Turning waves and breakdown for incompressible flows

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November 26, 2010

Abstract

We consider the evolution of an interface generated between two immiscible incompressible and irrotational fluids. Specifically we study the Muskat and water wave problems. We show that starting with a family of initial data given by $(\alpha, f_0(\alpha))$, the interface reaches a regime in finite time in which is no longer a graph. Therefore there exists a time t^* where the solution of the free boundary problem parameterized as $(\alpha, f(\alpha, t))$ blows-up: $\|\partial_\alpha f\|_{L^\infty}(t^*) = \infty$. In particular, for the Muskat problem, this result allows us to reach an unstable regime, for which the Rayleigh-Taylor condition changes sign and the solution breaks down.

1   Introduction

Here we study two problems of Fluids Mechanics concerning the evolution of two incompressible fluids of different characteristics in 2D. We consider that both fluids are immiscible and of different constant densities ρ^1 and ρ^2, modeling the dynamics of an interface that separates the domains $\Omega^1(t)$ and $\Omega^2(t)$. That is, the liquid density $\rho = \rho(x,t)$, $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, is defined by

$$\rho(x,t) = \begin{cases} \rho^1, & x \in \Omega^1(t) \\ \rho^2, & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases} \tag{1}$$

and satisfies the conservation of mass equation

$$\begin{aligned} \rho_t + v \cdot \nabla \rho &= 0, \\ \nabla \cdot v &= 0, \end{aligned} \tag{2}$$

where $v = (v_1(x,t), v_2(x,t))$ is the velocity field. With a free boundary parameterized by

$$\partial \Omega^j(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\},$$

we consider open curves vanishing at infinity

$$\lim_{\alpha \to \infty} (z(\alpha, t) - (\alpha, 0)) = 0,$$

or periodic in the space variable

$$z(\alpha + 2k\pi, t) = z(\alpha, t) + 2k\pi(1, 0).$$

1

The scalar vorticity, $\nabla^\perp \cdot v$, has the form

$$\nabla^\perp \cdot v(x,t) = \omega(\alpha,t)\delta(x - z(\alpha,t)), \tag{3}$$

i.e. the vorticity is a Dirac measure on z defined by

$$< \nabla^\perp \cdot v, \eta >= \int_{\mathbb{R}} \omega(\alpha,t)\eta(z(\alpha,t))d\alpha,$$

with $\eta(x)$ a test function. The system is closed by using one of the following fundamental fluid motion equations:

Darcy's law

$$\frac{\mu}{\kappa}v = -\nabla p - g\rho(0,1), \tag{4}$$

or

Euler equations

$$\rho(v_t + v \cdot \nabla v) = -\nabla p - g\rho(0,1). \tag{5}$$

Here $p = p(x,t)$ is pressure, g gravity, μ viscosity and κ permeability of the isotropic medium.

The Muskat problem [18] is given by (1,2,4) which considers the dynamics of two incompressible fluids of different densities throughout porous media and Hele-Shaw cells [22, 16]. In this last setting the fluid is trapped between two fixed parallel plates, that are close enough together, so that the fluid essentially only moves in two directions [15].

Taking $\rho^1 = 0$, equations (1,2,3,5) are known as the water waves problem (see [2] and references therein), modeling the dynamics of the contour between an inviscid fluid with density ρ^2 and vacuum (or air) under the influence of gravity.

Condition (3) (deduced by (4), assumed for (5)) allows us to write the evolution equation in terms of the free boundary as follows. One could recover the velocity field from (3) by means of Biot-Savart law

$$v(x,t) = \nabla^\perp \Delta^{-1}(\nabla^\perp \cdot v)(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(x - z(\alpha,t))^\perp}{|x - z(\alpha,t)|^2}\omega(\alpha,t)d\alpha,$$

applying the delta measure with amplitude ω. Taking limits on the above equation approaching the boundary in the normal direction inside Ω^j, the velocity is shown to be discontinuous in the tangential direction, but continuous in the normal, and given by the Birkhoff-Rott integral of the amplitude ω along the interface curve:

$$BR(z,\omega)(\alpha,t) = \frac{1}{2\pi}PV\int_{\mathbb{R}} \frac{(z(\alpha,t) - z(\beta,t))^\perp}{|z(\alpha,t) - z(\beta,t)|^2}\omega(\beta,t)d\beta.$$

Above PV denotes principal value. It yields the curve velocity from which one can subtract any term c in the tangential without modifying the geometry of the interface

$$z_t(\alpha,t) = BR(z,\omega)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t). \tag{6}$$

2

Understanding the problem as weak solutions of (1,2,4) or (1,2,3,5), the continuity of the pressure on the free boundary follows. Therefore, taking limits in Darcy's law from both sides and subtracting the results in the tangential direction, it is easy to close the system for Muskat (in this paper we consider two fluids with the same viscosity):

$$\omega(\alpha, t) = -(\rho^2 - \rho^1)\frac{\kappa g}{\mu}\partial_\alpha z_2(\alpha, t). \tag{7}$$

In a similar way for water waves, Euler equations yield

$$\omega_t(\alpha, t) = -2\partial_t BR(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - \partial_\alpha(\frac{|\omega|^2}{4|\partial_\alpha z|^2})(\alpha, t) + \partial_\alpha(c\omega)(\alpha, t)$$
$$+ 2c(\alpha, t)\partial_\alpha BR(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) + 2g\partial_\alpha z_2(\alpha, t). \tag{8}$$

Then, the two contour equations are set by (6,7) and (6,8).

