COMPOSITIO MATHEMATICA

JOSÉ IGNACIO BURGOS

A C^∞ logarithmic Dolbeault complex

Compositio Mathematica, tome 92, nº 1 (1994), p. 61-86.
<http://www.numdam.org/item?id=CM_1994__92_1_61_0>

Compositio Mathematica **92**: 61–86, 1994.

A C^∞ logarithmic Dolbeault complex

JOSÉ IGNACIO BURGOS*

Departamento de Algebra y Geometría, Facultad de Matemáticas, Universidad de Barcelona, 08071 Barcelona, Spain

Received 23 September 1992; accepted in final form 10 April 1993

Introduction

Let X be a complex manifold and Y a divisor with normal crossings. Today the utility of the holomorphic logarithmic complex, $\Omega_X^*(\log Y)$, in the study of the cohomology of quasi-projective varieties is well known. For example, it is a key ingredient of Deligne's construction of a mixed Hodge structure in the cohomology of $X-Y$. Of great importance in this construction are the weight and Hodge filtrations, noted W and F respectively.

One can construct acyclic bifiltered resolutions of $(\Omega_X^*(\log Y), W, F)$ by standard methods (cf. [D]); but it should be noted that, in general, we cannot except a real structure in these resolutions because $\Omega_X^*(\log Y)$ has none.

In [N] Navarro Aznar introduced an acyclic bifiltered resolution of $(\Omega_X^*(\log Y), W, F)$, the analytic logarithmic Dolbeault complex: $(\mathscr{A}_X^*(\log Y), W, F)$, which has a real structure. This complex is an algebra over the sheaf of real analytic functions, $\mathscr{A}_X$.

In [H-P] Harris and Phong constructed a resolution of $\mathscr{O}_X = \Omega_X^0(\log Y)$ by means of C^∞ functions over $X-Y$, imposing logarithmic growth conditions along Y. This suggested that it is possible to construct a bifiltered resolution, analogous to $(\mathscr{A}_X^*(\log Y), W, F)$ using C^∞ functions.

In this paper we introduce a C^∞ logarithmic Dolbeault complex, which we shall denote $(\mathscr{E}_X^*(\log Y), W, F)$, and we prove that it is also a bifiltered resolution of $\Omega_X^*(\log Y)$. To define this complex we do not use growth conditions, but give a definition similar to those of $\mathscr{A}_X^*(\log Y)$ substituting differentiable for real analytic. This complex also has a real structure and it is a bifiltered algebra over the sheaf of differentiable functions, $\mathscr{E}_X$. So it is fine, and hence acyclic. Furthermore, there is a natural inclusion

$$\mathscr{A}_X^*(\log Y) \to \mathscr{E}_X^*(\log Y)$$

which is a filtered quasi-isomorphism.

*Partially supported by CICYT PS90-0069.

To prove that $(\mathscr{E}_X^*(\log Y), W, F)$ is a bifiltered resolution of $(\Omega_X^*(\log Y), W, F)$ we follow the proof given in [N] of the corresponding fact for $(\mathscr{A}_X^*(\log Y), W, F)$. However, some facts about real analytic functions are not true for C^∞ functions. To avoid these technical complications we shall use Whitney's extension Theorem and some properties of flat functions.

One reason for studying this kind of complexes is that they are a natural place for Green functions to live. Green functions play a fundamental role in the theory of arithmetic intersections (cf. for example [L] or [G-S]). In particular, the usual Green functions belong to these complexes: to $\mathscr{A}_X^*(\log Y)$ if they are real analytic, and to $\mathscr{E}_X^*(\log Y)$ if they are differentiable. Moreover we shall show that belonging to these complexes is a good way to express the singularities usually required for Green functions and we shall give a proof of the existence of Green functions based on the properties of these complexes.

The paper is organized as follows: In section 1 we recall the definitions of $(\Omega_X^*(\log Y), W, F)$ and $(\mathscr{A}_X^*(\log Y), W, F)$. In section 2 we define $(\mathscr{E}_X^*(\log Y), W, F)$ and we start the proof of the main theorem. In section 3 we recall Whitney's extension Theorem and the definition and some properties of flat functions. In section 4 we relate the flat functions with the C^∞ logarithmic Dolbeault complex. In section 5 we complete the proof of the main theorem. Finally in section 6 we discuss the relationship between Green functions and the logarithmic Dolbeault complexes.

I am deeply indebted to V. Navarro Aznar for his guidance during the preparation of this work.

1. Preliminaries

Let X be a complex manifold. Let $Y \subset X$ be a divisor with normal crossings (in the sequel DNC). We shall write $V = X - Y$ and denote the inclusion by $j: V \hookrightarrow X$. Let $x \in X$. We shall say that U is a coordinate neighbourhood of x adapted to Y if x has coordinates $(0, \ldots, 0)$ and $Y \cap U$ is defined by the equation $z_{i_1} \cdots z_{i_M} = 0$. In particular if $x \notin Y$, then $Y \cap U = \varnothing$. When U and Y are fixed we shall write $I = \{i_1, \ldots, i_M\}$.

Let $\mathscr{O}_X$ be the structural sheaf of holomorphic functions and let Ω_X^* be the $\mathscr{O}_X$-module of holomorphic forms. Let us recall the definition of the holomorphic logarithmic complex, denoted $\Omega_X^*(\log Y)$ (cf. [D]). The sheaf $\Omega_X^*(\log Y)$ is the sub-$\mathscr{O}_X$-algebra of $j_*\Omega_V^*$ generated in each coordinate neighbourhood adapted to Y by the sections dz_i/z_i, for $i \in I$, and dz_i, for $i \notin I$.

There are two filtrations defined on $\Omega_X^*(\log Y)$. The Hodge filtration is the decreasing filtration defined by:

$$F^p\Omega_X^*(\log Y) = \bigoplus_{p' \geqslant p} \Omega_X^{p'}(\log Y).$$

The weight filtration is the multiplicative increasing filtration obtained by giving weight 0 to the sections of Ω_X^* and weight 1 to the sections dz_i/z_i, for $i \in I$.

Given a complex K^*, let $\tau_{\leqslant}$ be the canonical filtration:

$$\tau_{\leqslant p}(K)^n = \begin{cases} K^n, & \text{if } n < p, \\ \operatorname{Ker} d, & \text{if } n = p, \\ 0, & \text{if } n > p. \end{cases}$$

Deligne has proved in [D] the following theorem:

THEOREM 1.1. *Let X be a proper smooth algebraic variety over $\mathbb{C}$. There is a filtered quasi-isomorphism*

$$\alpha\colon (Rj_*\mathbb{R}, \tau_{\leqslant}) \otimes \mathbb{C} \to (\Omega_X^*(\log\ Y), W).$$

Moreover, the triple

$$((Rj_*\mathbb{R}, \tau_{\leqslant}), (\Omega_X^*(\log\ Y), W, F), \alpha)$$

is an $\mathbb{R}$-cohomological mixed Hodge complex which induces in $H^*(V, \mathbb{R})$ an $\mathbb{R}$-mixed Hodge structure functorial on V. This mixed Hodge structure is independent on X.

We refer the reader to [D] for the definitions and properties of mixed Hodge structures (MHS), mixed Hodge complexes (MHC) and cohomological mixed Hodge complexes (CMHC).

Remark. Actually, in [D] a stronger theorem is proven involving the rational and integer structures of $H^*(V)$. Nevertheless in this paper we shall deal only with the real structure.

