Self-Similar solutions for a transport equation with non-local flux.

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Abstract

In this paper we construct self-similar solutions for a N-dimensional transport equation where the velocity is given by the Riezs transform. These solutions implies non-uniqueness of weak solution. In addition we obtain self-similar solution for a one-dimensional conservative equation involving the Hilbert transform.

1 Introduction.

In this paper we shall construct self-similar solutions of the transport equation

$$\theta_t + R\theta \cdot \nabla\theta = 0 \quad \text{on} \quad \mathbb{R}^N \times \mathbb{R}^+, \tag{1.1}$$

$$\theta(x,0) = \theta_0(x), \tag{1.2}$$

where $\theta : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$, $N \geq 2$, $R\theta = (R_1\theta\,,\,...\,,\,R_N\theta)$ and $R_i\theta$ are the Riesz transform of θ in the i-th direction, i.e.

$$R_i\theta(x) = \Gamma\left(\frac{N+1}{2}\right)\pi^{-\frac{N+1}{2}} P.V. \int_{\mathbb{R}^N} \frac{x_i - y_i}{|x-y|^{N+1}}\theta(y)dy, \quad 1 \leq i \leq N. \tag{1.3}$$

The equation (1.1) was studied in [2] and the authors showed blow-up in finite time for all positive initial data. For a simple proof of the formation of singularities with radial initial data see [10] and for the viscous case see [13].

The technique used in this paper to construct self-similar solutions of the form

$$\theta(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N), \tag{1.4}$$

are based in a result of [11] where the author show that the function, $\theta(x,1)$ is such that $\Lambda\theta(x,1) = N$ in the unit ball (see section (2)). These are also self-similar solutions of the 1D transport equation

$$\theta_t + H\theta\theta_x = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^+ \tag{1.5}$$

$$\theta(x,0) = \theta_0(x), \tag{1.6}$$

*The authors were partially supported by the grant MTM2008-03754 of the MCINN (Spain) and the grant StG-203138CDSIF of the ERC.

where $H\theta$ is the Hilbert transform of θ, i.e

$$H\theta(x) = \frac{1}{\pi} P.V. \int \frac{\theta(y)}{x-y} dy$$

(for more details on this equation see [7], [8], [1] and [14]).

In section (3) we will see that this result can be used to show existence of self-similar solutions of the equation

$$\begin{aligned} \theta_t + (\theta H\theta)_x &= 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, & (1.7)\\ \theta(x,0) &= \theta_0(x), & (1.8) \end{aligned}$$

which was studied from completely different contexts (vortex sheet, water wave, 1D model of the quasi-geostrophic equation, dislocations dynamics in solids and complex Burgers equation) in [6], [4], [5], [1], [9], [12], [3] and reference there in. Nevertheless we will follow the ideas of [4] to construct the self-similar solutions.

Next we shall comment briefly the notation: the spaces $W^{k,p}$ are the classical Sobolev space (k derivatives in L^p). The operator Λ^α is defined by the operator $(-\Delta)^{\frac{\alpha}{2}}$ i.e in the Fourier space

$$\widehat{\Lambda^\alpha\theta}(\xi) = |\xi|^\alpha \hat{\theta}(\xi)$$

and we recall the identity

$$\widehat{R_j\theta}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{\theta}(\xi).$$

2 Riezs Transport Equation.

2.1 Self-Similar Solutions.

From the scaling invariance of equation (1.1), $\theta(x,t) \to \theta(\lambda x, \lambda t)$, with $\lambda > 0$, we will consider a self-similar function with the following form

$$\theta(x,t) = \Phi(x/t) = \Phi(\xi), \tag{2.1}$$

where $\xi = x/t$. The equalities

$$\begin{aligned} \partial_t\theta(x,t) &= \partial_t\Phi(x/t) = -\frac{\xi}{t}\nabla\Phi(\xi)\\ R\theta(x,t) &= R\Phi(\xi)\\ \nabla\theta(x,t) &= \nabla\big(\Phi(x/t)\big) = \frac{1}{t}\nabla\Phi(\xi) \end{aligned}$$

yields, from equation (1.1),

$$\nabla\Phi(\xi)\cdot\big(R\Phi(\xi) - \xi\big) = 0. \tag{2.2}$$

Now we shall show the existence of a solution of equation (2.2) by means of the following lemma.

