Morse theory for vector fields and the Witten Laplacian

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Abstract. In this paper we informally review some recent developments on the analytical approach to Morse-type inequalities for vector fields. Throughout this work we focus on the main ideas of this approach and emphasize the application of the theory to concrete examples.

1. A TRIVIAL OBSERVATION

Let us begin with a trivial observation that is nonetheless crucial for the ensuing developments. Consider the eigenvalue problem for the quantum harmonic oscillator in Euclidean n-space, namely

$$L_\varepsilon \psi := \varepsilon^2 \Delta_0 \psi(x) + |x|^2 \psi(x) = E\, \psi(x)\,. \tag{1.1}$$

Here $\Delta_0 \psi := -\operatorname{div}(\nabla \psi)$ is the positive Euclidean Laplacian, $\psi \in L^2(\mathbb{R}^n)$ is the wave function and ε is a positive real which plays the role of Planck's constant.

It is well known that the normalized ground state function of (1.1) is given by

$$\psi_0(x) := (\pi\varepsilon)^{-\frac{n}{4}}\, \mathrm{e}^{-\frac{|x|^2}{2\varepsilon}} \tag{1.2}$$

up to a global phase. The origin is obviously the only maximum of the Gaussian probability distribution $|\psi_0|^2$ in $\mathbb{R}^n$. In the *semiclassical limit* $\varepsilon \downarrow 0$, this Gaussian becomes sharply peaked at 0, so that

$$\lim_{\varepsilon\downarrow 0} \big|\psi_0(x)\big|^2\, \mathrm{d}x = \delta_0$$

in the sense of distributions. The interpretation of this result is clear: in the semiclassical limit, the ground state of the quantum Hamiltonian (1.1) concentrates on the classical stable equilibrium of the potential $V(x) := |x|^2$ with probability 1. Thus there exists a link between the topological properties of the potential V and the analytic properties of the operator L_ε. A natural yet somewhat imprecise question would be as follows:

Question 1 *Given a smooth function f on a closed manifold M, can one find a quantum Hamiltonian H_ε such that its low-lying eigenfunctions concentrate on all the critical points of f in the semiclassical limit and describe, in certain sense, the topological properties of M?*

2. WITTEN'S APPROACH TO MORSE THEORY

Before we continue, let us introduce some notation. Throughout this paper, M will denote a closed, orientable n-manifold of class C^∞. We shall say that a smooth function f on M is *Morse* if all its critical points are nondegenerate, i.e., if the Hessian of f at each critical point is an isomorphism. Given a critical point y of a Morse function f, we define the *Morse index* $\operatorname{ind}(y;f)$ as the number of negative eigenvalues of the Hessian of f at y. As is well known [10], the ultimate goal of Morse theory is to relate the critical points of a Morse function with the homology of the underlying manifold.

A striking connection between Morse theory and the semiclassical limit in quantum mechanics was laid bare by Edward Witten in the groundbreaking paper [15], where Question 1 was solved under the assumption that f is Morse. Witten's approach makes essential use of two ideas borrowed from quantum physics and Hodge theory:

(i) *Supersymmetry.* The desired operator H_ε should act on the differential forms, not only on functions, and should be the Laplacian associated to some coboundary operator.

(ii) *Isomorphic kernels.* In the limit $\varepsilon \to \infty$, H_ε should correspond to free motion on M (with respect to some Riemannian metric), which is described by the associated Hodge Laplacian. Moreover, the dimension of the kernel should not depend on ε.

Two general comments are now in order. The first one is that the operator H_ε will not be, in general, a Schrödinger operator of the form $\nabla^*\nabla + V$, where $\nabla^*\nabla$ is the rough Laplacian with respect to a certain Riemannian metric in M and V is a multiplication operator. The second one is that the fact that f is Morse is used in order to relate H_ε to a harmonic oscillator Hamiltonian. Whereas the Morse condition can be weakened, all the existing literature resorts at some point to some kind of nondegeneracy condition to control the spectral properties of H_ε.

