Global existence, Singularities and Ill-Possednes for a Nonlocal Flux.

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Abstract

In this paper we study a one dimensional model equation with a non-local flux given by the Hilbert transform that is related with the complex inviscid Burgers equation . This equation arises in different contexts to characterize non-local and non-linear behaviors. We show global existence, local existence, blow up in finite time and ill-posedness depending on the sign of the initial data for classical solutions.

1 Introduction.

We consider the following non-local equation:

$$\partial_t f + (fHf)_x = 0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^+ \tag{1.1}$$

$$f(x,0) = f_0, \tag{1.2}$$

where Hf is the Hilbert transform of the function f, which is defined by the expression

$$Hf(x) = \frac{1}{\pi} P.V \int \frac{f(y)}{x-y} dy.$$

One particular feature of equation (1.1) is the relation with the Burgers equation. Applying the Hilbert transform over equation (1.1) yields (for more details see [1]),

$$\partial_t(Hf) + HfHf_x - ff_x = 0 \tag{1.3}$$

$$Hf(x,0) = Hf_0(x). \tag{1.4}$$

Multiplying (1.1) by $-i$, adding (1.3) and defining the complex valued function $z(x,t) = Hf(x,t) - if(x,t)$ we get the equation

$$\partial_t z + zz_x = 0$$

$$z(x,0) = z_0(x) \equiv Hf_0 - if_0,$$

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which is known as the inviscid Burgers equation.

In [2] the authors displayed equation (1.1) as a one dimensional model of the 2D Vortex Sheet problem using the ideas of [5]. It is known that a Vortex Sheet is a layer of vorticity distributed as a delta function that evolves according to the incompressible Euler equations. If we parameterize the curve, by $y(\alpha, t)$, where the vorticiy is concentrated, we obtain (see [?]),

$$\partial_t y(\alpha, t) = PV \int_c \gamma(\alpha')K_2[y(\alpha, t) - y(\alpha', t)]|y_\alpha(\alpha', 0)|d\alpha'$$
$$y(\alpha, 0) = y_0(\alpha),$$

where $\gamma(\alpha)$ is the initial vortex sheet strength and K_2 is the kernel

$$K_2(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2}.$$

In a different context, the equation (1.3) can be obtained, in a first approximation, from the dynamics of the interface between two fluids, one with no viscosity satisfying Euler equations and the other satisfying Stock's equations (for more details see the appendix).

Another motivation comes from the analogy that equation (1.1) has with the 2D quasi-geostrophic equation, which was studied in [1].

In [6] the authors studied the equation

$$f_t + \delta(fHf)_x + (1-\delta)f_xHf = 0 \quad (1.5)$$
$$f(x, 0) = f_0(x), \quad (1.6)$$

and they show formation of singularities for $0 < \delta < 1/3$ and $\delta = 1$. Other proof of the existence of singularities for equation (1.1) can be found in [2] (notice that in this paper the authors take a different sign for the Hilbert transform).

The equation (1.5) is also studied in [1] where the authors showed blow up for $0 < \delta \leq 1$. By an hodograph transformation an explicit solution for $\delta = 1$ is obtained over the torus, with mean zero analytic initial data. In addition they analyzed the equation

$$f_t + (fHf)_x = -\nu Hf_x \quad (1.7)$$
$$f(x, 0) = f_0(x), \quad (1.8)$$

and they showed that the solutions to this equation may also develop singularities with mean zero analytic initial data and with the condition $\nu < ||f_0||_{L^\infty}$. We will study this equation in section 4.

The structure of the paper is the following. In section 2 we show, for equation (1.1), global existence for all initial data strictly positive in the class $C^{0,\delta}(\mathbb{R}) \cup \L^2(\mathbb{R})$. In section 3 we study the case where the initial data have different sign and we prove ill-possedness in Sobolev spaces, $H^s(\mathbb{R})$, with $s > 3/2$. Finally, in section 4 we show local existence and blow up in finite time for equation (1.1) when the initial data f_0 is positive and there exists a point $x_0 \in \mathbb{R}$ such that $f_0(x_0) = 0$. In order to obtain the last result we study the equation (1.7) and we show global existence when the sum of the viscosity and the minimum of the initial data is larger than zero. Ill-posedness occurs when this sum is smaller than zero.

Now we will give some comments about the notation.

We will set $H^s(\mathbb{R})$, with $s \in \mathbb{R}$ to the usual Sobolev space,

$$H^s(\mathbb{R}) = \{f \;:\; \hat{f} \text{ is a function and}$$

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+\xi^2)^s d\xi < \infty\}.$$

We denote by Λ the operator $(-\Delta)^{\frac{1}{2}}$. This operator can be defined, using the Fourier transform by

$$(\Lambda f)\hat{}(\xi) = |\xi| \hat{f}(\xi), \tag{1.9}$$

and we will use the representation

$$\Lambda f(x) = \frac{1}{\pi} P.V \int \frac{f(x)-f(y)}{(x-y)^2} dy = Hf_x(x). \tag{1.10}$$

We recall the following pointwise inequality (see [3]) for $f \in H^2(\mathbb{R})$.

$$f\Lambda f \geq \frac{1}{2}\Lambda(f^2), \tag{1.11}$$

that will be used in the proofs below.

