A SINGULAR INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR WAVE EQUATIONS AND HOLOGRAPHY IN ASYMPTOTICALLY ANTI-DE SITTER SPACES

ALBERTO ENCISO AND NIKY KAMRAN

Abstract. We analyze the initial value problem for semilinear Klein–Gordon equations on asymptotically anti-de Sitter spaces using energy methods adapted to the geometry of the problem at infinity. The key feature is that the coefficients become strongly singular at infinity, which leads to considering nontrivial data on the conformal boundary of the manifold. This question arises in Physics as the holographic prescription problem in string theory.

Contents

1. Introduction 1
2. Definitions and notation 7
3. Inequalities for twisted Sobolev spaces 9
4. Elliptic estimates at infinity 17
5. Wave propagation at infinity 21
6. Peeling off large solutions near the conformal boundary 25
7. Holography for the linear wave equation 29
8. Application to nonlinear wave equations 33
Acknowledgements 38
References 38

1. Introduction

Our goal in this paper is to study certain kinds of semilinear wave equations with non-constant coefficients that are of interest in string theory. These equations are defined by the fact that their principal part is close, in a certain sense, to the wave operator of the n-dimensional anti-de Sitter (AdS) space, and their salient feature is that the coefficients become very singular at infinity, so that the problem can be thought of as including some (nontrivial) boundary conditions at infinity. The analysis of the effect of the associated “boundary data at infinity” on the solutions of the wave equation, which is the main theme of this paper, is key in the analysis of these wave equations and, as shall see later on, has a very direct physical interpretation in the context of the celebrated AdS/CFT correspondence [23].
More specifically, we shall analyze wave equations of the form
\begin{equation}
\Box_g \phi - \mu \phi = 0,
\end{equation}
and their semilinear analog
\begin{equation}
\Box_g \phi - \mu \phi = F(x^a, \nabla \phi),
\end{equation}
where the non-linearity $F$ is allowed to depend on the space-time variables $x^a$, in Lorentzian manifolds whose geometry at infinity is close to that of the $n$-dimensional AdS space $\text{AdS}_n$. These manifolds are called asymptotically AdS, and their precise definition will be given in Section 7. In the equations, $\mu$ is a real constant, $\nabla$ stands for the space-time gradient. For notational simplicity, we will not always write the possible dependence of the nonlinearity $F$ on the space-time variables, simply writing the nonlinearity as $F(\nabla \phi)$, where it is understood that $F$ will also depend on the space-time variables.

To give a clearer idea of the structure of these equations, and also to set up some notation, it is useful to first recall the definition of AdS space as well as the form of the scalar wave equation therein. The $n$-dimensional AdS space is the simply connected Lorentzian space of constant negative sectional curvature $-K$. The metric in $\text{AdS}_n$ is thus given by
\begin{equation}
g_{\text{AdS}_n} = -K^{-2} \cosh^2(Kr) \, dt^2 + dr^2 + K^{-2} \sinh^2(Kr) \, g_{\mathbb{S}^{n-2}},
\end{equation}
where $t \in \mathbb{R}$ is the time variable, $r \in \mathbb{R}^+$, $\theta \in \mathbb{S}^{n-2}$ and $g_{\mathbb{S}^{n-2}}$ is the metric of the unit $(n-2)$-sphere. In what follows we will take $K := 1/2$.

To analyze the wave equation in $\text{AdS}_n$, it is convenient to replace the radial coordinate $r$ by a certain function thereof, $x$, which takes values in $(0, 1]$ and is defined through the relation
\begin{equation}
x := \left(2 \cosh \frac{r}{2} - 1\right)^{-\frac{1}{2}}.
\end{equation}
In terms of this variable, the AdS metric is given by
\begin{equation}
g_{\text{AdS}_n} = \frac{-(1+x^2)^2 \, dt^2 + dx^2 + (1-x^2)^2 \, g_{\mathbb{S}^{n-2}}}{x^2},
\end{equation}
and the wave equation (1.1) in $\text{AdS}_n$ takes the form
\begin{equation}
-\partial_t^2 \phi + \partial_x^2 \phi - \frac{n-2}{x} \partial_x \phi + \Delta_\theta \phi - \frac{\mu}{x^2} \phi + \cdots = 0,
\end{equation}
where $\Delta_\theta$ is the Laplacian in $\mathbb{S}^{n-2}$ and the dots stand for terms that are smaller, in a certain sense, for $x$ close to zero.

The point $x = 1$ corresponds to the spatial origin $r = 0$ while the set $\{x = 0\}$ (which is a cylinder $\mathbb{R}_t \times \mathbb{S}^{n-2}_0$ and is often called the conformal boundary or conformal infinity of the manifold) corresponds to the spatial infinity of the manifold. Notice that if we conformally rescale the metric (1.3) by the factor $x^2$ so that it becomes smooth when $x = 0$, and then compactify the manifold by adding the boundary set $\{x = 0\}$, then this boundary, which is homeomorphic to a cylinder, is a time-like hypersurface in the extended manifold endowed with the Lorentzian metric $-dt^2 + g_{\mathbb{S}^{n-2}}$. 
The coefficients of the equation are obviously singular at \( x = 0 \). A simple analysis of the singularities reveals that when the condition

\[
\mu > -\left(\frac{n-1}{2}\right)^2,
\]

is satisfied, the solutions to (1.4) are expected to behave at conformal infinity as

\[
x^{\frac{n-1}{2}+\alpha} [1 + O(x)] \quad \text{or} \quad x^{\frac{n-1}{2}-\alpha} [1 + O(x)]
\]

(see e.g. [22]), where we hereafter write

\[
\alpha := \left[\left(\frac{n-1}{2}\right)^2 + \mu\right]^{1/2}.
\]

Throughout this paper we will assume that the condition (1.5) is satisfied, which is equivalent to \( \alpha \) being strictly positive. (When case \( \alpha \) is a nonnegative integer, the solutions can have a logarithmic branch, so for simplicity we will assume that \( \alpha \notin \mathbb{N} \) in this paper.) Let us mention in passing that when \( \alpha < 1 \), both expansions at infinity are square-integrable, which means that one needs to impose certain boundary conditions at infinity to determine the evolution of the wave equation. This is directly related with the fact that \( \text{AdS}_n \), just as any other asymptotically \( \text{AdS} \) space, is not globally hyperbolic because null geodesics escape to infinity for finite values of their affine parameter. This has the effect that the evolution under the wave equation (1.1) in \( \text{AdS}_n \) of smooth, compactly supported initial data does not remain compactly supported in space for all values of time. It should be stressed, however, that we will be interested in the boundary behavior for all values of the parameter \( \alpha \), not just for \( \alpha < 1 \).

Our goal in this paper is to analyze how solutions to the Eqs. (1.1)–(1.2) in an asymptotically \( \text{AdS} \) space \( M \) are determined by prescribing initial conditions and a (compatible) boundary condition at infinity. We are mostly interested in the role of the latter, as this is the key property of this equation. To prescribe these boundary conditions at conformal infinity, we pick the dominant exponent and in \( \text{AdS}_n \) it is enough to impose

\[
x^{\alpha - \frac{n-1}{2}} \phi|_{x=0} = f(t, \theta),
\]

where \( f \) is the datum of the problem. In an arbitrary asymptotically \( \text{AdS} \) manifold \( M \) there is a direct counterpart of this condition,

\[
x^{\alpha - \frac{n-1}{2}} \phi|_{\partial M} = f,
\]

but we will not make this precise yet. Of course, the initial conditions we take are of the form

\[
\phi|_{M_0} = \phi_0, \quad \mathcal{T} \phi|_{M_0} = \phi_1,
\]

where \( M_0 \) is a time slice and the vector field \( \mathcal{T} \) roughly encodes the partial derivative with respect to time (details will be given in Section 7). For the purposes of this Introduction, one can take trivial initial conditions \( \phi_0 = \phi_1 = 0 \).

As we have mentioned, the problem under consideration arises naturally in the context of the \( \text{AdS}/\text{CFT} \) correspondence in string theory [23], so we shall elaborate a little on this topic. This is a conjectural relation which posits that a gravitational field on a Lorentzian \( n \)-manifold endowed with an Einstein metric metric close to the \( \text{AdS}_n \) metric at infinity can be recovered from a gauge field defined on the conformal boundary of the manifold. The gravitational field is modeled using some
The PDE of hyperbolic character in the manifold (typically, the Einstein equation) and the gauge field at conformal infinity plays the role of boundary datum through a relation analogous to (1.7). Since, in harmonic coordinates, the Einstein equation reduces to a nonlinear wave equation, at the heart of the AdS/CFT correspondence lies a boundary-value problem closely related to the one we have stated above for the scalar wave equation (1.2). Indeed, the holographic principle asserts that the boundary data (which here, in the context of scalar wave equations, would be the function $f$), defined on an $(n-1)$-dimensional boundary, propagates through a suitable $n$-manifold (which is referred to as the bulk and is denoted here by $\mathcal{M}$) to determine the field (here $\phi$) via a locally well posed problem.

It is important at this stage to note that most rigorous results related to the holographic principle have been obtained for the elliptic counterparts of these equations. As is well known, the Riemannian analog of AdS$_n$ is the hyperbolic space $\mathbb{H}^n$, whose metric we describe using coordinates $r \in [0, 1)$ and $\theta \in S^{n-1}$ as

$$g_{\mathbb{H}^n} = \frac{dr^2 + r^2 g_{S^{n-1}}}{1-r^2}.$$ 

Setting $x := 1 - r^2$, it is classical that the corresponding boundary value problem

$$\Delta_{\mathbb{H}^n} \phi - \mu \phi = 0, \quad x^{\alpha-n-1} \phi |_{x=0} = f(\theta),$$

has a unique solution. Lacking results on the Lorentzian analog of this problem, the boundary value problem (1.8) has been used as the basic model to understand holographic prescription, starting with Witten's paper [30]. Of course, the results for the hyperbolic space remain valid in much more general contexts. For instance, in the case $\mu = 0$, they have been extended by Anderson [1], Sullivan [26] and Anderson and Schoen [4] to treat harmonic functions on manifolds whose sectional curvature is not assumed to be a negative constant, as is the case of $\mathbb{H}^n$, but is pinched between two distinct negative constants.

For the full Einstein equations with Riemannian signature, the situation is already much more subtle [2]. A fundamental result in this direction, due to Graham and Lee [16], states that given a Riemannian metric $g_0$ on the sphere close enough to the round metric, there is a unique asymptotically hyperbolic metric in the ball close to $\mathbb{H}^n$ that has $g_0$ as its boundary value, in a suitable sense. In Riemannian signature, this shows how the holographic principle works when the boundary datum is small, which is the case of interest in cosmological applications. Significant refinements of the Graham–Lee theorem can be found in [3] and references therein.

The situation for the holographic prescription problem in Lorentzian signature is much less clear-cut, since both the available analytical techniques and the expected results are necessarily different. To the best of our knowledge, the wave equation on AdS$_4$ was first considered in [8], where separation of variables was used to discuss the behavior of the energy for all values of $\mu$ above the threshold value (1.5). Again for AdS$_4$, Choquet-Bruhat [9, 10] proved global existence for the Yang–Mills equation under a radiation condition, and Ishibashi and Wald [20] gave a proof of the well-posedness of the Cauchy problem for the Klein–Gordon equation (1.1) in AdS$_n$ using spectral theory. More refined results for the Klein–Gordon equation in an AdS space were developed by Bachelot [5, 6, 7], who used energy methods and dispersive estimates to study the decay of the solutions and prove Strichartz estimates and some results on the propagation of singularities.
An important paper is due to Vasy, [28], where fine results on the propagation of singularities are proved for the Klein–Gordon equation on asymptotically AdS spaces using microlocal analysis. Holzegel and Warnick, both independently and in joint work [18, 29, 19], used energy methods to show that the Cauchy problem for this equation in asymptotically AdS space-times is well posed in twisted Sobolev spaces (considering different “homogeneous boundary conditions” when the parameter $\mu$ is negative and above its threshold value), and discussed the uniform boundedness of solutions to the Klein–Gordon equation in stationary AdS black hole geometries.

In this paper, we prove that the boundary value problem at infinity for the wave equations (1.1)–(1.2) in an asymptotically AdS manifold is well posed in a certain scale of Sobolev spaces adapted to the geometry of the space-time. In the language of physics, this can be rephrased as saying that the holographic prescription problem is well posed for scalar fields with suitable nonlinearities. The prescription has the physically critical properties of being fully holographic and causal, which essentially means that, for trivial initial conditions and compactly supported datum $f$ on the conformal boundary, the field $\phi$ is purely controlled by $f$ and is identically zero for all times below the support of this function [13].

The precise statements of the well-posedness results proved in this paper are given in Theorems 7.2 and 8.1. Although we will not reproduce these statements here to avoid introducing too much notation, we shall nevertheless discuss the content of these theorems in some detail.

For the linear wave equation, we use energy estimates to prove that, given (compatible) initial conditions and a datum on the conformal boundary $f$, there is a unique solution of (1.1). This solution is defined for all time and can be estimated in terms of the data in suitable Sobolev spaces with norms that are defined in a way that compensates for the singular behavior of the metric at the conformal boundary. An interesting feature is that, due to the form of the energy, one does not simply get the usual weighted Sobolev spaces, but rather a twisted version thereof, given in (2.8), involving both a weight vanishing at the conformal boundary of the manifold and twisted derivatives, where the twist factor conjugating the derivative (see (2.6)) is directly related to the geometry of the asymptotically AdS space at infinity through the “renormalized energy” considered by Breitenlohner and Freedman [8]. One also shows that these Sobolev-type estimates imply the pointwise decay of the solutions by proving suitable Sobolev embedding theorems.

We would like to point out at this stage that Warnick [29] has proved in the range $\alpha < 1$ a well-posedness result of a nature similar to that of Theorem 7.2, obtaining first-order Sobolev estimates relating the norm of the unique weak solution of the initial-boundary value problem for the Klein–Gordon equation to the norms of the initial and boundary data. The results that we prove in Theorem 7.2 are somewhat stronger, as they apply to the full range $\alpha > 0$ and involve estimates on the higher-order Sobolev norms of the solution and its time derivatives. These stronger results turn out to be of crucial importance for the proof we give in Theorem 8.1 of the well-posedness for non-linear wave equations. Vasy’s microlocal approach to wave equations on asymptotically AdS spacetimes [28] also yields a closely related result for the Klein–Gordon equation, together with more powerful results on the propagation of singularities. However, the use of twisted Sobolev spaces together
with suitable embedding theorems seems to be better suited for the analysis of nonlinear wave equations on these spacetimes, which is the ultimate goal of this paper.

Indeed, for the nonlinear wave equation we prove a local well-posedness result (with analogous estimates) using a bootstrap argument. For concreteness we restrict our attention to quadratic nonlinearities of the form

\begin{equation}
F(\nabla \phi) := \Gamma g(\nabla \phi, \nabla \phi),
\end{equation}

where the function \( \Gamma \) behaves in a neighborhood of the conformal boundary as

\begin{equation}
\Gamma = x^q \widehat{\Gamma}(t, x, \theta),
\end{equation}

where \( q \) is a large enough power and the function \( \widehat{\Gamma} \) is smooth up to the conformal boundary. (Details will be given in Section 8.) The reason to consider a nonlinearity quadratic in the first derivatives is that the problem becomes a simplified version of the Einstein equation in which, in particular, the tensorial structure is disregarded. This is of particular relevance for the proof a Lorentzian analog of the Graham–Lee theorem for asymptotically AdS Einstein metrics, which will be considered elsewhere [14].