We shall denote by $\mathscr{A}_{X,\mathbb{R}}$ (resp. $\mathscr{E}_{X,\mathbb{R}}$) the sheaf of real analytic functions (resp. real C^∞ functions) over X, by $\mathscr{A}_{X,\mathbb{R}}^*$ (resp. $\mathscr{E}_{X,\mathbb{R}}^*$) the $\mathscr{A}_{X,\mathbb{R}}$-algebra (resp. $\mathscr{E}_{X,\mathbb{R}}$-algebra) of differential forms. We shall write $\mathscr{A}_X = \mathscr{A}_{X,\mathbb{R}} \otimes \mathbb{C}$ and $\mathscr{A}_X^* = \mathscr{A}_{X,\mathbb{R}}^* \otimes \mathbb{C}$. (resp. $\mathscr{E}_X = \mathscr{E}_{X,\mathbb{R}} \otimes \mathbb{C}$ and $\mathscr{E}_X^* = \mathrm{E}_{X,\mathbb{R}}^* \otimes \mathbb{C}$.) The complex structure of X induces bigradings:

$$\mathscr{A}_X^n = \bigoplus_{p+q=n} \mathscr{A}_X^{p,q}$$

and

$$\mathscr{E}_X^n = \bigoplus_{p+q=n} \mathscr{E}_X^{p,q}.$$

An example of bifiltered acyclic resolution of $\Omega_X^*(\log Y)$ is the following ([D]): Let K^* be the simple complex associated to the double complex formed by $\Omega_X^*(\log Y)\otimes_{\mathcal{O}}\mathscr{E}^{0,*}$. This complex is filtered by the subcomplexes $F^p(\Omega_X^*(\log Y))\otimes\mathscr{E}^{0,*}$ and $W_n(\Omega_X^*(\log Y))\otimes\mathscr{E}^{0,*}$. The sheaves $\mathrm{Gr}_F\,\mathrm{Gr}^W(K^n)$ are $\mathscr{E}_X$-modules, hence fine and therefore acyclic for the functor $\Gamma(X,\cdot)$. Using the fact that $\mathscr{E}_X$ is a flat $\mathcal{O}_X$-module ([M]) one can prove that

$$(\Omega_X^*(\log Y), W, F)\to(K^*, W, F)$$

is a bifiltered quasi-isomorphism. This construction is not symmetrical under conjugation. Hence this resolution does not have a real structure.

In [N] Navarro Aznar introduced the analytic logarithmic Dolbeault complex, denoted $\mathscr{A}_X^*(\log Y)$, of which we recall the definition. The sheaf $\mathscr{A}_{X,\mathbb{R}}^*(\log Y)$ is the sub-$\mathscr{A}_{X,\mathbb{R}}$-algebra of $j_*\mathscr{A}_{V,\mathbb{R}}^*$ generated in each coordinate neighbourhood U adapted to Y by the sections

$$\log z_i\bar{z}_i,\quad \mathrm{Re}\,\frac{dz_i}{z_i},\quad \mathrm{Im}\,\frac{dz_i}{z_i},\quad \text{for } i\in I, \text{ and}$$

$$\mathrm{Re}\,dz_i,\quad \mathrm{Im}\,dz_i,\quad \text{for } i\notin I.$$

The weight filtration of this complex, also noted W, is the multiplicative increasing filtration obtained by assigning weight 0 to the sections of $\mathscr{A}_{X,\mathbb{R}}^*$ and weight 1 to the sections

$$\log z_i\bar{z}_i,\quad \mathrm{Re}\,\frac{dz_i}{z_i},\quad \mathrm{Im}\,\frac{dz_i}{z_i},\quad \text{for } i\in I.$$

Consider the sheaf $\mathscr{A}_X^*(\log Y)=\mathscr{A}_{X,\mathbb{R}}^*(\log Y)\otimes\mathbb{C}$. It has a weight filtration induced by the weight filtration of $\mathscr{A}_{X,\mathbb{R}}^*(\log Y)$ and a bigrading induced by the bigrading of $j_*\mathscr{A}_V^*$:

$$\mathscr{A}_X^*(\log Y)=\bigoplus_{p+q=n}\mathscr{A}_X^{p,q}(\log Y)$$

where $\mathscr{A}_X^{p,q}(\log Y)=A_X^*(\log Y)\cap j_*\mathscr{A}_V^{p,q}$.

The Hodge filtration of $\mathscr{A}_X^*(\log Y)$ is defined by

$$F^p\mathscr{A}_X^*(\log Y)=\bigoplus_{p'\geqslant p}\mathscr{A}_X^{p',q}(\log Y).$$

It follows easily from the definitions that the inclusion

$$(\Omega_X^*(\log Y), W, F) \hookrightarrow (\mathscr{A}_X^*(\log Y), W, F)$$

is a bifiltered morphism.

In [N], the following result is proven:

THEOREM 1.2. (i) *There is a filtered quasi-isomorphism*

$$\beta\colon (Rj_*\mathbb{R}, \tau_{\leqslant}) \to (\mathscr{A}_{X,\mathbb{R}}^*(log\ Y), W).$$

(ii) *The inclusion*

$$\iota\colon (\Omega_X^*(\log Y), W, F) \hookrightarrow (\mathscr{A}_X^*(\log Y), W, F)$$

is a bifiltered quasi-isomorphism.

(iii) *The quasi-isomorphisms* ι, α *and* β *are compatible, i.e.* $\iota \circ \alpha = \beta \otimes \mathbb{C}$.

As a consequence of Theorem 1.2 we have

COROLLARY 1.3. *Let* X *be a smooth proper algebraic variety over* $\mathbb{C}$ *and let* Y *be a DNC. Then the triple*

$$((\mathscr{A}_{X,\mathbb{R}}^*(log\ Y), W), (\mathscr{A}_X^*(\log Y), W, F), \mathrm{Id})$$

is a $\mathbb{R}$-*CMHC which induces in* $H^*(V, \mathbb{R})$ *the* $\mathbb{R}$-*MHS given by Theorem* 1.1.

2. The C^∞ logarithmic Dolbeault complex

Throughout this section we shall use the notations of section 1.

Let us consider the sheaves

$$\mathscr{P}_{X,\mathbb{R}}^*(\log Y) := \mathscr{E}_{X,\mathbb{R}} \otimes_{\mathscr{A}_{X,\mathbb{R}}} \mathscr{A}_{X,\mathbb{R}}^*(\log Y)$$

and

$$\mathscr{P}_X^*(\log Y) := \mathscr{E}_X \otimes_{\mathscr{A}_X} \mathscr{A}_X^*(\log Y).$$

There is a natural morphism

$$\mu\colon \mathscr{P}_X^*(\log Y) \to j_*\mathscr{E}_V^*$$

given by multiplication: $\mu(f \otimes \omega) = f \cdot \omega$, for $\omega \in \mathscr{A}_X^*(\log Y)$ and $f \in \mathscr{E}_X$.

The C^∞ logarithmic Dolbeault complex, noted $\mathscr{E}_X^*(\log Y)$, is defined as the image of μ:

$$\mathscr{E}_X^*(\log\ Y) := \mu(\mathscr{P}_X^*(\log\ Y)).$$

This complex has a real structure given by

$$\mathscr{E}_{X,\mathbb{R}}^*(\log\ Y) := \mu(\mathscr{P}_{X,\mathbb{R}}^*(\log Y)).$$

The weight filtration of $\mathscr{A}_{X,\mathbb{R}}^*(\log Y)$ tensored with the trivial filtration of $\mathscr{E}_{X,\mathbb{R}}$ defines a weight filtration of $\mathscr{P}_{X,\mathbb{R}}^*(\log Y)$. The weight filtration of $\mathscr{E}_{X,\mathbb{R}}^*(\log Y)$ is defined by

$$W_n \mathscr{E}_X^*(\log\ Y) := \mu(W_n \mathscr{P}_X^*(\log\ Y)).$$

The complexes $\mathscr{P}_X^*(\log Y)$ and $j_* \mathscr{E}_V^*$ have bigradings induced by the complex structure of X and the morphism μ is a bigraded morphism. Hence the complex $\mathscr{E}_X^*(\log Y)$ has a natural bigrading:

$$\mathscr{E}_X^*(\log\ Y) = \bigoplus_{p+q=n} \mathscr{E}_X^{p,q}(\log Y),$$

where

$$\mathscr{E}_X^{p,q}(\log\ Y) = \mathscr{E}_X^*(\log\ Y) \cap j_* \mathscr{E}_V^{p,q} = \mu(\mathscr{P}_X^{p,q}(\log\ Y)).$$

The Hodge filtration of $\mathscr{E}_X^*(\log Y)$ is defined by

$$F^p \mathscr{E}_X^*(\log\ Y) = \bigoplus_{p' \geqslant p} \mathscr{E}_X^{p',q}(\log\ Y).$$

The sheaves $\mathrm{Gr}^W \mathscr{E}_X^*(\log Y)$ are acyclic because they are $\mathscr{E}_X$-modules, hence fine. We have the following diagram of bifiltered complexes and bifiltered morphisms

$$\begin{array}{ccc} (\Omega_X^*(\log\ Y),\ W,\ F) & \longrightarrow & (\mathscr{A}_X^*(\log\ Y),\ W,\ F) \\ \downarrow & \swarrow & \\ (\mathscr{E}_X^*(\log\ Y),\ W,\ F) & & \end{array}$$

where the upper arrow is a bifiltered quasi-isomorphism.