Lemma 2.1 *The function*

$$v(\xi) = Nk(N)\big((1-|\xi|^2)_+\big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N), \tag{2.3}$$

where $k(N) = \Gamma(N/2)\left(2^{1/2}\Gamma(3/2)\Gamma((2N+1)/2)\right)^{-1}$ *and* f_+ *is the positive part of the function* f*, satisfies the equalities:*

$$Rv(\xi) = \xi \quad \text{if } |\xi| < 1,$$

and

$$\nabla v(\xi) = 0 \quad \text{if } |\xi| > 1.$$

Proof: from [11] we know that $v(\xi)$ satisfies the following properties:

1. $\Lambda v(\xi) = N$ if $|\xi| < 1$.
2. $\Lambda v(\xi) \in L^1(R^N)$.
3. Λv is radial.

Since

$$Rv = \nabla\big(\Lambda^{-1}v\big) \equiv \nabla\Psi, \tag{2.4}$$
$$\nabla \cdot Rv = \Lambda v, \tag{2.5}$$

we have that $\Delta\Psi = \Lambda v$ and therefore Ψ is a radial function with $\Delta\Psi(\xi) = N$ if $|\xi| < 1$. This implies the following expression for Ψ,

$$\Psi(\xi) = \frac{|\xi|^2}{2} + a_0 \quad \text{if } |\xi| < 1,$$

where a_0 is constant. By using (2.4) we obtain

$$Rv(\xi) = \frac{\xi}{|\xi|}\frac{\partial}{\partial|\xi|}\Psi(\xi) = \xi \qquad \text{if } |\xi| < 1. \tag{2.6}$$

■

Thus, the function

$$\theta(x,t) = Nk(N)\Big(\Big(1-\Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N) \tag{2.7}$$

is a self-similar solution of equation (1.1) (almost everywhere).

Remark 2.2 *We can check that the functions* $\theta^T(x,t) = -\theta(x,(T-t))$*, with* $0 < T < \infty$ *are solutions with an initial data* $\theta^T(x,0) = -\theta(x,T)$ *which collapse in a point in finite time* T*.*

Remark 2.3 *The previous ideas can be easily adapted to prove that the function*

$$\theta(x,t) = k(1)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}), \tag{2.8}$$

is a self-similar solution of equation

$$\theta_t + H\theta\theta_x = 0 \quad on\ \mathbb{R}\times\mathbb{R}^+, \tag{2.9}$$

which is a one dimensional version of equation (1.1).

2.2 Formal Weak Solutions and Non-Uniqueness.

In this section we shall check that the previous functions are solutions of the equation (1.1) in the weak sense that we define below. In addition we will be able to show non-uniqueness.

Definition 2.4 *The function $\theta(x,t)$ is a weak solutions of equation (1.1) if*

$$\theta \in C\big((0,T), L^q(\mathbb{R}^{\mathbb{N}})\big) \cap C\big((0,T), W^{1,p}(\mathbb{R}^{\mathbb{N}})\big) \quad with\ 1 \le q < \infty\ and\ 1 \le p < 2,$$

$$\partial_t\theta \in W^{1,p}(\mathbb{R}^N) \quad \forall t > 0 \quad with\ 1 \le p < 2,$$

$$\int_{\mathbb{R}^N} \big(\theta(x,t)_t + R\theta(x,t)\cdot\nabla\theta(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t \in (0,T) \forall \phi \in C_c^\infty((0,T)\times\mathbb{R}^N),$$

and

$$\lim_{t\to 0^+} \theta(x,t) = \theta_0(x) \quad in\ L^q(\mathbb{R}^N)$$

Theorem 2.5 *(Non-Uniqueness). The function*

$$\Phi(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}}$$

is a global weak solution of the equation (1.1) in the sense of the definition (2.4) with zero initial data.

