$$

Conversely, let $v \in \mathrm{dom}(\mathrm{rec}(f))$ satisfying $P'(v) = 0$ and $u \in C_x$. On the one hand, by the properties of the function P_f, we have $P_{f,x}(u+v) \leq 0$. On the other hand, since $P' = \mathrm{rec}(P_{f,x})$,

$$P_{f,x}(u+v) - P_{f,x}(u) \geq P'(v) = 0.$$

Thus $P_{f,x}(u+v) \geq P_{f,x}(u) = 0$ and finally $P_{f,x}(u+v) = 0$. This implies that, if $u \in C_x$ then $u + v \in C_x$, showing $v \in \mathrm{rec}(C_x)$. Hence the claim is proved.

By definition $\Pi(f) = \{C_x\}_{x\in\mathrm{stab}(f)}$. Hence $\mathrm{rec}(\Pi(f)) = \{\mathrm{rec}(C_x)\}_{x\in\mathrm{stab}(f)}$. For each $x \in \mathrm{stab}(\mathrm{rec}(f))$, write

$$C'_x = \{v \in \mathrm{dom}(\mathrm{rec}(f)) \mid P'(v) = 0\}.$$

Then $\Pi(\mathrm{rec}(f)) = \{C'_x\}_{x\in\mathrm{stab}(\mathrm{rec}(f))}$. The result follows from the previous claim and the fact that $\mathrm{stab}(f) = \mathrm{stab}(\mathrm{rec}(f))$ by (3.62). □

Now we want to study the compatibility of Legendre-Fenchel duality and integral and rational structures. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n such that $N_\mathbb{R} = N \otimes \mathbb{R}$. Set $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ for its dual lattice, so $M_\mathbb{R} = M \otimes \mathbb{R}$. We also set $N_\mathbb{Q} = N \otimes \mathbb{Q}$ and $M_\mathbb{Q} = M \otimes \mathbb{Q}$.

Definition 3.73. A piecewise affine concave function f on $N_\mathbb{R}$ is an *H-lattice* (respectively, a *V-lattice*) concave function if it has an H-representation (respectively, a V-representation) with integral coefficients. We say that f is a *rational* piecewise affine concave function if it has an H-representation (or equivalently, a V-representation) with rational coefficients.

Observe that the domain of a V-lattice concave function is a lattice polyhedron, whereas the domain of an H-lattice concave function is a rational polyhedron.

Remark 3.74. The notion of H-lattice concave functions defined on the whole $N_\mathbb{R}$ coincides with the notion of tropical Laurent polynomials over the integers, that is, the elements of the group semi-algebra $\mathbb{Z}_{\mathrm{trop}}[N]$, where the arithmetic operations of the base semi-ring $\mathbb{Z}_{\mathrm{trop}} = (\mathbb{Z}, \oplus, \odot)$ are defined as $x \oplus y = \min(x,y)$ and $x \odot y = x + y$.

Proposition 3.75. *Let f be a piecewise affine concave function on $N_{\mathbb{R}}$.*

1. *f is an H-lattice concave function (respectively, a rational piecewise affine concave function) if and only if $f^\vee$ is a V-lattice concave function (respectively, a rational piecewise affine concave function).*
2. $\operatorname{rec}(f)$ *is an H-lattice concave function if and only if* $\operatorname{stab}(f)$ *is a lattice polyhedron.*

Proof. This follows easily from Proposition 3.64. □

Example 3.76. If Δ is a lattice polytope, its indicator function is a V-lattice function, its support function Ψ_Δ is an H-lattice function and, when Δ has maximal dimension, the fan Σ_Δ is a rational fan. In particular, if the isomorphism $N \simeq \mathbb{R}^n$ of Example 3.65 is given by the choice of an integral basis $e_1, \dots, e_n$ of N, then Δ^n is a lattice polytope, the function Ψ_{Δ^n} is an H-lattice concave function and Σ_{Δ^n} is a rational fan. If we write $e_0 = -\sum_{i=1}^n e_i$, this is the fan generated by the vectors $e_0, e_1, \dots, e_n$ in the sense that each cone of Σ_{Δ^n} is the cone generated by a strict subset of the above set of vectors. Figure 3 illustrates the case $n = 2$.

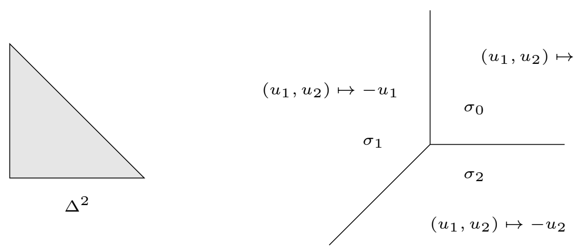

FIGURE 3. The standard simplex Δ^2, its associated fan and support function

Let Λ and Λ' be polyhedra in $N_{\mathbb{R}}$ and in $M_{\mathbb{R}}$, respectively. We set $\mathscr{P}(\Lambda, \Lambda')$ for the space of piecewise affine concave functions with effective domain Λ and stability set Λ'. We also set $\overline{\mathscr{P}}(\Lambda, \Lambda')$ for the closure of this space with respect to uniform convergence. We set

$$\mathscr{P}(\Lambda) = \bigcup_{\Lambda'} \mathscr{P}(\Lambda, \Lambda'), \quad \overline{\mathscr{P}}(\Lambda) = \bigcup_{\Lambda'} \overline{\mathscr{P}}(\Lambda, \Lambda')$$

for the space of piecewise affine concave functions with effective domain Λ and for its closure with respect to uniform convergence, respectively. We also set

$$\mathscr{P} = \bigcup_{\Lambda, \Lambda'} \mathscr{P}(\Lambda, \Lambda'), \quad \overline{\mathscr{P}} = \bigcup_{\Lambda, \Lambda'} \overline{\mathscr{P}}(\Lambda, \Lambda').$$

When we need to specify the vector space $N_{\mathbb{R}}$ we will denote it as a subindex as in $\mathscr{P}_{N_{\mathbb{R}}}$ or $\overline{\mathscr{P}}_{N_{\mathbb{R}}}$.

The following propositions contain the basic properties of the Legendre-Fenchel duality acting on $\overline{\mathscr{P}}$. The elements in $\overline{\mathscr{P}}$ are continuous functions on polyhedra. In particular, they are closed concave functions. Observe that when working with uniform limits of piecewise affine concave functions, the technical issues in §3.2 disappear.

Proposition 3.77. *The concave piecewise affine functions and their uniform limits satisfy the following properties.*

1. *Let $f \in \overline{\mathscr{P}}_{N_{\mathbb{R}}}$. Then $f^{\vee\vee} = f$.*
2. *If $f \in \mathscr{P}(\Lambda, \Lambda')$ (respectively $f \in \overline{\mathscr{P}}(\Lambda, \Lambda')$) then $f^\vee \in \mathscr{P}(\Lambda', \Lambda)$ (respectively $f^\vee \in \overline{\mathscr{P}}(\Lambda', \Lambda)$).*

(3) *If* $f \in \overline{\mathscr{P}}(\Lambda)$ *then* $\mathrm{dom}(\mathrm{rec}(f)) = \mathrm{rec}(\Lambda)$.
(4) *Let* $f_i \in \mathscr{P}(\Lambda_i, \Lambda_i')$ *(respectively* $f_i \in \overline{\mathscr{P}}(\Lambda_i, \Lambda_i')$*),* $i = 1, 2$*, with* $\Lambda_1 \cap \Lambda_2 \neq \emptyset$*. Then* $f_1 + f_2 \in \mathscr{P}(\Lambda_1 \cap \Lambda_2, \Lambda_1' + \Lambda_2')$ *(respectively* $f_1 + f_2 \in \overline{\mathscr{P}}(\Lambda_1 \cap \Lambda_2, \Lambda_1' + \Lambda_2')$*) and* $(f_1 + f_2)^\vee = f_1^\vee \boxplus f_2^\vee$*.*
(5) *Let* $f_i \in \mathscr{P}(\Lambda_i, \Lambda_i')$ *(respectively* $f_i \in \overline{\mathscr{P}}(\Lambda_i, \Lambda_i')$*),* $i = 1, 2$*, with* $\Lambda_1' \cap \Lambda_2' \neq \emptyset$*. Then* $f_1 \boxplus f_2 \in \mathscr{P}(\Lambda_1 + \Lambda_2, \Lambda_1' \cap \Lambda_2')$ *(respectively* $f_1 + f_2 \in \overline{\mathscr{P}}(\Lambda_1 + \Lambda_2, \Lambda_1' \cap \Lambda_2')$*) and* $(f_1 \boxplus f_2)^\vee = f_1^\vee + f_2^\vee$*.*
(6) *Let* $(f_i)_{i \geq 1} \subset \overline{\mathscr{P}}$ *be a sequence converging uniformly to a function* f*. Then* $f \in \overline{\mathscr{P}}$*.*

Proof. All the statements follow, either directly from the definition, or propositions 3.64 and 3.18. □

Proposition 3.78. *Let* $A\colon Q_\mathbb{R} \to N_\mathbb{R}$ *be an affine map defined as* $A = H + u_0$ *for a linear map* H *and a point* $u_0 \in N_\mathbb{R}$*. Let* $f \in \mathscr{P}_{N_\mathbb{R}}$ *(respectively* $f \in \overline{\mathscr{P}}_{N_\mathbb{R}}$*) with* $\mathrm{dom}(f) \cap \mathrm{im}(A) \neq \emptyset$ *and* $g \in \mathscr{P}_{Q_\mathbb{R}}$ *(respectively* $g \in \overline{\mathscr{P}}_{Q_\mathbb{R}}$*) such that* $\mathrm{stab}(g) \cap \mathrm{im}(H^\vee) \neq \emptyset$*. Then* $A^*f \in \mathscr{P}_{Q_\mathbb{R}}$ *(respectively* $A^*f \in \overline{\mathscr{P}}_{Q_\mathbb{R}}$*) and* $A_*g \in \mathscr{P}_{N_\mathbb{R}}$ *(respectively* $A_*g \in \overline{\mathscr{P}}_{N_\mathbb{R}}$*). Moreover,*
(1) $\mathrm{stab}(A^*f) = H^\vee(\mathrm{stab}(f))$*,* $(A^*f)^\vee = (H^\vee)_*(f^\vee - u_0)$ *and, for all* $y \in \mathrm{stab}(A^*f)$*,*
$$(A^*f)^\vee(y) = \max_{x \in (H^\vee)^{-1}(y)} (f^\vee(x) - \langle x, u_0 \rangle);$$
(2) $\mathrm{stab}(A_*g) = (H^\vee)^{-1}(\mathrm{stab}(g))$*,* $(A_*g)^\vee = (H^\vee)^*(g^\vee) + u_0$ *and, for all* $u \in \mathrm{dom}(A_*g)$*,*
$$A_*g(u) = \max_{v \in A^{-1}(u)} g(v).$$

Proof. These statements follow either from Proposition 3.46 or from [Roc70, Corollary 19.3.1]. □

We will be concerned mainly with functions in $\overline{\mathscr{P}}$ whose effective domain is either a polytope or the whole space $N_\mathbb{R}$. These are the kind of functions that arise when considering proper toric varieties. The functions in $\mathscr{P}(N_\mathbb{R})$ can be realized as the inverse image of the support function of the standard simplex, while the functions of $\mathscr{P}(\Delta)$ can be realized as direct images of the indicator function of the standard simplex.

Lemma 3.79. *Let* $f \in \mathscr{P}(N_\mathbb{R})$ *and let* $f(u) = \min_{0 \leq i \leq r}(a_i(u) + \alpha_i)$ *be an H-representation of* f*. Write* $\boldsymbol{\alpha} = (\alpha_i - \alpha_0)_{i=1,\dots,r}$*, and consider the linear map* $H\colon N_\mathbb{R} \to \mathbb{R}^r$ *given by* $H(u) = (a_i(u) - a_0(u))_{i=1,\dots,r}$ *and the affine map* $A = H + \boldsymbol{\alpha}$*. Then*
(1) $f = A^*\Psi_{\Delta^r} + a_0 + \alpha_0$;
(2) $f^\vee = \tau_{a_0}(H^\vee)_*(\iota_{\Delta^r} - \boldsymbol{\alpha}) - \alpha_0$.

This second function can be alternatively described as the function which parameterizes the upper envelope of the extended polytope
$$\mathrm{conv}((a_1, -\alpha_1), \dots, (a_l, -\alpha_l)) \subset M_\mathbb{R} \times \mathbb{R}.$$

Proof. Statement (1) follows from the explicit description of Ψ_{Δ^r} in Example 3.70. Statement (2) follows from Proposition 3.78. The last statement is a consequence of Proposition 3.64. □

The next proposition characterizes the elements of $\overline{\mathscr{P}}(N_\mathbb{R})$ and $\overline{\mathscr{P}}(\Delta)$ for a polytope Δ.

Proposition 3.80. *Let* Δ *be a convex polytope of* $M_\mathbb{R}$*.*

(1) *The space* $\overline{\mathscr{P}}(\Delta, N_{\mathbb{R}})$ *agrees with the space of all continuous concave functions on* Δ.
(2) *A concave function* f *belongs to* $\overline{\mathscr{P}}(N_{\mathbb{R}}, \Delta)$ *if and only if* $\operatorname{dom}(f) = N_{\mathbb{R}}$ *and* $|f - \Psi_{\Delta}|$ *is bounded.*

Proof. We start by proving (1). By the properties of uniform convergence, it is clear that any element of $\overline{\mathscr{P}}(\Delta, N_{\mathbb{R}})$ is concave and continuous. Conversely, a continuous function f on Δ is uniformly continuous because Δ is compact. Therefore, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in \Delta$ such that $\|u - v\| < \delta$. By compactness, we can find a triangulation $\Delta = \bigcup_i \Delta_i$ with $\operatorname{diam}(\Delta_i) < \delta$. Let $\{b_j\}_j$ be the vertices of this triangulation and consider the function $g \in \mathscr{P}(\Delta, N_{\mathbb{R}})$ defined as

$$g(u) = \sup\left\{ \sum_{j=1}^{l} \lambda_j f(b_j) \,\middle|\, \lambda_j \geq 0, \sum_j \lambda_j = 1 \sum_j \lambda_j a_j = x \right\}.$$

For $u \in \Delta$, let $b_{j_0}, \dots, b_{j_n}$ denote the vertices of an element of the triangulation containing u. We write $u = \lambda_{j_0} u_{j_0} + \dots + \lambda_{j_n} u_{j_n}$ for some $\lambda_{j_i} \geq 0$ and $\lambda_{j_0} + \dots + \lambda_{j_n} = 1$. By concavity, we have

$$f(u) \geq g(u) \geq \sum_{k=0}^{n} \lambda_{j_k} f(u_{j_k}) \geq f(u) - \varepsilon,$$

which shows that any continuous function on Δ can be arbitrarily approximated by elements of $\mathscr{P}(\Delta, N_{\mathbb{R}})$.

We now prove (2). Let $f \in \overline{\mathscr{P}}(N_{\mathbb{R}}, \Delta)$. By definition, for each $\varepsilon > 0$ we can find a function $g \in \mathscr{P}(N_{\mathbb{R}}, \Delta)$ with $\sup |f - g| \leq \varepsilon$. In particular, $|f - g|$ is bounded. Furthermore, $\operatorname{rec}(g) = \Psi_{\Delta}$ and $|g - \operatorname{rec}(g)|$ is bounded because $g \in \mathscr{P}(N_{\mathbb{R}})$. Hence $\operatorname{dom}(f) = \operatorname{dom}(g) = N_{\mathbb{R}}$ and $|f - \Psi_{\Delta}|$ is bounded.

Conversely, let f be a concave function such that $\operatorname{dom}(f) = N_{\mathbb{R}}$ and $|f - \Psi_{\Delta}|$ is bounded. Then $\operatorname{stab}(f) = \operatorname{stab}(\Psi_{\Delta}) = \Delta$ and $f^{\vee}$ is a continuous concave function on Δ. Hence we can apply (1) to $f^{\vee}$ to obtain functions $g_i \in \mathscr{P}(\Delta, N_{\mathbb{R}})$ approaching $f^{\vee}$ uniformly. We conclude that the functions $g_i^{\vee} \in \mathscr{P}(N_{\mathbb{R}}, \Delta)$ approach f uniformly and so $f \in \overline{\mathscr{P}}(N_{\mathbb{R}}, \Delta)$. □

Proposition 3.81. *Let* Δ *be a lattice polytope of* $M_{\mathbb{R}}$. *Then the subset of rational piecewise affine concave functions in* $\overline{\mathscr{P}}(\Delta, N_{\mathbb{R}})$ *(respectively, in* $\overline{\mathscr{P}}(N_{\mathbb{R}}, \Delta)$*) is dense with respect to uniform convergence.*

Proof. This follows from Proposition 3.80 and the density of rational numbers. □

3.6. **Differences of concave functions.** Let $C \subset N_{\mathbb{R}}$ be a convex set. A function $f\colon C \to \mathbb{R}$ is called a *difference of concave functions* or a *DC function* if it can be written as $f = g - h$ for concave functions $g, h\colon C \to \mathbb{R}$. DC functions play an important role in non-convex optimization and have been widely studied, see for instance [HT99] and the references therein. We will be interested in a subclass of DC functions, namely those which are a difference of uniform limits of piecewise affine concave functions.

Definition 3.82. For a convex polyhedron Λ in $N_{\mathbb{R}}$ we set

$$\mathscr{D}(\Lambda) = \{g - h \mid g, h \in \mathscr{P}(\Lambda)\}, \quad \overline{\mathscr{D}}(\Lambda) = \{g - h \mid g, h \in \overline{\mathscr{P}}(\Lambda)\}.$$

These spaces are closed under the operations of taking finite linear combinations, upper envelope and lower envelope.

Proposition 3.83. *Let* Λ *be a convex polyhedron in* $N_{\mathbb{R}}$ *and* $f_1, \dots, f_l$ *functions in* $\mathscr{D}(\Lambda)$ *(respectively, in* $\overline{\mathscr{D}}(\Lambda)$*). Then the functions*

(1) $\sum_i \lambda_i f_i$ *for any* $\lambda_i \in \mathbb{R}$,
(2) $\max_i\{f_i\}$, $\min_i\{f_i\}$

are also in $\mathscr{D}(\Lambda)$ *(respectively, in* $\overline{\mathscr{D}}(\Lambda)$*).*

Proof. Statement (1) is obvious. For the statement (2), write $f_i = g_i - h_i$ with g_i, h_i in $\mathscr{P}(\Lambda)$ (respectively, in $\overline{\mathscr{P}}(\Lambda)$). Then the upper envelope admits the DC decomposition $\max_i\{f_i\} = g - h$ with

$$g := \sum_j g_j, \quad h := \min_i \Big(h_i + \sum_{j\neq i} g_j\Big),$$

which are both concave functions in $\mathscr{P}(\Lambda)$ (respectively, in $\overline{\mathscr{P}}(\Lambda)$). This shows that $\max_i\{f_i\}$ is in $\mathscr{D}(\Lambda)$ (respectively, in $\overline{\mathscr{D}}(\Lambda)$). The statement for the lower envelope follows similarly. □

In particular, if f lies in $\mathscr{D}(\Lambda)$ or in $\overline{\mathscr{D}}(\Lambda)$, the same holds for the functions $|f|$, $\max(f, 0)$ and $\min(f, 0)$.

Corollary 3.84. *The space* $\mathscr{D}(\Lambda)$ *coincides with the space of piecewise affine functions on* Λ*.*

Proof. This follows from the max-min representation of piecewise affine functions in [Ovc02] and Proposition 3.83(2). □

Some constructions for concave functions can be extended to this kind of functions. In particular, we can define the recession of a functions in $\overline{\mathscr{D}}(\Lambda)$.

Definition 3.85. Let Λ be a polyhedron in $N_{\mathbb{R}}$ and $f \in \overline{\mathscr{D}}(\Lambda)$. The *recession function* of f is defined as

$$\operatorname{rec}(f)\colon \operatorname{rec}(\Lambda) \longrightarrow \mathbb{R}, \quad u \longmapsto \lim_{\lambda\to\infty} \frac{f(v_0+\lambda u) - f(v_0)}{\lambda} \tag{3.86}$$

for any $v_0 \in \Lambda$.

Write $f = g - h$ for any $g, h \in \overline{\mathscr{P}}(\Lambda)$. By (3.49), we have that, for all $u \in \operatorname{rec}(\Lambda)$, the limit (3.86) exists and

$$\operatorname{rec}(f)(u) = \operatorname{rec}(g)(u) - \operatorname{rec}(h)(u).$$

Observe that the recession function of a function in $\mathscr{D}(\Lambda)$ is a piecewise linear function on a subdivision of the cone $\operatorname{rec}(\Lambda)$ into polyhedral cones. Observe also that

$$|f - \operatorname{rec}(f)| \leq |g - \operatorname{rec}(g)| + |h - \operatorname{rec}(h)| = O(1).$$

We will be mostly interested in the case when $\Lambda = N_{\mathbb{R}}$.

Proposition 3.87. *Let* $\|\cdot\|$ *be any metric on* $N_{\mathbb{R}}$ *and* $f \in \overline{\mathscr{D}}(N_{\mathbb{R}})$*. Then there exists a constant* $\kappa > 0$ *such that, for all* $u, v \in N_{\mathbb{R}}$,

$$|f(u) - f(v)| \leq \kappa\|u - v\|.$$

A function which verifies the conclusion of this proposition is called *Lipchitzian.*

Proof. Let $f = g - h$ with $g, h \in \overline{\mathscr{P}}(N_{\mathbb{R}})$. The effective domain of the recessions of g and of h is the whole of $N_{\mathbb{R}}$. By [Roc70, Theorem 10.5], both g and h are Lipchitzians, hence so is f. □

Observe that $\overline{\mathscr{D}}(N_{\mathbb{R}})$ is *not* the completion of $\mathscr{D}(N_{\mathbb{R}})$ with respect to uniform convergence. It is easy to construct functions which are uniform limits of piecewise affine ones but do not verify the Lipschitz condition.

We will consider the integral and rational structures on the space of piecewise affine functions. We will use the notation previous to Definition 3.73.

Definition 3.88. Let Λ be a convex polyhedron and $f \in \mathscr{D}(\Lambda)$. We say that f is an *H-lattice* (respectively *V-lattice*) function if it can be written as the difference of two H-lattice (respectively V-lattice) concave functions. We say that f is a *rational piecewise affine* function if it is the difference of two rational piecewise affine concave functions.

Proposition 3.89. *If f is an H-lattice function (respectively a rational piecewise affine function) on $N_{\mathbb{R}}$, then there is a complete polyhedral complex Π in $N_{\mathbb{R}}$ such that, for every $\Lambda \in \Pi$,*

$$f|_{\Lambda}(u) = \langle m_{\Lambda}, u\rangle + l_{\Lambda},$$

with $(m_{\Lambda}, l_{\Lambda}) \in M \times \mathbb{Z}$ (respectively $(m_{\Lambda}, l_{\Lambda}) \in M_{\mathbb{Q}} \times \mathbb{Q}$). Conversely, every piecewise affine function on $N_{\mathbb{R}}$ such that its defining affine functions have integral (respectively rational) coefficients, is an H-lattice function (respectively a rational piecewise affine function).

Proof. We will prove the statement for lattice functions. The statement for rational piecewise affine functions is proved with the same argument. If f is an H-lattice function, we can write $f = g - h$, where g and h are H-lattice concave functions. We obtain Π as any common refinement of $\Pi(g)$ and $\Pi(h)$ to a polyhedral complex. Then the statement follows from the definition of H-lattice concave functions. The converse is an easy consequence of Corollary 3.84. □

Definition 3.90. Let f be a rational piecewise affine function on $N_{\mathbb{R}}$, and let Π and $\{(m_{\Lambda}, l_{\Lambda})\}_{\Lambda\in\Pi}$ be as in Proposition 3.89. The family $\{(m_{\Lambda}, l_{\Lambda})\}_{\Lambda\in\Pi}$ is called a set of *defining vectors* of f.

Proposition 3.91. *Let Π be a complete SCR polyhedral complex in $N_{\mathbb{R}}$ and f an H-lattice function on Π. Then $\operatorname{rec}(f)$ is a conic H-lattice function on the fan $\operatorname{rec}(\Pi)$.*

Proof. Let $\Lambda \in \Pi$ and $(m, l) \in M \times \mathbb{Z}$ such that $f(u) = \langle m, u\rangle + l$ for $u \in \Lambda$. Then, by the definition of $\operatorname{rec}(f)$, it is clear that $\operatorname{rec}(f)|_{\operatorname{rec}(\Lambda)}(u) = \langle m, u\rangle$. Hence, $\operatorname{rec}(f)$ is a conic H-lattice function on $\operatorname{rec}(\Pi)$. □

3.7. **Monge-Ampère measures.** Let $f\colon C \to \mathbb{R}$ be a concave function of class $\mathcal{C}^2$ on an open convex set $C \subset \mathbb{R}^n$. Its Hessian matrix

$$\operatorname{Hess}(f)(u) := \left(\frac{\partial^2 f}{\partial u_i \partial u_j}(u)\right)_{1\le i,j\le n}$$

is a non-positive definite matrix which quantifies the curvature of f at the point u. The real Monge-Ampère operator is defined as $(-1)^n$ times the determinant of this matrix. This notion can be extended as a measure to the case of an arbitrary concave function. A good reference for Monge-Ampère measures is [RT77].

Let μ be a Haar measure of $M_{\mathbb{R}}$. Assume that we choose linear coordinates $(x_1, \dots, x_n)$ of $M_{\mathbb{R}}$ such that μ is the measure associated to the differential form $\omega = \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ and the orientation of $M_{\mathbb{R}}$ defined by this system of coordinates. Let $(u_1, \dots, u_n)$ be the dual coordinates of $N_{\mathbb{R}}$.

Definition 3.92. Let f be a concave function on $N_{\mathbb{R}}$. The *real Monge-Ampère measure* of f with respect to μ is defined, for a Borel subset E of $N_{\mathbb{R}}$, as

$$\mathcal{M}_{\mu}(f)(E) = \mu(\partial f(E)).$$

It is a measure with support contained in $\operatorname{dom}(\partial f)$. The correspondence $f \mapsto \mathcal{M}_{\mu}(f)$ is called the *Monge-Ampère operator*.

When the measure μ is clear from the context, we will drop it from the notation. Moreover, since we are not going to consider complex Monge-Ampère measures, we will simply call $\mathcal{M}_{\mu}(f)$ the Monge-Ampère measure of f.

The total mass of $\mathcal{M}_\mu(f)$ is equal to $\mu(\mathrm{stab}(f))$. In particular, when $\mathrm{stab}(f)$ is bounded, $\mathcal{M}_\mu(f)$ is a finite measure.

Proposition 3.93. *The Monge-Ampère measure is a continuous map from the space of concave functions with the topology defined by uniform convergence on compact sets to the space of σ-finite measures on $N_\mathbb{R}$ with the weak topology.*

Proof. This is proved in [RT77, §3]. □

The two basic examples of Monge-Ampère measures that we are interested in are the ones associated to smooth functions and the ones associated to piecewise linear functions.

Proposition 3.94. *Let C be an open convex set in $N_\mathbb{R}$ and $f \in \mathcal{C}^2(C)$ a concave function. Then*

$$\mathcal{M}_\mu(f) = (-1)^n \det(\mathrm{Hess}(f))\, \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n,$$

where the Hessian matrix is calculated with respect to the coordinates $(u_1, \ldots, u_n)$.

Proof. This is [RT77, Proposition 3.4] □

By contrast, the Monge-Ampère measure of a piecewise affine concave function, is a discrete measure supported on the vertices of a polyhedral complex.

Proposition 3.95. *Let f be a piecewise affine concave function on $N_\mathbb{R}$ and $(\Pi(f), \Pi(f^\vee))$ the dual pair of polyhedral complexes associated to f. Denote by $\Lambda \mapsto \Lambda^*$ the correspondence $\mathcal{L}f$. Then*

$$\mathcal{M}_\mu(f) = \sum_{v \in \Pi(f)^0} \mu(\partial f(v))\delta_v = \sum_{v \in \Pi(f)^0} \mu(v^*)\delta_v = \sum_{\Lambda \in \Pi(f^\vee)^n} \mu(\Lambda)\delta_{\Lambda^*},$$

where δ_v is the Dirac measure supported on v.

Proof. This follows easily from the definition of $\mathcal{M}(f)$ and the properties of the Legendre correspondence of piecewise affine functions. □

Example 3.96. Let $\Delta \subset M_\mathbb{R}$ be a polytope and Ψ_Δ its support function. Then

$$\mathcal{M}_\mu(\Psi_\Delta) = \mu(\Delta)\delta_0.$$

The following relation between Monge-Ampère measure and Legendre-Fenchel duality is one of the key ingredients in the computation of the height of a toric variety. We will consider the $(n-1)$-differential form on $N_\mathbb{R}$

$$\lambda = \sum_{i=1}^n (-1)^{i-1} x_i\, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

It satisfies $\mathrm{d}\lambda = n\omega$.

Theorem 3.97. *Let $f\colon N_\mathbb{R} \to \mathbb{R}$ be a closed concave function, such that $D :=$ $\mathrm{stab}(f)$ is a compact convex set with piecewise smooth boundary ∂D. Then*

$$-\, n! \int_{N_\mathbb{R}} f \mathcal{M}_\mu(f) = (n+1)! \int_D f^\vee \, \mathrm{d}\mu - n! \int_{\partial D} f^\vee \lambda. \tag{3.98}$$

Proof. If the measure of D is zero then both sides of equation (3.98) are zero. Therefore, the theorem is trivially true in this case. Thus, we may assume that D has non-empty interior. Since $\mathrm{stab}(f)$ is compact, the right-hand side of (3.98) is continuous with respect to uniform convergence of functions, thanks to Proposition 3.18. Moreover, Proposition 3.93 and the fact that $\mathcal{M}_\mu(f)$ is finite imply that the left-hand side is also continuous with respect to uniform convergence. By the compacity of D, we can find a sequence of strictly concave smooth functions $(f_n)_{n\geq 1}$

that converges uniformly to f. Hence, we may assume that f is smooth and strictly concave. In this case, the Legendre transform $\nabla f\colon N_{\mathbb{R}} \to D^{\circ}$ is a diffeomeorphism.

By the definition of the Monge-Ampère measure,

$$-n!\int_{N_{\mathbb{R}}} f\mathcal{M}_{\mu}(f) = -n!\int_{D} f((\nabla f)^{-1}x)\,\mathrm{d}\mu(x), \tag{3.99}$$

which, in particular, shows that the integral on the left is convergent for smooth strictly concave functions with compact stability set. Therefore, it is convergent for any concave function within the hypothesis of the theorem.

By the properties of the Legendre transform,

$$-f((\nabla f)^{-1}(x)) = f^{\vee}(x) - \langle(\nabla f)^{-1}(x), x\rangle. \tag{3.100}$$

Moreover,

$$\begin{aligned} \mathrm{d}(f^{\vee}\lambda)(x) &= \mathrm{d}f^{\vee}\wedge\lambda(x) + f^{\vee}\,\mathrm{d}\lambda(x) \\ &= \langle\nabla f^{\vee}(x), x\rangle\omega + nf^{\vee}\omega \\ &= \langle(\nabla f)^{-1}(x), x\rangle\omega + nf^{\vee}\omega \end{aligned} \tag{3.101}$$

The result is obtained by combining equations (3.99), (3.100) and (3.101) with Stokes' theorem. □

We now particularize Theorem 3.97 to the case when the Haar measure comes from a lattice and the convex set is a lattice polytope of maximal dimension.

Definition 3.102. Let L be a lattice and set $L_{\mathbb{R}} = L\otimes\mathbb{R}$. We denote by vol_L the Haar measure on $L_{\mathbb{R}}$ normalized so that L has covolume 1.

Let N be a lattice of $N_{\mathbb{R}}$ and set $M = N^{\vee}$ for its dual lattice. For a concave function f, we denote by $\mathcal{M}_M(f)$ the Monge-Ampère measure with respect to the normalized Haar measure vol_M.

Notation 3.103. Let Λ be a rational polyhedron in $M_{\mathbb{R}}$ and $\mathrm{aff}(\Lambda)$ its affine hull. We denote by L_{Λ} the linear subspace of $M_{\mathbb{R}}$ associated to $\mathrm{aff}(\Lambda)$ and by $M(\Lambda)$ the induced lattice $M\cap L_{\Lambda}$. By definition, $\mathrm{vol}_{M(\Lambda)}$ is a measure on L_{Λ}, and we will denote also by $\mathrm{vol}_{M(\Lambda)}$ the measure induced on $\mathrm{aff}(\Lambda)$. If $v\in N_{\mathbb{R}}$ is orthogonal to L_{Λ}, we define $\langle v,\Lambda\rangle = \langle v,x\rangle$ for any $x\in\Lambda$. Furthermore, when $\dim(\Lambda) = n$ and F is a facet of Λ, we will denote by $v_F\in N$ the vector of minimal length that is orthogonal to L_F and satisfies $\langle v_F, F\rangle \le \langle v_F, x\rangle$ for each $x\in\Lambda$. In other words, v_F is the minimal inner integral orthogonal vector of F as a facet of Λ.

Corollary 3.104. *Let f be a concave function on $N_{\mathbb{R}}$ such that $\Delta = \mathrm{stab}(f)$ is a lattice polytope of dimension n. Then*

$$-n!\int_{N_{\mathbb{R}}} f\mathcal{M}_M(f) = (n+1)!\int_{\Delta} f^{\vee}\,\mathrm{d}\,\mathrm{vol}_M + \sum_F \langle v_F, F\rangle n!\int_F f^{\vee}\,\mathrm{d}\,\mathrm{vol}_{M(F)},$$

where the sum is over the facets F of Δ.

Proof. We choose $(m_1,\dots,m_n)$ a basis of M such that $(m_2,\dots,m_n)$ is a basis of $M(F)$ and m_1 points to the exterior direction. Expressing λ in this basis we obtain

$$\lambda|_F = -\langle v_F, F\rangle\,\mathrm{d}\,\mathrm{vol}_{M(F)}\,.$$

The result then follows from Theorem 3.97. □

In §6, we will see that we can express the height of a toric variety in terms of integrals of the form $\int_{\Delta} f^{\vee}\,\mathrm{d}\,\mathrm{vol}_M$ as in the above result. In some situations, it will be useful to translate those integrals to integrals on $N_{\mathbb{R}}$.

Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a concave function and $g\colon \operatorname{stab}(f) \to \mathbb{R}$ an integrable function. We consider the signed measure on $N_{\mathbb{R}}$ defined, for a Borel subset E of $N_{\mathbb{R}}$, as

$$\mathcal{M}_{M,g}(f)(E) = \int_{\partial f(E)} g \,\mathrm{d}\operatorname{vol}_M .$$

Clearly, $\mathcal{M}_{M,g}(f)$ is uniformly continuous with respect to $\mathcal{M}_M(f)$. By the Radon-Nicodym theorem, there is a $\mathcal{M}_M(f)$-measurable function, that we denote $g\circ \partial f$, such that

$$\int_E g\circ\partial f\, \mathcal{M}_M(f) = \int_E \mathcal{M}_{M,g}(f) = \int_{\partial f(E)} g\,\mathrm{d}\operatorname{vol}_M . \tag{3.105}$$

Example 3.106. When the function f is differentiable or piecewise affine, the measurable function $f^{\vee}\circ\partial f$ can be made explicit.

(1) Let $f\in \mathcal{C}^2(N_{\mathbb{R}})$. Proposition 3.94 and the change of variables formula imply $g\circ\partial f = g\circ\nabla f$. For the particular case when $g=f^{\vee}$, Theorem 3.52(4) implies, for $u\in N_{\mathbb{R}}$,

$$f^{\vee}\circ\partial f(u) = \langle \nabla f(u), u\rangle - f(u).$$

(2) Let f a piecewise affine concave function on $N_{\mathbb{R}}$. By Proposition 3.95, $\mathcal{M}_M(f)$ is supported in the finite set $\Pi(f)^0$ and so is $\mathcal{M}_{M,g}(f)$. For $v\in \Pi(f)^0$ write $v^*\in \Pi(f^{\vee})^n$ for the dual polyhedron. Then $g\circ\partial f(v) = \frac{1}{\operatorname{vol}_M(v^*)}\int_{v^*} g\,\mathrm{d}\operatorname{vol}_M$, which implies

$$f^{\vee}\circ\partial f(v) = \frac{1}{\operatorname{vol}_M(v^*)}\int_{v^*}\langle x,v\rangle\,\mathrm{d}\operatorname{vol}_M - f(v).$$

The function $f^{\vee}\circ\partial f$ is defined as a $\mathcal{M}_M(f)$-measurable function. Therefore, only its values at the points $v\in\Pi(f)^0$ are well defined. Nevertheless, we can extend the function $f^{\vee}\circ\partial f$ to the whole $N_{\mathbb{R}}$ by writing

$$f^{\vee}\circ\partial f(u) = \frac{1}{\operatorname{vol}_{\mu}(\partial f(u))}\int_{\partial f(u)}\langle x,u\rangle\,\mathrm{d}\mu - f(u)$$

for any Haar measure μ on the affine space determined by $\partial f(u)$.

The Monge-Ampère operator is homogeneous of degree n. It can be turned into a multi-linear operator which takes n concave functions as arguments.

Definition 3.107. Let $f_1,\dots,f_n$ be concave functions on $N_{\mathbb{R}}$. The *mixed Monge-Ampère measure* is defined by the formula

$$\mathcal{M}_M(f_1,\dots,f_n) = \frac{1}{n!}\sum_{j=1}^{n}(-1)^{n-j}\sum_{1\le i_1<\dots<i_j\le n}\mathcal{M}_M(f_{i_1}+\dots+f_{i_j}).$$

It is a measure on $N_{\mathbb{R}}$.

This operator was introduced by Passare and Rullgård [PR04]. It is multi-linear and symmetric in the variables f_i.

Proposition 3.108. *The mixed Monge-Ampère measure is a continuous map from the space of n-tuples of concave functions with the topology defined by uniform convergence on compact sets to the space of σ-finite measures on $N_{\mathbb{R}}$ with the weak topology.*

Proof. The general mixed case reduces to the unmixed case $f_1=\dots=f_n$, which is Proposition 3.93. □

Definition 3.109. The *mixed volume* of a family of compact convex sets $Q_1, \dots, Q_n$ of $M_{\mathbb{R}}$ is defined as

$$\mathrm{MV}_M(Q_1, \dots, Q_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \leq i_1 < \dots < i_j \leq n} \mathrm{vol}_M(Q_{i_1} + \dots + Q_{i_j}) \tag{3.110}$$

Since $\mathrm{MV}_M(Q, \dots, Q) = n! \, \mathrm{vol}_M(Q)$, the mixed volume is a generalization of the volume of a convex body. The mixed volume is symmetric and linear in each variable Q_i with respect to the Minkowski sum, and monotone with respect to inclusion [Ewa96, Chapter IV].

The next result generalizes [PR04, Proposition 3] and shows that the mixed Monge-Ampère measure can be defined in terms of mixed volumes if the effective domains of the functions overlap sufficiently.

Proposition 3.111. *Let $f_1, \dots, f_n$ be concave functions such that* $\mathrm{ri}(\mathrm{dom}(f_1)) \cap \dots \cap \mathrm{ri}(\mathrm{dom}(f_n)) \neq \emptyset$ *and $E \subset N_{\mathbb{R}}$ a Borel subset. Then*

$$\mathcal{M}_M(f_1, \dots, f_n)(E) = \frac{1}{n!} \mathrm{MV}_M(\partial f_1(E), \dots, \partial f_n(E)).$$

If $f_1, \dots, f_k$ are piecewise affine, this formula holds under the weaker hypothesis $\mathrm{dom}(f_1) \cap \dots \cap \mathrm{dom}(f_k) \cap \mathrm{ri}(\mathrm{dom}(f_{k+1})) \cap \dots \cap \mathrm{ri}(\mathrm{dom}(f_n)) \neq \emptyset$.

Proof. This follows from Proposition 3.43 and the definition of the mixed Monge-Ampère measures and of mixed volumes. □

In particular, this gives the total mass of the mixed Monge-Ampère measure.

Corollary 3.112. *In the setting of Proposition 3.111, we have*

$$\mathcal{M}_M(f_1, \dots, f_n)(N_{\mathbb{R}}) = \frac{1}{n!} \mathrm{MV}_M(\mathrm{stab}(f_1), \dots, \mathrm{stab}(f_n)).$$

Proof. This follows readily from the above proposition and (3.22). □

Following [PS08a], we introduce an extension of the notion of integral of a concave function.

Definition 3.113. Let Q_i, $i = 0, \dots, n$, be a family of compact convex subset of $M_{\mathbb{R}}$ and $g_i \colon Q_i \to \mathbb{R}$ a concave function on Q_i. The *mixed integral of $g_0, \dots, g_n$* is defined as

$$\mathrm{MI}_M(g_0, \dots, g_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \leq i_0 < \dots < i_j \leq n} \int_{Q_{i_0} + \dots + Q_{i_j}} g_{i_0} \boxplus \dots \boxplus g_{i_j} \, \mathrm{d}\,\mathrm{vol}_M .$$

For a compact convex subset $Q \subset M_{\mathbb{R}}$ and a concave function g on Q, we have $\mathrm{MI}_M(g, \dots, g) = (n+1)! \int_Q g \, \mathrm{d}\,\mathrm{vol}_M$. The mixed integral is symmetric and additive in each variable g_i with respect to the sup-convolution. For a scalar $\lambda \in \mathbb{R}_{\geq 0}$, we have $\mathrm{MI}_M(\lambda g_0, \dots, \lambda g_n) = \lambda \, \mathrm{MI}_M(g_0, \dots, g_n)$. We refer to [PS08a, PS08b] for the proofs and more information about this notion.

4. Toric varieties

In this section we recall some basic facts about the algebraic geometry of toric varieties and schemes. In the first place, we consider toric varieties over a field and then toric schemes over a DVR. We refer to [KKMS73, Oda88, Ful93, Ewa96] for more details.

We will use the notations of the previous section concerning concave functions and polyhedra, with the proviso that the vector space $N_{\mathbb{R}}$ will always be equipped with a lattice N and most of the objects we consider will be compatible with this integral structure, even if not said explicitly. In particular, from now on by a *fan* (Definition 3.13) we will mean a rational fan and by a *polytope* we will mean a lattice polytope.

4.1. **Fans and toric varieties.** Let K be a field and $\mathbb{T} \simeq \mathbb{G}_m^n$ a split torus over K. We alternatively denote it by $\mathbb{T}_K$ if we want to refer to its field of definition.

Definition 4.1. A *toric variety* is a normal variety X over K equipped with a dense open embedding $\mathbb{T} \hookrightarrow X$ and an action $\mu\colon \mathbb{T} \times X \to X$ that extends the action of $\mathbb{T}$ on itself by translations. When we want to stress the torus, we will call X a toric variety *with torus* $\mathbb{T}$.

Toric varieties can be described in combinatorial terms as we recall in the sequel. Let $N = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T}) \simeq \mathbb{Z}^n$ be the lattice of one-parameter subgroups of $\mathbb{T}$ and $M = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m) = N^\vee = \operatorname{Hom}(N, \mathbb{Z})$ its dual lattice of characters of $\mathbb{T}$. For a ring R we set $N_R = N \otimes R$ and $M_R = M \otimes R$.

To a fan Σ we associate a toric variety X_Σ over K by gluing together the affine toric varieties corresponding to the cones of the fan. For $\sigma \in \Sigma$, let $\sigma^\vee$ be the dual cone (Definition 3.67) and set

$$M_\sigma = \sigma^\vee \cap M = \{m \in M \mid \langle m, u\rangle \geq 0,\ \forall u \in \sigma\}$$

for the saturated semigroup of its lattice points. We consider the semigroup algebra

$$K[M_\sigma] = \Big\{ \sum_{m \in M_\sigma} \alpha_m \chi^m \Big| \alpha_m \in K, \alpha_m = 0 \text{ for almost all } m \Big\}$$

of formal finite sums of elements of M_σ with the natural ring structure. It is an integrally closed domain of Krull dimension n. We set $X_\sigma = \operatorname{Spec}(K[M_\sigma])$ for the associated affine toric variety. If τ is a face of σ we have that $K[M_\tau]$ is a localization of $K[M_\sigma]$. Hence there is an inclusion of open sets

$$X_\tau = \operatorname{Spec}(K[M_\tau]) \hookrightarrow X_\sigma = \operatorname{Spec}(K[M_\sigma]).$$

For $\sigma, \sigma' \in \Sigma$, the affine toric varieties X_σ, $X_{\sigma'}$ glue together through the open subset $X_{\sigma \cap \sigma'}$ corresponding to their common face. Thus these affine varieties glue together to form the toric variety

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma.$$

This is a normal variety over K of dimension n. When we need to specify the field of definition we will denote it as $X_{\Sigma,K}$. We denote by $\mathcal{O}_{X_\Sigma}$ its structural sheaf and by $\mathcal{K}_{X_\Sigma}$ its sheaf of rational functions. The open subsets $X_\sigma \subset X_\Sigma$ may be denoted by $X_{\Sigma,\sigma}$ when we want to include the ambient toric variety in the notation.

The cone $\{0\}$, that we denote simply by 0, is a face of every cone and its associated affine scheme

$$X_0 = \operatorname{Spec}(K[M])$$

is an open subset of all of the schemes X_σ. This variety is an algebraic group over K canonically isomorphic to $\mathbb{T}$. We identify this variety with $\mathbb{T}$ and call it the *principal open subset* of X_Σ.

For each $\sigma \in \Sigma$, the homomorphism

$$K[M_\sigma] \to K[M] \otimes K[M_\sigma], \quad \chi^m \mapsto \chi^m \otimes \chi^m$$

induces an action of $\mathbb{T}$ on X_σ. This action is compatible with the inclusion of open sets and so it extends to an action on the whole of X_Σ

$$\mu\colon \mathbb{T} \times X_\Sigma \longrightarrow X_\Sigma.$$

Thus we have obtained a toric variety in the sense of Definition 4.1. In fact, all toric varieties are obtained in this way.

Theorem 4.2. *The correspondence $\Sigma \mapsto X_\Sigma$ is a bijection between the set of fans in $N_\mathbb{R}$ and the set of isomorphism classes of toric varieties with torus $\mathbb{T}$.*

Proof. This result is [KKMS73, §I.2, Theorem 6(i)]. □

For each $\sigma \in \Sigma$, the set of K-rational points in X_σ can be identified with the set of semigroup homomorphisms from $(M_\sigma, +)$ to the semigroup $(K, \times) := K^\times \cup \{0\}$. That is,

$$X_\sigma(K) = \mathrm{Hom}_{\mathrm{sg}}(M_\sigma, (K, \times)).$$

In particular, the set of K-rational points of the algebraic torus can be written intrinsically as

$$\mathbb{T}(K) = \mathrm{Hom}_{\mathrm{sg}}(M_0, (K, \times)) = \mathrm{Hom}_{\mathrm{gp}}(M, K^\times) \simeq (K^\times)^n.$$

Every affine toric variety has a distinguished rational point: we will denote by $x_\sigma \in X_\sigma(K) = \mathrm{Hom}_{\mathrm{sg}}(M_\sigma, (K, \times))$ the point given by the semigroup homomorphism

$$M_\sigma \ni m \longmapsto \begin{cases} 1, & \text{if } -m \in M_\sigma, \\ 0, & \text{otherwise.} \end{cases}$$

For instance, the point $x_0 \in X_0 = \mathbb{T}$ is the unit of $\mathbb{T}$.

Most algebro-geometric properties of the toric scheme translate into combinatorial properties of the fan. In particular, X_Σ is proper if and only if the fan is *complete* in the sense that $|\Sigma| = N_\mathbb{R}$. The variety X_Σ is smooth if and only if every cone $\sigma \in \Sigma$ can be written as $\sigma = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_k$ with $v_1, \dots, v_k$ which are part of an integral basis of N.

Example 4.3. Let Σ_{Δ^n} be the fan in Example 3.70. The toric variety $X_{\Sigma_{\Delta^n}}$ is the projective space $\mathbb{P}^n_K$. More generally, to a polytope $\Delta \subset M_\mathbb{R}$ of maximal dimension we can associate a complete toric variety X_{Σ_Δ}, where Σ_Δ is the fan of Example 3.71.

4.2. **Orbits and equivariant morphisms.** The action of the torus induces a decomposition of a toric variety into disjoint orbits. These orbits are in one to one correspondence with the cones of the fan. Let $\sigma \in \Sigma$ and set

$$N(\sigma) = N/(N \cap \mathbb{R}\sigma), \quad M(\sigma) = N(\sigma)^\vee = M \cap \sigma^\perp, \tag{4.4}$$

where $\sigma^\perp$ denotes the orthogonal space to σ. We will denote by $\pi_\sigma \colon N \to N(\sigma)$ the projection of lattices. By abuse of notation, we will also denote by $\pi_\sigma \colon N_\mathbb{R} \to N(\sigma)_\mathbb{R}$ the induced projection of vector spaces.

The orthogonal space $\sigma^\perp$ is the maximal linear space inside $\sigma^\vee$ and $M(\sigma)$ is the maximal subgroup sitting inside the semigroup M_σ. Set

$$O(\sigma) = \mathrm{Spec}(K[M(\sigma)]),$$

which is a torus over K of dimension $n - \dim(\sigma)$. The surjection of rings

$$K[M_\sigma] \longrightarrow K[M(\sigma)], \quad \chi^a \longmapsto \begin{cases} \chi^a, & \text{if } a \in \sigma^\perp, \\ 0, & \text{if } a \notin \sigma^\perp, \end{cases}$$

induces a closed immersion $O(\sigma) \hookrightarrow X_\sigma$. In terms of rational points, the inclusion $O(\sigma)(K) \hookrightarrow X_\sigma(K)$ sends a group homomorphism $\gamma \colon M(\sigma) \to K^\times$ to the semigroup homomorphism $\widetilde{\gamma} \colon M_\sigma \to (K, \times)$ obtained by extending γ by zero. In particular, the distinguished point $x_\sigma \in X_\sigma(K)$ belongs to the image of $O(\sigma)(K)$ by the above inclusion. Composing with the open immersion $X_\sigma \hookrightarrow X_\Sigma$, we identify $O(\sigma)$ with a locally closed subvariety of X_Σ. For instance, the orbit associated to the cone 0 agrees with the principal open subset X_0. In fact, if we consider x_σ as a rational point of X_Σ, then $O(\sigma)$ agrees with the orbit of x_σ by $\mathbb{T}$.

We denote by $V(\sigma)$ the Zariski closure of $O(\sigma)$ with its induced structure of reduced closed subvariety of X_Σ. The subvariety $V(\sigma)$ has a natural structure of toric variety. To see it, we consider the fan on $N(\sigma)_\mathbb{R}$

$$\Sigma(\sigma) := \{\pi_\sigma(\tau) | \tau \supset \sigma\}. \tag{4.5}$$

This fan is called the *star* of σ in Σ. For each $\tau \in \Sigma$ with $\sigma \subset \tau$, set $\overline{\tau} = \pi_\sigma(\tau) \in \Sigma(\sigma)$. Then, $M(\sigma)_{\overline{\tau}} = M(\sigma) \cap M_\tau$. There is a surjection of rings

$$K[M_\tau] \longrightarrow K[M(\sigma)_{\overline{\tau}}], \quad \chi^m \longmapsto \begin{cases} \chi^m, & \text{if } m \in \sigma^\perp, \\ 0, & \text{if } m \notin \sigma^\perp, \end{cases}$$

that defines a closed immersion $X_{\overline{\tau}} \hookrightarrow X_\tau$. These maps glue together to give a closed immersion $\iota_\sigma \colon X_{\Sigma(\sigma)} \hookrightarrow X_\Sigma$.

Proposition 4.6. *The closed immersion ι_σ induces an isomorphism $X_{\Sigma(\sigma)} \simeq V(\sigma)$.*

Proof. Since the image of each $X_{\overline{\tau}}$ contains $O(\sigma)$ as a dense orbit, we deduce the result from the construction of ι_σ. □

In view of this proposition, we will identify $V(\sigma)$ with $X_{\Sigma(\sigma)}$ and consider it a toric variety.

We now discuss more general equivariant morphisms of toric varieties.