For these models the well-posedness turns out to be false for some settings. Rayleigh [21] and Saffman-Taylor [22] gave a condition that must be satisfied for the linearized model in order to exist a solution locally in time: the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh-Taylor condition. It reads

$$\sigma(\alpha, t) = -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t) > 0,$$

where $\nabla p^j(z(\alpha, t), t)$ denotes the limit gradient of the pressure obtained approaching the boundary in the normal direction inside $\Omega^j(t)$.

An easy linearization around a flat contour $(\alpha, f(\alpha, t))$, allows us to find

$$f_t = \frac{1}{2}H(\omega)$$

where H is the Hilbert transform which symbol on the Fourier side is given by $\widehat{H} = -i\,\text{sign}\,(\xi)$. The equations

$$\omega = -(\rho^2 - \rho^1)\frac{\kappa g}{\mu}\partial_\alpha f, \qquad\qquad \text{(linear Muskat)}$$

$$\omega_t = 2g\partial_\alpha f, \qquad\qquad \text{(linear water waves)}$$

show the parabolicity of the Muskat problem when the denser fluid is below $(\rho^2 > \rho^1)$ and the dispersive character of water waves.

1. There is a wide literature on the Muskat problem and the dynamics of two fluids in a Hele-Shaw cell. There are works considering the case of a viscosity jump neglecting the effect of gravity [23, 28]. Local-existence in a more general situation (with discontinuous viscosity and density) is shown in [7] and also treated in [1]. A different approach to prove local-existence can be found in [10] for the setting we are considering in this paper. The Rayleigh-Taylor stability depends upon the sign of $(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha, t)$ indicating that the heavier fluid has to be below in the stable case. If the lighter fluid is below, the problem have been shown to be ill-posed [10]. Global-existence results for small initial data can be

found in [6, 29, 23, 10, 13]. For large initial curves and parameterized by $(\alpha, f(\alpha,t))$, there are maximum principles for the L^∞ and L^2 norms of f, and decay rates, together with global-existence for Lipschitz curves if $\|\partial_\alpha f\|_{L^\infty}(0) < 1$ [11, 5].

2. The water waves problem have been extensively considered (see [8, 2] and references therein). For sufficiently smooth free boundary, the Rayleigh-Taylor condition remains positive with no bottom considerations [25], a fact that was used to prove local-existence [25]. The Rayleigh-Taylor stability can play a different role for the case of non "almost" flat bottom [17]. Recently, for small initial data, exponential time of existence has been proven in two dimensions [26], and global-existence in the three dimensional case (two dimensional interface) [14, 27].

The results that we announce below will be given in the forthcoming papers [3] and [4].

2   Rayleigh-Taylor breakdown for Muskat

This section is devoted to show the main ingredients to prove the Theorem 2.1 below. We consider the function

$$F(z_0)(\alpha, \beta) = \frac{|\beta|^2}{|z_0(\alpha) - z_0(\alpha - \beta)|^2}, \tag{9}$$

if $F(z_0) \in L^\infty(\mathbb{R}^2)$ then the curve z_0 satisfies the arc-chord condition. We say that the Rayleigh-Taylor of the solution of the Muskat problem breaks down in finite time if for initial data z_0 satisfying $\sigma(\alpha, 0) = (\rho^2 - \rho^1)\partial_\alpha z_1(\alpha, 0) > 0$ there exists a time $t^* > 0$ for which $\sigma(\alpha, t^*)$ is strictly negative in a nonempty open interval.

Theorem 2.1 *There exists a non-empty open set of initial data in H^4, satisfying Rayleigh-Taylor and arc-chord conditions, for which the Rayleigh-Taylor condition of the solution of the Muskat problem (1, 2, 4) breaks down in finite time.*

After choosing the appropriate tangential term and a integration by parts, the contour equation reads

$$z_t(\alpha, t) = \frac{\rho^2 - \rho^1}{2\pi} PV \int \frac{(z_1(\alpha,t) - z_1(\beta,t))}{|z(\alpha,t) - z(\beta,t)|^2}(\partial_\alpha z(\alpha,t) - \partial_\alpha z(\beta,t))d\beta. \tag{10}$$

The steps of the proof:

1. First, for any initial curve $z_0(\alpha) = z(\alpha, 0)$ in H^4 that satisfy R-T

$$(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha, 0) > 0$$

and the arc-chord condition then the solution to the Muskat problem $z(\alpha, t)$ becomes analytic for $0 < t < T$. Moreover, $z(\alpha, t)$ is real analytic in a strip

$$S(t) = \{\alpha + i\zeta : |\zeta| < ct\}$$

for $t \in (0, T)$ where c depends only on

$$\inf(0) = \inf_\alpha \frac{\partial_\alpha z_1(\alpha, 0)}{|\partial_\alpha z(\alpha, 0)|^2}.$$

4

The proof follows by controlling the quantities extended on $S(t)$:

$$f(\alpha + i\zeta, t) = \frac{\partial_\alpha z_1(\alpha + i\zeta, t)}{|\partial_\alpha z(\alpha + i\zeta, t)|^2},$$

$F(z)(\alpha + i\zeta, \beta, t)$ by using (9) and norms

$$\|F(z)\|_{L^\infty(S)}(t) = \sup_{\alpha + i\zeta \in S(t), \beta \in \mathbb{T}} |F(z)(\alpha + i\zeta, \beta)|,$$

$$\|z\|^2_{L^2(S)}(t) = \sum_{\pm} \int_{\mathbb{T}} |z(\alpha \pm ict, t)|^2 d\alpha,$$

$$\|z\|^2_{H^j(S)}(t) = \|z\|^2_{L^2(S)}(t) + \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^j z(\alpha \pm ict, t)|^2 d\alpha,$$

for $j \in \mathbb{N}$,

$$\inf(t) = \inf_{\alpha + i\zeta \in S(t)} \Re(f)(\alpha + i\zeta).$$

Then the quantity

$$\|z\|^2_{RT}(t) = \|z\|^2_{H^4(S)}(t) + \|F(z)\|_{L^\infty(S)}(t) + 1/(\inf(t) - c - K\|\Im(f)\|_{H^2(S)}(t))$$

satisfies

$$\frac{d}{dt}\|z\|_{RT}(t) \le C\|z\|^k_{RT}(t),$$

for C, K and k universal constants. It yields

$$\|z\|_{RT}(t) \le \frac{\|z\|_{RT}(0)}{(1 - C\|z\|^k_{RT}(0)t)^{1/k}},$$

providing control of the analyticity and $T = 1/(C\|z\|^k_{RT}(0))$.

2. Second, there is a lower bound on the strip of analyticity, which does not collapse to the real axis as long as the Rayleigh-Taylor is greater than or equal to 0. Then there is a time T and a solution of the Muskat problem $z(\alpha, t)$ defined for $0 < t \le T$ that continues analytically into a complex strip if $(\rho^2 - \rho^1)\partial_\alpha z_1 \ge 0$, where T is either a small constant or it is the first time a vertical tangent appears, whichever occurs first. We redefine the strip

$$S(t) = \{\alpha + i\zeta : |\zeta| < h(t), 0 < h(0)\},$$

and the quantity $\|z\|^2_S = \|z\|^2_{H^4(S)} + \|F(z)\|_{L^\infty(S)}$ with this new $S(t)$. For a $h(t)$ decreasing (the expression of $h(t)$ is chosen later), we consider the evolution of the most singular quantity

$$\sum_{\pm} \int |\partial_\alpha^4 z(\alpha \pm ih(t), t)|^2 d\alpha.$$

Taking a derivative in t one finds

$$\frac{d}{dt} \sum_{\pm} \int |\partial_\alpha^4 z(\alpha \pm ih(t))|^2 d\alpha \le \frac{h'(t)}{10} \sum_{\pm} \int \Lambda(\partial_\alpha^4 z)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z}(\alpha \pm ih(t)) d\alpha$$

5

$$-10h'(t)\int \Lambda(\partial_\alpha^4 z)(\alpha)\cdot\overline{\partial_\alpha^4 z}(\alpha)d\alpha + 2\sum_\pm \Re\int \partial_\alpha^4 z_t(\alpha\pm ih(t))\cdot\overline{\partial_\alpha^4 z}(\alpha\pm ih(t))d\alpha.$$

Estimating in a wise way one obtains

$$\frac{d}{dt}\sum_\pm\int |\partial_\alpha^4 z(\alpha\pm ih(t))|^2 d\alpha \leq C\|z\|_S^k(t) - 10h'(t)\int \Lambda(\partial_\alpha^4 z)(\alpha)\cdot\overline{\partial_\alpha^4 z}(\alpha)d\alpha$$

$$+(C\|z\|_S^k(t)h(t) + \frac{1}{10}h'(t))\int \Lambda(\partial_\alpha^4 z)(\alpha\pm ih(t))\cdot\overline{\partial_\alpha^4 z}(\alpha\pm ih(t))d\alpha.$$

Therefore, choosing

$$h(t) = h(0)\exp(-10C\int_0^t \|z\|_S^k(r)dr)$$

eliminates the most dangerous term. The other terms are easily controlled, giving finally

$$\frac{d}{dt}\sum_\pm\int |\partial_\alpha^4 z(\alpha\pm ih(t))|^2 d\alpha \leq C\|z\|_S^k(t).$$

This allows us to reach a regime for which the boundary z develops a vertical tangent at time T.