2 Global existence for strictly positive initial data.

In this section we study the equation (1.1) with initial data $f_0(x) > 0$, which imply that the solution will remain strictly positive, $f(x,t) > 0$. The main result is the following:

Theorem 2.1. *Let $f_0 \in L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})$, with $0 < \delta < 1$ and $f_0 > 0$ vanishing at infinity. Then there exits a global solution of equation (1.1) in $C^1((0,\infty]; Analytic)$ with $f(x,0) = f_0(x)$. Moreover, if $f_0 \in L^2(\mathbb{R}) \cap C^{1,\delta}(\mathbb{R})$ the solution is unique.*

Proof: We denote the upper half-plane by

$$M \equiv \{(x,y) \in \mathbb{R}^2 \;:\; y > 0\}$$

and the upper half-plane including the real axis by

$$\overline{M} \equiv \{(x,y) \in \mathbb{R}^2 \, : \, y \geq 0\}.$$

Let

$$P_y(x) \equiv \frac{1}{\pi}\frac{y}{y^2+x^2} \quad \text{and} \quad R_y(x) = \frac{1}{\pi}\frac{x}{y^2+x^2},$$

be the Poisson kernel and the conjugate Poisson kernel respectively. Then we denote the convolutions of a function, $f(x)$, with these kernels by

$$Pf(x,y) = (P_y * f)(x,y) \quad \text{and} \quad Rf(x,y) = (R_y * f)(x,y),$$

We recall that with this notation the complex function $g(x,y) = Pf(x,y) + iRf(x,y)$ is analytic on M and

$$\lim_{y\to 0^+} g(x,y) = f(x) + iHf(x). \tag{2.1}$$

Other properties of the Poisson kernel, that we will refer below, are the following:

- If $f \in L^2$, then

$$Rf(x,y) = PHf(x,y) \quad \text{on} \quad M$$

- If $f \in L^\infty$ and vanishing at infinity, then

$$\lim_{y\to +\infty} Pf(x,y) = 0 \quad \forall x \in \mathbb{R},$$

  and

$$\lim_{x\to \pm\infty} Pf(x,y) = 0 \quad \forall y \geq 0.$$

- If $f \in L^\infty$, then $Pf(x,y)$ is a bounded function on $\overline{M}$.

Consider the equation

$$\partial_t z(x,t) + z(x,t)z_x(x,t) = 0 \tag{2.2}$$
$$z(x,0) = z_0(x) \equiv Hf_0(x) - if_0(x), \tag{2.3}$$

where $z(x,t) = g(x,t) - if(x,t)$ is a complex valued function and g and $-f$ its the real and imaginary parts respectively. Equation (2.2) written in components can be read

$$\partial_t f + (gf)_x = 0 \quad f(x,0) = f_0(x) \tag{2.4}$$
$$\partial_t g + gg_x - ff_x = 0 \quad g(x,0) = Hf_0(x). \tag{2.5}$$

We set the inviscid complex Burgers equation on the upper half plane

$$\begin{aligned} \partial_t Z(w,t) + Z(w,t)Z_w(w,t) &= 0 \quad \text{on } \overline{M} \quad (w = x+iy) && (2.6)\\ Z(w,0) = Z_0(w) &\equiv Rf_0(x,y) - iPf_0(x,y), && (2.7) \end{aligned}$$

where $Z_0(w)$ is an analytic function over M, $\lim_{y\to 0^+} Z_0(w) = Hf_0 - iPf_0$ and $f_0 > 0$.

We introduce the complex trajectories

$$X(w,t) = Z_0(w)t + w, \tag{2.8}$$

which in components reads

$$\begin{aligned} X_1(x,y,t) &= Rf_0(x,y)t + x && (2.9)\\ X_2(x,y,t) &= -Pf_0(x,y)t + y. && (2.10) \end{aligned}$$

Thus, if $Z_0(w)$ is analytic in w_0 and $\frac{dX}{dw}(w_0,t) \neq 0$ we can define the function

$$Z(\alpha,t) = Z_0(X^{-1}(\alpha,t)),$$

with the property to be analytic on an open neighborhood of $X(w_0,t)$ and $Z(\alpha,0) = Z_0(\alpha)$. Consequently

$$Z(X(w_0,t),t) = Z_0(w_0).$$

Therefore,

$$\begin{aligned} \frac{dZ(X(w_0,t),t)}{dt} &= 0 = \partial_t Z(X,t) + Z_0(w_0)Z_X(X,t)\\ &= \partial_t Z(X,t) + Z(X,t)Z_X(X,t). \end{aligned}$$

Follows immediately that $Z(\alpha,t)$ is a solution of the complex inviscid Burgers equation on a neighborhood of $X(w_0,t)$.

Now we shall show that there exists a suitable analytic inverse function for the problem. First we prove the following lemma:

Lemma 2.2. *For all $(X_1,X_2) \in \overline{M}$ there exist a unique pair $(x,y) \in M$ such that (2.9) and (2.10) holds for all $t > 0$. In addition, if $X(w) \in \overline{M}$ then $(1 + t\partial_x R(x,y)) \neq 0$.*

By fixing $X_2 \geq 0$ and $t > 0$, for all $x \in \mathbb{R}$, there exist a point $y > 0$ such that the equation (2.10) holds. This is true since $Pf_0(x,y) > 0$ is bounded and we have that $y > X_2$. Now we will prove that this value y is unique by a contradiction argument:

Let us suppose that there exist $y_1 > y_2$ such that

$$\begin{aligned} y_1 - X_2 &= Pf_0(x,y_1)t\\ y_2 - X_2 &= Pf_0(x,y_2)t. \end{aligned}$$

Dividing both expressions we have

$$\frac{Pf_0(x,y_1)}{y_1 - X_2} = \frac{Pf_0(x,y_2)}{y_2 - X_2},$$

which is a contradiction since the product

$$\frac{y}{y - X_2} \cdot \frac{1}{y^2 + (x-s)^2}$$

is a decreasing function with respect to y (for $y > X_2$) and $f_0 > 0$. We will denote by $y_{X_2}(x)$ to be the solution of the equation (2.10) with fixed $X_2 \geq 0$, $t > 0$ and $x \in \mathbb{R}$ (the time dependence will be omitted), hence

$$y_{X_2}(x) - X_2 = tPf_0(x, y_{X_2}(x)) \tag{2.11}$$

Differentiating implicitly the expression (2.11) with respect to x (fixed X_2) we obtain

$$\frac{dy_{X_2}(x)}{dx} = \frac{\partial_x Pf_0(x, y_{X_2}(x))}{1 - t\partial_y Pf_0(x, y_{X_2}(x))}. \tag{2.12}$$