Definition 4.7. Let $\mathbb{T}_i \simeq \mathbb{G}_m^{n_i}$, $i = 1, 2$, be split tori over K, and $\rho \colon \mathbb{T}_1 \to \mathbb{T}_2$ a group morphism. Let X_i, $i = 1, 2$, be toric varieties with torus $\mathbb{T}_i$. A morphism $\varphi \colon X_1 \to X_2$ is *ρ-equivariant* if the diagram

$$\begin{tikzcd} \mathbb{T}_1 \times X_1 \arrow[r, "\mu"] \arrow[d, "\rho\times\varphi"'] & X_1 \arrow[d, "\varphi"] \\ \mathbb{T}_2 \times X_2 \arrow[r, "\mu"] & X_2 \end{tikzcd}$$

is commutative. A morphism $\varphi \colon X_1 \to X_2$ is *ρ-toric* if its restriction to $\mathbb{T}_1$ agrees with ρ. We say that φ is *equivariant* or *toric* if it is ρ-equivariant or ρ-toric, respectively, for some ρ.

Toric morphisms are equivariant. Indeed, a morphism is toric if and only if it is equivariant and sends the distinguished point $x_{1,0} \in X_1(K)$ to the distinguished point $x_{2,0} \in X_2(K)$.

The inclusion $V(\sigma) \to X_\Sigma$ is an example of equivariant morphism that is not toric. Moreover, the underlying morphism of tori depends on the choice of a section of the projection $\pi_\sigma \colon N \to N(\sigma)$.

Equivariant morphisms whose image intersects the principal open subset can be characterized in combinatorial terms. Let $\mathbb{T}_i$, $i = 1, 2$, be split tori over K. Put $N_i = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T}_i)$ and let Σ_i be fans in $N_{i,\mathbb{R}}$. Let $H \colon N_1 \to N_2$ be a linear map such that, for every cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subset \sigma_2$, and let $p \in X_{\Sigma_2,0}(K)$ be a rational point. The linear map induces a group homomorphism

$$\rho_H \colon \mathbb{T}_1 \to \mathbb{T}_2.$$

Let $\sigma_i \in \Sigma_i$, $i = 1, 2$, be cones such that $H(\sigma_1) \subset \sigma_2$. Let $H^\vee \colon M_2 \to M_1$ be the map dual to H. Then there is a homomorphism of semigroups $M_{2,\sigma_2} \to M_{1,\sigma_1}$ which we also denote by $H^\vee$. For a monomial $\chi^m \in K[M_{2,\sigma_2}]$ we denote by $\chi^{H^\vee m}$ its image in $K[M_{1,\sigma_1}]$. The assignment $\chi^m \mapsto \chi^m(p)\chi^{H^\vee m}$ induces morphisms of algebras $K[M_{2,\sigma_2}] \to K[M_{1,\sigma_1}]$, that, in turn, induce morphisms

$$X_{\sigma_1} = \operatorname{Spec}(K[M_{1,\sigma_1}]) \longrightarrow X_{\sigma_2} = \operatorname{Spec}(K[M_{2,\sigma_2}]).$$

These morphisms are compatible with the restriction to open subsets, and they glue together into a ρ_H-equivariant morphism

$$\varphi_{p,H}\colon X_{\Sigma_1} \longrightarrow X_{\Sigma_2}. \tag{4.8}$$

In case $p = x_{2,0}$, the distinguished point on the principal open subset of X_{Σ_2}, this morphism is a toric morphism and will be denoted as φ_H for short.

Theorem 4.9. *Let $\mathbb{T}_i$, N_i, and Σ_i, $i = 1, 2$, be as above. Then the correspondence $(p, H) \mapsto \varphi_{p,H}$ is a bijection between*

(1) *the set of pairs (p, H), where $H\colon N_1 \to N_2$ is a linear map such that for every cone $\sigma_1 \in \Sigma_1$ there exists a cone $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subset \sigma_2$, and p is a rational point of $X_{\Sigma_2,0}(K)$,*
(2) *the set of equivariant morphisms $\varphi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ whose image intersects the principal open subset of X_{Σ_2}.*

Proof. For a point $p \in X_{\Sigma_2,0}(K) = \mathbb{T}_2(K)$, let $t_p\colon X_2 \to X_2$ be the morphism induced by the toric action. Denote by $x_{1,0} \in X_{\Sigma_1}(K)$ the distinguished point of the principal open subset of X_{Σ_1}. The correspondence $\varphi \mapsto (t^{-1}_{\varphi(x_{1,0})} \circ \varphi, \varphi(x_{1,0}))$ establishes a bijection between the set of equivariant morphisms $\varphi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ whose image intersects the principal open subset of X_{Σ_2} and the set of pairs (ϕ, p), where $\phi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism and $p \in X_{\Sigma_2,0}(K)$ is a rational point in the principal open subset. Then the result follows from [Oda88, Theorem 1.13]. □

General equivariant morphisms are obtained composing an equivariant morphism of the form $\varphi_{p,H}\colon X_{\Sigma_1} \to X_{\Sigma_2}$ with the inclusion of X_{Σ_2} as a toric orbit of a third toric variety.

Example 4.10. The restriction of $\varphi_{p,H}$ to the principal open subset can be written in coordinates by choosing basis of N_1 and of N_2. Let n_i be the rank of N_i. The chosen basis determine isomorphisms $X_{\Sigma_i,0} \simeq \mathbb{G}_m^{n_i}$, which give coordinates $\boldsymbol{x} = (x_1, \dots, x_{n_1})$ and $\boldsymbol{t} = (t_1, \dots, t_{n_2})$ for $X_{\Sigma_1,0}$ and $X_{\Sigma_2,0}$, respectively. We write the the linear map H with respect to these basis as a matrix, and we denote its rows by a_i, $i = 1, \dots, n_2$. Write $p = (p_1, \dots, p_{n_2})$. In these coordinates, the morphism $\varphi_{p,H}$ is given by

$$\varphi_{p,H}(\boldsymbol{x}) = (p_1 \boldsymbol{x}^{a_1}, \dots, p_{n_2} \boldsymbol{x}^{a_{n_2}}).$$

We now show how to refine the Stein factorization for an equivariant morphism in terms of the combinatorial data. Let N_i, Σ_i H and p be as in Theorem 4.9. The linear map H factorizes as

$$N_1 \xrightarrow{H_{\mathrm{surj}}} N_3 := H(N_1) \xhookrightarrow{H_{\mathrm{sat}}} N_4 := \operatorname{sat}(N_3) \xhookrightarrow{H_{\mathrm{inj}}} N_2,$$

where N_3 is the image of H and N_4 is the saturation of N_3 with respect to N_2. Clearly $N_{3,\mathbb{R}} = N_{4,\mathbb{R}}$. By restriction, the fan Σ_2 induces a fan in this linear space. We will call this fan either Σ_3 or Σ_4, depending on the lattice we are considering. Applying the combinatorial construction of equivariant morphisms, we obtain a diagram

$$X_{\Sigma_1} \xrightarrow{\varphi_{H_{\mathrm{surj}}}} X_{\Sigma_3} \xrightarrow{\varphi_{H_{\mathrm{sat}}}} X_{\Sigma_4} \xrightarrow{\varphi_{p,H_{\mathrm{inj}}}} X_{\Sigma_2},$$

where the first morphism has connected fibres (see [Oda88, Proposition 1.14]), the second morphism is finite and surjective.

The third morphism is also finite and can be further factorized as a normalization followed by a closed immersion. In general, consider a saturated sublattice Q of N, Σ a fan in $N_{\mathbb{R}}$ and $p \in X_{\Sigma,0}(K)$. Let Σ_Q be the induced fan in $Q_{\mathbb{R}}$ and $\iota\colon Q \hookrightarrow N$ the inclusion of Q into N. Then, we have a finite equivariant morphism

$$\varphi_{p,\iota}\colon X_{\Sigma_Q} \longrightarrow X_{\Sigma}.$$

Set $P = Q^\vee = M/Q^\perp$ and let $\iota^\vee\colon M \to P$ be the dual of ι. Let $\sigma \in \Sigma$ and $\sigma' = \sigma \cap Q_\mathbb{R} \in \Sigma_Q$. The natural semigroup homomorphisms $M_\sigma \to P_{\sigma'}$ factors as

$$M_\sigma \twoheadrightarrow M_{Q,\sigma} := (M_\sigma + Q^\perp)/Q^\perp \hookrightarrow P_{\sigma'} := P \cap (\sigma')^\vee.$$

The first arrow is the projection and will be denoted as $m \mapsto [m]$, while the second one is the inclusion of $M_{Q,\sigma}$ into its saturation with respect to P. We have a diagram of K-algebra morphisms

$$K[M_\sigma] \twoheadrightarrow K[M_{Q,\sigma}] \hookrightarrow K[P_{\sigma'}],$$

where the left map is given by $\chi^m \mapsto \chi^m(p)\chi^{[m]}$, and the right map is given by $\chi^{[m]} \mapsto \chi^{\iota^\vee m}$. Let $Y_{\sigma,Q,p} \simeq \operatorname{Spec}(K[M_{Q,\sigma}])$ be the closed subvariety of X_σ given by the left surjection. Then we have induced maps

$$X_{\sigma'} \twoheadrightarrow Y_{\sigma,Q,p} \hookrightarrow X_\sigma.$$

These maps are compatible with the restriction to open subsets and so they glue together into maps

$$X_{\Sigma_Q} \twoheadrightarrow Y_{\Sigma,Q,p} \hookrightarrow X_\Sigma. \tag{4.11}$$

Then $Y_{\Sigma,Q,p}$ is the closure of the orbit of p under the action of the subtorus of $\mathbb{T}$ determined by Q, while the toric variety X_{Σ_Q} is the normalization of $Y_{\Sigma,Q,p}$.

When $p = x_0$, the subvariety $Y_{\Sigma,Q,p}$ will be denoted by $Y_{\Sigma,Q}$ for short.

Definition 4.12. A subvariety Y of X_Σ will be called a *toric subvariety* (respectively, a *translated toric subvariety*) if it is of the form $Y_{\Sigma,Q}$ (respectively, $Y_{\Sigma,Q,p}$) for a saturated sublattice $Q \subset N$ and $p \in X_{\Sigma,0}(K)$.

A translated toric subvariety is not necessarily a toric variety in the sense of Definition 4.1, since it may be non-normal.

Example 4.13. Let $N = \mathbb{Z}^2$, $(a,b) \in N$ with $\gcd(a,b) = 1$ and $\iota\colon Q \hookrightarrow N$ the saturated sublattice generated by (a,b). Let Σ be the fan in $N_\mathbb{R}$ of Example 3.70. Then $X_\Sigma = \mathbb{P}^2$ with projective coordinates $(x_0 : x_1 : x_2)$. The fan induced in $Q_\mathbb{R}$ has three cones: $\Sigma_Q = \{\mathbb{R}_{\le 0}, \{0\}, \mathbb{R}_{\ge 0}\}$. Thus $X_{\Sigma_Q} = \mathbb{P}^1$. Let $p = (1 : p_1 : p_2)$ be a point of $X_{\Sigma,0}(K)$. Then $\varphi_{p,\iota}((1:t)) = (1 : p_1t^a : p_2t^b)$. Therefore, $Y_{\Sigma,Q,p}$ is the curve of equation

$$p_2^a x_0^a x_1^b - p_1^b x_0^b x_2^a = 0.$$

In general, this curve is not normal. Hence it is not a toric variety.

4.3. **$\mathbb{T}$-Cartier divisors and toric line bundles.** When studying toric varieties, the objects that admit a combinatorial description are those that are compatible with the torus action. These objects are enough for many purposes. For instance, the divisor class group of a toric variety is generated by invariant divisors.

Let $\pi_2\colon \mathbb{T}\times X \to X$ denote the projection to the second factor and $\mu\colon \mathbb{T}\times X \to X$ the torus action. A Cartier divisor D is invariant if and only if

$$\pi_2^* D = \mu^* D.$$

Definition 4.14. Let X the a toric variety with torus $\mathbb{T}$. A Cartier divisor on X is called a $\mathbb{T}$-*Cartier divisor* if it is invariant under the action of $\mathbb{T}$ on X.

The combinatorial description of $\mathbb{T}$-Cartier divisors is done in terms of virtual support functions.

Definition 4.15. Let Σ be a fan in $N_\mathbb{R}$. A function $\Psi\colon |\Sigma| \to \mathbb{R}$ is called a *virtual support function* on Σ if it is a conic H-lattice function (Definition 3.88). Alternatively, a virtual support function is a function $\Psi\colon |\Sigma| \to \mathbb{R}$ such that, for every cone $\sigma \in \Sigma$, there exists $m_\sigma \in M$ with $\Psi(u) = \langle m_\sigma, u\rangle$ for all $u \in \sigma$. A set of functionals $\{m_\sigma\}_{\sigma\in\Sigma}$ as above is called a set of *defining vectors* of Ψ. A concave virtual support function on a complete fan will be called a *support function*.

A support function on a complete fan in the sense of the previous definition, is the support function of a polytope as in Example 3.16: it is the support function of the polytope

$$\operatorname{conv}(\{m_\sigma\}_{\sigma\in\Sigma^n}) \subset M_{\mathbb{R}},$$

where Σ^n is the subset of n-dimensional cones of Σ.

Two vectors $m, m' \in M$ define the same functional on a cone σ if and only if $m - m' \in \sigma^\perp$. Hence, for a given virtual support function Ψ on a fan Σ, each defining vector m_σ is unique up to the orthogonal space $\sigma^\perp$. In particular, $m_\sigma \in M$ is uniquely defined for $\sigma \in \Sigma^n$ and, in the other extreme, m_0 can be any point of M.

Let $\{m_\sigma\}_{\sigma\in\Sigma}$ be a set of defining vectors of Ψ. These vectors have to satisfy the compatibility condition

$$m_\sigma|_{\sigma\cap\sigma'} = m_{\sigma'}|_{\sigma\cap\sigma'} \text{ for all } \sigma, \sigma' \in \Sigma. \tag{4.16}$$

On each open set X_σ, the vector m_σ determines a rational function χ^{-m_σ}. For $\sigma, \sigma' \in \Sigma$, the above compatibility condition implies that $\chi^{-m_\sigma}/\chi^{-m_{\sigma'}}$ is a regular function on the overlap $X_\sigma \cap X_{\sigma'} = X_{\sigma\cap\sigma'}$ and so Ψ determines a Cartier divisor on X_Σ:

$$D_\Psi := \left\{(X_\sigma, \chi^{-m_\sigma})\right\}_{\sigma\in\Sigma}. \tag{4.17}$$

This Cartier divisor does not depend on the choice of defining vectors and it is a $\mathbb{T}$-Cartier divisor. All $\mathbb{T}$-Cartier divisors are obtained in this way.

Theorem 4.18. *Let Σ be a fan in $N_{\mathbb{R}}$ and X_Σ the corresponding toric variety. The correspondence $\Psi \mapsto D_\Psi$ is a bijection between the set of virtual support functions on Σ and the set of $\mathbb{T}$-Cartier divisors on X_Σ. Two Cartier divisors D_{Ψ_1} and D_{Ψ_2} are rationally equivalent if and only if the function $\Psi_1 - \Psi_2$ is linear.*

Proof. This is proved in [KKMS73, §I.2, Theorem 9]. □

We next recall the relationship between Cartier divisors and line bundles in the toric case.

Definition 4.19. Let X be a toric variety and L a line bundle on X. A *toric structure* on L is the choice of a non-zero vector z on the fibre $L_{x_0} = x_0^*L$ over the distinguished point. A *toric line bundle* is a pair (L, z), where L is a line bundle on X and z is a toric structure on L. A rational section s of a toric line bundle is a *toric section* if it is regular and nowhere vanishing on the principal open subset X_0, and $s(x_0) = z$. In order not to burden the notation, a toric line bundle will generally be denoted by L, the vector z being implicit.

Remark 4.20. The terminology "toric structure", "toric line bundle" and "toric section" comes from the fact that the total space of a toric line bundle $V(L) = \mathbf{Spec}_X(\operatorname{Sym}(L^\vee))$ admits a unique structure of toric variety satisfying the conditions:

(1) z is the distinguished point of the principal open subset;
(2) the structural morphism $V(L) \to X$ is a toric morphism;
(3) for each point $x \in X$ and vector $w \in L_x$, the morphism $\mathbb{G}_m \to V(L)$, given by scalar multiplication $\lambda \mapsto \lambda w$, is equivariant;
(4) every toric section s determines a toric morphism $U \to V(L)$, where U is the invariant open subset of regular points of s.

This can be shown using the construction of $V(L)$ as a toric variety in [Oda88, Proposition 2.1].

Remark 4.21. Every toric line bundle equipped with a toric section admits a unique structure of $\mathbb{T}$-equivariant line bundle such that the toric section becomes an invariant section. Conversely, every $\mathbb{T}$-equivariant toric line bundle admits a unique invariant toric section. Thus, there is a natural bijection between the space of $\mathbb{T}$-equivariant toric line bundles and the space of toric line bundles with a toric section. In particular, every line bundle admits a structure of $\mathbb{T}$-equivariant line bundle. This is not the case for higher rank vector bundles on toric varieties, nor for line bundles on other spaces with group actions like, for instance, elliptic curves.

To a Cartier divisor D, one associates an invertible sheaf of fractional ideals of $\mathcal{K}_X$, denoted $\mathcal{O}(D)$. When D is a $\mathbb{T}$-Cartier divisor given by a set of defining vectors, $\{m_\sigma\}_{\sigma\in\Sigma}$, the sheaf $\mathcal{O}(D)$ can be realized as the subsheaf of $\mathcal{O}_X$-modules generated, in each open subset X_σ, by the rational function χ^{m_σ}. The section $1 \in \mathcal{K}_X$ provides us with a distinguished rational section s_D such that $\mathrm{div}(s_D) = D$. Since D is supported on the complement of the principal open subset, s_D is regular and no-where vanishing on X_0. We set $z = s_D(x_0)$. This is a toric structure on $\mathcal{O}(D)$. From now on, we will assume that $\mathcal{O}(D)$ is equipped with this toric structure. Then $((\mathcal{O}(D), z), s_D)$ is a toric line bundle with a toric section.

Theorem 4.22. *Let X be a toric variety with torus $\mathbb{T}$. Then the correspondence $D \mapsto ((\mathcal{O}(D), s_D(x_0)), s_D)$ determines a bijection between the sets of*

1. *$\mathbb{T}$-Cartier divisors on X,*
2. *isomorphism classes of pairs (L, s) where L is a toric line bundle and s is a toric section.*

Proof. We have already shown that a $\mathbb{T}$-Cartier divisor produces a toric line bundle with a toric section. Let now $((L, z), s)$ be a toric line bundle equipped with a toric section and Σ the fan that defines X. Since every line bundle on an affine toric variety is trivial, for each $\sigma \in \Sigma$ we can find a section s_σ that generates L on X_σ and such that $s_\sigma(x_0) = z$. Since s is regular and nowhere vanishing on X_0 and $s(x_0) = z$, we can find elements $m_\sigma \in M$ such that $s = \chi^{-m_\sigma} s_\sigma$, because any regular nowhere vanishing function on a torus is a constant times a monomial. The elements m_σ glue together to define a virtual support function Ψ on Σ that does not depend on the chosen trivialization. It is easy to see that the correspondence $(L, s) \mapsto D_\Psi$ is the inverse of the previous one, which proves the theorem. □

Thanks to this result and Theorem 4.18, we can freely move between the languages of virtual support functions, $\mathbb{T}$-Cartier divisors, and toric line bundles with a toric section.

Notation 4.23. Let Ψ be a virtual support function. We will write $((L_\Psi, z_\Psi), s_\Psi)$ for the toric line bundle with toric section associated to the $\mathbb{T}$-Cartier divisor D_Ψ by Theorem 4.22. When we do not need to make explicit the vector z_Ψ, we will simply write (L_Ψ, s_Ψ). ??? `incluir la notacion` $\Psi_{L,s}$

We next recall the relationship between Cartier divisors and Weil divisors in the toric case.

Definition 4.24. A $\mathbb{T}$-*Weil divisor* on a toric variety X is a finite formal linear combination of hypersurfaces of X which are invariant under the torus action.

The invariant hypersurfaces of a toric variety are particular cases of the toric subvarieties considered in the previous section: they are the varieties of the form $V(\tau)$ for $\tau \in \Sigma^1$. Hence, a $\mathbb{T}$-Weil divisor is a finite formal linear combination of subvarieties of the form $V(\tau)$ for $\tau \in \Sigma^1$.

Since the toric variety X is normal, each Cartier divisor determines a Weil divisor. This correspondence associates to the $\mathbb{T}$-Cartier divisor D_Ψ, the $\mathbb{T}$-Weil divisor

$$[D_\Psi] = \sum_{\tau\in\Sigma^1} -\Psi(v_\tau)V(\tau), \tag{4.25}$$

where $v_\tau \in N$ is the smallest nonzero lattice point in τ.

Example 4.26. We continue with the notation of examples 3.76 and 4.3. The fan Σ_{Δ^n} has $n+1$ rays. For each $i = 0, \dots, n$, the closure of the orbit corresponding to the ray generated by the vector e_i is the standard hyperplane of $\mathbb{P}^n$

$$H_i := V(\langle e_i \rangle) = \{(p_0 : \cdots : p_n) \in \mathbb{P}^n \mid p_i = 0\}.$$

The function Ψ_{Δ^n} is a support function on Σ_{Δ^n} and the $\mathbb{T}$-Weil divisor associated to $D_{\Psi_{\Delta^n}}$ is $[D_{\Psi_{\Delta^n}}] = H_0$.

For a toric variety X_Σ of dimension n, we denote by $\mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ its group of $\mathbb{T}$-Cartier divisors, and by $Z_{n-1}^{\mathbb{T}}(X_\Sigma)$ its group of $\mathbb{T}$-Weil divisors. Recall that $\mathrm{Pic}(X_\Sigma)$, the Picard group of X_Σ, is the group of isomorphism classes of line bundles. Let $A_{n-1}(X_\Sigma)$ denote the Chow group of cycles of dimension $n-1$. The following result shows that these groups can computed in terms of invariant divisors.

Theorem 4.27. *Let Σ be a fan in $N_\mathbb{R}$ that is not contained in any hyperplane. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X_\Sigma) \arrow[r] \arrow[d, hook] & 0 . \\ 0 \arrow[r] & M \arrow[r] & Z_{n-1}^{\mathbb{T}}(X_\Sigma) \arrow[r] & A_{n-1}(X_\Sigma) \arrow[r] & 0 \end{tikzcd}$$

Proof. This is the first proposition in [Ful93, §3.4]. □

Remark 4.28. In the previous theorem, the hypothesis that Σ is not contained in any hyperplane is only needed for the injectivity of the second arrow in each row of the diagram.

In view of Theorem 4.22, the upper exact sequence of the diagram in Theorem 4.27 can be interpreted as follows.

Corollary 4.29. *Let X be a toric variety with torus $\mathbb{T}$.*

(1) *Every toric line bundle L on X admits a toric section. Moreover, if s and s' are two toric sections, then there exists $m \in M$ such that $s' = \chi^m s$.*
(2) *If the fan Σ that defines X is not contained in any hyperplane, and L and L' are toric line bundles on X, then there is at most one isomorphism between them.*

Proof. This follows from theorems 4.27 and 4.22. □

We next study the intersection of a $\mathbb{T}$-Cartier divisor with the closure of an orbit. Let Σ be a fan in $N_\mathbb{R}$ and Ψ the virtual support function on Σ given by the set of defining vectors $\{m_\tau\}_{\tau\in\Sigma}$. Let σ be a cone of Σ and $\iota_\sigma \colon V(\sigma) \hookrightarrow X_\Sigma$ the associated closed immersion. We consider first the case when $\Psi|_\sigma = 0$. Let $\tau \supset \sigma$ be another cone of Σ. For vectors $u \in \tau$ and $v \in \mathbb{R}\sigma$ such that $u + v \in \tau$, the condition $\Psi|_\sigma = 0$ implies

$$\Psi(u+v) = \langle m_\tau, u+v \rangle = \langle m_\tau, u \rangle = \Psi(u)$$

because $m_\tau\big|_{\mathbb{R}\sigma} = 0$. Hence, we can define a function

$$\Psi(\sigma) \colon N(\sigma)_\mathbb{R} \longrightarrow \mathbb{R}, \quad u + \mathbb{R}\sigma \longmapsto \Psi(u+v) \tag{4.30}$$

for any $v \in \mathbb{R}\sigma$ such that $u + v \in \bigcup_{\tau\supset\sigma} \tau$.

It is easy to produce a set of defining vectors of $\Psi(\sigma)$. For each cone $\tau \supset \sigma$ we denote by $\overline{\tau} = \pi_\sigma(\tau)$ the corresponding cone in $\Sigma(\sigma)$. Since $m_\tau\big|_{\mathbb{R}\sigma} = 0$, then $m_\tau \in M(\sigma) = M \cap \sigma^\perp$. We set $m_{\overline{\tau}} = m_\tau \in M(\sigma)$.

Proposition 4.31. *Let notation be as above. If* $\Psi|_\sigma = 0$*, then* D_Ψ *intersects* $V(\sigma)$ *properly and* $\iota_\sigma^* D_\Psi = D_{\Psi(\sigma)}$*. Moreover,* $\{m_{\overline{\tau}}\}_{\overline{\tau}\in\Sigma(\sigma)}$ *is a set of defining vectors of* $\Psi(\sigma)$*.*

Proof. The $\mathbb{T}$-Cartier divisor D_Ψ is given by $\{(X_\tau, \chi^{-m_\tau})\}_{\tau\in\Sigma}$. If $m_\sigma = 0$, the local equation of D_Ψ in X_σ is $\chi^0 = 1$. Therefore, the orbit $O(\sigma)$ does not meet the support of D_Ψ. Hence $V(\sigma)$ and D_Ψ intersect properly.

To see that $\{m_{\overline{\tau}}\}_{\overline{\tau}\in\Sigma(\sigma)}$ is a set of defining vectors, we pick a point $\overline{u} \in \overline{\tau}$ and we choose $u \in \tau$ such that $\pi_\sigma(u) = \overline{u}$. Then

$$\Psi(\sigma)(\overline{u}) = \Psi(u) = m_\tau(u) = m_{\overline{\tau}}(\overline{u}),$$

which proves the claim. Now, using the characterization of $\Psi(\sigma)$ in terms of defining vectors, we have

$$\iota_\sigma^* D_\Psi = \{(X_\tau \cap V(\sigma), \chi^{-m_\tau}\mid_{X_\tau\cap V(\sigma)})\}_{\overline{\tau}} = \{(X_{\overline{\tau}}, \chi^{-m_{\overline{\tau}}})\}_{\overline{\tau}} = D_{\Psi(\sigma)}.$$

□

When $\Psi|_\sigma \neq 0$, the cycles D_Ψ and $V(\sigma)$ do not intersect properly, and we can only intersect D_Ψ with $V(\sigma)$ up to rational equivalence. To this end, we choose any m'_σ such that $\Psi(u) = \langle m'_\sigma, u\rangle$ for every $u \in \sigma$. Then the divisor $D_{\Psi-m'_\sigma}$ is rationally equivalent to D_Ψ and $\Psi - m'_\sigma|_\sigma = 0$. By the above result, this divisor intersects $V(\sigma)$ properly, and its restriction to $V(\tau)$ is given by the virtual support function $(\Psi - m'_\sigma)(\sigma)$.

Example 4.32. We can use the above description of the restriction of a line bundle to an orbit to compute the degree of an orbit of dimension one. Let Σ be a complete fan and $\tau \in \Sigma^{n-1}$. Hence $V(\tau)$ is a toric curve. Let σ_1 and σ_2 be the two n-dimensional cones that have τ as a common face. Let Ψ be a virtual support function. Choose $v \in \sigma_1$ such that $\pi_\tau(v)$ is a generator of the lattice $N(\tau)$. Then, by (4.25) and (4.30),

$$\deg_{D_\Psi}(V(\tau)) = \deg(\iota_\tau^* D_\Psi) = m_{\sigma_2}(v) - m_{\sigma_1}(v). \tag{4.33}$$

Let now (L, z) be a toric line bundle on X_Σ and $\sigma \in \Sigma$. The line bundle $\iota_\sigma^* L$ on $V(\sigma)$ has an induced toric structure. Let s be a toric section of L that is regular and nowhere vanishing on X_σ, and set $z_\sigma = s(x_\sigma) \in L_{x_\sigma} \setminus \{0\}$. If s' is another such section, then $s' = \chi^m s$ for an $m \in M$ such that $m|_\sigma = 0$, by Corollary 4.29. Therefore $s'(x_\sigma) = s(x_\sigma)$. Hence, z_σ does not depend on the choice of section and $(\iota_\sigma^* L, z_\sigma)$ is the induced toric line bundle. The following result follows easily from the constructions.

Proposition 4.34. *Let* (L, z) *be a toric line bundle on* X_Σ *and* $\sigma \in \Sigma$*. Let* Ψ *be a virtual support function such that* $\Psi|_\sigma = 0$ *and* $(L, z) \simeq (L_\Psi, z_\Psi)$ *as toric line bundles. Then* $\iota_\sigma^*(L, z) \simeq (L_{\Psi(\sigma)}, z_{\Psi(\sigma)})$*.*

We next study the inverse image of a $\mathbb{T}$-Cartier divisor with respect to equivariant morphisms as those in Theorem 4.9. Let N_i, Σ_i, $i = 1, 2$, and let $H\colon N_1 \to N_2$ and $p \in X_{\Sigma_2,0}(K)$ be as in Theorem 4.9. Let $\varphi_{p,H}$ be the associated equivariant morphism, Ψ a virtual support function on Σ_2 and $\{m'_{\tau'}\}_{\tau'\in\Sigma_2}$ a set of defining vectors of Ψ. For each cone $\tau \in \Sigma_1$ we choose a cone $\tau' \in \Sigma_2$ such that $H(\tau) \subset \tau'$ and we write $m_\tau = H^\vee(m'_{\tau'})$. The following result follows easily from the definitions

Proposition 4.35. *The divisor* D_Ψ *intersects properly the image of* $\varphi_{p,H}$*. The function* $\Psi \circ H$ *is a virtual support function on* Σ_1 *and*

$$\varphi_{p,H}^* D_\Psi = D_{\Psi\circ H}.$$

Moreover, $\{m_\tau\}_{\tau\in\Sigma_1}$ *is a set of defining vectors of* $\Psi \circ H$*.*

Remark 4.36. If L is a toric line bundle on X_{Σ_2} and φ is a toric morphism, then φ^*L has an induced toric structure. Namely, $\varphi^*(L,z) = (\varphi^*L, \varphi^*z)$. By contrast, if $\varphi\colon X_{\Sigma_1} \to X_{\Sigma_2}$ is a general equivariant morphism that meets the principal open subset, there is no natural toric structure on φ^*L, because the image of the distinguished point $x_{1,0}$ does not need to agree with $x_{2,0}$. If (L,s) is a toric line bundle equipped with a toric section, then we set $\varphi^*(L,s) = ((\varphi^*L, (\varphi^*s)(x_{1,0})), \varphi^*s)$. However, the underlying toric bundle of $\varphi^*(L,s)$ depends on the choice of the toric section.

4.4. **Positivity properties of $\mathbb{T}$-Cartier divisors.** Let Σ be a fan in $N_{\mathbb{R}}$ and Ψ a virtual support function on Σ. In this section, we will assume that Σ is complete or, equivalently, that the variety X_Σ is proper.

Many geometric properties of the pair (X_Σ, D_Ψ) can be read directly from Ψ. For instance, $\mathcal{O}(D_\Psi)$ is generated by global sections if and only if the function Ψ is concave, and the line bundle $\mathcal{O}(D_\Psi)$ is ample if and only if Ψ is strictly concave on Σ. In the latter case, the fan Σ agrees with the polyhedral complex $\Pi(\Psi)$ (Definition 3.34) and the pair (X_Σ, D_Ψ) is completely determined by Ψ. Thus, the variety X_Σ is projective if and only if the fan Σ is complete and regular (Definition 3.60).

We associate to Ψ the subset of $M_{\mathbb{R}}$

$$\Delta_\Psi = \{x \in M_{\mathbb{R}} \mid \langle x, u\rangle \geq \Psi(u) \text{ for all } u \in N_{\mathbb{R}}\}.$$

This set is either empty or a lattice polytope. When $\mathcal{O}(D_\Psi)$ is generated by global sections, the polytope Δ_Ψ agrees with $\operatorname{stab}(\Psi)$, and Ψ is the support function of Δ_Ψ.

The polytope Δ_Ψ encodes a lot of information about the pair (X_Σ, D_Ψ). For instance, we can read from it the space of global sections of $\mathcal{O}(D_\Psi)$. A monomial rational section $\chi^m \in \mathcal{K}_{X_\Sigma}$, $m \in M$, is a regular global section of $\mathcal{O}(D_\Psi)$ if and only if $m \in \Delta_\Psi$. Moreover, the set $\{\chi^m\}_{m \in M \cap \Delta_\Psi}$ is a K-basis of the space of global sections $\Gamma(X_\Sigma, \mathcal{O}(D_\Psi))$. In the sequel we will see many more examples of this principle.

Proposition 4.37. *Let D_{Ψ_i}, $i = 1, \dots, n$, be $\mathbb{T}$-Cartier divisors on X_Σ generated by their global sections. Then*

$$(D_{\Psi_1} \cdot \dots \cdot D_{\Psi_n}) = \operatorname{MV}_M(\Delta_{\Psi_1}, \dots, \Delta_{\Psi_n}). \tag{4.38}$$

where MV_M denotes the mixed volume function associated to the Haar measure vol_M on $M_{\mathbb{R}}$ (Definition 3.109). In particular, for a $\mathbb{T}$-Cartier divisor D_Ψ generated by its global sections,

$$\deg_{D_\Psi}(X_\Sigma) = (D_\Psi^n) = n! \operatorname{vol}_M(\Delta_\Psi). \tag{4.39}$$

Proof. This follows from [Oda88, Proposition 2.10]. □

Remark 4.40. The intersection multiplicity and the degree in the above Proposition only depend on the isomorphism class of the line bundles $\mathcal{O}(D_{\Psi_i})$ and not on the $\mathbb{T}$-Cartier divisors themselves. It is easy to check directly that the right-hand sides of (4.38) and (4.39) only depends on the isomorphism class of the line bundles. In fact, let L be a toric line bundle generated by global sections and s_1, s_2 two toric sections. For $i = 1, 2$, set $D_i = \operatorname{div}(s_i)$ and let Ψ_i be the corresponding support function and Δ_i the associated polytope. Then $s_2 = \chi^m s_1$ for some $m \in M$. Thus $\Psi_2 = \Psi_1 - m$ and $\Delta_2 = \Delta_1 - m$. Since the volume and the mixed volume are invariant under translation, we see that these formulae do not depend on the choice of sections.

Definition 4.41. A *polarized toric variety* is a pair (X_Σ, D_Ψ), where X_Σ is a toric variety and D_Ψ is an ample $\mathbb{T}$-Cartier divisor.

Polarized toric varieties can be classified in terms of their polytopes.

Theorem 4.42. *The correspondence $(X_\Sigma, D_\Psi) \mapsto \Delta_\Psi$ is a bijection between the set of polarized toric varieties and the set of lattice polytopes of dimension n of M. Two ample $\mathbb{T}$-Cartier divisors D_Ψ and $D_{\Psi'}$ on a toric variety X_Σ are rationally equivalent if and only if $\Delta_{\Psi'}$ is the translated of Δ_Ψ by an element of M.*

Proof. If Ψ is a strictly concave function on Σ, then Δ_Ψ is an n-dimensional lattice polytope. Conversely, if Δ is a lattice polytope in $M_\mathbb{R}$, then Ψ_Δ, the support function of Δ, is a strictly concave function on the complete fan $\Sigma_\Delta = \Pi(\Psi_\Delta)$ (see examples 3.71 and 3.76). Therefore, the result follows from Theorem 4.18 and the construction of Remark 4.40. □

Remark 4.43. When D_Ψ is only generated by its global sections, the polytope Δ_Ψ may not determine the variety X_Σ, but it does determine a polarized toric variety that is the image of X_Σ by a toric morphism. Write $\Delta = \Delta_\Psi$ for short. Let $M(\Delta)$ be as in Notation 3.103 and choose $m \in \operatorname{aff}(\Delta) \cap M$. Set $N(\Delta) = M(\Delta)^\vee$. The translated polytope $\Delta - m$ has the same dimension as its ambient space $L_\Delta = M(\Delta)_\mathbb{R}$. By the theorem above, it defines a complete fan Σ_Δ in $N(\Delta)_\mathbb{R}$ together with a support function $\Psi_\Delta \colon N(\Delta) \to \mathbb{R}$. The projection $N \to N(\Delta)$ induces a toric morphism

$$\varphi\colon X_\Sigma \longrightarrow X_{\Sigma_\Delta},$$

the divisor D_{Ψ_Δ} is ample, and $D_\Psi = \varphi^* D_{\Psi_\Delta} + \operatorname{div}(\chi^{-m})$.

Example 4.44. The projective morphisms associated to $\mathbb{T}$-Cartier divisors generated by global sections can also be made explicit in terms of the lattice points of the associated polytope. Consider a complete toric variety X_Σ of dimension n equipped with a $\mathbb{T}$-Cartier divisor D_Ψ generated by global sections. Let $m_0, \dots, m_r \in \Delta_\Psi \cap M$ be such that $\operatorname{conv}(m_0, \dots, m_r) = \Delta_\Psi$. These vectors determine an H-representation $\Psi = \min_{i=0,\dots,r} m_i$. Let $H\colon N_\mathbb{R} \to \mathbb{R}^r$ be the linear map defined by $H(u) = (m_i(u) - m_0(u))_{i=1,\dots,r}$. By Lemma 3.79, $\Psi = H^*\Psi_{\Delta^r} + m_0$.

In $\mathbb{R}^r$ we consider the fan Σ_{Δ^r}, whose associated toric variety is $\mathbb{P}^r$. One easily verifies that, for each $\sigma \in \Sigma$, there is $\sigma' \in \Sigma_{\Delta^r}$ with $H(\sigma) \subset \sigma'$. Let $p = (p_0 : \dots : p_r)$ be an arbitrary rational point of the principal open subset of $\mathbb{P}^r$. The equivariant morphism $\varphi_{p,H}\colon X \to \mathbb{P}^r_K$ can be written explicitly as $(p_0\chi^{m_0} : \dots : p_r\chi^{m_r})$. Moreover, $D_\Psi = \varphi^*_{p,H} D_{\Psi_{\Delta^r}} + \operatorname{div}(\chi^{-m_0})$.

The orbits of a polarized toric variety (X_Σ, D_Ψ) are in one-to-one correspondence with the faces of Δ_Ψ.

Proposition 4.45. *Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a strictly concave function on Σ. The correspondence $F \mapsto O(\sigma_F)$ is a bijection between the set of faces of Δ_Ψ and the set of the orbits under the action of $\mathbb{T}$ on X_Σ.*

Proof. This follows from Example 3.71. □

Equation (4.25) gives a formula for the Weil divisor $[D_\Psi]$ in terms of the virtual support function Ψ. When the line bundle $\mathcal{O}(D_\Psi)$ is ample, we can interpret this formula in terms of the facets of the polytope Δ_Ψ.

Let D_Ψ be an ample line bundle on X_Σ. The polytope Δ_Ψ has maximal dimension n. For each facet F of Δ_Ψ, let v_F be as in Notation 3.103. The ray $\tau_F = \mathbb{R}_{\geq 0} v_F$ is a cone of Σ.

Proposition 4.46. *With the previous hypothesis,*

$$\operatorname{div}(s_\Psi) = [D_\Psi] = \sum_F -\langle v_F, F\rangle V(\tau_F),$$

where the sum is over the facets F of Δ.

Proof. Since Ψ is strictly concave on Σ, the Legendre-Fenchel correspondence shows that the set of rays of the form τ_F agrees with the set Σ^1. Moreover, $\Psi(v_F) = \langle v_F, F\rangle$, because Ψ is the support function of Δ. The proposition then follows from (4.25). □

For a $\mathbb{T}$-Cartier divisor generated by global sections, we can interpret its intersection with the closure of an orbit, and its inverse image with respect to an equivariant morphism, in terms of direct and inverse images of concave functions.

Proposition 4.47. *Let Σ be a complete fan in $N_{\mathbb{R}}$ and $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ a support function on Σ.*

(1) *Let $\sigma \in \Sigma$, F_σ the associated face of Δ_Ψ, and $m'_\sigma \in F_\sigma \cap M$. Let $\pi_\sigma\colon N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$ be the natural projection. Then*
$$(\Psi - m'_\sigma)(\sigma) = (\pi_\sigma)_*(\Psi - m'_\sigma). \tag{4.48}$$
In particular, the restriction of $D_{\Psi - m'_\sigma}$ to $V(\sigma)$ is given by the concave function $(\pi_\sigma)_*(\Psi - m'_\sigma)$. Moreover, the associated polytope is
$$\Delta_{(\Psi - m'_\sigma)(\sigma)} = F_\sigma - m'_\sigma \subset M(\sigma)_{\mathbb{R}} = \sigma^\perp. \tag{4.49}$$

(2) *Let $H\colon N' \to N$ be a linear map and $H^\vee\colon M \to M'$ its dual map, where $M' = (N')^\vee$. Let Σ' be a fan in $N'_{\mathbb{R}}$ such that, for each $\sigma' \in \Sigma'$ there is $\sigma \in \Sigma$ with $H(\sigma') \subset \sigma$, and let $p \in X_{\Sigma,0}(K)$. Then*
$$\varphi^*_{p,H} D_\Psi = D_{H^*\Psi}, \tag{4.50}$$
and the associated polytope is
$$\Delta_{H^*\Psi} = H^\vee(\Delta_\Psi) \subset M'_{\mathbb{R}}. \tag{4.51}$$

Proof. Equation (4.48) follows from (4.30), while equation (4.50) follows from Proposition 4.35. Then (4.49) and (4.51) follow from Proposition 3.78. □

As a consequence of the above construction, we can compute easily the degree of any orbit.

Corollary 4.52. *Let Σ be a complete fan in $N_{\mathbb{R}}$, $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ a support function on Σ, and $\sigma \in \Sigma$ a cone of dimension $n-k$. Then*
$$\deg_{D_\Psi}(V(\sigma)) = k! \operatorname{vol}_{M(F_\sigma)}(F_\sigma).$$

Proof. In view of equations (4.49) and (4.39), it is enough to prove that $M(\sigma) = M(F_\sigma)$. But this follows from the fact that $L_{F_\sigma} = \sigma^\perp$ (see Notation 3.103). □

Example 4.53. Let $\tau \in \Sigma^{n-1}$. The degree of the curve $V(\tau)$ agrees with the lattice length of F_τ.

We will also need the toric version of the Nakai-Moishezon criterion.

Theorem 4.54. *Let X_Σ be a proper toric variety and D_Ψ a $\mathbb{T}$-Cartier divisor on X_Σ.*

(1) *The following properties are equivalent:*
   (a) *D_Ψ is ample;*
   (b) *$(D_\Psi \cdot C) > 0$ for every curve C in X_Σ;*
   (c) *$(D_\Psi \cdot V(\tau)) > 0$ for every $\tau \in \Sigma^{n-1}$.*

(2) *The following properties are equivalent:*
   (a) *D_Ψ is generated by its global sections;*
   (b) *$(D_\Psi \cdot C) \geq 0$ for every curve C in X_Σ;*
   (c) *$(D_\Psi \cdot V(\tau)) \geq 0$ for every $\tau \in \Sigma^{n-1}$.*

Proof. This follows from [Oda88, Theorem 2.18] for non singular toric varieties, and from [Mav00] for the general case. □

4.5. **Toric schemes over a discrete valuation ring.** In this section we recall some basic facts about the algebraic geometry of toric schemes over a DVR. These toric schemes were introduced in [KKMS73, Chapter IV, §3], and we refer to this reference for more details. They are described and classified in terms of fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. In this section we will mostly consider proper toric schemes over a DVR. As a consequence of Corollary 3.15, proper toric schemes over a DVR can be described and classified in terms of complete SCR polyhedral complexes in $N_{\mathbb{R}}$ as, for instance, in [NS06].

Let K be a field equipped with a nontrivial discrete valuation $\mathrm{val}\colon K^{\times} \twoheadrightarrow \mathbb{Z}$. In this section we do not assume K to be complete. As usual, we denote by K° the valuation ring, by $K^{\circ\circ}$ its maximal ideal, by ϖ a generator of $K^{\circ\circ}$ and by k the residue field. We assume that $\mathrm{val}(\varpi) = 1$. We denote by S the base scheme $S = \mathrm{Spec}(K^{\circ})$, by η and o the generic and the special points of S and, for a scheme $\mathcal{X}$ over S, we set $\mathcal{X}_{\eta} = \mathcal{X} \times_S \mathrm{Spec}(K)$ and $\mathcal{X}_o = \mathcal{X} \times_S \mathrm{Spec}(k)$ for its generic and special fibre respectively. We will denote by $\mathbb{T}_S = \mathbb{T}_{K^{0}} \simeq \mathbb{G}^n_{m,S}$ a split torus over S. Let $\mathbb{T} = \mathbb{T}_K$, N and M be as in §4.1. We will write $\widetilde{N} = N \oplus \mathbb{Z}$ and $\widetilde{M} = M \oplus \mathbb{Z}$.

Definition 4.55. A *toric scheme over* S of relative dimension n is a normal integral separated S-scheme of finite type, $\mathcal{X}$, equipped with a dense open embedding $\mathbb{T}_K \hookrightarrow \mathcal{X}_{\eta}$ and an S-action of $\mathbb{T}_S$ over $\mathcal{X}$ that extends the action of $\mathbb{T}_K$ on itself by translations. If we want to stress the torus acting on $\mathcal{X}$ we will call them toric schemes *with torus* $\mathbb{T}_S$.

If $\mathcal{X}$ is a toric scheme over S, then $\mathcal{X}_{\eta}$ is a toric variety over K with torus $\mathbb{T}$.

Definition 4.56. Let X be a toric variety over K with torus $\mathbb{T}_K$ and let $\mathcal{X}$ be a toric scheme over S with torus $\mathbb{T}_S$. We say that $\mathcal{X}$ is a *toric model* of X over S if the identity of $\mathbb{T}_K$ can be extended to an isomorphism from X to $\mathcal{X}_{\eta}$.

If $\mathcal{X}$ and $\mathcal{X}'$ are toric models of X and $\alpha\colon \mathcal{X} \to \mathcal{X}'$ is an S-morphism, we say that α is a *morphism of toric models* if its restriction to $\mathbb{T}_K$ is the identity.

Since, by definition, a toric scheme is integral and contains $\mathbb{T}$ as a dense open subset, it is flat over S. Thus a toric model is a particular case of a model as in Definition 2.11.

Let $\widetilde{\Sigma}$ be a fan in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. To the fan $\widetilde{\Sigma}$ we associate a toric scheme $\mathcal{X}_{\widetilde{\Sigma}}$ over S. Let $\sigma \in \widetilde{\Sigma}$ be a cone and $\sigma^{\vee} \subset \widetilde{M}_{\mathbb{R}}$ its dual cone. Set $\widetilde{M}_{\sigma} = \widetilde{M} \cap \sigma^{\vee}$. Let $K^{\circ}[\widetilde{M}_{\sigma}]$ be the semigroup K°-algebra of $\widetilde{M}_{\sigma}$. By definition, $(0,1) \in \widetilde{M}_{\sigma}$. Thus $(\chi^{(0,1)} - \varpi)$ is an ideal of $K^{\circ}[\widetilde{M}_{\sigma}]$. There is a natural isomorphism

$$K^{\circ}[\widetilde{M}_{\sigma}]/(\chi^{(0,1)} - \varpi) \simeq \Big\{ \sum_{(m,l)\in\widetilde{M}_{\sigma}} \alpha_{m,l}\varpi^{l}\chi^{m} \mid \alpha_{m,l} \in K^{\circ} \text{ and, } \mathop{\forall}\limits^{\circ}(m,l), \alpha_{m,l} = 0 \Big\} \tag{4.57}$$

that we use to identify both rings. The ring $K^{\circ}[\widetilde{M}_{\sigma}]/(\chi^{(0,1)} - \varpi)$ is an integrally closed domain. We set

$$\mathcal{X}_{\sigma} = \mathrm{Spec}(K^{\circ}[\widetilde{M}_{\sigma}]/(\chi^{(0,1)} - \varpi))$$

for the associated affine toric scheme over S. For short we will use the notation

$$K^{\circ}[\mathcal{X}_{\sigma}] = K^{\circ}[\widetilde{M}_{\sigma}]/(\chi^{(0,1)} - \varpi). \tag{4.58}$$

For cones $\sigma, \sigma' \in \widetilde{\Sigma}$, with $\sigma \subset \sigma'$ we have a natural open immersion of affine schemes $\mathcal{X}_{\sigma} \hookrightarrow \mathcal{X}_{\sigma'}$. Using these open immersions as gluing data, we define the scheme

$$\mathcal{X}_{\widetilde{\Sigma}} = \bigcup_{\sigma\in\widetilde{\Sigma}} \mathcal{X}_{\sigma}.$$

This is a reduced and irreducible normal scheme of finite type over S of relative dimension n.

There are two types of cones in $\widetilde{\Sigma}$. The ones that are contained in the hyperplane $N_{\mathbb{R}}\times\{0\}$, and the ones that are not. If σ is contained in $N_{\mathbb{R}}\times\{0\}$, then $(0,-1)\in\widetilde{M}_\sigma$, and ϖ is invertible in $K^\circ[\mathcal{X}_\sigma]$. Therefore $K^\circ[\mathcal{X}_\sigma]\simeq K[M_\sigma]$; hence $\mathcal{X}_\sigma$ is contained in the generic fibre and it agrees with the affine toric variety X_σ. If σ is not contained in $N_{\mathbb{R}}\times\{0\}$, then $\mathcal{X}_\sigma$ is not contained in the generic fibre.

To stress the difference between both types of affine schemes we will follow the following notations. Let Π be the SCR polyhedral complex in $N_{\mathbb{R}}$ obtained by intersecting $\widetilde{\Sigma}$ by the hyperplane $N_{\mathbb{R}}\times\{1\}$ as in Corollary 3.15, and Σ the fan in $N_{\mathbb{R}}$ obtained by intersecting $\widetilde{\Sigma}$ with $N_{\mathbb{R}}\times\{0\}$. For $\Lambda\in\Pi$, the cone $\mathrm{c}(\Lambda)\in\widetilde{\Sigma}$ is not contained in $N\times\{0\}$. We will write $\widetilde{M}_\Lambda=\widetilde{M}_{\mathrm{c}(\Lambda)}$, $K^\circ[\widetilde{M}_\Lambda]=K^\circ[\widetilde{M}_{\mathrm{c}(\Lambda)}]$, $\mathcal{X}_\Lambda=\mathcal{X}_{\mathrm{c}(\Lambda)}$ and $K^\circ[\mathcal{X}_\Lambda]=K^\circ[\mathcal{X}_{\mathrm{c}(\Lambda)}]$.

Given polyhedrons $\Lambda,\Lambda'\in\Pi$, with $\Lambda\subset\Lambda'$, we have a natural open immersion of affine toric schemes $\mathcal{X}_\Lambda\hookrightarrow\mathcal{X}_{\Lambda'}$. Moreover, if a cone $\sigma\in\Sigma$ is a face of a cone $\mathrm{c}(\Lambda)$ for some $\Lambda\in\Pi$, then the affine toric variety X_σ, is also an open subscheme of $\mathcal{X}_\Lambda$. The open cover (4.58) can be written as

$$\mathcal{X}_{\widetilde{\Sigma}}=\bigcup_{\Lambda\in\Pi}\mathcal{X}_\Lambda\cup\bigcup_{\sigma\in\Sigma}X_\sigma.$$

We will reserve the notation $\mathcal{X}_\Lambda$, $\Lambda\in\Pi$ for the affine toric schemes that are not contained in the generic fibre and denote by X_σ, $\sigma\in\Sigma$ the affine toric schemes contained in the generic fibre, because they are toric varieties over K.