The choice of the exponent \( q \) appearing in our main theorem (see Theorem 8.1) is probably not sharp. We should also point out that the meaning of the exponent \( q \) in physically realistic models of asymptotically AdS space-times is not clear at this stage, and that further investigation of this question would be worthwhile. It should also be clear from the proofs that the methods employed in the paper are applicable to a much larger class of nonlinearities. Roughly speaking, the only requirement of the method is that nonlinearity must depend smoothly on the field and fall off fast enough at \( x = 0 \). The method of proof can thus be readily extended, with suitable modifications, to cover quasilinear equations where the coefficients of the second derivatives are those of an asymptotically AdS metric plus a smooth function of the field and its first-order derivatives that falls off fast enough at \( x = 0 \), in the same spirit as the nonlinearity considered in this paper. These questions will be considered in further detail in future contributions.

The proofs of these well-posedness results make up the substance of the rest of our paper. First of all, in Section 2 we provide precise definitions of most of the concepts that we have briefly described in this Introduction and devote Section 3 to derive suitable characterizations and Sobolev and Moser inequalities for the twisted Sobolev spaces that we use in this paper. These are not direct consequences of the standard proofs, while Hardy-type inequalities play an important role throughout. In Section 4 we prove estimates at infinity for the elliptic part of the wave equation in an asymptotically AdS space (see Theorem 4.1 and Corollary 4.2). Analogous estimates for asymptotically Minkowskian space-times, where the time slices are asymptotically Euclidean, were derived by Christodoulou and Choquet-Bruhat in [11]. The elliptic estimates are used in Section 5 to derive energy estimates for the Cauchy problem for the wave equation in an asymptotically AdS patch (see Theorems 5.1 and 5.2). To deal with the data on the conformal boundary, one uses an additional set of results obtained in Section 6, where the layers of the solution that are large at infinity are “peeled off” (see Theorem 6.3) and the remaining part of the solution is carefully controlled. In Section 7 we finally discuss the global structure of asymptotically AdS space-times and prove the global well-posedness
of the problem for the linear wave equation (1.1). The local well-posedness of the problem for the nonlinear wave equation (1.2) with the nonlinearity (1.9) is finally proved in Section 8 using an iterative argument.

2. Definitions and notation

In order to describe the geometry of an asymptotically AdS space-time at infinity (i.e., in a neighborhood of an end), in this section we define the notion of an asymptotically AdS patch. Most of our work in this paper will take place in asymptotically AdS patches. In this section we also introduce twisted Sobolev spaces which are adapted by definition to the boundary behavior at infinity of the metric in an asymptotically AdS patch. We shall see in Sections 4 and 5 that these function spaces are well adapted to the study of the the elliptic estimates at infinity and the energy estimates for wave propagation which are needed to prove the well-posedness of the holographic prescription problem.

We begin by introducing some basic notation to describe the asymptotics of functions. Consider the manifold \( \mathbb{R} \times (0,1) \times S^{n-2} \) with coordinates \((t,x,\theta)\). We shall commit a slight abuse of notation and think if the “coordinate” \( \theta \) as taking values in \( S^{n-2} \). Here we say that some quantity \( Q(t,x,\theta) \) is of order \( O(x^m) \) if there exist constants \( C_{jkl} \) such that
\[
|\partial^j t \partial^k x D^l \theta Q(t,x,\theta)| \leq C_{jkl} x^{m-k}
\]
for \( x \) close to 0, uniformly for all \((t,\theta) \in \mathbb{R} \times S^{n-2}\). Notice that, as is customary when considering functions on a manifold, the angular derivatives \( D^l \theta u \) must be interpreted either using local coordinates in the obvious way or, more intrinsically, using tangent vector fields, but we will omit this point whenever we find it notationally convenient. A similar abuse of notation will be often made when dealing with Sobolev spaces as in Eq. (2.8) below.

With this notation in place, we are now ready to define the concept of an asymptotically AdS patch. For convenience, we will include in the definition a small parameter \( a \) that describes the width of the patch.

\[\textbf{Definition 2.1.} \text{ As asymptotically AdS patch (of width } a \text{) in a Lorentzian manifold } \mathcal{M} \text{ is an open set } U \subset \mathcal{M} \text{ with smooth boundary that is covered by coordinates } (t,x,\theta) \in \mathbb{R} \times (0,a) \times S^{n-2} \text{ in which the coefficients of the metric are smooth and read as }\]
\[
g_{tt} = -x^{-2} - 1 + \mathcal{O}(x^2), \quad g_{xx} = x^{-2} - 1 + \mathcal{O}(x), \quad g_{tx} = \mathcal{O}(x),
\]
\[
g_{\theta i \theta j} = x^{-2} (g_{S^{n-2}})_{ij} + \mathcal{O}(x), \quad g_{\theta i} = \mathcal{O}(x), \quad g_{x^i x^j} = \mathcal{O}(x^2),
\]
where \( g_{S^{n-2}} \) stands for the metric of the unit \((n-2)\)-sphere.

In an asymptotically AdS patch, the wave equation (1.1) can be written, after dividing by the coefficient of \( \phi_{tt} \) (which is of the form \( x^2 + \mathcal{O}(x^4) \)), in the form
\[
(2.1) \quad - \partial^2_t \phi + \partial^2_x \phi - \frac{n-2}{x} \partial_x \phi + \Delta_\theta \phi - \frac{\mu}{x^2} \phi = \mathcal{O}(x) D_{1x\theta} \phi + \mathcal{O}(x^2) D_{x^2} D_{x\theta} D_{1x\theta} \phi,
\]
where the symbols \( D_{x\theta} \) and \( D_{1x\theta} \) stand for the derivatives of \( \phi \) with respect to all the space and space-time variables, respectively.
Following [29], it will be convenient to introduce the function
\[ u := x^{1-n/2} \phi, \]
in terms of which Eq. (2.1) becomes
\[ P_g u = 0, \]
where \( P_g \) is a second-order differential operator of the form
\[ P_g u := -\frac{\partial^2_t u + \partial^2_x u}{x} + \frac{1}{x} \partial_x u + \Delta_g u - \frac{\alpha^2 u}{x^2} + O(x) D_{tx} u + O(x^2) D_{x\theta} D_{tx} u. \]
Let us recall that the constant \( \alpha \) was introduced in Eq. (1.6) and record for future reference that the precise relationship between \( \Box_g \phi \) and \( P_g u \) is
\[ \Box_g \phi - \mu \phi = \left[ 1 + O(x^2) \right] x^{n+2} P_g u. \]
A key observation, due to Warnick [29], is that the terms of \( P_g u \) that are dominant for small \( x \) can be rewritten as
\[ -\frac{\partial^2_t u + \partial^2_x u}{x} + \partial_x u + \Delta_g u - \frac{\alpha^2 u}{x^2} = -\partial^2_t u - D^*_x D_x u + \Delta_g u, \]
where the twisted derivative
\[ D_x u := x^{-\alpha} \partial_x \left( x^\alpha u \right) = u_x + \frac{\alpha}{x} u, \]
is directly related to the “renormalized energy” considered by Breitenlohner and Freedman [8] and
\[ D^*_x u := -x^{\alpha-1} \partial_x \left( x^{1-\alpha} u \right) = -u_x + \frac{\alpha - 1}{x} u, \]
is the formal adjoint of \( D_x \) with respect to the scalar product of the space
\[ L^2 := L^2((0,a) \times S^{n-2}, x dx d\theta), \]
where \( d\theta \) denotes the standard measure on the unit sphere. It should be noticed that, on account of the relationship between \( \phi \) and \( u \) and of the coordinate expression for the metric in an asymptotically AdS patch, the \( L^2 \) norm of \( \phi \) with respect to the natural space-time measure,
\[ d\text{vol}_M := \sqrt{|\det g|} dx d\theta^1 \cdots d\theta^{n-2} dt, \]
is equivalent to the \( L^2 \) norm of \( u \) for small \( a \), since
\[ \int_{\mathcal{U}} \phi^2 d\text{vol}_M = \int_{(0,a) \times S^{n-2}} (1 + O(x)) u^2 x dx d\theta dt \\
= (1 + O(a)) \int_{-\infty}^{\infty} \|u(t, \cdot)\|^2_{L^2} dt. \]
To control functions defined on an asymptotically AdS patch, we will consider Sobolev spaces associated with the twisted derivatives, which, setting \( H^0 = L^2 \), can be recursively defined as
\[ H^{2j+1} := \{ v \in H^{2j} : D_{\theta} v \in H^{2j}, D_x (D^*_x D_x)^j v \in L^2 \}, \]
\[ H^{2j+2} := \{ v \in H^{2j+1} : D_{\theta} v \in H^{2j+1}, (D^*_x D_x)^{j+1} v \in L^2 \}, \]
with \( j \geq 0 \). Notice that the functions in \( H^k \) are defined on \((0,a) \times S^{n-2}\), which corresponds to the spatial part of an asymptotically AdS patch \( \mathcal{U} \). The norm
associated to $H^k$ (resp. the Lebesgue space $L^2$) will be denoted by $\| \cdot \|_{H^k}$ (resp. $\| \cdot \|_{L^2}$).

It should be noted that the $H^2$ twisted Sobolev spaces were first introduced and studied in the case $\alpha < 1$ by Warnick [29].

We will also need Sobolev spaces corresponding to Dirichlet boundary conditions. We will denote by $H^1_0$ the twisted Sobolev space on $(0, a) \times S^{n-2}$ with zero trace on the inner and outer boundaries $\{x = 0\} \cup \{x = a\}$. We shall not elaborate on the properties of the trace map here, since in the following section we will give a more direct characterization of these spaces (see Proposition 3.3 and Corollary 3.5).

3. Inequalities for twisted Sobolev spaces

Our goal in this section is to derive inequalities for the twisted Sobolev spaces defined above using some integral operators $A$ and $A^*$ which act as inverses of the twisted derivatives $D_x$ and $D^*_x$. This will be done via Hardy-type inequalities, which we now derive.

Consider the integral operator $A$ defined by

$$A\varphi(x) := x^{-\alpha} \int_0^x y\alpha \varphi(y) \, dy.$$ 

An immediate observation is that $A$ is a right inverse of the twisted derivative $D_x$, that is,

$$D_x(A\varphi) = \varphi.$$ 

We will be concerned with the mapping properties of $A$ for several weighted Lebesgue spaces. A special role will be played by the space

$$L^2_x := L^2((0, a), x \, dx).$$ 

The reason for which we will be interested in this space is that we have an obvious decomposition of $L^2$ of the form

$$L^2 = L^2_x L^2_\theta,$$

where $L^2_\theta := L^2(S^{n-2})$ is the usual $L^2$ space on the sphere. Therefore, the space $L^2_x$ can be interpreted to some extent as the “radial” part of the space of square-integrable functions in (the spacial part of) an asymptotically AdS patch.

The adjoint of $A$ with respect to the inner product of $L^2_x$ is given by

$$A^*\varphi(x) := x^{\alpha-1} \int_x^a y^{1-\alpha} \varphi(y) \, dy.$$ 

One can easily check that $A^*$ is a right inverse of $D^*_x$, that is,

$$D^*_x(A^*\varphi) = \varphi.$$ 

In the next theorem, we derive estimates for $A\varphi$ and $A^*\varphi$ assuming that we have $L^2_x$ bounds on $\varphi$. In some results that we will prove later on, it will be convenient to make explicit the dependence on the parameter $a$ of a few upper bounds. Therefore, here and in what follows we will use the capital letter $C$ to denote a positive constant that can depend on the parameter $a$ and the lower case $c$ to denote a positive, $a$-independent constant. As customary, both constants may vary from line to line.
Theorem 3.1. For any reals $s < \alpha$ and $r \geq s$, the operators $A, A^*$ have the following properties:

\begin{align}
(3.1a) & \quad \|x^{r-1}A\varphi\|_{L^2_x} \leq ca^{r-s}\|x^s\varphi\|_{L^2_x}, \\
(3.1b) & \quad \|A\varphi\|_{L^\infty_x} \leq c\|\varphi\|_{L^2_x}, \\
(3.1c) & \quad \|x^{-s}A^*\varphi\|_{L^2_x} \leq ca^{r-s}\|x^{1-r}\varphi\|_{L^2_x}, \\
(3.1d) & \quad |A^*\varphi(x)| \leq c\|\varphi\|_{L^2_x} \times \begin{cases} 
1 & \text{if } \alpha > 1, \\
\log(a/x) & \text{if } \alpha = 1, \\
(a/x)^{1-\alpha} & \text{if } \alpha < 1.
\end{cases}
\end{align}

Notice, in particular, that for $\alpha > 1$ one has $\|A^*\varphi\|_{L^\infty_x} \leq c\|\varphi\|_{L^2_x}$.

Proof. Let us begin by proving the inequality (3.1c), since the fact that $A^*\varphi(a) = 0$ simplifies the integration by parts. In view of the formula for $A^*$, we need the Hardy inequality

$$
\int_0^a x^{2\alpha - 2s - 1} \left( \int_x^a y^{1-\alpha} \varphi(y) \, dy \right)^2 \, dx \leq ca^{2r-2s} \int_0^a x^{3-2r} \varphi(x)^2 \, dx.
$$

To prove this, let us call $J^2$ the LHS of this inequality and set

$$
\psi(x) := \int_x^a y^{1-\alpha} \varphi(y) \, dy.
$$

Then integrating by parts and using the Cauchy–Schwarz inequality we find

$$
J^2 = \int_0^a x^{2\alpha - 2s - 1} \psi^2 \, dx = \frac{1}{\alpha - s} \int_0^a \psi \varphi x^{\alpha - 2s + 1} \, dx \\
= \frac{1}{\alpha - s} \int_0^a x^{-s} \left( x^{\alpha - s - \frac{1}{2}} \psi \right) (x^{3-2r} \varphi) \, dx \\
\leq \frac{a^{r-s}}{\alpha - s} J \left( \int_0^a x^{3-2r} \varphi^2 \, dx \right)^{1/2},
$$

where in the second and fourth lines we have used that $s < \alpha$ and $r \geq s$, respectively. This inequality is (3.1c).

Since this implies that

$$
A^* : L^2(x^{2-2r} x \, dx) \to L^2(x^{-2s} x \, dx)
$$

is bounded, by duality it stems that the adjoint with respect to $L^2_x = L^2(x \, dx)$ is a bounded map

$$
A : L^2(x^{2s} x \, dx) \to L^2(x^{2r-2} x \, dx)
$$

with the same norm, which immediately yields (3.1a).
Let us now pass to the pointwise bounds. To prove (3.1d) we utilize the Cauchy-Schwarz inequality to write (for \( \alpha \neq 1 \))

\[
|A^* \varphi(x)| = x^{\alpha-1} \left| \int_x^a y^{1-\alpha} \varphi(y) \, dy \right| \\
\leq x^{\alpha-1} \left( \int_x^a y^{1-2\alpha} \, dy \right)^{1/2} \left( \int_x^a y^{2\alpha} \varphi(y)^2 \, dy \right)^{1/2} \\
\leq \|\varphi\|_{L^2_x} \left[ \frac{1}{2-2\alpha} \left( \frac{a}{x} \right)^{2-2\alpha} - 1 \right]^{1/2}.
\]

The estimate (3.1d) follows from this inequality and the elementary observation

\[
\left( \frac{a}{x} \right)^{2-2\alpha} - 1 \right]^{1/2} \leq \begin{cases} 
1 & \text{if } \alpha > 1, \\
(a/x)^{1-\alpha} & \text{if } \alpha < 1.
\end{cases}
\]

The case \( \alpha = 1 \) and the inequality (3.1b) follow from an analogous argument. \( \square \)

In the following proposition we provide pointwise bounds for \( A\varphi \) and \( A^* \varphi \) under the assumption that we have pointwise bounds for \( \varphi \). Our goal here is to relate the fall off of the former at \( x = 0 \) to that of the latter using power laws. We shall use the notation \( x \wedge y := \min(x,y) \).