Since $f_0 \in C^{0,\delta} \cap L^2$ follows that $Hf_0 \in L^\infty$ and therefore $Rf_0(x,y)$ is a bounded function over $\overline{M}$. Furthermore

$$\lim_{x\to\pm\infty} Rf_0(x, y_{X_2}(x)) + x = \pm\infty \quad \forall X_2 \geq 0.$$

In addition, by differentiating with respect to x the expression

$$X_1(x, y_{X_2}(x)) = Rf_0(x, y_{X_2}(x))t + x,$$

and using Cauchy-Riemann equations

$$\begin{aligned} \partial_x Pf_0(x,y) &= \partial_y Rf_0(x,y) \\ \partial_y Pf_0(x,y) &= -\partial_x Rf_0(x,y), \end{aligned}$$

we obtain from (2.12)

$$\frac{dX(x, y_{X_2}(x))}{dx} = \frac{(1 + t\partial_x Rf_0(x, y_{X_2}(x)))^2 + (\partial_x Pf_0(x, y_{X_2}(x)))^2}{1 + t\partial_x Rf_0(x, y_{X_2}(x))}.$$

In the next step we shall prove that if $X_2 \geq 0$ then $0 < \frac{dX(x, y_{X_2}(x))}{dx} < \infty$, which is equivalent to show that

$$1 + t\partial_x Rf_0(x,y) \neq 0 \;\; \forall (X_1, X_2) \in \overline{M}. \tag{2.13}$$

Suppose that for $t > 0$ we have $1 + t\partial_x Rf_0(x,y) = 0$. Then

$$\partial_x Rf_0(x,y) < 0 \quad \text{and} \quad t = \frac{-1}{\partial_x Rf_0(x,y)}.$$

The time employed by the trajectory $X(x+iy,t)$ to reach the real axis, $X^2=0$, is

$$t_r = \frac{y}{Pf_0(x,y)},$$

On the other hand we have

$$-\partial_x Rf_0(x,y) = (\partial_x R_y * f_0)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{-y^2+(x-s)^2}{(y^2+(x-s)^2)^2} f_0(s)ds$$

$$< \frac{1}{\pi}\int_{\mathbb{R}} \frac{y^2+(x-s)^2}{(y^2+(x-s)^2)^2} f_0(s)ds = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(y^2+(x-s)^2)} f_0(s)ds = \frac{Pf_0(x,y)}{y}.$$

We now observe that on the hypothetical points $(x,y)\in M$ where $\partial_x Rf_0(x,y) < 0$ satisfies

$$\frac{-1}{\partial_x Rf_0(x,y)} > \frac{y}{Pf_0(x,y)}. \tag{2.14}$$

Hence, we have shown that $t_r < t$ and consequently $1 + t\partial_x Rf_0(x,y) \neq 0\ \forall (X_1,X_2)\in \overline{M}$. The lemma (2.2) is proved.

By the Complex Variable Inverse Function Theorem and by lemma (2.2), there exist an analytic inverse function, $X^{-1}(w,t)$ and an open set $O_t \subset \mathbb{C}$, time dependence, such that

$$X^{-1}(\cdot,t)\ :\ O_t \to M,$$

with $\overline{M}\subset O_t \quad \forall t>0$.

In fact, $Z(w,t) = Z_0(X^{-1}(w,t))$ is an analytic function that satisfies the complex inviscid Burgers equation over $\overline{M}$ and $Z(w,0)=Z_0(w)$. Furthermore $Z(w,t)$ vanishes at infinity $\forall t$ and the restriction $z(x,t)=Z(x,t)$ satisfies (2.2).

Note that the real part of $Z(w,t)$, $\Re Z(w,t)$, is a harmonic function, vanishing at infinity and with an analytic restriction to the real axis, $\Re z(x,t)$. But $P\Re z(x,y,t)$ is a harmonic function with restriction to the real axis equal to $\Re z(x,t)$ and vanishes at infinity. Then $\Re Z(w,t) = P\Re z$ and by unicity of harmonic conjugate we can write

$$z(x,t) = H\Re z(x,t) - i\Re z(x,t).$$

The proof of the existence follows from substituting the previous expression in equations (2.4) and (2.5).

In order to prove uniqueness we will suppose that f^1 and f^2 are two positive solutions of the equation(1.1) such that $f^1(x,0)=f^2(x,0)=f_0(x)\in C^{1,\delta}\cap L^2$. Then the difference, $d=f^1-f^2$ satisfies the equation,

$$\partial_t d + Hf^1_x d + f^2\Lambda d + Hf^1 d_x + f^2_x Hd = 0. \tag{2.15}$$

Multiplying by d and integrating over $\mathbb{R}$ we obtain,

$$\frac{d||d||^2_{L^2}}{dt} \leq C(||Hf^1_x||_{L^\infty} + ||Hf^2_x||_{L^\infty} + ||f^2_x||_{L^\infty})||d||^2_{L^2},$$

where we have integrated by parts and we have used inequality (1.11). By applying Grönwall's inequality uniqueness follows.

3 Ill-posedness for an initial data with different sign.

In this section we analyze the existence of solutions of the equation (1.1) for an initial data, which has positive and negative values.

We denote f^- to be the negative part of a function f. Let N_f be the set of points where a function f is strictly negative, i.e.

$$N_f \equiv \{x \in \mathbb{R} \ : \ f(x) < 0\},$$

and $|N_f|$ will denote its Lebesgue's measure. The theorem that we shall prove is the following:

Theorem 3.1. *Let $f_0 \in H^s$, with $s > \frac{3}{2}$ and $|N_{f_0}| \neq 0$. Then, if $f_0(x)$ is not C^∞ in a point $x_0 \in N_{f_0}$, there is no solution of equation (1.1), satisfying $f(x,0) = f_0(x)$, in the class $f \in C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$, with $s > \frac{3}{2}$ and $T > 0$. In addition, $f_0 \in C^\infty$ is not sufficiency to obtain existence.*

Proof: We will proceed by a contradiction argument.