The scheme $\mathcal{X}_0$ corresponding to the polyhedron $0:=\{0\}$ is a group S-scheme which is canonically isomorphic to $\mathbb{T}_S$. The S-action of $\mathbb{T}_S$ over $\mathcal{X}_{\widetilde{\Sigma}}$ is constructed as in the case of varieties over a field. Moreover there are open immersions $\mathbb{T}_K\hookrightarrow\mathcal{X}_\eta\hookrightarrow\mathcal{X}_{\widetilde{\Sigma}}$ of schemes over S and the action of $\mathbb{T}_S$ on $\mathcal{X}_{\widetilde{\Sigma}}$ extends the action of $\mathbb{T}_K$ on itself. Thus $\mathcal{X}_{\widetilde{\Sigma}}$ is a toric scheme over S. Moreover, the fan Σ defines a toric variety over K which coincides with the generic fibre $\mathcal{X}_{\widetilde{\Sigma},\eta}$. Thus, $\mathcal{X}_{\widetilde{\Sigma}}$ is a toric model of X_Σ. The special fibre $\mathcal{X}_{\widetilde{\Sigma},o}=\mathcal{X}_{\widetilde{\Sigma}}\underset{S}{\times}\operatorname{Spec}(k)$ has an induced action by $\mathbb{T}_k$, but, in general, it is not a toric variety over k, because it is not irreducible nor reduced. The reduced schemes associated to its irreducible components are toric varieties over k with this action.

Every toric scheme over S can be obtained by the above construction. Indeed, this construction gives a classification of toric schemes by fans in $N_{\mathbb{R}}\times\mathbb{R}_{\geq 0}$ [KKMS73, §IV.3(e)].

If the fan $\widetilde{\Sigma}$ is complete, then the scheme $\mathcal{X}_{\widetilde{\Sigma}}$ is proper over S. In this case the set $\{\mathcal{X}_\Lambda\}_{\Lambda\in\Pi}$ is an open cover of $\mathcal{X}_{\widetilde{\Sigma}}$. Proper toric schemes over S can also be classified by complete SCR polyhedral complexes in $N_{\mathbb{R}}$. This is not the case for general toric schemes over S as is shown in [BS10].

Theorem 4.59. *The correspondence $\Pi\mapsto\mathcal{X}_{\mathrm{c}(\Pi)}$, where $\mathrm{c}(\Pi)$ is the fan introduced in Definition 3.7, is a bijection between the set of complete SCR polyhedral complexes in $N_{\mathbb{R}}$ and the set of isomorphism classes of proper toric schemes over S of relative dimension n.*

Proof. Follows from [KKMS73, §IV.3(e)] and Corollary 3.15. □

If we are interested in toric schemes as toric models of a toric variety, we can restate the previous result as follows.

Theorem 4.60. *Let Σ be a complete fan in $N_{\mathbb{R}}$. Then there is a bijective correspondence between equivariant isomorphism classes of proper toric models over S of X_Σ and complete SCR polyhedral complexes Π in $N_{\mathbb{R}}$ such that $\operatorname{rec}(\Pi)=\Sigma$.*

Proof. Follows easily from Theorem 4.59. □

For the rest of the section we will restrict ourselves to the proper case and we will denote by Π a complete SCR polyhedral complex. To it we associate a complete fan $\mathrm{c}(\Pi)$ in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ and a complete fan $\mathrm{rec}(\Pi)$ in $N_{\mathbb{R}}$. For short, we will use the notation

$$\mathcal{X}_{\Pi} = \mathcal{X}_{\mathrm{c}(\Pi)}, \tag{4.61}$$

and we will identify the generic fibre $\mathcal{X}_{\Pi,\eta}$ with the toric variety $X_{\mathrm{rec}(\Pi)}$.

Example 4.62. We continue with Example 4.3. The fan Σ_{Δ^n} is in particular an SCR polyhedral complex and the associated toric scheme over S is $\mathbb{P}^n_S$, the projective space over S.

This example can be generalized to any complete fan Σ in $N_{\mathbb{R}}$.

Definition 4.63. Let Σ be a complete fan in $N_{\mathbb{R}}$. Then Σ is also a complete SCR polyhedral complex. Clearly $\mathrm{rec}(\Sigma) = \Sigma$. The toric scheme $\mathcal{X}_{\Sigma}$ is a model over S of X_{Σ} which is called the *canonical model.* Its special fibre

$$\mathcal{X}_{\Sigma,o} = X_{\Sigma,k}$$

is the toric variety over k defined by the fan Σ.

The description of toric orbits in the case of a toric scheme over a DVR is more involved than the case of toric varieties over a field, because we have to consider two kind of orbits.

In the first place, there is a bijection between $\mathrm{rec}(\Pi)$ and the set of orbits under the action of $\mathbb{T}_K$ on $\mathcal{X}_{\Pi,\eta}$, that sends a cone $\sigma \in \mathrm{rec}(\Pi)$ to the orbit $O(\sigma) \subset \mathcal{X}_{\Pi,\eta} = X_{\mathrm{rec}(\Pi)}$ as in the case of toric varieties over a field. We will denote by $\mathcal{V}(\sigma)$ the Zariski closure in $\mathcal{X}_{\Pi}$ of the orbit $O(\sigma)$ with its structure of reduced closed subscheme. Then $\mathcal{V}(\sigma)$ is a horizontal S-scheme, in the sense that the structure morphism $\mathcal{V}(\sigma) \to S$ is dominant, of relative dimension $n - \dim(\sigma)$.

Next we describe $\mathcal{V}(\sigma)$ as a toric scheme over S. As before, we write $N(\sigma) = N/(N \cap \mathbb{R}\sigma)$ and let $\pi_{\sigma}\colon N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$ be the linear projection. Each polyhedron Λ such that $\sigma \subset \mathrm{rec}(\Lambda)$ defines a polyhedron $\pi_{\sigma}(\Lambda)$ in $N(\sigma)_{\mathbb{R}}$. One verifies that these polyhedra form a complete SCR polyhedral complex in $N(\sigma)_{\mathbb{R}}$, that we denote $\Pi(\sigma)$. This polyhedral complex is called the *star* of σ in Π.

Proposition 4.64. *There is a canonical isomorphism of toric schemes*

$$\mathcal{X}_{\Pi(\sigma)} \longrightarrow \mathcal{V}(\sigma).$$

Proof. The proof is analogous to the proof of Proposition 4.6. □

In the second place, there is a bijection between Π and the set of orbits under the action of $\mathbb{T}_k$ on $\mathcal{X}_o$ over the closed point o. Given a polyhedron $\Lambda \in \Pi$, we set

$$\widetilde{N}(\Lambda) = \widetilde{N}/(\widetilde{N} \cap \mathbb{R}\mathrm{c}(\Lambda)), \quad \widetilde{M}(\Lambda) = \widetilde{N}(\Lambda)^{\vee} = \widetilde{M} \cap \mathrm{c}(\Lambda)^{\perp}.$$

We denote $O(\Lambda) = \mathrm{Spec}(k[\widetilde{M}(\Lambda)])$.This is a torus over the residue field k of dimension $n - \dim(\Lambda)$. There is a surjection of rings

$$K^{\circ}[\widetilde{M}_{\Lambda}] \longrightarrow k[\widetilde{M}(\Lambda)], \quad \chi^{(m,l)} \longmapsto \begin{cases} \chi^{(m,l)} & \text{if } (m,l) \in \widetilde{M}(\Lambda), \\ 0 & \text{if } (m,l) \notin \widetilde{M}(\Lambda). \end{cases}$$

Since the element $(0,1)$ does not belong to $\widetilde{M}(\Lambda)$, then this surjection sends the ideal $(\chi^{(0,1)} - \varpi)$ to zero. Therefore, it factorizes through a surjection $K^{\circ}[\mathcal{X}_{\Lambda}] \to k[\widetilde{M}(\Lambda)]$, that defines a closed immersion $O(\Lambda) \hookrightarrow \mathcal{X}_{\Lambda}$. The subscheme $O(\Lambda)$ is contained in the special fibre $\mathcal{X}_{\Pi,o}$, because the surjection sends ϖ to zero. By this reason, the orbits of this type will be called *vertical.*

We will denote by $V(\Lambda)$ the Zariski closure of the orbit $O(\Lambda)$. Then, $V(\Lambda)$ is a vertical cycle in the sense that its image by the structure morphism is the closed point o. We next describe its toric structure. For each polyhedron Λ' such that Λ is a face of Λ', the image of $\mathrm{c}(\Lambda')$ under the projection $\pi_\Lambda\colon \widetilde{N}_{\mathbb{R}} \to \widetilde{N}(\Lambda)_{\mathbb{R}}$ is a strongly convex rational cone that we denote $\sigma_{\Lambda'}$. The cones $\sigma_{\Lambda'}$ form a fan of $\widetilde{N}(\Lambda)_{\mathbb{R}}$ that we denote $\Pi(\Lambda)$. Observe that the fan $\Pi(\Lambda)$ is the analogue of the star of a cone defined in (4.5). For each cone $\sigma \in \Pi(\Lambda)$ there is a unique polyhedron $\Lambda_\sigma \in \Pi$ such that Λ is a face of Λ_σ and $\sigma = \pi_\Lambda(\mathrm{c}(\Lambda_\sigma))$.

Proposition 4.65. *There is a canonical isomorphism of toric varieties over k*

$$X_{\Pi(\Lambda),k} \longrightarrow V(\Lambda).$$

Proof. Again, the proof is analogous to the proof of Proposition 4.6. □

The description of the adjacency relations between orbits is similar to the one for toric varieties over a field. The orbit $V(\Lambda)$ is contained in $V(\Lambda')$ if and only if the polyhedron Λ' is a face of the polyhedron Λ. Similarly, $\mathcal{V}(\sigma)$ is contained in $\mathcal{V}(\sigma')$ if and only if σ' is a face of σ. Finally, $V(\Lambda)$ is contained in $\mathcal{V}(\sigma)$ if and only if σ is a face of the cone $\mathrm{rec}(\Lambda)$.

Remark 4.66. As a consequence of the above construction, we see that there is a one-to-one correspondence between the vertexes of Π and the components of the special fibre. For each $v \in \Pi^0$, the component $V(v)$ is a toric variety over k defined by the fan $\Pi(v)$ in $\widetilde{N}_{\mathbb{R}}/\mathbb{R}(v,1)$. The orbits contained in $V(v)$ correspond to the polyhedra $\Lambda \in \Pi$ containing v. In particular, the components given by two vertexes $v, v' \in \Pi^0$ share an orbit of dimension l if and only if there exists a polyhedron of dimension $n-l$ containing both v and v'.

To each polyhedron $\Lambda \in \Pi$, hence to each vertical orbit, we can associate a combinatorial invariant, which we call its multiplicity. For a vertex $v \in \Pi^0$, this invariant agrees with the order of vanishing of ϖ along the component $V(v)$ (see (4.87)).

Denote by $\jmath\colon N \to \widetilde{N}$ the inclusion $\jmath(u) = (u,0)$ and by $\mathrm{pr}\colon \widetilde{M} \to M$ the projection $\mathrm{pr}(m,l) = m$. We identify N with its image. We set

$$N(\Lambda) = N/(N \cap \mathbb{R}\mathrm{c}(\Lambda)), \quad M(\Lambda) = M \cap \mathrm{pr}(\mathrm{c}(\Lambda)^\perp).$$

Remark 4.67. The lattice $M(\Lambda)$ can also be described as $M(\Lambda) = M \cap L_\Lambda^\perp$. Therefore, for a cone $\sigma \subset N_{\mathbb{R}}$, the notation just introduced agrees with the one in (4.4). Here, the polytope Λ is contained in $N_{\mathbb{R}}$. By contrast, for a polyhedron $\Gamma \subset M_{\mathbb{R}}$, we follow Notation 3.103, so $M(\Gamma) = M \cap L_\Gamma$.

Then $\jmath$ and pr induce inclusions of lattices of finite index $N(\Lambda) \to \widetilde{N}(\Lambda)$ and $\widetilde{M}(\Lambda) \to M(\Lambda)$, that we denote also by $\jmath$ and pr, respectively. These inclusions are dual of each other and in particular, their indexes agree.

Definition 4.68. The *multiplicity* of a polyhedron $\Lambda \in \Pi$ is defined as

$$\mathrm{mult}(\Lambda) = [M(\Lambda) : \mathrm{pr}(\widetilde{M}(\Lambda))] = [\widetilde{N}(\Lambda) : \jmath(N(\Lambda))].$$

Lemma 4.69. *If* $\Lambda \in \Pi$*, then* $\mathrm{mult}(\Lambda) = \min\{n \geq 1 \mid \exists p \in \mathrm{aff}(\Lambda),\ np \in N\}$.

Proof. We consider the inclusion $\mathbb{Z} \to \widetilde{N}(\Lambda)$ that sends $n \in \mathbb{Z}$ to the class of $(0, n)$. There is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N(\Lambda) \cap \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/(N(\Lambda) \cap \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N(\Lambda) \arrow[r] \arrow[d] & \widetilde{N}(\Lambda) \arrow[r] \arrow[d] & \widetilde{N}(\Lambda)/N(\Lambda) \arrow[r] & 0 \\
 & N(\Lambda)/(N(\Lambda) \cap \mathbb{Z}) \arrow[r] \arrow[d] & \widetilde{N}(\Lambda)/\mathbb{Z} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

It is easy to see that the bottom arrow in the diagram is an isomorphism. By the Snake lemma the right vertical arrow is an isomorphism. Therefore

$$\operatorname{mult}(\Lambda) = [\mathbb{Z} : N(\Lambda) \cap \mathbb{Z}].$$

We verify that $N(\Lambda) \cap \mathbb{Z} = \{n \in \mathbb{Z} \mid \exists p \in \operatorname{aff}(\Lambda),\ np \in N\}$, from which the lemma follows. $\square$

We now discuss equivariant morphisms of toric schemes.

Definition 4.70. Let $\mathbb{T}_i$, $i = 1, 2$, be split tori over S and $\rho \colon \mathbb{T}_1 \to \mathbb{T}_2$ a morphism of algebraic group schemes. Let $\mathcal{X}_i$ be toric schemes over S with torus $\mathbb{T}_i$ and let μ_i denote the corresponding action. A morphism $\varphi \colon \mathcal{X}_1 \to \mathcal{X}_2$ is *ρ-equivariant* if the diagram

$$\begin{tikzcd}
\mathbb{T}_1 \times \mathcal{X}_1 \arrow[r, "\mu_1"] \arrow[d, "\rho \times \varphi"'] & \mathcal{X}_1 \arrow[d, "\varphi"] \\
\mathbb{T}_2 \times \mathcal{X}_2 \arrow[r, "\mu_2"'] & \mathcal{X}_2
\end{tikzcd}$$

commutes. A morphism $\varphi \colon \mathcal{X}_1 \to \mathcal{X}_2$ is *ρ-toric* if its restriction to $\mathbb{T}_{1,\eta}$, the torus over K, coincides with that of ρ.

It can be verified that a toric morphism of schemes over S is also equivariant. In the sequel, we extend the construction of equivariant morphisms in §4.2 to proper toric schemes. Before that, we need to relate rational points on the open orbit of the toric variety with lattice points in N.

Definition 4.71. The valuation map of the field, $\operatorname{val} \colon K^{\times} \to \mathbb{Z}$, induces a valuation map on $\mathbb{T}(K)$, also denoted $\operatorname{val} \colon \mathbb{T}(K) \to N$, by the identifications $\mathbb{T}(K) = \operatorname{Hom}(M, K^{\times})$ and $N = \operatorname{Hom}(M, \mathbb{Z})$.

Let $\mathbb{T}_{S,i}$, $i = 1, 2$, be split tori over S. For each i, let N_i be the corresponding lattice and Π_i a complete SCR polyhedral complex in $N_{i,\mathbb{R}}$. Let $A \colon N_1 \to N_2$ be an affine map such that, for every $\Lambda_1 \in \Pi_1$, there exists $\Lambda_2 \in \Pi_2$ with $A(\Lambda_1) \subset \Lambda_2$. Let $p \in \mathcal{X}_{\Pi_2,0}(K) = \mathbb{T}_2(K)$ such that $\operatorname{val}(p) = A(0)$. Write $A = H + \operatorname{val}(p)$, where $H \colon N_1 \to N_2$ is a linear map. H induces a morphism of algebraic groups

$$\rho_H \colon \mathbb{T}_{S,1} \longrightarrow \mathbb{T}_{S,2}.$$

Let $\Sigma_i = \mathrm{rec}(\Pi_i)$. For each cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subset \sigma_2$. Therefore H and p define an equivariant morphism $\varphi_{p,H}\colon X_{\Sigma_1} \to X_{\Sigma_2}$ of toric varieties over K as in Theorem 4.9.

Proposition 4.72. *With the above hypothesis, the morphism $\varphi_{p,H}$ can be extended to a ρ_H-equivariant morphism*

$$\Phi_{p,A}\colon \mathcal{X}_{\Pi_1} \longrightarrow \mathcal{X}_{\Pi_2}.$$

Proof. Let $\Lambda_i \in \Pi_i$ such that $A(\Lambda_1) \subset \Lambda_2$. Then the map $\widetilde{M}_2 \to \widetilde{M}_1$ given by $(m,l) \mapsto (H^\vee m, \langle \mathrm{val}(p), m\rangle + l)$ for $m \in M$ and $l \in \mathbb{Z}$ (which is just the dual of the linearization of A) induces a morphism of semigroups $\widetilde{M}_{2,\Lambda_2} \to \widetilde{M}_{1,\Lambda_1}$. Since $\chi^m(p)\varpi^{-\langle \mathrm{val}(p),m\rangle}$ belongs to K°, the assignment

$$\chi^{(m,l)} \longmapsto (\chi^m(p)\varpi^{-\langle \mathrm{val}(p),m\rangle})\chi^{(H^\vee m, \langle \mathrm{val}(p),m\rangle + l)}$$

defines a ring morphism $K^\circ[\widetilde{M}_{2,\Lambda_2}] \to K^\circ[\widetilde{M}_{1,\Lambda_1}]$. This morphism sends $\chi^{(0,1)} - \varpi$ to $\chi^{(0,1)} - \varpi$, hence induces a morphism $K^\circ[\mathcal{X}_{\Lambda_2}] \to K^\circ[\mathcal{X}_{\Lambda_1}]$ and a map $\mathcal{X}_{\Lambda_1} \to \mathcal{X}_{\Lambda_2}$. Varying Λ_1 and Λ_2 we obtain maps, that glue together into a map

$$\Phi_{p,A}\colon \mathcal{X}_{\Pi_1} \longrightarrow \mathcal{X}_{\Pi_2}.$$

By construction, this map extends $\varphi_{p,H}$ and is equivariant with respect to the morphism ρ_H. □

As an example of the above construction, we consider the toric subschemes associated to orbits under the action of subtori. Let N be a lattice, Π a complete SCR polyhedral complex in $N_\mathbb{R}$ and set $\Sigma = \mathrm{rec}(\Pi)$. Let $Q \subset N$ be a saturated sublattice and let $p \in X_{\Sigma,0}(K)$. We set $u_0 = \mathrm{val}(p)$. We consider the affine map $A\colon Q_\mathbb{R} \to N_\mathbb{R}$ given by $A(v) = v + u_0$. Recall that the sublattice Q and the point p induce maps of toric varieties (4.11)

$$X_{\Sigma_Q} \longrightarrow Y_{\Sigma_Q,p} \hookrightarrow X_\Sigma.$$

We want to identify the toric model of X_{Σ_Q} induced by the toric model $\mathcal{X}_\Pi$ of X_Σ. We define the complete SCR polyhedral complex $\Pi_{Q,u_0} = A^{-1}\Pi$ of $Q_\mathbb{R}$. Then, $\mathrm{rec}(\Pi_{Q,u_0}) = \Sigma_Q$. Applying the construction of Proposition 4.72, we obtain an equivariant morphism of schemes over S

$$\mathcal{X}_{\Pi_{Q,u_0}} \longrightarrow \mathcal{X}_\Pi. \tag{4.73}$$

The image of this map is the Zariski closure of $Y_{\Sigma_Q,p}$ and $\mathcal{X}_{\Pi_{Q,u_0}}$ is a toric model of X_{Σ_Q}. This map will be denoted either as $\Phi_{p,A}$ or $\Phi_{p,Q}$. Observe that the abstract toric scheme $\mathcal{X}_{\Pi_{Q,u_0}}$ only depends on Q and on $\mathrm{val}(p)$.

4.6. **$\mathbb{T}$-Cartier divisors on toric schemes.** The theory of $\mathbb{T}$-Cartier divisors carries over to the case of toric schemes over a DVR. Let $\mathcal{X}$ be a toric scheme over S with torus $\mathbb{T}_S$. There are two morphisms from $\mathbb{T}_S \times \mathcal{X}$ to $\mathcal{X}$: the toric action, that we denote by μ, and the second projection, that we denote by π_2. A Cartier divisor D on $\mathcal{X}$ is called a $\mathbb{T}$-*Cartier divisor* if $\mu^* D = \pi_2^* D$.

$\mathbb{T}$-Cartier divisors over a toric scheme can be described combinatorially. For simplicity, we will discuss only the case of proper schemes. So, let Π be a complete SCR polyhedral complex in $N_\mathbb{R}$, and $\mathcal{X}_\Pi$ the corresponding toric scheme. Let ψ be an H-lattice function on Π (Definitions 3.88 and 3.60). Then ψ defines a $\mathbb{T}$-Cartier divisor in a way similar to the one for toric varieties over a field. We recall that the schemes $\{\mathcal{X}_\Lambda\}_{\Lambda\in\Pi}$ form an open cover of $\mathcal{X}_\Pi$. Choose a set of defining vectors $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ of ψ. Then we set

$$D_\psi = \{(\mathcal{X}_\Lambda, \varpi^{-l_\Lambda}\chi^{-m_\Lambda})\}_{\Lambda\in\Pi},$$

where we are using the identification (4.57). The divisor D_ψ only depends on ψ and not on a particular choice of defining vectors.

We consider now toric varieties and $\mathbb{T}$-Cartier divisors over S as models of toric varieties and $\mathbb{T}$-Cartier divisors over K.

Definition 4.74. Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a virtual support function on Σ. Let (X_Σ, D_Ψ) be the associated toric variety and $\mathbb{T}$-Cartier divisor defined over K. A *toric model* of (X_Σ, D_Ψ) is a triple $(\mathcal{X}, D, e)$, where $\mathcal{X}$ is a toric model over S of X, D is a $\mathbb{T}$-Cartier divisor on $\mathcal{X}$ and $e > 0$ is an integer such that the isomorphism $\iota\colon X_\Sigma \to \mathcal{X}_\eta$ that extends the identity of $\mathbb{T}_K$ satisfies $\iota^*(D) = eD_\Psi$. When $e = 1$, the toric model $(\mathcal{X}, D, 1)$ will be denoted simply by $(\mathcal{X}, D)$. A toric model will be called *proper* whenever the scheme $\mathcal{X}$ is proper over S.

Example 4.75. We continue with Example 4.62. The function Ψ_{Δ^n} is an H-lattice concave function on Σ_{Δ^n} and $(\mathbb{P}^n_S, D_{\Psi_{\Delta^n}})$ is a proper toric model of $(\mathbb{P}^n_K, D_{\Psi_{\Delta^n}})$.

This example can be generalized as follows.

Definition 4.76. Let Σ be a complete fan in $N_\mathbb{R}$ and let Ψ be a virtual support function on Σ. Then Σ is a complete SCR polyhedral complex in $N_\mathbb{R}$ and Ψ is a rational piecewise affine function on Σ. Then $(\mathcal{X}_\Sigma, D_\Psi)$ is a model over S of (X_Σ, D_Ψ), which is called the *canonical model*.

Definition 4.77. Let $\mathcal{X}$ be a toric scheme and $\mathcal{L}$ a line bundle on $\mathcal{X}$. A *toric structure* on $\mathcal{L}$ is the choice of an element z of the fibre $\mathcal{L}_{x_0}$, where $x_0 \in \mathcal{X}_\eta$ is the distinguished point. A *toric line bundle* on $\mathcal{X}$ is a pair $(\mathcal{L}, z)$, where $\mathcal{L}$ is a line bundle over $\mathcal{X}$ and v is a toric structure on $\mathcal{L}$. Frequently, when the toric structure is clear from the context, the element z will be omitted from the notation and a toric line bundle will be denoted by the underlying line bundle. A *toric section* is a rational section that is regular and non vanishing over the principal open subset $X_0 \subset \mathcal{X}_\eta$ and such that $s(x_0) = z$. Exactly as in the case of toric varieties over a field, each $\mathbb{T}$-Cartier divisor defines a toric line bundle $\mathcal{O}(D)$ together with a toric section. When the $\mathbb{T}$-Cartier divisor comes from an H-lattice function ψ, the toric line bundle and toric section will be denoted $\mathcal{L}_\psi$ and s_ψ respectively.

In this section we will mainly use the language of $\mathbb{T}$-Cartier divisors, but in §6 we will prefer the language of toric line bundles.

The following result follows directly form the definitions.

Proposition 4.78. *Let (X_Σ, D_Ψ) be a toric variety with a $\mathbb{T}$-Cartier divisor. Every toric model $(\mathcal{X}, D, e)$ of (X_Σ, D_Ψ) induces a model $(\mathcal{X}, \mathcal{O}(D), e)$ of (X_Σ, L_Ψ), in the sense of Definition 2.16, where the identification of $\mathcal{O}(D)|_{X_\Sigma}$ with $L_\Psi^{\otimes e}$ matches the toric sections. Such models will be called* toric models.

Proposition-Definition 4.79. We say that two toric models $(\mathcal{X}_i, D_i, e_i)$, $i = 1, 2$, are *equivalent*, if there exists a toric model $(\mathcal{X}', D', e')$ of (X_Σ, D_Ψ) and morphisms of toric models $\alpha_i\colon \mathcal{X}' \to \mathcal{X}_i$, $i = 1, 2$, such that $e'\alpha_i^* D_i = e_i D'$. This is an equivalence relation.

Proof. Symmetry and reflexivity are straightforward. For transitivity assume that we have toric models $(\mathcal{X}_i, D_i, e_i)$, $i = 1, 2, 3$, that the first and second model are equivalent through $(\mathcal{X}', D', e')$ and that the second and the third are equivalent through $(\mathcal{X}'', D'', e'')$. Then, by Theorem 4.60, $\mathcal{X}'$ and $\mathcal{X}''$ are defined by SCR polyhedral complexes Π' and Π'' respectively, with $\operatorname{rec}(\Pi') = \operatorname{rec}(\Pi'') = \Sigma$. Let $\Pi''' = \Pi' \cdot \Pi''$. By Lemma 3.11, $\operatorname{rec}(\Pi''') = \Sigma$. Thus Π''' determines a model $\mathcal{X}'''$ of X_Σ. This model has morphisms β' and β'' to $\mathcal{X}'$ and $\mathcal{X}''$ respectively. We put $e''' = e'e''$ and $D''' = e''\beta'^* D' = e'\beta''^* D''$. Now it is easy to verify that $(\mathcal{X}''', D''', e''')$ provides the transitivity property. $\square$

We are interested in proper toric models and equivalence classes because, by Definition 2.17, a proper toric model of (X_Σ, D_Ψ) induces an algebraic metric on L_Ψ^{an}. By Proposition 2.18, equivalent toric models define the same algebraic metric.

We can classify proper models of $\mathbb{T}$-Cartier divisors (and therefore of toric line bundles) in terms of H-lattice functions. We first recall the classification of $\mathbb{T}$-Cartier divisors.

Theorem 4.80. *Let Π be a complete SCR polyhedral complex in $N_\mathbb{R}$ and let $\mathcal{X}_\Pi$ be the associated toric scheme over S. The correspondence $\psi \mapsto D_\psi$ is an isomorphism between the group of H-lattice functions on Π and the group of $\mathbb{T}$-Cartier divisors on $\mathcal{X}_\Pi$. Moreover, if ψ_1 and ψ_2 are two H-lattice functions on Π, then the divisors D_{ψ_1} and D_{ψ_2} are rationally equivalent if and only if $\psi_1 - \psi_2$ is affine.*

Proof. The result follows from [KKMS73, §IV.3(h)]. □

We next derive the classification theorem for models of $\mathbb{T}$-Cartier divisors.

Theorem 4.81. *Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a virtual support function on Σ. Then the correspondence $(\Pi, \psi) \mapsto (\mathcal{X}_\Pi, D_\psi)$ is a bijection between:*

- *the set of pairs (Π, ψ), where Π is a complete SCR polyhedral complex in $N_\mathbb{R}$ with* $\mathrm{rec}(\Pi) = \Sigma$ *and ψ is an H-lattice function on Π such that* $\mathrm{rec}(\psi) = \Psi$*;*
- *the set of isomorphism classes of toric models $(\mathcal{X}, D)$ of (X_Σ, D_Ψ).*

Proof. Denote by $\iota\colon X_\Sigma = X_{\mathrm{rec}(\Pi)} \to \mathcal{X}_\Pi$ the open immersion of the generic fibre. The recession function (Definition 3.85) determines the restriction of the $\mathbb{T}$-Cartier divisor to the fibre over the generic point. Therefore, when ψ is an H-lattice function on Π with $\mathrm{rec}(\psi) = \Psi$, we have that

$$\iota^* D_\psi = D_{\mathrm{rec}(\psi)} = D_\Psi. \tag{4.82}$$

Thus $(\mathcal{X}_\Pi, D_\psi)$ is a toric model of (X_Σ, D_Ψ). The statement follows from Theorem 4.60 and Theorem 4.80. □

Remark 4.83. Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a virtual support function on Σ. Let $(\mathcal{X}, D, e)$ be a toric model of (X_Σ, D_Ψ). Then, by Theorem 4.81, there exists a complete SCR polyhedral complex Π in $N_\mathbb{R}$ with $\mathrm{rec}(\Pi) = \Sigma$ and a rational piecewise affine function ψ on Π such that $e\psi$ is an H-lattice function, $\mathrm{rec}(\psi) = \Psi$ and $(\mathcal{X}, D, e) = (\mathcal{X}_\Pi, D_{e\psi}, e)$. Moreover, if $(\mathcal{X}', D', e')$ is another toric model that gives the function ψ', then both models are equivalent if and only if $\psi = \psi'$. Thus, to every toric model we have associated a rational piecewise affine function ψ on Π such that $\mathrm{rec}(\psi) = \Psi$. Two equivalent models give rise to the same function.

The converse is not true. Given a rational piecewise affine function ψ, with $\mathrm{rec}(\psi) = \Psi$, we can find a complete SCR polyhedral complex Π such that ψ is piecewise affine on Π. But, in general $\mathrm{rec}(\Pi)$ does not agree with Σ. What we can expect is that $\Sigma' := \mathrm{rec}(\Pi)$ is a refinement of Σ. Therefore the function ψ gives us an equivalence class of toric models of $(X_{\Sigma'}, D_\Psi)$. But ψ may not determine an equivalence class of toric models of (X_Σ, D_Ψ). In Corollary 5.43 in next section we will give a necessary condition for a function ψ to define an equivalence class of toric models of (X_Σ, D_Ψ) and in Example 5.44 we will exhibit a function that does not satisfy this necessary condition. By contrast, as we will see in Theorem 4.97, the concave case is much more transparent.

The correspondence between $\mathbb{T}$-Cartier divisors and $\mathbb{T}$-Weil divisors has to take into account that we have two types of orbits. Each vertex $v \in \Pi^0$ defines a vertical invariant prime Weil divisor $V(v)$ and every ray $\tau \in \mathrm{rec}(\Pi)^1$ defines a horizontal prime Weil divisor $\mathcal{V}(\tau)$. If $v \in \Pi^0$ is a vertex, by Lemma 4.69, its multiplicity $\mathrm{mult}(v)$ is the smallest positive integer $\nu \geq 1$ such that $\nu v \in N$. If τ is a ray, we denote by v_τ the smallest lattice point of $\tau \setminus \{0\}$.

Proposition 4.84. *Let ψ be an H-lattice function on Π. Let D_ψ be the associated $\mathbb{T}$-Cartier divisor. Then the corresponding $\mathbb{T}$-Weil divisor is given by*

$$[D_\psi] = \sum_{v\in\Pi^0} -\operatorname{mult}(v)\psi(v)V(v) + \sum_{\tau\in\operatorname{rec}(\Pi)^1} -\operatorname{rec}(\psi)(v_\tau)\mathcal{V}(\tau). \tag{4.85}$$

Proof. By Lemma 4.69, for $v\in\Pi^0$, the vector $\operatorname{mult}(v)v$ is the minimal lattice vector in the ray $\mathrm{c}(v)$. Now it is easy to adapt the proof of [Ful93, §3.3, Lemma] to prove this proposition. □

Example 4.86. Consider the constant H-lattice function $\psi_\varpi(u) = -1$. This function corresponds to the principal divisor $\operatorname{div}(\varpi)$. Then

$$\operatorname{div}(\varpi) = \sum_{v\in\Pi^0} \operatorname{mult}(v)V(v). \tag{4.87}$$

Thus, for a vertex v, the multiplicity of v agrees with the multiplicity of the divisor $V(v)$ in the special fibre $\operatorname{div}(\varpi)$. In particular, the special fibre $\mathcal{X}_{\Pi,o}$ is reduced if and only if all vertexes of Π^0 belong to N.

We next study the restriction of $\mathbb{T}$-Cartier divisors to orbits and their inverse image by equivariant morphisms. Let Π be a complete SCR polyhedral complex in $N_\mathbb{R}$, and ψ an H-lattice function on Π. Set $\Sigma = \operatorname{rec}(\Pi)$, and $\Psi = \operatorname{rec}(\psi)$. Choose sets of defining vectors $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ and $\{m_\sigma\}_{\sigma\in\Sigma}$ for ψ and Ψ, respectively.

Let $\sigma\in\Sigma$. We describe the restriction of D_ψ to $\mathcal{V}(\sigma)$, the closure of a horizontal orbit. As in the case of toric varieties over a field, we first consider the case when $\Psi|_\sigma = 0$. Recall that $\mathcal{V}(\sigma)$ agrees with the toric scheme associated to the polyhedral complex $\Pi(\sigma)$ and that each element of $\Pi(\sigma)$ is the image by $\pi_\sigma\colon N_\mathbb{R}\to N(\sigma)_\mathbb{R}$ of a polyhedron $\Lambda\in\Pi$ with $\sigma\subset\operatorname{rec}(\Lambda)$. The condition $\Psi|_\sigma = 0$ implies that we can define

$$\psi(\sigma)\colon N(\sigma)_\mathbb{R} \longrightarrow \mathbb{R},\quad u+\mathbb{R}\sigma \longmapsto \psi(u+v) \tag{4.88}$$

for any $v\in\mathbb{R}\sigma$ such that $u+v\in\bigcup_{\operatorname{rec}(\Lambda)\supset\sigma}\Lambda$. The function $\psi(\sigma)$ can also be described in terms of defining vectors. For each $\Lambda\in\Pi$ with $\sigma\subset\operatorname{rec}(\Lambda)$, we will denote $\overline{\Lambda}\in\Pi(\sigma)$ for its image by π_σ. For each Λ as before, the condition $\Psi|_\sigma = 0$ implies that $m_\Lambda\in M(\sigma)$. Hence we define $(m_{\overline{\Lambda}}, l_{\overline{\Lambda}}) = (m_\Lambda, l_\Lambda)$ for $\Lambda\in\Pi$ with $\operatorname{rec}(\Lambda)\supset\sigma$.

Proposition 4.89. *If $\Psi|_\sigma = 0$ then the divisor D_ψ and the horizontal orbit $\mathcal{V}(\sigma)$ intersect properly. Moreover, the set $\{(m_{\overline{\Lambda}}, l_{\overline{\Lambda}})\}_{\overline{\Lambda}\in\Pi(\sigma)}$ is a set of defining vectors of $\psi(\sigma)$ and the restriction of D_ψ to $\mathcal{V}(\sigma)$ is $D_{\psi(\sigma)}$.*

Proof. The proof is analogous to the proof of Proposition 4.31. □

If $\Psi|_\sigma \neq 0$, then $\mathcal{V}(\sigma)$ and D_ψ do not intersect properly and we can only restrict D_ψ with $\mathcal{V}(\sigma)$ up to rational equivalence. To this end, we consider the divisor $D_{\psi-m_\sigma}$, that is rationally equivalent to D_ψ and intersects properly with $\mathcal{V}(\sigma)$. The restriction of this divisor to $\mathcal{V}(\sigma)$ corresponds to the H-lattice function $(\psi-m_\sigma)(\sigma)$ as defined above.

Let now $\Lambda\in\Pi$ be a polyhedron. We will denote by $\widetilde{\pi}_\Lambda\colon \widetilde{N}\to\widetilde{N}(\Lambda)$ and $\pi_\Lambda\colon N\to N(\Lambda)$ the projections and by $\widetilde{\pi}_\Lambda^\vee\colon \widetilde{M}(\Lambda)\to\widetilde{M}$ and $\pi_\Lambda^\vee\colon M(\Lambda)\to M$ the dual maps. We will use the same notation for the linear maps obtained by tensoring with $\mathbb{R}$.

We first assume that $\psi|_\Lambda = 0$. If $u\in\widetilde{N}(\Lambda)_\mathbb{R}$, then there exists a polyhedron Λ' with Λ a face of Λ' and a point $(v,r)\in\mathrm{c}(\Lambda')$ that is sent to u under the projection $\widetilde{\pi}_\Lambda$. Then we set

$$\psi(\Lambda)\colon \widetilde{N}(\Lambda)_\mathbb{R} \longrightarrow \mathbb{R},\quad u\longmapsto r\psi(v/r) = m_{\Lambda'}(v) + rl_{\Lambda'}. \tag{4.90}$$

The condition $\psi|_\Lambda = 0$ implies that the above equation does not depend on the choice of (v, r).

We can describe also $\psi(\Lambda)$ in terms of defining vectors. For each cone $\sigma \in \Pi(\Lambda)$ let $\Lambda_\sigma \in \Pi$ be the polyhedron that has Λ as a face and such that $\mathrm{c}(\Lambda)$ is mapped to σ by $\widetilde{\pi}_\Lambda$. The condition $\psi|_\Lambda = 0$ implies that $(m_{\Lambda_\sigma}, l_{\Lambda_\sigma}) \in \widetilde{M}(\Lambda)$. We set $m_\sigma = (m_{\Lambda_\sigma}, l_{\Lambda_\sigma})$.

Proposition 4.91. *If $\psi|_\Lambda = 0$ then the divisor D_ψ intersects properly the orbit $V(\Lambda)$. Moreover, the set $\{m_\sigma\}_{\sigma\in\Pi(\Lambda)}$ is a set of defining vectors of $\psi(\Lambda)$ and the restriction of D_ψ to $V(\Lambda)$ is the divisor $D_{\psi(\Lambda)}$.*

Proof. The proof is analogous to that of Proposition 4.31. □

As before, when $\psi|_\Lambda \neq 0$, we can only restrict D_ψ to $V(\Lambda)$ up to rational equivalence. In this case we just apply the previous proposition to the function $\psi - m_\Lambda - l_\Lambda$.

Example 4.92. We particularize (4.90) to the case of one-dimensional vertical orbits. Let Λ be a $(n-1)$-dimensional polyhedron. Hence $V(\Lambda)$ is a vertical curve. Let Λ_1 and Λ_2 be the two n-dimensional polyhedron that have Λ as a common face. Let $v \in N_{\mathbb{Q}}$ such that the class $[(v, 0)]$ is a generator of the lattice $\widetilde{N}(\Lambda)$ and the affine space $(v, 0) + \mathbb{R}\,\mathrm{c}(\Lambda)$ meets $\mathrm{c}(\Lambda_1)$. This second condition fixes one of the two generators of $\widetilde{N}(\Lambda)$. Then, by equation (4.25)

$$\deg_{D_\psi}(V(\Lambda)) = \deg([D_\psi|_{V(\Lambda)}]) = m_{\Lambda_2}(v) - m_{\Lambda_1}(v). \tag{4.93}$$

We end this section discussing the inverse image of a $\mathbb{T}$-Cartier divisor by an equivariant morphisms. With the notation of Proposition 4.72, let ψ be an H-lattice function on Π_2, and $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi_2}$ a set of defining vectors of ψ. For each $\Gamma \in \Pi_1$ we choose a polyhedron $\Gamma' \in \Pi_2$ such that $A(\Gamma) \subset \Gamma'$. We set $m_\Gamma = H^\vee(m_{\Gamma'})$ and $l_\Gamma = m_{\Gamma'}(\mathrm{val}(p)) + l_{\Gamma'}$. The following proposition follows easily.

Proposition 4.94. *The divisor D_ψ intersects properly the image of $\Phi_{p,A}$. The function $\psi \circ A$ is an H-lattice function on Π_1 and*

$$\Phi_{p,A}^* D_\psi = D_{\psi\circ A}.$$

Moreover, $\{(m_\Gamma, l_\Gamma)\}_{\Gamma\in\Pi_1}$ is a set of defining vectors of $\psi \circ A$.

4.7. **Positivity on toric schemes.** The relationship between the positivity of the line bundle and the concavity of the virtual support function can be extended to the case of toric schemes over a DVR. In particular, we have the following version of the Nakai-Moishezon criterion.

Theorem 4.95. *Let Π be a complete SCR complex in $N_{\mathbb{R}}$ and $\mathcal{X}_\Pi$ its associate toric scheme over S. Let ψ be an H-lattice function on Π and D_ψ the corresponding $\mathbb{T}$-Cartier divisor on $\mathcal{X}_\Pi$.*

1. *The following properties are equivalent:*
   - (a) *D_ψ is ample;*
   - (b) *$D_\psi \cdot C > 0$ for every vertical curve C contained in $X_{\Pi,o}$;*
   - (c) *$D_\psi \cdot V(\Lambda) > 0$ for every $(n-1)$-dimensional polyhedron $\Lambda \in \Pi$;*
   - (d) *The function ψ is strictly concave on Π.*
2. *The following properties are equivalent:*
   - (a) *D_ψ is generated by global sections;*
   - (b) *$D_\psi \cdot C \geq 0$ for every vertical curve C contained in $X_{\Sigma,o}$;*
   - (c) *$D_\psi \cdot V(\Lambda) \geq 0$ for every $(n-1)$-dimensional polyhedron $\Lambda \in \Pi$;*
   - (d) *The function ψ is concave.*

Proof. In both cases, the fact that (a) implies (b) and that (b) implies (c) is clear. The fact that (c) implies (d) follows from equation (4.93). The fact that (1d) implies (1a) is [KKMS73, §IV.3(k)].

Finally, we prove that (2d) implies (2a). Let ψ be an H-lattice concave function. Each pair $(m,l) \in \widetilde{M}$ defines a rational section $\varpi^l \chi^m s_\psi$ of D_ψ. The section is regular if and only if the function $m(u)+l$ lies above ψ. Moreover, for a polyhedron $\Lambda \in \Pi$, this section does not vanish on $\mathcal{X}_\Lambda$ if and only if $\psi(u) = m(u) + l$ for all $u \in \Lambda$. Therefore, the affine pieces of the graph of ψ define a set of global sections that generate $\mathcal{O}(D_\psi)$. □

Definition 4.96. We will say that a $\mathbb{T}$-Cartier divisor on a toric scheme is *semipositive* if it is generated by global sections. Let X_Σ be a proper toric variety over K and let D_Ψ be a $\mathbb{T}$-Cartier divisor generated by global sections. A toric model $(\mathcal{X}, D, e)$ is called *semipositive* if D is semipositive.

Observe that, by Theorem 4.95, a toric model is semipositive if the associated metric is semipositive as in Definition 2.26. Equivalence classes of semipositive toric models are classified by rational concave functions.

Theorem 4.97. *Let Σ be a complete fan in $N_\mathbb{R}$. Let Ψ be a support function on Σ. Then the correspondence of Theorem 4.81 induces a bijective correspondence between the set of rational piecewise affine concave functions ψ with* $\operatorname{rec}(\psi) = \Psi$ *and the set of equivalence classes of semipositive toric models of (X_Σ, D_Ψ) over S.*

Proof. Let $(\mathcal{X}, D, e)$ be a semipositive toric model. By Theorem 4.81, to the pair $(\mathcal{X}, D)$ corresponds a pair (Π, ψ'), where ψ' is an H-lattice function on Π, $\operatorname{rec}(\Pi) = \Sigma$ and $\operatorname{rec}(\psi') = e\Psi$. By Theorem 4.95, the function ψ' is concave. We put $\psi = \frac{1}{e}\psi'$. It is clear that equivalent models produce the same function.

Conversely, let ψ be a rational piecewise affine concave function. Let $\Pi' = \Pi(\psi)$. This is a rational polyhedral complex. Let $\Sigma' = \operatorname{rec}(\Pi')$. This is a conic rational polyhedral complex. By Proposition 3.72, $\Sigma' = \Pi(\Psi)$. Since Ψ is a support function on Σ, we deduce that Σ is a refinement of Σ'. Put $\Pi = \Pi' \cdot \Sigma$ (Definition 3.10). Since Π' is a rational polyhedral complex and Σ is a fan, then Π is an SCR polyhedral complex. Moreover, by Lemma 3.11, we have

$$\operatorname{rec}(\Pi) = \operatorname{rec}(\Pi' \cdot \Sigma) = \operatorname{rec}(\Pi') \cdot \operatorname{rec}(\Sigma) = \Sigma' \cdot \Sigma = \Sigma.$$

Let $e > 0$ be an integer such that $e\psi$ is an H-lattice function. Then $(\mathcal{X}_\Pi, D_{e\psi}, e)$ is a toric model of (X_Σ, D_Ψ). Both procedures are inverse of each other. □

Recall that, for toric varieties over a field, a $\mathbb{T}$-Cartier divisor generated by global sections can be determined, either by the support function Ψ or by its stability set Δ_Ψ. In the case of toric schemes over a DVR, if ψ is a concave rational piecewise affine function on Π and $\Psi = \operatorname{rec}(\psi)$, then the stability set of ψ agrees with the stability set of Ψ. Then the equivalence class of toric models determined by ψ is also determined by the Legendre-Fenchel dual function $\psi^\vee$.

Corollary 4.98. *Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a support function on Σ. There is a bijection between equivalence classes of semipositive toric models of (X_Σ, D_Ψ) and rational piecewise affine concave functions on $M_\mathbb{R}$, with effective support Δ_Ψ.*

Proof. This follows from Theorem 4.97, Proposition 3.75 and Proposition 3.77. □

When D_ψ is generated by global sections, that is, when ψ is concave, we can interpret its restriction to toric orbits in terms of direct and inverse images of concave functions.

Proposition 4.99. *Let Π be a complete SCR polyhedral complex in $N_{\mathbb{R}}$ and ψ an H-lattice concave function on Π. Set $\Sigma = \operatorname{rec}(\Pi)$ and $\Psi = \operatorname{rec}(\psi)$. Let $\sigma \in \Sigma$ and $m_\sigma \in M$ such that $\Psi|_\sigma = m_\sigma|_\sigma$. Let $\pi_\sigma \colon N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$ be the projection and $\pi_\sigma^\vee \colon M(\sigma)_{\mathbb{R}} \to M_{\mathbb{R}}$ the dual inclusion. Then*

$$(\psi - m_\sigma)(\sigma) = (\pi_\sigma)_*(\psi - m_\sigma), \tag{4.100}$$

Hence the restriction of the divisor $D_{\psi - m_\sigma}$ to $\mathcal{V}(\sigma)$ corresponds to the H-lattice concave function $(\pi_\sigma)_*(\psi - m_\sigma)$. Dually,

$$(\psi - m_\sigma)(\sigma)^\vee = (\pi_\sigma^\vee + m_\sigma)^* \psi^\vee. \tag{4.101}$$

In other words, the Legendre-Fenchel dual of $(\psi - m_\sigma)(\sigma)$ is the restriction of $\psi^\vee$ to the face F_σ translated by $-m_\sigma$.

Proof. For equation (4.100), we suppose without loss of generality that $m_\sigma = 0$, and hence $\Psi|_\sigma = 0$. Let $u \in N(\sigma)_{\mathbb{R}}$. Then, the function $\psi|_{\pi_\sigma^{-1}(u)}$ is concave. Let $\Lambda \in \Pi$ such that $\operatorname{rec}(\Lambda) = \sigma$ and $\pi_\sigma^{-1}(u) \cap \Lambda \neq \emptyset$. Then, $\pi_\sigma^{-1}(u) \cap \Lambda$ is a polyhedron of maximal dimension in $\pi_\sigma^{-1}(u)$. The restriction of ψ to this polyhedron is constant and, by (4.88), agrees with $\psi(\sigma)(u)$. Therefore, by concavity,

$$(\pi_\sigma)_*\psi(u) = \max_{v \in \pi_\sigma^{-1}(u)} \psi(v),$$

agrees with $\psi(\sigma)(u)$. Thus we obtain equation (4.100). Equation (4.101) follows from the previous equation and Proposition 3.78(2). To prove equation (4.101) when $m_\sigma \neq 0$ we use Proposition 3.40(4). □

We now consider the case of a vertical orbit. For a function ψ as before, with $\Psi = \operatorname{rec}(\psi)$, we denote by $\mathrm{c}(\psi) \colon \widetilde{N}_{\mathbb{R}} \to \underline{\mathbb{R}}$ the concave function given by

$$\mathrm{c}(\psi)(u, r) = \begin{cases} r\psi(u/r), & \text{if } r > 0, \\ \Psi(u), & \text{if } r = 0, \\ -\infty, & \text{if } r < 0. \end{cases}$$

The function $\mathrm{c}(\psi)$ is a support function on $\mathrm{c}(\Pi)$.

Lemma 4.102. *The stability set of $\mathrm{c}(\psi)$ is the epigraph $\operatorname{epi}(-\psi^\vee) \subset \widetilde{M}_{\mathbb{R}}$.*

Proof. The H-representation of $\mathrm{c}(\psi)$ is

$$\operatorname{dom}(\mathrm{c}(\psi)) = \{(u, r) \in \widetilde{N}_{\mathbb{R}} \mid r \geq 0\},$$
$$\mathrm{c}(\psi)(u, r) = \min_{\Lambda}(m_\Lambda(u) + l_\Lambda r).$$

By Proposition 3.64

$$\operatorname{stab}(\mathrm{c}(\psi)) = \mathbb{R}_{\geq 0}(0, 1) + \operatorname{conv}(\{(m_\Lambda, l_\Lambda)\}_{\Lambda \in \Pi}).$$

Furthermore, by the same proposition, for $x \in \operatorname{stab}(\psi)$,

$$\psi^\vee(x) = \sup\left\{ \sum_\Lambda -\lambda_\Lambda l_\Lambda \,\middle|\, \lambda_\Lambda \geq 0, \sum_\Lambda \lambda_\Lambda = 1, \sum_\Lambda \lambda_\Lambda m_\Lambda = x \right\}.$$

Hence $\operatorname{epi}(-\psi^\vee) = \mathbb{R}_{\geq 0}(0, 1) + \operatorname{conv}(\{(m_\Lambda, l_\Lambda)\}_{\Lambda \in \Pi})$, which proves the statement. □

Proposition 4.103. *Let Π and ψ be as before and let $\Lambda \in \Pi$. Let $m_\Lambda \in M$ and $l_\Lambda \in \mathbb{Z}$ be such that $\psi|_\Lambda = (m_\Lambda + l_\Lambda)|_\Lambda$. Let $\widetilde{\pi}_\Lambda \colon \widetilde{N}_{\mathbb{R}} \to \widetilde{N}(\Lambda)_{\mathbb{R}}$ be the projection, and $\widetilde{\pi}_\Lambda^\vee \colon \widetilde{M}(\Lambda)_{\mathbb{R}} \to \widetilde{M}_{\mathbb{R}}$ the dual map. Then*

$$(\psi - m_\Lambda - l_\Lambda)(\Lambda) = (\widetilde{\pi}_\Lambda)_*(\mathrm{c}(\psi - m_\Lambda - l_\Lambda)). \tag{4.104}$$

Moreover, this is a support function on the fan $\Pi(\Lambda)$. Its stability set is the polytope $\Delta_{\psi,\Lambda} := (\widetilde{\pi}_\Lambda^\vee + (m_\Lambda, l_\Lambda))^{-1}\operatorname{epi}(-\psi^\vee)$. Hence, the restriction of the divisor $D_{\psi-m_\Lambda-l_\Lambda}$ to the variety $V(\Lambda)$ is the divisor associated to the support function of $\Delta_{\psi,\Lambda}$

Proof. To prove equation (4.104) we may assume that $m_\Lambda = 0$ and $l_\Lambda = 0$. Let $u \in \widetilde{N}(\Lambda)_\mathbb{R}$. Then, the function $\mathrm{c}(\psi)|_{\widetilde{\pi}_\Lambda^{-1}(u)}$ is concave. Let $\Lambda' \in \Pi$ such that Λ is a face of Λ' and $\widetilde{\pi}_\Lambda^{-1}(u)\cap \mathrm{c}(\Lambda') \neq \emptyset$. Then, $\widetilde{\pi}_\Lambda^{-1}(u)\cap\mathrm{c}(\Lambda')$ is a polyhedron of maximal dimension of $\widetilde{\pi}_\Lambda^{-1}(u)$ and the restriction of $\mathrm{c}(\psi)$ to this polyhedron is constant and, by equation (4.90), agrees with $\psi(\Lambda)(u)$. Therefore, by concavity,

$$(\widetilde{\pi}_\Lambda)_*\,\mathrm{c}(\psi)(u) = \max_{v\in\pi_\sigma^{-1}(u)} \mathrm{c}(\psi)(v),$$

agrees with $\psi(\Lambda)(u)$. This proves equation (4.104).