Let us now outline Witten's construction. We shall endow M with an arbitrary Riemannian metric g and introduce a deformation of the exterior derivative as

$$\mathrm{d}_\varepsilon := e^{-f/\varepsilon}\mathrm{d}e^{f/\varepsilon} = \mathrm{d} + \varepsilon^{-1} e_{\mathrm{d}f}\,, \tag{2.1}$$

where $e_\alpha\omega := \alpha \wedge \omega$ and $i_X\omega := \omega(X,\cdot)$ ($\omega \in \Omega^\bullet(M)$) respectively denote the exterior product by a form α and the contraction with a vector field X. The operator H_ε can be defined as (the unique self-adjoint extension of) the Laplacian associated to the coboundary operator d_ε,

$$H_\varepsilon := \varepsilon(\mathrm{d}_\varepsilon \mathrm{d}_\varepsilon^* + \mathrm{d}_\varepsilon^* \mathrm{d}_\varepsilon) = \varepsilon\Delta + \varepsilon^{-1}|\mathrm{d}f|^2 + W\,, \tag{2.2}$$

acting on the space of L^2 differential forms on M. Here Δ stands for the Hodge Laplacian,

$$\mathrm{d}_\varepsilon^* := e^{f/\varepsilon}\mathrm{d}^* e^{-f/\varepsilon} = \mathrm{d}^* + \varepsilon^{-1} i_{\nabla f}$$

is the adjoint of d_ε and W is the zeroth order operator defined by

$$W := \nabla\mathrm{d}f(u_i,u_j)(e_{u_i^\flat} i_{u_j} - i_{u_j} e_{u_i^\flat})\,,$$

with $\{u_i\}$ any local orthonormal basis and $u_i^\flat := g(u_i,\cdot)$.

Since

$$(\psi, H_\varepsilon \psi) = \varepsilon\big(\|\mathrm{d}_\varepsilon \psi\|^2 + \|\mathrm{d}_\varepsilon^* \psi\|^2\big)\,,$$

from Eq. (2.1) it is standard that the dimension of the kernel of H_ε is independent of ε. When $\varepsilon \to \infty$ we recover the Hodge Laplacian, so that $\dim\ker(H_\varepsilon|_{\Omega^p(M)})$ is independent of the particular metric g and is given by the p-th Betti number $b_p(M)$ of M, that is, by the rank of the p-th cohomology group $H^p(M;\mathbb{R})$ of M.

Witten's heuristic argument (later completed by the rigorous work of Helffer and Sjöstrand [6]) now goes as follows. As f is Morse, there exist local coordinates $\{x_i\}$ in a neighborhood U of each critical point y such that $f|_U$ is given by

$$f(y) - \sum_{j=1}^{\operatorname{ind}(y;f)} x_j^2 + \sum_{j=\operatorname{ind}(y;f)+1}^{n} x_j^2\,. \tag{2.3}$$

If we let T_ε be the analogue of Witten's Laplacian H_ε in Euclidean n-space with the normal form (2.3) playing the role of the function f entering Eq. (2.2), we immediately see that T_ε simply reads as

$$T_\varepsilon := \varepsilon\Delta_0 + \frac{|x|^2}{\varepsilon} + W\,, \tag{2.4}$$

W being a matrix multiplication operator which does not depend on x. Hence the spectrum of the latter operator acting on p-forms can be readily computed to be

$$\operatorname{spec}(T_\varepsilon|_{\Omega^p(\mathbb{R}^n)}) = \left\{ \sum_{j=1}^{n}(1+2N_j) - \sum_{j=1}^{\operatorname{ind}(y;f)} n_j + \sum_{j=\operatorname{ind}(y;f)+1}^{n} n_j \right\},$$

where N_j are nonnegative integers and $n_j \in \{\pm 1\}$ are such that $\sum_{j=1}^n n_j = 2p - n$. This last condition means that there are exactly p values of j such that $n_j = 1$. Witten's observation is that $\ker(T_\varepsilon|_{\Omega^p(\mathbb{R}^n)})$ is trivial unless $\operatorname{ind}(y;f) = p$, in which case it has dimension 1.