The main result of this paper is:

THEOREM 2.1. *The inclusion*

$$(\Omega_X^*(\log\ Y), W, F) \hookrightarrow (\mathscr{E}_X^*(\log\ Y), W, F)$$

is a bifiltered quasi-isomorphism.

Remark. By the notations it may seem that Theorem 2.1 contradicts [N, (8.1)]. However it should be noted that the sheaf called $\mathscr{E}_X^*(\log Y)$ in [N] is called here $\mathscr{P}_X^*(\log Y)$. And, with our notations, the morphism $\mu: \mathscr{P}_X^*(\log Y) \to \mathscr{E}_X^*(\log Y)$ is not an isomorphism (see Corollary 4.2 below).

As a consequence of Theorem 2.1 the morphism

$$(\mathscr{A}_X^*(\log\ Y), W, F) \to (\mathscr{E}_X^*(\log\ Y), W, F)$$

is a bifiltered quasi-isomorphism and, $\mathbb{C}$ being a faithfully flat $\mathbb{R}$-module,

$$(\mathscr{A}_{X,\mathbb{R}}^*(\log\ Y), W) \to (\mathscr{E}_{X,\mathbb{R}}^*(\log\ Y), W)$$

is a filtered quasi-isomorphism. Thus, we have

COROLLARY 2.2. *Let X be a smooth proper algebraic variety over $\mathbb{C}$ and let Y be a DNC. Then the triple*

$$((\mathscr{E}_{X,\mathbb{R}}^*(\log\ Y), W), (\mathscr{E}_X^*(\log\ Y), W, F), \mathrm{Id})$$

is a $\mathbb{R}$-CMHC which induces in $H^*(V, \mathbb{R})$ the $\mathbb{R}$-MHS given by Theorem 1.1.

In the rest of this section, in order to prove Theorem 2.1, we shall follow the proof of part (ii) of Theorem 1.2 given in [N] and point out where some modifications are needed. The result is that Theorem 2.1 is a consequence of two key lemmas whose proof will be delayed until section 5.

By definition of bifiltered quasi-isomorphism, Theorem 2.1 is equivalent to

PROPOSITION 2.3. *The sequence*

$$0 \to W_n\Omega_X^p(\log Y) \xrightarrow{i} W_n\mathscr{E}_X^{p,0}(\log Y) \xrightarrow{\bar\partial} W_n\mathscr{E}_X^{p,1}(\log Y) \xrightarrow{\bar\partial} \cdots$$

is an exact sequence of sheaves.

Proof. Let $x \in X$. Let U be a coordinate neighbourhood of x adapted to Y. Put $I = \{i_1, \ldots, i_M\}$ as in section 1. We shall prove the exactness on stalks.

Let n, p, $q \geqslant 0$. For each $J \subset I$ we denote by $W_{n,J}^{p,q}$ the intersection of

$W_n\mathscr{E}_X^{p,q}(\log Y)_x$ with the algebra generated by

$$\frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i}, \log z_i\bar{z}_i, \quad \text{for } i\in J,$$

$$\frac{dz_i}{z_i}, d\bar{z}_i, \quad \text{for } i\notin J,\ i\in I \text{ and}$$

$$dz_i, d\bar{z}_i, \quad \text{for } i\notin I.$$

If there is no danger of confusion we shall omit the superindexes p, q. Let $W_{n,J,k}$ be the subset of $W_{n,J}$ composed by the elements of $W_{n,J}$ such that, for at least one $m\in J$, their weight on $d\bar{z}_m/\bar{z}_m$ and $\log z_m\bar{z}_m$ is less than or equal to k.

One has the following relations:

$$\begin{aligned} W_n &= W_{n,I}, \\ W_{n,J} &= \bigcup_{k\geq 0} W_{n,J,k}, \\ W_{n,J,0} &= \bigcup_{K\subsetneq J} W_{n,K} \end{aligned} \tag{1}$$

Let $\omega\in W_n\mathscr{E}^{p,q}(\log Y)_x$ be such that $\bar{\partial}\omega = 0$. We need to prove that $\omega = \bar{\partial}\eta$ with $\eta\in W_n\mathscr{E}^{p,q-1}(\log Y)_x$. There is $J\subset I$ and $k\in\mathbb{Z}$ such that $\omega\in W_{n,J,k}$. We shall make the proof by induction over k and over the cardinal of J. If $J=\varnothing$ the result follows from the next lemma.

LEMMA 2.4. *The sequence*

$$0\to W_n\Omega_X^p(\log Y)_x \xrightarrow{i} W_{n,\varnothing}^{p,0} \xrightarrow{\bar{\partial}} W_{n,\varnothing}^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

is exact.

Proof. By definition one has

$$W_{n,\varnothing}^{p,q} = W_n\Omega_x^p(\log Y)_x \otimes \mathscr{E}_{X,x}^{0,q}.$$

The exactness of this sequence has already been discussed after Theorem 1.1.

Let us continue the proof of Proposition 2.3. After Lemma 2.4 and the relations (1) it is enough to prove that, if $k>0$, then there exists an element $\eta\in W_n\mathscr{E}_X^{p,q-1}(\log Y)_x$ such that $\omega-\bar{\partial}\eta\in W_{n,J,k-1}$.

Assume that $1\in J$ and that the weight of ω on $d\bar{z}_1/\bar{z}_1$ and $\log z_1\bar{z}_1$ is less than or equal to k. For simplicity we shall write $\lambda_1 = \log z_1\bar{z}_1$. We have a decomposition

$$\omega = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \gamma \wedge \lambda_1^{k-1} \frac{d\bar{z}_1}{\bar{z}_1} + \rho,$$

where α, β, $\gamma \in W_{n-k,J-\{1\}}$ do not contain $d\bar{z}_1$ and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$. We must show that, adding to ω elements of $\bar{\partial} W_n$, we can eliminate the first three terms.

For the first step we have that $(1/k)\gamma\lambda_1^k \in W_n \mathscr{E}_X^{p,q-1}(\log Y)_x$, and

$$\bar{\partial}\left(\frac{1}{k}\gamma\lambda_1^k\right) = \frac{1}{k}\bar{\partial}\gamma\lambda_1^k + (-1)^{p+q-1}\gamma \wedge \lambda_1^{k-1} \frac{d\bar{z}_1}{\bar{z}_1}.$$

Hence we can write

$$\omega = \alpha'\lambda_1^k + \beta' \wedge \lambda_1^k d\bar{z}_1 + \rho, \quad \operatorname{mod} \bar{\partial} W_n,$$

where α' and β' satisfy the same conditions that α and β.

For the next step we need the following lemma which will be proven in section 5:

LEMMA 2.5. *Let $\beta \in W_{n-k,J-\{1\}}$ be a form which does not contain $d\bar{z}_1$, then there exists a form $\varphi \in W_{n-k}$ such that*

$$\bar{\partial}(\bar{z}\varphi\lambda_1^k) = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \rho,$$

where $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$.

Using this lemma one has that

$$\omega = \alpha''\lambda_1^k + \rho', \quad \operatorname{mod} \bar{\partial} W_n,$$

where α'' and ρ' satisfy the same conditions as α and ρ respectively.