Back in the general case when m_Λ and l_Λ may be different from zero, by Proposition 3.78, Proposition 3.40(4) and Lemma 4.102 we have

$$\begin{aligned}
\operatorname{stab}((\widetilde{\pi}_\Lambda)_*(\mathrm{c}(\psi-m_\Lambda-l_\Lambda))) &= (\widetilde{\pi}_\Lambda^\vee)^{-1}\operatorname{stab}(\mathrm{c}(\psi-m_\Lambda-l_\Lambda))\\
&= (\widetilde{\pi}_\Lambda^\vee)^{-1}(\operatorname{stab}(\mathrm{c}(\psi))-(m_\Lambda,l_\Lambda))\\
&= (\widetilde{\pi}_\Lambda^\vee+(m_\Lambda,l_\Lambda))^{-1}\operatorname{stab}(\mathrm{c}(\psi))\\
&= (\widetilde{\pi}_\Lambda^\vee+(m_\Lambda,l_\Lambda))^{-1}\operatorname{epi}(-\psi^\vee).
\end{aligned}$$

The remaining statements are clear. □

We next interpret the above result in terms of dual polyhedral complexes. Let $\Pi(\psi)$ and $\Pi(\psi^\vee)$ be the pair of dual polyhedral complexes associated to ψ. Since ψ is piecewise affine on Π, then Π is a refinement of $\Pi(\psi)$. For each $\Lambda\in\Pi$ we will denote by $\overline{\Lambda}\in\Pi(\psi)$ the smallest element of $\Pi(\psi)$ that contains Λ. It is characterized by the fact that $\operatorname{ri}(\Lambda)\cap\operatorname{ri}(\overline{\Lambda})\neq\emptyset$. Let $\Lambda^*\in\Pi(\psi^\vee)$ be the polyhedron $\Lambda^* = \mathcal{L}\psi(\overline{\Lambda})$. This polyhedron agrees with $\partial\psi(u_0)$ for any $u_0\in\operatorname{ri}(\Lambda)$. Then the function $\psi^\vee|_{\Lambda^*}$ is affine. The polyhedron Λ^*-m_Λ is contained in $M(\Lambda)_\mathbb{R}$. The polyhedron

$$\widetilde{\Lambda^*} = \{(x,-\psi^\vee(x))|x\in\Lambda^*\}$$

is a face of $\operatorname{epi}(-\psi^\vee)$ and it agrees with the intersection of the image of $\pi_\Lambda^\vee+(m_\Lambda,l_\Lambda)$ with this epigraph. We consider the commutative diagram of lattices

$$\begin{tikzcd}
\widetilde{M}(\Lambda) \arrow[r, "\widetilde{\pi}_\Lambda^\vee+(m_\Lambda,l_\lambda)"] \arrow[d, "\mathrm{pr}"'] & \widetilde{M} \arrow[d, "\mathrm{pr}"'] \\
M(\Lambda) \arrow[r, "\pi_\Lambda^\vee+m_\Lambda"] & M,
\end{tikzcd}$$

where $\pi_\Lambda^\vee$ is the inclusion $M(\Lambda)\subset M$, and the corresponding commutative diagram of real vector spaces obtained by tensoring with $\mathbb{R}$. This diagram induces a commutative diagram of polytopes

$$\begin{tikzcd}
\Delta_{\psi,\Lambda} \arrow[r, "\widetilde{\pi}_\Lambda^\vee+(m_\Lambda,l_\lambda)"] \arrow[d, "\mathrm{pr}"'] & \widetilde{\Lambda^*} \arrow[d, "\mathrm{pr}"'] \\
\Lambda^*-m_\Lambda \arrow[r, "\pi_\Lambda^\vee+m_\Lambda"] & \Lambda^*,
\end{tikzcd}$$

where all the arrows are isomorphisms.

In other words, the polytope $\Delta_{\psi,\Lambda}$ associated to the restriction of $D_{\psi-m_\Lambda-l_\Lambda}$ to $V(\Lambda)$ is obtained as follows. We include $\widetilde{M}(\Lambda)_\mathbb{R}$ in $\widetilde{M}_\mathbb{R}$ throughout the affine map

$\widetilde{\pi}_\Lambda^\vee + (m_\Lambda, l_\lambda)$. The image of this map intersects the polyhedron $\mathrm{epi}(-\psi^\vee)$ in the face of it that lies above Λ^*. The inverse image of this face agrees with $\Delta_{\psi,\Lambda}$.

Since we have an explicit description of the polytope $\Delta_{\psi,\Lambda}$, we can easily calculate the degree with respect to D_ψ of an orbit $V(\Lambda)$.

Proposition 4.105. *Let Π be a complete SCR polyhedral complex in $N_\mathbb{R}$ and ψ an H-lattice concave function on Π. Let $\Lambda \in \Pi$ be a polyhedron of dimension $n-k$, $u_0 \in \mathrm{ri}(\Lambda)$ and $\Lambda^* = \partial\psi(u_0)$. Then*

$$\mathrm{mult}(\Lambda)\deg_{D_\psi}(V(\Lambda)) = k!\,\mathrm{vol}_{M(\Lambda)}(\Lambda^*), \tag{4.106}$$

where $\mathrm{mult}(\Lambda)$ *is the multiplicity of* Λ *(see Definition 4.68).*

Proof. From the description of $D_\psi|_{V(\Lambda)}$ and Proposition 4.37, we know that

$$\deg_{D_\psi}(V(\Lambda)) = k!\,\mathrm{vol}_{\widetilde{M}(\Lambda)}(\Delta_{\psi,\Lambda}).$$

Since

$$\mathrm{vol}_{\widetilde{M}(\Lambda)}(\Delta_{\psi,\Lambda}) = \frac{1}{[M(\Lambda):\widetilde{M}(\Lambda)]}\,\mathrm{vol}_{M(\Lambda)}(\Lambda^*),$$

the result follows from the definition of the multiplicity. □

Remark 4.107. If $\dim(\Lambda^*) < k$, then both sides of (4.106) are zero. If $\dim(\Lambda^*) = k$, then $M(\Lambda) = M(\Lambda^*)$ and $\mathrm{vol}_{M(\Lambda)}(\Lambda^*)$ agrees with the lattice volume of Λ^*.

We now interpret the inverse image of a semipositive $\mathbb{T}$-Cartier divisor by an equivariant morphism in terms of direct and inverse images of concave functions.

Proposition 4.108. *With the hypothesis of Proposition 4.72, let ψ_2 be an H-lattice concave function on Π_2 and let D_{ψ_2} be the corresponding semipositive $\mathbb{T}$-Cartier divisor. Then $\Phi_{p,A}^* D_{\psi_2}$ is the semipositive $\mathbb{T}$-Cartier divisor associated to the H-lattice concave function $\psi_1 = A^*\psi_2$. Moreover the Legendre-Fenchel dual is given by*

$$\psi_1^\vee = (H^\vee)_*(\psi_2^\vee - \mathrm{val}(p)).$$

Proof. The first statement is Proposition 4.94. The second statement follows from Proposition 3.78(1). □

Example 4.109. Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a support function on Σ. By Theorem 4.97, any equivalence class of semipositive models of (X_Σ, D_Ψ) is determined by a rational piecewise affine concave function ψ with $\mathrm{rec}(\psi) = \Psi$. By Lemma 3.79, any such function can be realized as the inverse image by an affine map of the support function of a standard simplex. Using the previous proposition, any equivalence class of semipositive toric models can be induced by an equivariant projective morphism.

More explicitly, let $e > 0$ be an integer such that $e\psi$ is an H-lattice concave function. Let Π be a complete SCR complex in $N_\mathbb{R}$ compatible by $e\psi$ and such that $\mathrm{rec}(\Pi) = \Sigma$ (see the proof of Theorem 4.97). Then, $(\mathcal{X}_\Pi, D_{e\psi}, e)$ is a toric model of (X_Σ, D_Ψ) in the class determined by ψ.

Choose an H-representation $e\psi(u) = \min_{0\leq i\leq r}(m_i(u) + l_i)$ with $(m_i, l_i) \in \widetilde{M}$ for $i = 0, \dots, r$. Put $\boldsymbol{\alpha} = (l_1 - l_0, \dots, l_r - l_0)$. Let H and A be as in Lemma 3.79. In our case, H is a morphism of lattices and

$$e\psi = A^*\Psi_{\Delta^r} + m_0 + l_0. \tag{4.110}$$

We follow examples 4.3, 4.26, 4.44 and 4.75, and consider $\mathbb{P}^r_S$ as a toric scheme over S. Let $p = (p_0 : \dots : p_r)$ be a rational point in the principal open subset of $\mathbb{P}^r_K$ such that $\mathrm{val}(p) = \boldsymbol{\alpha}$. One can verify that the hypothesis of Proposition 4.72 are satisfied. Let $\Phi_{p,A} \colon \mathcal{X}_\Pi \to \mathbb{P}^r_S$ be the associated morphism. Then

$$D_{e\psi} = \Phi_{p,A}^* D_{\Psi_{\Delta^r}} + \mathrm{div}(\varpi^{-l_0}\chi^{-m_0}).$$

5. Metrics and measures on toric varieties

The aim of this section is to characterize the metrics on a toric line bundle over a toric variety that are, at the same time, invariant under the action of the compact torus and approachable or integrable. Moreover we study the associated measures.

5.1. **The variety with corners $X_\Sigma(\mathbb{R}_{\geq 0})$.** Let K be either $\mathbb{R}$, $\mathbb{C}$ or a complete field with respect to an absolute value associated to a nontrivial discrete valuation. When $K=\mathbb{R}$ we will use the technique of Remark 2.5 and in the non-Archimedean case we will use the notations of §2.3. Let $\mathbb{T}$ be an n-dimensional split torus over K and let N and $M=N^\vee$ be the corresponding lattices. Let Σ be a fan in $N_\mathbb{R}$. For each cone $\sigma\in\Sigma$, we will denote by X_σ^{an} the complex analytic space $X_\sigma(\mathbb{C})$ in Archimedean case or the Berkovich analytic space associated to the scheme $X_{\sigma,K}$ in the non-Archimedean case. These analytic spaces glue together in an analytic space X_Σ^{an}.

Given any cone $\sigma\in\Sigma$, we write

$$X_\sigma(\mathbb{R}_{\geq 0})=\operatorname{Hom}_{\mathrm{sg}}(M_\sigma,(\mathbb{R}_{\geq 0},\times)).$$

On $X_\sigma(\mathbb{R}_{\geq 0})$, we put the coarsest topology such that, for each $m\in M_\sigma$, the map $X_\sigma(\mathbb{R}_{\geq 0})\to\mathbb{R}_{\geq 0}$ given by $\gamma\mapsto\gamma(m)$ is continuous. Observe that if τ is a face of σ, then there is a dense open immersion $X_\tau(\mathbb{R}_{\geq 0})\hookrightarrow X_\sigma(\mathbb{R}_{\geq 0})$. Hence the topological spaces $X_\sigma(\mathbb{R}_{\geq 0})$ glue together to define a topological space $X_\Sigma(\mathbb{R}_{\geq 0})$. This is the *variety with corners* associated to X_Σ. Analogously to the algebraic case, one can prove that this topological space is Hausdorff and that the spaces $X_\sigma(\mathbb{R}_{\geq 0})$ can be identified with open subspaces of $X_\Sigma(\mathbb{R}_{\geq 0})$ satisfying

$$X_\sigma(\mathbb{R}_{\geq 0})\cap X_{\sigma'}(\mathbb{R}_{\geq 0})=X_{\sigma\cap\sigma'}(\mathbb{R}_{\geq 0}).$$

For each $\sigma\in\Sigma$ there is a continuous map $\rho_\sigma\colon X_\sigma^{\mathrm{an}}\to X_\sigma(\mathbb{R}_{\geq 0})$. This map is given, in the Archimedean case, by

$$X_\sigma^{\mathrm{an}}=\operatorname{Hom}_{\mathrm{sg}}(M_\sigma,(\mathbb{C},\times))\xrightarrow{|\cdot|}\operatorname{Hom}_{\mathrm{sg}}(M_\sigma,(\mathbb{R}_{\geq 0},\times))=X_\sigma^{\mathrm{an}}(\mathbb{R}_{\geq 0}).$$

While, in the non-Archimedean case, since a point $p\in X_\sigma^{\mathrm{an}}$ corresponds to a multiplicative seminorm on $K[M_\sigma]$ and a point in $X_\sigma(\mathbb{R}_{\geq 0})$ corresponds to a semigroup homomorphism from M_σ to $(\mathbb{R}_{\geq 0},\times)$, we can define $\rho_\sigma(p)$ as the semigroup homomorphism that, to an element $m\in M_\sigma$, corresponds $|\chi^m(p)|$. These maps glue together to define a continuous map $\rho_\Sigma:X_\Sigma^{\mathrm{an}}\to X_\Sigma(\mathbb{R}_{\geq 0})$.

Lemma 5.1. *The map ρ_Σ satisfies $\rho_\Sigma^{-1}(X_\sigma(\mathbb{R}_{\geq 0}))=X_\sigma^{\mathrm{an}}$.*

Proof. By definition $X_\sigma^{\mathrm{an}}\subset\rho_\Sigma^{-1}(X_\sigma(\mathbb{R}_{\geq 0}))$. For the reverse inclusion we will write only the non-Archimedean case. Assume that $p\in\rho_\Sigma^{-1}(X_\sigma(\mathbb{R}_{\geq 0}))$. There is a σ' with $p\in X_{\sigma'}^{\mathrm{an}}$. Let $\tau=\sigma\cap\sigma'$ be the common face. Then p is a multiplicative seminorm of $K[M_{\sigma'}]$ and we show next that it can be extended to a multiplicative seminorm of $K[M_\tau]$. By [Ful93, §1.2 Proposition 2] there is an element $u\in M_{\sigma'}$ such that $M_\tau=M_{\sigma'}+\mathbb{Z}_{\geq 0}(-u)$. Hence $K[M_\tau]=K[M_{\sigma'}+\mathbb{Z}_{\geq 0}(-u)]$. Since $\rho_\Sigma(p)\in X_\tau(\mathbb{R}_{\geq 0})$ we have that $|\chi^u(p)|\neq 0$. Therefore p extends to a multiplicative seminorm of $K[M_\tau]$. Hence $p\in X_\tau^{\mathrm{an}}\subset X_\sigma^{\mathrm{an}}$. □

When Σ is complete, the analytic space X_Σ^{an} is compact, and the map ρ_Σ is proper. By Lemma 5.1, for each cone $\sigma\in\Sigma$, the map ρ_σ is proper. Since every rational cone belongs to a complete fan, the map ρ_Σ is proper even if Σ is not complete. Of particular interest is the case when $\sigma=\{0\}$. Then $\mathbb{T}^{\mathrm{an}}:=X_0^{\mathrm{an}}$ is an Abelian analytic group, that is, an Abelian group object in the category of analytic spaces. In particular, for any field extension K' of K, the set $X_0^{\mathrm{an}}(K')$ is an Abelian group. Also $\mathbb{T}(\mathbb{R}_{\geq 0}):=X_0(\mathbb{R}_{\geq 0})\simeq(\mathbb{R}_{\geq 0})^n$ is a topological Abelian group.

Moreover, $\mathbb{T}^{\mathrm{an}}$ acts on X_Σ^{an}, $\mathbb{T}(\mathbb{R}_{\geq 0})$ acts on $X_\Sigma(\mathbb{R}_{\geq 0})$ and the map ρ_Σ is equivariant with respect to these actions. The kernel of the map ρ_0 is a closed subgroup, that we call the *compact torus* of $\mathbb{T}^{\mathrm{an}}$ and we denote by $\mathbb{S}^{\mathrm{an}}$. In the Archimedean case it is isomorphic to $(S^1)^n$, while in the non-Archimedean case it is the compact torus of Example 2.8. In fact, the fibres of the map ρ_Σ are orbits under the action of $\mathbb{S}^{\mathrm{an}}$. Therefore the space $X_\sigma(\mathbb{R}_{\geq 0})$ is the quotient of X_σ^{an} by the action of the closed subgroup $\mathbb{S}^{\mathrm{an}}$. We warn the reader that the compact topological space underlying $\mathbb{S}^{\mathrm{an}}$ is not an abstract group (see [Ber90, Chapter 5]).

The maps ρ_σ, $\sigma \in \Sigma$, have canonical sections that we denote θ_σ. These sections glue together to give a section θ_Σ of ρ_Σ. In the Archimedean case θ_σ is induced by the semigroup inclusion $\mathbb{R}_{\geq 0} \subset \mathbb{C}$. In the non-Archimedean case θ_σ is defined by the following result.

Proposition-Definition 5.2. Assume that we are in the non-Archimedean case. For each $\gamma \in \operatorname{Hom}_{\mathrm{sg}}(M_\sigma, \mathbb{R}_{\geq 0})$, the seminorm that, to a function $\sum \alpha_m \chi^m \in K[M_\sigma]$ assigns the value $\sup_m(|\alpha_m|\gamma(m))$, is a multiplicative seminorm on $K[M_\sigma]$ that extends the norm of K. Therefore it determines a point of X_σ^{an} that we denote as $\theta_\sigma(\gamma)$. The maps θ_σ are injective, continuous and proper. Moreover, they glue together to define a map

$$\theta_\Sigma\colon X_\Sigma(\mathbb{R}_{\geq 0}) \longrightarrow X_\Sigma^{\mathrm{an}}$$

that is injective, continuous and proper. Every point in the image of θ_Σ is fixed under the action of $\mathbb{S}^{\mathrm{an}}$.

Proof. The fact that the seminorm $\theta_\sigma(\gamma)$ extends the norm of K is clear. Let now $f = \sum_m \alpha_m \chi^m$ and $g = \sum_l \beta_l \chi^l$ and write $fg = \sum_k \varepsilon_k \chi^k$ with $\varepsilon_k = \sum_{m+l=k} \alpha_m \beta_l$. Then, since the absolute value of K is ultrametric,

$$\sup_{k \in M_\sigma} (|\varepsilon_k|\gamma(k)) \leq \sup_{m \in M_\sigma} (|\alpha_m|\gamma(m)) \sup_{l \in M_\sigma} (|\beta_l|\gamma(l)).$$

Let $M_f = \{m \in M_\sigma | \sup_{m'}(|\alpha_{m'}|\gamma(m')) = |\alpha_m|\gamma(m)\}$. We define M_g analogously. Let r be a vertex of the Minkowski sum $\operatorname{conv}(M_f) + \operatorname{conv}(M_g)$. Then there is a unique decomposition $r = m_r + l_r$ with $m_r \in M_f$ and $l_r \in M_g$. Hence $\varepsilon_r = \alpha_{m_r}\beta_{l_r}$. Thus

$$\sup_{k \in M_\sigma} (|\varepsilon_k|\gamma(k)) \geq |\varepsilon_r|\gamma(r) = \sup_{m \in M_\sigma} (|\alpha_m|\gamma(m)) \sup_{l \in M_\sigma} (|\beta_l|\gamma(l)).$$

Thus $\theta_\sigma(\gamma)(fg) = \theta_\sigma(\gamma)(f)\theta_\sigma(\gamma)(g)$. Hence, it is a multiplicative.

We show next that the map θ_σ is continuous. The topology of X_σ^{an} is the coarsest topology that makes the functions $p \to |f(p)|$ continuous for all $f \in K[M_\sigma]$. Thus to show that θ_σ is continuous it is enough to show that the map $\gamma \to |f(\theta_\sigma(\gamma))|$ is continuous on $X_\sigma(\mathbb{R}_{\geq 0}) = \operatorname{Hom}_{\mathrm{sg}}(M_\sigma, \mathbb{R}_{\geq 0})$. The topology of $X_\sigma(\mathbb{R}_{\geq 0})$ is the coarsest topology such that, for each $m \in M_\sigma$, the map $\gamma \to \gamma(m)$ is continuous. Since, for $f = \sum_{m \in M_\sigma} \alpha_m \chi^m$, we have that

$$|f(\theta_\sigma(\gamma))| = \max(|\alpha_m|\gamma(m)),$$

we obtain that θ_σ is continuous. Since each θ_σ is a section of ρ_σ, they are injective.

The fact that the maps θ_σ glue together to give a continuous map θ_Σ and that θ_Σ is a section of ρ_Σ follows easily from the definitions. This implies in particular that θ_Σ is injective. When Σ is complete, since $X_\Sigma(\mathbb{R}_{\geq 0})$ is compact and X_Σ^{an} is Hausdorff, the map θ_Σ is proper. We deduce that the map θ_Σ is proper in general, by using the same argument that shows that the function ρ_Σ is proper.

The last assertion is clear from the definition of $\theta_\sigma(\gamma)$. □

Let now

$$\lambda_K = \begin{cases} 1, & \text{if } K = \mathbb{R},\, \mathbb{C}, \\ -\log|\varpi|, & \text{otherwise.} \end{cases} \tag{5.3}$$

and denote by $\mathbf{e}_K\colon \mathbb{R} \to \mathbb{R}_{>0}$ the map $u \mapsto \exp(-\lambda_K u)$. This map induces an homeomorphism $N_\mathbb{R} \to X_0(\mathbb{R}_{>0})$ that we also denote by $\mathbf{e}_K$.

In the non-Archimedean case, the map $\operatorname{val}\colon \mathbb{T}(K) \to N$ of Definition 4.71, can be extended to a map $\mathbb{T}^{\mathrm{an}} \to N_\mathbb{R}$ that we denote val_K or, when K is clear from the context by val. For each $p \in X_0^{\mathrm{an}}$ we denote by $\operatorname{val}_K(p) \in \operatorname{Hom}_{\mathrm{sg}}(M,\mathbb{R}) = N_\mathbb{R}$ the morphism

$$m \longmapsto \langle m, \operatorname{val}_K(p)\rangle = \frac{-\log|\chi^m(p)|}{\lambda_K}. \tag{5.4}$$

In the Archimedean case we will denote by $\operatorname{val}_\mathbb{C}$ or simply by val the map defined by the same equation. Then, the diagram

$$\begin{tikzcd} & X_0^{\mathrm{an}} \arrow[dl, "\operatorname{val}_K"'] \arrow[d, "\rho_0"] \\ N_\mathbb{R} \arrow[r, "\mathbf{e}_K"'] & X_0(\mathbb{R}_{\geq 0}) \end{tikzcd} \tag{5.5}$$

is commutative.

The map $\mathbf{e}_K$ allows us to see $X_\Sigma(\mathbb{R}_{\geq 0})$ as a partial compactification on $N_\mathbb{R}$. Following [AMRT75, Chapter I, §1] we can give another description of the topology of $X_\Sigma(\mathbb{R}_{\geq 0})$. For $\sigma \in \Sigma$, we denote

$$N_\sigma = \coprod_{\tau \text{ face of } \sigma} N(\tau)_\mathbb{R}.$$

We choose a positive definite bilinear pairing in $N_\mathbb{R}$. Hence we can identify the quotient spaces $N(\tau)_\mathbb{R}$ with subspaces of $N_\mathbb{R}$, that, for simplicity, we will denote also by $N(\tau)_\mathbb{R}$. For a point $u \in N(\tau)_\mathbb{R}$, let $U \subset N(\tau)_\mathbb{R}$ be a neighbourhood of u. For each τ' face of τ, τ induces a cone $\pi_{\tau'}(\tau)$ contained in $N(\tau')_\mathbb{R}$. If $p \in \tau$ its image $\pi_{\tau'}(p)$ in $N(\tau')_\mathbb{R}$, is contained in $\pi_{\tau'}(\tau)$. We write

$$W(\tau, U, p) = \coprod_{\tau' \text{ face of } \tau} \pi_{\tau'}(U + p + \tau). \tag{5.6}$$

Moving U and p we obtain a basis of neighbourhoods of u in N_σ. This defines a topology on N_σ such that the map $\mathbf{e}_K\colon N_\mathbb{R} \to X_\sigma(\mathbb{R}_{\geq 0})$ extends to a homeomorphism $N_\sigma \to X_\sigma(\mathbb{R}_{\geq 0})$.

We write

$$N_\Sigma = \coprod_{\sigma\in\Sigma} N(\sigma)_\mathbb{R},$$

and put in N_Σ the topology that makes $\{N_\sigma\}_{\sigma\in\Sigma}$ an open cover. Then the map $\mathbf{e}_K$ extends to a homeomorphism between N_Σ and $X_\Sigma(\mathbb{R}_{\geq 0})$ and the map val_K extends to a proper continuous map $X_\Sigma^{\mathrm{an}} \to N_\Sigma$ such that the diagram

$$\begin{tikzcd} & X_\Sigma^{\mathrm{an}} \arrow[dl, "\operatorname{val}_K"'] \arrow[d, "\rho_\Sigma"] \\ N_\Sigma \arrow[r, "\mathbf{e}_K"'] & X_\Sigma(\mathbb{R}_{\geq 0}) \end{tikzcd} \tag{5.7}$$

is commutative.

Remark 5.8. In case we are given a strictly concave support function Ψ on a fan Σ, then N_Σ is homeomorphic to the polytope Δ_Ψ introduced in §4.4. An homeomorphism is obtained as the composition of $\mathbf{e}_K$ with the moment map $\mu\colon X_\Sigma(\mathbb{R}_{\geq 0}) \to \Delta_\Psi$ induced by Ψ:

$$\begin{array}{ccccc} N_\Sigma & \xrightarrow{\mathbf{e}_K} & X_\Sigma(\mathbb{R}_{\geq 0}) & \xrightarrow{\mu} & \Delta_\Psi \\ u & \longmapsto & \mathbf{e}_K(u) & \longmapsto & \frac{\sum \exp(-\lambda_K\langle m,u\rangle)m}{\sum \exp(-\lambda_K\langle m,u\rangle)} \end{array}$$

where the sums in the last expression are over the elements $m \in M \cap \Delta_\Psi$.

We end this section stating the functorial properties of the space $X_\Sigma(\mathbb{R}_{\geq 0})$. The proofs are left to the reader. Let N and Σ be as before and $\sigma \in \Sigma$. Recall that the associated closed subvariety $V(\sigma)$ is canonically isomorphic to the toric variety $X_{\Sigma(\sigma)}$.

Proposition 5.9. *The natural map $N(\sigma)_\mathbb{R} \hookrightarrow N_\sigma$ extends to a continuous map $X_{\Sigma(\sigma)}(\mathbb{R}_{\geq 0}) \to X_\Sigma(\mathbb{R}_{\geq 0})$. Moreover, there are commutative diagrams*

$$\begin{tikzcd} X^{\mathrm{an}}_{\Sigma(\sigma)} \arrow[r] \arrow[d, "\rho_{\Sigma(\sigma)}"'] & X^{\mathrm{an}}_{\Sigma} \arrow[d, "\rho_\Sigma"] \\ X_{\Sigma(\sigma)}(\mathbb{R}_{\geq 0}) \arrow[r] & X_\Sigma(\mathbb{R}_{\geq 0}), \end{tikzcd} \qquad \begin{tikzcd} X^{\mathrm{an}}_{\Sigma(\sigma)} \arrow[r] & X^{\mathrm{an}}_{\Sigma} \\ X_{\Sigma(\sigma)}(\mathbb{R}_{\geq 0}) \arrow[r] \arrow[u, "\theta_{\Sigma(\sigma)}"] & X_\Sigma(\mathbb{R}_{\geq 0}). \arrow[u, "\theta_\Sigma"'] \end{tikzcd}$$

Let N_1 and N_2 be lattices and let Σ_1 and Σ_2 be complete fans in $N_{1,\mathbb{R}}$ and $N_{2,\mathbb{R}}$ respectively. Let $H\colon N_1 \to N_2$ be a linear map such that, for each cone $\sigma_1 \in \Sigma_1$, there is a cone $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subset \sigma_2$. Let $p \in X_{\Sigma_2,0}(K)$ and let $A\colon N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$ be the affine map $A = H + \mathrm{val}(p)$.

Proposition 5.10. *The affine map $A\colon N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$ extends to a continuous map $X_{\Sigma_1}(\mathbb{R}_{\geq 0}) \to X_{\Sigma_2}(\mathbb{R}_{\geq 0})$ that we also denote by $\varphi_{p,H}$. Moreover, there are commutative diagrams*

$$\begin{tikzcd} X^{\mathrm{an}}_{\Sigma_1} \arrow[r, "\varphi_{p,H}"] \arrow[d, "\rho_{\Sigma_1}"'] & X^{\mathrm{an}}_{\Sigma_2} \arrow[d, "\rho_{\Sigma_2}"] \\ X_{\Sigma_1}(\mathbb{R}_{\geq 0}) \arrow[r, "\varphi_{p,H}"] & X_{\Sigma_2}(\mathbb{R}_{\geq 0}), \end{tikzcd} \qquad \begin{tikzcd} X^{\mathrm{an}}_{\Sigma_1} \arrow[r, "\varphi_{p,H}"] & X^{\mathrm{an}}_{\Sigma_2} \\ X_{\Sigma_1}(\mathbb{R}_{\geq 0}) \arrow[r, "\varphi_{p,H}"] \arrow[u, "\theta_{\Sigma_1}"] & X_{\Sigma_2}(\mathbb{R}_{\geq 0}). \arrow[u, "\theta_{\Sigma_2}"'] \end{tikzcd}$$

5.2. **Toric metrics.** From now on we assume that Σ is complete. Let L be a toric line bundle on X_Σ and let s be a toric section of L (Definition 4.19). By Theorem 4.22 and Theorem 4.18, we can find a virtual support function Ψ on Σ such that there is an isomorphism $L \simeq \mathcal{O}(D_\Psi)$ that sends s to s_Ψ. The algebraic line bundle L defines an analytic line bundle L^{an} on X^{an}_Σ. Let $\overline{L} = (L, \|\cdot\|)$, where $\|\cdot\|$ is a metric on L^{an}.

Every toric object has a certain invariance property with respect to the action of $\mathbb{T}$. This is also the case for metrics. Since $\mathbb{T}^{\mathrm{an}}$ is non compact, we can not ask for a metric to be $\mathbb{T}^{\mathrm{an}}$-invariant, but we can impose $\mathbb{S}^{\mathrm{an}}$-invariance. We need a preliminary result.

Proposition 5.11. *Let L be a toric line bundle on X_Σ and let $\|\cdot\|$ be a metric on L^{an}. If there is a toric section s_0 such that the function $p \mapsto \|s_0(p)\|$ is $\mathbb{S}^{\mathrm{an}}$-invariant, then, for every toric section s, the function $p \mapsto \|s(p)\|$ is $\mathbb{S}^{\mathrm{an}}$-invariant.*

Proof. If s and s' are two toric sections, then there is an element $m \in M$ such that $s' = \chi^m s$. Since for any element $t \in \mathbb{S}^{\mathrm{an}}$ we have $|\chi^m(t)| = 1$, if the function $\|s(p)\|$ is $\mathbb{S}^{\mathrm{an}}$-invariant, then the function $\|s'(p)\| = \|\chi^m(p)s(p)\|$ is also $\mathbb{S}^{\mathrm{an}}$-invariant. $\square$

Definition 5.12. Let L be a toric line bundle on X_Σ. A metric on L^{an} is called *toric* if, for any toric section s of L over X_0, the function $p \longmapsto \|s(p)\|$ is $\mathbb{S}^{\mathrm{an}}$-invariant.

To the metrized line bundle $\overline{L}$ and the section s we associate the function $g_{\overline{L},s}\colon X_0^{\mathrm{an}} \to \mathbb{R}$ given by $g_{\overline{L},s}(p) = \log(\|s(p)\|)/\lambda_K$. In the Archimedean case, the function $g_{\overline{L},s}$ is $-1/2$ times the usual Green function associated to the metrized line bundle $\overline{L}$ and the section s. The metric $\|\cdot\|$ is toric if and only if the function

$g_{\overline{L},s}$ is $\mathbb{S}^{\mathrm{an}}$-invariant. In this case we can form the commutative diagram

$$\begin{tikzcd} X_0^{\mathrm{an}} \arrow[r, "g_{\overline{L},s}"] \arrow[d, "\mathrm{val}_K"'] & \mathbb{R} \\ N_{\mathbb{R}} \arrow[ur, dashed] & \end{tikzcd} \tag{5.13}$$

The dashed arrow exists as a continuous function because ρ_0, hence val_K, is a proper surjective map and, by $\mathbb{S}^{\mathrm{an}}$-invariance, $g_{\overline{L},s}$ is constant along the fibres. This justifies the following definition.

Definition 5.14. Let L be a toric line bundle, s a toric section of L and let $\|\cdot\|$ be a toric metric. Denote $\overline{L}=(L,\|\cdot\|)$. We define the function $\psi_{\overline{L},s}\colon N_{\mathbb{R}}\to\mathbb{R}$ by

$$\psi_{\overline{L},s}(u)=\frac{\log\|s(p)\|}{\lambda_K} \tag{5.15}$$

for any $p\in X_0^{\mathrm{an}}$ with $\mathrm{val}_K(p)=u$. When the line bundle and the section are clear from the context, we will alternatively denote this function as $\psi_{\|\cdot\|}$.

Proposition 5.16. *Let Ψ be a virtual support function on Σ, $L=\mathcal{O}(D_\Psi)$ and $s=s_\Psi$. Then the correspondence $\|\cdot\|\mapsto\psi_{\|\cdot\|}$ determines a bijection between the set of toric metrics on L^{an} and the set of continuous functions ψ on $N_{\mathbb{R}}$ with the property that $\psi-\Psi$ can be extended to a continuous function on N_Σ. The metric associated to a function ψ will be denoted $\|\cdot\|_\psi$.*

Proof. Let $\|\cdot\|$ be a toric metric on L^{an}. Since s is a regular nowhere vanishing section on X_0^{an}, $\psi_{\|\cdot\|}$ is a well defined continuous function on $N_{\mathbb{R}}$. Let $\{m_\sigma\}$ be a set of defining vectors of Ψ. For each cone $\sigma\in\Sigma$, the section $\chi^{m_\sigma}s$ is a regular nowhere vanishing section on X_σ^{an}. Therefore $\log(\|(\chi^{m_\sigma}s)(p)\|)$ is a continuous function on X_σ^{an} that is $\mathbb{S}^{\mathrm{an}}$-invariant. So it defines a continuous function on $X_\sigma(\mathbb{R}_{\geq 0})$. By equation (5.4),

$$\begin{aligned}\psi_{\|\cdot\|}(\mathrm{val}(p))-m_\sigma(\mathrm{val}(p))&=\frac{1}{\lambda_K}\big(\log(\|s(p)\|)-\log(|\chi^{-m_\sigma}(p)|)\big)\\&=\frac{1}{\lambda_K}\log(\|(\chi^{m_\sigma}s)(p)\|).\end{aligned}$$

Therefore $\psi_{\|\cdot\|}-m_\sigma$ extends to a continuous function on $N_\sigma\simeq X_\sigma(\mathbb{R}_{\geq 0})$. If we see that $\Psi-m_\sigma$ extends also to a continuous function on N_σ we will be able to extend $\psi_{\|\cdot\|}-\Psi$ to a continuous function on N_σ for every $\sigma\in\Sigma$ and therefore to N_Σ.

Let τ be a face of σ and let $u\in N(\tau)_{\mathbb{R}}$. Let $W(\tau,U,p)$ be a neighbourhood of u as in (5.6). By taking U small enough and p big enough we can assume that $W(\tau,U,p)\cap N_{\mathbb{R}}$ is contained in the set of cones that have τ as a face. Since Ψ and m_σ agree when restricted to σ (hence when restricted to τ) it follows that, if $w+t\in W(\tau,U,p)\cap N_{\mathbb{R}}$ with $w\in U$ and $t\in p+\tau$, then $(\Psi-m_\sigma)(w+t)$ only depends on w and not on t. Hence it can be extended to a continuous function on the whole $W(\tau,U,p)$. By moving τ, u, U and p we see that it can be extended to a continuous function on N_σ.

Let now ψ be a function on $N_{\mathbb{R}}$ such that $\psi-\Psi$ extends to a continuous function on N_Σ. We define a toric metric $\|\cdot\|_\psi$ on L^{an} over the set X_0^{an} by the formula

$$\|s(p)\|_\psi=\exp(\lambda_K\psi(\mathrm{val}_K(p))).$$

Then, by the argument before, $\psi-m_\sigma$ extends to a continuous function on N_σ, which proves that $\|\cdot\|_\psi$ extends to a metric over X_σ^{an}. Varying $\sigma\in\Sigma$ we obtain that $\|\cdot\|_\psi$ extends to a metric over X_Σ^{an}. □

Corollary 5.17. *For any toric metric $\|\cdot\|$, the function $|\psi_{\|\cdot\|}-\Psi|$ is bounded.*

Proof. Since we are assuming that Σ is complete, the space $N_\Sigma \simeq X_\Sigma(\mathbb{R}_{\geq 0})$ is compact. Thus the corollary follows from Proposition 5.16. $\square$

Example 5.18. With the notation in Example 3.65, consider the standard simplex Δ^n with fan $\Sigma = \Sigma_{\Delta^n}$ and support function $\Psi = \Psi_{\Delta^n}$. The corresponding toric variety is $X_\Sigma = \mathbb{P}^n$ with toric line bundle $L_\Psi = \mathcal{O}(1)$ and toric section $s_\Psi = s_\infty$.

(1) The canonical metrics $\|\cdot\|_{\mathrm{can}}$ in examples 2.25 and 2.32 are toric and both correspond to the function $\psi_{\|\cdot\|_{\mathrm{can}}} = \Psi$.
(2) The Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$ in Example 2.2 is also toric and corresponds to the differentiable function $\psi_{\|\cdot\|_{\mathrm{FS}}} = f_{\mathrm{FS}}$ introduced in Example 3.53.

Proposition 5.19. *The correspondence* $(\overline{L}, s) \mapsto \psi_{\overline{L},s}$ *satisfies the following properties.*

(1) *Let* $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 1, 2$, *be toric line bundles equipped with toric metrics and let* s_i *be a toric section of* L_i. *Then*
$$\psi_{\overline{L}_1\otimes\overline{L}_2, s_1\otimes s_2} = \psi_{\overline{L}_1,s_1} + \psi_{\overline{L}_2,s_2}.$$
(2) *Let* $\overline{L} = (L, \|\cdot\|)$ *be a toric line bundle equipped with a toric metric and let* s *be a toric section of* L. *Then*
$$\psi_{\overline{L}^{-1},s^{-1}} = -\psi_{\overline{L},s}.$$

Proof. This follows easily from the definitions. $\square$

A consequence of Proposition 5.16 is that every toric line bundle has a distinguished metric.

Proposition-Definition 5.20. Let Σ be a complete fan, X_Σ the corresponding toric variety, and L a toric line bundle on X_Σ. Let s be a toric section of L and Ψ the virtual support function on Σ associated to (L, s) by theorems 4.22 and 4.18. The metric on L^{an} associated to the function Ψ by Proposition 5.16 only depends on the structure of toric line bundle of L. This metric is called the *canonical metric* of L^{an} and is denoted $\|\cdot\|_{\mathrm{can}}$. We write $\overline{L}^{\mathrm{can}} = (L, \|\cdot\|_{\mathrm{can}})$.

Proof. Let s' be another toric section of L. Then there is an element $m \in M$ such that $s' = \chi^m s$. The corresponding virtual support function is $\Psi' = \Psi - m$. Denote by $\|\cdot\|$ and $\|\cdot\|'$ the metrics associated to s, Ψ and to s', Ψ' respectively. Then
$$\|s(p)\|' = \|\chi^{-m}s'(p)\|' = \mathrm{e}^{\lambda_K(m+\Psi')(\mathrm{val}(p))} = \mathrm{e}^{\lambda_K\Psi(\mathrm{val}(p))} = \|s(p)\|.$$
Thus both metrics agree. $\square$

The canonical metrics $\|\cdot\|_{\mathrm{can}}$ in examples 2.25 and 2.32 are particular cases of the canonical metric of Proposition-Definition 5.20.

Proposition 5.21. *The canonical metric is compatible with the tensor product of line bundles.*

(1) *Let* L_i, $i = 1, 2$, *be toric line bundles. Then* $\overline{L_1 \otimes L_2}^{\mathrm{can}} = \overline{L_1}^{\mathrm{can}} \otimes \overline{L_2}^{\mathrm{can}}$.
(2) *Let* L *be a toric line bundle. Then* $\overline{L^{-1}}^{\mathrm{can}} = (\overline{L}^{\mathrm{can}})^{-1}$.

Proof. This follows easily from the definitions. $\square$

Next we describe the behaviour of the correspondence of Proposition 5.16 with respect to equivariant morphisms. We start with the case of orbits. Let Σ be a complete fan in N and Ψ a virtual support function on Σ. Let L and s be the associated toric line bundle and toric section, and $\{m_\sigma\}_{\sigma\in\Sigma}$ a set of defining vectors of Ψ. Let $\sigma \in \Sigma$ and let $V(\sigma)$ be the corresponding closed subvariety. As in Proposition 4.34, the restriction of L to $V(\sigma)$ is a toric line bundle. Since $V(\sigma)$ and $\mathrm{div}(s)$ may not intersect properly we can not restrict s directly to $V(\sigma)$. By contrast, $D_{\Psi-m_\sigma} = \mathrm{div}(\chi^{m_\sigma}s)$ intersects properly $V(\sigma)$ and we can restrict the section $\chi^{m_\sigma}s$ to $V(\sigma)$ to obtain a toric section of $\mathcal{O}(D_{(\Psi-m_\sigma)(\sigma)}) \simeq L\mid_{V(\sigma)}$. Denote $\iota\colon V(\sigma) \to X_\Sigma$ the closed immersion. For short, we write $s' = \chi^{m_\sigma}s$. Then ι^*s' is a nowhere vanishing section on $O(\sigma)$. Recall that $V(\sigma)$ has a structure of toric variety given by the fan $\Sigma(\sigma)$ on $N(\sigma)$ (Proposition 4.6). The principal open subset of $V(\sigma)$ is the orbit $O(\sigma)$.

Let $\|\cdot\|$ be a toric metric on L^{an} and write $\overline{L} = (L, \|\cdot\|)$. By the proof of Proposition 5.16, the function $\psi_{\overline{L},s} - m_\sigma = \psi_{\overline{L},s'}$ can be extended to a continuous function on N_σ that we denote $\overline{\psi}_{\overline{L},s'}$.

Proposition 5.22. *The function $\psi_{\iota^*\overline{L},\iota^*s'}\colon N(\sigma)_\mathbb{R} \to \mathbb{R}$ agrees with the restriction of $\overline{\psi}_{\overline{L},s'}$ to $N(\sigma)_\mathbb{R} \subset N_\sigma$.*

Proof. The section s' is a nowhere vanishing section over $X_{\Sigma,\sigma}$. Therefore, the function $g_{\overline{L},s'}\colon X^{\mathrm{an}}_{\Sigma,\sigma} \to \mathbb{R}$ of diagram (5.13) can be extended to a continuous function on $X_{\Sigma,\sigma}$ that we also denote $g_{\overline{L},s'}$. By the definition of the inverse image of a metric, there is a commutative diagram

$$\begin{tikzcd} O(\sigma)^{\mathrm{an}} \arrow[r, "\iota"] \arrow[dr, "g_{\iota^*\overline{L},\iota^*s'}"'] & X^{\mathrm{an}}_{\Sigma,\sigma} \arrow[d, "g_{\overline{L},s'}"] \\ & \mathbb{R} \end{tikzcd}$$

Then the result is a consequence of the definition of $\psi_{\iota^*\overline{L},\iota^*s'}$ and of the commutativity of the diagram

$$\begin{tikzcd} O(\sigma)^{\mathrm{an}} \arrow[r] \arrow[d] & X^{\mathrm{an}}_{\Sigma,\sigma} \arrow[d] \\ N(\sigma)_\mathbb{R} \arrow[r] & N_\sigma, \end{tikzcd}$$

that follows from Proposition 5.9. □

Corollary 5.23. *Let $\overline{L}$ be a toric line bundle on X_Σ equipped with the canonical metric, let $\sigma \in \Sigma$ and $\iota\colon V(\sigma) \to X_\Sigma$ the closed immersion. Then the restriction $\iota^*\overline{L}$ is a toric line bundle equipped with the canonical metric.*

Proof. Choose a toric section s of L whose divisor meets $V(\sigma)$ properly. Let Ψ be the corresponding virtual support function. The condition of proper intersection is equivalent to $\Psi|_\sigma = 0$. Then Ψ extends to a continuous function $\overline{\Psi}$ on N_σ and the restriction of $\overline{\Psi} \to N(\sigma)$ is equal to $\Psi(\sigma)$. Hence the result follows from Proposition 5.22. □

We end with the case of an equivariant morphism whose image intersect the principal open subset. Let N_i, Σ_i, $i = 1, 2$, H, p and A be as in Proposition 5.10. Let Ψ_2 be a virtual support function on Σ_2 and let $\Psi_1 = \Psi_2 \circ H$. This is a virtual support function on Σ_1. Let (L_i, s_i) be the corresponding toric line bundles and sections. By Proposition 4.35 and Theorem 4.22, there is an isomorphism $\varphi^*_{p.H}L_2 \simeq L_1$ that sends $\varphi^*_{p.H}s_2$ to s_1. We use this isomorphism to identify them. Let $\|\cdot\|$ be a toric metric on L_2^{an} and write $\overline{L}_2 = (L_2, \|\cdot\|)$, $\overline{L}_1 = (L_1, \varphi^*_{p.H}\|\cdot\|)$. The following result follows from Proposition 5.10 and is left to the reader.

Proposition 5.24. *The equality $\psi_{\overline{L}_1,s_1} = \psi_{\overline{L}_2,s_2} \circ A$ holds.*

In the case of toric morphism, the canonical metric is stable by inverse image. The following result follows easily from the definitions.

Corollary 5.25. *Assume furthermore that $p = x_0$ and so the equivariant morphism $\varphi_{p,H} = \varphi_H\colon X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism. If $\overline{L}$ is a toric line bundle on X_{Σ_2} equipped with the canonical metric, then $\varphi_H^*\overline{L}$ is a toric line bundle equipped with the canonical metric.*

The inverse image of the canonical metric by an equivariant map does not need to be the canonical metric. In fact, the analogue of Example 4.109 in terms of metrics shows that many different metrics can be obtained as the inverse image of the canonical metric on the projective space.

Example 5.26. Let Σ be a complete fan in $N_{\mathbb{R}}$ and X_Σ the corresponding toric variety. Recall the description of the projective space $\mathbb{P}^r$ as a toric variety given in Example 4.3. Let $H\colon N \to \mathbb{Z}^r$ be a linear map such that, for each $\sigma \in \Sigma$ there exist $\tau \in \Sigma_{\Delta^r}$ with $H(\sigma) \subset \tau$. Let $p \in \mathbb{P}^r_0(K)$. Then we have an equivariant morphism $\varphi_{p,H}\colon X_\Sigma \to \mathbb{P}^r$. Consider the support function Ψ_{Δ^r} on Σ_{Δ^r}. Then $L_{\Psi_{\Delta^r}} = \mathcal{O}_{\mathbb{P}^r}(1)$. Write $L = \varphi_{p,H}^* L_{\Psi_{\Delta^r}}$, $s = \varphi_{p,H}^* s_{\Psi_{\Delta^r}}$ and $\Psi = H^*\Psi_{\Delta^r}$. Thus $(L, s) = (L_\Psi, s_\Psi)$.

Set $A = H + \mathrm{val}_K(p)$ for the affine map. Let $\|\cdot\|$ be the metric on L^{an} induced by the canonical metric of $\mathcal{O}(D_{\Psi_{\Delta^r}})^{\mathrm{an}}$ and let ψ be the function associated to it by Proposition 5.16. By Proposition 5.24, $\psi = A^*\Psi_{\Delta^r}$. This is a piecewise affine concave function on $N_{\mathbb{R}}$ with $\mathrm{rec}(\psi) = \Psi$ that can be made explicit as follows.

Let $\{e_1, \dots, e_r\}$ be the standard basis of $\mathbb{Z}^r$ and let $\{e_1^\vee, \dots, e_r^\vee\}$ be the dual basis. Write $m_i = e_i^\vee \circ H \in M$ and $l_i = e_i^\vee(\mathrm{val}_K(p)) \in \mathbb{R}$. Then

$$\Psi = \min\{0, m_1, \dots, m_r\}$$
$$\psi = \min\{0, m_1 + l_1, \dots, m_r + l_r\}$$

We want to characterize all the functions that can be obtained with a slight generalization of the previous construction.

Proposition 5.27. *Let Σ be a complete fan in N and Ψ a support function on Σ. Write $L = L_\Psi$ and $s = s_\Psi$. Let $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ a piecewise affine concave function with $\mathrm{rec}(\psi) = \Psi$, that has an H-representation*

$$\psi = \min_{i=0,\dots,r}\{m_i + l_i\},$$

with $m_i \in M_{\mathbb{Q}}$ and $l_i \in \mathbb{R}$ in the Archimedean case and $l_i \in \mathbb{Q}$ in the non-Archimedean case. Then there is an equivariant morphism $\varphi\colon X_\Sigma \to \mathbb{P}^r$, an integer $e > 0$ and an isomorphism $L^{\otimes e} \simeq \varphi^*\mathcal{O}(1)$ such that the metric induced on L^{an} by the canonical metric of $\mathcal{O}(1)^{\mathrm{an}}$ agrees with $\|\cdot\|_\psi$.

Proof. First observe that the condition $l_i \in \mathbb{R}$ in the Archimedean case and $l_i \in \mathbb{Q}$ in the non-Archimedean case is equivalent to the condition $l_i \in \mathbb{Q}\,\mathrm{val}_K(K^\times)$. Let $e > 0$ be an integer such that $em_i \in M$ and $el_i \in \mathrm{val}_K(K^\times)$ for $i = 0, \dots, r$.

Consider the linear map $H\colon N_{\mathbb{R}} \to \mathbb{R}^r$ given by $H(u) = (em_i(u) - em_0(u))_{i=1,\dots,r}$ and the affine map $A = H + \boldsymbol{l}$ with $\boldsymbol{l} = (el_i - el_0)_{i=1,\dots,r}$. By Lemma 3.79,

$$e\psi = A^*\Psi_{\Delta^r} + em_0 + el_0.$$

We claim that, for each $\sigma \in \Sigma$ there exists $\sigma_{i_0} \in \Sigma_{\Delta^r}$ such that $H(\sigma) \subset \sigma_{i_0}$. Indeed, $\Psi(u) = \min_i\{m_i(u)\}$. Since Ψ is a support function on Σ, for each $\sigma \in \Sigma$, there exists an i_0 such that $\Psi(u) = m_{i_0}(u)$ for all $u \in \sigma$. Writing $e_0^\vee = 0$, this condition implies

$$\min_{0\le i\le r}\{e_i^\vee(H(u))\} = e_{i_0}^\vee(H(u)) \quad \text{for all } u \in \sigma.$$

Hence, $H(\sigma) \subset \sigma_{i_0}$, where $\sigma_{i_0} \in \Sigma_{\Delta^r}$ is the cone $\{v \mid \min_{0\le i\le r}\{e_i^\vee(v)\} = e_{i_0}^\vee(v)\}$ and the claim is proved.