Let us suppose that there exist a solution of equation (1.1) in the class $C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$ with $f(x,0) = f_0(x)$.

Taking the Hilbert transform on equation (1.1) yields

$$\partial_t Hf + HfHf_x - ff_x = 0.$$

Now we define the complex valued function $z(x,t) = Hf(x,t) - if(x,t)$, that satisfies

$$\partial_t z + zz_x = 0.$$

We set the complex function $Z(x,y,t)$ by the expression (we omit the time dependence):

$$Z(x,y,t) = Rf(x,y) - iPf(x,y) = P(Hf - if)(x,y),$$

so that, $Z(x,y,t) = Z(w,t)$ ($w = x + iy$) is an analytic function on M. Now we shall prove that this function satisfies the complex Burgers equation on M. In order to do that, we take the time derivative of Z

$$\partial_t Z = \partial_t Pz = P(\partial_t z) = -P(zz_x).$$

Now we have to check that $Pz\partial_x Pz = P(zz_x)$. We note that the restriction of the function $P(zz_x)$ and $PzPz_x$ is the same, namely zz_x. In addition both functions have the same behavior at infinity. Taking modules:

$$|Pzz_x| \leq P(|z||z_x|) \leq (|Hf_x|^2_{L^\infty} + |f_x|^2_{L^\infty})^{\frac{1}{2}} P(|z|)$$
$$|PzPz_x| \leq (|Hf_x|^2_{L^\infty} + |f_x|^2_{L^\infty})^{\frac{1}{2}} P(|z|),$$

yields that both functions vanish at infinity.

Since Hf and $f \in L^\infty$ we have $Pz_x = \partial_x Pz$ and

$$PzPz_x = \frac{1}{2}((Pz)^2)_x.$$

Since the function $P(zz_x)$ is harmonic on M, we have to prove that $(Pz)^2$ is also harmonic on M. Applying the Laplacian operator we obtain

$$\frac{1}{2}\Delta((Pz)^2) = Pz\Delta(Pz) + (\partial_x Pz)^2 + (\partial_y Pz)^2$$

$$= (\partial_x Rf)^2 - (\partial_x Pf)^2 + (\partial_y Rf)^2 - (\partial_y Pf)^2 - 2i(\partial_x Pf \partial_x Rf + \partial_y Pf \partial_y Rf) = 0,$$

where we have used the Cauchy-Riemann equations. Therefore the analytic function on M, $Z(w,t)$, satisfies the inviscid complex Burgers equation,

$$\partial_t Z(w,t) + Z(w,t) Z_w(w,t) = 0 \quad \text{over } M$$
$$Z(w,0) = Z_0(w)$$

Let us define the complex trajectories

$$\frac{dX(w,t)}{dt} = Z(X(w,t),t)$$
$$X(w,0) = w = x + iy,$$

where we choose w so that $y > 0$ and $Pf_0(x,y) < 0$. For sufficiently small t, by Picard's Theorem, these trajectories exist and $X(w,t) \in M$. Therefore,

$$\frac{dZ(X(w,t),t)}{dt} = \partial_t Z(X,t) + Z_X(X,t) Z(X,t) = 0,$$

and we obtain that $X(w,t) = Z_0(w)t + w$.

Now we take a sequence $w^\epsilon = x + iy^\epsilon$ for each $x \in N_{f_0}$ such that $y^\epsilon > 0$, $Pf_0(x, y^\epsilon) < 0$ and $\lim_{\epsilon \to 0} y^\epsilon = 0$. Since $X(x,t) = Z_0(x)t + x \in M$, $\forall t > 0$, then

$$Z(Z_0(x)t + x, t) = \lim_{\epsilon \to 0} Z(X(w^\epsilon, t)) = \lim_{\epsilon \to 0} Z_0(w^\epsilon) = Hf_0(x) - if_0(x).$$

Taking one derivative respect to x in both sides of the previous equality we obtain

$$\frac{dZ(X(x,t),t)}{dX} = \frac{\frac{dZ_0(x)}{dx}}{1 + t\frac{dZ_0(x)}{dx}}.$$

Taking two derivatives we have the equation

$$\frac{d^2Z(X(x,t),t)}{(dX)^2} = \frac{\frac{d^2Z_0(x)}{dx^2}}{(1+t\frac{dZ_0(x)}{dx})^3}.$$

And for n-th derivatives,

$$\begin{aligned}\frac{d^nZ(X(x,t),t)}{(dX)^n} &= \frac{\frac{d^nZ_0(x)}{dx^n}}{(1+t\frac{dZ_0(x)}{dx})^{n+1}} + \text{lower terms in derivatives}\\ &= \frac{\frac{d^nHf_0(x)}{dx^n} - i\frac{d^nf_0(x)}{dx^n}}{(1+t\frac{dZ_0(x)}{dx})^{n+1}} + \text{lower terms in derivatives}\end{aligned}$$

Indeed, if on $x_0 \in N_{f_0}$ the n-th derivative of f_0 or Hf_0 is not continuous we get a contradiction. In addition, if $x_0 \in N_{f_0}$ and $f_0^{(n)}(x_0) = 0$ $\forall n$ but f_0 is not constant we have that

$$\begin{aligned}\frac{dZ(X(x,t),t)}{dX} = \frac{dZ(X(x,t),t)}{dX^1} &= \frac{d\Re Z(X^1,X^2,t)}{dX^1} + i\frac{d\Im Z(X^1,X^2,t)}{dX^1}\\ &= \frac{\frac{dHf_0(x)}{dx} - i\frac{df_0(x)}{dx}}{(1+t\frac{dHf_0}{dx}) - i\frac{df_0}{dx}}.\end{aligned}$$

Therefore

$$\frac{d\Im Z(X^1(x_0),X^2(x_0),t)}{dX^1} = 0.$$

Continuing this process we obtain that all derivatives satisfy

$$\frac{d^n\Im Z(X^1(x_0),X^2(x_0))}{(dX^1)^n} = 0.$$

But $\Im Z(x,y,t)$ is analytic on $(x,y) = (X^1(x_0),X^2(x_0))$, then $\Im Z(x,y,t)$ is constant over the line $y = X^2(x_0)$ and this is a contradiction. Similar result can be obtained if $x_0 \in N_{f_0}$ and $\frac{d^nHf(x_0)}{dx^n} = 0$ $\forall n$ but the initial data f_0 is not a constant.