**Proposition 3.2.** Let \( \varphi, \psi \) be functions satisfying

\[
|\varphi(x)| \leq C_1 x^s \quad \text{and} \quad |\psi(x)| \leq C_2 x^r,
\]

with \( s > -1 - \alpha \) and an arbitrary real \( r \). Then \( A\varphi \) and \( A^* \varphi \) obey the pointwise bounds

\[
|A\varphi(x)| \leq CC_1 x^{s+1},
\]

\[
|A^* \varphi(x)| \leq CC_2 x^{(r+1)\wedge (\alpha-1)},
\]

where \( C \) does not depend on the functions \( \varphi \) or \( \psi \).

**Proof.** It is easy to find that

\[
|A\varphi(x)| \leq x^{-\alpha} \int_0^x C_1 y^{\alpha+s} \, dy = CC_1 x^{s+1}.
\]

Likewise,

\[
|A^* \psi(x)| \leq C_2 x^{\alpha-1} \int_x^a y^{r-\alpha+1} \, dy \\
= CC_2 x^{\alpha-1} \left( a^{r-\alpha+2} - x^{r-\alpha+2} \right) \leq CC_2 x^{(r+1)\wedge (\alpha-1)}.
\]

\( \square \)

In the rest of this section we shall see how the previous estimates for the operators \( A \) and \( A^* \) are of use in the analysis of the Sobolev spaces \( H^k \). Hence we now focus our attention on the action of the operators \( A, A^* \) not only on functions of the variable \( x \), but also on functions that depend on \( x \) and \( \theta \). An easy result that we will need later on is the following. We recall that we are using the shorthand notation \( L^2_\theta \) for the space of square-integrable functions on \( S^{n-2} \).
Proposition 3.3. Let $v(x, \theta)$ be a function in $\mathbf{L}^2$ such that $D_x v$ is also in $\mathbf{L}^2$. Then

$$v = \begin{cases} A(D_x v) & \text{if } \alpha \geq 1, \\ A(D_x v) + x^{-\alpha} G(\theta) & \text{if } \alpha < 1. \end{cases}$$

Here $G(\theta)$ is some function in $L^2_0$, which corresponds to the trace of the function $x^\alpha v$ on the set $\{x = 0\}$ and satisfies

$$\|G\|_{L^2_0} \leq C(\|v\|_{\mathbf{L}^2} + \|D_x v\|_{\mathbf{L}^2}).$$

If $w(x, \theta)$ is a function in $\mathbf{L}^2$ such that $D_x^* w \in \mathbf{L}^2$, then

$$w = A^*(D_x w) + x^{\alpha-1} H(\theta)$$

for some function $H(\theta) \in L^2_0$ with

$$\|H\|_{L^2_0} \leq C(\|w\|_{\mathbf{L}^2} + \|D_x^* w\|_{\mathbf{L}^2}).$$

Proof. Since $A$ is a right inverse of $D_x$ and $A$ maps $\mathbf{L}^2$ to itself by Proposition 3.1, the expression for $v$ follows from the fact that the only functions in the kernel of $D_x$ are those of the form $x^{-\alpha} G(\theta)$, and that they are in $\mathbf{L}^2$ if and only if $\alpha < 1$. The estimate for the norm of $G$ follows from

$$\|x^{-\alpha} G\|_{\mathbf{L}^2} = \frac{a^{1-\alpha}}{\sqrt{2 - 2\alpha}} \|G\|_{L^2_0} \leq \|v\|_{\mathbf{L}^2} + \|A(D_x v)\|_{\mathbf{L}^2} \leq \|v\|_{\mathbf{L}^2} + C\|D_x v\|_{\mathbf{L}^2},$$

where we have used (3.1a) to estimate the norm of $A(D_x v)$. By construction, $G(\theta)$ is the trace of $x^\alpha v(x, \theta)$ on $\{x = 0\}$. The proof of the formula for $w$ follows the same lines. \qed

Remark 3.4. Notice that, for $\alpha < 1$, a function $v \in H^1$ with zero trace on $\{x = a\}$ is in $H^1_0$ if and only if the function $G(\theta)$ considered in Proposition 3.3 is zero, that is, if $v = A(D_x v)$. When $v \in H^1$ one can show that the trace $G(\theta)$ to $\{x = 0\}$ actually belongs to $H^0(\mathbb{S}^{n-2})$, but we will not need this fact.

Since $C^\infty_c((0, a) \times \mathbb{S}^{n-2})$ is clearly dense in $\mathbf{L}^2$, a corollary of the previous proposition is the following characterization of the twisted Sobolev space $H^1_0$ with zero trace. We recall that a function is differentiable in a closed interval if it differentiable in a open interval containing it.

Corollary 3.5. The Sobolev space $H^1_0$ is the completion in the $H^1$-norm of the space $C^\infty_c((0, a) \times \mathbb{S}^{n-2})$ and any $v \in H^1_0$ satisfies

$$\|v\|_{\mathbf{L}^2} \leq c a \|D_x v\|_{\mathbf{L}^2},$$

where the constant $c$ is independent of $a$. For $\alpha < 1$, the space $x^{-\alpha} C^\infty_c((0, a) \times \mathbb{S}^{n-2})$ is dense in the space of $H^1$ functions with zero trace on $\{x = a\}$.

Proof. The bound of the $\mathbf{L}^2$ norm of $v$ in terms of that of $D_x v$ is an immediate consequence of the expression $v = A(D_x v)$ and the estimates for $A$ proved in Theorem 3.1. Since $C^\infty_c((0, a) \times \mathbb{S}^{n-2})$ is clearly dense in $\mathbf{L}^2$, there is a smooth, compactly supported function $F(x, \theta)$ that approximates $D_x v$ in the $\mathbf{L}^2$ norm. By the estimates proved in Theorem 3.1 and the expression of $A$, it then follows that $A(F)$ is in $C^\infty_c((0, a] \times \mathbb{S}^{n-2})$ and approximates $A(D_x u)$ in the $\mathbf{L}^2$ norm.
By Proposition 3.3, for $\alpha \geq 1$ this proves that $C_\infty^\omega((0,a] \times \mathbb{S}^{n-2})$ is dense in the space of $H^1$ functions whose trace to $\{x = 0\}$ is zero. Since the $H^1$-norm is equivalent to the standard $H^1$-norm in a neighborhood of the other component of the boundary, $\{x = a\}$, it then follows that $C_\infty^\omega((0,a] \times \mathbb{S}^{n-2})$ is dense in $H^1_0$ for $\alpha \geq 1$. For $\alpha < 1$, it suffices to write 

$$u = A(D_x u) + x^{-\alpha} G(\theta)$$

and combine the discussion of $A(D_x u)$ with the observation that the function $G(\theta)$ can be approximated by a smooth function $\tilde{G} \in C_\infty^\omega(\mathbb{S}^{n-2})$. □

It should be noted that in the case $\alpha < 1$, Proposition 3.3 and Corollary 3.5 were proved by Warnick in [29].

To pass from estimates in twisted Sobolev spaces to pointwise estimates, the key is the following Morrey-type inequality for $H^m$. It is worth noticing that, although the functions in this space are defined on an $(n-1)$-dimensional manifold, we will only get pointwise estimate for $m > \frac{n}{2}$, instead of for $m > \frac{n-1}{2}$ as one would in standard Sobolev spaces $H^m(\mathbb{R}^{n-1})$. This can be understood to be a consequence of the singular behavior of the measure and of the twist factor at the conformal boundary. Likewise, we are interested in controlling the growth of functions near the conformal boundary. Before stating the result, it is convenient to introduce some notation for ordered twisted derivatives and write

$$D^{(m)}_x v := \begin{cases} (D^*_x D_x)^{\frac{m}{2}} v & \text{if } m \text{ is even}, \\ D_x (D^*_x D_x)^{\frac{m-1}{2}} v & \text{if } m \text{ is odd}. \end{cases}$$

This will be used throughout this paper.

**Theorem 3.6.** Let $v \in H^m$ with $m > \frac{n}{2}$. Then for any non-negative integers $i, j$

$$i + j \leq m - \frac{n}{2},$$

the following inequalities hold:

(i) If $\alpha > 1$,

$$|D^j_\theta D^{(j)}_x v(x, \theta)| \leq C \|v\|_{H^m} x^{(m-i-j-\frac{n+1}{2})\wedge \alpha}.$$ 

(ii) If $\alpha < 1$,

$$|D^j_\theta D^{(j)}_x v(x, \theta)| \leq C \|v\|_{H^m} \begin{cases} x^{-\alpha} & \text{if } j \text{ is even}, \\ x^{\alpha-1} & \text{if } j \text{ is odd}. \end{cases}$$

(iii) If $\alpha = 1$, the estimate in (i) still holds after replacing $m$ by $m - \delta$ in the exponent, with any $\delta > 0$.

**Remark 3.7.** When $v \in H^1_0 H^s_\sigma \cap H^1_0$ with $\sigma > s + \frac{n}{2} - 1$ for some nonnegative integer $s$, the estimate for $v$ replaced by the uniform bound

$$\|D^j_\theta v\|_{L^\infty} \leq C_\sigma \|v\|_{H^1_0 H^s_\sigma}$$

for $0 \leq l \leq s$.

By the Sobolev embedding theorem, Theorem 3.6 and the subsequent remark are a straightforward consequence of the following
Lemma 3.8. Let \( v \in H^m_x L^\infty_\theta \), with \( m \geq 1 \). Then, for any integer \( j \leq m - 1 \), the following inequalities hold:

(i) If \( \alpha > 1 \),
\[
\|D_j^{(j)} v(x, \theta)\| \leq C \|v\|_{H^m_x L^\infty_\theta} \cdot x^{(m-j-1)\wedge \alpha}.
\]

(ii) If \( \alpha < 1 \),
\[
\|D_j^{(j)} v(x, \theta)\| \leq C \|v\|_{H^m_x L^\infty_\theta} \begin{cases} x^{-\alpha} & \text{if } j \text{ is even,} \\ x^{\alpha-1} & \text{if } j \text{ is odd.} \end{cases}
\]

If \( v \in H^1_x L^\infty_\theta \cap H^1_0 \), the estimate for \( v \) can be refined to
\[
\|v\|_{L^\infty} \leq C \|v\|_{H^1_x L^\infty_\theta}.
\]

(iii) If \( \alpha = 1 \), the estimate (i) still holds after replacing \( m \) by \( m - \delta \) in the exponent, with any \( \delta > 0 \).

Proof. In the proof of this theorem we will use the notation
\[
v_j := D_x^{(j)} v.
\]
To keep things concrete, we will see how the result is proved for small values of \( m \).

When \( m = 1 \), we have that \( v_1 \in L^2_x L^\infty_\theta \), so it follows from Theorem 3.1 that
\[
\|Av_1(x, \theta)\| \leq C \sup\|v_1(\cdot, \theta)\|_{L^\infty_\theta} \leq C \sup\|v(\cdot, \theta)\|_{L^\infty_\theta} = C \|v\|_{L^2_x L^\infty_\theta} \leq C \|v\|_{H^1_x L^\infty_\theta},
\]
where \( \sup \) stands for the essential supremum. Proposition 3.3 ensures that \( v = Av_1 + x^{-\alpha} G_0(\theta) \), the second summand being absent if \( \alpha \geq 1 \) or \( v \in H^1_0 \).

When these conditions are not satisfied, we can estimate the second summand by noticing that
\[
\|G_0\|_{L^\infty_\theta} = C \|x^{-\alpha} G_0(\theta)\|_{L^2_x L^\infty_\theta} \leq C \|v\|_{L^2_x L^\infty_\theta} + C \|Av_1\|_{L^2_x L^\infty_\theta} \leq C \|v\|_{L^2_x L^\infty_\theta} + C \|v\|_{H^1_x L^\infty_\theta} \leq C \|v\|_{H^1_x L^\infty_\theta}
\]
using again Theorem 3.1. The estimate for \( v \) then follows.

When \( m = 2 \), we have an \( L^2_x L^\infty_\theta \) bound on \( v_2 \), so Theorem 3.1 ensures that
\[
\|A^* v_2(x, \theta)\| \leq C \|v\|_{H^2_x L^\infty_\theta} x^{-\beta},
\]
where
\[
\beta := \begin{cases} 0 & \text{if } \alpha > 1, \\ \delta & \text{if } \alpha = 1, \\ 1 - \alpha & \text{if } \alpha = 1. \end{cases}
\]

Here \( \delta \) is an arbitrarily small positive constant that we introduce to take care of the logarithmic term in Theorem 3.1. Proposition 3.3 then gives
\[
v_1 = A^* v_2 + x^{\alpha - 1} G_1(\theta),
\]
so with the above bound for \( A^* v_2 \) and using that
\[
\|G_1\|_{L^\infty_\theta} = C \|x^{\alpha - 1} G_1(\theta)\|_{L^2_x L^\infty_\theta} \leq C \|v_1\|_{L^2_x L^\infty_\theta} + C \|A^* v_2\|_{L^2_x L^\infty_\theta} \leq C \|v\|_{H^2_x L^\infty_\theta},
\]
we find
\[
\|v_1(x, \theta)\| \leq C \|v\|_{H^2_x L^\infty_\theta} x^{(\alpha - 1)\wedge (-\beta)}.
\]
We can use this pointwise estimate for \( v_1 \) and Proposition 3.2 to infer that
\[
|Av_1(x, \theta)| \leq C\|v\|_{H^2_x L^\infty_x} x^{\alpha \wedge (1-\beta)}
\]
Since
\[
v = Av_1 + x^{-\alpha} G_0(\theta)
\]
and \( G_0 \) can be controlled as in the case \( m = 1 \) to find that it does not appear for \( \alpha \geq 1 \) or \( v \in H^1_0 \) and satisfies \( \|G_0\|_{L^\infty} \leq C\|v\|_{H^2_x L^\infty_x} \) otherwise, we arrive at the desired estimate in the case \( m = 2 \).

Now that we have established the cases \( m = 1 \) and \( m = 2 \), the general case follows from a totally analogous reasoning and a simple induction argument. □

Later on in the paper we will need to estimate the norms of products of functions. This will be accomplished using the following result, which is a Moser estimate for the twisted Sobolev space \( H^m \). Just as in Theorem 3.6, we will require functions in \( H^m \) with \( m > \frac{n}{2} \), rather than \( m > \frac{n-1}{2} \) as in the Euclidean case.

**Proposition 3.9.** Given \( m > \frac{n}{2} \), let us consider nonnegative integers \( j_1, \ldots, j_l \) and \( k_1, \ldots, k_l \) with total sum
\[
j_1 + \cdots + j_l + k_1 + \cdots + k_l \leq m.
\]
Furthermore, let us set
\[
\eta := \begin{cases} 0 & \text{if } \alpha > 1, \\ \alpha \wedge (1-\alpha) & \text{if } \alpha < 1. \end{cases}
\]
Then the inequality
\[
\left\| x^{(l-1)\eta} D_x^{(k_1)} D_\theta^{j_1} u_1 \cdots D_x^{(k_l)} D_\theta^{j_l} u_l \right\|_{L^2_x} \leq C\|u_1\|_{H^m} \cdots \|u_l\|_{H^m}
\]
holds for any functions \( u_1, \ldots, u_l \) in \( H^m \). For \( \alpha = 1 \), the result is still true for any positive (but arbitrarily small) \( \eta \).