Let us go back to the compact manifold M. It is not difficult to see that the eigenfunctions of H_ε whose energy is not unbounded as $\varepsilon \to 0$ must concentrate on the critical set of f. As we can assume that the metric g is given by $\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$ in a neighborhood of each critical point of f, in this case H_ε should not be too different from T_ε, in some sense, when acting on functions supported in a small neighborhood of a critical point of index p. By comparing the kernel of N operators of the form (2.4) and that of H_ε, Witten's conclusion was that the dimension of the kernel of $H_\varepsilon|_{\Omega^p(M)}$ must be bounded above by the number of critical points of f with Morse index p, i.e., that

$$\operatorname{card}\big\{y \in \operatorname{Cr}(f) : \operatorname{ind}(y;f) = p\big\} \geqslant b_p(M)\,. \tag{2.5}$$

These are the celebrated weak Morse inequalities [10]. With some more work one can also recover the strong Morse inequalities and construct a model for the cohomology of M in terms of the critical points of f.

3. A MORSE–WITTEN THEORY FOR VECTOR FIELDS

We have just reviewed Witten's analytic approach to Morse inequalities (2.5). It is clear that it would be desirable to have an analogous result for a class of vector fields wider than simply the gradient of Morse functions. This would yield lower bounds for the number of nonwandering sets of such vector fields, which is extremely useful in the study of the kind of problems illustrated by the following

Question 2 *Does any non-vanishing Morse-Smale vector field on a closed 3-manifold possess a periodic orbit?*

In fact, there does exist a satisfactory Morse theory for vector fields satisfying certain technical properties: the Conley index theory [4], which unifies and extends previous work of Smale [14] and Rosenberg [12]. This theory is entirely based on topological techniques. Our goal in this section is to present an operator-theoretic approach à la Witten yielding a Morse theory for a class of vector fields which is strictly smaller than that of Conley, but considerably larger than those of Smale and Rosenberg. The advantage of this approach over a purely topological one is that it provides a detailed description of the cohomology complex of M in terms of certain invariant sets of the vector fields. Technical details can be consulted in [5].

Definition 3 *We shall denote by $\Upsilon(M)$ the set of smooth vector fields on M satisfying the following conditions:*

(i) X has a finite number of invariant submanifolds β_a, $a = 1, \cdots, N$, which are compact and without boundary and contain all the nonwandering points.
(ii) X is r-normally hyperbolic [8], $r \geqslant 2$, at each β_a.
(iii) The set $\Theta := \{\beta_a\}_{a=1}^N$ is partially ordered by the relation $\beta_a \succ \beta_b$ if and only if $a \neq b$ and there exist elements $\beta_a \equiv \beta_{c_1}, \ldots, \beta_{c_{k+1}} \equiv \beta_b$ in Θ and orbits $\gamma_1, \ldots, \gamma_k$ of X such that the α-limit of γ_j belongs to β_{c_j} and its ω-limit belongs to $\beta_{c_{j+1}}$ for all $1 \leqslant j \leqslant k$.

Let us remark that the invariant sets β_a are not required to be minimal, i.e., there may exist a proper closed subset of β_a invariant under the flow of X. Quite informally, Condition (ii) is an appropriate nondegeneracy assumption imposed on the normal dynamics of X at the invariant sets, and Condition (iii) is used to obtain a filtration (or Morse decomposition) $\Theta_0 \subset \cdots \subset \Theta_m = \Theta$ of the invariant sets of X, Θ_0 consisting of the local attractors. This filtration induces a decomposition of the manifold analogous to the handlebody decomposition obtained using the gradient flow of a Morse function.

In order to state the main theorem we need to introduce the *Morse index* of the invariant set β_a, which is

$$\operatorname{ind}(\beta_a, X) := \dim(T_x W^s(\beta_a) \cap W),$$

where x is some point in β_a, $W \subset T_x M$ is some subspace transversal to $T_x\beta_a$ and $W^s(\beta_a)$ stands for the stable manifold of the invariant set β_a.

Theorem 4 *For any $X \in \Upsilon(M)$,*

$$\sum_{a=1}^{N} b_{p-\mathrm{ind}(\beta_a,X)}(\beta_a) \geqslant b_p(M)\,. \tag{3.1}$$

Eq. (3.1) is the analogous to the weak Morse inequalities; a version of the strong Morse inequalities can be found in [5]. Before sketching the analytic proof of Theorem 4 we shall present a couple of examples.

Example 5 *A function $f \in C^\infty(M)$ is* Morse–Bott *if its critical set is the union of a finite number of submanifolds β_a, compact and without boundary, and the Hessian of f at any critical point $x \in \beta_a$ is a nondegenerate bilinear form on $T_xM/T_x\beta_a$. It is not difficult to see that $\nabla f \in \Upsilon(M)$ when f is Morse–Bott, and that in this case* (3.1) *yields the weak Morse–Bott inequalities [2]. In the particular case in which f is Morse, Eq.* (3.1) *simply reproduces the well known relation* (2.5).