4 Local existence and singularities for positive initial data.

The aim of this section is to prove local existence for equation (1.1) with positive initial data. Furthermore, we shall prove blow up in finite time if there exist $x_0 \in \mathbb{R}$ such that the initial data satisfies $f_0(x_0) = 0$.

The argument of the proof requires the introduction of a viscous term. The equation that we shall study is the following

$$f_t + (Hff)_x = -\nu Hf_x \tag{4.1}$$
$$f(x,0) = f_0(x), \tag{4.2}$$

where $\nu > 0$.

We will divide this study in two subsections. First we analyze the case $f_0 + \nu > 0$ and we will show global existence. In the second part we study the case $f_0 + \nu \geq 0$ and we will show local existence and blow up in finite time.

4.1 Global existence for $f_0 + \nu > 0$.

Here we shall prove the following result:

Theorem 4.1. *Let be $f_0 \in L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})$, with $0 < \delta < 1$ and $f_0 + \nu > 0$ vanishing at the infinity. Then there exits a global solution of equation (4.1) in $C^1((0,\infty]; Analytic)$ with $f(x,0) = f_0(x)$. Moreover, if $f_0 \in L^2(\mathbb{R}) \cap C^{1,\delta}(\mathbb{R})$ the solution is unique.*

Proof: This proof is essentially based on the proof of the (2.1)which we sketch below. The complex transport equation that we have to consider is the following:

$$\partial_t Z(w,t) + (Z(w,t) - i\nu)Z_w(w,t) = 0$$

$$Z(w,0) = Z_0(w) = Rf_0(x,y) - iPf_0(x,y).$$

In order to obtain global existence of this equation over $\overline{M}$ the only modification respect (2.1) is the inequality (2.14). In this case we find that

$$-\partial_x Rf_0(x,y) = \frac{1}{\pi}\int \frac{-y + (x-s)^2}{(y^2 + (x-s)^2)^2} f_0(s)ds$$

$$= \frac{1}{\pi}\int \frac{-y^2 + (x-s)^2}{(y^2 + (x-s)^2)^2}(f_0(s) + \nu)ds < \frac{P(f_0+\nu)(x,y)}{y},$$

where we have used

$$\int \frac{-y^2 + s^2}{(y^2+s^2)^2} ds = 0.$$

Remark 4.2. Ill-possedness occurs in equation (4.1) if the addition of the minimum of the initial data f_0 plus the viscosity is larger than zero. In fact, we have the following theorem:

Theorem 4.3. *Let $f_0 \in H^s$, with $s > \frac{3}{2}$ and $|N_{f_0+\nu}| \neq 0$. Then, if $f_0(x)$ or $Hf_0(x)$ is not C^∞ in a point $x_0 \in N_{f_0+\nu}$, there is no solution of equation (1.1), satisfying $f(x,0) = f_0(x)$, in the class $f \in C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$, with $s > \frac{3}{2}$ and $T > 0$. In addition, f_0 analytic in all point where $f_0 + \nu < 0$ is not sufficiency to obtain existence.*

Proof: The proof follows the steps of the theorem (4.3).

4.2 Local existence in the limit case.

In this subsection we analyze the equation (4.1) with initial data $f_0 \geq -\nu$. We will use energy estimates and the techniques used in the article [1] for the control of the L^∞-norm of the solutions. The theorem that we shall prove is the following:

Theorem 4.4. *Let $f_0 \in H^2$ and*

$$m_0 \equiv min_x f_0(x) \leq 0,$$

such that $m_0 + \nu = 0$. Then there exits a time $T > 0$ such that the equation (4.1) has a unique solution in $C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; H^1(\mathbb{R}))$ with $f(x,0) = f_0(x)$.

Remark 4.5. In the case $\nu = 0$ this theorem asserts local existence of solutions of equation (1.1) in the class $C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; H^1(\mathbb{R}))$ when the initial data $f_0(x) \geq 0$.

Proof: By theorem (4.1) we consider global solutions of the equation

$$f_t + (Hff)_x = -\epsilon H f_x \tag{4.3}$$

$$f(x,0) = f_0(x) \in H^2 \quad \text{and} \quad \epsilon + m_0 > 0, \tag{4.4}$$

First we will compute two estimates of the L^∞-norm of f_x and Hf_x which are uniform with respect to $\epsilon > -m_0$

Lemma 4.6. *Let $f_0 \in H^2$, with $m_0 \equiv \min_{x\in\mathbb{R}} f_0(x) \leq 0$ and $m_0 + \epsilon > 0$. Let f be the solution of equation (4.3) given by the theorem (4.1). Then, if we define*

$$m(t) = \min_{x\in\mathbb{R}} f(x,t),$$

we have that

$$m(t) + \epsilon > 0 \quad \forall t \geq 0.$$

Proof: From theorem (4.1) we know that $f \in C^1([0,\infty) \times \mathbb{R})$, in particular the function $m(t)$ is differentiable almost every t. There always exist a point $x_m \in \mathbb{R}$ (which depends on t) such that $m(t) = f(x_m(t), t)$. Using the same argument as [4] we obtain