Therefore, we can apply Theorem 4.9 and given a point $p \in \mathbb{P}^r_r(K)$ such that $\mathrm{val}_K(p) = \boldsymbol{l}$, there is an equivariant map $\varphi_{p,H}\colon X_\Sigma \to \mathbb{P}^r$. By Example 4.44, there is an isomorphism $L^{\otimes e} \simeq \varphi^*_{p,H}\mathcal{O}(1)$ and $a \in K^\times$ with $\mathrm{val}_K(a) = l_0$ such that $(a^{-1}\chi^{-m_0}s)^{\otimes e}$ corresponds to $\varphi^*_{p,H}(s_{\Psi_{\Delta^r}})$.

Let $\overline{L}$ be the line bundle L equipped with the metric induced by the above isomorphism and the canonical metric of $\mathcal{O}(1)^{\mathrm{an}}$. Then

$$\psi_{\overline{L},s} = \psi_{\overline{L},a^{-1}\chi^{-m_0}s} + m_0 + l_0 = \frac{1}{e}A^*\Psi_{\Delta^r} + m_0 + l_0 = \psi,$$

as stated. □

Corollary 5.28. *Let ψ be as in Proposition 5.27. Then the metric $\|\cdot\|_\psi$ is approachable.*

Proof. This follows readily from the previous result together with Example 2.32 in the Archimedean case and Example 2.25 in the non-Archimedean case and the fact that the inverse image of an approachable metric is also approachable. □

5.3. **Smooth metrics and their associated measures.** We now discuss the relationship between semipositivity of smooth metrics and concavity of the associated function in the Archimedean case. Moreover we will determine the associated measure.

In this section K is either $\mathbb{R}$ or $\mathbb{C}$ and we fix a lattice N of rank n, a complete fan Σ in $N_\mathbb{R}$ and a virtual support function Ψ on Σ, with L and s the corresponding toric line bundle and section. Let X^{an}_Σ be the complex analytic space associated to X_Σ and L^{an} the analytic line bundle associated to L.

Proposition 5.29. *Let $\|\cdot\|$ be a smooth toric metric on L^{an}. Then $\|\cdot\|$ is semipositive if and only if the function $\psi = \psi_{\|\cdot\|}$ is concave.*

Proof. Since the condition of being semipositive is closed, it is enough to check it in the open set X_0^{an}. We choose an integral basis of $M = N^\vee$. This determines isomorphisms

$$X_0^{\mathrm{an}} \simeq (\mathbb{C}^\times)^n, \quad X_0(\mathbb{R}_{\geq 0}) \simeq (\mathbb{R}_{>0})^n, \quad N_\mathbb{C} \simeq \mathbb{C}^n, \quad N_\mathbb{R} \simeq \mathbb{R}^n.$$

Let $z_1, \dots, z_n$ be the coordinates of X_0^{an} and $u_1, \dots, u_n$ the coordinates of $N_\mathbb{R}$ determined by these isomorphisms. With these coordinates the map

$$\mathrm{val}\colon X_0^{\mathrm{an}} \to N_\mathbb{R}$$

is given by

$$\mathrm{val}(z_1, \dots, z_n) = \frac{-1}{2}(\log(z_1\bar{z}_1), \dots, \log(z_n\bar{z}_n)).$$

As usual, we denote $\overline{L} = (L, \|\cdot\|)$. Set $g = g_{\overline{L},s} = \log\|s\|$. Then, the integral valued first Chern class is given by

$$\frac{1}{2\pi i}c_1(\overline{L}) = \frac{1}{\pi i}\partial\bar{\partial}g = \frac{-i}{\pi}\sum_{k,l}\frac{\partial^2 g}{\partial z_k \partial \bar{z}_l}\,\mathrm{d}z_k \wedge \mathrm{d}\bar{z}_l. \tag{5.30}$$

The standard orientation of the unit disk $\mathbb{D} \subset \mathbb{C}$ is given by $\mathrm{d}x \wedge \mathrm{d}y = (i/2)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Hence, the metric of $\overline{L}$ is semipositive if and only if the matrix $G = (\frac{\partial^2 g}{\partial z_k \partial \bar{z}_l})_{k,l}$ is semi-negative definite. Since

$$\frac{\partial^2 g}{\partial z_k \partial \bar{z}_l} = \frac{1}{4 z_k \bar{z}_l}\frac{\partial^2 \psi}{\partial u_k \partial \bar{u}_l}, \tag{5.31}$$

if we write $\mathrm{Hess}(\psi) = (\frac{\partial^2 \psi}{\partial u_k \partial \bar{u}_l})_{k,l}$ and $Z = \mathrm{diag}((2z_1)^{-1}, \dots, (2z_n)^{-1})$, then $G = \bar{Z}^t\,\mathrm{Hess}(\psi)Z$. Therefore G is semi-negative definite if and only if $\mathrm{Hess}(\psi)$ is semi-negative definite, hence, if and only if ψ is concave. □

The line bundle L^{an} admits a semipositive metric is and only if Ψ is concave. Thus, from now on we assume that Ψ is a support function, that is, a concave support function.

Definition 5.32. Let $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ be a concave function such that $|\Psi - \psi|$ is bounded. Let $\mathcal{M}_M(\psi)$ be the Monge-Ampère measure associated to ψ and the lattice M. We will denote by $\overline{\mathcal{M}}_M(\psi)$ the measure on N_Σ given by

$$\overline{\mathcal{M}}_M(\psi)(E) = \mathcal{M}_M(\psi)(E \cap N_{\mathbb{R}})$$

for any Borel subset of N_Σ.

By its very definition, the measure $\overline{\mathcal{M}}_M(\psi)$ is bounded with total mass

$$\overline{\mathcal{M}}_M(\psi)(N_\Sigma) = \mathrm{vol}_M(\Delta_\Psi)$$

and the set $N_\Sigma \setminus N_{\mathbb{R}}$ has measure zero.

Theorem 5.33. *Let $\|\cdot\|$ be a semipositive smooth toric metric on L^{an}. Let $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ be the measure defined by $\overline{L}$. Then,*

$$\mathrm{val}_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}) = n!\overline{\mathcal{M}}_M(\psi), \tag{5.34}$$

where val *is the map of diagram* (5.7). *In addition, this measure is uniquely characterized by equation* (5.34) *and the property of being $\mathbb{S}^{\mathrm{an}}$-invariant.*

Proof. Since the measure $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ is given by a smooth volume form and $X_\Sigma^{\mathrm{an}} \setminus X_0^{\mathrm{an}}$ is a set of Lebesgue measure zero, the measure $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ is determined by its restriction to the dense open subset X_0^{an}. Thus, to prove equation (5.34) it is enough to show that

$$\mathrm{val}_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}|_{X_0^{\mathrm{an}}}) = n!\mathcal{M}_M(\psi). \tag{5.35}$$

We use the coordinate system of the proof of Proposition 5.29. We denote by $\widetilde{\mathbf{e}}\colon N_{\mathbb{C}} \to X_0(\mathbb{C})$ the map induced by the morphism $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto \exp(-z)$. We write $u_k + iv_k$ for the complex coordinates of $N_{\mathbb{C}}$. Then

$$\widetilde{\mathbf{e}}^*\left(\frac{\mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k}{z_k\bar{z}_k}\right) = (-2i)\,\mathrm{d}u_k \wedge \mathrm{d}v_k. \tag{5.36}$$

Using now equations (5.30), (5.31) and (5.36), we obtain that,

$$\begin{aligned}\frac{1}{(2\pi i)^n}\widetilde{\mathbf{e}}^* c_1(\overline{L})^n &= \widetilde{\mathbf{e}}^*\left(\frac{1}{(i\pi)^n} n! \det G\, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \dots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n\right)\\ &= \frac{(-1)^n}{(2\pi)^n} n! \det \mathrm{Hess}(\psi)\, \mathrm{d}u_1 \wedge \mathrm{d}v_1 \wedge \dots \wedge \mathrm{d}u_n \wedge \mathrm{d}u_n.\end{aligned}$$

Since the map val is the composition of $\widetilde{\mathbf{e}}^{-1}$ with the projection $N_{\mathbb{C}} \to N_{\mathbb{R}}$, integrating with respect to the variables $v_1, \dots, v_n$ in the domain $[0, 2\pi]^n$, taking into account the natural orientation of $\mathbb{C}^n$ and the orientation of $N_{\mathbb{R}}$ given by the coordinate system, and the fact that the normalization factor $1/(2\pi i)^n$ is implicit in the current δ_{X_Σ}, we obtain

$$\mathrm{val}_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}|_{X_0^{\mathrm{an}}}) = (-1)^n n! \det \mathrm{Hess}(\psi)\, \mathrm{d}u_1 \wedge \dots \wedge \mathrm{d}u_n.$$

Thus equation (5.35) follows from Proposition 3.94. Finally, the last statement follows from the fact that, in a compact Abelian group there is a unique Haar measure with fixed total volume. $\square$

We end this section recalling how to obtain a toric metric from a non-toric one. Let L be a toric line bundle on the toric variety X_Σ and let s be a toric section. If $\|\cdot\|$ is a smooth, non-necessarily toric, metric, we can average it to obtain a toric metric. This averaging process preserves smoothness and semipositivity. Let μ_{Haar}

be the Haar measure of $\mathbb{S}^{\mathrm{an}}$ of total volume 1. Then we define the metric $\|\cdot\|_{\mathbb{S}}$ over X_0^{an} by

$$\log \|s(p)\|_{\mathbb{S}} = \int_{\mathbb{S}^{\mathrm{an}}} \log \|s(t\cdot p)\| \, \mathrm{d}\mu_{\mathrm{Haar}}(t). \tag{5.37}$$

Proposition 5.38. *The metric $\|\cdot\|_{\mathbb{S}}$ extends to a toric smooth metric over X_Σ^{an}. Moreover, if $\|\cdot\|$ is semipositive then $\|\cdot\|_{\mathbb{S}}$ is semipositive.*

Proof. Let $\|\cdot\|'$ be any toric smooth metric. Then $\|\cdot\|_{\mathbb{S}}$ extends to a smooth metric if and only if $\log(\|s\|_{\mathbb{S}}/\|s\|')$ can be extended to a smooth function on X_Σ^{an}. But we have

$$\log(\|s\|_{\mathbb{S}}/\|s\|') = \int_{\mathbb{S}^{\mathrm{an}}} \log(\|s(t\cdot p)\|/\|s(t\cdot p)\|') \, \mathrm{d}\mu_{\mathrm{Haar}}(t)$$

and the right-hand side can be extended to a smooth function on the whole X_Σ^{an}. Clearly the metric $\|\cdot\|_{\mathbb{S}}$ is toric. Moreover

$$c_1(L, \|\cdot\|_{\mathbb{S}}) = \int_{\mathbb{S}^{\mathrm{an}}} t^* c_1(L, \|\cdot\|) \, \mathrm{d}\mu_{\mathrm{Haar}}(t).$$

Therefore, if $(L, \|\cdot\|)$ is semipositive, then $(L, \|\cdot\|_{\mathbb{S}})$ is semipositive. □

5.4. **Algebraic metrics from toric models.** Next we study some properties of the algebraic metrics that arise from toric models. This kind of metrics will be called *toric algebraic metrics.* Thus, we assume that K is a complete field with respect to an absolute value associated to a nontrivial discrete valuation. We keep the usual notations. We fix a complete fan Σ in $N_{\mathbb{R}}$.

We begin by studying the relationship between the maps val and red.

Lemma 5.39. *Let Π be a complete SCR polyhedral complex of $N_{\mathbb{R}}$ such that $\operatorname{rec}(\Pi) = \Sigma$. Let $\mathcal{X} := \mathcal{X}_\Pi$ be the model of X_Σ determined by Π. Let $\Lambda \in \Pi$ and $p \in X_0^{\mathrm{an}}$. Then $\operatorname{red}(p) \in \mathcal{X}_\Lambda$ if and only if $\operatorname{val}_K(p) \in \Lambda$.*

Proof. By the definition of the semigroup $\widetilde{M}_\Lambda$, the condition $\operatorname{val}(p) \in \Lambda$ holds if and only in $\langle m, \operatorname{val}(p)\rangle + l \geq 0$ for all $(m,l) \in \widetilde{M}_\Lambda$. This is equivalent to $\log|\chi^{-m}(p)| + \log|\varpi|^{-l} \geq 0$ for all $(m,l) \in \widetilde{M}_\Lambda$. In turn, this is equivalent to $|\chi^m(p)\varpi^l| \leq 1$, for all $(m,l) \in \widetilde{M}_\Lambda$. Hence, $\operatorname{val}(p) \in \Lambda$ if and only if $|a(p)| \leq 1$ for all $a \in K^\circ[\mathcal{X}_\Lambda]$, which is exactly the condition $\operatorname{red}(p) \in \mathcal{X}_\Lambda$ (see (2.12)). □

Corollary 5.40. *With the same hypothesis as Lemma 5.39,* $\operatorname{red}(p) \in O(\Lambda)$ *if and only if* $\operatorname{val}(p) \in \operatorname{ri}(\Lambda)$.

Proof. This follows from Lemma 5.39 and the fact that the special fibre is

$$\mathcal{X}_{\Lambda,o} = \coprod_{\Lambda' \text{ face of } \Lambda} O(\Lambda'),$$

and $\operatorname{ri}(\Lambda) = \Lambda \setminus \bigcup_{\Lambda' \text{ proper face of } \Lambda} \Lambda'$. □

Let Ψ be a virtual support function on Σ, and (L, s) the corresponding toric line bundle and section. Let Π be a complete SCR polyhedral complex in $N_{\mathbb{R}}$ such that $\operatorname{rec}(\Pi) = \Sigma$ and let ψ be a rational piecewise affine function on Π with $\operatorname{rec}(\psi) = \Psi$. Let $e > 0$ be an integer such that $e\psi$ is an H-lattice function. By Theorem 4.81, the pair $(\Pi, e\psi)$ determines a toric model $(\mathcal{X}_\Pi, \mathcal{L}_{e\psi}, e)$ of (X_Σ, L). We will write $\mathcal{L} = \mathcal{L}_{e\psi}$. Definition 2.17 gives us an algebraic metric $\|\cdot\|_{\mathcal{L}}$ on L^{an}. In its turn, the metric $\|\cdot\|_{\mathcal{L}}$ defines a function $\psi_{\|\cdot\|_{\mathcal{L}}}$. The following proposition closes the circle.

Proposition 5.41. *The equality $\psi_{\|\cdot\|_{\mathcal{L}}} = \psi$ holds. Hence $\psi - \Psi$ extends to a continuous function on N_Σ and the metric $\|\cdot\|_\psi$ associated to ψ by Proposition 5.16 agrees with $\|\cdot\|_{\mathcal{L}}$.*

Proof. The tensor product $s^{\otimes e}$ defines a rational section of $\mathcal{L}$. Let $\Lambda \in \Pi$ and choose $m_\Lambda \in M$, $l_\Lambda \in \mathbb{Z}$ such that $e\psi|_\Lambda = m_\Lambda + l_\Lambda|_\Lambda$. Let $u \in \Lambda$ and $p \in X_\Sigma^{\mathrm{an}}$ with $u = \mathrm{val}(p)$. Then $\mathrm{red}(p) \in \mathcal{X}_\Lambda$. But in $\mathcal{X}_\Lambda$ the section $\chi^{m_\Lambda}\varpi^{l_\Lambda}s^{\otimes e}$ is regular and non-vanishing. Therefore, by Definition 2.17,

$$\|\chi^{m_\Lambda}(p)\varpi^{l_\Lambda}s^{\otimes e}(p)\|_{\mathcal{L}} = 1.$$

Thus

$$\begin{aligned}\psi_{\|\cdot\|_{\mathcal{L}}}(u) &= \frac{1}{\lambda_K}\log(\|s(p)\|_{\mathcal{L}})\\ &= \frac{1}{e\lambda_K}\log(|\chi^{-m_\Lambda}(p)\varpi^{-l_\Lambda}|)\\ &= \frac{1}{e}(\langle m_\Lambda, u\rangle + l_\Lambda)\\ &= \psi(u).\end{aligned}$$

Therefore ψ agrees with the function associated to the metric $\|\cdot\|_{\mathcal{L}}$. Hence $\psi - \Psi$ extends to a continuous function on N_Σ and the metric $\|\cdot\|_\psi$ agrees with $\|\cdot\|_{\mathcal{L}}$. □

Example 5.42. In the non-Archimedean case, the canonical metric of Proposition-Definition 5.20 is the toric algebraic metric induced by the canonical model of Definition 4.76.

Proposition 5.41 imposes a necessary condition for a rational piecewise affine function to determine a model of (X_Σ, L_Ψ).

Corollary 5.43. *Let Ψ be a virtual support function on Σ and let ψ be a rational piecewise affine function on $N_\mathbb{R}$, with $\mathrm{rec}(\psi) = \Psi$, such that there exists a complete SCR polyhedral complex Π with $\mathrm{rec}(\Pi) = \Sigma$ and ψ piecewise affine on Π. Then $\psi - \Psi$ can be extended to a continuous function on N_Σ.*

Proof. If there exists such a SCR polyhedral complex Π, then Π and ψ determine a model of $\mathcal{O}(D_\Psi)$ and hence a toric algebraic metric $\|\cdot\|$. By Proposition 5.41, $\psi = \psi_{\|\cdot\|}$ and, by the classification of toric metrics in Proposition 5.16, the function $\psi_{\|\cdot\|} - \Psi$ extends to a continuous function on N_Σ. □

Example 5.44. Let $N = \mathbb{Z}^2$ and consider the fan Σ generated by $e_0 = (-1,-1)$, $e_1 = (1,0)$ and $e_2 = (0,1)$. Then $X_\Sigma = \mathbb{P}^2$. The virtual support function $\Psi = 0$ corresponds to the trivial line bundle $\mathcal{O}_{\mathbb{P}^2}$. Consider the function

$$\psi(x,y) = \begin{cases} 0, & \text{if } x \leq 0,\\ x, & \text{if } 0 \leq x \leq 1,\\ 1, & \text{if } 1 \leq x.\end{cases}$$

Then $\mathrm{rec}(\psi) = \Psi$, but ψ does not extend to a continuous function on N_Σ and therefore it does not determine a model of $(X_\Sigma, \mathcal{O})$. By contrast, let Σ' be the fan obtained subdividing Σ by adding the edge corresponding to $e' = (0,-1)$. Then $X_{\Sigma'}$ is isomorphic to a blow-up of $\mathbb{P}^2$ at one point. The function ψ extends to a continuous function on $N_{\Sigma'}$ and it corresponds to a toric model of $(X_{\Sigma'}, \mathcal{O})$.

Question 5.45. Is the condition in Corollary 5.43 also sufficient? In other words, let N, Σ and Ψ be as before and let ψ be a rational piecewise affine function on $N_\mathbb{R}$ such that $\psi - \Psi$ can be extended to a continuous function on N_Σ. Does it exists a complete SCR polyhedral complex Π with $\mathrm{rec}(\Pi) = \Sigma$ and ψ is piecewise affine on Π?

Remark 5.46. By the proof of Theorem 4.97 and Corollary 5.43, when ψ is concave, the conditions

(1) $|\psi-\Psi|$ is bounded;
(2) $\psi-\Psi$ can be extended to a continuous function on N_Σ;
(3) there exist a complete SCR polyhedral complex Π with $\operatorname{rec}(\Pi)=\Sigma$ and ψ piecewise affine on Π;

are equivalent. In particular, the answer to the above question is positive when ψ is concave.

By Theorem 4.97, a rational piecewise affine concave function ψ with $\operatorname{rec}(\psi)=\Psi$ determines an equivalence class of semipositive toric models of $(X_{\Sigma,K},L_\Psi)$. As before, every toric model in this class defines an algebraic metric on L_Ψ^{an}. Since, by Proposition 2.18, equivalent models give rise to the same metric, this metric only depends on ψ. Then Proposition 5.41 has the following direct consequence.

Corollary 5.47. *Let Σ be a complete fan and let Ψ be a support function on Σ. Let ψ be a rational piecewise affine concave function on $N_\mathbb{R}$ with $\operatorname{rec}(\psi)=\Psi$ and let $\|\cdot\|$ be the metric defined by any model of (X_Σ,L_Ψ) in the equivalence class determined by ψ. Then the equality $\psi_{\|\cdot\|}=\psi$ holds. So the metric $\|\cdot\|$ agrees with the metric $\|\cdot\|_\psi$ of Proposition 5.16. Moreover, the algebraic metric $\|\cdot\|$ is semipositive.*

Proof. The equation $\psi_{\|\cdot\|}=\psi$ is just Proposition 5.41 in the concave case. By the definition of semipositive algebraic metrics and Theorem 4.95 we obtain that ψ concave implies $\|\cdot\|$ semipositive. □

We have seen that rational piecewise affine functions give rise to toric algebraic metrics. We now study the converse. Let g be a rational function on X_Σ. Then we denote by $\psi_g\colon N_\mathbb{R}\to\mathbb{R}$ the function $\psi_g(u)=\frac{1}{\lambda_K}\log|g(\theta_0(\mathbf{e}_{\lambda_K}(u)))|$.

Lemma 5.48. *Let g be a rational function on X_Σ. Then the function ψ_g is an H-lattice function (Definition 3.88). In particular it is a piecewise affine function.*

Proof. The function g can be written as $g=\frac{\sum_{m\in M}\alpha_m\chi^m}{\sum_{m\in M}\beta_m\chi^m}$. Then

$$\begin{aligned}\psi_g(u)&=\frac{1}{\lambda_K}\log|g(\theta_0(\mathbf{e}_{\lambda_K}(u)))|\\&=\frac{1}{\lambda_K}\log\Big|\sum_{m\in M}\alpha_m\chi^m(\theta_0(\mathbf{e}_{\lambda_K}(u)))\Big|-\frac{1}{\lambda_K}\log\Big|\sum_{m\in M}\beta_m\chi^m(\theta_0(\mathbf{e}_{\lambda_K}(u)))\Big|\\&=\max_{m\in M}\Big(\frac{\log|\alpha_m|}{\lambda_K}-\langle m,u\rangle\Big)-\max_{m\in M}\Big(\frac{\log|\beta_m|}{\lambda_K}-\langle m,u\rangle\Big)\\&=\max_{m\in M}(-\operatorname{ord}(\alpha_m)-\langle m,u\rangle)-\max_{m\in M}(-\operatorname{ord}(\beta_m)-\langle m,u\rangle)\\&=\min_{m\in M}(\langle m,u\rangle+\operatorname{ord}(\beta_m))-\min_{m\in M}(\langle m,u\rangle+\operatorname{ord}(\alpha_m)).\end{aligned}$$

Thus, it is the difference of two H-lattice concave functions. □

Theorem 5.49. *Let Σ be a complete fan, Ψ a virtual support function on Σ and (L,s) the corresponding toric line bundle and section. Let $\|\cdot\|$ be a toric algebraic metric on L^{an}. Then the function $\psi_{\|\cdot\|}$ is rational piecewise affine. If moreover $\psi_{\|\cdot\|}$ is concave, the toric algebraic metric $\|\cdot\|$ is semipositive and it comes from a toric model.*

Proof. Since the metric is algebraic, there exist a proper K°- scheme $\mathcal{X}$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that the base change of $(\mathcal{X},\mathcal{L})$ to K is isomorphic to $(X_\Sigma,L^{\otimes e})$. Let $\{\mathcal{U}_i,s_i\}$ be a trivialization of $\mathcal{L}$. Let $C_i=\operatorname{red}^{-1}(\mathcal{U}_i\cap\mathcal{X}_o)$. The subsets C_i form a finite closed cover of X_Σ^{an}. On $\mathcal{U}_i$ we can write $s_\Psi^{\otimes e}=g_is_i$ for certain rational function g_i. Therefore, on C_i, we have $\log\|s_\Psi(p)\|=\frac{\log|g(p)|}{e}$. By Lemma 5.48, it follows that there is a finite closed cover of $N_{\mathbb{R}}$ and the restriction of $\psi_{\|\cdot\|}$ to each of these closed subsets is rational piecewise affine. Therefore $\psi_{\|\cdot\|}$ is rational piecewise affine. The second statement follows from the first and Corollary 5.47. □

The next point we study is how to turn a non-toric metric into a toric one. Since the image of θ_0 consists of fixed points under the action of $\mathbb{S}^{\mathrm{an}}$ (see Proposition-Definition 5.2), we may think of it as the analogue, in the non-Archimedean case, of a Haar measure of volume 1 on the compact torus $\mathbb{S}^{\mathrm{an}}$.

Let Ψ be a virtual support function on Σ. Write $L = \mathcal{O}(D_\Psi)$ and $s = s_\Psi$. Let $\|\cdot\|$ be a metric on L^{an}, non-necessarily toric. Then we define $\psi_{\|\cdot\|}\colon N_{\mathbb{R}} \to \mathbb{R}$ by

$$\psi_{\|\cdot\|}(u) = \frac{1}{\lambda_K} \log \|s(\theta_0(\mathbf{e}_K(u)))\|. \tag{5.50}$$

Note that, if $\|\cdot\|$ is a toric metric, the definition of $\psi_{\|\cdot\|}$ we have just given agrees with the one given in §5.2. This is clear because, if the metric is toric, then $\|s(p)\| = \|s(\theta_0\rho_0(p))\|$.

Proposition 5.51. *The assignment that, to a local section s of L gives the function defined as $\|s(\theta_0\rho_0(p)\|$ for $p \in X_\Sigma^{\mathrm{an}}$, is a toric metric on L^{an}, that we denote $\|\cdot\|_{\mathbb{S}}$. Moreover, $\psi_{\|\cdot\|} = \psi_{\|\cdot\|_{\mathbb{S}}}$.*

Proof. As in the proof of Proposition 5.16, we can verify that the function $\psi_{\|\cdot\|} - \Psi$ can be extended to a continuous function on N_Σ. Using that θ_0 is a section of ρ_0 and the image of θ_0 consists of points which are fixed under the action of $\mathbb{S}^{\mathrm{an}}$, we also verify that $\|\cdot\|_{\mathbb{S}}$ is the toric metric associated to $\psi_{\|\cdot\|}$ by the same proposition. □

The relationship between toric algebraic metrics and rational piecewise functions of Theorem 5.49 can be extended to the case when the metric is non-toric.

Proposition 5.52. *Let $\|\cdot\|$ be an algebraic metric. Then the function $\psi_{\|\cdot\|}$ is rational piecewise affine.*

Proof. Just observe that in the proof of Theorem 5.49 one does not use the fact that the metric is toric. □

We now study the effect of taking a field extension. Let $K \subset H$ be a finite extension of fields that are complete with respect to an absolute value associated to a nontrivial discrete valuation. We assume that the absolute value of H is an extension of the absolute value of K. Let H° be the valuation ring of H, $H^{\circ\circ}$ the maximal ideal, ϖ' a generator of the maximal ideal, $\lambda_H = \log(|\varpi'|^{-1})$. Let $e_{H/K}$ be the ramification degree of the extension. Hence $\lambda_K = e_{H/K}\lambda_H$.

Proposition 5.53. *Let Σ be a complete fan in $N_{\mathbb{R}}$ and let Π be a complete SCR polyhedral complex in $N_{\mathbb{R}}$ with $\Sigma = \mathrm{rec}(\Pi)$.*

(1) *Let $X_{\Sigma,K}$ and $X_{\Sigma,H}$ denote the toric varieties defined by Σ over K and H respectively. Then*

$$X_{\Sigma,H} = \mathrm{Spec}(H) \times X_{\Sigma,K}.$$

Moreover there is a commutative diagram

$$\begin{tikzcd} X_{\Sigma,H}^{\mathrm{an}} \arrow[r] \arrow[dr, "\rho_{\Sigma,H}"'] & X_{\Sigma,K}^{\mathrm{an}} \arrow[d, "\rho_{\Sigma,K}"] \\ & X_\Sigma(\mathbb{R}_{\geq 0}), \end{tikzcd}$$

where the horizontal map is induced by the restriction of seminorms.

(2) *Let* Π' *be the polyhedral complex in* $N_{\mathbb{R}}$ *obtained from* Π *by applying a homothety of ratio* $e_{H/K}$. *Then*

$$\mathcal{X}_{\Pi',H^\circ} = \mathrm{Nor}(\mathrm{Spec}(H^\circ) \times \mathcal{X}_{\Pi,K^\circ}),$$

where Nor *denotes the normalization of a scheme.*

(3) *Let* ψ *be a rational piecewise linear function on* Π *and denote* $\Psi = \mathrm{rec}(\psi)$. *Let* $L = \mathcal{O}(D_\Psi)$ *be the line bundle on* $X_{\Sigma,K}$ *determined by* Ψ *and let* $\|\cdot\|$ *be the metric on* L^{an} *determined by* ψ. *Let* L' *be the line bundle obtained by base change and* $\|\cdot\|'$ *the metric obtained by inverse image. Then*

$$\psi_{\|\cdot\|'}(u) = (\psi e_{H/K})(u) = e_{H/K}\psi(e_{H/K}^{-1}u).$$

(4) *There is a commutative diagram*

$$\begin{tikzcd} X^{\mathrm{an}}_{\Sigma,H} \arrow[r] & X^{\mathrm{an}}_{\Sigma,K} \\ & X_\Sigma(\mathbb{R}_{\geq 0}). \arrow[ul, "\theta_{\Sigma,H}"] \arrow[u, "\theta_{\Sigma,K}"'] \end{tikzcd}$$

Proof. The statement (1) can be checked locally. Let σ be a cone of Σ. Then

$$X_{\sigma,H} = \mathrm{Spec}(H[M_\sigma]) = \mathrm{Spec}(K[M_\sigma] \underset{K}{\otimes} H) = \mathrm{Spec}(H) \underset{K}{\times} X_{\sigma,K}.$$

This proves the first assertion. The commutativity of the diagram follows from the fact that the map $X^{\mathrm{an}}_{\Sigma,H} \to X^{\mathrm{an}}_{\Sigma,K}$ is given by the restriction of seminorms.

The statement (2) can also be checked locally. Let Λ be a polyhedron of Π. Let $\Lambda' = e_{K'/K}\Lambda$. Then it is clear that

$$K^\circ[\mathcal{X}_\Lambda] \underset{K^\circ}{\otimes} H^\circ \subset H^\circ[\mathcal{X}_{\Lambda'}].$$

Since the right-hand side ring is integrally closed, the integral closure of the left side ring is contained in the right side ring. Therefore we need to prove that $H[\widetilde{M}_{\Lambda'}]$ is integral over the left side ring. Let $(a,l) \in \widetilde{M}_{\Lambda'}$. Thus $(e_{H/K}a, l) \in \widetilde{M}_\Lambda$. Then the monomial $\chi^a\varpi'^l \in H^\circ[\mathcal{X}_{\Lambda'}]$ satisfies

$$(\chi^a\varpi'^l)^{e_{H/K}} = (\chi^{e_{H/K}a}\varpi^l) \in K^\circ[\mathcal{X}_\Lambda] \underset{K^\circ}{\otimes} H^\circ.$$

Hence $\chi^a\varpi'^l$ is integral over $K^\circ[\mathcal{X}_\Lambda] \underset{K^\circ}{\otimes} H^\circ$. Since these monomials generate $H^\circ[\mathcal{X}_{\Lambda'}]$, we obtain the result.

To prove (3), let $p' \in X^{\mathrm{an}}_{0,H}$ and let $p \in X^{\mathrm{an}}_{0,K}$ be the corresponding point. Then $\mathrm{val}_K(p) = \frac{\mathrm{val}_H(p')}{e_{H/K}}$. Therefore, if we write $u = \mathrm{val}_K(p)$ and $u' = \frac{\mathrm{val}_H(p')}{e_{H/K}}$, we have

$$\psi_{\|\cdot\|'}(u') = \frac{1}{\lambda_H}\log\|s(p')\|' = \frac{e_{H/K}}{\lambda_K}\log\|s(p)\| = e_{H/K}\psi(u) = e_{H/K}\psi(u'/e_{H/K}).$$

Finally, statement (4) follows directly from the definition of θ_Σ because the horizontal arrow is given by the restriction of seminorms. □

5.5. **The one-dimensional case.** We now study in detail the one-dimensional case. Besides being a concrete example of the relationship between functions, models, metrics, and measures, it is also a crucial step in the proof that a toric metric is semipositive if and only if the corresponding function is concave. Of this equivalence, up to now we have proved only one implication and the reverse implication will be proved in the next section.

The only complete one-dimensional toric variety over a field is the projective line. Since, by Proposition 5.53 and Proposition 2.35 we know the effect of taking finite extensions of the field K, we can use the following result to reduce any model of $\mathbb{P}^1$ to a simpler form.

Definition 5.54. Let K be a field complete with respect to an absolute value associated to a nontrivial discrete valuation. Let K° be the ring of integers. Let X be a proper curve over K. A *semi-stable* model of X is a flat proper regular scheme $\mathcal{X}$ of finite type over $\operatorname{Spec}(K^{\circ})$ with an isomorphism $X \to \mathcal{X}_{\eta}$, such that the special fibre $\mathcal{X}_o$ is a reduced normal crossing divisor.

Proposition 5.55. *Let K be a field complete with respect to an absolute value associated to a nontrivial discrete valuation. Let K° be the ring of integers. Let $\mathcal{X}$ be a proper model over K° of $\mathbb{P}^1_K$. Then there exists a finite extension H of K with ring of integers H°, a semi-stable model $\mathcal{X}'$ of $\mathbb{P}^1_H$, and a proper morphism of models $\mathcal{X}' \to \mathcal{X} \times \operatorname{Spec}(H^{\circ})$.*

Proof. This follows, for instance, from [Liu06, Corollary 2.8]. □

Consider the toric variety $X_{\Sigma} \simeq \mathbb{P}^1$. We can choose an isomorphism $N \simeq \mathbb{Z}$ and $N_{\mathbb{R}} \simeq \mathbb{R}$. Then $\Sigma = \{\mathbb{R}_-, \{0\}, \mathbb{R}_+\}$. Let 0 denote the invariant point of $\mathbb{P}^1_K$ corresponding to the cone $\mathbb{R}_+$ and ∞ the invariant point corresponding to the cone $\mathbb{R}_-$. Let t denote the absolute coordinate of $\mathbb{P}^1$ given by the monomial χ^1.

Let $\mathcal{X}$ be a semi-stable model of $\mathbb{P}^1_K$. By extending scalars if necessary, we may suppose that all the components of the special fibre are defined over $k = K^{\circ}/K^{\circ\circ}$ and contain a rational point. Since the special fibre $\mathcal{X}_o$ is connected and of genus zero, we deduce that the special fibre is a tree of rational curves, each isomorphic to $\mathbb{P}^1_k$. Let D_0 and D_{∞} denote the horizontal divisors corresponding to the point 0 and ∞ of $\mathbb{P}^1_K$. Then, there is a chain of rational curves that links the divisor D_0 with D_{∞} that is contained in the special fibre. We will denote the irreducible components of the special fibre that form this chain by $E_0, \dots, E_k$, in such a way that the component E_0 meets D_0, the component E_k meets D_{∞} and, for $0 < i < k$, the component E_i meets only E_{i-1} and E_{i+1}. The other components of $\mathcal{X}_o$ will be grouped in branches, each branch has its root in one of the components E_i. We will denote by $F_{i,j}$, $j \in \Theta_i$ the components that belong to a branch with root in E_i. We are not giving any particular order to the sets Θ_i.

We denote by $E \cdot F$ the intersection product of two 1-cycles of $\mathcal{X}$. Since the special fibre is reduced, we have

$$\operatorname{div}(\varpi) = \sum_{i=0}^{k} \left(E_i + \sum_{j \in \Theta_i} F_{i,j} \right).$$

Again by the assumption of semi-stability, the intersection product of two different components of $\mathcal{X}_o$ is either 1, if they meet, or zero, if they do not meet. Since the intersection product of $\operatorname{div}(\varpi)$ with any component of $\mathcal{X}_o$ is zero, we deduce that, if E is any component of $\mathcal{X}_o$, the self-intersection product $E \cdot E$ is equal to minus the number of components that meet E. In particular, all components $F_{i,j}$ that are terminal, are (-1)-curves. By Castelnuovo Criterion, we can successively blow-down all the components $F_{i,j}$ to obtain a new semi-stable model of $\mathbb{P}^1_K$ whose special fibre consist of a chain of rational curves. For reasons that will become apparent later we denote this model as $\mathcal{X}_{\mathbb{S}}$.

Lemma 5.56. *If we view t as a rational function on $\mathcal{X}$, then there is an integer a such that*

$$\operatorname{div}(t) = D_0 - D_{\infty} + \sum_{i=0}^{k} (a - i) \left(E_i + \sum_{j \in \Theta_i} F_{i,j} \right).$$

Proof. It is clear that

$$\operatorname{div}(t) = D_0 - D_\infty + \sum_{i=0}^{k} a_i E_i + \sum_{j\in\Theta_i} a_{i,j} F_{i,j}$$

for certain coefficients a_i and $a_{i,j}$ that we want to determine as much as possible.

If a component E of $\mathcal{X}_0$, with coefficient a, does not meet D_0 nor D_∞, but meets $r \geq 1$ other components, and the coefficients of $r-1$ of these components are equal to a, while the coefficient of the remaining component is b, we obtain that

$$0 = \operatorname{div}(t)\cdot E = aE\cdot E + a(r-1) + b = -ra + a(r-1) + b = b - a$$

Thus $b = a$. Starting with the components $F_{i,j}$ that are terminal, we deduce that, for all i and $j \in \Theta_i$, $a_i = a_{i,j}$. Therefore,

$$\operatorname{div}(t) = D_0 - D_\infty + \sum_{i=0}^{k} a_i \left(E_i + \sum_{j\in\Theta_i} F_{i,j} \right).$$

In particular, the lemma is proved for $k = 0$. Assume now that $k > 0$.

It only remains to show that $a_i = a_0 - i$, that we prove by induction. For $i = 1$, we compute

$$0 = \operatorname{div}(t)\cdot E_0 = D_0\cdot E_0 + a_0 E_0\cdot E_0 + a_0 \sum_{j\in\Theta_0} F_{0,j}\cdot E_0 + a_1 E_1\cdot E_0 = 1 - a_0 + a_1.$$

Thus $a_1 = a_0 - 1$. For $1 < i \leq k$, by induction hypothesis, $a_{i-1} = a_{i-2} - 1$. Then

$$0 = \operatorname{div}(t)\cdot E_{i-1} = a_{i-2} - 2a_{i-1} + a_i = 1 - a_{i-1} + a_i.$$

Thus $a_i = a_{i-1} - 1 = a_0 - i$, proving the lemma. □

The determination of $\operatorname{div}(t)$ allows us to give a partial description of the map $\operatorname{red}\colon X_\Sigma^{\mathrm{an}} \to \mathcal{X}_o$. For us, the most interesting points of $\mathcal{X}_o$ are the points $q_0 := D_0 \cap E_0$, $q_i := E_{i-1}\cap E_i$, $i = 1,\dots,k$, $q_{k+1} := E_k \cap D_\infty$ and the generic points of the components E_i that we denote η_i, $i = 0,\dots,k$.

Lemma 5.57. *Let $p \in X_\Sigma^{\mathrm{an}}$. Then*

$$\operatorname{red}(p) = \begin{cases} q_0, & \text{if } |t(p)| < |\varpi|^a \\ q_i,\ i = 1\dots,k, & \text{if } |\varpi|^{a-i+1} < |t(p)| < |\varpi|^{a-i} \\ q_{k+1}, & \text{if } |\varpi|^{a-k} < |t(p)| \\ \eta_i,\ i = 0\dots,k, & \text{if } |t(p)| = |\varpi|^{a-i} \text{ and } p \in \operatorname{im}(\theta_\Sigma). \end{cases}$$

Proof. Let $1 \leq i \leq k$. The rational function $x := t\varpi^{-a+i}$ has a zero of order one along the component E_{i-1} and the support of its divisor does not contain the component E_i. On the other hand, the rational function $y := t^{-1}\varpi^{a-i+1}$ has a zero of order one along the component E_i and the support of its divisor does not contain the component E_{i-1}. Thus $\{x, y\}$ is a system of parameters in a neighbourhood of q_i. We denote

$$A = K^\circ[t\varpi^{-a+i}, t^{-1}\varpi^{a-i+1}] \simeq K^\circ[x,y]/(xy - \varpi).$$

The local ring at the point q_i is $A_{(x,y)}$. Let p be a point such that $|\varpi|^{a-i+1} < |t(p)| < |\varpi|^{a-i}$. Therefore, for $f \in A$ we have $|f(p)| \leq 1$. Moreover, if $f \in (x,y)$, then $|f(p)| < 1$. Since the ideal (x,y) is maximal, we deduce that, for $f \in A$, the condition $|f(p)| < 1$ is equivalent to the condition $f \in (x,y)$. This implies that $\operatorname{red}(p) = q_i$. A similar argument works for q_0 and q_{k+1}.

Assume now that $p \in \operatorname{im}(\theta_\Sigma)$ and that $|t(p)| = |\varpi|^{a-i}$. If $i \neq 0$ we consider again the ring A, but in this case $|x(p)| = |t(p)\varpi^{-a+i}| = 1$. Let $I = \{f \in A \mid |f(p)| < 1\}$. It is clear that $(y, \varpi) \subset I$. For $f = \sum_{m\in\mathbb{Z}} \beta_m t^m \in A$, since $p \in \operatorname{im}(\theta_\Sigma)$, we have

$$|f(p)| = \sup_m (|\beta_m||t(p)|^m).$$

This implies that $I \subset (y, \varpi)$. Hence I is the ideal that defines the component E_i and this is equivalent to $\operatorname{red}(p) = \eta_i$. The case $i = 0$ is analogous. □

The image by red of the remaining points of X_Σ^{an} is not characterized only by the value of $|t(p)|$. Using a proof similar to that of the lemma, one can show that, if $|t(p)| = |\varpi|^{a-i}$ then $\operatorname{red}(p)$ belongs either to E_i or to any of the components $F_{i,j}$, $j \in \Theta_i$.

We denote by ξ_i (resp. $\xi_{i,j}$) the point of X_Σ^{an} corresponding to the component E_i (resp. $F_{i,j}$). That is, $\operatorname{red}(\xi_i) = \eta_i$ and $\operatorname{red}(\xi_{i,j}) = \eta_{i,j}$, where $\eta_{i,j}$ is the generic point of $F_{i,j}$ (see (2.15) and (2.14)).

Lemma 5.58. *Let $0 \leq i \leq k$. Then, for every $j \in \Theta_i$,*

$$\operatorname{val}_K(\xi_i) = \operatorname{val}_K(\xi_{i,j}) = a - i,$$

where a is the integer of Lemma 5.56.

Proof. We consider the rational function $\varpi^{-a+i}t$. Since the support of $\operatorname{div}(\varpi^{-a+i}t)$ does not contain the component E_i nor any of the components $F_{i,j}$, we have that

$$|\varpi^{-a+i}t(\xi_i)| = |\varpi^{-a+i}t(\xi_{i,j})| = 1.$$

Since $t = \chi^1$, we deduce, using equation (5.4), that

$$\operatorname{val}_K(\xi_i) = \frac{-\log|\chi^1(x_i)|}{\lambda_K} = \frac{-\log|\varpi^{a-i}|}{-\log|\varpi|} = a - i.$$

□

Let now Ψ be a virtual support function on Σ. It can be written as

$$\Psi(u) = \begin{cases} m_\infty u, & \text{if } u \leq 0, \\ m_0 u, & \text{if } u \geq 0. \end{cases}$$

for some $m_0, m_\infty \in \mathbb{Z}$. Then, $L = \mathcal{O}(D_\Psi) \simeq \mathcal{O}(m_\infty - m_0)$, and $\operatorname{div}(s_\Psi) = -m_0[0] + m_\infty[\infty]$. Let $\mathcal{L}$ be a model over $\mathcal{X}$ of $L^{\otimes e}$. If we consider $s_\Psi^{\otimes e}$ as a rational section of $\mathcal{L}$, then

$$\operatorname{div}(s_\Psi^{\otimes e}) = -em_0 D_0 + em_\infty D_\infty + \sum_{i=0}^{k}\left(\alpha_i E_i + \sum_{j\in\Theta_i} \alpha_{i,j} F_{i,j}\right) \tag{5.59}$$

for certain coefficients α_i and $\alpha_{i,j}$. Let $\|\cdot\|$ be the metric on L^{an} determined by this model.

Lemma 5.60. *The function $\psi_{\|\cdot\|}$ is given by*

$$\psi_{\|\cdot\|}(u) = \begin{cases} m_0 u - m_0 a - \frac{\alpha_0}{e}, & \text{if } u \geq a, \\ \frac{(\alpha_{i+1}-\alpha_i)u - (\alpha_{i+1}-\alpha_i)(a-i) - \alpha_i}{e}, & \text{if } a-i \geq u \geq a-i-1, \\ m_\infty u - m_\infty(a-k) - \frac{\alpha_k}{e}, & \text{if } a-k \geq u. \end{cases}$$

In other words, if Π is the polyhedral complex in $N_\mathbb{R}$ given by the intervals

$$(-\infty, a-k], \quad [a-i, a-i+1],\ i = 1, \dots, k, \quad [a, \infty),$$

then $\psi_{\|\cdot\|}$ is the rational piecewise affine function on Π characterized by the conditions

(1) $\operatorname{rec}(\psi_{\|\cdot\|}) = \Psi$,

(2) *the value of* $\psi_{\|\cdot\|}$ *at the point* $a-i$ *is* $-\alpha_i/e$.

Proof. Let $p \in \operatorname{im}\theta_\Sigma$ be such that $\operatorname{val}_K(p) > a$, hence $|t(p)| < |\varpi|^a$. By Lemma 5.57, this implies that $\operatorname{red}(p) = q_0$. In a neighbourhood of q_0, the divisor of the rational section $s_\Psi^{\otimes e} t^{em_0} \varpi^{-\alpha_0 - em_0 a}$ is zero, and so

$$\|s_\Psi^{\otimes e}(p) t^{em_0}(p) \varpi^{-\alpha_0 - em_0 a}\| = 1.$$

Set $u = \operatorname{val}(p)$. Then,

$$\begin{aligned}
\psi_{\|\cdot\|}(u) &= \frac{\log \|s_\Psi^{\otimes e}(p)\|}{e\lambda_K} \\
&= \frac{-em_0 \log|t(p)| + (\alpha_0 + em_0 a)\log|\varpi|}{-e\log|\varpi|} \\
&= m_0(u-a) - \frac{\alpha_0}{e}.
\end{aligned}$$

The other cases are proved in a similar way. □

Since $\operatorname{rec}(\Pi) = \Sigma$, this polyhedral complex defines a toric model $\mathcal{X}_\Pi$ of X_Σ.

Proposition 5.61. *The identity map of* X_Σ *extend to an isomorphism of models* $\mathcal{X}_{\mathbb{S}} \to \mathcal{X}_\Pi$.

Proof. The special fibre of $\mathcal{X}_\Pi$ is a chain of rational curves E_i, $i = 0, \dots, k$, corresponding to the points $a - i$. The monomial χ^1 is a section of the trivial line bundle and corresponds to the function $\psi(u) = -u$. Using Proposition 4.84 we obtain that

$$\operatorname{div}(\chi^1) = D_0 - D_\infty + \sum_{i=0}^{k} (a-i)E_i,$$

where D_0 and D_∞ are again the horizontal divisors determined by the points 0 and ∞.

Since the vertices of the polyhedral complex Π are integral, by equation (4.87), we deduce that $\operatorname{div}(\varpi)$ is reduced.

Then the result follows from [Lic68, Corollary 1.13] using an explicit description of the local rings at the points of the special fibre as in the proof of Lemma 5.57. □

From Proposition 5.61 we obtain a proper morphism $\pi\colon \mathcal{X} \to \mathcal{X}_\Pi$. On $\mathcal{X}$ we had a line bundle $\mathcal{L}$ and $s_\Psi^{\otimes e}$ was considered as a rational section of this line bundle. Let $D = \operatorname{div}(s_\Psi^{\otimes e})$ be the divisor given by equation (5.59). We denote

$$D_{\mathbb{S}} = \pi_* D = -em_0 D_0 + em_\infty D_\infty + \sum_{i=0}^{k} \alpha_i E_i. \tag{5.62}$$

By Proposition 4.84 and Lemma 5.60 we see that $D_{\mathbb{S}} = D_{e\psi_h}$. Thus $\mathcal{O}(D_{\mathbb{S}})$ is a toric model of $L^{\otimes e}$. Recall that $\|\cdot\|$ denoted the metric associated to the model $\mathcal{O}(D)$. Let $\|\cdot\|_{\mathbb{S}}$ be the toric metric obtained from $\|\cdot\|$ as in Proposition 5.51. By this proposition and equation (5.62), the metric $\|\cdot\|_{\mathbb{S}}$ agrees with the metric defined by the model $\mathcal{O}(D_{\mathbb{S}})$. Thus, we have identified a toric model that corresponds to the metric $\|\cdot\|_{\mathbb{S}}$. This allows us to compute directly the associated measure.

Proposition 5.63. *Let* $X_\Sigma \simeq \mathbb{P}^1_K$ *be a one-dimensional toric variety over* K. *Let* $L \simeq \mathcal{O}(D_\Psi)$ *be a toric line bundle and let* $\|\cdot\|$ *be an algebraic metric defined by a semi-stable model and let* $\|\cdot\|_{\mathbb{S}}$ *be the associated toric metric. Then*

$$c_1(L, \|\cdot\|_{\mathbb{S}}) \wedge \delta_{X_\Sigma} = (\theta_\Sigma)_*(\rho_\Sigma)_*\left(c_1(L, \|\cdot\|) \wedge \delta_{X_\Sigma}\right).$$

Proof. Since the special fibre is reduced, by equation (2.29)

$$c_1(L,\|\cdot\|)\wedge\delta_{X_\Sigma}=\frac{1}{e}\sum_{i=0}^{k}\left(\deg_{\mathcal{L}}E_i\delta_{\xi_i}+\sum_{j\in\Theta_i}\deg_{\mathcal{L}}F_{i,j}\delta_{\xi_{i,j}}\right).$$

Denote this measure temporarily by μ. Then

$$\begin{aligned}(\theta_\Sigma)_*(\rho_\Sigma)_*\mu&=\frac{1}{e}\sum_{i=0}^{k}\left(\deg_{\mathcal{L}}E_i+\sum_{j\in\Theta_i}\deg_{\mathcal{L}}F_{i,j}\right)\delta_{\xi_i}\\&=\frac{1}{e}\sum_{i=0}^{k}\left(D\cdot E_i+\sum_{j\in\Theta_i}D\cdot F_{i,j}\right)\delta_{\xi_i}\\&=\frac{1}{e}\sum_{i=0}^{k}\sum_{l=0}^{k}\left(\alpha_lE_l+\sum_{s\in\Theta_l}\alpha_{l,s}F_{l,s}\right)\cdot\left(E_i+\sum_{j\in\Theta_i}F_{i,j}\right)\delta_{\xi_i}\\&=\frac{1}{e}\sum_{i=0}^{k}(\alpha_{i-1}E_{i-1}+\alpha_iE_i+\alpha_{i+1}E_{i+1})\cdot\left(E_i+\sum_{j\in\Theta_i}F_{i,j}\right)\delta_{\xi_i}\\&=\frac{1}{e}\sum_{i=0}^{k}(\alpha_{i-1}-2\alpha_i+\alpha_{i+1})\delta_{\xi_i}.\end{aligned}$$

In the previous computation, we have used that, since $E_l\cdot\operatorname{div}(\varpi)=F_{l,s}\cdot\operatorname{div}(\varpi)=0$, then

$$F_{l,s}\cdot(E_i+\sum_{j\in\Theta_i}F_{i,j})=0,\ \text{for all } i,j,l,s,$$

$$E_l\cdot(E_i+\sum_{j\in\Theta_i}F_{i,j})=\begin{cases}0, & \text{if } l\neq i-1,i,i+1,\\ 1, & \text{if } l=i-1,i+1,\\ -2, & \text{if } l=i.\end{cases}$$

An analogous computation shows that

$$c_1(L,\|\cdot\|_{\mathbb{S}})\wedge\delta_{X_\Sigma}=\frac{1}{e}\sum_{i=0}^{k}(\alpha_{i-1}-2\alpha_i+\alpha_{i+1})\delta_{\xi_i}. \tag{5.64}$$

□

Using Proposition 5.55 we can extend the above result to the case when the model is not semi-stable.