For the last step we need another lemma which will also be proved in section 5:

LEMMA 2.6. *Let $\omega = \alpha\lambda_1^k + \rho$ be a form such that $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$ and $\bar{\partial}\omega = 0$. Then $\omega \in W_{n,J,k-1}$.*

Clearly this lemma concludes the proof of Theorem 2.1.

Remarks. (a) In the case of analytic functions, Lemma 2.5 is proven in [N] solving the equation

$$\frac{\partial}{\partial \bar{z}_1}(\bar{z}_1\varphi) = \beta \tag{2}$$

integrating the power series that defines the components of β in a neighbourhood of x. In general, the equation (2) cannot be solved in the case of C^∞ functions. For example ([N]), let $\tilde{f} \in \mathbb{C}[\![z]\!]$ be a non-convergent formal power series. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a differentiable function which has $\tilde{f}$ as Taylor series at 0. This function exists by Borel's relative extension Theorem (see Theorem 3.3 below). Then the equation

$$\frac{\partial}{\partial \bar{z}}(\bar{z}g) = \frac{\partial}{\partial \bar{z}} f$$

does not have any solution: If g were a solution, then $f - \bar{z}g$ would be a holomorphic function with non-convergent Taylor series.

(b) On the other hand, in the real analytic case, Lemma 2.6 can be strengthened saying that α is actually 0. This is a consequence of the following: Let $\{f_i\}$ be a finite family of real analytic functions in a neighbourhood of x such that

$$\sum_i f_i \lambda_1^i = 0, \tag{3}$$

then, for all i, the functions f_i are zero. But this is not true in the case of C^∞ functions (cf. Corollary 4.2 below).

Roughly speaking, the idea of the proof of Lemma 2.5 and of Lemma 2.6 in the differentiable case is, first, to obtain a solution of (2) up to a flat function using Borel's relative extension Theorem; second, to prove that equation (3) implies, in the differentiable case, that the functions f_i are flat and finally, to show that the smoother property of flat functions gives us the proof.

3. Whitney functions

In this section we recall some results of the theory of Whitney functions that we shall use throughout this paper. A complete treatment of this subject, including the proofs omitted here, can be found in [M] or in [T]. The notations that we shall use differ slightly from those of these texts. In fact all the constructions given here depend only on the differentiable structure, thus they can be formulated in terms of real differentiable manifolds. We state them in a complex setting due to the use we shall give them in the rest of the paper.

The results we shall need are Borel's relative extension Theorem, Theorem 3.3 below, and Theorem 3.4 which relates the ideal of flat functions on the intersection of two analytic sets with the ideals of flat functions on each set.

Consider the space $\mathbb{C}^d$ with complex coordinates $(z_1, \ldots, z_d)$. We shall use double multi-index notation. Let $\alpha = (\alpha_1, \ldots, \alpha_d, \alpha'_1, \ldots, \alpha'_d)$, α_i, $\alpha'_i \in \mathbb{Z}_{\geqslant 0}$, then we shall note $\sigma = \{1, \ldots, d\}$,

$$|\alpha| = \sum_{i \in \sigma} \alpha_i + \alpha'_i, \quad z^\alpha = \prod_{i \in \sigma} z^{\alpha_i} \bar{z}^{\alpha'_i}, \quad \alpha! = \prod_{i \in \sigma} \alpha_i! \alpha'_i! \quad \text{and} \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial z^\alpha}.$$

Let $U \subset \mathbb{C}^d$ be an open set, and let A be a closed subset of U. The space of (complex) jets of order m over A, $J^m(A)$, is defined as the set of all sequences $F = (F^\alpha)_{|\alpha| \leqslant m}$, where the F^α are continuous complex functions over A.

The space of jets over A is defined as

$$J(A) = \varprojlim J^m(A).$$

Let $\mathscr{E}_{\mathbb{C}^d}(U) = \Gamma(U, \mathscr{E}_{\mathbb{C}^d})$ be the ring of complex C$^\infty$ functions over U. For each $m \in \mathbb{Z}_{\geqslant 0}$, there is a morphism

$$J^m : \mathscr{E}_{\mathbb{C}^d}(U) \to J^m(A)$$

defined by

$$J^m(f)^\alpha = \frac{1}{\alpha!} \partial^\alpha f \bigg|_A .$$

Taking limits, they give a morphism

$$J : \mathscr{E}_{\mathbb{C}^d}(U) \to J(A).$$

If it is necessary to precise the closed set over which the jets are defined we shall write J_A.

Let $x \in A$ and $F \in J^m(A)$. The Taylor polynomial of F, of degree m, centred at x is the polynomial

$$T^m_x F(z) = \sum_{|\alpha| \leqslant m} (z - x)^\alpha F^\alpha(x).$$

The remainder of F at x of degree m is the jet over A defined by

$$R_x^m(F) = F - J^m(T_x^m F) \in J^m(A).$$

If $F \in J(A)$ then there are obvious definitions of Taylor polynomial and remainder of F of all degrees.

The Taylor approximation Theorem implies that if $F = J^m(f)$ is the jet of a C^∞ function then it satisfies the condition:

W. For all compact $K \subset A$ then

$$(R_x^m F)^\alpha(z) = o(|z - x|^{m-|\alpha|}),$$

for $x, z \in K$ and $|\alpha| \leqslant m$, when $|z - x| \to 0$.

A jet $F \in J^m(A)$ is said to be a Whitney function of order m, denoted $F \in \mathscr{W}^m(A)$, if it satisfies the condition **W**.

The space of Whitney functions is defined as

$$\mathscr{W}(A) = \varprojlim \mathscr{W}^m(A).$$

Thus the jet of a C^∞ function is a Whitney function. The interest of the Whitney functions is given by the following theorem which says that they are exactly the image of J.

THEOREM 3.1. (Whitney's extension Theorem, see [M] or [T]) *The morphism*

$$J_A : \mathscr{E}_{\mathbb{C}^d}(U) \to \mathscr{W}(A)$$

is an epimorphism.

Given a jet $F \in J(A)$, after Theorem 3.1, to know whether it is the jet of a differentiable function we must check the conditions **W** for all m. If A is a hyperplane, using C^∞ functions over A instead of continuous functions, we can give a different definition of Whitney functions avoiding the condition **W**. In this case Whitney's extension Theorem specializes in a relative version of Borel's extension Theorem.

Let Y_1 be the hyperplane of equation $z_1 = 0$. We define the morphism

$$J_1 : \mathscr{E}_{\mathbb{C}^d}(U) \to \mathscr{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!]$$

$$f \mapsto \sum_{i,j} \frac{1}{i!j!} \left.\frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\right|_{Y_1} z_1^i \bar{z}_1^j,$$

and the morphism

$$\delta : \mathscr{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!] \to J(T_1)$$

$$\sum_{i,j} F^{i,j} z_1^i \bar{z}_1^j \mapsto \left(\frac{1}{\hat{\alpha}_1!} \partial^{\hat{\alpha}_1} F^{\alpha_1, \alpha'_1} \right)_\alpha ,$$

where, if $\alpha = (\alpha_1, \ldots, \alpha_d, \alpha'_1, \ldots, \alpha'_d)$, then $\hat{\alpha}_1 = (0, \alpha_2, \ldots, \alpha_d, 0, \alpha'_2, \ldots, \alpha'_d)$.

It follows from the definitions that δ is injective and that

$$J_{Y_1} = \delta \circ J_1.$$

Hence $\mathscr{W}(Y_1 \cap U) \subset \operatorname{Im} \delta$.