**Proof.** Let us prove the statement for \( l = 2 \), which is the case that will be needed in this paper. The general result follows from an analogous argument using the generalized Cauchy–Schwarz inequality.

Since
\[
\|x^\eta u_1\|_{L^\infty_x} \leq C\|u_1\|_{H^m}
\]
by Theorem 3.6, we have
\[
\|x^\eta u_1 D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x} \leq C\|x^\eta u_1\|_{L^\infty_x} \|D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x}
\]
\[
\leq C\|u_1\|_{H^m} \|D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x}
\]
\[
\leq C\|u_1\|_{H^m} \|u_2\|_{H^m},
\]
which proves the result when \( j_1 \) and \( k_1 \) are zero. Hence, by symmetry we can henceforth assume that both \( j_1 + k_1 \) and \( j_2 + k_2 \) are nonzero.

It follows from Proposition 3.3 that
\[
D_x^{(k_1)} D_\theta^{j_1} u_1 = A^\# (D_x^{(k_1+1)} D_\theta^{j_1} u_1) + x^{-\eta_1} F(\theta),
\]
where \( A^\# \) (resp. \( \eta_1 \)) stands for \( A \) or \( A^* \) (resp. \( \alpha \) or \( 1-\alpha \)) depending on the parity of \( k_1 \). The terms with exponent \( \eta_1 = -\alpha \) do not appear if \( \alpha \geq 1 \). Taking \( s \) derivatives
with respect to $\theta$ in the identity (3.5) and using the estimates for the operators $A$ and $A^*$ proved in Theorem 3.1, we immediately find that

$$
\|x^n D_x^{(k_1)} D_\theta^{j_1} u_1\|_{L^2_x L^p_\theta} \leq C\left(\|D_x^{(k_1)} D_\theta^{j_1} u_1\|_{L^2_x} + \|D_x^{(k_1+1)} D_\theta^{j_1} u_1\|_{L^2_x}\right)
$$

$$
\leq C\|u_1\|_{H^{1+k_1+j_1+1}}.
$$

Since $j_1 + k_1$ is now at most $m - 1$, by the Sobolev embedding theorem this yields

$$
\|x^n D_x^{(k_1)} D_\theta^{j_1} u_1\|_{L^2_x L^p_\theta} \leq C\|x^n D_x^{(k_1)} D_\theta^{j_1} u_1\|_{L^2_x H^{m-j_1-k_1-1}} \leq C\|u_1\|_{H^m},
$$

where the exponent $p_1 \in [2, \infty)$ is $\infty$ if $m - k_1 - j_1 > \frac{n}{2}$ and the reciprocal of

$$
\frac{1}{2} - \frac{m - k_1 - j_1 - 1}{n - 2}
$$

if $m - k_1 - j_1 < \frac{n}{2}$. In the limiting case, $m - k_1 - j_1 = \frac{n}{2}$, we can take any finite value of $p_1$.

In the first case, namely, $m - k_1 - j_1 > \frac{n}{2}$, we then have an $L^\infty$ bound for $x^n D_x^{(k_1)} D_\theta^{j_1} u_1$, so the estimate follows just as in (3.4), reversing the roles of $u_1$ and $u_2$. Hence let us assume that we are not in this case and notice that the inequality

$$
\|D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x L^p_\theta} \leq C\|D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x H^{m-j_2-k_2}} \leq C\|u_2\|_{H^m}
$$

holds provided that the exponent $p_2$ is chosen as

$$
p_2 := \begin{cases} 
\infty & \text{if } m - k_2 - j_2 > \frac{n}{2}, \\
\left(\frac{1}{2} - \frac{m-k_2-j_2}{n-2}\right)^{-1} & \text{if } m - k_2 - j_2 < \frac{n}{2}.
\end{cases}
$$

When $m - k_2 - j_2 = \frac{n}{2}$, one can take any finite $p_2$. Since

$$
j_1 + j_2 + k_1 + k_2 \leq m \quad \text{and} \quad m > \frac{n}{2},
$$

a straightforward computation shows that

$$
\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{2}.
$$

(In the limiting case, of course, one has to choose the exponent large enough.)

Hence one can use the Cauchy–Schwarz inequality to arrive at

$$
\|x^n D_x^{(k_1)} D_\theta^{j_1} u_1 D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x} = \left(\int x^{2n} (D_x^{(k_1)} D_\theta^{j_1} u_1)^2 (D_x^{(k_2)} D_\theta^{j_2} u_2)^2 x \, dx \, d\theta\right)^{\frac{1}{2}}
$$

$$
\leq \|x^n D_x^{(k_1)} D_\theta^{j_1} u_1\|_{L^2_x L^p_\theta} \|D_x^{(k_2)} D_\theta^{j_2} u_2\|_{L^2_x L^p_\theta}
$$

$$
\leq C\|u_1\|_{H^m} \|u_2\|_{H^m},
$$

as claimed.

For completeness, we shall conclude this section with the statement of a compactness result for twisted Sobolev spaces. We will omit the proof, which is a minor modification of the one given in [19] for a more general statement in the case $\alpha < 1$.

**Proposition 3.10.** Let $u_j$ be a sequence bounded in $H^1$ that converges weakly to some $u \in H^1$. Then $u_j$ also converges strongly, that is, $\|u_j - u\|_{L^2} \to 0$. 
4. Elliptic estimates at infinity

One of the key ingredients in our proof of the well-posedness property for the mixed initial-boundary value problem corresponding to holography in asymptotically AdS manifolds consists in estimates in twisted Sobolev spaces for the elliptic part $L_g$ of the hyperbolic operator $P_g$. These are an analog for asymptotically AdS spaces of the estimates for asymptotically Euclidean spaces derived by Christodoulou and Choquet-Bruhat in [11]. The statements and method of proof, however, are totally different: in the asymptotically Euclidean setting, the proof hinges on an integral inequality due to Nirenberg and Walker [24] and it is enough to consider standard Sobolev spaces with dimension-dependent polynomial weights. For more general elliptic operators, related estimates in weighted Sobolev spaces can be found in [25]. It should be noticed that, in view of our future applications to energy inequalities for the wave equations, here we need a different approach yielding estimates in twisted Sobolev spaces $H^m$.

To define what we mean by the elliptic part of $P_g$, observe that, by Eq. (2.4), the operator $P_g$ reads as

$$P_g w = -\partial_t^2 w + b_i \partial_t \partial_{\theta_i} w + b_0 \partial_t \partial_x w + b \partial_t w + L_g w.$$ 

The operator $L_g$, which we define as the elliptic part of $P_g$, is an elliptic operator of second order in the space variables $(x, \theta)$ (whose coefficients also depend on time) and which is of the form

$$L_g w = -D^2_x D_x w + \Delta_\theta w + O(x) D_{x\theta} w + O(x^2) D^2_{x\theta} w.$$  

We will consider the equation

$$L_g w = F \text{ in } (0, a) \times S^{n-2}$$  

with suitable Dirichlet boundary conditions

$$x^\alpha w|_{x=0} = w|_{x=a} = 0.$$  

When considering weak solutions to the equations, these conditions will simply translate as the requirement that the solution $w$ lies in $H^1_0$. Notice that the leading part of the operator $L_g$, namely $-D^2_x D_x + \Delta_\theta$, defines a self-adjoint operator in $L^2$ with domain $H^1_0$ for all positive $\alpha$. In fact, for $\alpha \geq 1$ this operator is essentially self-adjoint on the space of smooth, compactly supported functions, while for $\alpha \in (0, 1)$ it admits more than one self-adjoint extensions [19].

There are two important aspects we need to pay attention to in the derivation of the estimates at infinity for the equation (4.2): the way the twisted derivative and its adjoint enter the estimates, and the dependence of the various constants on the small parameter $a$. We will first consider the simpler problem

$$D^*_x D_x w - \Delta_\theta w = F(x, \theta) \text{ in } (0, a) \times S^{n-2},$$  

again with the boundary conditions (4.3), with the obvious notion of weak solutions $w \in H^1_0$. To spell out the details in the simplest case, we will say that a function $w \in H^1_0$ is a weak solution of the problem (4.4) if

$$\int (D_x w D_x v + \nabla_\theta w \cdot \nabla_\theta v) x \, dx \, d\theta = \int F v x \, dx \, d\theta.$$
for all \( v \in H^1_0 \). Here \( \nabla_\theta \) corresponds to the covariant derivative on the \((n-2)\)-sphere, the dot product is also taken with respect to the sphere metric and hereafter all integrals are taken over \((0,a) \times S^{n-2} \) unless otherwise stated.

Some arguments will be made slightly easier by keeping track of the dependence of some constants on the width \( a \) of this chart. For this, it is convenient to replace the variable \( x \in (0,a) \) by a rescaled variable

\[ \xi := x/a, \]

which takes values in \((0,1)\). With some abuse of notation, let us denote the twisted derivatives with respect to \( \xi \) by

\[ D_\xi w := \partial_\xi w + \frac{\alpha}{\xi} w, \quad D_\xi^\alpha w = -\partial_\xi w + \frac{\alpha - 1}{\xi} w \]

and define the ordered \( m \)th twisted derivative \( D^{(m)}_\xi \) similarly (cf. Eq. (3.3)). In terms of this new variable, Eq. (4.4) reads as

\[
\frac{D_\xi^2 D_{\xi} w}{a^2} - \Delta_\theta w = F \quad \text{in} \quad (0,1)\xi \times S^{n-2}_\theta, \quad \xi^\alpha w|_{\xi=0} = w|_{\xi=1} = 0,
\]

so that

\[
\int \left( \frac{D_\xi w D_{\xi} v}{a^2} + \nabla_\theta w \cdot \nabla_\theta v \right) \xi \, d\xi \, d\theta = \int F v \, d\xi \, d\theta
\]

for any \( v \in H^1_0 \).

The dependence of the various estimates on the small constant \( a \) will be described then in terms of the norms

\[
\|w\|_{H^2} := \|w\|_{L^2},
\]

\[
\|w\|_{H^1_0} := \frac{\|w\|_{L^2}}{a} + \frac{\|D_\xi w\|_{L^2}}{a} + \|D_\theta w\|_{L^2},
\]

\[
\|w\|_{H^i_0} := \frac{\|w\|_{L^2}}{a} + \sum_{l=1}^m \left( \|D_\theta^l w\|_{L^2} + \frac{\|D_\theta^{l-1} D_\xi w\|_{L^2}}{a} \right) + \sum_{l=0}^{m-2} \frac{\|D_\theta^l D_\xi^{(m-l)} w\|_{L^2}}{a^2},
\]

where \( m \geq 2 \). We will also need the norms \( \|\cdot\|_{H^m_0} \), which are defined as above after substituting the parameter \( a \) by 1.

The basic estimate for the simplified problem (4.4) is the following, which generalizes the \( H^2 \) estimate obtained by Warnick [29] in the case \( \alpha < 1 \) to \( H^{m+2} \) estimates in the full range \( \alpha > 0 \):

**Theorem 4.1.** Suppose \( w \in H^1_0 \) is a weak solution of the problem (4.4), with \( F \in H^m \) for some \( m \geq 0 \). Then \( w \in H^{m+2} \) and satisfies the estimate

\[ \|w\|_{H^{m+2}} \leq c_m \|F\|_{H^m_0}, \]

where the constant \( c_m \) does not depend on \( a \).
Proof. The $\mathcal{H}_a^1$ estimate

\begin{equation}
\frac{\|D\xi w\|_{L^2}}{a} + \|D\theta w\|_{L^2} \leq c\|F\|_{L^2}
\end{equation}

follows immediately from the identity (4.7) after taking $v = w$. This implies that $\|w\|_{L^2} \leq c\|F\|_{L^2}$ by Corollary 3.5.

Let us now prove the $\mathcal{H}_a^2$ estimates

\begin{equation}
\frac{\|D_{\xi}D_{\xi}w\|_{L^2}}{a^2} + \frac{\|D_{\theta}D_{\xi}w\|_{L^2}}{a} + \|D_{\theta}^2 w\|_{L^2} \leq c\|F\|_{L^2}.
\end{equation}

To explain the gist of the method, suppose we formally take $v = -Y^*_j w$ in the identity (4.7), where $Y_j := \partial_{\theta_j}$ and $Y^*_j$ is its formal adjoint, and sum over $j$. After integrating by parts we find that

\[ I := \sum_{j=1}^{n-2} \int \left( \frac{D\xi w D\xi (Y^*_j Y_j w)}{a^2} + \nabla_{\theta_j} w \cdot \nabla_{\theta_j} (Y^*_j Y_j w) \right) \xi d\xi d\theta \]

\[ = \sum_{j=1}^{n-2} \int \left( \frac{D\xi Y_j w}{a^2} + |\nabla_{\theta_j} (Y^*_j w)|^2 + \Gamma_j (D\theta w, D\theta Y_j w) \right) \xi d\xi d\theta, \]

where $\Gamma_j, \tilde{\Gamma}_j$ are bilinear functions of their entries that depend smoothly on $x$ and $\theta$. Therefore we find

\begin{equation}
I \geq \frac{\|D\theta D\xi w\|_{L^2}^2}{a^2} + \|D\theta^2 w\|_{L^2}^2 - c\|D\theta w\|_{L^2} \|D\theta^2 w\|_{L^2} - c\|D\xi w\|_{L^2} \frac{\|D\theta D\xi w\|_{L^2}}{a} \frac{\|D\theta^2 w\|_{L^2}}{a}
\end{equation}

\begin{equation}
\geq c \left( \frac{\|D\theta D\xi w\|_{L^2}^2}{a^2} + \|D\theta^2 w\|_{L^2}^2 - c\|F\|_{L^2}^2 \right)
\end{equation}

where we have used the estimate (4.9) for the first derivatives of $w$ and the elementary identity $AB \leq \delta A^2 + B^2/(4\delta)$. Since, by (4.7),

\[ I = \sum_{j=1}^{n-2} \int F Y^*_j Y_j w \xi d\xi d\theta \leq c\|F\|_{L^2} \|D\theta^2 w\|_{L^2}, \]

using (4.10) we arrive at

\begin{equation}
\frac{\|D\theta D\xi w\|_{L^2}}{a} + \|D\theta^2 w\|_{L^2} \leq c\|F\|_{L^2}.
\end{equation}

Of course, this calculation does not make sense in this form. First of all, one cannot take $v = Y^*_j Y_j w$ because $w$ is not, a priori, differentiable enough. However, it is standard that this difficulty can be overcome by replacing the partial derivatives $\partial_{\theta_j}$ in the expression of $Y_j$ by finite differences (see e.g. [15]). The second problem is that the derivatives $\partial_{\theta_j}$ are not well-defined globally, as the coordinate $\theta^j$ is just local. It is well known too that this can be circumvented either by considering a family of vector fields of the form

\[ Y_j = \sum_{k=1}^{n-2} Y_{jk}(\theta) \partial_{\theta^k}, \]
where $Y_{jk}$ is a smooth function supported in a coordinate patch of the sphere $S^{n-2}$, and taking as many vector fields of this form as necessary to ensure that they span the whole tangent plane at each point of the sphere, or by using a partition of unity. For brevity, we will skip these cumbersome but standard details.