Corollary 5.65. *Let $X_\Sigma\simeq\mathbb{P}^1_K$ be a one-dimensional toric variety over K. Let $L\simeq\mathcal{O}(D_\Psi)$ be a toric line bundle, $\|\cdot\|$ an algebraic metric, and $\|\cdot\|_{\mathbb{S}}$ the associated toric metric. Then*

$$c_1(L,\|\cdot\|_{\mathbb{S}})\wedge\delta_{X_\Sigma}=(\theta_\Sigma)_*(\rho_\Sigma)_*\left(c_1(L,\|\cdot\|)\wedge\delta_{X_\Sigma}\right).$$

Proof. Let $(\mathcal{X},\mathcal{L})$ be a model of $(X_\Sigma,L^{\otimes e})$ that realizes the algebraic metric $\|\cdot\|$. For short, denote $\mu=c_1(L,\|\cdot\|)\wedge\delta_{X_\Sigma}$ and $\mu_{\mathbb{S}}=c_1(L,\|\cdot\|_{\mathbb{S}})\wedge\delta_{X_\Sigma}$. By Proposition 5.55 there is a non-Archimedean field H over K and a semi-stable model $\mathcal{X}'$ of $X_{\Sigma,H}$. We may further assume that all the components of the special fibre of $\mathcal{X}'$ are defined over $H^\circ/H^{\circ\circ}$. Let $(L',\|\cdot\|')$ be the metrized line bundle obtained by base change to H. Then $(\|\cdot\|')_{\mathbb{S}}$ is obtained from $\|\cdot\|_{\mathbb{S}}$ by base change. We denote by $\pi\colon X^{\mathrm{an}}_{\Sigma,H}\to X^{\mathrm{an}}_{\Sigma,K}$ the map of analytic spaces. Be will denote by μ', $\mu'_{\mathbb{S}}$, θ'_Σ and

ρ'_Σ the corresponding objects for $X_{\Sigma,H}$. Then, by Proposition 2.35 and Proposition 5.53,

$$\mu_{\mathbb{S}} = \pi_*\mu'_{\mathbb{S}} = \pi_*(\theta'_\Sigma)_*(\rho'_\Sigma)_*\mu' = (\theta_\Sigma)_*(\rho_\Sigma)_*\pi_*\mu' = (\theta_\Sigma)_*(\rho_\Sigma)_*\mu.$$

□

We can now relate semipositivity of the metric with concavity of the associated function on the one-dimensional case.

Corollary 5.66. *Let $X_\Sigma \simeq \mathbb{P}^1_K$ be a one-dimensional toric variety over K. Let (L,s) be a toric line bundle with a toric section and let $\|\cdot\|$ be a semipositive algebraic metric. Then $\|\cdot\|_{\mathbb{S}}$ is a semipositive toric algebraic metric and $\psi_{\|\cdot\|}$ is concave.*

Proof. Since $\|\cdot\|$ is semipositive, $c_1(L,\|\cdot\|)\wedge\delta_{X_\Sigma}$ is a positive measure. By Corollary 5.65, $c_1(L,\|\cdot\|_{\mathbb{S}})\wedge\delta_{X_\Sigma}$ is a positive measure. Hence $\|\cdot\|_{\mathbb{S}}$ is a semipositive toric metric. By equation (5.64) and Lemma 5.60, the positivity of $c_1(L,\|\cdot\|_{\mathbb{S}})\wedge\delta_{X_\Sigma}$ implies that the function $\psi_{\|\cdot\|} = \psi_{\|\cdot\|_{\mathbb{S}}}$ is concave. □

5.6. **Algebraic metrics and their associated measures.** We come back to the case of general dimension. Let Σ be a complete fan, Ψ a support function on Σ and $(L,s)=(L_\Psi,s_\Psi)$. Since Ψ is a support function, the line bundle L is generated by global sections.

Proposition 5.67. *Let $\|\cdot\|$ be a semipositive algebraic metric on L^{an}. Then the function $\psi_{\|\cdot\|}$ is concave.*

Proof. Assume that $\|\cdot\|$ is semipositive. Let u_0 be a point of $N_{\mathbb{Q}}$ and let $v_0\in N$ be primitive. Since the condition of being concave is closed, if we prove that, for all choices of $u_0\in N_{\mathbb{Q}}$ and $v_0\in N$, the restriction of $\psi_{\|\cdot\|}$ to the line $u_0+\mathbb{R}v_0$ is concave, we will deduce that the function $\psi_{\|\cdot\|}$ is concave. Let $e\in\mathbb{N}^\times$ such that $eu_0\in N$. Then $H=K(\varpi^{1/e})$ is a finite extension of K and there is a unique extension of the absolute value of K to H. We will denote with $'$ the objects obtained by base change to H. Let $p\in X_{0,H}(H)$ such that $\mathrm{val}_H(p)=eu_0$. We consider the affine map $A\colon \mathbb{Z}\to N$ given by $l\mapsto v_0 l + eu_0$, and let H be the linear part of A. We consider the equivariant morphism $\varphi=\varphi_{p,H}\colon \mathbb{P}^1_H\to X_{\Sigma,H}$ of Theorem 4.9. The metric $\|\cdot\|$ induces an algebraic semipositive metric $\varphi^*\|\cdot\|'$ on the restriction of L' (the line bundle obtained from L by base change to H) to $\mathbb{P}^1_H$. By propositions 5.24 and 5.53(3) we obtain that

$$\psi_{\varphi^*\|\cdot\|}(u) = e\psi_{\|\cdot\|}(u_0+e^{-1}uv_0).$$

By Corollary 5.66 the left-hand side function is concave. Thus the restriction of $\psi_{\|\cdot\|}$ to $u_0+\mathbb{R}v_0$ is concave. We conclude that $\psi=\psi_{\|\cdot\|}$ is concave. □

Corollary 5.68. *Let $\|\cdot\|$ be a semipositive algebraic metric on L^{an}. Then the toric metric $\|\cdot\|_{\mathbb{S}}$ is a semipositive toric algebraic metric.*

Proof. By Proposition 5.67, the function $\psi_{\|\cdot\|}$ is concave. By Proposition 5.52 it is also rational piecewise affine. By Corollary 5.47, the metric $\|\cdot\|_{\mathbb{S}} = \|\cdot\|_\psi$ is toric algebraic and semipositive. □

Putting together Proposition 5.67 and Theorem 5.49, we see that the relationship between semipositivity of the metric and concavity of the associated function given in the Archimedean case by Proposition 5.29 carries over to the non-Archimedean case.

Corollary 5.69. *Let $\|\cdot\|$ be a toric algebraic metric and $\psi_{\|\cdot\|}$ the associated function. Then the metric is semipositive if and only if the function $\psi_{\|\cdot\|}$ is concave.*

We can now characterize the Chambert-Loir measure associated to a toric semipositive algebraic metric.

Theorem 5.70. *Let $\|\cdot\|$ be a toric semipositive algebraic metric on L^{an} and let $\psi = \psi_{\|\cdot\|}$ be the associated function on $N_{\mathbb{R}}$. Let $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ be the associated measure. Then*

$$(\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}) = n!\overline{\mathcal{M}}_M(\psi), \tag{5.71}$$

where $\overline{\mathcal{M}}(\psi)$ is the measure of Definition 5.32. Moreover,

$$c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = (\theta_\Sigma)_*(\mathbf{e}_K)_* n!\overline{\mathcal{M}}_M(\psi). \tag{5.72}$$

Proof. Since the metric is semipositive and toric, by Proposition 5.67 the function ψ is concave. Since, moreover it is algebraic, by Theorem 5.49 it is defined by a toric model $(\mathcal{X}_\Pi, D_\psi, e)$ of (X_Σ, D_Ψ) in the equivalence class determined by ψ. As in Remark 4.66, the irreducible components of $\mathcal{X}_{\Pi,o}$ are in bijection with the vertices of Π. For each vertex $v \in \Pi^0$, let ξ_v be the point of X_Σ^{an} corresponding to the generic point of $V(v)$ defined by equation (2.15). Then, by equation (2.29),

$$c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = \frac{1}{e^n} \sum_{v\in\Pi^0} \nu_v \deg_{D_\psi} V(v)\delta_{\xi_v}.$$

Thus, by Corollary 5.40,

$$(\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}) = \frac{1}{e^n} \sum_{v\in\Pi^0} \nu_v \deg_{D_\psi} V(v)\delta_v.$$

But, using Proposition 3.95 and Proposition 4.105, the Monge-Ampère measure is given by

$$\begin{aligned}\mathcal{M}_M(\psi) &= \frac{1}{e^n}\mathcal{M}_M(e\psi)\\ &= \frac{1}{e^n}\sum_{v\in\Pi^0} \mathrm{vol}_M(v^*)\delta_v\\ &= \frac{1}{n!e^n}\sum_{v\in\Pi^0} \nu_v \deg_{D_\psi} V(v)\delta_v.\end{aligned}$$

Since $\mathcal{M}_M(\psi)$ is a finite sum of Dirac deltas, we obtain that

$$\overline{\mathcal{M}}_M(\psi) = \frac{1}{n!e^n}\sum_{v\in\Pi^0} \nu_v \deg_{D_\psi} V(v)\delta_v.$$

Hence we have proved (5.71). To prove equation (5.72) we just observe that $x_v = (\theta_0 \circ \mathbf{e}_K)(v)$. □

5.7. **Approachable and integrable metrics.** We are now in position to characterize the approachable metrics. In this section K is either $\mathbb{R}$, $\mathbb{C}$ or a complete field with respect to an absolute value associated to a nontrivial discrete valuation. We fix a complete fan Σ of $N_{\mathbb{R}}$, so that X_Σ is proper. Let Ψ be a support function on Σ, Δ_Ψ the corresponding polytope, and (L_Ψ, s_Ψ) the corresponding toric line bundle and section. For short, write $X = X_\Sigma$, $L = L_\Psi$ and $s = s_\Psi$.

Theorem 5.73. *Assume the previous hypothesis.*

(1) *The assignment $\|\cdot\| \mapsto \psi_{\|\cdot\|}$ is a bijection between the space of approachable toric metrics on L^{an} and the space of continuous concave functions ψ on $N_{\mathbb{R}}$ such that $|\psi - \Psi|$ is bounded.*
(2) *The assignment $\|\cdot\| \mapsto \psi^\vee_{\|\cdot\|}$ is a bijection between the space of approachable toric metrics on L^{an} and the space of continuous concave functions on Δ_Ψ.*

Proof. By Proposition 3.77(2) and Proposition 3.80, the statements (1) and (2) are equivalent.

Let $\|\cdot\|$ be an approachable toric metric. By Corollary 5.17 the function $|\psi_{\|\cdot\|}-\Psi|$ is bounded. By approachability there is a sequence $\|\cdot\|_l$ of smooth (resp. algebraic) semipositive metrics that converges to $\|\cdot\|$. Since $\|\cdot\|$ is toric, $\|\cdot\|_{\mathbb{S}}=\|\cdot\|$. Hence, the sequence of toric metrics $(\|\cdot\|_l)_{\mathbb{S}}$ also converges to $\|\cdot\|$. We denote $\psi_l=\psi_{(\|\cdot\|_l)_{\mathbb{S}}}$. By Proposition 5.38 and Proposition 5.67 the functions ψ_l are concave. Since the sequence $(\psi_l)_l$ converge uniformly to $\psi_{\|\cdot\|}$, the latter is concave.

Let now ψ be a concave function on $N_{\mathbb{R}}$ such that $|\Psi-\psi|$ is bounded. Then ψ determines a metric $\|\cdot\|$ on the restriction of L^{an} to X_0^{an}. Since $\mathrm{stab}(\psi)=\mathrm{stab}(\Psi)=\Delta_\Psi$, by Proposition 3.81 there is a sequence of rational piecewise affine concave functions ψ_l that converge uniformly to ψ and with $\mathrm{rec}(\psi_l)=\Psi$. By Remark 5.46, the functions $\Psi-\psi_l$ can be extended to continuous functions on N_Σ. Therefore, $\Psi-\psi$ can be extended to a continuous function on N_Σ. Consequently the metric $\|\cdot\|$ can be extended to X^{an}. Let $\|\cdot\|_l$ be the metric associated to ψ_l. Then the sequence of metrics $\|\cdot\|_l$ converges to $\|\cdot\|$. By Corollary 5.28, the metrics $\|\cdot\|_l$ are approachable. We deduce that $\|\cdot\|$ is approachable. □

Remark 5.74. For the case $K=\mathbb{C}$, statement (2) in the above result is related to the Guillemin-Abreu classification of Kähler structures on symplectic toric varieties as explained in [Abr03]. By definition, a symplectic toric variety is a compact symplectic manifold of dimension $2n$ together with a Hamiltonian action of the compact torus $\mathbb{S}^{\mathrm{an}}\simeq(S^1)^n$. These spaces are classified by Delzant polytopes of $M_{\mathbb{R}}$, see for instance [Gui95]. For a given Delzant polytope $\Delta\subset M_{\mathbb{R}}$, the possible $(S^1)^n$-invariant Kähler forms on the symplectic toric variety corresponding to Δ are classified by smooth convex functions on Δ° satisfying some conditions near the border of Δ. Several differential geometric invariants of a Kähler toric variety can be translated and studied in terms of this convex function, also called the "symplectic potential".

For a smooth positive toric metric $\|\cdot\|$ on $L_{\Psi_\Delta}(\mathbb{C})$, the Chern form defines a Kähler structure on the complex toric variety $X_{\Sigma_\Delta}(\mathbb{C})$. It turns out that the corresponding symplectic potential coincides with minus the function $\psi^\vee_{\|\cdot\|}$. It would be most interesting to explore further this connection.

We now study the compatibility of the restriction of approachable toric metrics to toric orbits and its inverse image by equivariant maps with direct and inverse image of concave functions. This is an extension of propositions 4.99 and 4.108. We start with the case of orbits, and we state a variant of Proposition 5.22 for approachable metrics.

Proposition 5.75. *Let $\|\cdot\|$ be an approachable toric metric on L^{an}, and denote $\overline{L}=(L,\|\cdot\|)$ and $\psi=\psi_{\overline{L},s}$ the associated concave function on $N_{\mathbb{R}}$. Let $\sigma\in\Sigma$ and $m_\sigma\in M$ such that $\Psi|_\sigma=m_\sigma|_\sigma$. Let $\pi_\sigma\colon N_{\mathbb{R}}\to N(\sigma)_{\mathbb{R}}$ be the projection, $\pi_\sigma^\vee\colon M(\sigma)_{\mathbb{R}}\to M_{\mathbb{R}}$ the dual inclusion and $\iota\colon V(\sigma)\to X$ the closed immersion. Set $s'=\chi^{m_\sigma}s$. Then*

$$\psi_{\iota^*\overline{L},\iota^*s'}=(\pi_\sigma)_*(\psi-m_\sigma). \tag{5.76}$$

Dually, we have that

$$\psi^\vee_{\iota^*\overline{L},\iota^*s'}=(\pi_\sigma^\vee+m_\sigma)^*\psi^\vee. \tag{5.77}$$

In other words, the Legendre-Fenchel dual of $\psi_{\iota^*\overline{L},\iota^*s'}$ is the restriction of $\psi^\vee$ to the face F_σ translated by $-m_\sigma$.

Proof. As in the proof of Proposition 4.99, it is enough to prove equation (5.76). By replacing ψ by $\psi-m_\sigma$, we can assume without loss of generality that $m_\sigma=0$. By the continuity of the metric, the function ψ can be extended to a continuous function $\overline{\psi}_\sigma$ on N_σ. Fix $u_0 \in N(\sigma)_{\mathbb{R}}$, write $s = \overline{\psi}_\sigma(u_0)$ and let $u \in N_{\mathbb{R}}$ such that $\pi_\sigma(u) = u_0$. By definition

$$(\pi_\sigma)_*(\psi)(u_0) = \sup_{p\in\mathbb{R}\sigma} \psi(u+p).$$

It is clear that $\sup_{p\in\mathbb{R}\sigma} \psi(u+p) \geq s$. Suppose that $\sup_{p\in\mathbb{R}\sigma} \psi(u+p) > s$. Let $q \in \mathbb{R}\sigma$ such that $\psi(u+q) > s$ and let $\varepsilon = (\psi(u+q) - s)/2$. By the definition of the topology of N_σ, there exists a $p \in \mathbb{R}\sigma$ such that

$$s - \varepsilon < \psi(u + p + \sigma) < s + \varepsilon. \tag{5.78}$$

Since σ is a cone of maximal dimension in $\mathbb{R}\sigma$, there exists a point $r \in (q+\sigma)\cap(p+\sigma)$. By the right inequality of equation (5.78) $\psi(u+r) < \psi(u+q)$. By concavity of ψ this implies that

$$\lim_{\lambda\to\infty} \psi(u + r + \lambda(r - q)) = -\infty. \tag{5.79}$$

Since, by construction $u + r + \mathbb{R}_{\geq 0}(r - q)$ is contained in $u + p + \sigma$, equation (5.79) contradicts the left inequality of equation (5.78). Hence $\sup_{p\in\mathbb{R}\sigma} \psi(u+p) = s$, which proves equation (5.76). □

We now interpret the inverse image of an approachable toric metric by an equivariant map whose image intersects the principal open subset in terms of direct and inverse images of concave functions.

Proposition 5.80. *Let N_1 and N_2 be lattices and Σ_i a complete fan in $N_{i,\mathbb{R}}$, $i = 1, 2$. Let $H\colon N_1 \to N_2$ be a linear map such that, for each $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subset \sigma_2$. Let $p \in X_{\Sigma_2,0}(K)$ and write $A\colon N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$ for the affine map $A = H + \mathrm{val}(p)$. Let $\|\cdot\|$ be an approachable toric metric on $\mathcal{O}(D_{\Psi_2})^{\mathrm{an}}$. Then*

$$\psi_{\varphi^*_{p.H}\|\cdot\|} = A^*\psi_{\|\cdot\|}.$$

Moreover, the Legendre-Fenchel dual of this function is given by

$$\psi^\vee_{\varphi^*_{p.H}\|\cdot\|} = (H^\vee)_*\big(\psi^\vee_{\|\cdot\|} - \mathrm{val}(p)\big).$$

Proof. The first statement is a direct consequence of Proposition 5.24 while the second one follows from Proposition 3.78(1). □

We next characterize the measures associated to an approachable metric.

Theorem 5.81. *Let Σ be a complete fan of $N_{\mathbb{R}}$, let Ψ be a support function on Σ and let $L = \mathcal{O}(D_\Psi)$. Let $\|\cdot\|$ be an approachable metric on L^{an} and let $\psi = \psi_{\|\cdot\|}$ be the corresponding concave function. Then*

$$(\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}) = n!\overline{\mathcal{M}}_M(\psi). \tag{5.82}$$

Moreover, the measure $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ is characterized, in the Archimedean case, by equation (5.82) *and the fact of being toric, while in the non-Archimedean case it is given by*

$$c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = (\theta_\Sigma)_*(\mathbf{e}_K)_* n!\overline{\mathcal{M}}_M(\psi).$$

Proof. For short, denote $\mu = (\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma})$. Let $\|\cdot\|_l$ be a sequence of semipositive smooth (respectively algebraic) metrics converging to $\|\cdot\|$. By Proposition 2.33, the measures $c_1(L, \|\cdot\|_l)^n \wedge \delta_{X_\Sigma}$ converge to $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$. Therefore, the measures $(\mathrm{val}_K)_*(c_1(L, \|\cdot\|_l)^n \wedge \delta_{X_\Sigma})$ converge to the measure μ on N_Σ. Proposition 2.38 implies that the measure of $X^{\mathrm{an}}_\Sigma \setminus X^{\mathrm{an}}_0$ with respect to $c_1(\overline{L})^n \wedge \delta_{X_\Sigma}$ is zero. Therefore $N_\Sigma \setminus N_{\mathbb{R}}$ has μ-measure zero. Denote $\psi_l = \psi_{(\|\cdot\|_l)_{\mathbb{S}}}$. By Proposition 3.108, the measures $\mathcal{M}_M(\psi_l)$ converge to the measure $\mathcal{M}_M(\psi)$. Thus $\mu|_{N_{\mathbb{R}}} = n!\mathcal{M}_M(\psi)$. If we add to this that the measure of $N_\Sigma \setminus N_{\mathbb{R}}$ is zero, we deduce equation (5.82). The last statement of the theorem is clear from Theorem 5.33 and Theorem 5.70. □

We end this section by characterizing integrable metrics.

Corollary 5.83. *Let Σ be a complete fan. Then the map $\|\cdot\| \mapsto \psi_{\|\cdot\|}$ is a bijection between the space of integrable toric metrics on $\mathcal{O}(D_\Psi)^{\mathrm{an}}$ and the space of functions $\psi \in \overline{\mathscr{D}}(N_{\mathbb{R}})$ such that* $\operatorname{rec}(\psi) = \Psi$, *were $\overline{\mathscr{D}}(N_{\mathbb{R}})$ is the space of functions of Definition 3.82.*

5.8. **Adelic toric metrics.** We now turn to the global case. Let $(\mathbb{K}, M_{\mathbb{K}})$ be an adelic field (Definition 2.48). We fix a complete fan Σ in $N_{\mathbb{R}}$ and a virtual support function Ψ on Σ. Let (L, s) be the associated toric line bundle and section. If X is a variety over $\mathbb{K}$ and $v \in M_{\mathbb{K}}$ we will denote by $X^{\mathrm{an},v}$ its analytification with respect to v. Analogously $\mathbb{S}^{\mathrm{an},v}$ will denote the compact subtorus of $\mathbb{T}^{\mathrm{an},v}$.

Definition 5.84. A *toric metric* on L is a family $(\|\cdot\|_v)_{v\in M_{\mathbb{K}}}$, where $\|\cdot\|_v$ is a toric metrics on L_v^{an}. A toric metric is called *adelic* if $\psi_{\|\cdot\|_v} = \Psi$ for all but finitely many v.

Theorem 5.85. *Let $(\mathbb{K}, M_{\mathbb{K}})$ be a global field. A toric metric on L is quasi-algebraic (Definition 2.53) if and only if it is an adelic toric metric.*

Proof. Let $(\|\cdot\|_v)_{v\in M_{\mathbb{K}}}$ be a metric on L and write $\overline{L} = (L, (\|\cdot\|_v)_{v\in M_{\mathbb{K}}})$. Suppose first that $\overline{L}$ is toric and quasi-algebraic. Let $S \subset M_{\mathbb{K}}$ be a finite set containing the Archimedean places, $\mathbb{K}_S^\circ$ as in Definition 2.52, $e \geq 1$ an integer and $(\mathcal{X}, \mathcal{L})$ a proper model over $\mathbb{K}_S^\circ$ of $(X_\Sigma, L^{\otimes e})$ so that $\|\cdot\|_v$ is induced by the localization $\mathcal{L}_v$ for all $v \notin S$. Over $\mathbb{K}$, there is an isomorphism from $(\mathcal{X}, \mathcal{L})$ to the canonical model $(\mathcal{X}_\Sigma, \mathcal{L}_{e\Psi})$. Since $\mathbb{K}_S^\circ$ is Noetherian, this isomorphism and its inverse are defined over $\mathbb{K}_{S'}^\circ$ for certain finite subset S' containing S. Thus, enlarging the finite set S if necessary, we can suppose without loss of generality that $(\mathcal{X}, \mathcal{L})$ agrees with the canonical model $(\mathcal{X}_\Sigma, \mathcal{L}_{e\Psi})$. Hence, $\|\cdot\|_v = \|\cdot\|_{v,e\Psi}^{1/e} = \|\cdot\|_{v,\Psi}$ for all places $v \notin S$. In consequence, it is an adelic toric metric.

Conversely, suppose that $\overline{L}$ is a toric adelic metrized line bundle. Let S be the union of the set of Archimedean places and $\{v \in M_{\mathbb{K}} | \psi_v \neq \Psi\}$. By definition, this is a finite set. Let $(\mathcal{X}_\Sigma, \mathcal{L}_\Psi)$ be the canonical model over $\mathbb{K}_S^\circ$ of (X_Σ, L). Then $\|\cdot\|_v$ is the metric induced by this model, for all $v \notin S$. Hence $\overline{L}$ is quasi-algebraic. □

Corollary 5.86. *Let L be as before.*

(1) *There is a bijection between the set of approachable adelic toric metric on L and the set of families of continuous concave functions $\{\psi_v\}_v$ on $N_{\mathbb{R}}$ such that $|\psi_v - \Psi|$ is bounded and $\psi_v = \Psi$ for all but finitely many v.*
(2) *There is a bijection between the set of approachable adelic toric metric on L and the set of families of continuous concave functions $\{\psi_v^\vee\}_v$ on Δ_Ψ such that $\psi_v^\vee = 0$ for all but finitely many v.*

Proof. This follows from Theorem 5.85 and Theorem 5.73. □

6. Height of toric varieties

In this section we will state and prove a formula to compute the height of a toric variety with respect to a toric line bundle.

6.1. **Local heights of toric varieties.** ??? `behaviour of the roof function with respect to change of field, ditto for global height`

Let K be either $\mathbb{R}$, $\mathbb{C}$ or a complete field with respect to an absolute value associated to a nontrivial discrete valuation. Let $N \simeq \mathbb{Z}^n$ be a lattice and $M = N^\vee$ the dual lattice. We will use the notations of §4 and we recall the definition of λ_K in (5.3).

Let Σ be a complete fan on $N_{\mathbb{R}}$ and X_Σ the corresponding proper toric variety. In Definition 2.39 we recalled the definition of local heights. These local heights depend, not only on cycles and metrized line bundles, but also on the choice of sections of the involved line bundles. For toric line bundles, Proposition-Definition 5.20, provides us with a distinguished choice of a toric metric, the canonical metric. This metric is integrable and, if the line bundle is generated by global sections, it is approachable. By comparing any integrable metric to the canonical metric, we can define a local height for toric line bundles that is independent from the choice of sections.

Definition 6.1. Let $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, d$, be a family of toric line bundles, with integrable toric metrics. Denote by $\overline{L}_i^{\mathrm{can}}$ the same line bundles equipped with the canonical metric. Let Y be a d-dimensional cycle of X_Σ. Then the *toric local height* of Y with respect to $\overline{L}_0, \dots, \overline{L}_d$ is

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y) = \mathrm{h}_{\varphi^*\overline{L}_0,\dots,\varphi^*\overline{L}_d}(Y'; s_0, \dots, s_d) - \mathrm{h}_{\varphi^*\overline{L}_0^{\mathrm{can}},\dots,\varphi^*\overline{L}_d^{\mathrm{can}}}(Y'; s_0, \dots, s_d), \tag{6.2}$$

where Σ' is a regular refinement of Σ (hence $X_{\Sigma'}$ is projective), $\varphi\colon X_{\Sigma'} \to X_\Sigma$ is the corresponding proper toric morphism, Y' is a cycle of X' such that $\varphi_* Y' = Y$ and $s_0, \dots, s_d$ are sections meeting Y' properly. When $\overline{L}_0 = \dots = \overline{L}_d = \overline{L}$ we will denote

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = \mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y).$$

Remark 6.3. Even if the toric local height in the above definition differs from the local height of Definition 2.39, we will be able to use them to compute global heights because, for toric subvarieties and closures of orbits, the sum over all places of the local canonical heights is zero (see Proposition 6.35). This is the case, in particular, for the height of the total space X_Σ.

By Theorem 2.47(4), the right-hand side of equation (6.2) does not depend on the choice of refinement nor on the choice of sections, but the toric local height depends on the toric structure of the line bundles (see Definition 4.19), because the canonical metric depends on the toric structure.

Proposition 6.4. *The toric local height is symmetric and multilinear with respect to tensor product of metrized toric line bundles. In particular, let Σ be a complete fan, $\overline{L}_i$ a family of $d+1$ toric line bundles with integrable toric metrics and Y an algebraic cycle of X_Σ of dimension d. Then*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y) = \sum_{j=0}^{d} (-1)^{d-j} \sum_{1\le i_0 < \dots < i_j \le d} \mathrm{h}^{\mathrm{tor}}_{\overline{L}_{i_0}\otimes\dots\otimes\overline{L}_{i_j}}(Y). \tag{6.5}$$

Proof. It suffices to prove the statement for the case when X_Σ is projective, as the general case reduces to this one by taking a suitable refinement of the fan.

The symmetry of the toric local height follows readily from the analogous property for the local height, see Theorem 2.47(1). For the multilinearity, let $\overline{L}_d'$ be a further metrized line bundle. By the moving lemma, there are sections s_i of L_i, $0 \le i \le d$ meeting properly on Y and s_d' of L_d' such that $s_0, \dots, s_{d-1}, s_d'$ meets properly on Y too. By Theorem 2.47(1),

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_{d-1},\overline{L}_d\otimes\overline{L}_d'}(Y; s_0, \dots, s_{d-1}, s_d \otimes s_d') = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y; s_0, \dots, s_d) + \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_{d-1},\overline{L}_d'}(Y; s_0, \dots, s_{d-1}, s_d')$$

and a similar formula holds for the canonical metric. By the definition of the toric local height, $\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_{d-1},\overline{L}_d\otimes\overline{L}'_d}(Y)=\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y)+\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_{d-1},\overline{L}'_d}(Y)$. The inclusion-exclusion formula follows readily from the symmetry and the multilinearity of the local toric height. □

Theorem 6.6. *Let Σ be a complete fan on $N_{\mathbb{R}}$. Let $\overline{L}=(L,\|\cdot\|)$ be a toric line bundle on X_Σ, generated by global sections, and equipped with an approachable toric metric. Choose any toric section s of L; let Ψ be the associated support function on Σ, and put $\Delta_\Psi=\operatorname{stab}(\Psi)$ for the associated polytope. Then, the toric local height of X_Σ with respect to $\overline{L}$ is given by*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma)=(n+1)!\lambda_K\int_{\Delta_\Psi}\psi^\vee_{\overline{L},s}\,\mathrm{d}\,\mathrm{vol}_M\,. \tag{6.7}$$

where $\mathrm{d}\,\mathrm{vol}_M$ is the unique Haar measure of $M_{\mathbb{R}}$ such that the co-volume of M is one and $\psi^\vee_{\overline{L},s}$ is the Legendre-Fenchel dual to the function $\psi_{\overline{L},s}$ associated to $(\overline{L},s)$ in Definition 5.14.

We note that, by Theorem 5.73(2), the function $\psi_{\overline{L},s}$ is concave because the metric $\|\cdot\|$ on L^{an} is approachable. We also introduce the function

$$f_{\overline{L},s}(u)=(\psi_{\overline{L},s}\lambda_K)(u)=\lambda_K\psi_{\overline{L},s}(u/\lambda_K).$$

Definition 6.8. Let $(\overline{L},s)$ be a metrized toric line bundle with a toric section as in the theorem above. Then the *roof function* associated to $(\overline{L},s)$ is the concave function $\vartheta_{\overline{L},s}\colon\Delta_\Psi\to\mathbb{R}$ defined as

$$\vartheta_{\overline{L},s}=f^\vee_{\overline{L},s}=\lambda_K\psi^\vee_{\overline{L},s}.$$

The concave function $\psi^\vee_{\overline{L},s}$ will be called the *rational roof function*. When the toric section s is clear form the context, we will denote $f_{\overline{L},s}$ and $\vartheta_{\overline{L},s}$ by $f_{\|\cdot\|}$ and $\vartheta_{\|\cdot\|}$ respectively.

The function $\psi_{\|\cdot\|}$ is not invariant under field extensions (see Proposition 5.53(3)) but it has the advantage that, if the metric $\|\cdot\|$ is algebraic, then it is rational with respect to the lattice N. By contrast, the function $f_{\|\cdot\|}$ is invariant under field extensions. It is not rational, but it takes values in $\lambda_K\mathbb{Q}$ on $\lambda_K N_{\mathbb{Q}}$. This is the function that appears in [BPS09].

In case $\psi_{\|\cdot\|}$ is a piecewise affine concave function, $\vartheta_{\|\cdot\|}$ and $\psi^\vee_{\|\cdot\|}$ parameterize the upper envelope of some extended polytope, as explained in Lemma 3.79, hence the terminology "roof function". In case K is non-Archimedean and $||\cdot||$ is algebraic, the function $\psi^\vee_{\|\cdot\|}$ is a rational concave function.

Alternatively, we can express the toric height in terms of the roof function as

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma)=(n+1)!\int_{\Delta_\Psi}\vartheta_{\overline{L},s}\,\mathrm{d}\,\mathrm{vol}_M\,. \tag{6.9}$$

Proof of Theorem 6.6. For short, we set $\Delta=\Delta_\Psi$ and $\psi=\psi_{\|\cdot\|}$. Let Σ_Δ be the fan associated to Δ as in Remark 4.43. There is a toric morphism $\varphi\colon X_\Sigma\to X_{\Sigma_\Delta}$. The function $\psi^\vee$ defines an approachable metric $\|\cdot\|'$ on $\mathcal{O}(D_{\Psi_\Delta})^{\mathrm{an}}$. We denote $\overline{L}'=(\mathcal{O}(D_{\Psi_\Delta}),\|\cdot\|')$. Then there is an isometry $\varphi^*(\overline{L}')=\overline{L}$. By Corollary 5.25 there is an isometry $\varphi^*(\overline{L}'^{\mathrm{can}})=\overline{L}^{\mathrm{can}}$.

If the dimension of Δ is less than n, then the right-hand side of equation (6.7) is zero. Moreover, $n=\dim(X_\Sigma)>\dim(X_{\Sigma_\Delta})$ and the metrized line bundles $\overline{L}$ and $\overline{L}^{\mathrm{can}}$ come from a variety of smaller dimension. Therefore, by Theorem 2.47(2), the left-hand side of equation (6.7) is also zero, because φ_*X_Σ is the cycle zero. If Δ has dimension n then φ is a birational morphism, so, by Theorem 2.47(2),

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \mathrm{h}^{\mathrm{tor}}_{\overline{L}'}(X_{\Sigma_\Delta}).$$

Therefore it is enough to prove the theorem for X_{Σ_Δ}. By construction, the fan Σ_Δ is regular; hence the variety X_{Σ_Δ} is projective and L' is ample. Thus we are reduced to prove the theorem in the case when Σ is regular and L is ample.

Now the proof is done by induction on n, the dimension of X_Σ. If $n = 0$ then $X_\Sigma = \mathbb{P}^0$, $\Psi = 0$, $\Delta = \{0\}$ and $L = \mathcal{O}(D_0) = \mathcal{O}_{\mathbb{P}^0}$. By equation (5.15), $\log\|s\| = \lambda_K\psi(0)$ and $\log\|s\|_{\mathrm{can}} = \lambda_K\Psi(0) = 0$. The Legendre-Fenchel dual of ψ satisfies $\psi^\vee(0) = -\psi(0)$. By equation (2.40), $\mathrm{h}_{\overline{L}}(X_\Sigma; s) = -\lambda_K\psi(0)$ and $\mathrm{h}_{\overline{L}^{\mathrm{can}}}(X_\Sigma; s) = 0$. Therefore

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = -\lambda_K\psi(0) = \lambda_K\psi^\vee(0) = 1!\lambda_K\int_\Delta \psi^\vee \,\mathrm{d}\,\mathrm{vol}_M .$$

Let $n \geq 1$ and let $s_0, \dots, s_{n-1}$ be rational sections of $\mathcal{O}(D_\Psi)$ such that $s_0, \dots, s_{n-1}, s$ intersect X_Σ properly. By the construction of local heights (Definition 2.39),

$$\begin{aligned}\mathrm{h}_{\overline{L}}(X_\Sigma; s_0, \dots, s_{n-1}, s) = \mathrm{h}_{\overline{L}}(&\mathrm{div}(s); s_0, \dots, s_{n-1}) \\ &- \int_{X_\Sigma^{\mathrm{an}}} \log\|s\| c_1(\overline{L})^n \wedge \delta_{X_\Sigma}\end{aligned} \tag{6.10}$$

and a similar formula holds for the canonical metric.

For each facet F of Δ let v_F be as in Notation 3.103. Since L is ample, Proposition 4.46 implies

$$\mathrm{h}_{\overline{L}}(\mathrm{div}(s); s_0, \dots, s_{n-1}) = \sum_F -\langle v_F, F\rangle\, \mathrm{h}_{\overline{L}}(V(\tau_F); s_0, \dots, s_{n-1}), \tag{6.11}$$

where the sum is over the facets F of Δ. Observe that the local height of $V(\tau_F)$ with respect to the metrized line bundle $\overline{L}$ coincides with the local height associated to the restriction of $\overline{L}$ to this subvariety. Moreover by Corollary 5.23, the restriction of the canonical metric of L^{an} to this subvariety agrees with the canonical metric of $L^{\mathrm{an}}|_{V(\tau_F)}$. Hence, by substracting from equation (6.11) the analogous formula for the canonical metric, we obtain

$$\begin{aligned}\sum_F -\langle m_F, v_F\rangle\, \mathrm{h}^{\mathrm{tor}}_{\overline{L}|_{V(\tau_F)}}(V(\tau_F)) = \mathrm{h}_{\overline{L}}(&\mathrm{div}(s); s_0, \dots, s_{n-1}) \\ &- \mathrm{h}_{\overline{L}^{\mathrm{can}}}(\mathrm{div}(s); s_0, \dots, s_{n-1}).\end{aligned} \tag{6.12}$$

Moreover, Theorem 2.38 implies that

$$\int_{X_\Sigma^{\mathrm{an}}} \log\|s\| c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = \int_{X_{\Sigma,0}^{\mathrm{an}}} \log\|s\| c_1(\overline{L})^n \wedge \delta_{X_\Sigma}.$$

By equation (5.15), $\log\|s\| = (\mathrm{val}_K)^*(\lambda_K\psi)$. Moreover

$$\int_{X_{\Sigma,0}^{\mathrm{an}}} (\mathrm{val}_K)^*(\lambda_K\psi) c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = \int_{N_\mathbb{R}} \lambda_K\psi (\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma})$$

and by Theorem 5.81, $(\mathrm{val}_K)_*(c_1(\overline{L})^n \wedge \delta_{X_\Sigma}) = n!\mathcal{M}_M(\psi)$. Hence

$$\int_{X_\Sigma^{\mathrm{an}}} \log\|s\| c_1(\overline{L})^n \wedge \delta_{X_\Sigma} = n!\lambda_K \int_{N_\mathbb{R}} \psi\mathcal{M}_M(\psi). \tag{6.13}$$

By Example 3.96, $\mathcal{M}_M(\Psi) = \mathrm{vol}_M(\Delta)\delta_0$. Therefore, in the case of the canonical metric, equation (6.13) reads as

$$\int_{X_\Sigma^{\mathrm{an}}} \log\|s\|_{\mathrm{can}} c_1(\overline{L}^{\mathrm{can}})^n \wedge \delta_{X_\Sigma} = n!\lambda_K \,\mathrm{vol}_M(\Delta)\Psi(0) = 0. \tag{6.14}$$

Thus, substracting from equation (6.10) the analogous formula for the canonical metric and using equations (6.12), (6.13) and (6.14), we obtain

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \sum_F -\langle v_F, F\rangle\, \mathrm{h}^{\mathrm{tor}}_{\overline{L}|_{V(\tau_F)}}(V(\tau_F)) - n!\lambda_K \int_{N_\mathbb{R}} \psi \mathcal{M}_M(\psi). \tag{6.15}$$

By the inductive hypothesis and equation (5.77)

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}|_{V(\tau_F)}}(V(\tau_F)) = n!\lambda_K \int_F \psi^\vee \,\mathrm{d}\,\mathrm{vol}_{M(F)}\,.$$

Hence, by Corollary 3.104,

$$\begin{aligned}\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) &= -n!\lambda_K \sum_F \langle v_F, F\rangle \int_F \psi^\vee \,\mathrm{d}\,\mathrm{vol}_{M(F)} - n!\lambda_K \int_{N_\mathbb{R}} \psi \mathcal{M}_M(\psi)\\ &= (n+1)!\lambda_K \int_\Delta \psi^\vee \,\mathrm{d}\,\mathrm{vol}_M,\end{aligned}$$

proving the theorem □

Remark 6.16. The left-hand side of equation (6.7) only depends on the structure of toric line bundle of L and not on a particular choice of toric section, while the right-hand side seems to depend on the section s. We can see directly that the right hand side actually does not depend on the section. If we pick a different toric section, say s', then the corresponding support function Ψ' differs from Ψ by a linear functional. The polytope $\Delta_{\Psi'}$ is the translated of $\Delta_{\Psi'}$ by the corresponding element of M. The function $\psi_{\overline{L},s'}$ differs from $\psi_{\overline{L},s}$ by the same linear functional and $\psi^\vee_{\overline{L},s'}$ is the translated of $\psi^\vee_{\overline{L},s}$ by the same element of M. Thus the integral on the right has the same value whether we use the section s of the section s'.

Theorem 6.6 can be reformulated in terms of an integral over $N_\mathbb{R}$.

Corollary 6.17. *Let notation be as in Theorem 6.6 and write $\psi = \psi_{\overline{L},s}$ for short. Then*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \lambda_K (n+1)! \int_{N_\mathbb{R}} (\psi^\vee \circ \partial\psi)\, \mathcal{M}_M(\psi), \tag{6.18}$$

where $\psi^\vee \circ \partial\psi$ is the integrable function defined by (3.105). *When $\psi \in \mathcal{C}^2(N_\mathbb{R})$,*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = (-1)^n (n+1)! \int_{N_\mathbb{R}} (\langle \nabla\psi(u), u\rangle - \psi(u)) \det(\mathrm{Hess}(\psi))\,\mathrm{d}\,\mathrm{vol}_N\,. \tag{6.19}$$

When ψ is piecewise affine,

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = (n+1)! \sum_{v\in\Pi(\psi)^0} \int_{v^*} (\langle x, v\rangle - \psi(v))\,\mathrm{d}\,\mathrm{vol}_M(x). \tag{6.20}$$

Proof. Equation (6.18) follows readily from Theorem 6.6 and (3.105). The second statement follows from Proposition 3.94 and Example 3.106(1) while the third one follows from Proposition 3.95 and Example 3.106(2). □

Theorem 6.6 can be extended to compute the local toric height associated to distinct line bundles in term of the mixed integral of the associated roof functions.

Corollary 6.21. *Let Σ be a complete fan on $N_\mathbb{R}$ and $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, n$, be toric line bundles on X_Σ generated by global sections and equipped with approachable toric metrics. Choose toric sections s_i of L_i and let Ψ_i be the corresponding support functions. Then the toric height of X_Σ with respect to $\overline{L}_0, \dots, \overline{L}_n$ is given by*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_n}(X_\Sigma) = \mathrm{MI}_M(\vartheta_{\|\cdot\|_0}, \dots, \vartheta_{\|\cdot\|_n}) = \lambda_K\, \mathrm{MI}_M(\psi^\vee_{\|\cdot\|_0}, \dots, \psi^\vee_{\|\cdot\|_n}).$$

Proof. Let $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 1, 2$, be toric line bundles equipped with toric metrics and let s_i be a toric section of L_i. By propositions 5.19 (1) and 3.38 (3)

$$(\psi_{\overline{L}_1\otimes\overline{L}_2,s_1\otimes s_2})^{\vee} = \psi^{\vee}_{\overline{L}_1,s_1} \boxplus \psi^{\vee}_{\overline{L}_2,s_2}. \tag{6.22}$$

The result then follows from (6.5), the definition of the mixed integral (Definition 3.113) and Theorem 6.6. □

Remark 6.23. In the integrable case, the toric height can be expressed as an alternating sum of mixed integrals as follows. Let $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, n$, be toric line bundles on X_Σ equipped with integrable toric metrics and set $\overline{L}_i = \overline{L}_{i,+} \otimes (\overline{L}_{i,-})^{-1}$ for some approachable metrized toric line bundles $\overline{L}_{i,+}$, $\overline{L}_{i,-}$. Choose a toric section for each line bundle and write $\vartheta_{i,+}$ and $\vartheta_{i,-}$ for the corresponding roof functions. Then

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_n}(X_\Sigma) = \sum_{\varepsilon_0,\dots,\varepsilon_n\in\{\pm1\}} \varepsilon_0 \dots \varepsilon_n \, \mathrm{MI}_M(\vartheta_{0,\varepsilon_0}, \dots, \vartheta_{n,\varepsilon_n}).$$

We have defined and computed the local height of a toric variety. We now will compute the toric height of toric subvarieties. We start with the case of orbits.

Proposition 6.24. *Let Σ be a complete fan on $N_{\mathbb{R}}$ and $\sigma \in \Sigma$ a cone of codimension d. To it, we have associated the dimension d closed subvariety $V(\sigma)$ and the closed immersion $\iota_\sigma \colon X_{\Sigma(\sigma)} \to X_\Sigma$ whose image is $V(\sigma)$. Let L be a toric line on X_Σ generated by global sections, s a toric section, Ψ the corresponding support function, and $\|\cdot\|$ an approachable toric metric on L^{an}. As usual write $\overline{L} = (L, \|\cdot\|)$. Then*

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(V(\sigma)) = \mathrm{h}^{\mathrm{tor}}_{\iota_\sigma^*\overline{L}}(X_{\Sigma(\sigma)}) = (d+1)! \int_{F_\sigma} \vartheta_{\overline{L},s} \,\mathrm{d}\,\mathrm{vol}_{M(F_\sigma)},$$

where F_σ is the face of Δ_Ψ corresponding to σ, $M(F_\sigma)$ is the lattice induced by M on the linear space associated to F_σ and $\iota_\sigma^* L$ has the structure of toric line bundle of Proposition 4.34.

Proof. By Corollary 5.23 the restriction of the canonical metric of L^{an} is the canonical metric of $\iota_\sigma^* L^{\mathrm{an}}$. Therefore, the equality $\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(V(\sigma)) = \mathrm{h}^{\mathrm{tor}}_{\iota_\sigma^*\overline{L}}(X_{\Sigma(\sigma)})$ follows from Theorem 2.47(2).

To prove the second equality, choose $m_\sigma \in F_\sigma \cap M$. We will follow the notation of Proposition 5.75. By Theorem 6.6,

$$\mathrm{h}^{\mathrm{tor}}_{\iota_\sigma^*\overline{L}}(X_{\Sigma(\sigma)}) = (d+1)! \int_{\Delta_{(\Psi-m_\sigma)(\sigma)}} \vartheta_{\|\cdot\|_\sigma} \,\mathrm{d}\,\mathrm{vol}_{M(\sigma)}.$$

By Proposition 4.47, $\Delta_{(\Psi-m_\sigma)(\sigma)} = (\pi_\sigma^\vee + m_\sigma)^{-1} F_\sigma$. By Proposition 5.75

$$\vartheta_{\|\cdot\|_\sigma} = \lambda_K \varphi^\vee_{\|\cdot\|_\sigma} = \lambda_K (\pi_\sigma^\vee + m_\sigma)^* \varphi^\vee_{\|\cdot\|} = (\pi_\sigma^\vee + m_\sigma)^* \vartheta_{\|\cdot\|}.$$

Since $M(F_\sigma) = M(\sigma)$, we obtain

$$\int_{\Delta_{(\Psi-m_\sigma)(\sigma)}} \vartheta_{\|\cdot\|_\sigma} \,\mathrm{d}\,\mathrm{vol}_{M(\sigma)} = \int_{F_\sigma} \vartheta_{\|\cdot\|} \,\mathrm{d}\,\mathrm{vol}_{M(F_\sigma)},$$

proving the result. □

We now study the behaviour of the toric local height with respect to toric morphisms. Let N_1 be a lattice of rank d and M_1 the dual lattice. Let $H \colon N_1 \to N$ be a linear map and Σ_1 a fan on $N_{1,\mathbb{R}}$ such that, for each cone $\sigma \in \Sigma_1$, $H(\sigma)$ is contained in a cone of Σ. Let $\varphi \colon X_{\Sigma_1} \to X_\Sigma$ be the associated morphism. Denote $Q = H(N_1)^{\mathrm{sat}}$ the saturated sublattice of N and let Y_Q be the image of X_{Σ_1} under φ. Then Y_Q is equal to the toric subvariety $Y_{\Sigma,Q} = Y_{\Sigma,Q,x_0}$ of Definition 4.12, where we recall that x_0 denote the distinguished point of the principal orbit of X_Σ.

Proposition 6.25. *With the previous notation, let $\overline{L}$ be a toric line bundle on X_Σ generated by global sections, equipped with an approachable toric metric. We put on $\varphi^* L$ the structure of toric line bundle of Remark 4.36. Choose a toric section s of L and let Ψ be the associated support function.*

(1) *If H is not injective, then $\mathrm{h}^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma_1}) = 0$.*

(2) *If H is injective, then $\mathrm{h}^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma_1}) = [Q : H(N_1)]\, \mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y_Q)$. Moreover*

$$\mathrm{h}^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma_1}) = (d+1)! \int_{H^\vee(\Delta_\Psi)} H^\vee_*(\vartheta_{\|\cdot\|})\, \mathrm{d}\,\mathrm{vol}_{M_1}. \tag{6.26}$$

Proof. By Corollary 5.25, the inverse image of the canonical metric by a toric morphism is the canonical metric. Thus (1) and the first statement of (2) follow from 2.47 (2).

By Proposition 5.24 and Theorem 6.6 we deduce

$$\begin{aligned}\mathrm{h}^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma_1}) &= (d+1)!\lambda_K \int_{\Delta_{\Psi\circ H}} (H^*\psi_{\|\cdot\|})^\vee\, \mathrm{d}\,\mathrm{vol}_{M_1} \\ &= (d+1)! \int_{H^\vee(\Delta_\Psi)} H^\vee_*(\vartheta_{\|\cdot\|})\, \mathrm{d}\,\mathrm{vol}_{M_1},\end{aligned}$$

proving the result. □

We now study the case of an equivariant morphism. Let N, N_1, d, H, Σ and Σ_1 as before. For simplicity, we assume that $H\colon N_1 \to N$ is injective and that $Q = H(N_1)$ is a saturated sublattice, because the effect of a non-injective map or a non-saturated sublattice is explained in Proposition 6.25. Let $p \in X_{\Sigma,0}(K)$ be a point of the principal open subset and $u = \mathrm{val}_K(p) \in N_\mathbb{R}$. Then, in the non-Archimedean case, $u \in N$. Denote $\varphi = \varphi_{p,H}$ the equivariant morphism determined by H and p, also denote $Y = Y_{\Sigma,Q,p}$ the image of X_{Σ_1} by φ, and $A = H + u$ the associated affine map.

Let $\overline{L}$ be a toric line bundle generated by global sections, equipped with an approachable toric metric. Recall that there is no natural structure of toric line bundle in the inverse image $\varphi^* L$. Therefore we have to choose a toric section s of L. Let $\overline{L}_1$ denote the line bundle $\varphi^* L$ with the metric induced by $\|\cdot\|$ and the toric structure induced by the section s. We denote by Ψ the support function associated to (L, s).

Proposition 6.27. *With the previous hypothesis and notations, the equality*

$$\begin{aligned}\mathrm{h}^{\mathrm{tor}}_{\overline{L}_1}(X_{\Sigma_1}) &= (d+1)!\lambda_K \int_{H^\vee(\Delta_\Psi)} (A^*\psi_{\overline{L},s})^\vee\, \mathrm{d}\,\mathrm{vol}_{M_1} \\ &= (d+1)!\lambda_K \int_{H^\vee(\Delta_\Psi)} H^\vee_*(\psi^\vee_{\overline{L},s} - u)\, \mathrm{d}\,\mathrm{vol}_{M_1}\end{aligned} \tag{6.28}$$

holds. Moreover

$$\begin{aligned}\mathrm{h}^{\mathrm{tor}}_{\overline{L}_1}(X_{\Sigma_1}) - \mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) &= (d+1)!\lambda_K \int_{H^\vee(\Delta_\Psi)} (A^*\Psi)^\vee\, \mathrm{d}\,\mathrm{vol}_{M_1} \\ &= (d+1)! \int_{H^\vee(\Delta_\Psi)} H^\vee_*(\iota_{\Delta_\Psi} - \lambda_K u)\, \mathrm{d}\,\mathrm{vol}_{M_1},\end{aligned} \tag{6.29}$$

where ι_{Δ_Ψ} is the indicator function of Δ_Ψ (Example 3.16).

Proof. By Proposition 5.24, $\psi_{\overline{L}_1,\varphi^* s} = A^*\psi_{\overline{L},s}$. By Proposition 3.46(3) we obtain that $\mathrm{stab}(A^*\psi_{\overline{L},s}) = H^\vee(\Delta_\Psi)$ and that

$$(A^*\psi_{\overline{L},s})^\vee = H^\vee_*(\psi_{\overline{L},s} - u),$$

from which equation (6.28) follows.