Using Taylor's approximation Theorem it is easy to show that an element of $\operatorname{Im} \delta$ satisfies the condition **W** for all m. And so $\operatorname{Im} \delta \subset \mathscr{W}(Y_1 \cap U)$. Thus we obtain

PROPOSITION 3.2. *The morphism δ is an isomorphism between $\mathscr{E}_{Y_1}(Y_1 \cap U)[\![z_1, \bar{z}_1]\!]$ and $\mathscr{W}(U \cap Y_1)$.*

In this situation Theorem 3.1 can be restated as follows:

THEOREM 3.3. (Borel's relative extension Theorem) *The morphism J_1 is an epimorphism, i.e. if*

$$F = \sum_{i,j} F^{i,j} z_1^i \bar{z}_1^j$$

is a formal power series, where the coefficients $F^{i,j}$ are complex C^∞ functions over $Y_1 \cap U$, then there exists a complex C^∞ function f over U, such that

$$\frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j} \bigg|_{Y_1} = F^{i,j}.$$

Now that we have a characterization of $\operatorname{Im} J$ let us look at $\operatorname{Ker} J$. Recall that a function f on U is said to be flat on A if $J_A(f) = 0$. The flat functions form an ideal which we denote by $\mathfrak{m}_A^\infty(U)$.

A useful property of flat functions is the following theorem by Łojasiewicz.

THEOREM 3.4. *Let A_1 and A_2 be two closed analytic subsets of U. Then*

$$\mathfrak{m}_{A_1 \cap A_2}^\infty(U) = \mathfrak{m}_{A_1}^\infty(U) + \mathfrak{m}_{A_2}^\infty(U).$$

Proof. Usually this result is formulated in other terms which we recall here.

Let $B_1 \subset B_2$ be two closed subsets of U. There is an obvious restriction morphism $\mathscr{W}(B_2) \to \mathscr{W}(B_1)$ and a commutative diagram

$$\begin{array}{ccc} \mathscr{E}_{\mathbb{C}^d}(U) & \longrightarrow & \mathscr{W}(B_2) \\ & \searrow & \downarrow \\ & & \mathscr{W}(B_1). \end{array}$$

Theorem 3.1 implies that all such restriction morphisms are epimorphisms.

Let now B_1 and B_2 be two different closed subsets. We can construct the following sequence

$$0 \to \mathscr{W}(B_1 \cup B_2) \xrightarrow{\rho} \mathscr{W}(B_1) \oplus \mathscr{W}(B_2) \xrightarrow{\pi} \mathscr{W}(B_1 \cap B_2) \to 0,$$

where $\rho(F) = (F|_{B_1}, F|_{B_2})$ and $\pi(F, G) = F|_{B_1 \cap B_2} - G|_{B_1 \cap B_2}$. It is clear that ρ is injective, π is surjective and that $\pi \circ \rho = 0$. But in general this sequence is not exact. It is said that B_1 and B_2 are regularly situated if this sequence is exact.

The usual formulation of the Łojasiewicz result is the following.

THEOREM 3.5. (Łojasiewicz, see [M]) *If A_1 and A_2 are two real analytic closed sets of U, then they are regularly situated.*

Theorem 3.5 is equivalent to Theorem 3.4 because, by Theorem 3.1,

$$\mathscr{W}(A) \cong \frac{\mathscr{E}_{\mathbb{C}^d}(U)}{\mathfrak{m}_A^\infty(U)}.$$

4. Flat functions and logarithmic singularities

Recall the notations of section 1. Let X be a complex manifold of dimension d and let Y be a DNC. Let $x \in X$. From now on we shall fix a coordinate neighbourhood U of x adapted to Y, with coordinates $(z_1, \ldots, z_d)$. If Y is defined by the equation $z_{i_1} \cdots z_{i_M} = 0$ set $I = \{i_1 \cdots i_M\}$. For shorthand let us write $\lambda_i = \log z_i \bar{z}_i$. We denote by Y_i the hyperplane of equation $z_i = 0$.

In this section we shall relate the kernel of the morphism $\mu: \mathscr{P}_X^*(\log Y) \to \mathscr{E}_X^*(\log Y)$ with the flat functions. The results we shall need in the sequel are Proposition 4.1 and Proposition 4.3.

Roughly speaking, the flat functions act as smoothers: let h be a differentiable function, singular along a closed set A, let f be a function flat on A. If the singularity of h is not "too bad", then $f \cdot h$ can be extended to a smooth function flat over A. (cf. for example [T, IV. 4.2] for a precise statement.) In particular, we have the following easy result.

PROPOSITION 4.1. *Let f be a complex C^∞ function on U, flat on Y_i, then for all $k \geqslant 0$ the function*

$$f \cdot \frac{\lambda_i^k}{P(z_i, \bar{z}_i)},$$

where $P(z_i, \bar{z}_i)$ is a monomial, can be extended to a C^∞ function flat on Y_i.

Proposition 4.1 is the reason for the morphism μ not being an isomorphism. For instance, let us consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = e^{-1/z\bar{z}}.$$

It is a function flat on 0. By Proposition 4.1 the function $f(z) \cdot \log z\bar{z}$ is a C^∞ function over $\mathbb{C}$. Thus

$$s = f \otimes \log z\bar{z} - f \cdot \log z\bar{z} \otimes 1$$

is a nonzero section of $\mathscr{P}_X^0(\log 0)$ and $\mu(s) = 0$. Generalizing this example we obtain the following result.

COROLLARY 4.2. *The ideal* Ker μ *contains the elements*

$$f \otimes \lambda_i - f \cdot \lambda_i \otimes 1,$$

where $i \in I$ and f is flat on Y_i.

Let us introduce a notation. A single multi-index of length d is an ordered set $a = (a_1, a_2, \ldots, a_d)$, with $a_i \in \mathbb{Z}_{\geqslant 0}$. The set of all single multi-indexes of length d is $\mathbb{Z}^d_{\geqslant 0}$. This is a partially ordered set: Put $b \geqslant a$ if $b_i \geqslant a_i$, $\forall i$. For $a \in \mathbb{Z}^d_{\geqslant 0}$ we shall write

$$\lambda^a = \prod_i \lambda_i^{a_i}, \quad \text{and} \quad |a| = \sum_i a_i.$$

We define the support of $a \in \mathbb{Z}^d_{\geqslant 0}$ as

$$\operatorname{supp}(a) = \{i \mid a_i \neq 0\}$$

let $\Lambda \subset \mathbb{Z}^d_{\geqslant 0}$ be a finite subset. We define the support of Λ as

$$\operatorname{supp}(\Lambda) = \bigcup_{a \in \Lambda} \operatorname{supp}(a).$$

If $J = \{j_1, \dots, j_N\}$ is a subset of I we put

$$Y_J = \bigcap_{j \in J} Y_j.$$

If $a \in \Lambda$ let us write

$$Y_{a,\Lambda} = \bigcap_{\substack{b \in \Lambda \\ b \geqslant a}} Y_{\mathrm{supp}(b)}.$$

Note that if $\Lambda' \subset \Lambda$ then $Y_{a,\Lambda} \subset Y_{a,\Lambda'}$, that $Y_{a,\Lambda} \subset Y_{\mathrm{supp}(a)}$ and that if a is maximal in Λ then $Y_{a,\Lambda} = Y_{\mathrm{supp}(a)}$.

PROPOSITION 4.3. *Let $\Lambda \subset \mathbb{Z}^d_{\geqslant 0}$ be a finite set of multi-indexes. Let $\{f_a\}_{a \in \Lambda}$ be a family of C^∞ functions on U. Then the equation*

$$\sum_{a \in \Lambda} f_a \lambda^a = 0$$

implies that the functions f_a are flat on $Y_{a,\Lambda}$. In particular, if a is maximal in Λ then f_a is flat on $Y_{\mathrm{supp}(a)}$.

Proof. Let us prove first the case $\#\mathrm{supp}(\Lambda) = 1$. We can assume that $\mathrm{supp}(\Lambda) = \{1\}$. In this case we have to prove that, if $f_k \in \mathscr{E}_X(U)$ and

$$\sum_k f_k(z_1, \dots, z_d) \cdot \lambda_1^k = 0, \tag{4}$$

then the functions f_k are flat on Y_1.