Once we have established the estimates \((4.11)\), the estimate for $D^* \xi D \xi w$ is easy: it suffices to isolate this term in the equation to write

$$\|D^* \xi D \xi w\|_{L^2} = \|F + \Delta_\theta w\|_{L^2} \leq \|F\|_{L^2} + c\|D^2 \theta w\|_{L^2} \leq c\|F\|_{L^2}.$$ 

The proof of the higher order estimates uses the same set of ideas. Basically, additional regularity in $\theta$ is recovered by integrating by parts in the identity \((4.5)\). For example, if $F \in H^1$ one would obtain the estimate

$$\|D^2 \xi w\|_{L^2} + \|\theta D \xi w\|_{L^2} \leq c(\|F\|_{L^2} + \|\theta D w\|_{L^2})$$

from the identity \((4.5)\) after taking $v = Y_{jk} Y_{lk} Y_{jk} Y_{jl} w$, with the same caveats as above. On the other hand, additional (twisted) derivatives with respect to $x$ are recovered by isolating

$$\|D^3 \xi w\|_{L^2} = F + \Delta_\theta w$$

and then taking as many derivatives as needed in this equation:

$$\|D^3 \xi w\|_{L^2} = \|\xi F + \Delta_\theta D^3 \xi w\|_{L^2} \leq \|D^2 \xi F\|_{L^2} + \|\Delta_\theta D^3 \xi w\|_{L^2} \leq c(\|F\|_{L^2} + \|\xi D^2 \xi F\|_{L^2} + \|\xi D^3 \xi F\|_{L^2}).$$

For $F$ with a larger number $m$ of derivatives in $L^2$, one would repeat this process $m$ times, increasing by two the number of angular derivatives taken in the function $v$ and differentiating the equation with respect to $\xi$ after that using $D^2 \xi$ or $D^3 \xi$ alternatively. The only things one has to pay attention to is that the twisted derivative $D^2 \xi$ and its adjoint $D^* \xi$ appear in the right places and that the powers of $a$ do appear as in the $\|\cdot\|_{H^{m+2}}$ norm. Details are largely straightforward and will be omitted. \hfill $\square$

The energy estimate at infinity for the full elliptic operator $L_\theta$ arises now as an easy corollary to Theorem 4.1. We can safely assume that the parameter $a$ is small.

**Corollary 4.2.** Let $w \in H^1_\theta$ be a weak solution of the equation $L_\theta w = F$ in $(0,a) \times S^{n-2}$, with $F \in H^m$ for some $m \geq 0$. Then $w \in H^{m+2}$ and satisfies the estimate

$$\|w\|_{H^{m+2}} \leq c_m \|F\|_{H^1_\theta}$$

with a constant $c_m$ independent of $a$.

**Proof.** It follows from Eq. \((4.1)\) that

$$L_\theta w = -\frac{D_\xi^* D_\xi w}{a^2} + \Delta_\theta w + O(1) D_\xi w + O(1) D^2_\xi D_\xi w$$

$$+ O(a) D_\theta D_\xi w + O(a) D_\theta w + O(a^2) D^2_\theta w + O(1) w.$$
From this expression and the definition of the $H^m_a$ norms, it then follows that the function $w$ satisfies the equation

$$-\frac{D^*_{\xi}D\xi w}{a^2} + \Delta w = \tilde{F}$$

with a function satisfying

$$\|\tilde{F}\|_{H^m} \leq \|F\|_{H^m} + ca\|w\|_{H^{m+2}_a}.$$  

Hence Theorem 4.1 yields the estimate

$$\|w\|_{H^{m+2}_a} \leq c\|F\|_{H^m} + ca\|w\|_{H^{m+2}_a},$$  

which proves the result provided $a$ is small enough (that is, $a$ smaller than some positive constant independent of $F$).

5. Wave propagation at infinity

Now that we have proved the estimates at infinity for the elliptic part $L_g$ of the wave operator $P_g$ in an asymptotically AdS chart, we are ready to move on to the next step in our proof of well-posedness, which is concerned with solving the following Cauchy problem in the region $\{|t| < T\}$ of an asymptotically AdS patch:

\begin{align}
(5.1a) \quad & P_g v = F(t, x, \theta) \quad \text{in } (-T, T) \times (0, a) \times S^{n-2}, \\
(5.1b) \quad & v(0) = v_0, \quad \partial_t v(0) = v_1.
\end{align}

Here $T$ is an arbitrary real constant and we will supplement this problem with the homogeneous Dirichlet boundary conditions (4.3). We will use the notation

$$H^k_t H^l_t := H^k_t((-T, T), H^l)$$

and similarly for other mixed Sobolev or Lebesgue spaces. We will say a function $F(t, x, \theta)$ belongs to the spacetime Sobolev space $H^m_{tx\theta}$ of order $m$ if $F \in H^k_t H^{m-k}_{tx\theta}$ for all $0 \leq k \leq m$. The corresponding norm will be denoted by $\|\cdot\|_{H^m_{tx\theta}}$.

Obviously, the non-standard part of Eq. (5.1) is hidden in the twisted derivatives $D_x$. The key point is that, with some work, the energy associated with the twisted Sobolev spaces $H^m$ allows us to overcome this difficulty. In what follows we will see how one can implement this idea. Again, without loss of generality we assume throughout this section that $a$ is suitably small.

**Theorem 5.1.** For any $F \in L^1_t L^2$, there is a unique weak solution $v \in L^2_t H^1_0 \cap H^1_t L^2$ to the problem (5.1). Moreover, it satisfies the energy estimate

$$\|v\|_{L^\infty L^1} + \|\partial_t v\|_{L^\infty L^2} \leq c e^{caT} (\|v_0\|_{H^1} + \|v_1\|_{L^2} + \|F\|_{L^1_t L^2})$$

with a constant $c$ independent of $a$ and $T$.

**Proof.** It follows from Eq. (2.4) that the equation $P_g v = F$ can be written as

$$\partial_t^2 v + \frac{D^2 \xi (bD\xi v)}{a^2} + \partial_\theta^2 (b^{ij} \delta_{ij}) v + a\partial_\theta (b^i \partial_i v) + a\partial^\xi (b^i \delta_i v) = F + a\partial_\xi (b^0 \partial_0 v) + a\partial_\theta (b^i \partial_i v) + \mathcal{O}(1)D\xi v + \mathcal{O}(a) D_\theta v + \mathcal{O}(1)v,$$
where \( \partial^*_\xi \) and \( \partial^*_\eta \) are the formal adjoints of the partial derivatives \( \partial_\eta, \partial_\xi \) with respect to the measure \( \xi \, d\xi \, d\theta \) and

\[
b = 1 + O(a^2), \quad \gamma^{ij} = (g_{S^{n-2}})^{ij} + O(a^2), \quad b^0, b^i, \tilde{b}^i = O(1).
\]

Notice that \( b^0, b^i \) and \( \tilde{b}^i \) vanish on \( \{ \xi = 0 \} \). Let us define the energy as

\[
E[v](t) := \int_{(0,1) \times S^{n-2}} \left( v^2 + \frac{b}{a^2}(D_\xi v)^2 + \gamma^{ij} \partial_{\eta} v \, \partial_{\eta} v + 2ab^i \, \partial_\xi v \, \partial_\eta v \right) \, \xi \, d\xi \, d\theta.
\]

Since \( a \) is small, a simple computation shows that \( E(t) \) is equivalent to the (squared) \( H^1_a \) norm of \( v(t) \) in the sense that

\[
c_1 E[v](t)^{\frac{1}{2}} \leq \|v(t)\|_{H^1_a} \leq c_2 E[v](t)^{\frac{1}{2}}
\]

with constants independent of \( a \).

To derive the energy estimates, in this section we will take derivatives of \( v \) as if it were smooth. It is standard that this can be justified either by considering a smooth \( v \) and the density property proved in Corollary 3.5 or an argument using Garlerkin's method, in which we expand \( v \) (e.g.) in the basis of eigenfunctions of the operator \( a^{-2} D_\xi D_\xi - \Delta_0 \) in \( (0, a) \times S^{n-2} \) with Dirichlet boundary conditions.

Using the boundary conditions to integrate by parts and the equation satisfied by \( v \), a straightforward computation shows that

\[
\partial_t E[v](t) = 2 \int v_t \left( F - a \partial_\xi (b^0 v_t) - a \partial_\eta (\tilde{b}^i v_t) \right)
+ O(1) D_\xi v + O(a) D_\eta v + O(1)v \right) \, \xi \, d\xi \, d\theta.
\]

All the terms but the second and third ones can be immediately controlled as

\[
\int v_t \left( F + O(1) D_\xi v + O(a) D_\eta v + O(1)v \right) \, \xi \, d\xi \, d\theta
\leq \|F(t)\|_{L^2} E[v](t)^{\frac{1}{2}} + ca E[v](t),
\]

where we have used that \( \|v(t)\|_{L^2} \leq C \|D_\xi v(t)\|_{L^2} \leq ca E(t)^{1/2} \) by Proposition 3.5. The second term can be estimated as

\[
\int v_t \partial_\xi (b^0 v_t) \, \xi \, d\xi \, d\theta = \int \partial_\xi b^0 v^2_t \, \xi \, d\xi \, d\theta + \frac{1}{2} \int b^0 \partial_\xi (v^2_t) \, \xi \, d\xi \, d\theta
= \int \left[ \xi \partial_\xi b^0 - \frac{1}{2} \partial_\xi (\xi b^0) \right] v^2_t \, \xi \, d\xi \, d\theta \leq c E[v](t),
\]

where we have used that \( b^0 \) vanishes at \( \xi = 0 \). The third term can be controlled in a similar fashion, so we arrive at

\[
(5.6) \quad \partial_t E[v](t) \leq c \|F(t)\|_{L^2} E[v](t)^{\frac{1}{2}} + ca E[v](t),
\]

which is well known to imply that

\[
E[v](t)^{\frac{1}{2}} \leq c \left( E[v](0)^{\frac{1}{2}} + \int_0^t \|F(\tau)\|_{L^2} d\tau \right) e^{cat}
\]

by Gronwall’s inequality. (Here and in what follows, we follow the convention that the above integral is nonnegative independently of the sign of \( t \).)

Due to the equivalence of norms (5.5), this yields the desired energy estimate (5.2). It is standard that this energy estimate readily implies that there is at most a unique
for all $j$ source function $F$ to obtain higher order estimates we need to impose some compatibility conditions on the functions. In the case of $H^v(5.7)$ the vanishing of the trace to the boundary of the functions proves the existence of a unique solution to the problem. □

Garlerkin’s method [21] converges to a weak solution of the problem (5.1), thereby weak solution in $L^2$ written in terms of the initial conditions $v$ derivatives of $v$ looking at the algebraic structure of these functions reveals that $v$, $\sum_{l=0}^{j-2} \|F\|_{W^{l,\infty}} v^{k+j-2-l} + \|v_0\|_{H^{k+j}} + \|v_1\|_{H^{k+j-1}}$.)

In the next theorem, we obtain mixed energy estimates on the higher-order time derivatives of $v$ solving Eq. (5.1), assuming that the source term $F$, the boundary data $v_0$, $v_1$ and their higher time derivatives belong to suitably chosen Sobolev classes.

**Theorem 5.2.** Let us fix an integer $m \geq 1$. Assume that $F \in H^m_{l=\theta}$, $v_0 \in H^{m+1}$, $v_1 \in H^m$ and that $v_j \in H^0_\theta$ for all $0 \leq j \leq m$. Then the solution to the problem (5.1) is of class

$$v \in H^m_{l=\theta} \cap H^m_\theta$$

and satisfies

$$\sum_{l=0}^{m+1} \|\partial_{\xi}^l v\|_{L^2(H^{m+1-l})} \leq C e^{C T} \left( \sum_{j=0}^{m} \|\partial_{\xi}^j F\|_{L^2(H^{m-l})} + \|v_0\|_{H^{m+1}} + \|v_1\|_{H^m} \right).$$

**Proof.** In fact, we will prove the more precise estimate

$$\sum_{l=0}^{m+1} \|\partial_{\xi}^l v\|_{L^2(H^{m+1-l})} \leq c e^{T} \left( \sum_{j=0}^{m} \|\partial_{\xi}^j F\|_{L^2(H^{m-l})} + \sum_{j=0}^{m-1} \|v_j\|_{H^0_\theta} + \|v_{m+1}\|_{L^2} \right),$$

where the constant $c$ is independent of $a$ and $T$. Let us first prove the result for $m = 1$, using the same notation as in the proof of Proposition 5.1 without further notice.

Consider the energy

$$E_1(t) := E[v](t) + E[v_1](t).$$

The equation (5.1) implies that for a.e. $t$ the function $v$ satisfies the elliptic equation

$$L_\theta v = -v_{tt} + F + a\partial_\xi(b^\theta \partial_\xi v) + a\partial_{\theta}(\hat{b}^i \partial_i v) + \mathcal{O}(1)D_\xi v + \mathcal{O}(a) D_{\theta \theta} v + \mathcal{O}(1)v,$$

whose RHS can be estimated in $L^2$ norm by

$$c(E_1(t)\frac{1}{2} + \|F(t)\|_{L^2}).$$
on account of the equivalence (5.5) and the bounds for the coefficients of the equation. By Theorem 4.1, this yields the a priori estimate

\[(5.10) \quad \|v(t)\|_{H^2} \leq c(E_1(t)^\frac{1}{2} + \|F(t)\|_{L^2})\]

Let us now compute the variation of $E_1(t)$. One can easily check that

\[
\partial_t E_1(t) = 2\int v_t \left( \partial_t F - [\partial_t, P_g] v - a \partial_k (\partial^0 v_{tt}) - a \partial_k (\partial^1 v_{tt}) \right)
\]

\[
+ \mathcal{O}(1) D_\xi v_t + \mathcal{O}(a) D_{1\theta} v_t + \mathcal{O}(1) v_t \right) \xi d \xi dt
\]

Since

\[
\left\| [\partial_t, P_g] v(t) \right\|_{L^2} \leq c a \left\| v(t) \right\|_{H^2} \leq c (E_1(t)^\frac{1}{2} + \|F(t)\|_{L^2})
\]

by Eq. (5.10) and the other terms are basically as in Proposition 5.1, one can then use Eq. (5.6) and argue as in the proof of the aforementioned proposition, mutatis mutandis, to find that

\[(5.11) \quad \partial_t E_1(t) \leq c a E_1(t) + c (\|F(t)\|_{L^2} + \|\partial_t F(t)\|_{L^2}) E_1(t)^\frac{1}{2},\]

This readily yields

\[E_1(t)^\frac{1}{2} \leq c e^{ca t} \left( E_1(0)^\frac{1}{2} + \int_0^t \left( \|F(\tau)\|_{L^2} + \|\partial_t F(\tau)\|_{L^2} \right) d \tau \right),\]

which, when combined with the elliptic estimate (5.10) and the norm equivalence (5.5), proves the estimate (5.9) for $m = 1$.

The general case is proved using the same ideas. Basically, one considers the energies

\[E_k(t) := \sum_{j=0}^k E[\partial^j v](t),\]

with $k \leq m$. By taking time derivatives in the equation and using elliptic estimates as in (5.10), one finds that for a.e. $t$ we have

\[\sum_{j=0}^{k-1} \|\partial^j v(t)\|_{H^{k+1-j}} \leq c \left( E_k(t) + \sum_{j=0}^{k-1} \|\partial^j F(t)\|_{H^{k+1-j}} \right).\]

Arguing now as in the proof of (5.11), one then finds that $E_k$ satisfies the estimate

\[(5.12) \quad \partial_t E_k(t) \leq c a E_k(t) + c E_k(t)^{\frac{1}{2}} \sum_{j=0}^{k-1} \|\partial^j F(t)\|_{H^{k+j}},\]

to which we can apply Gronwall’s inequality. This yields the desired estimates arguing as above. The details are omitted.