To prove equation (6.29), we observe that, by the definition of $\mathrm{h}^{\mathrm{tor}}$,

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_1}(X_{\Sigma_1}) - \mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = \mathrm{h}^{\mathrm{tor}}_{\varphi^*(\overline{L}^{\mathrm{can}})}(X_{\Sigma_1}),$$

where $\varphi^*(\overline{L}^{\mathrm{can}})$ has the toric structure induced by s and the metric induced by the canonical metric of L^{an}. We remark here that this metric differs from the canonical metric of $\varphi^* L^{\mathrm{an}}$. Now equation (6.29) follows from equation (6.28) and the definition of the canonical metric. □

Corollary 6.30. *With the previous hypothesis*

$$\mathrm{h}^{\mathrm{tor}}_{\varphi^*(\overline{L}^{\mathrm{can}})}(X_{\Sigma_1}) = (d+1)!\lambda_K \int_{H^\vee(\Delta_\Psi)} (A^*\Psi)^\vee \,\mathrm{d}\,\mathrm{vol}_{M_1}.$$

Example 6.31. We continue with Example 5.26. Let $\mathbb{Z}^r$ be the standard lattice of rank r, Δ^r the standard simplex of dimension r and Σ_{Δ^r} the fan of $\mathbb{R}^r$ associated to Δ^r. The corresponding toric variety is $\mathbb{P}^r$. Let $H\colon N \to \mathbb{Z}^r$ be an injective linear morphism such that $H(N)$ is a saturated sublattice. Denote $m_i = e_i^\vee \circ H \in M$, $i = 1, \dots, r$. Let Σ the regular fan on N defined by H and Σ_{Δ^r}. Let Ψ_{Δ^r} be the support function of Δ^r and let $\Psi = \Psi_{\Delta^r} \circ H$. Explicitly,

$$\Psi(v) = \min(0, m_1(v), \dots, m_r(v)).$$

Let $p \in \mathbb{P}^r_0(K)$ and $u = \mathrm{val}_K(p) \in \mathbb{R}^r$. Write $u = (u_1, \dots, u_r)$. If $p = (1 : \alpha_1 : \dots : \alpha_r)$, then $u_i = \frac{-\log(|\alpha_i|)}{\lambda_K}$. There is an equivariant morphism $\varphi := \varphi_{p,H}\colon X_\Sigma \to \mathbb{P}^r$. Consider the toric line bundle with toric section determined by Ψ_{Δ^r} with the canonical metric and denote by $(\overline{L}, s)$ the induced toric line bundle with toric section on X_Σ equipped with the induced metric. Then

$$\psi_{\overline{L},s}(v) = \min(0, m_1(v) + u_1, \dots, m_r(v) + u_r).$$

Thus $\Delta = \mathrm{stab}(\psi_{\overline{L},s}) = \mathrm{conv}(0, m_1, \dots, m_r) = H^\vee(\Delta^r)$. By Proposition 3.64 the Legendre-Fenchel dual $\psi^\vee_{\overline{L},s}\colon \Delta \to \mathbb{R}$ is given by

$$\psi^\vee_{\overline{L},s}(x) = \sup\left\{ \sum_{j=1}^r -\lambda_j u_j \,\middle|\, \lambda_j \geq 0, \sum_{j=1}^r \lambda_j \leq 1,\ \sum_{j=1}^r \lambda_j a_j = x \right\} \text{ for } x \in \Delta.$$

This function is the upper envelope of the extended polytope of $M_\mathbb{R} \times \mathbb{R}$,

$$\mathrm{conv}\left((0,0), (m_1, -u_1), \dots, (m_r, -u_r)\right),$$

Similarly, the roof function $\vartheta_{\overline{L},s} = \lambda_K \psi^\vee_{\overline{L},s}$ is the upper envelope of the extended polytope

$$\mathrm{conv}\left((0,0), (m_1, \log|\alpha_1|), \dots, (m_r, \log|\alpha_r|)\right).$$

6.2. **Global heights of toric varieties.** In this section we prove the integral formula for the global height of a toric variety.

Let $(\mathbb{K}, \mathfrak{M}_\mathbb{K})$ be an adelic field. Let Σ be a complete fan on $N_\mathbb{R}$ and Ψ_i, $i = 0, \dots, d$, be virtual support functions on Σ. For each i, let $L_i = L_{\Psi_i}$ and s_{Ψ_i} be the associated toric line bundle and toric section, and $\|\cdot\|_i = (\|\cdot\|_{i,v})_{v \in \mathfrak{M}_\mathbb{K}}$ an integrable adelic toric metric on L_i. Write $\overline{L}_i = (L_i, \|\cdot\|_i)$ and, for each $v \in \mathfrak{M}_\mathbb{K}$, also $\overline{L}_i^v = (L_i, ||\cdot||_{i,v})$. Write also $\overline{L}_i^{\mathrm{can}}$ for the same line bundles equipped with the canonical adelic toric metric. This is also an integrable adelic toric metric.

From the local toric height we can define a toric (global) height for adelic toric metrics as follows.

Definition 6.32. Let Y be a d-dimensional cycle of X_Σ. The *toric height* of Y with respect to $\overline{L}_0, \dots, \overline{L}_d$ is

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y) = \sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \,\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0^v,\dots,\overline{L}_d^v}(Y) \in \mathbb{R}.$$

Remark 6.33. Definition 6.32 makes sense because the condition of the metrics being adelic imply that only a finite number of terms in the sum are non-zero. Moreover, the value of the toric height depends on the toric structure of the involved line bundle, but its class in $\mathbb{R}/\operatorname{def}(\mathbb{K}^\times)$ does not.

Remark 6.34. In general, the toric height is not a global height in the sense of Definition 2.57. It is the difference between the global height with respect to the given metric and the global height with respect to the canonical metric. Nevertheless, the next result shows that the global height of the closure of an orbit or of a toric subvariety agrees with the toric height defined above.

Proposition 6.35. *With notations as above, let Y be either the closure of an orbit or a toric subvariety. Then Y is integrable with respect to $\overline{L}_0, \dots, \overline{L}_d$ in the sense of Definition 2.54. Moreover, its global height is given by*

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_d}(Y) = \left[\mathrm{h}^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_d}(Y)\right] \in \mathbb{R}/\operatorname{def}(\mathbb{K}^\times).$$

Proof. In view of propositions 6.25, 6.27 and the fact that the restriction of the canonical metric to closures of orbits and to toric subvarieties is the canonical metric (corollaries 5.23 and 5.25), we are reduced to treat the case $Y = X_\Sigma$.

Thus we assume that X_Σ has dimension d. We next prove that X_Σ is integrable with respect to $\overline{L}_0^{\mathrm{can}}, \dots, \overline{L}_d^{\mathrm{can}}$ and that the corresponding global height is zero. By a polarization argument, we can reduce to the case $\Psi_0 = \dots = \Psi_d = \Psi$. The proof is done by induction on d. For short, write $L = \mathcal{O}(D_\Psi)$ and $s = s_\Psi$.

Let $d = 0$. By equation (2.40), for each $v \in \mathfrak{M}_\mathbb{K}$,

$$\mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(X_\Sigma; s) = -\log\|s\|_{v,\Psi} = \Psi(0) = 0.$$

Furthermore, $\mathrm{h}_{\overline{L}^{\mathrm{can}}}(X_\Sigma; s) = \sum_v n_v \,\mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(X_\Sigma; s) = 0$.

Now let $d \geq 1$. By the construction of local heights, for each $v \in \mathfrak{M}_\mathbb{K}$,

$$\begin{aligned}\mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(X_\Sigma; s_0, \dots, s_{d-1}, s) = {}&\mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(\operatorname{div}(s); s_0, \dots, s_{d-1}) \\ &- \int_{X_\Sigma^{v,\mathrm{an}}} \log\|s\|_{v,\Psi} c_1(\overline{L}^{v,\mathrm{can}})^d \wedge \delta_{X_\Sigma}.\end{aligned} \tag{6.36}$$

As shown in (6.14), the last term in the equality above vanishes. Hence

$$\mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(X_\Sigma; s_0, \dots, s_{d-1}, s) = \mathrm{h}_{\overline{L}^{v,\mathrm{can}}}(\operatorname{div}(s); s_0, \dots, s_{d-1}).$$

The divisor $\operatorname{div}(s)$ is a linear combination of subvarieties of the form $V(\tau)$, $\tau \in \Sigma^1$, and the restriction of the canonical metric to these varieties coincides with their canonical metrics. With the inductive hypothesis, this shows that X_Σ is integrable with respect to $\overline{L}^{\mathrm{can}}$. Adding up the resulting equalities over all places,

$$\mathrm{h}_{\overline{L}^{\mathrm{can}}}(X_\Sigma; s_0, \dots, s_{d-1}, s) = \mathrm{h}_{\overline{L}^{\mathrm{can}}}(\operatorname{div}(s); s_0, \dots, s_{d-1}).$$

Using again the inductive hypothesis, $\mathrm{h}_{\overline{L}^{\mathrm{can}}}(X_\Sigma; s_0, \dots, s_{d-1}, s) \in \operatorname{def}(\mathbb{K}^\times)$.

We now prove the statements of the theorem. Again by a polarization argument, we can also reduce to the case when $\overline{L}_0 = \dots = \overline{L}_d = \overline{L}$. By the definition of approachable adelic toric metrics, X_Σ is also integrable with respect to $\overline{L}$. Furthermore,

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \mathrm{h}_{\overline{L}}(X_\Sigma; s_0, \dots, s_d) - \mathrm{h}_{\overline{L}^{\mathrm{can}}}(X_\Sigma; s_0, \dots, s_d)$$

for any choice of sections s_i intersecting X_Σ properly. Hence, the classes of $\mathrm{h}_{\overline{L}}(X_\Sigma)$ and of $\mathrm{h}_{\overline{L}}(X_\Sigma; s_0, \dots, s_d)$ agree up to $\mathrm{def}(\mathbb{K}^\times)$. But the latter is the global height of X_Σ with respect to $\overline{L}$, hence the second statement. □

Summing up the preceding results we obtain a formula for the height of a toric variety.

Theorem 6.37. *Let Σ be a complete fan on $N_\mathbb{R}$. Let $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 0, \dots, n$, be toric line bundles on X_Σ generated by its global sections and equipped with approachable adelic toric metrics. For each i, let s_i be a toric section of L_i. Then the height of X_Σ with respect to $\overline{L}_0, \dots, \overline{L}_n$ is*

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_n}(X_\Sigma) = \left[\sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \,\mathrm{MI}_M(\vartheta_{\overline{L}_0^v,s_0}, \dots, \vartheta_{\overline{L}_n^v,s_n})\right] \in \mathbb{R}/\mathrm{def}(\mathbb{K}^\times). \tag{6.38}$$

In particular, if $\overline{L}_0 = \dots = \overline{L}_n = \overline{L}$, let s be a toric section and put $\Delta = \mathrm{stab}(\psi_{\overline{L}^v,s})$. Then

$$\mathrm{h}_{\overline{L}}(X_\Sigma) = \left[(n+1)! \sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \int_\Delta \vartheta_{\overline{L}^v,s} \,\mathrm{d}\,\mathrm{vol}_M\right].$$

Proof. This follows readily from Corollary 6.21 and Proposition 6.35. □

Corollary 6.39. *Let $H\colon N \to \mathbb{Z}^r$ be an injective map such that $H(N)$ is a saturated sublattice of $\mathbb{Z}^r$, $p \in \mathbb{P}^r(\mathbb{K})$ and $Y \subset \mathbb{P}^r$ the closure of the image of the map $\varphi_{H,p}\colon \mathbb{T} \to \mathbb{P}^r$. Let $m_0 \in M$ and $m_i = e_i^\vee \circ H + m_0 \in M$, $i = 1, \dots, r$, and write $p = (p_0 : \dots : p_r)$ with $p_i \in \mathbb{K}^\times$. Let $\Delta = \mathrm{conv}(m_0, \dots, m_r) \subset M_\mathbb{R}$ and $\vartheta_v\colon \Delta \to \mathbb{R}$ the function parameterizing the upper envelope of the extended polytope* $\mathrm{conv}\left((m_0, \log|p_0|_v), \dots, (m_r, \log|p_r|_v)\right) \subset M_\mathbb{R}\times\mathbb{R}$. *Then Y is integrable and*

$$\mathrm{h}_{\overline{\mathcal{O}(1)}^{\mathrm{can}}}(Y) = \left[(n+1)! \sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \int_\Delta \vartheta_v \,\mathrm{d}\,\mathrm{vol}_M\right] \in \mathbb{R}/\mathrm{def}(\mathbb{K}^\times).$$

Proof. By the definition of adelic field, $\mathrm{val}_{\mathbb{K}_v}(p) = 0$ for almost all $v \in \mathfrak{M}_\mathbb{K}$. Therefore, the integrability of Y follows as in the proof of Proposition 6.35.

Let Σ be the complete regular fan of $N_\mathbb{R}$ induced by H and Σ_{Δ^r}, and let X_Σ be the associated toric variety. Write $\varphi = \varphi_{H,p}$ for short. The fact that $H(N)$ is saturated implies that φ has degree 1 and so $Y = \varphi_* X_\Sigma$. By the functoriality of the global height (Theorem 2.58(2)),

$$\mathrm{h}_{\overline{\mathcal{O}(1)}^{\mathrm{can}}}(Y) = \mathrm{h}_{\varphi^*(\overline{\mathcal{O}(1)}^{\mathrm{can}})}(X_\Sigma).$$

Let $v \in \mathfrak{M}_\mathbb{K}$. Using the results in Example 6.31, it follows from Theorem 6.37 that

$$\mathrm{h}_{\varphi^*(\overline{\mathcal{O}}(1)^{\mathrm{can}})}(X_\Sigma) = \left[(n+1)! \sum_v \int_{\overline{\Delta}} n_v \overline{\vartheta}_v \,\mathrm{d}\,\mathrm{vol}_M\right].$$

where $\overline{\Delta} = \mathrm{conv}(0, m_1 - m_0, \dots, m_r - m_0) \subset M_\mathbb{R}$ and $\overline{\vartheta}_v$ is the function parameterizing the upper envelope of the extended polytope

$$\mathrm{conv}\left((0,0), (m_1 - m_0, \log|p_1/p_0|_v), \dots, (m_r - m_0, \log|p_r/p_0|_v)\right) \subset M_\mathbb{R}\times\mathbb{R}.$$

We have that $\overline{\Delta} = \Delta - m_0$ and $\overline{\vartheta}_v = \tau_{-m_0}\vartheta_v - \log|p_0|_v$. Hence,

$$\int_{\overline{\Delta}} \overline{\vartheta}_v \,\mathrm{d}\,\mathrm{vol}_M = \int_\Delta \vartheta_v \,\mathrm{d}\,\mathrm{vol}_M - \log|p_0|_v \,\mathrm{vol}_M(\Delta).$$

Since $\sum_v n_v \log|p_0|_v \in \mathrm{def}(\mathbb{K}^\times)$, we deduce the result. □

Remark 6.40. The above corollary can be easily extended to the mixed case by using an argument similar to that in the proof of Corollary 6.21. Applying the obtained result to the case when $\mathbb{K}$ is a number field (respectively, the field of rational functions of a complete curve) we recover [PS08a, Théorème 0.3] (respectively, [PS08b, Proposition 4.1]).

7. Metrics from polytopes

7.1. **Integration on polytopes.** In this section, we present a closed formula for the integral over a polytope of a function of one variable composed with a linear form, extending in this direction Brion's formula for the case of a simplex [Bri88], see Proposition 7.3 and Corollary 7.14 below. In the next section, these formulae will allow us to compute the height of toric varieties with respect to some interesting metrics arising from polytopes.

Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n and $u \in \mathbb{R}^n$ a vector. An *aggregate of* Δ *in the direction* u is defined as the union of the faces of Δ lying in some affine hyperplane orthogonal to u, provided that the union is non-empty. We write $\Delta(u)$ for the set of aggregates of Δ in the direction u. Note that, for $V \in \Delta(u)$ and x a point in the affine space spanned by V, the value $\langle u, x\rangle$ is independent of x. We denote this common value by $\langle u, V\rangle$. For any two aggregates $V_1, V_2 \in \Delta(u)$, we have $V_1 = V_2$ if and only if $\langle u, V_1\rangle = \langle u, V_2\rangle$.

In each facet F of Δ we choose a point m_F. Let L_F be the linear hyperplane defined by F. Hence, $F - m_F$ is a polytope in L_F of full dimension $n-1$. Observe that, for $V \in \Delta(u)$, the intersection $V \cap F$ is an aggregate of F. We write π_F for the orthogonal projection of $\mathbb{R}^n$ onto L_F. We also denote by u_F the vector inner normal to F of norm 1.

Definition 7.1. For each aggregate $V \in \Delta(u)$, we define the polynomial

$$C(\Delta, u, V) = \sum_{k=0}^{\dim(V)} \frac{k!}{\dim(V)!} C_k(\Delta, u, V) z^{\dim(V)-k} \in \mathbb{R}[z]$$

recursively. For $k > \dim(V)$ we set $C_k(\Delta, u, V) = 0$. For convenience, we set $C(\Delta, u, \emptyset) = 0$ for all Δ and u. If $u = 0$, then $V = \Delta$ and we define $C_n(\Delta, 0, \Delta)$ as the Lebesgue measure of Δ and $C_k(\Delta, 0, \Delta) = 0$, for $k < n$. If $u \neq 0$, we set

$$C_k(\Delta, u, V) = -\sum_F \frac{\langle u_F, u\rangle}{\|u\|^2} C_k(F, \pi_F(u), V \cap F), \tag{7.2}$$

where the sum is over the facets F of Δ.

As usual, we write $\mathscr{C}^n(\mathbb{R})$ for the space of functions of one real variable which are n-times continuously differentiable. For $f \in \mathscr{C}^n(\mathbb{R})$ and $k \geq 0$, we write $f^{(k)}$ for the k-th derivative of f. Write vol_n for the Lebesgue measure of $\mathbb{R}^n$.

We want to give a formula that, for $f \in \mathscr{C}^n(\mathbb{R})$, computes $\int_\Delta f^{(n)}(\langle u, x\rangle) \,\mathrm{d}\,\mathrm{vol}_n$ in terms of the values of $f \circ u$ at the vertices of Δ. However, when u is orthogonal to some faces of Δ of positive dimension, such a formula necessarily depends on the values of the derivatives of f.

Proposition 7.3. *Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n and $u \in \mathbb{R}^n$. Then, for any $f \in \mathscr{C}^n(\mathbb{R})$,*

$$\begin{aligned}\int_\Delta f^{(n)}(\langle u, x\rangle) \,\mathrm{d}\,\mathrm{vol}_n &= \sum_{V \in \Delta(u)} \big(C(\Delta, u, V)(z) \cdot f(z + \langle u, V\rangle)\big)^{(\dim(V))}(0) \\ &= \sum_{V \in \Delta(u)} \sum_{k \geq 0} C_k(\Delta, u, V) f^{(k)}(\langle u, V\rangle). \end{aligned} \tag{7.4}$$

The coefficients $C_k(\Delta, u, V)$ are uniquely determined by this identity.

Proof. In view of Definition 7.1 both formulae in the above statement are equivalent and so it is enough to prove the second one. In case $u = 0$, we have $\Delta(u) = \{\Delta\}$ and formula (7.4) holds because

$$\int_\Delta f^{(n)}(\langle 0, x\rangle)\,\mathrm{d}\,\mathrm{vol}_n = \mathrm{vol}(\Delta) f^{(n)}(0) = \sum_{k\geq 0} C_k(\Delta, 0, \Delta) f^{(k)}(0),$$

We prove (7.4) by induction on the dimension n. In case $n = 0$, we have $u = 0$ and so the verification reduces to the above one. Hence, we assume $n \geq 1$ and $u \neq 0$. For short, we write $\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Choose any vector $v \in \mathbb{R}^n$ of norm 1 and such that $\langle u, v\rangle \neq 0$. Performing an orientation-preserving orthonormal change of variables, we may assume $v = (1, 0, \dots, 0)$. We have

$$f^{(n)}(\langle u, x\rangle)\,\mathrm{d}x = \frac{1}{\langle u, v\rangle}\,\mathrm{d}\left(f^{(n-1)}(\langle u, x\rangle)\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n\right).$$

With Stokes' theorem, we obtain

$$\begin{aligned}\int_\Delta f^{(n)}(\langle u, x\rangle)\,\mathrm{d}\,\mathrm{vol}_n &= \int_\Delta f^{(n)}(\langle u, x\rangle)\,\mathrm{d}x \\ &= \frac{1}{\langle u, v\rangle} \sum_F \int_F f^{(n-1)}(\langle u, x\rangle)\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n.\end{aligned} \tag{7.5}$$

where the sum is over the facets F of Δ, and we equip each facet with the induced orientation.

For each facet F of Δ, we let $\iota_{u_F}(\mathrm{d}x)$ be the differential form of order $n-1$ obtained by contracting $\mathrm{d}x$ with the vector u_F. The form $\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$ is invariant under translations and its restriction to the linear hyperplane L_F coincides with $\langle u_F, v\rangle \iota_{u_F}(\mathrm{d}x)$. Therefore,

$$\int_F f^{(n-1)}(\langle u, x\rangle)\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n = \langle u_F, v\rangle \int_{F-m_F} f^{(n-1)}(\langle u, x + m_F\rangle)\iota_{u_F}(\mathrm{d}x).$$

Let vol_{n-1} denote the Lebesgue measure on L_F. We can verify that vol_{n-1} coincides with the measure induced by integration of $-\iota_{u_F}(\mathrm{d}x)$ along L_F. Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function defined as $g(z) = f(z + \langle u, m_F\rangle)$. Then $f^{(n-1)}(\langle u, x + m_F\rangle) = g^{(n-1)}(\langle \pi_F(u), x\rangle)$ for all $x \in L_F$. Hence,

$$\int_{F-m_F} f^{(n-1)}(\langle u, x + m_F\rangle)\iota_{u_F}(\mathrm{d}x) = -\int_{F-m_F} g^{(n-1)}(\langle \pi_F(u), x\rangle)\,\mathrm{d}\,\mathrm{vol}_{n-1}.$$

Applying the inductive hypothesis to F and the function g we obtain

$$\begin{aligned}\int_F g^{(n-1)}(\langle \pi_F(u), x\rangle)\,\mathrm{d}\,\mathrm{vol}_{n-1} &= \sum_{V' \in F(\pi_F(u))} \sum_{k\geq 0} C_k(F, \pi_F(u), V') g^{(k)}(\langle \pi_F(u), V'\rangle) \\ &= \sum_{V' \in F(\pi_F(u))} \sum_{k\geq 0} C_k(F, \pi_F(u), V') f^{(k)}(\langle u, V'\rangle).\end{aligned}$$

Each aggregate $V' \in F(\pi_F(u))$ is contained in a unique $V \in \Delta(u)$ and it coincides with $V \cap F$. Therefore, we can transform the right-hand side of the last equality in

$$\sum_{V \in \Delta(u)} \sum_{k\geq 0} C_k(F, \pi_F(u), V \cap F) f^{(k)}(\langle u, V\rangle),$$

where, for simplicity, we have set $C_k(F, \pi_F(u), V \cap F) = 0$ whenever $V \cap F = \emptyset$. Plugging the resulting expression into (7.5) and exchanging the summations on V

and F, we obtain that $\int_\Delta f^{(n)}(\langle x,u\rangle)\,\mathrm{d}\,\mathrm{vol}_n$ is equal to

$$\sum_{V\in\Delta(u)}\sum_{k\geq 0}\Big(-\sum_F \frac{\langle u_F,v\rangle}{\langle u,v\rangle}C_k(F,\pi_F(u),V\cap F)f^{(k)}(\langle u,V\rangle)\Big). \tag{7.6}$$

Specialising this identity to $v=u$, we readily derive formula (7.4) from Definition 7.1 of the coefficients $C_k(\Delta,u,V)$.

For the last statement, observe that the values $f^{(k)}(\langle u,V\rangle)$ can be arbitrarily chosen. Hence, the coefficients $C_k(\Delta,u,V)$ are uniquely determined from the linear system obtained from the identity (7.4) for enough functions f. □

Corollary 7.7. *Let $\Delta\subset\mathbb{R}^n$ be a polytope of dimension n and $u\in\mathbb{R}^n$. Then,*

$$\mathrm{vol}_n(\Delta)=\sum_{V\in\Delta(u)}\sum_{k=0}^{\dim(V)}C_k(\Delta,u,V)\frac{\langle u,V\rangle^{n-k}}{(n-k)!}.$$

Proof. This follows from formula (7.4) applied to the function $f(z)=z^n/n!$. □

Proposition 7.8. *Let $\Delta\subset\mathbb{R}^n$ be a polytope of dimension n and $u\in\mathbb{R}^n$. Let $V\in\Delta(u)$ and $k\geq 0$.*

(1) *The coefficient $C_k(\Delta,u,V)$ is homogeneous of weight $k-n$, in the sense that, for $\lambda\in\mathbb{R}^\times$,*

$$C_k(\Delta,\lambda u,V)=\lambda^{k-n}C_k(\Delta,u,V).$$

(2) *The coefficients $C_k(\Delta,u,V)$ satisfy the vector relation*

$$C_k(\Delta,u,V)\cdot u=-\sum_F C_k(F,\pi_F(u),V\cap F)\cdot u_F, \tag{7.9}$$

where the sum is over the facets F of Δ.

(3) *Let $\Delta_1,\Delta_2\subset\mathbb{R}^n$ be two polytopes of dimension n intersecting along a common facet and such that $\Delta=\Delta_1\cup\Delta_2$. Then $V\cap\Delta_i=\emptyset$ or $V\cap\Delta_i\in\Delta_i(u)$ and*

$$C_k(\Delta,u,V)=C_k(\Delta_1,u,V\cap\Delta_1)+C_k(\Delta_2,u,V\cap\Delta_2).$$

Proof. Statement (1) follows easily from the definition of $C_k(\Delta,u,V)$. For statement (2), we use that, from (7.6), the integral formula in Proposition 7.3 also holds for the choice of coefficients

$$-\sum_F\frac{\langle u_F,v\rangle}{\langle u,v\rangle}C_k(F,\pi_F(u),V\cap F)$$

for any vector v of norm 1 such that $\langle u,v\rangle\neq 0$. But the coefficients satisfying that formula are unique. Hence, this choice necessarily coincides with $C_k(\Delta,u,V)$ for all such v. Hence,

$$\langle u,v\rangle C_k(\Delta,u,V)=-\sum_F\langle u_F,v\rangle C_k(F,\pi_F(u),V\cap F)$$

and formula (7.9) follows. Statement (3) follows from Formula (7.4) applied to Δ, Δ_1 and Δ_2 together with the additivity of the integral and the fact that the coefficients $C_k(\Delta,u,V)$ are uniquely determined. □

Example 7.10. In case Δ is a simplex, its aggregates in a given direction $u\in\mathbb{R}^n$ are some of its faces and the corresponding coefficients can be made explicit. Indeed, they satisfy the linear system

$$\sum_{V\in\Delta(u)}\sum_{k=0}^{\min\{i,\dim(V)\}}C_k(\Delta,u,V)\frac{\langle u,V\rangle^{i-k}}{(i-k)!}=\begin{cases}0 & \text{for } i=0,\dots,n-1,\\ \mathrm{Vol}_n(\Delta) & \text{for } i=n.\end{cases}$$

This system has as many unknowns as equations and might be solved using Cramer's rule. These coefficients admit the closed formula below, which the reader might check using the recurrence relation (7.9):

$$C_k(\Delta,u,V)=(-1)^{\dim(V)-k}\frac{n!}{k!}\mathrm{vol}_n(\Delta)\sum_{|\beta|=\dim(V)-k}\prod_{\nu\notin V}\langle V-\nu,u\rangle^{-\beta_\nu-1}, \quad (7.11)$$

where the products are over the vertices ν of Δ not lying in V and the sum is over the tuples β of non negative integers of length $\dim(V)-k$, indexed by those same vertices of Δ that are not in V, that is, $\beta\in\mathbb{N}^{n-\dim(V)}$ and $|\beta|=\dim(V)-k$. In case $V=\nu_0$ is a vertex of Δ, the above formula reduces to

$$C_0(\Delta,u,\nu_0)=n!\mathrm{vol}_n(\Delta)\prod_{\nu\neq\nu_0}\langle\nu_0-\nu,u\rangle^{-1}. \quad (7.12)$$

Suppose that the simplex is presented as the intersection of $n+1$ halfspaces as $\Delta=\bigcap_{i=0}^n\{x\in\mathbb{R}^n|\,\langle u_i,x\rangle-\lambda_i\geq 0\}$ for some $u_i\in\mathbb{R}^n$ and $\lambda_i\in\mathbb{R}$. Up to a reordering, we can assume that u_0 is normal to the unique face of Δ not containing ν_0 and that $\det(u_1,\dots,u_n)>0$. Then the above coefficient can be alternatively written as

$$C_0(\Delta,u,\nu_0)=\frac{\det(u_1,\dots,u_n)^{n-1}}{\prod_{i=1}^n\det(u_1,\dots,u_{i-1},u,u_{i+1},\dots,u_n)}. \quad (7.13)$$

We obtain the following extension of Brion's "short formula" for the case of a simplex [Bri88, Théorème 3.2], see also [BBDL+11].

Corollary 7.14. *Let $\Delta\subset\mathbb{R}^n$ be a simplex of dimension n that is the convex hull of points ν_i, $i=0,\dots,n$, and let $u\in\mathbb{R}^n$ such that $\langle u,\nu_i\rangle\neq\langle u,\nu_j\rangle$ for $i\neq j$. Then, for any $f\in\mathscr{C}^n(\mathbb{R})$,*

$$\int_\Delta f^{(n)}(\langle u,x\rangle)\,\mathrm{d}\,\mathrm{vol}_n=n!\mathrm{vol}_n(\Delta)\sum_{i=0}^n\frac{f(\langle u,\nu_i\rangle)}{\prod_{j\neq i}\langle\nu_i-\nu_j,u\rangle}.$$

Proof. This follows from Proposition 7.3 and equation (7.12). □

In the next section, we will have to compute integrals over a polytope of functions of the form $\ell(x)\log(\ell(x))$ where ℓ is an affine function. The following result gives the value of such integral for the case of a simplex.

Proposition 7.15. *Let $\Delta\subset\mathbb{R}^n$ be a simplex of dimension n and let $\ell\colon\mathbb{R}^n\to\mathbb{R}$ be an affine function which is non-negative on Δ. Write $\ell(x)=\langle u,x\rangle-\lambda$ for some vector u and constant λ. Then $\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta\ell(x)\log(\ell(x))\,\mathrm{d}\,\mathrm{vol}_n$ equals*

$$\sum_{V\in\Delta(u)}\sum_{\beta'}\binom{n}{n-|\beta'|}\frac{\ell(V)\left(\log(\ell(V))-\sum_{j=2}^{|\beta'|+1}\frac{1}{j}\right)}{(|\beta'|+1)\prod_{\nu\notin V}\left(-\left(\frac{\ell(\nu)}{\ell(V)}-1\right)^{\beta'_\nu}\right)}, \quad (7.16)$$

where the second sum is over $\beta'\in(\mathbb{N}^\times)^{n-\dim(V)}$ with $|\beta'|\leq n$ and the product is over the $n-\dim(V)$ vertices ν of Δ not in V. In case $\ell(x)$ is the defining equation of a hyperplane containing a facet F of Δ,

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta\ell(x)\log(\ell(x))\,\mathrm{d}x=\frac{\ell(\nu_F)}{n+1}\left(\log(\ell(\nu_F))-\sum_{j=2}^{n+1}\frac{1}{j}\right), \quad (7.17)$$

where ν_F denotes the unique vertex of Δ not contained in F.

Proof. This follows from formulae (7.4) and (7.11) with the function $f^{(n)}(z) = (z-\lambda)\log(z-\lambda)$, a $(n-k)$-th primitive of which is

$$f^{(k)}(z) = \frac{(z-\lambda)^{n-k+1}}{(n-k+1)!}\left(\log(z-\lambda) - \sum_{j=2}^{n-k+1}\frac{1}{j}\right).$$

□

We end this section with a lemma specific to integration on the standard simplex.

Lemma 7.18. *Let Δ^r be the standard simplex of $\mathbb{R}^r$ and $\beta = (\beta_0, \dots, \beta_{r-1}) \in \mathbb{N}^r$. Let $f \in \mathscr{C}^{|\beta|+r}(\mathbb{R})$ where $|\beta| = \beta_0 + \dots + \beta_{r-1}$. For $(w_1, \dots, w_r) \in \Delta^r$ write $w_0 = 1 - w_1 - \dots - w_r$. Then*

$$\int_{\Delta^r}\left(\prod_{i=0}^{r-1}\frac{w_i^{\beta_i}}{\beta_i!}\right) f^{(|\beta|+r)}(w_r)\,\mathrm{d}w_1\wedge\dots\wedge\mathrm{d}w_r = f(1) - \sum_{j=0}^{|\beta|+r-1}\frac{f^{(j)}(0)}{j!}.$$

Proof. We proceed by induction on r. Let $r=1$. Applying β_0+1 successive integrations by parts, the integral computes as

$$\sum_{j=0}^{\beta_0}\left[\frac{(1-w_1)^j}{j!}f^{(j)}(w_1)\right]_0^1 = f(1) - \sum_{j=0}^{\beta_0}\frac{f^{(j)}(0)}{j!},$$

as stated. Let $r \geq 2$. Applying the case $r-1$ to the function $f(z) = \frac{z^{|\beta|+r-1}}{(|\beta|+r-1)!}$,

$$\frac{1}{\beta_0!\dots\beta_{r-1}!}\int_{\Delta_{r-1}} w_0^{\beta_0}w_1^{\beta_1}\dots w_{r-1}^{\beta_{r-1}}\,\mathrm{d}w_1\wedge\dots\wedge\mathrm{d}w_{r-1} = \frac{1}{(|\beta|+r-1)!}$$

and, after rescaling,

$$\frac{1}{\beta_0!\dots\beta_{r-1}!}\int_{(1-w_r)\Delta_{r-1}} w_0^{\beta_0}w_1^{\beta_1}\dots w_{r-1}^{\beta_{r-1}}\,\mathrm{d}w_1\wedge\dots\wedge\mathrm{d}w_{r-1} = \frac{(1-w_r)^{|\beta|+r-1}}{(|\beta|+r-1)!}.$$

Therefore, the left-hand side of the equality to be proved reduces to

$$\frac{1}{(|\beta|+r-1)!}\int_0^1 (1-w_r)^{|\beta|+r-1}f^{(|\beta|+r)}(w_r)\,\mathrm{d}w_r.$$

Applying the case $r=1$ and index $|\beta|+r-1 \in \mathbb{N}$, we find that this integral equals $f(1) - \sum_{j=0}^{|\beta|+r-1} f^{(j)}(0)/j!$, which concludes the proof. □

Corollary 7.19. *Let $\alpha \in \mathbb{N}^{r+1}$. For $(w_1, \dots, w_r) \in \Delta^r$, write $w_0 = 1 - w_1 - \dots - w_r$. Then*

$$\int_{\Delta^r} w_0^{\alpha_0}w_1^{\alpha_1}\dots w_r^{\alpha_r}\,\mathrm{d}w_1\wedge\dots\wedge\mathrm{d}w_r = \frac{\alpha_0!\dots\alpha_r!}{(|\alpha|+r)!}$$

and, for $i = 0, \dots, r$,

$$\int_{\Delta^r} w_0^{\alpha_0}w_1^{\alpha_1}\dots w_r^{\alpha_r}\log(w_i)\,\mathrm{d}w_1\wedge\dots\wedge\mathrm{d}w_r = -\frac{\alpha_0!\dots\alpha_r!}{(|\alpha|+r)!}\sum_{j=\alpha_i+1}^{|\alpha|+r}\frac{1}{j}.$$

Proof. The first integral follows from Lemma 7.18 applied to $\beta = (\alpha_0, \dots, \alpha_{r-1})$ and $f(z) = \frac{z^{|\alpha|+r}}{(|\alpha|+r)!}$. The second one follows similarly, applying Lemma 7.18 to the function $f(z) = \frac{z^{|\alpha|+r}}{(|\alpha|+r)!}\left(\log(z) - \sum_{j=\alpha_i+1}^{|\alpha|+r}\frac{1}{j}\right)$, after some possible permutation (for $i = 1, \dots, r-1$) or linear change of variables (for $i = 0$). □

7.2. **Metrics, heights and entropy.** In this section we will consider some metrics arising from polytopes. We will use the notation of §4 and §5. In particular, we consider a split torus over the field of rational numbers $\mathbb{T} \simeq \mathbb{G}_{m,\mathbb{Q}}^n$ and we denote by $N, M, N_{\mathbb{R}}, M_{\mathbb{R}}$ the lattices and dual spaces corresponding to $\mathbb{T}$.

Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope of dimension n. Let ℓ_i, $i = 1, \dots, r$, be affine functions on $M_{\mathbb{R}}$ defined as $\ell_i(x) = \langle u_i, x\rangle - \lambda_i$ for some $u_i \in N_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$ such that $\ell_i \geq 0$ on Δ and let also $c_i > 0$. Write $\ell = (\ell_1, \dots, \ell_r)$ and $c = (c_1, \dots, c_r)$. We consider the function $\vartheta_{\Delta,\ell,c}\colon \Delta \to \mathbb{R}$ defined, for $x \in \Delta$, by

$$\vartheta_{\Delta,\ell,c}(x) = -\sum_{i=1}^{r} c_i \ell_i(x) \log(\ell_i(x)). \tag{7.20}$$

When Δ, ℓ, c are clear from the context, we write for short $\vartheta = \vartheta_{\Delta,\ell,c}$.

Lemma 7.21. *Let notation be as above.*

(1) *The function $\vartheta_{\Delta,\ell,c}$ is concave.*
(2) *If the family $\{u_i\}_i$ generates $N_{\mathbb{R}}$, then $\vartheta_{\Delta,\ell,c}$ is strictly concave.*
(3) *If $\Delta = \bigcap_i \{x \in M_{\mathbb{R}} | \ell_i(x) \geq 0\}$, then the restriction of $\vartheta_{\Delta,\ell,c}$ to Δ° is of Legendre type (Definition 3.51).*

Proof. Let $1 \leq i \leq r$ and consider the affine map $\ell_i\colon \Delta \to \mathbb{R}_{\geq 0}$. We have that $-z\log(z)$ is a strictly concave function on $\mathbb{R}_{\geq 0}$ and $-\ell_i \log(\ell_i) = \ell_i^*(-z\log(z))$. Hence, each function $-c_i\ell_i(x)\log(\ell_i(x))$ is concave and so is ϑ, as stated in (1)

For statement (2), let x_1, x_2 be two different points of Δ. The assumption that $\{u_i\}_i$ generates $N_{\mathbb{R}}$ implies that $\ell_{i_0}(x_1) \neq \ell_{i_0}(x_2)$ for some i_0. Hence, the affine map ℓ_{i_0} gives an injection of the segment $\overline{x_1x_2}$ into $\mathbb{R}_{\geq 0}$. We deduce that $-c_{i_0}\ell_{i_0}\log(\ell_{i_0})$ is strictly concave on $\overline{x_1x_2}$ and so is ϑ. Varying x_1, x_2, we deduce that ϑ is strictly concave on Δ.

For statement (3), it is clear that $\vartheta|_{\Delta^{\circ}}$ is differentiable. Moreover, the assumption that Δ is the intersection of the halfspaces defined by the ℓ_i's implies that the u_i's generate $N_{\mathbb{R}}$ and so ϑ is strictly concave. The gradient of ϑ is given, for $x \in \Delta^{\circ}$, by

$$\nabla\vartheta(x) = -\sum_{i=1}^{r} c_i u_i (\log(\ell_i(x) + 1). \tag{7.22}$$

Let $\|\cdot\|$ be a fixed norm on $M_{\mathbb{R}}$ and $(x_j)_{j\geq 0}$ a sequence in Δ° converging to a point in the border. Then there exists some i_1 such $\ell_{i_1}(x_j) \overset{j}{\to} 0$. Thus, $\|\nabla\vartheta(x)\| \overset{j}{\to} \infty$ and the statement follows. □

Definition 7.23. Let Σ_{Δ} and Ψ_{Δ} be the fan and the support function on $N_{\mathbb{R}}$ induced by Δ. Let $(X_{\Sigma_{\Delta}}, D_{\Psi_{\Delta}})$ be the associated polarized toric variety over $\mathbb{Q}$ and write $L = \mathcal{O}(D_{\Psi_{\Delta}})$. By Lemma 7.21(1), ϑ is a concave function on Δ. By Theorem 5.73, it corresponds to some approachable toric metric on $L(\mathbb{C})$. We denote this metric by $\|\cdot\|_{\Delta,\ell,c}$. We write $\overline{L}$ for the line bundle L equipped with the metric $\|\cdot\|_{\Delta,\ell,c}$ at the Archimedean place of $\mathbb{Q}$ and with the canonical metric at the non-Archimedean places. This is an example of an adelic toric metric.

Example 7.24. Following the notation in Example 3.53, consider the standard simplex Δ^n and the concave function $\vartheta = \frac{1}{2}\varepsilon_n$ on Δ^n. From examples 3.53 and 5.18(1), we deduce that the corresponding metric is the Fubini-Study metric of $\mathcal{O}(1)^{\mathrm{an}}$.

In case Δ is the intersection of the halfspaces defined by the ℓ_i's, Lemma 7.21(3) shows that $\vartheta|_{\Delta^{\circ}}$ of Legendre type (Definition 3.51). By Theorem 3.52 and equation (7.22), the gradient of ϑ gives a homeomorphism between Δ° and $N_{\mathbb{R}}$ and, for $x \in \Delta^{\circ}$,

$$\vartheta^{\vee}(\nabla\vartheta(x)) = -\sum_{i=1}^{r} c_i\lambda_i \log(\ell_i(x)) + c_i\langle u_i, x\rangle. \tag{7.25}$$

This gives an explicit expression of the function $\psi_{\|\cdot\|_{\Delta,\ell,c}} = \vartheta^\vee$, and *a fortiori* of the metric $\|\cdot\|_{\Delta,\ell,c}$, in the coordinates of the polytope. Up to our knowledge, there is no simple expression for ψ in linear coordinates of $N_{\mathbb{R}}$, except for special cases like Fubini-Study.

Remark 7.26. This kind of metrics are interesting when studying the Kähler geometry of toric varieties. Given a Delzant polytope $\Delta \subset M_{\mathbb{R}}$, Guillemin has constructed a "canonical" Kähler structure on the associated symplectic toric variety [Gui95]. The corresponding symplectic potential is the function $-\vartheta_{\Delta,\ell,c}$, for the case when r is the number of facets of Δ, $c_i = 1/2$ for all i, and u_i is a primitive vector in N and λ_i is an integer such that $\Delta = \{x \in M_{\mathbb{R}} | \langle u_i, x\rangle \geq \lambda_i, i = 1, \dots, r\}$, see [Gui95, Appendix 2, (3.9)].

In this case, the metric $\|\cdot\|_{\Delta,\ell,c}$ on the line bundle $\mathcal{O}(D_\Psi)^{\mathrm{an}}$ is smooth and positive and, as explained in Remark 5.74, its Chern form gives this canonical Kähler form.

We obtain the following formula for the height of X_{Σ_Δ} with respect to the adelic metrized line bundle $\overline{L}$, in terms of the coefficients $C_k(\Delta, u_i, V)$.

Proposition 7.27. *Let notation be as in Definition 7.23. Then* $\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta})$ *equals*

$$(n+1)!\sum_{i=1}^{r} c_i \sum_{V\in\Delta(u_i)} \sum_{k=0}^{\dim(V)} C_k(\Delta, u_i, V)\frac{\ell_i(V)^{n-k+1}}{(n-k+1)!}\left(\sum_{j=2}^{n-k+1}\frac{1}{j} - \log(\ell_i(V))\right).$$

Suppose furthermore that $\Delta\subset\mathbb{R}^n$ *is a simplex,* $r = n+1$ *and that* ℓ_i, $i = 1, \dots, n+1$, *are affine functions such that* $\Delta = \bigcap_i \{x \in M_{\mathbb{R}} | \ell_i(x) \geq 0\}$. *Then*

$$\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta}) = n!\,\mathrm{vol}_M(\Delta)\sum_{i=1}^{n+1} c_i\ell_i(\nu_i)\Big(\sum_{j=2}^{n+1}\frac{1}{j} - \log(\ell_i(\nu_i))\Big). \tag{7.28}$$

where ν_i *is the unique vertex of* Δ *not contained in the facet defined by* ℓ_i.

Proof. The first statement follows readily from Theorem 6.37 and Proposition 7.3 applied to the functions $f_i(z) = \left(\log(z-\lambda_i) - \sum_{j=2}^{n+1}\frac{1}{j}\right)(z-\lambda_i)^{n+1}/(n+1)!$. The second statement follows similarly from Proposition 7.15. □

Example 7.29. Let $\mathcal{O}(1)$ be the universal line bundle of $\mathbb{P}^n$. The Fubini-Study metric of $\mathcal{O}(1)^{\mathrm{an}}$ corresponds to the case of the standard simplex, $\ell_i(x) = x_i$, $i = 1, \dots, n$ and $\ell_{n+1}(x) = 1 - \sum_{i=1}^n x_i$ and the choice $c_i = 1/2$ for all i. Hence we recover from (7.28) the well known expression for the height of $\mathbb{P}^n$ with respect to the Fubini-Study metric in [BGS94, Lemma 3.3.1]:

$$\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n) = \frac{n+1}{2}\sum_{j=2}^{n+1}\frac{1}{j}.$$

Example 7.30. In dimension 1, a polytope is an interval of the form $\Delta = [m_0, m_1]$ for some $m_i \in \mathbb{Z}$. The corresponding roof function in (7.20) writes down, for $x \in [m_0, m_1]$, as

$$\vartheta(x) = -\sum_{i=1}^{r} c_i\ell_i(x)\log(\ell_i(x)) \tag{7.31}$$

for affine function $\ell_i = u_i x - \lambda_i$ which take non negative values on the Δ and $c_i > 0$

The polarized toric variety corresponding to Δ is $\mathbb{P}^1$ together with the ample divisor $m_1[(0:1)] - m_0[(1:0)]$. Write $L = \mathcal{O}_{\mathbb{P}^1}(x_1 - x_0)$ for the associate line bundle and $\overline{L}$ for the adelic metrized line bundle corresponding to the function ϑ.

The Legendre-Fenchel dual to $-c_i\ell_i(x)\log(\ell_i(x))$ is the function $f_i\colon \mathbb{R}\to\mathbb{R}$ defined, for $v\in\mathbb{R}$, by

$$f_i(v)=\frac{\lambda_i}{u_i}v-c_i\mathrm{e}^{-1-\frac{v}{c_iu_i}}.$$

Therefore, the function $\psi=\vartheta^\vee$ is the sup-convolution of these function, namely $\psi=f_1\boxplus\cdots\boxplus f_m$ For the height, a simple computation shows that

$$\mathrm{h}_{\overline{L}}(\mathbb{P}^1)=\int_{m_0}^{m_1}\vartheta\,\mathrm{d}x=\sum_{i=1}^{r}\frac{c_i}{4u_i}\Big[\ell_i(x)^2\left(1-2\log(\ell_i(x))\right)\Big]_{m_0}^{m_1}$$

In some cases, the height of a toric variety with respect to the metrics constructed above has an interpretation in terms of the average entropy of some natural random processes. Let Γ be an arbitrary polytope containing Δ. For a point $x\in\mathrm{ri}(\Delta)$, we consider the partition Π_x of Γ which consists of the cones $\eta_{x,F}$ of vertex x and base the relative interior of each proper face F of Γ.

We consider Γ as a probability space endowed with the uniform probability distribution and β_x the random variable which, for a point $y\in\Gamma$, returns the base F of the unique cone $\eta_{x,F}$ it belongs to. Clearly, the probability that a given face F is returned is the ratio of the volume of the cone based on F to the volume of Γ. We have $\mathrm{vol}_n(\eta_{x,F})=n^{-1}\mathrm{dist}(x,F)\mathrm{vol}_{n-1}(F)$ where, as before, vol_n and vol_{n-1} denote the Lebesgue measure on $\mathbb{R}^n$ and on L_F, respectively. Hence,

$$P(\beta_x=F)=\begin{cases}\dfrac{\mathrm{dist}(x,F)\mathrm{vol}_{n-1}(F)}{n\mathrm{vol}_n(\Gamma)} & \text{if } \dim(F)=n-1,\\ 0 & \text{if } \dim(F)\le n-2.\end{cases}\tag{7.32}$$

The entropy of the random variable β_x is

$$\mathcal{E}(x)=-\sum_F P(\beta_x=F)\log(P(\beta_x=F)),$$

where the sum is over the facets F of Γ.

For each facet F of Γ we let $u_F\in\mathbb{R}^n$ be the inner normal vector to F of Euclidean norm $(n-1)!\,\mathrm{vol}_{n-1}(F)$ and $\lambda_F=\Psi_\Gamma(u_F)\in\mathbb{R}$ and consider the affine form ℓ_F defined as $\ell_F(x)=\langle u_F,x\rangle-\lambda_F$. Hence, $\Gamma=\{x\in M_\mathbb{R}|\ell_F(x)\ge 0\}$. Let also $c_F=c$ for some constant $c>0$. By the Minkowski condition, $\sum_F u_F=0$. Hence $\sum_F\ell_F=-\sum_F\lambda_F$.

Remark 7.33. Suppose that Γ is a lattice polytope and let F be a facet of Γ. Recall that $M(F)$ is the lattice $L_F\cap M$ and let $M(F)'$ be the sublattice of $M(F)$ generated by the differences of the lattice points in F. Then the vector u_F can be alternatively defined as $[M(F):M(F)']$ times the primitive inner normal vector to the facet F.

The concave function $\vartheta=-\sum_F c\,\ell_F(x)\log(\ell_F(x))$ belongs to the class of functions considered in Definition 7.23. Thus, we obtain a line bundle with an adelic toric metric $\overline{L}$ on X_Δ. For short, we write $X=X_\Delta$. The following result shows that the average entropy of the random variable β_x with respect to the uniform distribution on Δ can be expressed in terms of the height of the toric variety X with respect to $\overline{L}$.

Proposition 7.34. *With the above notation,*

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta\mathcal{E}(x)\,\mathrm{d}\,\mathrm{vol}_n=\frac{1}{n!\,\mathrm{vol}_n(\Gamma)}\bigg(\frac{\mathrm{h}_{\overline{L}}(X)}{c(n+1)\deg_L(X)}-\log(n!\,\mathrm{vol}_n(\Gamma))\Big(\sum_F\lambda_F\Big)\bigg)$$

where the sum is over the facets F of Γ. In particular, if $\Gamma=\Delta$,

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta\mathcal{E}(x)\,\mathrm{d}\,\mathrm{vol}_n=\frac{\mathrm{h}_{\overline{L}}(X)}{c(n+1)\deg_L(X)^2}-\frac{\log(\deg_L(X))}{\deg_L(X)}\Big(\sum_F\lambda_F\Big).$$

Proof. For $x \in \operatorname{ri}(\Delta)$ and F a facet of Γ, we deduce from equation (7.32) that $P(\beta_x = F) = \ell_F(x)/(n!\operatorname{vol}_n(\Gamma))$. Hence,

$$\begin{aligned}\mathcal{E}(x) &= -\sum_F \frac{\ell_F(x)}{n!\mathrm{vol}_n(\Gamma)} \log\Big(\frac{\ell_F(x)}{n!\mathrm{vol}_n(\Gamma)}\Big)\\ &= \frac{1}{n!\mathrm{vol}_n(\Gamma)}\Big(-\sum_F \ell_F(x)\log(\ell_F(x)) - \log(n!\mathrm{vol}_n(\Gamma))\Big(\sum_F \lambda_F\Big)\Big)\\ &= \frac{1}{n!\mathrm{vol}_n(\Gamma)}\Big(\frac{\vartheta(x)}{c} - \log(n!\mathrm{vol}_n(\Gamma))\Big(\sum_F \lambda_F\Big)\Big).\end{aligned}$$

The result then follows from Theorem 6.37. □

Example 7.35. The Fubini-Study metric of $\mathcal{O}(1)^{\mathrm{an}}$ corresponds to the case when Γ and Δ are the standard simplex Δ^n and $c = 1/2$. In that case, the average entropy of the random variable β_x is

$$\frac{1}{n!}\int_{\Delta^n} \mathcal{E}(x)\,\mathrm{d}\operatorname{vol}_n = \frac{2\,\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)}{(n+1)} = \sum_{j=2}^{n+1}\frac{1}{j}.$$

Remark 7.36. In case Δ is a Delzant polytope whose facets have lattice volume 1, $\Gamma = \Delta$, and $c = 1/2$, the roof function ϑ coincides with the symplectic potential of Guillemin canonical Kähler metric, see Remark 7.26.