Let $y = (0, x_2, \dots, x_d)$ be a point of $Y_1 \cap U$. Consider the functions

$$h_k(z) = f_k(z, x_2, \dots, x_d).$$

We shall write $r^2 = z_1 \bar{z}_1$. If we see that, for all n, $h_k = O(r^n)$, i.e. that $h(z)/r^n$ is bounded when $r \to 0$, then the functions f_k and all their derivatives with respect to z_1 and $\bar{z}_1$ will be zero in y. Varying the point y, we shall obtain that f_k is flat on Y_1.

Let $n, l \geqslant 0$. Suppose that, for $k > l$ one has $h_k = O(r^n)$ and, for $k \leqslant l$ one has $h_k = O(r^{n-1})$. This is true for $n = 1$ and l large enough. Making the quotient of (4) by $r^{n-1} \log^l r^2$ we obtain

$$0 = h_0 \frac{1}{r^{n-1} \log^l r^2} + \cdots + h_{l-1} \frac{1}{r^{n-1} \log r^2} + h_l \frac{1}{r^{n-1}} + h_{l+1} \frac{\log r^2}{r^{n-1}} + \cdots.$$

In this equation all terms tend to zero when z tends to zero except perhaps the l-th term. Therefore it also tends to zero, i.e.

$$\lim_{z \to 0} h_l \frac{1}{r^{n-1}} = 0.$$

Thus $h_l = O(r^n)$. By inverse induction over l and induction over n we have that $h_l = O(r^n)$ for all n and all l.

Suppose now that $\#\mathrm{supp}(\Lambda) > 1$. We shall prove first, by induction over $\#\mathrm{supp}(\Lambda)$, that the functions f_a with a maximal are flat on $Y_{\mathrm{supp}(a)}$. Let $a' \in \Lambda$ be a maximal element and assume that $1 \in \mathrm{supp}(a')$.

Let us write

$$\sum_a f_a \lambda^a = \sum_k \left(\sum_b f_{k,b} \lambda^b \right) \lambda_1^k = 0.$$

Put $V = U - \bigcup_{i \in I - \{1\}} Y_i$. For each k, the functions

$$\sum_b f_{k,b} \lambda^b$$

are C^∞ functions on V. By the case $\#\mathrm{supp}(\Lambda) = 1$, they are flat on $Y_1 \cap V$. Hence, for all $p, q \in \mathbb{Z}_{\geqslant 0}$

$$\left. \frac{\partial^{p+q}}{\partial z_1^p \partial \bar{z}_1^q} \sum_b f_{k,b} \lambda^b \right|_{Y_1} = 0.$$

By induction hypothesis, for b maximal, the functions

$$\left. \frac{\partial^{p+q}}{\partial z_1^p \partial \bar{z}_1^q} f_{k,b} \right|_{Y_1}$$

are flat on $Y_{\mathrm{supp}(b)}$. If $a' = (k', b')$ is maximal in Λ, then b' is maximal in the set $\{b \mid (k', b) \in \Lambda\}$. Therefore the function $f_{a'}$ is flat on $Y_{\mathrm{supp}(a')} = Y_{\mathrm{supp}(b')} \cap Y_1$.

Finally let us prove the general statement by induction over $\max\{|a| \mid a \in \Lambda\}$.

Set $\Lambda' = \{a \in \Lambda \mid a$ is not maximal$\}$. Then $\max\{|a| \mid a \in \Lambda'\} < \max\{|a| \mid a \in \Lambda\}$. For each $a \in \Lambda$ maximal, f_a is flat on $Y_{\mathrm{supp}(a)}$. By Theorem 3.4 we can write

$$f_a = \sum_{i \in \mathrm{supp}(a)} f_{a,i}$$

where $f_{a,i}$ is flat on Y_i. Using Proposition 4.1 and reorganizing terms we obtain

$$\sum_{a\in\Lambda} f_a\lambda^a = \sum_{a\in\Lambda'} g_a\lambda^a = 0.$$

By construction $f_a - g_a$ is flat on $Y_{a,\Lambda}$ and by induction hypothesis g_a is flat on $Y_{a,\Lambda'} \supset Y_{a,\Lambda}$. Hence f_a is flat on $Y_{a,\Lambda}$.

Now we can give a precise characterization of $\operatorname{Ker}\mu$.

PROPOSITION 4.4. *The ideal* $\operatorname{Ker}\mu$ *is generated by the elements*

$$f\otimes\lambda_i - f\cdot\lambda_i\otimes 1,$$

where $i\in I$ *and* f *is flat on* Y_i.

Proof. We shall denote by $\mathscr{I}$ the ideal generated by the elements

$$f\otimes\lambda_i - f\cdot\lambda_i\otimes 1,$$

with $i\in I$ and f flat on Y_i.

Let $\eta\in\operatorname{Ker}\mu$. We can assume that $\eta\in\mathscr{P}^0_X(\log Y)_x$. Let us write

$$\eta = \sum_{a\in\Lambda} g_a\otimes\lambda^a.$$

We shall do the proof by induction over the weight w of η: $w(\eta) = \max\{|a| \mid a\in\Lambda\}$. If $w = 0$ then $\eta = 0$ because $\mu(\eta) = \eta$.

If $w > 0$ it is enough to show that adding elements of $\mathscr{I}$ we can lower the weight of η. Let $a\in\Lambda$ with $|a| = w$. Then a is a maximal element of Λ. Hence, by Proposition 4.3, g_a is flat on $Y_{\operatorname{supp}(a)}$. Thus we can write

$$g_a = \sum_{i\in\operatorname{supp}(a)} g_{a,i},$$

where $g_{a,i}$ is flat on Y_i. Let $\hat{a}_i = (a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_d)$. Then

$$\begin{aligned} g_a\otimes\lambda^a &= \sum_{i\in\operatorname{supp}(a)} g_{a,i}\otimes\lambda^a \\ &= \sum_{i\in\operatorname{supp}(a)} g_{a,i}\cdot\lambda^{a_i}\otimes\lambda^{\hat{a}_i}, \quad \operatorname{mod}\mathscr{I}. \end{aligned}$$

Repeating this process for each a with $|a| = w$ we have the inductive step.

5. Proof of Lemmas 2.5 and 2.6

In this section we shall end the proof of Theorem 2.1. We follow the notations of section 2 and of section 4. We also use the following notation:

$$\xi_i = \frac{dz_i}{z_i}, \quad \text{for } i \in I$$

$$\xi_i = dz_i, \quad \text{for } i \notin I.$$

If $L = (\{l_1, \ldots, l_p\}, \{l'_1, \ldots, l'_q\})$ is a pair of ordered subsets of $[1, d]$ we shall note

$$\xi^L = \xi_{l_1} \wedge \cdots \wedge \xi_{l_p} \wedge \bar{\xi}_{l'_1} \wedge \cdots \wedge \bar{\xi}_{l'_q}.$$

Let us recall Lemma 2.5:

LEMMA. *Let $\beta \in W^{p,q-1}_{n-k,J-\{1\}}$ be a form which does not contain $d\bar{z}_1$, then there exists a form $\varphi \in W_{n-k}$ such that*

$$\bar{\partial}(\bar{z}\varphi\lambda_1^k) = \alpha\lambda_1^k + \beta \wedge \lambda_1^k d\bar{z}_1 + \rho,$$

where $\alpha \in W_{n-k,J-\{1\}}$ does not contain $d\bar{z}_1$, and $\rho \in W_{n,J,k-1}$ has weight on $d\bar{z}_1/\bar{z}_1$ and λ_1 less than or equal to $k-1$.

Proof. We shall see first that we can solve the equation

$$\frac{\partial}{\partial \bar{z}_1}(\bar{z}_1 g) = f$$

up to a flat function.

LEMMA 5.1. *Let $f: U \to \mathbb{C}$ be a C^∞ function, then there exists a C^∞ function $g: U \to \mathbb{C}$ such that the function*

$$f - \frac{\partial}{\partial \bar{z}_1}(\bar{z}_1 g)$$

is flat on Y_1.