Example 6 *A smooth vector field in M is* Morse–Smale *if its nonwandering set is the union of a finite number of hyperbolic limit cycles and hyperbolic singular points such that the stable and unstable manifolds of any two distinct elements are transverse. In this case, Theorem 4 gives the weak inequalities proven by Smale in [14]. Particularly, if M is three-dimensional, Theorem 4 can be used to solve Question 2. If we denote by M_j ($j = 0,1,2$) the number of limit cycles of X of index j, then we find that*

$$\begin{aligned} M_0 &\geqslant 1\,, \qquad & M_0 + M_1 &\geqslant b_1(M)\,, \\ M_2 &\geqslant 1\,, \qquad & M_1 + M_2 &\geqslant b_1(M)\,. \end{aligned}$$

This readily implies that X has, at least, two limit cycles: one attractor and one repeller.

Let us now sketch the proof of Theorem 4. The key idea is to construct a C^r Morse–Bott *Lyapunov function*, i.e., a Morse–Bott function $f \in C^r(M)$ such that $X(f) > 0$ in $M\backslash\Theta$ whose critical set is exactly Θ. This result is interesting in itself and extends a theorem of Meyer [9] in which such a function was constructed for any Morse–Smale vector field. This construction has a local part, which rests on the r-normal hyperbolicity of the invariant submanifolds, and a global part, which makes essential use of the filtration of the set Θ to consistently drag local Lyapunov functions along the flow of X.

Since by construction we have that

$$\mathrm{ind}(\beta_a, X) = \mathrm{ind}(\beta_a, \nabla f)\,,$$

the problem is then reduced to prove Morse inequalities for Morse–Bott functions using the Witten Laplacian (2.2) associated to the Lyapunov function. Our demonstration thereof depends on operator-theoretic techniques and ideas due to Simon [13]; different proofs of the Morse–Bott inequalities with an analytic flavor had been previously given by Bismut [1], Helffer and Sjöstrand [7], Braverman and Farber [3] and Prokhorenkov [11].

4. APPLICATIONS AND OPEN PROBLEMS

The Morse inequalities for vector fields presented in the previous section can be used to obtain different proofs of several emblematic results. We shall next present some examples to illustrate this fact.

Example 7 *Let X be a Morse–Smale vector field. When* $p = 0$*, Eq.* (3.1) *states that the total number of singular points or limit cycles of index zero is bounded below by* $1 = b_0(M)$*. Hence the flow of X has at least one repelling element, and thus in any closed n-manifold there cannot exist a Morse–Smale vector field which preserves volume.*

Example 8 *A classical theorem of Reeb asserts that if all the limit sets of X are attracting or repelling singular points, then M is homeomorphic to a sphere and X is* C^0*-orbitally conjugate to the north–south flow. To prove this result, it suffices to observe that X is* C^0*-orbitally conjugate to a Morse–Smale vector field Y. From the proof of Theorem 4 it stems that Y has a global Morse Lyapunov function f with critical points of index* 0 *and n, and thus it can be easily proved using the flow of* ∇f *that M is homeomorphic to a sphere and the flow of f is* C^0*-orbitally conjugate to the north–south flow.*

Despite the vast literature dealing with Witten's approach to Morse inequalities, there are still several open problems. For instance, our proof of Theorem 4 depends crucially on the existence of a global Lyapunov function, which is then used to construct an analogue of Witten's Laplacian. It would be highly desirable to obtain a direct proof of this theorem using an operator constructed using only the vector field X, not another auxiliary object. If this were accomplished, possibly one could extract some finer information on the dynamics of the vector field. Quite remarkably, the normal hyperbolicity of the flow at the invariant sets and the existence of a partial order are required in all the existing literature on this problem. Probably some progress in the former question would shed some light on the latter, and viceversa.

Another interesting problem is the following. It is well known that the operator (2.2) is the quantization of the classical Witten Lagrangian, which is defined on a certain supermanifold. Is it possible to obtain Morse inequalities directly working at the classical level? Results in this direction would provide an interesting connection between classical supermechanics and Morse theory.

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