In later sections it will be useful to have the following estimate for the time evolution of the $H^k$ norm of $v$ and $\partial_t v$ again in terms of $F$ and the initial data $v_0$, $v_1$:

**Proposition 5.3.** Under the same hypotheses of Theorem 5.2, there is a constant $c$ that does not depend on $T$ such that for a.e. $t$

\[\|v(t)\|_{H^{m+1}} + \|\partial_t v(t)\|_{H^{m}} \leq c e^{c|a|T} \left( \|v_0\|_{H^{m+1}} + \|v_1\|_{H^{m}} + \int_0^t \|F(\tau)\|_{H^{m}} d \tau \right).\]
Proof. With the notation of the proof of Theorem 5.1 (cf. Eq. (5.3)), let us set
\[ e_m(t) := \sum_{j+|\beta| \leq m} \int_{(0,1) \times S^{n-2}} \left[ (D_\xi^{(j)} D_\theta^\beta v_t)^2 + \frac{b}{a^2} (D_\xi^{(j+1)} D_\theta^\beta v)^2 ight. \]
\[ + \gamma^{ij} D_\xi^{(j)} D_\theta^\beta \partial_{\theta^i} v D_\xi^{(j)} D_\theta^\beta \partial_{\theta^j} v + 2ab^i D_\xi^{(j)} D_\theta^\beta \partial_{\xi^i} v D_\xi^{(j)} D_\theta^\beta \partial_{\theta^j} v \left. \right] d\xi d\theta, \]
where \( \beta = (\beta_1, \ldots, \beta_{n-2}) \) denotes a multiindex and, as usual, we are abusing the notation for angular derivatives. One should compare this expression with that of the energy \( E[v](t) \), see Eq. (5.4). Arguing as in the case of \( E[v](t) \), it is easy to see that \( e_m(t) \) satisfies
\[ \frac{e_m(t)}{c} \leq \|v(t)\|_{H^{m+1}}^2 + \|\partial_t v(t)\|_{H^m}^2 \leq c e_m(t). \]

A tedious but straightforward computation similar to the ones carried out in the proof of Theorem 5.2 shows that the derivative of \( e_m \) can be estimated by
\[ \partial_t e_m(t) \leq c e_m(t) + c e_m(t) \|F(t)\|_{H^m}. \]
Since the commutator
\[ [D_\xi, D_\xi^\rho w] = \frac{1-2\alpha}{\xi^2} w \]
is singular at \( \xi = 0 \), the key point to check in order to derive this inequality is that all the twisted derivatives appear with the right ordering. Once the differential inequality has been established, the proposition follows from the norm equivalence (5.13) and Gronwall’s inequality. \( \square \)

6. Peeling off large solutions near the conformal boundary

We now move on the case of solutions of the wave equations (1.1) and (1.2) with nontrivial boundary conditions at the conformal infinity. Again, we work in an asymptotically AdS chart and our specific goal for this section is to show how we can reduce the analysis of a solution with nontrivial boundary conditions at infinity to that of the sum of certain terms that are large at infinity, but essentially controlled, and another solution that depends on the boundary datum in a complicated way but is smaller at infinity. We refer to this method of passing from the boundary datum to this sum of large but controlled terms and a smaller function, which is described in Theorem 6.3, as a peeling off of the large solutions near the conformal boundary.

We recall that the nonlinearities that we are considering in Eq. (1.2) are of the form (1.9). Throughout this paper we will assume that the exponent \( q \) of this nonlinearity satisfies
\[ q \geq \alpha + 2. \]
As discussed in the Introduction, choosing a large enough value of \( q \) aims at ensuring that the nonlinearity decays fast enough at timelike infinity. The decay condition (6.1) is probably not sharp.

The nonlinear wave equation (1.2) with nonlinearity (1.9) reads in an asymptotically AdS chart as
\[ P_g u = Q(u, u), \]
where we have used the relationship (2.5) to write the nonlinear term as

$$Q(u, v) := x^{q - \frac{n - 2}{2} - t} \tilde{F}(t, x, \theta) g^{\mu\nu} \partial_\mu (x^{\frac{n - 2}{2}} u) \partial_\nu (x^{\frac{n - 2}{2}} v).$$

Observe that $\tilde{F}$ is smooth up to the boundary (that is, in $\mathbb{R} \times [0, a] \times S^{n-2}$) and that Greek subscripts run over the set $\{t, x, \theta_1, \ldots, \theta^{n-2}\}$. The linear wave equation corresponds to the case where $Q$ is identically zero, cf. Eq. (1.1).

To describe the kind of terms that appear in the equations, we will introduce the notation $\mathcal{O}_p(x^s h^{\leq k})$ to denote functions that are of order $x^s$ and whose dependence on $(t, \theta)$ can be controlled in terms of a polynomial of the first $k$ derivatives of certain function $h(t, \theta)$. Specifically, given a smooth enough function $h(t, \theta)$, a real $s$ and an integer $k$, we will say that a function $H(t, x, \theta)$ is of order $\mathcal{O}_p(x^s h^{\leq k})$ if it can be written as a finite sum

$$H(t, x, \theta) = \sum_{i=1}^{N_H} a_i(x) H_i(t, \theta)$$

in which the summands obey the bounds

$$|\partial^\mu_a_i(x)| \leq C_{H,i} x^{s-i}, \quad |D_{\theta_i} H_i(t, \theta)| \leq C_{H,i} \sum_{|J| \leq l+k} \prod_{i=1}^M |D_{\theta_i}^J h(t, \theta)|.$$  

Here

$$J = (J_1, \ldots, J_M)$$

is a set of $M$ nonnegative integers, with $M = M_{H,1}$ also depending on $H$ and the number of derivatives considered, and $|J|$ stands for the sum of these integers. A crucial property of the summands is that, by the Moser inequalities, whenever $l + k > \frac{n-2}{2}$ we have

$$\|H_i\|_{H^{\mu}_{\theta_i}} \leq C_1(M) C_{H,i} \left( \|h\|_{H^{i+k}_{\theta_i}} + \|h\|_{H^{\mu}_{\theta_i}}^M \right).$$

When the bounds (6.2) hold with $M = 1$, we drop the subscript to write $\mathcal{O}(x^s h^{\leq k})$. This case is particularly relevant because it appears in the analysis of the linear wave equation. Of course, in this case the estimate (6.3) is valid for all $l \geq 0$.

In the following lemma we describe the basic step of the peeling off procedure. The point here is to observe the powers of $x$ that appear in the different terms and how the implicit constants (namely $C_{H,i}, N_H$ and $M = M_{H,i}$, in the above notation for the function $H \in \mathcal{O}_p(x^s h^{\leq k})$) are controlled in terms of the initial functions $F$ and $G$. To be more precise, in this section we will say that certain quantity $H \in \mathcal{O}_p(x^s h^{\leq k})$ is controlled in terms of $F \in \mathcal{O}_p(x^{s'} h^{\leq k'})$ if the constants of $H$ can be estimated in terms of those of $F$ as

$$C_{H,i} + M_{H,i} + N_H \leq C_{\sup_{0 \leq j \leq l} C_{F,j}, \sup_{0 \leq j \leq l} M_{F,j}, N_F},$$

where the function $C(\cdot, \cdot, \cdot)$ is independent of $H$ and $F$.

**Lemma 6.1.** Suppose that the function $v$ satisfies an equation of the form

$$P_g v + Q(v, G) - Q(v, v) = F,$$

with

$$F \in \mathcal{O}_p(x^{-\alpha + \tau} h), \quad G \in \mathcal{O}_p(x^{-\alpha + \nu} h),$$

then
Using that we can now write the RHS of this equation as

\[ v_1 := v + O_p(x^{-\alpha+r+2}h) + O_p(x^{-\alpha+r+2}\tilde{h}) \]

that satisfies an equation of the form

\[ P_p v_1 + Q(v_1, O_p(x^{-\alpha+s}\tilde{h}) + O_p(x^{-\alpha+r+2}h)) - Q(v_1, v_1) = O_p(x^{-\alpha+r+2}\tilde{h}^{\leq 2}) + O_p(x^{-\alpha+r+2}\tilde{h}^{\leq 2}). \]

Furthermore:

(i) All the terms of the form \( O_p(x^s\tilde{h}^{\leq k}) \) (resp. \( O_p(x^s\tilde{h}^{\leq k}) \)) are controlled in terms of \( F \) (resp. \( G \)) in the sense specified above.

(ii) If \( v \) satisfies the boundary condition \( x^\alpha v|_{x=0} = 0 \), then we also have \( x^\alpha v_1|_{x=0} = 0 \).

(iii) If \( F \) and \( G \) are supported in the region \( \{ x < a_0 \} \), then one can ensure that the various terms \( O_p(\cdots) \) arising in Eqs. (6.6) and (6.7) are also supported in this region.

(iv) If the equation is linear (i.e., \( Q := 0 \)), the statement remains valid with \( O_p(\cdots) \) replaced by \( O(\cdots) \).

Proof. It is clear that \( F \) can be written as

\[ F = x^{-\alpha+r}H_1(t, \theta) + x^{-\alpha+r+1}H_2(t, \theta) + O_p(x^{-\alpha+r+2}h), \]

where the functions \( H_j \) are bounded by powers of \( h \) as in Eq. (6.2). Let us denote by \( \chi(x) \) a smooth function supported in \( \{ x < a_0 \} \) and equal to 1 in \( \{ x < a_0/2 \} \).

Setting

\[ v_0 := v + \frac{\chi(x) x^{-\alpha+r+2}H_1(t, \theta)}{\alpha^2 - (\alpha - r - 2)^2} + \frac{\chi(x) x^{-\alpha+r+3}H_2(t, \theta)}{\alpha^2 - (\alpha - r - 3)^2} \]

and making use of the elementary identity

\[ D_x^* D_x(x^s) = (\alpha^2 - s^2)x^{s-2}, \]

we find after a short computation that

\[
\begin{align*}
P_p v_0 - Q(v_0, O_p(x^{-\alpha+s}\tilde{h}) + O_p(x^{-\alpha+r+2}h)) + Q(v_0, v_0) &= O_p(x^{-\alpha+r+2}\tilde{h}^{\leq 2}) + O_p(x^{-\alpha+r+s}\tilde{h} + O_p(x^{-\alpha+r+s+2}(\tilde{h})^{\leq 1}).
\end{align*}
\]

Using that

\[ O_p(x^s(\tilde{G} H)^{\leq k}) \in O_p(x^s G^{\leq k}) + O_p(x^s H^{\leq k}) \]

we can now write the RHS of this equation as

\[ x^{-\alpha+r}H_2(t, \theta) + x^{-\alpha+r+1}H_3(t, \theta) + O_p(x^{-\alpha+r+2}h^{\leq 2}) + O_p(x^{-\alpha+r+s+2}\tilde{h}^{\leq 1}), \]

where \( H_2 \) and \( H_3 \) (which can be identically zero) can be controlled pointwise in terms of powers of \( h \) and \( \tilde{h} \). If we now define

\[ v_1 := v_0 + \frac{\chi(x) x^{-\alpha+r+2}H_2(t, \theta)}{\alpha^2 - (\alpha - r - 2)^2} + \frac{\chi(x) x^{-\alpha+r+3}H_3(t, \theta)}{\alpha^2 - (\alpha - r - 2)^2}, \]

we readily find that \( v_1 \) satisfies an equation of the form (6.7). The statement about the boundary conditions of \( v_1 \) and the support of the various terms \( O_p(\cdots) \) stems
from the construction, as does the fact that the statement remains valid when \( Q \) is zero and all the subscripts are dropped in the terms \( \mathcal{O}_p(\cdots) \).

**Remark 6.2.** The argument is actually also valid when the condition (6.5) does not hold, the only difference being that when \( r = 2\alpha - m \) for some nonnegative integer \( m \in \{2, 3\} \), the summand that we use to construct \( v_1 \) or \( v_2 \) whose denominator would apparently be singular is replaced by a term \( \log x \alpha H_j(t,\theta) \). In particular, the case \( n = 5, \mu = 0, \alpha = 2 \) is important as it encompasses the case of gravitational waves in AdS\(_5\), see [6]. It would be an interest project to examine the structure of these logarithmic terms in further detail.

After describing the basic step of the peeling off procedure, we are ready to state and prove the main result of this section. For simplicity of exposition, we assume that \( 2\alpha \) is not an integer, although it will be clear from the proofs that all the results we prove in this section remain valid when \( 2\alpha \in \mathbb{N} \) (and for real values of \( r,s \)) provided one includes suitable logarithmic terms in the statements when necessary (see Remark 6.2 above).

**Theorem 6.3.** Suppose that \( u \) satisfies the equation

\[
P_g u = Q(u,u)
\]

with boundary condition \( x^\alpha u|_{x=0} = f(t,\theta) \). Then, for any positive integer \( k \), one can take a function

\[
u_k = u - \sum_{j=0}^{k-1} \mathcal{O}_p(x^{-\alpha+2j}f^{\leq 2j})
\]

that satisfies an equation of the form

\[
P_g u_k = Q(u_k,u_k) + \sum_{j=0}^{k-1} \mathcal{O}_p(x^{-\alpha+2j}f^{\leq 2j}) + \mathcal{O}_p(x^{-\alpha+2k-2}f^{\leq 2k})
\]

and the homogeneous boundary condition \( x^\alpha u_k|_{x=0} = 0 \). Furthermore,

(i) All the terms \( \mathcal{O}_p(x^{-\alpha+r}f^{\leq 2j}) \) that appear can be assumed to be supported in the region \( \{x < a_0\} \) for any fixed \( a_0 \).

(ii) The implicit constants that appear in the terms \( \mathcal{O}_p(x^s f^{\leq k}) \) admit bounds independent of \( f \).

(iii) If the equation is linear (\( Q := 0 \)), the statement remains valid with \( \mathcal{O}_p(\cdots) \) replaced by \( \mathcal{O}(\cdots) \).

**Proof.** Let \( \chi(x) \) be a smooth function supported in \( \{x < a_0\} \) that is identically equal to 1 is \( \{x < a_0/2\} \). Setting

\[
u_1 := u - \chi(x)x^{-\alpha}f(t,\theta)
\]

we immediately find that \( x^\alpha u_1|_{x=0} = 0 \) and

\[
P_g u_1 = Q(u_1,u_1) + \mathcal{O}_p(x^{-\alpha}f) + \mathcal{O}_p(x^{-\alpha}f^{\leq 2}),
\]

where the terms \( \mathcal{O}_p(x^{-\alpha}f) \) and \( \mathcal{O}_p(x^{-\alpha}f^{\leq 2}) \) are supported in \( \{x < a_0\} \). This proves the statement for \( k = 1 \). The general case then follows by repeatedly applying Lemma 6.1 to this equation. \( \square \)
7. Holography for the linear wave equation

We are now ready to show the well-posedness of the Klein–Gordon equation in a general asymptotically AdS manifold. Our well-posedness theorem corresponding to the holographic prescription in this geometric setting (Theorem 7.2) will follow by combining the energy estimates obtained in Theorems 5.1 and 5.2 for the wave propagation at infinity, the peeling properties obtained for the solutions of the linear problem near the conformal boundary obtained in Theorem 6.3, and standard estimates for the wave equation away from the conformal boundary.