Proof. The jet of f on Y_1 is the formal power series

$$J_1(f) = \sum_{i,j} \frac{1}{i!j!} \left.\frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\right|_{Y_1} z_1^i \bar{z}_1^j.$$

Integrating this series with respect to $\bar{z}_1$ and dividing by $\bar{z}_1$ we get the series

$$\hat{g} = \sum_{i,j} \frac{1}{i!(j+1)!} \left.\frac{\partial^{i+j} f}{\partial z_1^i \partial \bar{z}_1^j}\right|_{Y_1} z_1^i \bar{z}_1^j.$$

By Theorem 3.3 there exists a function g on U whose jet on Y_1 is $\hat{g}$. This is the desired function.

Let us continue the proof of Lemma 2.5. Set

$$\beta = \sum_{a,L} f_{a,L} \lambda^a \xi^L.$$

Applying Lemma 5.1 to the functions $f_{a,L}$ we obtain functions $g_{a,L}$. With them we can write

$$\varphi = \sum_{a,L} g_{a,L} \lambda^a \xi^L.$$

Notice that $\varphi \in W^{p,q-1}_{n-k,J-\{1\}}$ because $\beta \in W^{p,q-1}_{n-k,J-\{1\}}$.

Let l be the degree of φ, i.e. $l = p + q - 1$. We have

$$\begin{aligned}\beta \wedge \lambda_1^k d\bar{z}^1 - (-1)^l \bar{\partial}(\bar{z}_1 \varphi \lambda_1^k) &= \left(\beta - \frac{\partial(\bar{z}_1\varphi)}{\partial \bar{z}_1}\right) \wedge \lambda_1^k d\bar{z}_1 \\ &\quad - \left((-1)^l \bar{\partial}(\bar{z}_1 \varphi) - \frac{\partial(\bar{z}_1\varphi)}{\partial \bar{z}_1} \wedge d\bar{z}_1\right) \lambda_1^k \\ &\quad - k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1.\end{aligned}$$

By construction $\beta - \partial(\bar{z}_1\varphi)/\partial\bar{z}_1$ is flat on Y_1. Hence, by Proposition 4.1, the weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ of $(\beta - \partial(\bar{z}_1\varphi)/\partial\bar{z}_1) \wedge \lambda_1^k d\bar{z}_1$ is zero.

The weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ of $k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1$ is $k - 1$. Thus we can write

$$\rho = \left(\beta - \frac{\partial \bar{z}_1 \varphi}{\partial \bar{z}_1}\right) \wedge \lambda_1^k d\bar{z}_1 - k\varphi \wedge \lambda_1^{k-1} d\bar{z}_1.$$

On the other hand the form

$$\alpha = (-1)^l \bar{\partial}(\bar{z}_1 \varphi) - \frac{\partial \bar{z}_1 \varphi}{\partial \bar{z}_1} \wedge d\bar{z}_1$$

does not contain $d\bar{z}_1$ and belongs to $W_{n-k,J-\{1\}}$. Therefore $(-1)^l \varphi$ is the form we are looking for.

Recall now Lemma 2.6:

LEMMA. *Let* $\omega = \alpha\lambda_1^k + \rho \in W_{n,J,k}^{p,q}$ *be a form such that* $\alpha \in W_{n-k,J-\{1\}}$ *does not contain* $d\bar{z}_1$, $\rho \in W_{n,J,k-1}$ *has weight on* $d\bar{z}_1/\bar{z}_1$ *and* λ_1 *less than or equal to* $k-1$ *and* $\bar{\partial}\omega = 0$. *Then* $\omega \in W_{n,J,k-1}$.

Proof. Set

$$\alpha = \sum_{a\in\Lambda}\sum_{L} f_{a,L}\lambda^a\xi^L.$$

We shall do the proof by induction over $\max\{|a| \mid a\in\Lambda\}$, the weight of α on λ. Let $V = U - \bigcup_{i\in I-\{1\}} Y_i$. By hypothesis

$$\bar{\partial}\omega = \lambda_1^k\bar{\partial}\alpha + (-1)^l k\alpha\wedge\lambda_1^{k-1}\frac{d\bar{z}_1}{\bar{z}_1} + \bar{\partial}\rho = 0,$$

where $l = p+q$ is the degree of α.

For each L, the function

$$h_L = \frac{\partial}{\partial\bar{z}_1}\left(\sum_{a\in\Lambda} f_{a,L}\lambda^a\right)$$

is C^∞ in V and is the coefficient of $\lambda_1^k d\bar{z}_1 \wedge \xi^L$ in $\bar{\partial}\omega$. So by Proposition 4.3 h^L is flat on $Y_1 \cap V$.

Look now at the terms with λ_1^{k-1}. Since ρ has weight on λ_1 and $d\bar{z}_1/\bar{z}_1$ less than or equal to $k-1$, the coefficient of $\lambda_1^{k-1}\bar{\xi}_1$ in $\bar{\partial}\rho$ must be divisible by $\bar{z}_1$. Applying Proposition 4.3 to the coefficient of $\lambda_1^{k-1}\bar{\xi}_1 \wedge \xi^L$ we have that, for some function g,

$$\sum_{a\in\Lambda} f_{a,L}\lambda^a + \bar{z}_1 g$$

is flat on $Y_1 \cap V$. This fact and the corresponding fact for h_L implies that

$$\sum_{a\in\Lambda} f_{a,L}\lambda^a$$

is flat on $Y_1 \cap V$. Considering the partial derivatives of this function as in the case $\#\mathrm{supp}(\Lambda) > 1$ of the proof of Proposition 4.3, we obtain that the functions $f_{a,L}$, with a maximal, are flat on $Y_{\mathrm{supp}(a)} \cap Y_1$.

By Theorem 3.4 we can write, for a maximal in Λ,

$$f_{a,L} = \sum_{i\in\mathrm{supp}(a)} f_{i,a,L} + f_{1,a,L},$$

where $f_{i,a,L}$ is flat on Y_i. Hence, for $i \in \mathrm{supp}(a)$ we have

$$f_{i,a,L}\lambda^a = (f_{i,a,L}\lambda_i^{a_i})\lambda^{\hat{a}_i}$$

and $(f_{i,a,L}\lambda_i^{a_i})$ is a C^∞ function on U. On the other hand $f_{1,a,L}\lambda_1^k\lambda^a \in W_{n,J-\{1\}} \subset W_{n,J,k-1}$. Therefore we can write $\alpha = \alpha' + \alpha''$, where the weight of α' on λ is less than those of α and $\alpha'' \in W_{n,J,k-1}$. Thus we have $\omega = \alpha' + \rho'$, where $\rho' = \rho + \alpha''$ satisfies the same conditions as ρ and α' the same as α but with less weight on λ. This concludes the inductive step.

If $\max\{|a| \mid a \in \Lambda\} = 0$ we proceed in the same way but now we obtain $\alpha' = 0$, hence the result.

This finishes the proof of Theorem 2.1.

6. Green functions and logarithmic Dolbeault complexes

Since the fundamental work of Néron and Arakelov in Arithmetic Intersection Theory, Green functions have been widely used in the study at infinity of arithmetic divisors.

In this section we shall examine the relationships between Green functions and logarithmic Dolbeault complexes, suggesting that these complexes may be a useful tool in the study and generalization of Green functions.

Let X be a complex manifold and let D be an irreducible divisor. We shall denote by $|D|$ the support of D. Let ω be a real $(1, 1)$ form which represents the cohomology class of D. Then a Green function for D with respect to ω is a function

$$g_D \in \Gamma(X - |D|, \mathscr{E}^0_{X,\mathbb{R}})$$

with logarithmic singularities along $|D|$ and such that

$$dd^c g_D = \omega,$$

where d^c is the real differential operator defined by $d^c = \dfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

The meaning of the words logarithmic singularities may vary from one work to another, ranging from logarithmic growth conditions to a more precise description of the singularity.