$$m'(t) = f_t(x_m(t), t) \quad \text{at almost every } t.$$

Therefore

$$m'(t) = -Hf(x_m(t),t) f_x(x_m(t),t) - Hf_x(x_m,t)(f(x_m(t),t) + \epsilon)$$

$$= -Hf_x(x_m(t),t)(m(t) + \epsilon).$$

an by integrating

$$(m(t)+\epsilon) = (m_0+\epsilon)\exp(-\int_0^t Hf_x(x_m(\tau),\tau)d\tau).$$

Since

$$Hf_x(x) = \frac{1}{\pi}P.V.\int \frac{f(x)-f(y)}{(x-y)^2}dy,$$

we have that $-Hf_x(x_m(t),t) \geq 0$. Therefore

$$m(t)+\epsilon \geq m_0+\epsilon > 0.$$

Lemma 4.7. *Let $f_0 \in H^2$, with $m_0 \equiv \min_{x\in\mathbb{R}} f_0(x) \leq 0$ and $m_0+\epsilon > 0$. Let f be the solution of equation (4.3) given by theorem (4.1). Then, if we define*

$$\begin{aligned}
m(t) &= \min_{x\in\mathbb{R}} f_x(x,t) = f_x(x_m(t),t)\\
M(t) &= \max_{x\in\mathbb{R}} f_x(x,t) = f_x(x_M(t),t)\\
j(t) &= \min_{x\in\mathbb{R}} Hf_x(x,t) = Hf_x(x_j(t),t)\\
J(t) &= \max_{x\in\mathbb{R}} Hf_x(x,t) = f_x(x_J(t),t)
\end{aligned}$$

we have that

$$\begin{aligned}
m(t) &\geq \frac{m(0)}{(1+j(0)t)^2}\\
M(t) &\leq \frac{M(0)}{(1+j(0)t)^2}\\
j(t) &\geq \frac{j(0)}{1+j(0)t}\\
J(t) &\leq J(0) - \frac{3}{j(0)}\frac{M^2(0)}{(1+j(0)t)^3}.
\end{aligned}$$

Therefore

$$\sup_{t\in[0,T]} (||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty}) < \infty$$

$$\textit{if} \quad T < T_e \equiv -\frac{1}{j(0)} \quad \forall\epsilon > -m_0.$$

Proof: We know that $f_x \in C^1([0,\infty)\times\mathbb{R})$ and $Hf_x \in C^1([0,\infty)\times\mathbb{R})$. Therefore

$$\begin{aligned}
m'(t) = \partial_t fx(x_m(t),t) \quad M'(t) = \partial_t f_x(x_M(t),t)\\
j'(t) = \partial_t Hf_x(x_j(t),t) \quad J'(t) = \partial_t Hf_x(x_J(t),t).
\end{aligned}$$

and

$$j'(t) = -j^2(t) + (f_x)^2(x_j(t),t) + (f(x_j(t),t)+\epsilon)f_{xx}(x_j(t),t).$$

Using the following representation

$$f_{xx}(x) = \frac{1}{\pi}P.V\int \frac{Hf_x(y)-Hf_x(x)}{(x-y)^2}dy,$$

we have that $f_{xx}(x_j(t),t) \geq 0$. Since $j(t)$ is negative yiels

$$j(t) \geq \frac{j(0)}{1+tj(0)}.$$

Now we shall study the evolution of $m(t)$.

$$m'(t) = -m(t)Hf_x(x_m(t),t) - (f(x_m(t),t)+\epsilon)Hf_{xx}(x_m(t),t).$$

Since

$$Hf_{xx}(x) = \frac{1}{\pi}\int \frac{f_x(x)-f_x(y)}{(x-y)^2}dy,$$

we have that $Hf_{xx}(x_m(t),t) \leq 0$. We know that $m(t) < 0$ then

$$m'(t) \geq -2m(t)Hf_x(x_m(t),t) \geq -2m(t)j(t),$$

and the following inequality holds

$$m(t) \geq \frac{m(0)}{(1+j(0)t)^2}.$$

Operating in a similar way we obtain that

$$M(t) \leq \frac{M(0)}{(1+j(0)t)^2}.$$

Finally we have that

$$J'(t) = -J^2(t) + (f_x)^2(x_J(t),t) + (f(x_J(t),t)+\epsilon)f_{xx}(x_J(t),t) \leq M^2(t).$$

Therefore

$$J(t) \leq J(0) - \frac{3}{j(0)}\frac{M^2(0)}{(1+j(0)t)^3}$$

Next, we shall check that the L^2-norm of f is bounded. Multiplying equation (1.1) by f and integrating over $\mathbb{R}$ we have

$$\frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} + \int_{\mathbb{R}}(fHf)_x f + \epsilon\int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} - \int_{\mathbb{R}} fHff_x dx + \epsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} - \frac{1}{2}\int_{\mathbb{R}} Hf(f^2)_x dx + \epsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} + \frac{1}{2}\int_{\mathbb{R}} Hf_x f^2 dx + \epsilon \int_{\mathbb{R}} Hf_x f dx = 0.$$

By the expression (1.10) we can estimate the last two terms of the equality

$$\int_{\mathbb{R}} Hf_x(x) f^2(x) dx = \frac{1}{\pi}\int_{\mathbb{R}} f^2(x) P.V \int \frac{f(x)-f(y)}{(x-y)^2} dydx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} P.V. \int \frac{f^2(x)(f(x)-f(y)) + f^2(y)(f(y)-f(x))}{(x-y)^2} dydx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} P.V. \int \frac{(f(x)+f(y))(f(x)-f(y))^2}{(x-y)^2} dydx,$$

and

$$\epsilon\int_{\mathbb{R}} Hf_x f dx = \frac{\epsilon}{\pi}\int_{\mathbb{R}} f(x) P.V \int \frac{f(x)-f(y)}{(x-y)^2} dydx$$

$$= \frac{\epsilon}{2\pi}\int_{\mathbb{R}} P.V \int \frac{f(x)(f(x)-f(y)) + f(y)(f(y)-f(x))}{(x-y)^2} dydx$$

$$= \frac{\epsilon}{\pi}\int_{\mathbb{R}} P.V \int \frac{(f(x)-f(y))^2}{(x-y)^2} dydx$$