3. Third, it is shown the existence of a large class of analytic curves for which there exist a point where the tangent vector is vertical and the velocity indicates that the curve is going to turn up and reach the unstable regime. That is

$$a.\ \partial_\alpha z_1(\alpha) > 0 \text{ if } \alpha \neq 0, \qquad b.\ \partial_\alpha z_1(0) = 0,$$

$$c.\ \partial_\alpha z_2(0) > 0, \qquad\qquad d.\ \partial_\alpha v_1(0) < 0,$$

for analytic functions $z_1(\alpha)$ and $z_2(\alpha)$ such that $z(\alpha)$ satisfies the arc-chord condition. Here we consider an open curve vanishing at infinity (being analogous in the periodic case). We assume that $z(\alpha)$ is a smooth odd curve satisfying the properties a, b and c. Differentiating the expression (10) for the horizontal component of the velocity it is easy to obtain

$$(\partial_\alpha v_1)(\alpha) = \int_{-\infty}^\infty \frac{(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))^2 + (z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha^2 z_1(\alpha) - \partial_\alpha^2 z_1(\alpha-\beta))}{|z(\alpha) - z(\alpha-\beta)|^2}d\beta$$

$$-2\int_{-\infty}^\infty (z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))\frac{(z(\alpha) - z(\alpha-\beta))\cdot(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha-\beta))}{|z(\alpha) - z(\alpha-\beta)|^4}d\beta.$$

At $\alpha = 0$ it yields

$$(\partial_\alpha v_1)(0) = \int_{-\infty}^\infty \frac{(\partial_\alpha z_1(\beta))^2 + z_1(\beta)\partial_\alpha^2 z_1(\beta)}{|z(\beta)|^2}d\beta$$

$$-2\int_{-\infty}^\infty z_1(\beta)\partial_\alpha z_1(\beta)\frac{z_1(\beta)\partial_\alpha z_1(\beta) - z_2(\beta)(\partial_\alpha z_2(0) - \partial_\alpha z_2(\beta))}{|z(\beta)|^4}d\beta.$$

Integration by parts provides

$$\int_{-\infty}^\infty \frac{z_1(\beta)\partial_\alpha^2 z_1(\beta)}{|z(\beta)|^2}d\beta = -\int_{-\infty}^\infty \frac{(\partial_\alpha z_1(\beta))^2}{|z(\beta)|^2}d\beta$$

$$+2\int_{-\infty}^{\infty} z_1(\beta)\partial_\alpha z_1(\beta)\frac{z_1(\beta)\partial_\alpha z_1(\beta)+z_2(\beta)\partial_\alpha z_2(\beta)}{|z(\beta)|^4}d\beta.$$

Therefore it is easy to obtain that

$$(\partial_\alpha v_1)(0) = 4\partial_\alpha z_2(0)\int_0^\infty \frac{z_1(\beta)z_2(\beta)}{|z(\beta)|^4}\partial_\alpha z_1(\beta)d\beta. \tag{11}$$

Expression (11) allows us to determine the sign of $(\partial_\alpha v_1)(0)$. One could take

$$z_1(\beta) = \frac{\beta^3}{(1+\beta^2)},$$

and construct the function $z_2(\beta)$ in the following way: Let β_1, β_2 and β_3 be real increasing numbers. We pick $z_2(\beta) < c < 0$ for $\beta_2 < \beta < \infty$ and $z_2^*(\beta)$ a smooth function with the following properties

    a. $z_2^*(\beta)$ is odd,   *b.* $(\partial_\beta z_2^*)(0) > 0,$

    c. $z_2^*(\beta) > 0$ if $\beta \in (0,\beta_1),$ *d.* $z_2^*(\beta) < 0$ if $\beta \in (\beta_1,\beta_2].$

For $z_2(\beta) = bz_2^*(\beta)$, $0 \le \beta \le \beta_2$ and $b > 0$, the velocity satisfies

$$(\partial_\alpha v_1)(0) < 4(\partial_\alpha z_2)(0)\left(\int_0^{\beta_1}\frac{z_1(\beta)z_2(\beta)}{|z(\beta)|^4}\partial_\alpha z_1(\beta)d\beta + \int_{\beta_3}^\infty \frac{z_1(\beta)z_2(\beta)}{|z(\beta)|^4}\partial_\alpha z_1(\beta)d\beta\right)$$

$$= 4(\partial_\alpha z_2)(0)\left(\int_0^{\beta_1}\frac{z_1(\beta)bz_2^*(\beta)}{(z_1(\beta)^2+b^2z_2^*(\beta)^2)^2}\partial_\alpha z_1(\beta)d\beta + A\right),$$

where $A < 0$. The constant b large enough yields $(\partial_\alpha v_1)(0) < 0$. Rectifying the curve on the interval $[\beta_2,\beta_3]$ it is easy to obtain a smooth curve. Finally, convolving with the heat kernel the vertical component, the curve $z(\alpha)$ is approximated by an analytic one.