First of all, let us give a precise definition of what an asymptotically AdS space-time is. We recall that a Lorentzian manifold $M$ is said to be asymptotically AdS (cf. e.g. [17]) if there is a spatially compact set $K$ such that $M\setminus K$ is the union of asymptotically AdS patches. Therefore, $M$ is covered by a finite number of coordinate charts $U_1, \ldots, U_N, V_1, \cdots, V_M$, where the patches $U_j$ are asymptotically AdS in the sense of Definition 2.1 and the patches $V_k$ are contained in $K$ (and therefore “regular”). We also assume that $M$ is time-oriented, meaning that there is a nowhere vanishing timelike vector field $T$, and that this vector field coincides with the partial derivative $\partial_t$ in each asymptotically AdS chart. For simplicity, we will also assume that the time function $t$ is globally defined in $M$, satisfies $T t = 1$ and foliates $M$ as

$$M = \bigcup_{\tau \in \mathbb{R}} M_{\tau}, \quad M_{\tau} := \{ t = \tau \}.$$ 

We will also use the notation

$$M_{(T_1, T_2)} = \bigcup_{T_1 < \tau < T_2} M_{\tau}$$

for the part of the manifold sandwiched between two time slices. Notice that we are not making the assumption that the defining function $x$ of the conformal boundary is defined in the whole manifold.

When solving the linear wave equation in $M$, the finite speed of propagation ensures that, if $\phi$ solves the equation

$$\Box_g \phi - \mu \phi = F \quad \text{in } M,$$

$$\phi|_{M_0} = \phi_0, \quad T \phi|_{M_0} = \phi_1$$

with $F, \phi_0, \phi_1$ supported in the coordinate chart $V_k$ (resp. $U_j$), for time $t$ in some small enough interval $(-T, T)$ the function $\phi(t)$ is supported in a lightly larger set $\tilde{V}_k$ that is still contained in $K$ (resp. in another slightly larger asymptotically AdS chart $U'_j$). To exploit this propagation property to immediately derive global estimates for the wave equation in $M$, let us define the twisted Sobolev norm in $M$ at time $t$, denoted

$$\|\phi(\tau)\|_{H^k(M_{\tau})},$$

as the sum of the usual $H^k$ norm of $\phi(t)$ in the spacelike regions $V_k \cap M_{\tau}$ and the twisted norm $\|u(\tau)\|_{H^k(U_{\tau})}$ corresponding to the asymptotically AdS charts (again, the relationship between $\phi$ and $u$ in the asymptotically AdS chart is given by (2.2)). The key here is that, near the conformal boundary, the norm we take for $\phi$ is obtained by transplanting the norm of the associated function $u$ which we can control with our estimates in an asymptotically AdS chart. By the propagation properties of the wave equation, it then follows that the estimates proved in Theorems 5.1
and 5.2, together with the well-known results for the wave equation in globally hyperbolic spaces, yield $H^k(M)$-energy estimates for the linear wave equation in $M$ analogous to those proved in asymptotically AdS charts.

It should be stressed that our manifold $M$ has been obtained by patching together “regular” coordinate patches $V_j$ and asymptotically AdS patches $U_k$, so we have not allowed for the presence of stationary black holes in $M$. Notice, however, that the problem of wave propagation near the black hole and close to timelike infinity are, in a way, effectively separated. Hence if one can derive suitable estimates near certain stationary black hole using a suitable redshift argument (see [12, 27]), then one would naively expect to be able to incorporate charts with this kind of black holes too [19].

Let us summarize the conclusion of the preceding discussion in the following theorem, where of course $H^0_0(M_0)$ stands for the completion of $C^\infty(M_0)$ in the $H^1(M_0)$ norm. The corresponding spacetime norm $H^k(M)$ is defined in the obvious fashion and we set

$$
\|\phi\| \equiv (\int_{\tau_1}^{\tau_2} \|\phi\|^p_{H^k(M_\tau)} d\tau)^{1/p},
$$

$$
\|\phi\|_{L^\infty H^k(M)} := \sup_{\tau_1 < \tau < \tau_2} \|\phi\|_{H^k(M_\tau)}.
$$

In order to derive higher-order estimates, we need to assume that the functions $v_j$ defined in Eq. (5.7) vanish at the conformal boundary, which is the set $\{x = 0\}$ in each asymptotically AdS chart. An equivalent, more convenient way of doing this more invariantly is to demand that $T\phi|_{M_0} \in H^k_0(M_0)$. Of course, this quantity can be written in terms of the initial conditions $\phi_0, \phi_1$ and the function $F$ alone.

**Theorem 7.1.** Let $m$ be an integer $m \geq 0$. Assume that $F \in H^m(M)$, $\phi_0 \in H^{m+1}(M_0)$, $\phi_1 \in H^m(M_0)$ and that $T\phi|_{M_0} \in H^1_0(M_0)$ for all $0 \leq j \leq m$. Then the problem (7.1) has a unique solution in $M$, which is of class $H^{m+1}(M_{(-T,T)})$ for any $T$ and satisfies the estimates

$$
\sum_{\tau_1}^{\tau_2} \|T\phi\|_{L^\infty H^{m+1}(M_{(-T,T)})} \leq C e^{CT} \left( \sum_{j=0}^{m} \|T^j F\|_{L^1 H^{m-j}(M_{(-T,T)})} \right.
$$

$$
+ \|\phi_0\|_{H^{m+1}(M_0)} + \|\phi_1\|_{H^{m}(M_0)} \bigg).
$$

Furthermore, for a.e. $\tau$ we also have

$$
\|\phi\|_{H^{m+1}(M_\tau)} + \|T\phi\|_{H^{m}(M_\tau)} \leq C e^{C|\tau|} \left( \|\phi_0\|_{H^{m+1}(M_0)} + \|\phi_1\|_{H^{m}(M_0)} \right.
$$

$$
\left. + \int_{\tau}^{\tau} \|F\|_{H^{m}(M_\tau)} d\tau \right).
$$

**Proof.** The existence of a unique solution to the problem (7.1) in $M_{(-T,T)}$ and the fact that it satisfies the estimate (7.2) in this set follows from the above argument provided $T$ is small enough. Notice that the smallness of $T$ is only used to ensure that for all times $|t| < T$, the support of the solution to the problem (7.1) with $\phi_0, \phi_1$, and $F$ supported in the set $V_\kappa$ is contained in the slightly larger set $V_\kappa'$, and similarly with data supported in $U_j$. This means that we can choose $T$ to be...
independent of the particular choice of the data. Therefore, it is standard that we can repeat the argument, now using the functions \((\phi|_{\mathcal{M}_T}, T \phi|_{\mathcal{M}_T})\) as initial conditions on \(\mathcal{M}_T\), to eventually derive that the solution exists globally and satisfies the bound (7.2) for any finite \(T\).

The estimate (7.3) is then an immediate consequence of Proposition 5.3 (and of the standard energy estimates for wave equations in the non-asymptotically AdS charts \(\mathcal{V}_k\)). □

Note that the RHS of the inequality (7.2) is clearly finite for \(F \in H^m(\mathcal{M})\). Our next objective is to consider the case where we have nontrivial boundary conditions at some of the ends of the asymptotically AdS manifold, as in Eq. (1.7).

To begin with, let us make precise the meaning of these boundary conditions. To this end, let us recall that \(\mathcal{M}\) has \(N\) asymptotically AdS ends, which are covered by patches \(\mathcal{U}_1, \ldots, \mathcal{U}_N\). Given a function

\[
(7.4) \quad f = \left(f_1(t, \theta), \ldots, f_N(t, \theta)\right),
\]

we will say that the field \(\phi\) satisfies the boundary condition

\[
(7.5) \quad x^{\alpha - \frac{n-1}{2}} \phi|_{\partial \mathcal{M}} = f
\]

in the manifold if at each asymptotically AdS patch \(\mathcal{U}_j\) we have

\[
x^{\alpha - \frac{n-1}{2}} \phi|_{x=0} = f_j.
\]

Notice that the variables \((t, \theta)\) take values in \(\mathbb{R} \times S^{n-2}\) and that \(f\) can be understood as a function defined on \(\partial \mathcal{M}\). In fact, we will use the notation

\[
\|f\|_{H^k(\partial \mathcal{M})} := \sum_{j=1}^{N} \|f_j\|_{H^k_{t\theta}}.
\]

We are now ready to prove that, given a smooth enough datum \(f\) on the conformal boundary, there is a unique solution to the Klein–Gordon equation that satisfies the boundary condition (7.5), and that one can satisfactorily estimate this solution in terms of the boundary datum. In the following theorem, we show how the solution decomposes as the sum of terms that arise from the peeling-off procedure described in Section 6 and another function that is, in a way, smaller at infinity. We emphasize that, although the estimates are presented in twisted Sobolev spaces, Theorem 3.6 allows us to convert them into pointwise bounds.

We will state this result for the case where there is no source term, the boundary datum \(f\) is supported in the region \(\{t > 0\}\) and the equation has vanishing initial conditions, as this is the case of direct interest in cosmology and the statement is simpler. We are thus led to the problem

\[
(7.6a) \quad \Box g \phi - \mu \phi = 0 \quad \text{in} \ \mathcal{M},
\]
\[
(7.6b) \quad \phi|_{\mathcal{M}_0} = 0, \quad T \phi|_{\mathcal{M}_0} = 0.
\]

Of course, adding general initial conditions (compatible with the chosen boundary datum) and source terms is immediate in view of Theorem 7.1 and the linearity of the equation.
Theorem 7.2. Given a nonnegative integer $m$, let us take any integer
\[
(7.7) \quad k > \frac{m + 1 + \alpha}{2}
\]
and a boundary datum $f = (f_1, \ldots, f_N)$ as in (7.4), where we assume that $f_j \in H^{m+2k}_{\text{tot}}$ is supported in the region $\{ t > 0 \}$. Then the problem (7.6) has a unique solution, which is in $H^{m+1}_{\text{loc}}(M)$ and can be written as
\[
\phi = \sum_{j=0}^{k-1} \psi_j.
\]
For $j < k$, the function $\psi_j(t, x, \theta)$ is supported in the asymptotically AdS patches $\bigcup_{i=1}^{N} U_i$ and reads as
\[
(7.8) \quad \psi_j(t, x, \theta) = x^{\alpha - n \frac{2k-j}{2}} \chi_j(x) H_j(t, \theta),
\]
where $\chi_j(x)$ is a smooth function that is identically 1 in a neighborhood of $x = 0$ and
\[
(7.9) \quad \| H_j \|_{H^{m+2k-2j}_{\text{tot}}} \leq C \| f \|_{H^{m+2k}(\partial M)}
\]
for some constant independent of $f$. The function $\psi_k(t, x, \theta)$ is in $H^{m+1}(M(\mathcal{T}, \mathcal{T}))$ for all $T$ and is bounded as
\[
(7.10) \quad \sum_{l=0}^{m+1} \| T^l \psi_k \|_{L^\infty H^{m+1-l}(M(\mathcal{T}, \mathcal{T}))} \leq C_T \| f \|_{H^{m+2k}(\partial M)}.
\]

Proof. Let us recall that, in an asymptotically AdS patch, the equations $\Box_g \phi - \mu \phi = 0$ and $P_g u = 0$ are related by the transformation (2.2). Therefore, by applying Theorem 6.3 in each asymptotically AdS chart of $M$, we infer that there are functions $u_j$ of order $O(x^{-\alpha + 2j + \frac{m+1}{2}})$ supported in these charts and such that the function
\[
\psi_k := \phi - \sum_{j=0}^{k-1} x^{\alpha - n \frac{2k-j}{2}} u_j
\]
\[
\Box_g \psi_k - \mu \psi_k = x^{\frac{3-n}{2}} F
\]
and the boundary condition
\[
x^{\alpha - n \frac{2k-j}{2}} \psi_k |_{\partial M} = 0.
\]
Here $F$ is a function supported in the asymptotically AdS charts and of order $O(x^{m+2k-2} f^{\leq 2k})$. Notice that the functions $\psi_j := x^{n \frac{m+1}{2}} u_j$ and $x^{\frac{3-n}{2}} F$ are obviously well defined globally because $u_j$ is zero outside the asymptotically AdS charts, and the expression $f^{\leq k}$ for the $N$-component function $f$ has the obvious meaning. Furthermore, all the implicit functions in the terms $O(\cdots)$ are uniformly bounded, and the fact that the estimates (7.8) and (7.9) are satisfied is an immediate consequence of Theorem 6.3 and the definition of $\psi_j$.

Since $u_j = O(x^{-\alpha + 2j} f^{\leq 2j})$ and $f$ is supported in the set $\{ t > 0 \}$, it follows from Theorem 6.3 that $u_j$ is also supported in this set, so $\psi_k$ also satisfies the initial condition
\[
\psi_k |_{M_0} = 0, \quad T \psi_k |_{M_0} = 0.
\]
In an asymptotically AdS chart, the equation satisfied by $u_k$ reads as

$$P_g u_k = F$$

with $F = O(x^{-\alpha+k-2}f \lesssim 2k)$.

Since the function $x^{-\alpha+2k-2}$ belongs to the Sobolev space $H^m$ when the condition (7.7) holds, an easy computation shows that the function $\tilde{F}$ belongs to the space $H^m(M)$ with norm

$$\|\tilde{F}\|_{H^m(M)} \lesssim C\|f\|_{H_{10}^{m+2k}}.$$

Hence we can apply Theorem 7.1 to show that there is a unique solution $\psi_k$ to the above initial-boundary value problem, which satisfies the bound (7.10).

8. Application to nonlinear wave equations

Our goal in this section is to prove the local well-posedness of the nonlinear wave equation (1.2) in an asymptotically AdS manifold with a nontrivial boundary datum $f$ at conformal infinity (Eq. (1.7)). As before, we will assume that the nonlinearity is of the form (1.9). To derive this result, which is stated below as Theorem 8.1, we will make use of the full range of results obtained in the previous sections on the behavior of the linear wave equation and the properties of twisted Sobolev spaces.

Specifically, let us fix some positive number $T$ and consider the problem

$$\Box_g \phi - \mu \phi = \Gamma(\nabla \phi, \nabla \phi) \quad \text{in } M(-T,T),$$

$$\phi\mid_{M_0} = \phi_0, \quad T\phi\mid_{M_0} = \phi_1$$

with the boundary condition

$$x^{\alpha-n-1/2}\phi\mid_{\partial M(-T,T)} = f,$$

with $\partial M(-T,T)$ denoting the portion of the boundary $\partial M$ where $|t| < T$ and $f$ is as in Eq. (7.4). As before, $\Gamma$ has the behavior

$$\Gamma = x^q \tilde{\Gamma}(t, x, \theta)$$

in each asymptotically AdS chart, where the function $\tilde{\Gamma}$ is smooth up to the boundary, and the exponent $q$ can be different in each asymptotically AdS chart. Just as in Theorem 7.2, we can safely assume that $f$ is identically zero in the region $\{t \leq 0\}$ and that the initial conditions fall off at infinity.

The main result of this section is the following theorem, which proves that the problem (8.1) is locally well posed in suitable twisted Sobolev spaces. Notice that the estimate (8.2) below makes sense because $\psi_j$ is supported in the asymptotically AdS charts. By Theorem 3.6, the Sobolev estimates established here immediately yield pointwise estimates for the solution.

To prove this theorem, we start by peeling off the layers of the solution that are large at infinity, and then use a suitable bootstrap argument to control the remaining part of the solution. In the second argument we need to use the properties of the twisted Sobolev spaces to prove that certain nonlinear function of the solution is locally Lipschitz continuous, which is done in Lemma 8.3 below.
Theorem 8.1. Let us fix an integer \( m > \frac{n}{2} \). Take nonnegative integers \( k, l \) such that, together with the exponents \( q \) of the nonlinearity \( \Gamma \) (which may be different in each asymptotically AdS chart), satisfy the following conditions:

1. \( k > \frac{m + 1 + \alpha}{2} \).
2. \( l \geq 2k + m + \frac{n - 1}{2} \).
3. \( q > \alpha + m + \eta + \frac{5 - n}{2} \), where \( \eta \) is defined as in Proposition 3.9.