Proof: Given a function $\phi(x,t) \in C_c^\infty((0,\infty)\times\mathbb{R}^N)$ and a fixed time $t > 0$ we have that

$$\int_{\mathbb{R}^N} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$= \int_{|x|<t} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$= \int_{\varepsilon<|x|<t} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$+ \int_{|x|<\varepsilon} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx,$$

where $0 < \varepsilon < t$. The second term on the right hand side of the last expression is equal to zero. In addition we have the following identities,

$$\nabla\Phi(x,t) = \begin{cases} 0 & |x| > t \\ Nk(N)\frac{\frac{x}{t^2}}{(1-\frac{|x|^2}{t^2})^{1/2}} & |x| < t \end{cases} \tag{2.10}$$

$$\partial_t\Phi(x,t) = \begin{cases} 0 & |x| > t \\ Nk(N)\frac{-\frac{|x|^2}{t^3}}{(1-\frac{|x|^2}{t^2})^{1/2}} & |x| < t \end{cases} \tag{2.11}$$

Thus, if $p < 2$, we obtain,

$$||\nabla\Phi(\cdot,t)||_{L^p(\mathbb{R}^{\mathbb{N}})} = Nk(N)\left(\int_{|x|<t}\frac{\frac{|x|^p}{t^{2p}}}{(1-\frac{|x|^2}{t^2})^{p/2}}dx\right)^{\frac{1}{p}} \tag{2.12}$$

$$= Nk(N)t^{\frac{N}{p}-p}\left(\int_{|x|<t}\frac{|x|^p}{(1-|x|^2)^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= C(N)t^{\frac{N}{p}-p}\left(\int_0^1\frac{r^{N-1+p}}{(1-r^2)^{p/2}}dr\right)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p},$$

$$||\partial_t\Phi(\cdot,t)||_{L^1(\mathbb{R}^N)} = Nk(N)\left(\int_{|x|<t}\frac{\frac{|x|^{2p}}{t^{3p}}}{(1-\frac{|x|^2}{t^2})^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= Nk(N)t^{\frac{N}{p}-p}\left(\int_{|x|<1}\frac{|x|^{2p}}{(1-|x|^2)^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= C(N)t^{\frac{N}{p}-p}\left(\int_0^1\frac{r^{2p+N-1}}{(1-r^2)^{p/2}}dr\right)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p}$$

Therefore,

$$\int_{\mathbb{R}^N}\partial_t\Phi(x,t)\phi(x,t)dx \le ||\partial_t\Phi(\cdot,t)||_{L^1(\mathbb{R}^{\mathbb{N}})}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)} = C(N,1)t^{N-1}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)}$$

and

$$\int_{\mathbb{R}^N}R\Phi(x,t)\cdot\nabla\Phi(x,t)\phi(x,t)dx \le ||R\Phi(\cdot,t)||_{L^q(\mathbb{R}^N)}||\nabla\Phi(\cdot,t)||_{L^p(\mathbb{R}^N)}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)} \le$$

$$C(N,q,p)t^{N-p}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)},$$

where $1 < p < 2$, $1/p + 1/q = 1$ and $t > 0$. Then, we can conclude that

$$\lim_{\varepsilon\to t}\int_{\varepsilon<|x|<t}\big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t>0,$$

and

$$\int_{\mathbb{R}^N} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t>0 \quad \forall \phi \in C_c^\infty((0,\infty)\times\mathbb{R}^{\mathbb{N}}).$$

In addition is easy to check that

$$\lim_{t\to 0^+} \Phi(x,t) = 0 \quad \text{in } L^p(\mathbb{R}^{\mathbb{N}}) \text{ with } 1 \le p < \infty.$$

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3 One Dimensional Conservative Equation

In this section we will construct self-similar solutions for the equation

$$\begin{aligned} \theta_t + (\theta H\theta)_x &= 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, & (3.1)\\ \theta(x,0) &= \theta_0(x), & (3.2) \end{aligned}$$

where $\theta:\mathbb{R}\to\mathbb{R}$ and $H\theta$ is the Hilbert transform of the function θ.

We will use the techniques developed in [4] to obtain formally a self-similar solution.

We sketch the mean features of the equation (3.1) in the following lemma.