4. Fourth, with the initial data found in 3. and no assumption on the R-T condition, we use a modification of Cauchy-Kowalewski theorems [19, 20] to show that there exists an analytic solution for the Muskat problem in some interval $[-T,T]$ for a small enough $T > 0$. Here we are forced to change substantially the method in [24] because in this case the curve can not be parameterized as a graph, so we have to deal with the arc-chord condition. Then, with $\{X_r\}_{r>0}$ a scale of Banach spaces given by real functions that can be extended analytically on the complex strip $S_r = \{\alpha + i\zeta \in \mathbb{C} : |\zeta| < r\}$ with norm

$$\|f\|_r = \sum_\pm \int |f(\alpha \pm ir)|^2 d\alpha + \int |\partial_\alpha^4 f(\alpha \pm ir)|^2 d\alpha,$$

and $z^0(\alpha)$ a curve satisfying the arc-chord condition and $z^0(\alpha) \in X_{r_0}$ for some $r_0 > 0$, we prove the existence of a time $T > 0$ and $0 < r < r_0$ so that there is a unique solution to the Muskat problem in $C([-T,T];X_r)$. This allows us to find solutions that do not satisfy the R-T but shrink the strip of analyticity. We extend equation (10) as follows:

$$z_t(\alpha + i\zeta, t) = G(z(\alpha + i\zeta, t)),$$

7

with

$$G(z)(\alpha + i\zeta, t) = \frac{\rho^2 - \rho^1}{2\pi} \int \frac{z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta)}{|z(\alpha + i\zeta) - z(\alpha + i\zeta - \beta)|^2} (\partial_\alpha z(\alpha + i\zeta) - \partial_\alpha z(\alpha + i\zeta - \beta)) d\beta.$$

For $0 < r' < r$ and the open set O in S_r given by

$$O = \{z \in X_r : \|z\|_r < R, \quad \|F(z)\|_{L^\infty(S_r)} < R^2\}, \tag{12}$$

the function G for $G : O \to X_{r'}$ is a continuous mapping and there is a constant C_R (depending on R only) such that

$$\|G(z)\|_{r'} \leq \frac{C_R}{r - r'} \|z\|_r, \tag{13}$$

$$\|G(z^2) - G(z^1)\|_{r'} \leq \frac{C_R}{r - r'} \|z^2 - z^1\|_r, \tag{14}$$

and

$$\sup_{\alpha + i\zeta \in S_r, \beta \in \mathbb{T}} |G(z)(\alpha + i\zeta) - G(z)(\alpha + i\zeta - \beta)| \leq C_R |\beta|, \tag{15}$$

for $z, z^j \in O$. For initial data $z^0 \in X_{r_0}$ satisfying arc-chord, we can find a $0 < r'_0 < r_0$ and a constant R_0 such that $\|z^0\|_{r'_0} < R_0$ and

$$\left| \frac{(z_1^0(\alpha + i\zeta) - z_1^0(\alpha + i\zeta - \beta))^2 + (z_2^0(\alpha + i\zeta) - z_2^0(\alpha + i\zeta - \beta))^2}{\beta^2} \right| > \frac{1}{R_0^2}, \tag{16}$$

for $\alpha + i\zeta \in S_{r'_0}$. We take $0 < r < r'_0$ and $R_0 < R$ to define the open set O as in (12). Therefore we can use the classical method of successive approximations:

$$z^{n+1}(t) = z^0 + \int_0^t G(z^n(s)) ds,$$

for $G : O \to X_{r'}$ and $0 < r' < r$. We assume by induction that

$$\|z^k\|_r(t) < R, \qquad \text{and} \qquad \|F(z^k)\|_{L^\infty(S_r)}(t) < R$$

for $k \leq n$ and $0 < t < T$ with $T = \min(T_A, T_{CK})$ and T_{CK} the time obtaining in the proofs in [19] and [20]. We get $\|z^{n+1}\|_r(t) < R$ that follows using (13,14). The time T_A is to yield $\|F(z^{n+1})\|_{L^\infty(S_r)}(t) < R$. Then, using the induction hypothesis and (15) we can control the quantity taking $0 < T_A < (R_0^{-2} - R^{-2})(C_R^2 + 2R_0 C_R)^{-1}$.