Take a function \( f \in H^l(\partial M_{(-T,T)}) \) (which we assume to vanish identically for \( t \leq 0 \)) and initial data \( \phi_0 \in H^{m+1}_0(M_0), \phi_1 \in H^m(M_0) \). For all \( T \) smaller than some positive constant depending only on the norms \( \|f\|_{H^l(\partial M_{(-T,T)})}, \|\phi_0\|_{H^{m}(M_0)} \) and \( \|\phi_1\|_{H^m(M_0)} \), the problem (8.1) has a unique solution \( \phi \), which can be written as

\[
\phi = \sum_{j=0}^{k} \psi_j.
\]

For \( j \leq k-1 \) and all \( s \), the terms \( \psi_j \) are supported in the asymptotically AdS charts and are bounded as

\[
\|x^{\alpha - 2j - \frac{n+1}{2}} \psi_j\|_{W_s^{\infty, \infty} H^{m+1}} \leq C_s \|f\|_{H^l(\partial M_{(-T,T)})},
\]

while the term \( \psi_k \) can be estimated as

\[
\sum_{l=0}^{m+1} \|T^l \psi_k\|_{L^\infty H^{m+1-l}(M_{(-T,T)})} \leq C_T \left( \|f\|_{H^l(\partial M_{(-T,T)})} \right)
+ \|\phi_0\|_{H^{m+1}(M_0)} + \|\phi_1\|_{H^m(M_0)}.
\]

Proof. Let us start by peeling off the large behavior of the function \( \phi \) near the conformal boundary \( \partial M_{(-T,T)} \). Since in an asymptotically AdS patch the equations \( \Box_g \phi - \mu \phi = 0 \) and \( P_\mu u = 0 \) are related by the transformation (2.2), we can invoke Theorem 6.3 in each asymptotically AdS chart of \( M \) to show that there are \( k-1 \) functions of the form

\[
\psi_j = x^{\frac{n-1}{2}} u_j,
\]

with \( u_j \) of order \( \mathcal{O}_b(x^{-\alpha + 2j + \frac{n-1}{2}} f^{\leq 2j}) \) and supported in these charts, such that the function

\[
\psi_k := \phi - \sum_{j=0}^{k-1} \psi_j
\]

satisfies the equation

\[
\Box_g \psi_k - \mu \psi_k = \tilde{N}(\psi_k) + \rho_k,
\]

the boundary condition

\[
x^{\alpha - \frac{n-1}{2}} \psi_k|_{\partial M} = 0
\]

and the initial conditions

\[
\psi_k|_{M_0} = \phi_0, \quad T\psi_k|_{M_0} = \phi_1.
\]
In this proof, all the implicit constants that appear in terms of order \( \mathcal{O}_p(x^r f^{\leq p}) \) are bounded independently of \( f \). The fact that the function \( \psi_j \) satisfies (8.2) is immediate in view of Theorem 6.3.

Let us discuss the various terms that appear in Eq. (8.4). The term \( \rho_k \) is supported in the asymptotically AdS charts and of the form
\[
\rho_k = x^{-\frac{n+1}{2}} \mathcal{O}_p(x^{-\alpha+2k-2} f^{\leq 2k}),
\]
so
\[
\|\rho_k\|_{L^\infty H^m(\mathcal{M}(\partial M_{\mathcal{M}}))} \leq C \|f\|_{H^i(\partial M(\mathcal{M}_{\mathcal{M}}))}.
\]
When evaluated a point outside the asymptotically AdS charts, the nonlinear term \( N(\psi) \) looks simply as
\[
N(\psi) = \Gamma g(\nabla \phi, \nabla \psi),
\]
that is, as the nonlinear term that we have introduced in the wave equation. In an asymptotically AdS chart and setting \( u_k := x^{-\frac{n}{2}} \psi_k \), Eq. (8.4) reads as
\[
P_g u_k = \tilde{N}(u_k) + \tilde{\rho}_k,
\]
where \( \tilde{\rho}_k \) is of the form
\[
\tilde{\rho}_k = \mathcal{O}_p(x^{-\alpha+2k-2} f^{\leq 2k})
\]
and the nonlinearity is
\[
\tilde{N}(u_k) := Q(u_k, u_k) + Q(R_k, u_k)
\]
for some function
\[
R_k = \sum_{j=0}^{k-1} \mathcal{O}_p(x^{-\alpha+2j} f^{\leq 2j}).
\]

To prove that the function \( \psi_k \) is uniquely determined, we will use a fixed point argument. For this, let us consider the Banach space
\[
\mathcal{X} := L^\infty H^{m+1}(\mathcal{M}(\partial M_{\mathcal{M}})) \cap W^{1,\infty} H^m(\mathcal{M}(\partial M_{\mathcal{M}})),
\]
endowed with its natural norm \( \| \cdot \|_{\mathcal{X}} \). The set of functions in this space of norm less than \( r \) will be denoted by
\[
\mathcal{X}_r := \{ \phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} < r \}.
\]
By Lemma 8.3 below, the inequality
\[
\|N(\psi) - N(\Psi)\|_{L^\infty H^m(\mathcal{M}(\partial M_{\mathcal{M}}))} \leq C_1 \|\psi - \Psi\|_{\mathcal{X}}
\]
is valid for all \( \psi, \Psi \in \mathcal{X}_r \), with a constant that depends on \( r \) and \( \|f\|_{H^i(\partial M(\mathcal{M}_{\mathcal{M}}))} \).

Let us now consider the solution map \( S \) of the problem
\[
\begin{align*}
\Box_g \phi - \mu \phi &= F &\text{in } \mathcal{M}(\partial M_{\mathcal{M}}), \\
\phi|_{\mathcal{M}_0} &= \phi_0, & T \phi|_{\mathcal{M}_0} &= \phi_1
\end{align*}
\]
with boundary condition
\[
x^{\alpha -\frac{n+1}{2}} \phi|_{\partial M} = 0,
\]
which maps each \( F \in L^\infty H^m(\mathcal{M}(\mathcal{M}_{\mathcal{M}})) \) to the unique solution \( \phi =: S(F) \) of the problem. Theorem 7.1 implies that \( S \) defines a continuous affine map from \( L^\infty H^m(\mathcal{M}(\mathcal{M}_{\mathcal{M}})) \) to \( \mathcal{X} \) with
\[
\|S(F)\|_{L^\infty H^m(\mathcal{M}(\mathcal{M}_{\mathcal{M}}))} \leq C_2 (Tr + \|\phi_0\|_{H^{m+1}(\mathcal{M}_0)} + \|\phi_1\|_{H^m(\mathcal{M}_0)}).
\]
for all $F \in \mathcal{X}$, and all $T$ smaller than some fixed constant $T_0$ (roughly speaking, $C_2 = C_0 CT_0$, in the notation of Theorem 7.1 and the factor of $T$ comes from the trivial estimate $\|F\|_{L^1_\tau H^m} \leq T \|F\|_{L^\infty_\tau H^m}$). We will use the notation $S_0 : L^\infty H^m(\mathcal{M}_{(-T,T)}) \to \mathcal{X}$ for the solution map of Eq. (8.8) with trivial initial data $\phi_0 = \psi_0 = 0$, which is a bounded linear map. Notice, in particular, the elementary identity

$$S(F) - S(G) = S_0(F - G).$$

To construct the function $\psi_k$, let us recursively define the functions

$$\Psi_0 := S\rho_k, \Psi_i := S(\rho_{k+i} + \mathcal{N}(\Psi_{i-1})), \quad i \geq 1.$$

A standard argument then shows that if $T$ is smaller than some $T_0(\|f\|_{H^1(\partial \mathcal{M}_{(-T,T)})})$, then $\Psi_i$ converges to the only solution $\psi_k$ to our problem, which satisfies the bound (8.3). To see this, notice that the fact that $\mathcal{N}(0) = 0$ and the estimate (8.7) imply that

$$\|\mathcal{N}(\Psi)\|_{L^\infty H^m(\mathcal{M}_{(-T,T)})} \leq C_1 r$$

for all $\Psi \in \mathcal{X}$. Hence taking $r$ large enough (e.g., larger than $2\|S\rho_k\|_{\mathcal{X}}$) and using a standard bootstrap argument we can ensure that

$$\|\Psi_i\|_{\mathcal{X}} \leq \|S\rho_k\|_{\mathcal{X}} + \|S_0\mathcal{N}(\Psi_{i-1})\|_{\mathcal{X}} \leq \frac{r}{2} + C_2 T \|\mathcal{N}(\Psi_{i-1})\|_{L^\infty H^m(\mathcal{M}_{(-T,T)})}$$

remains smaller than $r$ for small enough $T$ and all $i$. This allows us to apply the inequality (8.7) and the bound (8.9) to estimate

$$\|\Psi_{i+1} - \Psi_i\|_{\mathcal{X}} = \|S_0[\mathcal{N}(\Psi_i) - \mathcal{N}(\Psi_{i-1})]\|_{\mathcal{X}} \leq C_1 C_2 T \|\Psi_i - \Psi_{i+1}\|_{\mathcal{X}}.$$

Choosing $T$ small enough so that $C_1 C_2 T < 1$, the contraction mapping theorem yields the desired function $\psi_k \in \mathcal{X}$ as the limit of $\Psi_i$ as $i$ tends to infinity. To conclude, the estimates for the wave equation established in Theorem 7.1 and the estimate for the norm of $\rho_k$ (Eq. (8.5) can then be easily bootstrapped to obtain the bound (8.3).

Remark 8.2. As required in Physics, the construction ensures that when the support of the function $f$ is contained in the set $t \geq T_0$, then the solution $\phi$ is identically zero for $t \leq T_0$ and, after becoming nonzero, is still a regular solution to the problem for a time $T$ that depends only on the norm of the boundary datum $f$. This time tends to infinity as the $H^1$ norms of $f$, $\phi_0$ and $\phi_1$ tend to zero because in this case the Lipschitz constant $C_1$ above becomes arbitrarily small.

To conclude, the following lemma contains the proof of the local Lipschitz continuity of the function $\mathcal{N}$, which we have used in the proof of Theorem 8.1 above. We borrow the notation from the demonstration of this result without further mention.

Lemma 8.3. Under the hypotheses of Theorem 8.1, the function $\mathcal{N}$ satisfies the estimate

$$\|\mathcal{N}(\phi) - \mathcal{N}(\psi)\|_{L^\infty H^m(\mathcal{M}_{(-T,T)})} \leq C_0 \|\phi - \psi\|_{\mathcal{X}}$$

for all $\phi, \psi \in \mathcal{X}$, with the constant $C_0$ depending only on $r$ and $\|f\|_{H^1(\partial \mathcal{M}_{(-T,T)})}$. 
Proof. When evaluated outside the asymptotically AdS charts, the function $N$ is simply
$$N(\phi) = \Gamma (\nabla \phi, \nabla \phi)$$
and the estimate is standard. Therefore, let us evaluate $N$ in an asymptotically AdS chart and introduce the notation
$$u := x^{-\frac{1-n}{2}} \phi, \quad v := x^{-\frac{1-n}{2}} \psi.$$ The estimate for $N$ is then equivalent to showing
$$\|\tilde{N}(u, v)\|_{L^\infty_t \mathbf{H}^m} \leq C \|u - v\|_{\hat{X}}$$
for all $u, v \in \hat{X}_r$, where the function
$$\tilde{N}(u, v) = Q(u, u) + Q(R_k, u)$$
was defined in Eq. (8.6). Here $\hat{X} := L^\infty_t \mathbf{H}^{m+1} \cap \mathcal{W}^{1, \infty}_t \mathbf{H}^m$ and
$$\hat{X}_r := \{ u \in \hat{X} : \|u\|_{\hat{X}} < r \}$$
with the obvious definition of the norm.

To prove the inequality (8.10), let us notice that the bilinear function $Q$ can be written as
$$Q(u, v) = x^{q + \frac{2-n}{2}} \left[ c_{ij} \partial_i u \partial_j v + \frac{c_i}{x} (\partial_i u v + u \partial_i v) + \frac{c_0}{x^2} u v \right],$$
where the functions $c_{ij}, c_i, c_0$ are smooth up to the boundary and the indices $i, j$ (which are summed over) take values in the set $\{ t, x, \theta^1, \ldots, \theta^n \}$. Since the twisted derivative $D^{(x)}_x$ can be written as
$$D^{(x)}_x = \sum_{r_1 + r_2 = r} b_{r_1 r_2} x^{-r_1} \partial_x^r,$$
one can check that
$$D^{(x)}_x D^{(x)}_y Q(u, v) = \sum_{r_1 + r_2 + r_3 \leq s} \sum_{r_1 + r_2 + r_3} x^{r + \frac{2-n}{2} - r_3} \left[ \tilde{c}_{ij} D^{(r_1)}_x D^{(r_2)}_y u D^{(r_3)}_x \partial_i u \partial_j v \right. \left. + \frac{\tilde{c}_i}{x} (\partial_i D^{(r_1)}_x D^{(r_2)}_y u D^{(r_3)}_x \partial_i u + u \partial_i v) + \frac{\tilde{c}_0}{x^2} D^{(r_1)}_x D^{(r_2)}_y D^{(r_3)}_x u \right]$$
with some coefficients that are smooth up to the boundary. (In fact, these coefficients also depend on the indices $r_1, r_2, s_1, s_2$, but in order to keep the notation simple we are not making explicit this dependence.)

A simple application of Proposition 3.9 then ensures that
$$\|D^{(x)}_x D^{(x)}_y Q(u, v)\|_{L^2} \leq C \|u\|_{\hat{X}} \|v\|_{\hat{X}}$$
whenever $r + s \leq m$ and the exponent $q$ is larger than the quantity specified in Theorem 8.1. This shows that
$$\|Q(u, u) - Q(v, v)\|_{L^\infty_t \mathbf{H}^m} \leq \|Q(u, u - v)\|_{L^\infty_t \mathbf{H}^m} + \|Q(v, u - v)\|_{L^\infty_t \mathbf{H}^m} \leq C \|u - v\|_{\hat{X}}.$$ (8.11)

An analogous argument can be used to take care of the term $Q(R_k, u)$, finding that
$$\|D^{(x)}_x D^{(x)}_y Q(R_k, u)\|_{L^2} \leq C(\|f\|_{H^1(\partial_M (-T, T))}) \|u\|_{\hat{X}}.$$
under our hypothesis on the exponent $q$. By the definition of $\tilde{N}$, from this inequality and the bound (8.11) we derive the desired estimate for $\tilde{N}$, thereby completing the proof of the lemma.

Acknowledgements

A.E. is financially supported by the Ramón y Cajal program of the Spanish Ministry of Science and thanks McGill University for hospitality and support. A.E.’s research is supported in part by the Spanish MINECO under grants FIS2011-22566 and the ICMAT Severo Ochoa grant SEV-2011-0087. The research of N.K. is supported by NSERC grant RGPIN 105490-2011. Finally, we would like to thank the referees for their careful reading of the manuscript and their valuable suggestions.

References


Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, 28049 Madrid, Spain

E-mail address: aenciso@icmat.es

Department of Mathematics and Statistics, McGill University, Montréal, Québec, Canada H3A 2K6

E-mail address: nkmran@math.mcgill.ca