Addendum to “Geometrical and topological aspects of Electrostatics on Riemannian manifolds”

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In our recent paper [1] in this journal we have studied some qualitative aspects of the electric fields generated by a configuration of point charges on a complete Riemannian manifold using techniques from the theory of dynamical systems. When there is only one charge, the resulting electric field is a useful tool in the study of the geometric and topological properties of the underlying space. The integral curves of the electric field, which are generally not geodesics, are called electric lines.

In this addendum we intend to correct some mistakes in our paper [1] and address some of the open problems that we posed therein. We follow directly the notation of [1]. In particular, \((M, g)\) stands for a boundaryless \(C^\omega\) non-compact, complete \(n\)-manifold of finite type, and \(V_p \in C^\omega(M - p)\) is a Li–Tam Green function [5], satisfying

\[
-\Delta V_p = \delta_p .
\] (1)

If \(\mathcal{C} = \{(q_i, p_i)\}_{i=1}^N \subset \mathbb{R}^- \times M\) is a configuration of \(N\) negative point charges, its associated potential and electric field are

\[
V = \sum_{i=1}^N q_i V_{p_i} \quad \text{and} \quad E = -\nabla V .
\]

Eq. (2) in [1] is correct, but it is certainly not true that the vector field \(W\) be analytic in the whole manifold \(M\). It is analytic in \(M - p\), and it can be

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readily shown that its norm satisfies [3]
\[
\limsup_{r \to 0} \left( r^{n-2} |W| \right) < \infty.
\]

Consequently, the local desingularized electric field \( \tilde{E} = r^n E \) and the vector field \( X = (1 + |E|^2)^{-1} \tilde{E} \) are generally of class \( C^1(M) \cap C^\omega(M - p) \), and Property 2 in Proposition 1 does not, a priori, necessarily hold.

Theorem 3, which gives a topological bound to the number of critical points \( N_Z \) and the branch number \( N_B \) of a Li–Tam Green function on a Riemannian open surface, can be restated as

**Theorem 1** Let \( (M, g) \) be a complete, finitely generated surface. Then \( V \) has a finite number of critical points, and we have the topological upper bounds

\[
N_Z \leq 2g + h + N - 2, \quad N_B \leq 2(2g + h + N - 2).
\]

Moreover, the upper bounds are attained if and only if all the critical points of \( V \) are hyperbolic.

**PROOF.** We showed in [1] that the number of critical points and the branch number of \( V \) satisfy these bounds when \( N_Z \) is finite. The fact that \( N_Z \) is finite has been recently established in [2] by means of a detailed study of the possible saddle connections. The proof is quite lengthy and technical and will be omitted.

**Remark 2** The lower bounds in Theorem 3 are incorrect. The idea is that it is possible to remove the loop boundary with a single critical point with \( N + 2g + h - 1 \) branches. In fact, standard techniques in dynamical systems allow to construct a gradient-like vector field on a surface of genus \( g \) with only one (degenerate) saddle, \( h \) maxima and \( N \) minima. As a consequence of this, the statement that \( N_Z = 2g \) in Corollary 3 does not hold either (it is only an upper bound).

Concerning the question of genericity of Morse–Smale electric fields, raised in Section 7, we can prove the following result. The proof uses the explicit forms of the electric field and its first integral, showing the difficulty of extending this result to more general Riemannian manifolds. An extension to the hyperbolic plane, however, should be feasible. Let us recall that a subset of a topological space is *generic* if it is open and dense.

**Theorem 3** Let \( C = \{(q_i, p_i)\}_{i=1}^N \) be a configuration of charges in the Euclidean plane \( (\mathbb{R}^2, \delta) \). Then the electric field \( E \) is Morse–Smale for generic values of \( q_1 \) with fixed \( (q_2, \ldots, q_N, p_1, \ldots, p_N) \), and for generic values of \( p_1 \) with fixed \( (q_1, \ldots, q_N, p_2, \ldots, p_N) \).
PROOF. We start by proving that the critical points of $V$ are generically nondegenerate. That this is the case for generic $p_1$ was proven in [6], so we will only consider generic values of $q_1$. Let us identify $\mathbb{R}^2$ with the complex plane, writing the electric field generated by $\mathcal{C}$ as

$$E = \frac{1}{2\pi} \sum_{i=1}^{N} \frac{q_i(z - p_i)}{|z - p_i|^2},$$

so that the critical points are the zeros of the polynomial

$$P(z, q_1) = \sum_{i=1}^{N} q_i \prod_{j \neq i} (z - p_j).$$

If there were an open subset $U \subset \mathbb{R}$ such that $P$ had a multiple zero for all $q_1 \in U$, there would be a $C^\omega$ complex-valued function $f$, defined on a subset $U'$ of $U$, such that [4]

$$P(z, q_1) = (z - f(q_1))^2 Q(z, q_1)$$

for all $q_1 \in U'$. Here $Q(z, q_1)$ is a polynomial in $z$. It stems that $f(q_1)$ is a zero of $\frac{\partial P}{\partial q_1}$, contradicting the fact that

$$\frac{\partial P(z, q_1)}{\partial q_1} = \prod_{j=2}^{N} (z - p_j) \neq 0 \text{ for all } z \neq p_j.$$

In what follows we shall assume that the critical points of $E$ are nondegenerate and show that generically there are no saddle connections. The main tool here will be the first integral

$$I(z) = \sum_{i=1}^{N} q_i \tan^{-1} \left( \frac{x - x_i}{y - y_i} \right),$$

introduced in Section 6 in [1], which can be assumed to be univalued in a region not containing any charges. Here we are writing $z = x + iy$ and $p_i = x_i + iy_i$. Let us suppose that for a given value $q_1 = q$ there is a saddle connection between two critical points $Z_j = X_j + iY_j \ (j = 1, 2)$ which implies that $I(Z_1) = I(Z_2)$. These critical points being hyperbolic, for $q_1 = q + \epsilon$ (with $|\epsilon|$ small enough) there are critical points $Z_j(\epsilon)$ of $V$ depending analytically on $\epsilon$. If the saddle connection is not broken, we necessarily have that $I(Z_1(\epsilon)) = I(Z_2(\epsilon))$. A first order expansion in $\epsilon$ yields that

$$\frac{X_1 - x_1}{Y_1 - y_1} = \frac{X_2 - x_1}{Y_2 - y_1},$$

i.e., that $Z_1$, $Z_2$ and $p_1$ must be collinear. This condition fails to be satisfied for a generic set of parameters, proving the statement for generic $q_1$. If we
perform the same analysis for generic $p_1$ we also arrive at Eq. (2), and the theorem follows.

Another open problem was to ascertain whether there is a metric in $\mathbb{R}^n$ ($n \geq 3$) with an associated Li–Tam Green function having critical points. In Ref. [2] we have constructed analytic, asymptotically flat metrics on $\mathbb{R}^n$ whose minimal Green function has an arbitrary number of critical points (possibly nondegenerate). In particular, no topological upper bound exists in higher dimensions.

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