5. Fifth, all the results above allow us to prove that there is a non empty set of initial data in H^4 satisfying the arc-chord and R-T conditions such that the solution of the Muskat problem reaches the unstable regime: the R-T becomes strictly negative on a non-empty interval. We pick initial data as in 3. We apply the local-existence result in 4 to get an analytic solution $z(\alpha, t)$ on $[-T, T]$. Then we consider a time $0 < \delta < T$ and a curve $\omega_\delta^\varepsilon(\alpha, t)$, solving the Muskat problem with initial datum $z(\alpha, -\delta) + \eta_\delta^\epsilon(\alpha)$. The function η_δ^ϵ has a small H^4 norm, i.e.

$$\|\omega_\delta^\varepsilon(\cdot, -\delta) - z(\cdot, -\delta)\|_{H^4} = \|\eta_\delta^\epsilon\|_{H^4} \leq \varepsilon.$$

The time δ is small enough so that $\omega_\delta^\varepsilon(\alpha, -\delta)$ satisfies R-T: $(\rho^2 - \rho^1)\partial_\alpha(\omega_\delta^\varepsilon)_1(\alpha, -\delta) > 0$. Then we apply the local existence result in 1. that $\omega_\delta^\varepsilon(\alpha, t)$ becomes analytic for some time $-\delta < t$. With 2. we assure the existence and analyticity of the solution even if $\partial_\alpha(\omega_\delta^\varepsilon)_1(\alpha, t) \leq 0$ for some time t. Then, we show that both solutions are close in the H^4 topology as time evolves. We can apply to $\omega_\delta^\varepsilon$ the local-existence result in 4. if it is needed. Then, with δ and ε small enough we find the desired result.

3   Turning water waves

In this section we prove for the water wave problem ($\rho^1 = 0$ and (1,2,3,5)) that with initial data given by a graph $(\alpha, f_0(\alpha))$, the interface reaches a regime in finite time where it only can be parameterized as $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t))$, for $\alpha \in \mathbb{R}$, with $\partial_\alpha z_1(\alpha, t) < 0$ for $\alpha \in I$, a non-empty interval. Therefore there exists a time t^* where the solution of the free boundary problem reparameterized by $(\alpha, f(\alpha, t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$.

Theorem 3.1 *There exists a non-empty open set of initial data $(\alpha, f_0(\alpha))$, with $f_0 \in H^5$, such that in finite time t^* the solution of the water waves problem ($\rho^1 = 0$ and (1,2,3,5)) given by $(\alpha, f(\alpha, t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$. The solution can be continued for $t > t^*$ as $z(\alpha, t)$ with $\partial_\alpha z_1(\alpha, t) < 0$ for $\alpha \in I$, a non-empty interval.*

In order to prove this theorem we consider a curve $z^*(\alpha) \in H^5$ with the same properties as in point 3. of previous section. Then, we pick $z(\alpha, t^*) = z^*(\alpha)$ and $\omega(\alpha, t^*) = \partial_\alpha z_1^*(\alpha)$ as a datum for the initial value problem. It is easy to find the same properties for the velocity, since the tangential direction does not affect the evolution. Picking the appropriate $c(\alpha, t)$ and applying the local-existence result in [8], there exists a solution of the water waves problem with $z(\alpha, t) \in C([t^* - \delta, t^* + \delta]; H^5)$, $\omega(\alpha, t) \in C([t^* - \delta, t^* + \delta]; H^4)$ and $\delta > 0$ small enough. Then, the initial datum $(z_0(\alpha), \omega_0(\alpha)) = (\alpha, f_0(\alpha), \omega_0(\alpha))$ is given by $(z(\alpha, t^* - \delta), \omega(\alpha, t^* - \delta))$.

4   Muskat breakdown

In this section we show that there exists a smooth initial data in the stable regime for the Muskat problem such that the solution turns to the unstable regime and later it breaks down. The outline of the proof is to construct a curve in the unstable regime which is analytic except in a single point. We show that as we evolve backwards in time the curve becomes analytic and is as close as we desired (in the H^k topology with k large enough) to the curve from part 3. of section 2.

Here we will work in the periodic setting and will consider the equation

$$\partial_t z(\zeta, t) = \int_{w \in \Gamma_+(t)} \frac{\sin(z_1(\zeta, t) - z_1(w, t))}{\cosh(z_2(\zeta, t) - z_2(w, t)) - \cos(z_1(\zeta, t) - z_1(w, t))}(\partial_\zeta z(\zeta, t) - \partial_\zeta z(w, t))dw,$$

(17)

where $\zeta \in \Omega(t)$,

$$\Omega(t) = \{\zeta \in \mathbb{C}/2k\pi \ : \ |\Im\zeta| < h(\Re z, t)\},$$

9

$h(x,t)$ is a positive periodic function with period 2π and smooth for fixed time t and

$$\Gamma_{\pm}(t) = \{\zeta \in \mathbb{C}/2k\pi \ : \ \zeta = x + ih(x,t)\}.$$

This equation is equivalent to (1,2,4) for holomorphic functions.