Therefore

$$\frac{1}{2}\int_{\mathbb{R}} Hf_x f^2 dx + \epsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}} P.V \int \frac{(f(x)+f(y)+2\epsilon)(f(x)-f(y))^2}{(x-y)^2} dydx \geq 0,$$

and we can conclude

$$||f(t)||_{L^2} \leq ||f_0||_{L^2}$$

In addition we have one a priori estimate of L^2-norm of f_{xx}. Taking two derivatives to the equation (1.1), multiplying by f_{xx} and integrating by f_{xx} we obtain

$$\frac{1}{2}||f_{xx}(t)||_{L^2} = -\int_{\mathbb{R}} (fHf)_{xxx} f_{xx} dx - \epsilon \int_{\mathbb{R}} Hf_{xxx} f_{xx} dx = 0. \qquad (4.5)$$

All the terms of the right side of (4.5), except

$$-\int_{\mathbb{R}} (f+\epsilon) Hf_{xxx} f_{xx} dx, \qquad (4.6)$$

can be controlled in a simple way by

$$C(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})||f_{xx}||^2_{L^2}.$$

To bound (4.6) we will use the inequality (1.11) as follows

$$-\int_{\mathbb{R}} (f+\epsilon)Hf_{xxx}f_{xx}dx = -\int_{\mathbb{R}} (f+\epsilon)\Lambda f_{xx}f_{xx}dx$$

$$\leq \frac{-1}{2}\int_{\mathbb{R}} (f+\epsilon)\Lambda((f_{xx})^2)dx \leq \frac{1}{2}||Hf_x||_{L^\infty}||f_{xx}||^2_{L^2}.$$

Therefore

$$\frac{d||f_{xx}||^2_{L^2}}{dt} \leq C(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})||f_{xx}||^2_{L^2}.$$

Finally, integrating in time, we get the estimate

$$||f_{xx}(t)||_{L^2} \leq ||f_{0\,xx}||_{L^2} \exp(C \sup_{t\in[0,T]} (||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})T)$$

Indeed we have that the H^2-norm of f is bounded over $[0,T]$ with $T < T_e$ and this estimate is uniform in ϵ. The rest of the proof is straightforward . We take the approximating problems

$$\begin{aligned} f^\epsilon_t + (Hf^\epsilon f^\epsilon)_x &= -\epsilon Hf^\epsilon_x \\ f^\epsilon(x,0) &= f_0(x) \in H^2 \quad \text{and} \quad \epsilon + m_0 > 0, \end{aligned}$$

and using the established uniform estimates and the Compacity Rellich Theorem we obtain the existence of a solution in the class $C([0,T];H^2(\mathbb{R})) \cap C^1([0,T];H^1(\mathbb{R}))$ of the equation

$$\begin{aligned} f_t + (Hff)_x &= m_0 Hf_x \\ f(x,0) &= f_0(x) \in H^2 \end{aligned}$$

by taking the limit $\epsilon \to -m_0^+$.

Theorem 4.8. *Let $f_0 \in H^2$ and $f_0(x) \geq -\nu$ such that there exist a point x_0 where $f(x_0) = -\nu$. Then the solution $f(x,t)$ of the equation (4.1) given by theorem (4.4), develops a singularity in finite time.*

Remark 4.9. Theorem (4.8) assert blow up in finite time for equation (1.1) when the initial data is positive and there exist a point $x_0 \in \mathbb{R}$ such that $f_0(x_0) = 0$.

Proof: We proceed by a contradiction argument. Let us suppose, that given an initial data $f_0 \in H^2$, the solution of equation (1.1), by theorem (4.4), exist for all $t > 0$. We set the trajectory $X(x_0, t)$ by

$$\frac{dX(x_0,t)}{dt} = Hf(X(x_0,t),t)$$
$$X(x_0,0) = x_0.$$

This trajectory exist for sufficiently short time by Picard's Theorem. If we evaluate the solution over that trajectory we obtain

$$\frac{df(X(x_0,t),t)}{dt} = \partial_t f(X,t) + \frac{dX}{dt} f_x(X,t)$$
$$= -(f(X(x_0,t),t) + \nu) Hf_x(X(x_0,t),t).$$

Therefore,

$$f(X(x_0,t),t) + \nu = (f(X(x_0,0),0) + \nu)\exp(-\int_0^t Hf_x(X(x_0,\tau),\tau)d\tau) = 0.$$

Evaluating the Hilbert transform of f over that trajectory we obtain,

$$\frac{dHf(X(x_0,t),t)}{dt} = \partial_t Hf(X,t) + HfHf_x(X,t)$$
$$= (f(X(x_0,t),t) + \nu) f_x(X(x_0,t),t) = 0.$$

So that

$$Hf(X(x_0,t),t) = Hf_0(x_0),$$

and

$$X(x_0,t) = Hf_0(x_0)t + x_0.$$

If we evaluate the first derivative of the solution over that trajectory we get

$$\frac{df_x(X(x_0,t),t)}{dt} = \partial_t f_x(X,t) + Hf(X,t)f_{xx}(X,t)$$
$$= -2f_x(X(x_0,t),t)Hf_x(X(x_0,t),t)$$

Since $f_{0x}(x_0) = 0$, by the characteristics of f_0, yields

$$f_x(X(x_0,t),t) = f_x(X(x_0,0),0)\exp(-\int_0^t Hf_x(X(x_0,\tau),\tau)d\tau) = 0.$$

Finally we evaluate Hf_x over the trajectory and we obtain

$$\frac{dHf_x(X(x_0,t),t)}{dt} = \partial_t Hf_x(X,t) + Hf(X,t)Hf_{xx}(X,t)$$
$$= -(Hf_x(X(x_0,t),t))^2,$$

Which implies

$$Hf_x(X(x_0,t),t) = \frac{Hf_{0\,x}(x_0)}{1 + tHf_{0\,x}(x_0)}.$$

Moreover, we can write

$$Hf_{0\,x}(x_0) = \frac{1}{\pi}P.V \int \frac{f_0(x_0) - f_0(y)}{(x_0 - y)^2}dy = \frac{1}{\pi}P.V \int \frac{m_0 - f_0(y)}{(x_0 - y)^2}dy < 0,$$

if $f_0 \neq 0$.