8. Variations on Fubini-Study metrics

8.1. **Height of toric projective curves.** In this section, we study the Arakelov invariants of curves which are the image of an equivariant map into a projective space. In the Archimedean case we equip the projective space with the Fubini-Study metric, while in the non-Archimedean case we equip it with the canonical metric. For each of these curves, the metric, measure and toric local height can be computed in terms of the roots of a univariate polynomial associated to the relevant equivariant map.

Let K be either $\mathbb{R}, \mathbb{C}$ or a complete field with respect to an absolute value associated to a nontrivial discrete valuation. On $\mathbb{P}^r$, we consider the universal line bundle $\mathcal{O}(1)$ equipped with the Fubini-Study metric in the Archimedean case, and with the canonical metric in the non-Archimedean case. We write $\overline{\mathcal{O}(1)}$ for the resulting metrized line bundle. We also consider the toric section s_∞ of $\mathcal{O}(1)$ whose Weil divisor is the hyperplane at infinity. Next result gives the induced function ψ for a subvariety of $\mathbb{P}^r$ which is the image of an equivariant map.

Proposition 8.1. *Let $H\colon N \to \mathbb{Z}^r$ be an injective map such that $H(N)$ is a saturated sublattice of $\mathbb{Z}^r$, $p \in \mathbb{P}^r_0(K)$. Consider the map $\varphi_{H,p} : \mathbb{T} \to \mathbb{P}^r$, and set $\overline{L} = \varphi_{H,p}^*\overline{\mathcal{O}(1)}$ and $s = \varphi^*_{H,p}s_\infty$. Let $\psi_{\overline{L},s}\colon N_{\mathbb{R}} \to \mathbb{R}$ be the associated concave function, $m_i = e_i^\vee \circ H \in M$, $i = 1,\dots,r$, and $p = (1 : p_1 : \dots : p_r)$ with $p_i \in K^\times$. Then, for $u \in N_{\mathbb{R}}$,*

$$\psi_{\overline{L},s}(u) = \begin{cases} -\frac{1}{2}\log(1 + \sum_{i=1}^r |p_i|^2\,\mathrm{e}^{-2\langle m_i,u\rangle}), & \textit{in the Archimedean case,}\\ \min_{1\le i\le r}\{0, \langle m_i,u\rangle + \operatorname{val}_K(p_i)\} & \textit{in the non-Archimedean case.}\end{cases}$$

Proof. In the Archimedean case, the expression for the concave function ψ follows from that for $\mathbb{P}^r_K$ (Example 5.18(2)) and Proposition 5.24. The non-Archimedean case follows from Example 5.26. □

Let $Y \subset \mathbb{P}^r$ be the closure of the image of the map $\varphi_{H,p}$. In this situation, the roof function seems difficult to calculate. Hence it is difficult to use it directly to compute the toric local height (see Example 3.57). A more promising approach is to apply the formula of Corollary 6.17. Writing $\psi = \psi_{\overline{L},s}$ this formula reads

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = \lambda_K (n+1)! \int_{N_{\mathbb{R}}} \psi^{\vee} \circ \partial\psi\, \mathcal{M}_M(\psi). \tag{8.2}$$

To make this formula more explicit in the Archimedean case, we choose a basis of N, hence coordinate systems in $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ and we write

$$g = (g_1, \dots, g_n) := \nabla\psi \colon N_{\mathbb{R}} \longrightarrow \Delta,$$

where $\Delta = \operatorname{stab}(\psi)$ is the associated polytope. Then, from Proposition 3.94 and Example 3.106(1), we derive

$$\begin{aligned} \mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) &= (n+1)! \int_{N_{\mathbb{R}}} (\langle \nabla\psi(u), u\rangle - \psi(u))\, (-1)^n \det(\mathrm{Hess}(\psi))\, \mathrm{d}\operatorname{vol}_N \\ &= (n+1)! \int_{N_{\mathbb{R}}} (\langle g(u), u\rangle - \psi(u))\, (-1)^n\, \mathrm{d}g_1 \wedge \dots \wedge \mathrm{d}g_n. \end{aligned} \tag{8.3}$$

When K is not Archimedean, we have $\mathcal{M}_M(\psi) = \sum_{v \in \Pi^0(\psi)} \delta_v$ and, for $v \in \Pi(\psi)^0$,

$$\psi^{\vee} \circ \partial\psi(v) = \frac{1}{\operatorname{vol}_M(v^*)} \int_{v^*} \langle x, v\rangle\, \mathrm{d}\operatorname{vol}_M - \psi(v),$$

see Proposition 3.95 and Example 3.106(2). Thus, if now we denote by $g \colon N_{\mathbb{R}} \to M_{\mathbb{R}}$ the function that sends a point u to the barycentre of $\partial\psi(u)$, then

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = \lambda_K (n+1)! \sum_{v \in \Pi^0(\psi)} (\langle g(v), v\rangle - \psi(v)). \tag{8.4}$$

In the case of curves, the integral of equation (8.2), can be transformed into another integral that will prove useful for explicit computations. We introduce a notation for derivatives of concave functions of one variable. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a concave function. We write

$$f'(u) = \frac{1}{2}(D_+ f(u) + D_- f(u)), \tag{8.5}$$

where $D_+ f$ and $D_- f$ denote the right and left derivatives of f respectively, that exist always. Then f' is monotone and is continuous almost everywhere (with respect to the Lebesgue measure). The associated distribution agrees with the derivative of f in the sense of distributions. This implies that, if $\{f_n\}_n$ is a sequence of concave functions converging uniformly to f on compacts, then $\{f'_n\}$ converges to f' almost everywhere.

Lemma 8.6. *Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be a concave function whose stability set is an interval $[a, b]$. Then*

$$2 \int_{\mathbb{R}} \psi^{\vee} \circ \partial\psi\, \mathcal{M}_{\mathbb{Z}}(\psi) = (b-a)(\psi^{\vee}(a) + \psi^{\vee}(b)) + \int_{\mathbb{R}} (\psi'(u) - a)(b - \psi'(u))\, \mathrm{d}u.$$

Proof. By the properties of the Monge-Ampère measure (Proposition 3.93) and of the Legendre-Fenchel dual (Proposition 3.18) the left-hand side is continuous with respect to uniform convergence of functions. Again by Proposition 3.18 and the discussion before the lemma, the right-hand side is also continuous with respect to uniform convergence of functions. Therefore it is enough to treat the case when ψ is smooth and strictly concave. Then

$$2 \int_{\mathbb{R}} \psi^{\vee} \circ \partial\psi\, \mathcal{M}_{\mathbb{Z}}(\psi) = 2 \int_{\mathbb{R}} (\psi(u) - u\psi'(u))\psi''(u)\, \mathrm{d}u.$$

Consider the function

$$\begin{aligned}\gamma(u) &= (\psi'(u) - \frac{a+b}{2})\psi(u) - u\frac{(\psi')^2}{2} + u\frac{ab}{2}\\ &= -(\psi'(u) - \frac{a+b}{2})\psi^\vee(\psi'(u)) - \frac{u}{2}(\psi'(u)-a)(b-\psi'(u)).\end{aligned}$$

Then

$$\lim_{u\to\infty}\gamma(u) = \frac{b-a}{2}\psi^\vee(a), \qquad \lim_{u\to-\infty}\gamma(u) = \frac{a-b}{2}\psi^\vee(b),$$

and

$$\mathrm{d}\gamma = (\psi - u\psi')\psi''\,\mathrm{d}u - \frac{1}{2}(\psi'-a)(b-\psi')\,\mathrm{d}u,$$

from which the result follows. □

With the notation in Proposition 8.1, assume that $N=\mathbb{Z}$. The elements $m_j \in N^\vee$ can be identified with integer numbers and the hypothesis that the image of H is a saturated sublattice is equivalent to $\gcd(m_1,\dots,m_r)=1$. Moreover, by reordering the variables of $\mathbb{P}^r$ and multiplying the expression of $\varphi_{H,p}$ by a monomial (which does not change the equivariant map), we may assume that $0\le m_1\le\dots\le m_r$. We make the further hypothesis that $0<m_1<\dots<m_r$. With these conditions, we next obtain explicit expressions for the concave function ψ and the associated measure and toric local height in terms of the roots of a univariate polynomial. We consider the absolute value $|\cdot|$ of the algebraic closure $\overline{K}$ extending the absolute value of K. For $\xi\in\overline{K}^\times$, we set $\mathrm{val}_{\overline{K}}(\xi) = -\frac{\log|\xi|}{\lambda_K}$.

Theorem 8.7. *Let $0<m_1<\dots<m_r$ be integer numbers with $\gcd(m_1,\dots,m_r)=1$, and $p_1,\dots,p_r\in K^\times$. Let $\varphi\colon \mathbb{T}\to\mathbb{P}^r$ be the map given by $\varphi(t)=(1:p_1t^{m_1}:\dots:p_rt^{m_r})$ and let Y be the closure of the image of φ. Consider the polynomial $q\in K[z]$ defined as*

$$q = \begin{cases} 1+\sum_{j=1}^r |p_j|^2 z^{m_j}, & \text{in the Archimedean case,}\\ 1+\sum_{j=1}^r p_j z^{m_j}, & \text{in the non-Archimedean case.}\end{cases}$$

Let $\{\xi_i\}_i\subset\overline{K}^\times$ be the set of roots of q and, for each i, let $\ell_i\in\mathbb{N}$ be the multiplicity of ξ_i. Let $\overline{L}$ and s be as in Proposition 8.1. Then, in the Archimedean case,

(1) $\psi_{\overline{L},s}(u) = -\log|p_r| - \frac{1}{2}\sum_i \ell_i\log|\mathrm{e}^{-2u}-\xi_i|$ *for* $u\in\mathbb{R}$,

(2) $\mathcal{M}_{\mathbb{Z}}(\psi_{\overline{L},s}) = -2\sum_i \ell_i \dfrac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})^2}\,\mathrm{d}u$,

(3) $\mathrm{h}_{\overline{L}}^{\mathrm{tor}}(Y) = m_r\log|p_r| + \frac{1}{2}\sum_i \ell_i^2 + \frac{1}{2}\sum_{i<j}\ell_i\ell_j\dfrac{\xi_i+\xi_j}{\xi_i-\xi_j}(\log(-\xi_i)-\log(-\xi_j))$,

where log *is the principal determination of the logarithm.*

While in the non-Archimedean case,

(4) $\psi_{\overline{L},s}(u) = \mathrm{val}_K(p_r) + \sum_i \ell_i\min\{u,\mathrm{val}_{\overline{K}}(\xi_i)\}$ *for* $u\in\mathbb{R}$,

(5) $\mathcal{M}_{\mathbb{Z}}(\psi_{\overline{L},s}) = \sum_i \ell_i\delta_{\mathrm{val}_{\overline{K}}(\xi_i)}$,

(6) $\mathrm{h}_{\overline{L}}^{\mathrm{tor}}(Y) = m_r\log|p_r| + \sum_{i<j}\ell_i\ell_j\log(\max\{1,|\xi_i|/|\xi_j|\})$.

Remark 8.8. The real roots of the polynomial q are all negative, this allows the use of the principal determination of the logarithm in (3). Introducing the argument $\theta_i \in]-\pi,\pi[$ of $-\xi_i$, the last sum in (3) can be rewritten

$$\frac{1}{2}\sum_{i<j}\ell_i\ell_j\frac{(|\xi_i|^2-|\xi_j|^2)\log|\xi_i/\xi_j|+2|\xi_i||\xi_j|(\theta_i-\theta_j)\sin(\theta_i-\theta_j)}{|\xi_i|^2+|\xi_j|^2-2|\xi_i||\xi_j|\cos(\theta_i-\theta_j)}$$

showing that it is real.

Proof. Write $\psi=\psi_{\overline{L},s}$ for short. First we consider the Archimedean case. We have that $q=|p_r|^2\prod_i(z-\xi_i)^{\ell_i}$. By Proposition 8.1,

$$\psi(u)=-\frac{1}{2}\log(q(\mathrm{e}^{-2u}))=-\log|p_r|-\frac{1}{2}\sum_i\ell_i\log|\mathrm{e}^{-2u}-\xi_i|,$$

which proves (1). Hence,

$$\psi'(u)=\sum_i\ell_i\frac{1}{1-\xi_i\,\mathrm{e}^{2u}}\quad\text{and}\quad\psi''(u)=\sum_i 2\ell_i\frac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})^2}.$$

The Monge-Ampère measure of ψ is given by $-\psi''\,\mathrm{d}u$, and so the above proves (2). To prove (3) we apply Lemma 8.6. We have that $\operatorname{stab}(\psi)=[0,m_r]$, $\psi^\vee(0)=0$, and $\psi^\vee(m_r)=\log|p_r|$. Thus,

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y)=m_r\log|p_r|+\int_{-\infty}^{\infty}(m_r-\psi')\psi'\,\mathrm{d}u. \tag{8.9}$$

We have $m_r-\psi'(u)=\sum_i\ell_i\left(1-\frac{1}{1-\xi_i\,\mathrm{e}^{2u}}\right)=-\sum_i\ell_i\frac{\xi_i\,\mathrm{e}^{2u}}{1-\xi_i\,\mathrm{e}^{2u}}$. Hence,

$$\begin{aligned}(m_r-\psi'(u))\psi'(u)&=-\left(\sum_i\ell_i\frac{\xi_i\,\mathrm{e}^{2u}}{1-\xi_i\,\mathrm{e}^{2u}}\right)\left(\sum_j\ell_j\frac{1}{1-\xi_j\,\mathrm{e}^{2u}}\right)\\&=-\sum_i\ell_i^2\frac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})^2}-\sum_{i\neq j}\ell_i\ell_j\frac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})(1-\xi_j\,\mathrm{e}^{2u})}.\end{aligned}$$

Moreover $\int_{-\infty}^{\infty}\frac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})^2}\,\mathrm{d}u=\left[\frac{1}{2(1-\xi_i\,\mathrm{e}^{2u})}\right]_{-\infty}^{\infty}=-\frac{1}{2}$ and

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\xi_i\,\mathrm{e}^{2u}}{(1-\xi_i\,\mathrm{e}^{2u})(1-\xi_j\,\mathrm{e}^{2u})}\,\mathrm{d}u&=\left[\frac{\xi_i}{2(\xi_i-\xi_j)}(\log(1-\xi_j\,\mathrm{e}^{2u}))-\log(1-\xi_i\,\mathrm{e}^{2u})\right]_{-\infty}^{\infty}\\&=\frac{\xi_i}{2(\xi_i-\xi_j)}(\log(-\xi_i)-\log(-\xi_j)),\end{aligned}$$

for the principal determination of log. These calculations together with equation (8.9) imply that

$$\begin{aligned}\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y)&=m_r\log|p_r|+\frac{1}{2}\sum_i\ell_i^2+\frac{1}{2}\sum_{i\neq j}\ell_i\ell_j\frac{\xi_i}{\xi_i-\xi_j}(\log(-\xi_i)-\log(-\xi_j))\\&=m_r\log|p_r|+\frac{1}{2}\sum_i\ell_i^2+\frac{1}{2}\sum_{i<j}\ell_i\ell_j\frac{\xi_i+\xi_j}{\xi_i-\xi_j}(\log(-\xi_i)-\log(-\xi_j)),\end{aligned}$$

which proves (3).

Next we consider the non-Archimedean case. Let $U\subset\overline{K}^\times$ be a sufficiently small open subset and $\zeta\in U$. For short, write $v_i=\operatorname{val}_{\overline{K}}(\xi_i)$. By Proposition 8.1, the genericity of ζ, and the condition $m_i\neq m_j$ for $i\neq j$, imply

$$\psi(\operatorname{val}_{\overline{K}}(\zeta))=\min_i\{0,m_i\operatorname{val}_{\overline{K}}(\zeta)+\operatorname{val}_K(p_i)\}=\operatorname{val}_{\overline{K}}(q(\zeta)).$$

By the factorization of q,

$$\operatorname{val}_{\overline{K}}(q(\zeta)) = \operatorname{val}_K(p_r) + \sum_i \ell_i \operatorname{val}_{\overline{K}}(\zeta - \xi_i) = \operatorname{val}_K(p_r) + \sum_i \ell_i \min\{\operatorname{val}_{\overline{K}}(\zeta), v_i\}.$$

The image of $\operatorname{val}_{\overline{K}}\colon \overline{K}^\times \to \mathbb{R}$ is a dense subset. We deduce that, $u \in \mathbb{R}$,

$$\psi(u) = \operatorname{val}_K(p_r) + \sum_i \ell_i \min\{u, \operatorname{val}_K(\xi_i)\},$$

which proves (4). The gradient of this function is, for $u \in \mathbb{R}$,

$$\partial\psi(u) = \begin{cases} \left[\sum_{j:v_j>v_i} \ell_j, \sum_{j:v_j\geq v_i} \ell_j\right] & \text{if } u = v_i \text{ for some } i,\\ \sum_{j:v_j>x} \ell_j & \text{otherwise.}\end{cases}$$

Hence, the associated Monge-Ampère measure is $\sum_i \ell_i\delta_{v_i}$, which proves (5). The derivative of ψ in the sense of (8.5) is, for $u \in \mathbb{R}$,

$$\psi'(u) = \begin{cases} \sum_{j:v_j>v_i} \ell_j + \frac{1}{2}\sum_{j:v_j=v_i} \ell_j & \text{if } u = v_i \text{ for some } i,\\ \sum_{j:v_j>x} \ell_j & \text{otherwise.}\end{cases}$$

Moreover, $\operatorname{stab}(\psi) = [0, m_r]$, $\psi^\vee(0) = 0$ and $\psi^\vee(m_r) = -\operatorname{val}_K(p_r)$. By Lemma 8.6

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = -m_r\lambda_K \operatorname{val}_K(p_r) + \lambda_K \int_{-\infty}^{\infty} (m_r - \psi')\psi' \,\mathrm{d}u. \tag{8.10}$$

If we write

$$f_i(u) = \begin{cases} 0, & \text{if } x \leq v_i\\ \ell_i, & \text{if } x > v_i,\end{cases}$$

then, we have that, almost everywhere $\psi'(u) = \sum_i \ell_i - f_i(u)$ and $m_r - \psi'(u) = \sum_i f_i$. Therefore

$$\int_{-\infty}^{\infty} (m_r - \psi')\psi' \,\mathrm{d}u = \sum_{i,j} \int_{-\infty}^{\infty} f_i(\ell_j - f_j)\,\mathrm{d}u = \sum_{i,j} \ell_i\ell_j \max\{0, v_j - v_i\}. \tag{8.11}$$

Thus, joining together (8.10), (8.11) and the relation $\log(|\zeta|) = -\lambda_K \operatorname{val}_K(\zeta)$ we deduce

$$\mathrm{h}^{\mathrm{tor}}_{\overline{L}}(Y) = m_r \log|p_r| + \sum_{i,j} \ell_i\ell_j \max\{0, \log(|\xi_i|/|\xi_j|)\},$$

finishing the proof of the theorem. □

We now treat the global case.

Corollary 8.12. *Let $(\mathbb{K}, \mathfrak{M}_{\mathbb{K}})$ be a global field. Let $0 < m_1 < \cdots < m_r$ be integer numbers with $\gcd(m_1, \dots, m_r) = 1$, and $p_1, \dots, p_r \in \mathbb{K}^\times$. Let $\varphi\colon \mathbb{T} \to \mathbb{P}^r$ be the map given by $\varphi(t) = (1 : p_1t^{m_1} : \cdots : p_rt^{m_r})$, Y the closure of the image of φ, and $\overline{L} = \varphi^*\overline{\mathcal{O}(1)}$, where $\overline{\mathcal{O}(1)}$ is equipped with the Fubini-Study metric for the Archimedean places and with the canonical metric for the non-Archimedean places. For $v \in \mathfrak{M}_K$, set*

$$q_v = \begin{cases} 1 + \sum_{j=1}^r |p_j|_v^2 z^{m_j}, & \textit{if } v \textit{ is Archimedean},\\ 1 + \sum_{j=1}^r p_j z^{m_j}, & \textit{if } v \textit{ is not Archimedean.}\end{cases}$$

Let $\{\xi_{v,i}\} \subset \overline{\mathbb{K}}^{\times}$ be the set of roots of q_v and, for each i, let $\ell_{v,i} \in \mathbb{N}$ denote the multiplicity of $\xi_{v,i}$. Then

$$\mathrm{h}_{\overline{L}}(Y) = \sum_{v|\infty} n_v \left(\frac{1}{2} \sum_i \ell_{v,i}^2 + \frac{1}{2} \sum_{i<j} \ell_{v,i}\ell_{v,j} \frac{\xi_{v,i}+\xi_{v,j}}{\xi_{v,i}-\xi_{v,j}} (\log(-\xi_{v,i}) - \log(-\xi_{v,j})) \right) + \sum_{v\nmid\infty} n_v \left(\sum_{i<j} \ell_{v,i}\ell_{v,j} \log(\max\{1, |\xi_{v,i}|_v/|\xi_{v,j}|_v\}) \right).$$

Proof. This follows readily from Proposition 6.35, Theorem 8.7, and the product formula. □

Corollary 8.13. *Let $C_r \subset \mathbb{P}^r_{\mathbb{Q}}$ be the Veronese curve of degree r and $\overline{\mathcal{O}(1)}$ the universal line bundle on $\mathbb{P}^r_{\mathbb{Q}}$ equipped with the Fubini-Study metric at the Archimedean place and with the canonical metric at the non-Archimedean ones. Then*

$$\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r) = \frac{r}{2} + \pi \sum_{j=1}^{\lfloor r/2 \rfloor} \left(1 - \frac{2j}{r+1}\right) \cot\left(\frac{\pi j}{r+1}\right) \in \frac{r}{2} + \pi\, \overline{\mathbb{Q}}. \tag{8.14}$$

Proof. The curve C_r coincides with the closure of the image of the map $\varphi\colon \mathbb{T} \to \mathbb{P}^r$ given by $\varphi(t) = (1 : t : t^2 : \dots : t^r)$. With the notation in Corollary 8.12, this map correspond to $m_i = i$ and $p_i = 1$, for $i = 1, \dots, r$. Then $q_v = \sum_{j=0}^r z^j$ for all $v \in \mathfrak{M}_{\mathbb{Q}}$. Consider the primitive $(r+1)$-th root of unity $\omega = \mathrm{e}^{\frac{2\pi i}{r+1}}$. The polynomial q_v is separable and its set of roots is $\{\omega^l\}_{l=1,\dots,r}$. Since $|\omega^l|_v = 1$ for all v, Corollary 8.12 implies that

$$\mathrm{h}_{\overline{L}}(Y) = \frac{r}{2} + \frac{1}{2} \sum_{l<j} \frac{\omega^l + \omega^j}{\omega^l - \omega^j} (\log(-\omega^l) - \log(-\omega^j)) = \frac{r}{2} + \frac{1}{2} \sum_{l\neq j} \frac{\omega^l + \omega^j}{\omega^l - \omega^j} \log(-\omega^l). \tag{8.15}$$

We have that

$$\sum_{j=1}^r \frac{\omega^j + 1}{\omega^j - 1} = \sum_{j=1}^r \frac{\omega^j}{\omega^j - 1} + \sum_{j=1}^r \frac{1}{\omega^j - 1} = \sum_{j=1}^r \frac{1}{1 - \omega^{-j}} + \sum_{j=1}^r \frac{1}{\omega^j - 1} = 0.$$

This implies that, for $l = 1, \dots, r$,

$$\sum_{1\leq j\leq r, j\neq l} \frac{\omega^l + \omega^j}{\omega^l - \omega^j} = -\frac{\omega^l + 1}{\omega^l - 1} = i \cot\left(\frac{\pi l}{r+1}\right)$$

Hence,

$$\frac{1}{2} \sum_{l\neq j} \frac{\omega^l + \omega^j}{\omega^l - \omega^j} \log(-\omega^l) = -\frac{i}{2} \sum_{l=1}^r \cot\left(\frac{\pi l}{r+1}\right) \log(-\omega^l) = \pi \sum_{l=1}^{\lfloor r/2 \rfloor} \cot\left(\frac{\pi l}{r+1}\right)\left(1 - \frac{2l}{r+1}\right),$$

since $\cot(\frac{\pi(r+1-l)}{r+1}) \log(-\omega^{r+1-l}) = \cot(\frac{\pi l}{r+1}) \log(-\omega^l)$ for $l = 1, \dots, \lfloor r/2 \rfloor$ and $\log(-\omega^{\frac{r+1}{2}}) = 0$ whenever r is odd. The statement follows from this calculations together with (8.15). □

Here follow some special values:

r	1	2	3	5	7
$\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r)$	$\dfrac{1}{2}$	$1+\dfrac{1}{3\sqrt{3}}\pi$	$\dfrac{3}{2}+\dfrac{1}{2}\pi$	$\dfrac{5}{2}+\dfrac{7}{3\sqrt{3}}\pi$	$\dfrac{7}{2}+(1+\sqrt{2})\pi$

Corollary 8.16. *With the notation of Corollary 8.13,* $\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r) = r\log r + O(r)$ *for* $r \to \infty$.

Proof. We have that $\pi\cot(\pi x) = \dfrac{1}{x} + O(1)$ for $x \to 0$. Hence,

$$\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r) = \sum_{j=1}^{\lfloor r/2\rfloor}\left(1-\frac{2j}{r+1}\right)\frac{j}{r+1} + O(r) = r\Bigg(\sum_{j=1}^{\lfloor r/2\rfloor}\frac{1}{j}\Bigg) + O(r) = r\log r + O(r).$$

□

By the theorem of algebraic successive minima [Zha95a],

$$\mu^{\mathrm{ess}}(C_r) \le \frac{\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r)}{\deg_{\mathcal{O}(1)}(C_r)} \le 2\mu^{\mathrm{ess}}(C_r)$$

The essential minimum of C_r is $\mu^{\mathrm{ess}}(C_r) = \frac{1}{2}\log(r+1)$ [Som05]. Hence, the quotient $\frac{\mathrm{h}_{\overline{\mathcal{O}(1)}}(C_r)}{\deg_{\mathcal{O}(1)}(C_r)}$ is asymptotically closer to the upper bound than to the lower bound.

8.2. **Height of toric bundles.** Let $n \ge 0$ and write $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Q}}$ for short. Given $a_r \ge \cdots \ge a_0 \ge 1$, consider the bundle $\mathbb{P}(E) \to \mathbb{P}^n$ of hyperplanes of the vector bundle

$$E = \mathcal{O}(a_0) \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) \longrightarrow \mathbb{P}^n,$$

where $\mathcal{O}(a_j)$ denotes the a_j-th power of the universal line bundle of $\mathbb{P}^n$. Equivalently, $\mathbb{P}(E)$ can be defined as the bundle of lines of the dual vector bundle $E^\vee$. The fibre of the map $\pi\colon \mathbb{P}(E) \to \mathbb{P}^n$ over each point $p \in \mathbb{P}^n(\overline{\mathbb{Q}})$ is a projective space of dimension r. This bundle is a smooth toric variety over $\mathbb{Q}$ of dimension $n+r$, see [Oda88, pp. 58-59], [Ful93, p. 42]. The particular case $n = r = 1$ corresponds to Hirzebruch surfaces: for $b \ge 0$, we have $\mathbb{F}_b = \mathbb{P}(\mathcal{O}(0)\oplus\mathcal{O}(b)) \simeq \mathbb{P}(\mathcal{O}(a_0)\oplus\mathcal{O}(a_0+b))$ for any $a_0 \ge 1$.

The *tautological line bundle* of $\mathbb{P}(E)$, denoted $\mathcal{O}_{\mathbb{P}(E)}(-1)$, is defined as a subbundle of $\pi^*E^\vee$. Its fibre over a point of $\mathbb{P}(E)$ is the inverse image under π of the line in $E^\vee$ which is dual to the hyperplane of E defining the given point. The *universal line bundle* $\mathcal{O}_{\mathbb{P}(E)}(1)$ of $\mathbb{P}(E)$ is defined as the dual of the tautological one. Since $\mathcal{O}(a_j)$, $j = 0,\dots,r$, is ample, the universal line bundle is also ample [Har66]. This is the line bundle corresponding to the Cartier divisor $a_0D_0 + D_1$, where D_0 denotes the inverse image in $\mathbb{P}(E)$ of the hyperplane at infinity of $\mathbb{P}^n$ and $D_1 = \mathbb{P}(0 \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. Observe that, although $\mathbb{P}(E)$ is isomorphic to the bundle associated to the family of integers $a_i + c$ for any $c \in \mathbb{N}$, this is not the case for the associated universal line bundle, that depends on the choice of c.

Following Example 4.3, we regard $\mathbb{P}^n$ as a toric variety over $\mathbb{Q}$ equipped with the action of the split torus $\mathbb{G}_m^n$. Let s be the toric section of $\mathcal{O}(1)$ which corresponds to the hyperplane at infinity H_0 and let $s_j = s^{\otimes -a_j}$, which is a section of $\mathcal{O}(-a_j)$. Let $U = \mathbb{P}^n \setminus H_0$. The restriction of $\mathbb{P}(E)$ to U is isomorphic to $U \times \mathbb{P}^r$ through the map φ defined, for $p \in U$ and $q \in \mathbb{P}^r$, as

$$(p,q) \longmapsto (p, q_0s_0(p) \oplus \cdots \oplus q_rs_r(p)).$$

The torus $\mathbb{T} := \mathbb{G}_m^{n+r}$ can then be included as an open subvariety of $\mathbb{P}(E)$ through the map φ composed with the standard inclusion of $\mathbb{G}_m^{n+r}$ into $U \times \mathbb{P}^r$. The action of $\mathbb{T}$ on itself by translation extends to an action of the torus on the whole of $\mathbb{P}(E)$. Hence $\mathbb{P}(E)$ is a toric variety over $\mathbb{Q}$. With this action the divisor $a_0D_0 + D_1$ is a $\mathbb{T}$-Cartier divisor.

By abuse of notation, we also denote $E^\vee$ the total space associated to the vector bundle $E^\vee$. The map $\mathbb{G}_m^{n+r} \to E^\vee$ defined as

$$(z,w) \longmapsto ((1:z),(s_0(1:z)\oplus w_1s_1(1:z)\oplus\cdots\oplus w_rs_r(1:z)))$$

induces a no-where vanishing section of the tautological line bundle of $\mathbb{P}(E)$ over the open subset $\mathbb{T}$. Its inverse, denoted s, is a no-where vanishing section of $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{T}$. In particular, this section induces a structure of toric line bundle on $\mathcal{O}_{\mathbb{P}(E)}(1)$. The divisor of the section s is precisely the $\mathbb{T}$-Cartier divisor $a_0D_0+D_1$ considered above.

We now introduce an adelic toric metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. For $v=\infty$, we consider the complex vector bundle $E(\mathbb{C})$ that can be naturally metrized by the direct sum of the Fubiny-Study metric on each factor $\mathcal{O}(a_j)(\mathbb{C})$. By duality, this gives a metric on $E^\vee(\mathbb{C})$, which induces by restriction a metric on the tautological line bundle. Applying duality once more time, we obtain a smooth metric, denoted $\|\cdot\|_\infty$, on $O_{\mathbb{P}(E)(\mathbb{C})}(1)$. For $v\in M_\mathbb{Q}\setminus\{\infty\}$, we equip $O_{\mathbb{P}(E)}(1)$ with the canonical metric (Proposition-Definition 5.20). We write $\overline{\mathcal{O}_{\mathbb{P}(E)}(1)} = (\mathcal{O}_{\mathbb{P}(E)}(1),(\|\cdot\|_v)_{v\in M_\mathbb{Q}})$ for the obtained adelic metrized toric line bundle.

We have made a choice of splitting of $\mathbb{T}$ and therefore a choice of an identification $N=\mathbb{Z}^{n+r}$. Thus we obtain a system of coordinates in the real vector space associated to the toric variety $\mathbb{P}(E)$, $N_\mathbb{R}=\mathbb{R}^{n+r}=\mathbb{R}^n\times\mathbb{R}^r$. Since the metric considered at each non-Archimedean place is the canonical one, the only nontrivial contribution to the global height will come from the Archimedean place. The restriction to the principal open subset $\mathbb{P}(E)_0(\mathbb{C})\simeq(\mathbb{C}^\times)^{n+r}=(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^r$ of the valuation map is expressed, in these coordinates, as the map $\operatorname{val}\colon(\mathbb{C}^\times)^{n+r}\to N_\mathbb{R}$ defined by

$$\operatorname{val}(z,w)=(-\log|z_1|,\dots,-\log|z_n|,-\log|w_1|,\dots,-\log|w_r|).$$

Let θ_0 be the natural inclusion of real variety $\mathbb{P}(E)_0(\mathbb{R}_{\geq 0})\simeq(\mathbb{R}_{>0})^{n+r}$ in $\mathbb{P}(E)_0(\mathbb{C})$ and let $\mathbf{e}_\mathbb{C}$ be the homeomorphism $N_\mathbb{R}\to\mathbb{P}(E)_0(\mathbb{R}_{\geq0})$, both defined in §5.1. In these coordinates, the composition map $\theta_0\circ\mathbf{e}_\mathbb{C}\colon N_\mathbb{R}\to\mathbb{P}(E)_0(\mathbb{C})$ is given by $(u,v)\mapsto(\mathrm{e}^{-u_1},\dots,\mathrm{e}^{-u_n},\mathrm{e}^{-v_1},\dots,\mathrm{e}^{-v_r})$.

Write $\psi_\infty\colon N_\mathbb{R}\to\mathbb{R}$ for the function corresponding to the metric $\|\cdot\|_\infty$ and the toric section s defined above.

Lemma 8.17. *The function ψ_∞ is defined, for $u\in\mathbb{R}^n$ and $v\in\mathbb{R}^r$, as*

$$\psi_\infty(u,v)=-\frac{1}{2}\log\left(\sum_{j=0}^r \mathrm{e}^{-2v_j}\left(\sum_{i=0}^n\mathrm{e}^{-2u_i}\right)^{a_j}\right),$$

with the convention $u_0=v_0=0$. It is a strictly concave function.

Proof. The metric on $E^\vee$ is given, for $p\in\mathbb{P}^n(\mathbb{C})$ and $q_0,\dots,q_r\in\mathbb{C}$, by

$$||q_0s_0(p)\oplus\cdots\oplus q_rs_r(p)||_\infty^2=|q_0|^2||s_0(p)||^2+\cdots+|q_r|^2||s_r(p)||^2,$$

where $||s_j(p)||$ is the norm of $s_j(p)$ with respect to the Fubini-Study metric on $\mathcal{O}(-a_j)^{\mathrm{an}}$. By Example 2.2,

$$||s_j(p)||^2=\left(\frac{|p_0|^2}{|p_0|^2+\cdots+|p_n|^2}\right)^{-a_j}.$$

Let $s^{\otimes-1}$ be the monomial section of the tautological line bundle defined by s. Then

$$\|s^{\otimes-1}\circ\theta_0\circ\mathbf{e}_\mathbb{C}(u,v)\|^2=\sum_{j=0}^r\mathrm{e}^{-2v_j}\left(\sum_{i=0}^n\mathrm{e}^{-2u_i}\right)^{a_j}. \tag{8.18}$$

By Proposition 5.19(2), ψ_∞ is $-1/2$ times the logarithm of the above expression.

For the last statement, observe that the functions $\mathrm{e}^{-2v_j}(\sum_{i=0}^{n}\mathrm{e}^{-2u_i})^{a_j}$ are log-strictly convex, because $-1/2$ times their logarithm is the function associated to the Fubini-Study metric on $\mathcal{O}(a_j)^{\mathrm{an}}$, which is a strictly concave function. Their sum is also log-strictly convex [BV04, §3.5.2]. Hence, ψ_∞ is strictly concave. $\square$

Corollary 8.19. *The metric $\|\cdot\|_\infty$ is a semipositive smooth toric metric.*

The following result summarizes the toric structure of $\mathbb{P}(E)$ and of $\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}$.

Proposition 8.20.

(1) *Let e_i, $1 \le i \le n$, and f_j, $1 \le j \le r$, be the i-th and $(n+j)$-th vectors of the standard basis of $N = \mathbb{Z}^{n+r}$. Set $f_0 = -f_1 - \dots - f_r$ and $e_0 = a_0 f_0 + \dots + a_r f_r - e_1 - \dots - e_n$. The fan Σ corresponding to $\mathbb{P}(E)$ is the fan in $N_{\mathbb{R}}$ whose maximal cones are the convex hull of the rays generated by the vectors*

$$e_0, \cdots, e_{k-1}, e_{k+1}, \cdots, e_n, f_0, \cdots, f_{\ell-1}, f_{\ell+1}, \cdots, f_r$$

for $0 \le k \le n, 0 \le \ell \le r$. This is a complete regular fan.

(2) *The support function $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ corresponding to the universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is defined, for $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^r$, as*

$$\Psi(u,v) = \min_{\substack{0\le k\le n\\ 0\le \ell\le r}} (a_\ell u_k + v_\ell),$$

where, for short, we have set $u_0 = v_0 = 0$.

(3) *The polytope Δ in $M_{\mathbb{R}} = \mathbb{R}^n \times \mathbb{R}^r$ associated to (Σ, Ψ) is*

$$\Big\{(x,y)|y_1,\dots,y_r \ge 0,\ \sum_{\ell=1}^{r} y_\ell \le 1,\ x_1,\dots,x_n \ge 0,\ \sum_{k=1}^{n} x_k \le L(y)\Big\}$$

with $L(y) = a_0 + \sum_{\ell=1}^{r}(a_\ell - a_0)y_\ell$. Using the convention $y_0 = 1 - \sum_{\ell=1}^{r} y_\ell$ and $x_0 = L(y) - \sum_{k=1}^{n} x_k$, then $L(y) = \sum_{\ell=0}^{r} a_\ell y_\ell$ and the polytope Δ can be written as

$$\Big\{(x,y)|y_0,\dots,y_r \ge 0,\ x_0,\dots,x_n \ge 0\Big\}.$$

(4) *The Legendre-Fenchel dual of ψ_∞ is the concave function $\psi_\infty^\vee\colon \Delta \to \mathbb{R}$ defined, for $(x,y) \in \Delta$, as*

$$\psi_\infty^\vee(x,y) = -\frac{1}{2}\left(\varepsilon_r(y_1,\dots,y_r) + L(y)\cdot \varepsilon_n\left(\frac{x_1}{L(y)},\dots,\frac{x_n}{L(y)}\right)\right),$$

where, for $k \ge 0$, ε_k is the function defined in (3.54). For $v \ne \infty$, the concave function $\psi_v^\vee$ is the indicator function of Δ.

Proof. By Corollary 5.17, we have $\Psi = \mathrm{rec}(\psi_\infty)$. By equation (3.49), we have $\mathrm{rec}(\psi_\infty) = \lim_{\lambda\to\infty} \lambda^{-1}\psi_\infty(\lambda(u,v))$. Statement (2) follows readily from this and from the expression for ψ_∞ in Lemma 8.17.

The function Ψ is strictly concave on Σ, because $\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle. Hence $\Sigma = \Pi(\Psi)$ and this is the fan described in statement (1).

Let $(e_1^\vee, \dots, e_n^\vee, f_1^\vee, \dots, f_r^\vee)$ be the dual basis of M induced by the basis of N. By Proposition 3.64 and statement (2), we have

$$\Delta = \mathrm{conv}\left(0, (a_0 e_k^\vee)_{1\le k\le n}, (f_\ell^\vee)_{1\le \ell\le r}, (a_\ell e_k^\vee + f_\ell^\vee)_{\substack{1\le k\le n\\ 1\le \ell\le r}}\right).$$

Statement (3) follows readily from this.

For the first part of statement (4), it suffices to compute the Legendre-Fenchel dual of ψ_∞ at a point (x,y) in the interior of the polytope. Lemma 8.17 shows that ψ_∞ is strictly concave. Hence, by Theorem 3.52(3), $\nabla\psi_\infty$ is a homeomorphism between $N_{\mathbb{R}}$ and Δ°. Thus, there exist a unique $(u,v)\in N_{\mathbb{R}}$ such that, for $i=1,\dots,n$ and $j=1,\dots,r$,

$$x_i=\frac{\partial\psi_\infty}{\partial u_i}(u,v),\quad y_j=\frac{\partial\psi_\infty}{\partial v_j}(u,v).$$

We use the conventions $x_0=L(y)-\sum_{i=1}^n x_i$, $y_0=1-\sum_{j=1}^r y_j$, and $u_0=v_0=0$ as before, and also $\eta=\sum_{i=0}^n \mathrm{e}^{-2u_i}$ and $\psi=\psi_\infty$, so that $-2\psi=\log\big(\sum_{j=0}^r \mathrm{e}^{-2v_j}\,\eta^{a_j}\big)$. Computing the gradient of ψ, we obtain, for $i=1,\dots,n$ and $j=1,\dots,r$,

$$x_i\,\mathrm{e}^{-2\psi}=\Big(\sum_{j=0}^r a_j\eta^{a_j-1}\,\mathrm{e}^{-2v_j}\Big)\,\mathrm{e}^{-2u_i},\quad y_j\,\mathrm{e}^{-2\psi}=\eta^{a_j}\,\mathrm{e}^{-2v_j}\,.$$

Combining these expressions, we obtain, for $i=0,\dots,n$ and $j=0,\dots,r$,

$$\frac{x_i}{L(y)}=\frac{\mathrm{e}^{-2u_i}}{\eta},\quad y_j=\frac{\mathrm{e}^{-2v_j+2\psi}}{\eta^{a_j}}.$$

From the case $i=0$ we deduce $\eta=L(y)/x_0$ and from the case $j=0$ it results $2\psi=\log(y_0)+a_0\log(x_0/L(y))$. From this, one can verify

$$u_i=\frac12\log\Big(\frac{x_0}{x_i}\Big),\quad v_j=\frac12\log\Big(\frac{y_0}{y_j}\Big)+\frac{a_0-a_j}{2}\log\Big(\frac{x_0}{L(y)}\Big).$$

From Theorem 3.52(4), we have $\psi^\vee(x,y)=\langle x,u\rangle+\langle y,v\rangle-\psi(u,v)$. Inserting the expressions above for ψ, u_i and v_j in terms of x,y, we obtain the stated formula.

For $v\neq\infty$, we have $\psi_v=\Psi$. The last statement follows from Example 3.16. □

Proposition 4.37 and Theorem 6.37 imply

$$\begin{aligned}\deg_{\mathcal{O}_{\mathbb{P}(E)}(1)}(\mathbb{P}(E))&=(n+r)!\operatorname{vol}(\Delta),\\ \mathrm{h}_{\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}}(\mathbb{P}(E))&=(n+r+1)!\int_\Delta \psi_\infty^\vee\ \mathrm{d}x\,\mathrm{d}y,\end{aligned}\tag{8.21}$$

where, for short, $\mathrm{d}x$ and $\mathrm{d}y$ stand for $\mathrm{d}x_1\dots\mathrm{d}x_n$ and $\mathrm{d}y_1\dots\mathrm{d}y_r$, respectively.

We now compute these volume and integral giving the degree and the height of $\mathbb{P}(E)$. We show, in particular, that the height is a rational number. Recall that Δ^r and Δ^n are the standard simplexes of $\mathbb{R}^r$ and $\mathbb{R}^n$, respectively.

Lemma 8.22. *With the above notation, we have*

$$\deg_{\mathcal{O}_{\mathbb{P}(E)}(1)}(\mathbb{P}(E))=\frac{(n+r)!}{n!}\int_{\Delta^r}L(y)^n\,\mathrm{d}y\tag{8.23}$$

$$\begin{aligned}\mathrm{h}_{\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}}(\mathbb{P}(E))=\frac{(n+r+1)!}{(n+1)!}\,&\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)\int_{\Delta^r}L(y)^{n+1}\,\mathrm{d}y\\ &-\frac{(n+r+1)!}{2\,n!}\int_{\Delta^r}L(y)^n\varepsilon_r(y)\,\mathrm{d}y,\end{aligned}\tag{8.24}$$

where $\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)=\sum_{h=1}^n\sum_{j=1}^h\frac{1}{2j}$ *is the height of the projective space relative to the Fubini-Study metric.*

Proof. Equation (8.21) shows that the degree of $\mathbb{P}(E)$ is equal to $(n+r)!\operatorname{vol}(\Delta)$. The same equation together with Proposition 8.20(4) gives that the height of $\mathbb{P}(E)$ is equal to :

$$-\frac{(n+r+1)!}{2}\Big(\int_\Delta\varepsilon_r(y)\,\mathrm{d}x\,\mathrm{d}y+\int_\Delta L(y)\cdot\varepsilon_n(L(y)^{-1}x)\,\mathrm{d}x\,\mathrm{d}y\Big).\tag{8.25}$$

Let I_1 and I_2 be the two above integrals. Observe $\Delta = \bigcup_{y\in\Delta^r}(\{y\}\times L(y)\cdot\Delta^n)$. Then

$$\mathrm{vol}(\Delta) = \int_{\Delta^r}\left(\int_{L(y)\cdot\Delta^n} dx\right)\mathrm{d}y = \frac{1}{n!}\int_{\Delta^r} L(y)^n\,\mathrm{d}y,$$

$$I_1 = \int_{\Delta^r}\left(\int_{L(y)\cdot\Delta^n}\mathrm{d}x\right)\varepsilon_r(y)\,\mathrm{d}y = \frac{1}{n!}\int_{\Delta^r} L(y)^n\varepsilon_r(y)\,\mathrm{d}y,$$

since $\int_{L(y)\cdot\Delta^n}\mathrm{d}x = L(y)^n/n!$. And, for the second integral,

$$\begin{aligned} I_2 &= \int_{\Delta^r} L(y)\left(\int_{L(y)\cdot\Delta^n}\varepsilon_n(L(y)^{-1}x)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \left(\int_{\Delta^r} L(y)^{n+1}\,\mathrm{d}y\right)\cdot\left(\int_{\Delta^n}\varepsilon_n(x)\,\mathrm{d}x\right) = -\frac{2\,\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)}{(n+1)!}\int_{\Delta^r} L(y)^{n+1}\,\mathrm{d}y. \end{aligned}$$

since $\int_{L(y)\cdot\Delta^n}\varepsilon_n(L(y)^{-1}x)\,\mathrm{d}x = L(y)^n\int_{\Delta^n}\varepsilon_n(x)\,\mathrm{d}x$ and

$$\int_{\Delta^n}\varepsilon_n(x)\,\mathrm{d}x = \frac{-1}{(n+1)!}\cdot\sum_{h=1}^{n}\sum_{j=1}^{h}\frac{1}{j} = -\frac{2\,\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)}{(n+1)!}.$$

The expression for $\mathrm{vol}(\Delta)$ gives the formula for the degree. Carrying the expressions of I_1 and I_2 in (8.25) concludes the proof of Lemma 8.22. $\square$

Proposition 8.26. *In the above setting, one has :*

$$\deg_{\mathcal{O}_{\mathbb{P}(E)}(1)}(\mathbb{P}(E)) = \sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n}} a_0^{i_0}\dots a_r^{i_r}$$

$$\mathrm{h}_{\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}}(\mathbb{P}(E)) = \left(\sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n+1}} a_0^{i_0}\dots a_r^{i_r}\right)\mathrm{h}_{\overline{\mathcal{O}_{\mathbb{P}^n}(1)}}(\mathbb{P}^n) + \sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n}} a_0^{i_0}\dots a_r^{i_r}A_{n,r}(i_0,\dots,i_r),$$

where $A_{n,r}(i_0,\dots,i_r) = \sum_{m=0}^{r}(i_m+1)\sum_{j=i_m+2}^{n+r+1}\frac{1}{2j}$. *In particular, the height of* $\mathbb{P}(E)$ *is a positive rational number.*

Proof. To prove this result it suffices to compute the two integrals appearing in Lemma 8.22. However

$$L(y) = a_0 + \sum_{\ell=1}^{r}(a_\ell - a_0)y_\ell = a_0y_0 + \dots + a_ry_r,$$

with $y_0 = 1 - y_1 - \dots - y_r$, and therefore

$$L(y)^n = \sum_{\substack{\alpha\in\mathbb{N}^{r+1}\\ |\alpha|=n}}\binom{n}{\alpha_0,\dots,\alpha_r}\prod_{\ell=0}^{r}(a_\ell y_\ell)^{\alpha_\ell}$$

and similarly for $L(y)^{n+1}$. Now, Corollary 7.19 gives :

$$\int_{\Delta^r} y_0^{\alpha_0}y_1^{\alpha_1}\dots y_r^{\alpha_r}\,\mathrm{d}y = \frac{\alpha_0!\dots\alpha_r!}{(|\alpha|+r)!},$$

$$\int_{\Delta^r} y_0^{\alpha_0}y_1^{\alpha_1}\dots y_r^{\alpha_r}\log(y_j)\,\mathrm{d}y = -\frac{\alpha_0!\dots\alpha_r!}{(|\alpha|+r)!}\sum_{\ell=\alpha_j+1}^{|\alpha|+r}\frac{1}{\ell},$$

which, combined with the above expression for $L(y)^n$ and $L(y)^{n+1}$, gives

$$\int_{\Delta^r} L(y)^n dy = \sum_{\substack{\alpha\in\mathbb{N}^{r+1}\\ |\alpha|=n}} \frac{n!}{(n+r)!}\prod_{\ell=0}^{r} a_\ell^{\alpha_\ell} = \sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n}} \prod_{\ell=0}^{r} a_\ell^{i_\ell}$$

$$\int_{\Delta^r} L(y)^{n+1} dy = \sum_{\substack{\alpha\in\mathbb{N}^{r+1}\\ |\alpha|=n+1}} \frac{(n+1)!}{(n+1+r)!}\prod_{\ell=0}^{r} a_\ell^{\alpha_\ell} = \sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n+1}} \prod_{\ell=0}^{r} a_\ell^{i_\ell}$$

$$\begin{aligned}\int_{\Delta^r} L(y)^n \varepsilon_r(y) dy &= -\sum_{m=0}^{r}\sum_{\substack{\alpha\in\mathbb{N}^{r+1}\\ |\alpha|=n}} \frac{n!(\alpha_m+1)}{(n+1+r)!}\Big(\prod_{\ell=0}^{r} a_\ell^{\alpha_\ell}\Big)\sum_{\ell=\alpha_m+2}^{n+1+r}\frac{1}{\ell}\\ &= -\frac{n!}{(n+1+r)!}\sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n}}\Big(\prod_{\ell=0}^{r} a_\ell^{i_\ell}\Big)\sum_{m=0}^{r}(i_m+1)\sum_{\ell=i_m+2}^{n+1+r}\frac{1}{\ell}\\ &= -\frac{2\,n!}{(n+1+r)!}\sum_{\substack{i_0,\dots,i_r\in\mathbb{N}\\ i_0+\dots+i_r=n}}\Big(\prod_{\ell=0}^{r} a_\ell^{i_\ell}\Big)A_{n,r}(i_0,\dots,i_r).\end{aligned}$$

The statement follows from these expressions together with Lemma 8.22. □

Remark 8.27. We check $A_{1,1}(0,1) = A_{1,1}(1,0) = 3/4$. Let $b \geq 0$ and let $\overline{\mathcal{O}_{\mathbb{F}_b}(1)}$ the adelic line bundle on $\mathbb{F}_b$ associated to $a_0 = 1$ and $a_1 = b+1$. Putting $n = r = 1$, $a_0 = 1$ and $a_1 = b+1$ in Proposition 8.26, we recover the expression for the height of Hirzebruch surfaces established in [Mou06]: $\mathrm{h}_{\overline{\mathcal{O}_{\mathbb{F}_b}(1)}}(\mathbb{F}_b) = \frac{1}{2}b^2 + \frac{9}{4}b + 3$.

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