In order to prove the result we will need the following theorem:

Theorem 4.1 *Let $h(x,t)$ be a positive, smooth and periodic function with period 2π for fixed time $t \in [t_0 - \delta, t_0]$. Let $z(x,t_0)$ be a curve satisfying the following properties:*

- *$z_1(x,t_0) - x$ and $z_2(x,t_0)$ are periodic with period 2π.*

- *$z(\zeta, t_0)$ is real for ζ real.*

- *$z(\zeta, t_0)$ is analytic in $\zeta \in \Omega(t_0)$.*

- *$z(\zeta, t) \in H^k(\Gamma_{\pm}(t_0))$ with k a large enough integer.*

- *Complex Arc-Chord condition.*

  $$|\cosh(z_2(\zeta, t_0) - z_2(w, t_0)) - \cos(z_1(\zeta, t_0) - z_1(w, t_0))| \geq [||\Re(\zeta - w)|| + |\Im(\zeta - w)|]^2,$$

  for $\zeta, w \in \overline{\Omega}(t_0)$, where $||x|| = distance(x, 2k\pi)$.

- *Generalized Rayleigh-Taylor condition: $RT(\zeta, t_0) > 0$, where*

  $$RT(\zeta, t) = \Re\left(\frac{-2\pi\partial_{\zeta}z_1(\zeta, t)}{(\partial_{\zeta}z_1(\zeta, t))^2 + (\partial_{\zeta}z_2(\zeta, t))^2}(1 + i\partial_x h(\Re\zeta, t))^{-1}\right)$$

  $$+\Im\left(\left\{P.V.\int_{w\Gamma_+(t)}\frac{\sin(z_1(\zeta, t) - z_1(w, t))}{\cosh(z_2(\zeta, t) - z_2(w, t)) - \cos(z_1(\zeta, t) - z_1(w, t))}dw + i\partial_t h(\zeta, t)\right\}\right.$$

  $$\left.\times(1 + i\partial_x h(\Re\zeta, t))^{-1}\right)$$

Then, for small enough δ, there exist a solution for the equation (17) in the time interval $t \in [t_0 - \delta, t_0]$ satisfying

- *$z_1(x,t) - x$ and $z_2(x,t)$ are periodic with period 2π.*

- *$z(\zeta, t)$ is real for ζ real.*

- *$z(\zeta, t)$ is analytic in $\zeta \in \Omega(t_0)$.*

- *$z(\zeta, t) \in H^k(\Gamma_{\pm}(t))$ with k a large enough integer.*

Now, let $\underline{z}(x,t)$ be the solution of the Muskat problem with $\underline{z}(x,0) = \underline{z}^0(x)$, where $\underline{z}^0(x)$ is the particular initial data from part 3. of the section 2. We shall define this solution as the unperturbed solution. Let us denote the Rayleigh-Taylor function

$$\sigma_1^0(x,t) \equiv \frac{-2\pi\partial_x\underline{z}_1(x,t)}{(\partial_x\underline{z}_1(x,t))^2 + (\partial_x\underline{z}_2(x,t))^2}.$$

Notice the minus sign in the right-hand side of the previous expression. One can check the following properties of this Rayleigh-Taylor function:

1. $\sigma_1^0(\cdot, t)$ is analytic on $\{x + iy : x \in \mathbb{T}, |y| \le c_b\}$ with $|\sigma_1^0(x + iy, t)| \le C$, for all $x + iy$ as above and for all $t \le [0, \tau]$.

2. $\sigma_1^0(0, 0)$ is real for $x \in \mathbb{T}$, $t \in [0, \tau]$.

3. σ_1^0 has a priori bounded C^{k_0} norm as a function of $(x, t) \in \mathbb{T} \times [0, \tau]$ (k_0 large enough).

4. $\sigma_1^0(0, 0) = 0$.

5. $\partial_x \sigma_1^0(0, 0) = 0$.

6. $\partial_x^2 \sigma_1^0(0, 0) = -c_2 < 0$.

7. $\partial_t \sigma_1^0(0, 0) = c_1 > 0$.

In this setting we define the following weight functions

$$h(x, t) = A^{-1}(\tau^2 - t^2) + (A^{-1} - (\tau - t))\sin^2\left(\frac{x}{2}\right) \quad \text{for } t \in [\tau^2, \tau]. \tag{18}$$

$$\hbar(x, t) = \frac{1}{4}\left(A^{-1}\tau^2 + A^{-1}\sin\left(\frac{x}{2}\right)\right) + A^{-2}\tau t + At\sin\left(\frac{x}{2}\right) \quad t \in [0, \tau^2]., \tag{19}$$

with $x \in \mathbb{T}$. First we choose the parameters A large enough and then τ small enough, then one can show that

$$\sigma_1^0(x, t) + \partial_t h(x, t) - A^{\frac{1}{2}} h(x, t) \ge c\tau^2 \quad \text{for } x \in \mathbb{T}, \, t \in [\tau^2, \tau] \tag{20}$$

and

$$\sigma_1^0(x, t) + \partial_t \hbar(x, t) - A^{\frac{1}{2}} \hbar(x, t) \ge \frac{1}{2}A^{-2}\tau \quad \text{for } x \in \mathbb{T}, \, t \in [0, \tau^2]. \tag{21}$$