Then $Hf_x(x,t)$ blow up at time $t = -(Hf_x(x_0))^{-1}$ at the point $x = Hf(x_0)t + x_0$.

5 Appendix.

Here we give, in a first approximation, the formulation of the dynamics of the interface between two fluids, one with no viscosity satisfying Euler equations and the other satisfying Stock's equations. We assume that the interface can be parameterize as $y(x,t) = (x_1, h(x_1,t))$. We locate the viscous fluid above of the curve $y(x_1,t)$ satisfying

$$\begin{aligned} \nu\Delta v + \nabla p_v &= 0 \\ \nabla \cdot v &= 0 \end{aligned}$$

where v is the velocity of the viscous fluid, p_v the pressure and $\nu > 0$ the viscosity. The ideal fluid, which is underneath of $y(x,t)$, satisfies

$$\begin{aligned} u_t + (u \cdot \nabla)u &= -\nabla p_u \\ \nabla \cdot u = 0, & \end{aligned}$$

where u is the velocity of the ideal fluid and p_u the pressure. As a contour condition over the interface we will impose the conservation of the normal component of the stress tensor, i.e

$$-p_u\vec{n} = -p_v\vec{n} + \nu T\vec{n}, \tag{5.1}$$

where all functions are evaluated over the interface, $\vec{n}$ is the normal vector to the interface and $T = \nabla v + (\nabla v)^t$. We take a initial condition where the vorticity is concentrated over the initial curve $y_0(x)$, i.e.

$$w_0(x) = \gamma_0(x_1)\delta(x - y_0(x_1)),$$

where $y_0(x) = y(x,0)$ separates both fluids. We will suppose that the solution of the problem continues being a delta distribution over the curve $y(x_1,t)$,

$$w(x,t) = \gamma(x_1,t)\delta(x - y(x_1,t)).$$

Therefore, using the Biot-Savart law, we obtain that the velocity of the total fluid V is

$$V(x) = \int_C \gamma(x_1) K_2(x - y(x_1,t)|y(x_1,t)|dx_1.$$

Under our hypothesis both viscous and ideal fluid are potentials, then

$$\begin{aligned} v &= \nabla\Psi \\ u &= \nabla\Phi, \end{aligned}$$

and the incompressible conditions provide

$$\Delta\Psi = 0 \quad \text{and} \quad \Delta\Phi = 0.$$

Therefore we have that,

$$p_v = 0 \tag{5.2}$$

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + p_u = 0. \tag{5.3}$$

Evaluating (5.3) over the interface and using (5.1) we get

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 = \nu\,\vec{n}T\vec{n}. \tag{5.4}$$

Finally we obtain the system of equations

$$\frac{d}{dt}(x_1(t), h(x_1(t),t)) = V(x_1(t), h(x_1(t),t),t) \tag{5.5}$$

$$x(0) = x_0 \quad h(x_0,0) = h_0(x_0) \tag{5.6}$$

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 = \nu\,\vec{n}T\vec{n} \quad \text{on the interface} \tag{5.7}$$

$$\gamma(x_1,0) = \gamma_0(x_1) \tag{5.8}$$

where

$$\begin{aligned} V(x) = \int_C \gamma(x_1) K_2(x - y(x_1,t)|y(x_1,t)|dx_1 \\ \nabla\Phi = V \quad \text{underneath the interface} \\ T = \nabla V + (\nabla V)^t \quad \text{above the interface} \end{aligned}$$

Conversely a solution of this system provides a solution of the equation

$$\begin{aligned} \Delta v = 0 \quad v \equiv V \quad \text{above the interface} \\ \partial_t u + (u\cdot\nabla)u = -\nabla p, \quad u \equiv V \quad \text{underneath the interface} \\ \nabla\cdot V = 0 \quad \text{except in the interface} \\ \nabla\times V = \gamma(x_1,t)\delta(x - y(x_1,t)) \\ p = -\nu\vec{n}T\vec{n} \quad \text{on the interface.} \end{aligned}$$

We study this problem, in a first approximation, despising the terms in $|h_{x_1}(x_1,t)|$, we obtain for the velocity $u = (u^1, u^2)$ in the interface of the ideal fluid

$$\begin{aligned} u^1 &\sim -\frac{1}{2}\gamma \\ u^2 &\sim \frac{1}{2}H\gamma. \end{aligned}$$

In addition, since (5.5) we have

$$\partial_t h(x_1,t) = u^2(x_1,t) - u^1(x_1,t)h_{x_1}(x_1,t) \sim u^2(x_1,t).$$

We denote the function $\zeta(x_1,t)$ by the expression

$$\zeta(x_1,t) = \Phi(x_1, h(x_1,t), t),$$

therefore

$$\zeta_{x_1}(x_1,t) = u^1(x_1,t) + u^2(x_1,t)h_{x_1}(x_1,t) \sim u^1(x_1,t).$$

Furthermore

$$\Phi_t = \zeta_t - u^2 h_t \sim \zeta_t - (u^2)^2.$$

Introducing these relations in (5.4) and differentiating with respect to x_1 we obtain

$$\partial_t u^1 - \frac{1}{2}((Hu^1)^2 - (u^1)^2)_x = \nu\partial_x(\vec{n}T\vec{n}).$$

Approximating the velocity in the interface given by the viscous fluid and defining $u^1 = Hf$ we get finally

$$Hf_t + HfHf_x - ff_x = \nu Hf_{xx},$$

which is equation (1.3) in the limit $\nu \to 0$.

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