A well known method to construct Green functions is the following. Let L be the line bundle associated to D. Let $\|\cdot\|$ be a hermitian metric in L and s

be a section of L such that $D = (s)$. Then a Green function for D is

$$g_D = -\log\|s\|^2. \tag{5}$$

It is also well known that $\omega = dd^c g_D$ is the first Chern form of $(L, \|\cdot\|)$ and that ω represents the cohomology class of D. To obtain Green functions with respect to another form in the same cohomology class, say ω', it is enough to apply the $\partial\bar{\partial}$-Lemma to the exact form $\omega - \omega'$.

From now on, we shall use the following convention. The global sections of a sheaf will be denoted by the same letter as the sheaf but in script instead of italic, for instance

$$E_X^{p,q}(\log Y) := \Gamma(X, \mathscr{E}_X^{p,q}(\log Y)).$$

Let Y be a divisor with normal crossings, $Y = \bigcup Y_k$ with Y_k a smooth divisor for each k. Set $V = X - Y$. A first relationship between logarithmic Dolbeault complexes and Green forms is that $E_X^*(\log Y)$ can be characterized as being the minimum sub-E_X-algebra of E_V^*, closed under ∂ and $\bar{\partial}$, that contains a Green function of the type (5) for each smooth divisor Y_k.

Let us examine specifically the case of curves, noting that the same type of reasoning will work in the general case. Let C be a compact Riemann surface. Choose a point x of C and assume that ω is a differentiable (resp. real analytic) volume form on C normalized in such a way that

$$\int_C \omega = 1.$$

By De Rham duality this is equivalent to saying that ω represents the cohomology class of x viewed as a divisor. In this case the usual definition of Green functions is the following (cf. for example [L] or [G]):

A Green function for x with respect to the form ω is a differentiable (resp. real analytic) function

$$g_x : C - \{x\} \to \mathbb{R}$$

such that

G1. $dd^c g_x = \omega$.

G2. If z is a local parameter for x in a neighbourhood U of x then

$$g_x(z) = -\log z\bar{z} + \varphi(z),$$

where φ is a real differentiable (resp. real analytic) function defined in the whole U.

G3. It satisfies

$$\int_C g_x \omega = 0.$$

It is well known that the conditions **G1** and **G2** determine g_x up to an additive constant and that this constant is fixed by **G3**.

The condition **G2** obviously implies the condition

G2′. The function g_x belongs to $E^0_C(\log\{x\})$.

In fact, in the presence of **G1**, the statements **G2** and **G2′** are equivalent, i.e. we have the following regularity lemma.

PROPOSITION 6.1. *Let $g \in E^0_C(\log\{x\})$ be a solution of the differential equation* **G1**. *Then, up to an additive constant, g is a Green function for x with respect to ω.*

Proof. We only need to show that g satisfies **G2** in a neighbourhood U of x. Let z be a local parameter for x in U. Put $\lambda = \log z\bar{z}$. We have a (non-unique) decomposition

$$g = \sum_{k=0}^{n} f_k \lambda^k,$$

where the functions f_k are smooth on x. The fact that g satisfies **G1** and Proposition 4.3 implies that the functions f_k are flat on x for $k > 1$, and that there exists a constant a such that $f_1 - a$ is flat on x. Hence, by Proposition 4.1, we can write

$$g = a\lambda + \varphi,$$

where φ is a C^∞ function in the whole U.

It only remains to determine the value of the constant a. This constant is determined by the cohomology class of ω. Let us consider in U the standard metric of $\mathbb{C}$. Let S_ε be the sphere of centre x and radius ε. We have, using Stokes' Theorem, that

$$1 = \int_C \omega = \int_C dd^c g = -\lim_{\varepsilon \to 0} \int_{S_\varepsilon} d^c g = -a.$$

Therefore, $a = -1$, concluding the proof of the lemma.

In view of Proposition 6.1, to prove the existence of a Green function for x it is enough to solve the equation **G1** in the complex $E^*_C(\log\{x\})$. Let us give a proof of the existence of such solutions which does not depend on the existence of metrics on line bundles (see also [L] and [G]).

PROPOSITION 6.2. *Let ω be a real* $(1,1)$ *form on C. Then, for each $x \in C$, there exists a real function $g \in E^0_C(\log\{x\})$ such that $dd^c g = \omega$. This function is unique up to an additive constant.*

Proof. The uniqueness follows from the fact that g satisfies **G1** and **G2.**

The form ω is exact in the complex $E^*_C(\log\{x\})$ because $H^2(C - \{x\}, \mathbb{C}) = \{0\}$. Since the spectral sequence of $E^*_C(\log\{x\})$ with the filtration F degenerates at E_1, the differential d is strictly compatible with F. Hence there exists an element $\varphi \in F^1 E^1_C(\log\{x\})$ such that $d\varphi = \omega$. Then $\varphi \in E^{1,0}_C(\log x)$, $\partial\varphi = 0$ and $\bar\partial\varphi = \omega$. Thus we have that $\bar\partial\bar\varphi = 0$ and, ω being real, that $\partial\bar\varphi = \omega$.

Now the form $\bar\varphi - \varphi$ is closed and represents an element of $H^1(C - \{x\}, \mathbb{C})$. Since C is smooth the Hodge filtration of this complex satisfies ([D]):

$$H^1(C - \{x\}, \mathbb{C}) = F^1 + \bar{F}^1.$$

Therefore there exist forms $\psi_1 \in E^{1,0}_C(\log\{x\})$ and $\psi_2 \in E^{0,1}_C(\log\{x\})$, with $d\psi_1 = d\psi_2 = 0$ and a function $f \in E^{0,0}_C(\log\{x\})$ such that

$$df + \psi_1 + \psi_2 = \bar\varphi - \varphi.$$

Hence $\partial\bar\partial f = \omega$. Writing

$$g = \frac{\pi}{\sqrt{-1}}(f - \bar f)$$

we have the desired function.

Remarks. (a) This proposition is a version of the $\partial\bar\partial$-Lemma. (Compare for example with [D-G-M-S]). The properties of elliptic differential operators usually used to prove the existence of Green functions are hidden here in the mixed Hodge structure of the cohomology groups of C and in the degeneracy of the spectral sequence associated with the filtration F.

(b) All the results of this section remain true if we replace the C^∞ complexes for real analytical ones. In particular, if ω is a real analytic (1, 1) form then there exists a real analytic Green function with respect to ω for any point $x \in C$. In this case, by uniqueness, any green function with respect to ω is real analytic.

(c) The definition of Green function has been generalized by Gillet and Soulé (cf. [G-S]) in the concept of Green forms and Green currents associated with arithmetic cycles. They also introduced the star product of Green currents which corresponds to the intersection product of cycles. The techniques of this section can be generalized giving alternative definitions of Green forms for cycles and of the star product between them. They can also be used to prove the existence of these Green forms.

References

[D] Deligne, P., Théorie de Hodge II, *Publ. Math. IHES* 40 (1972), 5–57; III, *Publ. Math. IHES* 44 (1975), 5–77.

[D-G-M-S] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., Real Homotopy Theory of Kähler Manifolds, *Inventiones Math.* 29 (1975), 245–274.

[G-S] Gillet, H. and Soulé, C., Arithmetic Intersection Theory, *Publ. Math. IHES* 72 (1990), 93–174.

[G] Gross, B. H., *Local Heights on Curves*, in "Arithmetic Geometry", (Edited by G. Cornell and J. H. Silverman), Springer-Verlag, New York, 1986, pp. 327–339.

[H-P] Harris, M. and Phong, D. H., Cohomologie de Dolbeault à croissance logarithmique à l'infini, *Comp. Rend. Acad. Sci. Paris* 302 (1986), 307–310.

[L] Lang, S., *Introduction to Arakelov Theory*, Springer-Verlag, Berlin, 1988.

[M] Malgrange, B., *Ideals of Differentiable Functions*, Oxford University Press, 1966.

[N] Navarro Aznar, V., Sur la théorie de Hodge-Deligne, *Invent. Math.* 90 (1987), 11–76.

[T] Tougeron, J. C., *Idéaux de fonctions differentibles*, Springer-Verlag, Berlin, 1972.