The inequalities (20) and (21) are one of the main ingredients of the proof of the following results

Theorem 4.2 *Let $z(x, t)$ be a solution of the Muskat equation in the interval $t \in [0, \tau]$. Let $h(x, t)$ and $\hbar(x, t)$ as in the expressions (18) and (19) and k a large enough integer. Assume that $z(x, t)$ satisfies*

- *$z_1(x, t) - x$ and $z_2(x, t)$ are periodic with period 2π.*

- *$z(\zeta, t)$ is real for ζ real.*

- *$z(\zeta, t)$ is analytic in $\zeta \in \Omega(t)$.*

- *$z(\zeta, t) \in H^k(\Gamma_\pm(t))$ with k a large enough integer.*

- *Complex Arc-Chord condition.*

  $$|\cosh(z_2(\zeta, t) - z_2(w, t)) - \cos(z_1(\zeta, t) - z_1(w, t))| \ge [||\Re(\zeta - w)|| + |\Im(\zeta - w)|]^2,$$

  for $\zeta, w \in \overline{\Omega}(t)$.

11

Here in the definition of $\Omega(t)$ and $\Gamma_{\pm}(t)$ we use $h(x,t)$ if $t \in [\tau^2, \tau]$ and $\hbar(x,t)$ if $t \in [0, \tau^2]$. Then

$$\frac{1}{2}\frac{d}{dt}\left(\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)\right|^2 d\Re\zeta\right) \geq -C(A)\lambda^2,$$

if $t \in [\tau^2, \tau]$

$$\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)\right|^2 d\Re\zeta \leq \lambda^2$$

and $\lambda \leq \tau^{50}$.

In addition

$$\frac{1}{2}\frac{d}{dt}\left(\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)\right|^2 d\Re\zeta\right) \geq -C(A)\tau^{-1}\lambda^2,$$

if $t \in [0, \tau^2]$

$$\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)\right|^2 d\Re\zeta \leq \lambda^2$$

and $\lambda \leq \tau^{50}$.

This theorem implies that for all $\gamma > 0$ there is $\varepsilon > 0$ such that

$$\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)\right|^2 d\Re\zeta \leq \gamma$$

for $t \in [0, \tau]$ if

$$\int_{w \in \Gamma_+(t)} \left|\partial_\zeta^k z(\zeta,\tau) - \partial_\zeta^k \underline{z}(\zeta,\tau)\right|^2 d\Re\zeta \leq \varepsilon$$

and $z(x,t)$ satisfies the requirements of the theorem.

Lemma 4.3 *Let $z(x,t)$ be a solution of the Muskat problem satisfying the requirements of theorem (4.2) and close enough to the unperturbed solution in $t \in [0, \tau]$. Let $h(x,t)$ and $\hbar(x,t)$ be as in (18) and (19) with a suitable choice of A and τ. Then $z(x,t)$ satisfies the generalized Rayleigh-Taylor condition in $t \in [0, \tau]$. In particular the unperturbed solution satisfies the generalized Rayleigh-Taylor condition in $t \in [0, \tau]$.*

Theorems (4.1), (4.2) and lemma (4.3) allow us to achieve the desired result. Indeed we can choose a curve $z(x, \tau)$ such that

$$\int_{\zeta \in \Gamma_\pm} \left|\partial_\zeta^k z(\zeta,\tau) - \partial_\zeta^k \underline{z}(\zeta,\tau)\right|^2 d\Re\zeta \leq \varepsilon,$$

with $0 < \varepsilon < \varepsilon_0$ (ε_0 small enough), satisfying the generalized Rayleigh-Taylor condition by lemma (4.3) and satisfying the rest of the hypothesis of theorem (4.1). Since $h(0, \tau) = 0$, $z(x,t)$ is allow to be no analytic at $x = 0$ (maybe $z(x, \tau) \in H^k(\mathbb{T})$ but $z(x, \tau) \notin H^{k+1}(\mathbb{T})$). By theorem (4.1) there is a solution $z(x, t)$, analytic in $\Omega(t)$, for some interval $t \in [\tau - \delta, \tau]$

12

with small enough δ and for all ε. By theorem (4.2), we can choose ε small enough in such a way that, by lemma (4.3), $z(x, \tau - \delta)$ satisfies the generalized Rayleigh-Taylor condition. Then we can go further the time $\tau - \delta$. Iterating this argument, we find we can extend $z(x, t)$ to be a solution of the Muskat problem, analytic in $\Omega(t)$ for all $t \in [0, \tau]$ and as close as we want to the unperturbed solution.

Acknowledgements

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