Lemma 3.1 *Let $Z(w,t)$ be a complex function, $Z: M \to \mathbb{C}$, where $M = \{w = x+iy \;:\; y > 0\}$ such that*

$$\begin{aligned} Z_t + ZZ_w &= 0 \quad \text{on } M & (3.3)\\ Z(w,0) &= R\theta_0(x,y) - iP\theta_0(x,y). & (3.4) \end{aligned}$$

$P\theta(x,y)$ is the convolution with the Poisson kernel and $R\theta(x,y)$ is the convolution with the harmonic conjugate Poisson kernel, i.e.

$$P\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{y^2+(x-s)^2}\theta(s)ds \qquad R\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-s}{y^2+(x-s)^2}\theta(s)ds. \quad (3.5)$$

Then, if $Z(w,t)$ is analytic on M and vanishing at infinity

$$\theta(x,t) = -\Im\left(Z(w,t)|_{y=0}\right) \quad (3.6)$$

is a solution of equation (3.1), with $\theta(x,0) = \theta_0(x)$ on the points where θ and $H\theta$ are differentiable.

Proof: If $Z(w,t)$ satisfies the statements of lemma (3.1) we can write it in the following way

$$Z(w,t) = R\theta(x,y;t)) - iP\theta(x,y;t) \quad (3.7)$$

where $\theta(x,t) = -\Im\left(Z(w,t)|_{y=0}\right)$. In addition we know that

$$Z_t + ZZ_x = 0 \quad \text{on } M,$$

and from (3.7) follows $Z(w,t)|_{y=0} = H\theta(x,t) - i\theta(x,t)$. By taking the limit $y \to 0^+$ in equation (3.7) we have the desired result. ∎

Next we shall use the previous lemma to prove the following theorem.

Theorem 3.2 *The function*

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}} \left(\left(1 - \frac{\pi x^2}{4t} \right)_+ \right)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R})$$

is a self-similar solution (at least in a weak sense) of equation (3.1) with the initial data $\theta_0 = \delta_0$, where δ_0 is the Dirac Delta.

Proof: By the lemma (3.1), we have to study the solutions of the equation,

$$Z_t + ZZ_w = 0 \quad \text{on } M \tag{3.8}$$

$$Z(w,0) = \frac{1}{\pi}\frac{x}{x^2+y^2} - i\frac{1}{\pi}\frac{y}{x^2+y^2}, \tag{3.9}$$

A standard argument yields that the solution is constant along the following complex trajectories

$$X^1(x,y,t) = \frac{1}{\pi}\frac{x}{x^2+y^2}t + x \tag{3.10}$$

$$X^2(x,y,t) = -\frac{1}{\pi}\frac{y}{x^2+y^2}t + y. \tag{3.11}$$

Thus

$$Z(X^1(x,y,t), X^2(x,y,t), t) = Z_0(x,y),$$

and one can check that the solution, $Z(w,t)$, satisfies the requirements of the lemma (3.1). In addition

$$\theta(X^1,t) = -\Im\left(Z(X^1,X^2,t)|_{X^2=0}\right) = P\theta_0(x,y,t)|_{X^2=0} = \frac{y}{\pi t}|_{X^2=0}.$$

The function

$$y = \sqrt{\pi t}\left(\left(1 - \frac{\pi x^2}{t} \right)_+ \right)^{\frac{1}{2}}$$

satisfies equation (3.11) with $X^2 = 0$ and by the equation (3.11) we have that

$$X^1 = \begin{cases} 2x & |x| < \sqrt{t/\pi} \\ \frac{t}{\pi x} + x & |x| > \sqrt{t/\pi} \end{cases}$$

Furthermore we can conclude that,

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}} \left(\left(1 - \frac{\pi x^2}{4t} \right)_+ \right)^{\frac{1}{2}}.$$

∎

Remark 3.3 *This solution was obtained in [3] and by using the techniques of section (2). In fact they constructed self-similar solutions for the equation*

$$\begin{aligned} u_t + \Lambda^\alpha u u_x &= 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^+, && (3.12) \\ u(x,0) &= H(x), && (3.13) \end{aligned}$$

where $H(x)$ is the Heaviside function and $0 < \alpha \leq 2$.

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